Electronic Journal of Probability

Vol. 6 (2001) Paper no. 7, pages 1–22.

Journal URL
http://www.math.washington.edu/~ejpecp/
Paper URL
http://www.math.washington.edu/~ejpecp/EjpVol6/paper7.abs.html

FINITELY POLYNOMIALLY DETERMINED LÉVY PROCESSES

Arindam Sengupta
Stat-Math Unit, Indian Statistical Institute
203 B. T. Road, Calcutta 700035, India
arindam_v@isical.ac.in

Anish Sarkar
Stat-Math Unit, Indian Statistical Institute
Delhi Centre, 7 S. J. S. Sansanwal Marg, New Delhi 110016, India
anish@isid.ac.in

Abstract A time-space harmonic polynomial for a continuous-time process $X = \{X_t : t \geq 0\}$ is a two-variable polynomial P such that $\{P(t, X_t) : t \geq 0\}$ is a martingale for the natural filtration of X. Motivated by Lévy's characterisation of Brownian motion and Watanabe's characterisation of the Poisson process, we look for classes of processes with reasonably general path properties in which a characterisation of those members whose laws are determined by a finite number of such polynomials is available. We exhibit two classes of processes, the first containing the Lévy processes, and the second a more general class of additive processes, with this property and describe the respective characterisations.

Keywords Lévy process, additive process, Lévy's characterisation, Lévy measure, Kolmogorov measure

AMS subject classification 60G44, 60J30

Submitted to EJP on July 10, 2000. Final version accepted on August 30, 2000.

1 Introduction

The famous characterisation due to Paul Lévy for Brownian motion contained in Doob ([D], page 384, Theorem 11.9), for instance, says that if $X = (X_t)_{t\geq 0}$ is a martingale with continuous paths such that $X_t^2 - t$ is also a martingale, then (X_t) is the standard Brownian motion.

This characterisation may be restated as: for a process X with continuous paths, a certain pair of polynomials in t and X_t being martingales, determines the distribution of X uniquely, and in fact as that of a Brownian motion. We call such polynomials *time-space harmonic* for the process X concerned; an exact definition follows. It is well-known that all the two-variable Hermite polynomials

$$H_k(t,x) = (-t)^k e^{x^2/2t} \frac{\partial^k}{\partial x^k}(e^{-x^2/2t}), \quad k \geq 1,$$

of which x and $x^2 - t$ are the first two, are time-space harmonic for Brownian motion. P. McGill, B. Rajeev and B. V. Rao demonstrated in [MRR] that actually, any two of these determines Brownian motion, modulo the aforesaid continuous-path property, thereby generalising Lévy's characterisation.

Both these results depend crucially on the assumption of the continuity of the paths of the process. Indeed, if we remove this assumption, then there are several other processes, including the standard compensated Poisson and Gamma processes, which will also satisfy both the martingale conditions involved in Lévy's characterisation. We cite more examples of this kind in the final section.

A similar characterisation, referred to sometimes as Watanabe's characterisation [W], also exists for the standard Poisson process. This characterisation uses, in fact, only a single polynomial. The assumption that one imposes here is that the given process be a simple point process. This condition too, as one can see from the example of Brownian motion, cannot be dropped.

Both the conditions imposed on the aforementioned processes to achieve uniqueness of their respective laws can be seen as 'path properties' of the process. A natural question that arises, then, is the following : can one somehow render these 'path properties' inessential by specifying, if necessary, some more two-variable polynomials? Or at least, can one have a class of processes such that they are the only members of this class with a certain number of time-space harmonic polynomials specified?

Besides these two processes, there are also other processes of interest about which we may ask a similar question; namely, whether it is possible to characterise the gamma process, for instance, through a finite sequence of time-space harmonic polynomials of the process. More generally, our aim in this paper is to exhibit a class of processes, rich enough to contain all the three processes cited above, in which a characterisation of those members which are determined by finitely many time-space harmonic polynomials is available. Indeed, a class of processes which seems a very natural choice, in the sense that all the three are prominent examples of its members, suffices to achieve this end. This is the class of the familiar Lévy processes.

In [S2] (Theorem 6), it was proved that a whole (infinite) sequence of time-space harmonic polynomials always determines all the moments of a Lévy process. Even this, as a counterexample we present in the last section shows, does not necessarily characterise its distribution. The point of this paper is that for certain Lévy processes, which we describe fully, finitely many time-space harmonic polynomials are enough to determine its law completely. This also poses another interesting question for such processes, namely, what is the minimum number of time-space harmonic polynomials required to obtain a characterisation? We determine the number of polynomials required for such a characterisation in terms of the support of the Lévy measure of the process. We also extend our results to a more general class of additive processes.

Our characterization shows that the answer to the question we raise is in the affirmative for the first two examples and in the negative for the third one. These and other examples constitute the last section. A key ingredient of our proofs is the characterisation of finite measures on $\mathbb{R}$ which are determined by finitely many of its moments, interpreted as integrals of respective powers. This and the other necessary preparatory results are derived in the third section. The next section contains the basic definitions and the statements of the theorems, while the fourth is devoted to their proofs.

2 Definitions and Statements of Theorems

We start with an additive process X starting at 0, that is, $\{X_t : t \geq 0\}$ is a process with independent increments such that $X_0 = 0$. We also assume that X has no fixed time of discontinuity and r.c.l.l. (right continuous with left limits) paths. X is referred to as a Lévy process if it is homogeneous as a Markov process; in other words, if its increments are stationary ($X_{t+h} - X_t \stackrel{d}{=} X_h$) apart from being independent.

Let us denote by $(\mathcal{F}_t)_{t\geq 0}$ the natural filtration of the process X, that is, for $t \geq 0$, $\mathcal{F}_t = \sigma\!<\! X_u : 0 \leq u \leq t \!>$.

Before we proceed further, we recall the main definitions an some facts from [S1] and [S2].

Definition 2.1 *A time-space harmonic polynomial for a process X is a polynomial P in two variables with $P(0,0) = 0$, such that the process $\{Z_t := P(t, X_t); t \geq 0\}$ is an $\mathcal{F}_t$-martingale.*

We denote by $\mathcal{P} = \mathcal{P}(X)$, the vector space of all time-space harmonic polynomials for X with $P(0,0) = 0$. Define, for each fixed $k \geq 1$, the subspace

$$\mathcal{P}_k = \mathcal{P}_k(X) = \{P \in \mathcal{P} : P(t,x) \text{ is of degree } k \text{ in the variable } x\}.$$

Clearly, then, $\mathcal{P} = \cup_{k\geq 1}\mathcal{P}_k$.

For every fixed $k \geq 1$, a necessary and sufficient condition for $\mathcal{P}_j(X)$ to be non-empty for every j, $1 \leq j \leq k$, for an additive process X, was obtained in [S1] (Theorem 5.6), which we state below as Proposition 2.1. When X is a Lévy process, since this condition is trivially satisfied, for every $k \geq 1$, we have $\mathcal{P}_k(X) \neq \emptyset$ (provided, of course, $\mathbf{E}(|X_1|^k) < \infty$). This is recorded as Corollary 2.1.

We shall throughout assume that additive processes X under consideration satisfy what is known as the 'support condition'; namely, that for every positive integer k, there exists a $t \geq 0$ such that the support of X_t contains at least k points. This condition again is trivially satisfied by nondeterministic Lévy processes.

Proposition 2.1 *If X is an additive process with $\mathbf{E}|X_t|^k < \infty$ for each $t \geq 0$, then $\mathcal{P}_j(X) \neq \emptyset$ for each $1 \leq j \leq k$ if and only if $\mathbf{E}X_t^j$ is a polynomial in t fo $1 \leq j \leq k$.*

In this case, any sequence $\{P_j \in \mathcal{P}_j : 1 \leq j \leq k\}$ spans $\cup_{1 \leq j \leq k} \mathcal{P}_j$ and determines the first k moment functions of X.

Corollary 2.1 *For a Lévy process X with $\mathbf{E}|X_1|^k < \infty$, $\mathcal{P}_j \neq \emptyset$ for all $1 \leq j \leq k$. Any sequence $\{P_j \in \mathcal{P}_j : 1 \leq j \leq k\}$ spans $\cup_{1 \leq j \leq k} \mathcal{P}_j$ and determines the first k moment functions of X (or equivalently, the first k moments of X_1).*

Our point of departure comes with the question : when do $\{\mathcal{P}_j : 1 \leq j \leq k\}$ for some $k \geq 1$ determine the law of a Lévy process completely? For this we introduce the following definition.

Definition 2.2 *Given a class $\mathcal{C}$ of additive processes, an element X of $\mathcal{C}$ is called finitely polynomially determined (fpd for short) if for some $k \geq 1$, the classes $\mathcal{P}_j(X)$, $1 \leq j \leq k$ are each nonempty and together, determine the law of the process within $\mathcal{C}$. The subcollection consisting of fpd members of $\mathcal{C}$ is denoted $\tilde{\mathcal{C}}$.*

To be precise,

$$\begin{aligned} \tilde{\mathcal{C}} &= \{X \in \mathcal{C} : \text{ there exists } k \geq 1 \text{ such that } Y \in \mathcal{C}, \\ &\qquad \mathcal{P}_j(Y) = \mathcal{P}_j(X) \text{ for all } 1 \leq j \leq k \Rightarrow Y \stackrel{d}{=} X\}. \end{aligned}$$

Our aim in this paper is to present a couple of examples of $\mathcal{C}$ for which a complete description of its fpd members, or equivalently, of $\tilde{\mathcal{C}}$, available. The first is the class of the Lévy processes and the second a more general class. Although the former class is already contained in the latter, it is much simpler to treat this case separately. Besides, the method here is quite instructive, and suggests the line of attack for the more general case.

Clearly, if $\{\mathcal{P}_j : 1 \leq j \leq k\}$ determine the law of the process, then so do $\{\mathcal{P}_j : 1 \leq j \leq k+i\}$ for any $i \geq 1$. The natural question that arises now is : how many polynomials do we need to determine a fpd process? This motivates the next definition.

Definition 2.3 *An process X in $\tilde{\mathcal{C}}$ is called k-polynomially determined if k is the minimum number for which $\mathcal{P}_j$, $1 \leq j \leq k$ determines the law of X, i.e., $\{\mathcal{P}_j : 1 \leq j \leq k\}$ determines the law of X in $\mathcal{C}$ while $\{\mathcal{P}_j : 1 \leq j \leq k-1\}$ does not.*

Clearly we can not expect a process to be fpd unless at least $\mathcal{P}_2 \neq \emptyset$, since any two additive processes whose mean functions match will have the same $\mathcal{P}_1$ (and there are infinitely many such, since adding another independent 0-mean process leaves the mean function unchanged). In the sequel, we assume throughout that any process X under consideration satisfies $\mathbf{E}X_t^2 < \infty$ for all t.

The distribution of each X_t for a Lévy process X being infinitely divisible, its characteristic function (c.f.) admits a 'Lévy-Khintchine' representation (see [GK], page 76):

$$\log(\mathbf{E}(e^{i\alpha X_t})) = i\alpha\beta t - \frac{\alpha^2\sigma^2 t}{2} + t\int_{\mathbb{R}\setminus\{0\}} \left(e^{i\alpha u} - 1 - \frac{i\alpha u}{1+u^2}\right) l(du)$$

where $\beta \in \mathbb{R}$, $\sigma \geq 0$ and l is a measure having no mass at 0. This measure l is called the Lévy measure for the process X. Under our assumption on the finiteness of second moments, the c.f. has an alternative form:

$$\log(\mathbf{E}(e^{i\alpha X_t})) = i\alpha mt + t\int_{\mathbb{R}}\left(\frac{e^{i\alpha u}-1-i\alpha u}{u^2}\right)\eta(du) \tag{2.1}$$

where $m \in \mathbb{R}$ is the mean of the random variable X_1 and η is a finite measure, called the 'Kolmogorov measure'. The integrand is defined by the limiting value for $u = 0$. The explicit relation between the measures η and l is given by

$$\eta(A) = \sigma^2 1_A(0) + \int_{A\setminus\{0\}} u^2 dl(u) \tag{2.2}$$

where $A \in \mathcal{B}(\mathbb{R})$. It is immediate from the relation (2.2) that whenever η has finite support, so has l and vice versa. We now state

Theorem 2.1 *A Lévy process is fpd if and only if its Lévy measure, or equivalently, Kolmogorov measure is finitely supported.*

The next theorem gives the number of polynomials required to determine an fpd Lévy process X. This will in particular imply that this number is always even. For a Borel measure μ on the line, let $\mathrm{Card}(\mu)$ denote the cardinality of its support. No distinction is made here between countably infinite and uncountable cardinalities, and both are treated as $+\infty$.

Theorem 2.2 *A Lévy process X with Kolmogorov measure η is k-polynomially determined if and only if $k = 2\mathrm{Card}(\eta) + 2$.*

While the previous Theorem 2.1 follows as a consequence of the last one, it is inconvenient to try to prove the latter directly, bypassing the former, since it plays a crucial role in first ensuring that the Lévy measure or equivalently, the Kolmogorov measure is finitely supported. As the proof will bear out, only after this has been established is it easy to find the exact relationship between k and $\mathrm{Card}(\eta)$.

Now we turn to the more general case. In analogy to Lévy processes, the c.f. of a general additive process X also has a similar 'Lévy-Khintchine' representation

$$\begin{aligned} \log \mathbf{E}(e^{i\alpha X_t}) &= i\alpha\beta(t) - \frac{\alpha^2\sigma^2(t)}{2} - \int\left(e^{i\alpha u} - 1 - \frac{i\alpha u}{1+u^2}\right)L([0,t]\otimes du) \\ &= i\alpha m(t) + \int\left(\frac{e^{i\alpha u}-1-i\alpha u}{u^2}\right)K([0,t]\otimes du) \end{aligned} \tag{2.3}$$

where β, m and $\sigma \geq 0$ are continuous functions, the last also being increasing, and L and K are σ-finite Borel measures on $[0,\infty)\times\mathbb{R}$, called respectively the 'Lévy measure' and 'Kolmogorov measure' of X. Actually, it is not quite L and K but their 'derivatives' in a certain sense with respect to t, if they exist, that are the exact analogue to the Lévy measure and Kolmogorov measure for the homogeneous case, and might be considered more deserving candidates for those names.

In view of the above, $\mathcal{C}$, the class of processes that we consider for the general case consists of those additive processes for which the Kolmogorov measure admits a 'derivative' κ in the sense that κ is a transition measure on $[0,\infty) \times \mathcal{B}$ satisfying

$$K([0,t] \times A) = \int_0^t \kappa(s,A)ds \quad \forall\, t \geq 0,\ A \in \mathcal{B}.$$

In this case, it can be checked that moreover,

$$\int f dK = \int f(t,u)\kappa(t,du)dt \tag{2.4}$$

whenever $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is a function such that either side is well-defined. We designate κ as 'derivative measure' of K.

This class $\mathcal{C}$ is fairly rich, containing such examples as Gaussian additive processes, nonhomogeneous compound Poisson processes etc. Naturally, any additive process which arises as the independent sum of two processes of the above kinds, also belongs to $\mathcal{C}$. It should also be noted that $\mathcal{C}$ contains all Lévy processes, for which $\kappa(t,\cdot) \equiv \eta(\cdot)$. The necessary and sufficient condition characterising elements of $\tilde{\mathcal{C}}$ that we get would therefore naturally be expected to be in terms of the derivative measure $\kappa(\cdot,\cdot)$ It appears quite difficult even to formulate this, or a simila condition, in terms of the Kolmogorov measure K (or the Lévy measure L) instea of κ. In fact, this seems to be precisely the stumbling block in obtaining a characterisation, among all additive processes, of those which are fpd.

The characterisation in this case is that the Kolmogorov measure be supported on the graphs of finitely many measurable functions. In similarity to the homogeneous case, it is convenient to first show this fact, and only after establishing this, to give the exact relationship between the number of functions and number of polynomials.

Theorem 2.3 *Let X be an additive process of the class $\mathcal{C}$. Then*

(a) If there exists an $n \geq 0$ and a measurable function $(x_1,\ldots,x_n,p_1,\ldots,p_n) : [0,\infty) \to \mathbb{R}^n \times [0,\infty)^n$ such that

- *for each $0 \leq j \leq 2n$, $\sum_{i=1}^n p_i(t)\{x_i(t)\}^j$ is a polynomial in t almost everywhere, and*
- *$\kappa(t,du) = \sum_{i=1}^n p_i(t)\,\delta_{x_i(t)}(du)$, $t \geq 0$, is a version of the derivative measure,*

and $m(\cdot)$ is a polynomial, then X is fpd in $\mathcal{C}$.

(b) Conversely, suppose X is fpd in $\mathcal{C}$, and is determined by $\{P_j(X), 1 \leq j \leq k\}$. If κ is a version of the derivative measure associated with X, then there exists a measurable function $(x_1,x_2,\ldots,x_k,p_1,p_2,\ldots,p_k) : [0,\infty) \to \mathbb{R}^k \times [0,\infty)^k$ such that

$$\kappa(t,du) = \sum_{i=1}^k p_i(t)\,\delta_{x_i(t)}(du) \qquad \text{for almost every } t.$$

Further, this function has the property that for all j, $0 \leq j \leq k-2$,

$$\sum_{i=1}^n p_i(t)\{x_i(t)\}^j$$

is a polynomial for almost every t.

Here and in the sequel, when we say that a function is a polynomial in t almost everywhere, we mean the existence of a polynomial in t with which the stated function agrees for almost every $t \geq 0$.

In analogy with Theorem 2.2 of the homogeneous case, in the general case too the same relation between the cardinality of the support of the derivative measure and the number of polynomials required to determine the underlying process obtains. Here, since the derivative measure $\kappa(t,\cdot)$ depends on t, the relation turns out to involve the maximum value of $\mathrm{Card}(\kappa(t,\cdot))$ with positive measure. Let ℓ denote the lebesgue measure on the line.

Theorem 2.4 *Suppose X is an additive process and κ a version of its derivative measure. Then X is k-p.d. if and only if $k = 2n+2$, where for Lebesgue almost every $t \geq 0$, $\mathrm{Card}(\kappa(t,\cdot)) \leq n$ and for a set of t of positive Lebesgue measure, $\mathrm{Card}(\kappa(t,\cdot)) = n$.*

Theorem 2.4 essentially says that X is k-p.d. if and only if k is even and

$$\frac{k}{2} - 1 = \max\{j : \ell\{t : \mathrm{Card}(\kappa(t,\cdot)) = j\} > 0\}.$$

We show later that for every j, the set $\{t : \mathrm{Card}(\kappa(t,\cdot)) = j\}$ is Borel.

What do our results exactly mean for the underlying process? In the first case, it means that a Lévy process is fpd if, and only if, its jumps, when they occur, can have sizes only in a fixed finite set. For the Lévy measure (see Ito [I], page 145) is nothing but a multiple of the "jump distribution" of the process. However, finitely supported jump distributions conform to the definition as the distribution of the "first jump".

In the non-homogeneous case too, our result implies that jumps at any time t can take values in a finite set, but there is a difference; namely, that this set now depends on the time t. For, the form of the derivative of L we referred to earlier will also be the same as that of K, only, whereas the former puts no mass at $[0,\infty) \times \{0\}$, the latter will in general do so, unless the Gaussian part of the process is deterministic.

3 A Key Lemma

The key result that is used in proving the results is the following lemma. Recall that by the k-th moment of a finite measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ we mean the integral, provided it exists, of the function $x \mapsto x^k$ with respect to that measure.

Lemma 3.1 *A finite measure μ on $\mathbb{R}$ is determined by finitely many of its moments if and only if it is finitely supported. In such a case, the minimal number of moments (excluding the zeroth moment) that are required to determine μ is $2\mathrm{Card}(\mu)$.*

Proof : Suppose first that $\mathrm{Card}(\mu)$ is finite. Then it can be written as follows:

$$\mu = \sum_{i=1}^{n} p_i \delta_{r_i} \tag{3.5}$$

for some $n \geq 1$, where $p_i > 0$ for $i = 1, \ldots, n$, δ_a is the Dirac measure at the point a and r_i's are distinct real numbers. We show that $2n$ many moments determine the measure μ. Suppose that there exist another finite measure $\overline{\mu}$ such that

$$\int_{\mathbb{R}} u^j \mu(du) = \int_{\mathbb{R}} u^j \overline{\mu}(du) \quad \text{for } j = 0, 1, \ldots, 2n. \tag{3.6}$$

Further, from (3.5), we have

$$\int_{\mathbb{R}} \prod_{j=1}^{n} (u - r_i)^2 \mu(du) = \sum_{i=1}^{n} p_i \prod_{j=1}^{n} (r_i - r_j)^2 = 0.$$

The above integral can be expressed as a function of the constants $r_1, \ldots, r_n$ and the first $2n$ moments of μ only. Since $\int_{\mathbb{R}} u^j \mu(du) = \int_{\mathbb{R}} u^j \overline{\mu}(du)$ for $j = 0, 1, \ldots, 2n$, we have

$$\int_{\mathbb{R}} \prod_{j=1}^{n} (u - r_i)^2 \overline{\mu}(du) = 0.$$

But the function $f(u) = \prod_{j=1}^{n} (u - r_i)^2$ is a non-negative function vanishing only at the points $r_1, r_2, \ldots, r_n$, therefore we have

$$\overline{\mu}(\{r_1, \ldots, r_n\}^c) = 0.$$

In other words, $\overline{\mu}$ can be written as

$$\overline{\mu} = \sum_{i=1}^{n} p_i' \delta_{r_i}$$

where $p_i' \geq 0$ for $i \geq 0$. It remains to show $p_i' = p_i$, $1 \leq i \leq n$. Now, from the form of μ and $\overline{\mu}$, we have the following sets of linear equations:

$$\sum_{i=1}^{n} p_i r_i^j = \int_{\mathbb{R}} u^j \mu(du) \qquad \text{and} \qquad \sum_{i=1}^{n} p_i' r_i^j = \int_{\mathbb{R}} u^j \overline{\mu}(du), \quad j = 0, 1, \ldots, n-1.$$

This can be expressed as

$$\mathbf{A}\mathbf{p} = \mathbf{C} = \mathbf{A}\mathbf{p}'$$

where

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ r_1 & r_2 & \ldots & r_n \\ \vdots & \vdots & & \vdots \\ r_1^{n-1} & r_2^{n-1} & \ldots & r_n^{n-1} \end{pmatrix}, \mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix}, \mathbf{p}' = \begin{pmatrix} p_1' \\ p_2' \\ \vdots \\ p_n' \end{pmatrix}$$

$$\text{and} \quad \mathbf{C} = \begin{pmatrix} \int_{\mathbb{R}} \mu(du) \\ \int_{\mathbb{R}} u\mu(du) \\ \vdots \\ \int_{\mathbb{R}} u^{n-1}\mu(du) \end{pmatrix} = \begin{pmatrix} \int_{\mathbb{R}} \overline{\mu}(du) \\ \int_{\mathbb{R}} u\overline{\mu}(du) \\ \vdots \\ \int_{\mathbb{R}} u^{n-1}\overline{\mu}(du) \end{pmatrix}.$$

Since the matrix $\mathbf{A}$ is a non-singular (Vandermonde) matrix, we have $\mathbf{p}' = \mathbf{p}$. This completes the proof of the if part of the first statement.

For the only if part, suppose that μ is not finitely supported. Fix any $k \geq 1$. We exhibit another finite measure $\overline{\mu}$ such that the first $(k-1)$ moments and total measures of μ and $\overline{\mu}$ match.

Since the support of μ is not finite, we can pick $(k+1)$ distinct points $r_1, r_2, \ldots, r_k, r_{k+1}$ in the support of μ and take open neighbourhoods A_i of r_i, $i = 1, 2, \ldots, k+1$ with $\mu(A_i) > 0$ and $A_i \cap A_j = \emptyset$. Now consider the following real vector space of signed measures.

$$\mathcal{V} = \left\{ \nu : \nu(A) = \sum_{i=1}^{k+1} c_i \mu(A \cap A_i), c_i \in \mathbb{R}, A \in \mathcal{B}(\mathbb{R}) \right\}.$$

This clearly has dimension $k+1$. Define now the linear map $\Lambda : \mathcal{V} \to \mathbb{R}^k$ as

$$\Lambda(\nu) = \left(\int \nu(du), \int u\nu(du), \ldots, \int u^{k-1}\nu(du) \right).$$

Since the range of Λ (a subspace of $\mathbb{R}^k$) has dimension at most k, the nullity of Λ must be at least 1. Choose a non-zero element in the null space of Λ, say

$$\nu'(A) = \sum_{i=1}^{k} c_i \mu(A \cap A_i),$$

and scale it suitably so that $|c_i| < 1$ for $i \geq 1$. Let us call the resulting signed measure ν. We now define $\overline{\mu}$ by

$$\overline{\mu}(A) = \mu(A) + \nu(A).$$

Clearly $\overline{\mu}$ is a positive measure and since ν is non-zero $\overline{\mu} \neq \mu$. Further, for each $j = 0, 1, \ldots, k-1$, we have

$$\int u^j \overline{\mu}(du) = \int u^j \mu(du) + \int u^j \nu(du) = \int u^j \mu(du).$$

To prove the second statement, we observe that dividing by the whole mass of the measure μ, we can reduce the question to a problem about probability measures, so we now consider only the case of probability measures. We require two subsidiary lemmas for this.

Lemma 3.2 *Given a symmetric matrix A of order $n \times n$, we can associate a probability measure μ_A supported on at most n points such that for each $j \geq 1$, the j-th moment of μ_A is given by the first diagonal entry of the matrix A^j.*

Proof : Let $\{e_0, e_1, \ldots, e_{n-1}\}$ be the standard basis of $\mathbb{R}^n$ and $(\cdot, \cdot)$ the Euclidean inner product. Since A is symmetric, its eigenvalues, say $\lambda_1, \lambda_2, \ldots, \lambda_n$ are real, and we can write its spectral decomposition as $A = \sum \lambda_j P_j$ where P_j's are the orthogonal projections of rank 1 onto eigenspaces corresponding to the eigenvalues λ_j.

Now we define the probability measure μ_A supported on $\lambda_1, \lambda_2, \ldots, \lambda_n$ with masses $(e_0, P_1 e_0)$, $(e_0, P_2 e_0)$, $\ldots$, $(e_0, P_n e_0)$ respectively. Since $(e_0, P_j e_0) \geq 0$ and $\sum_i (e_0, P_i e_0) = (e_0, \sum_i P_i e_0) = (e_0, I e_0) = 1$, this indeed defines a probability measure. Moreover, $\forall j \geq 1$, we have that $A^j = \sum_j \lambda_i^j P_i$ and hence

$$\int x^j \mu_a(dx) = \sum_i \lambda_i^j (e_0, P_j e_0) = (e_0, \sum_j \lambda_i^j P_i e_0) = (e_0, A^j e_0).$$

This completes the proof of the lemma. $\square$

Lemma 3.3 *Given a probability measure μ supported on a set of n distinct points there exists a symmetric tridiagonal matrix A of order $n \times n$ such that $\mu = \mu_A$ with the property that if its off-diagonal entries are denoted by $b_0, b_1, \ldots, b_{n-2}$, then $b_j \neq 0$ for $j = 0, 1, \ldots, n-2$.*

Proof : Since μ is supported on n points, $\mathbb{L}^2(\mu)$ is an n-dimensional vector space. Let $\{f_0, f_1, \ldots, f_{n-1}\}$ be the orthonormal basis of orthogonal polynomials obtained by applying the Gram-Schmidt process on $\{1, x, x^2, \ldots, x^{n-1}\}$, which can easily be seen to be linearly independent. Notice that $f_0 = 1$.

Now consider the linear operator A on $\mathbb{L}^2(\mu)$ defined as $Af(x) = xf(x)$. We claim that the matrix representation of A with respect to the basis $\{f_0, f_1, \ldots, f_{n-1}\}$ is tridiagonal.

If $\mathbb{S}_j = \text{span}\{1, x, \ldots, x^{j-1}\}$ then the span of $\{f_0, f_1, \ldots, f_{j-1}\}$ is also $\mathbb{S}_j$. Thus, $A(\mathbb{S}_j) \subseteq \mathbb{S}_{j+1}$. This means that the representation of A with respect to $\{f_0, f_1, \ldots, f_{n-1}\}$ will be of the form:

$$A = \begin{pmatrix} \star & \star & \star & \ldots & \star & \star & \star & \star \\ \star & \star & \star & \ldots & \star & \star & \star & \star \\ 0 & \star & \star & \ldots & \star & \star & \star & \star \\ 0 & 0 & \star & \ldots & \star & \star & \star & \star \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & \star & \star & \star \\ 0 & 0 & 0 & \ldots & 0 & 0 & \star & \star \end{pmatrix}$$

where $\star$ denotes a possibly non-zero entry. Since A is a self-adjoint operator, the resulting matrix is symmetric. Thus, it will be tridiagonal, that is,

$$A = \begin{pmatrix} a_0 & b_0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ b_0 & a_1 & b_1 & \ldots & 0 & 0 & 0 & 0 \\ 0 & b_1 & a_2 & \ldots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & b_{n-3} & a_{n-2} & b_{n-2} \\ 0 & 0 & 0 & \ldots & 0 & 0 & b_{n-2} & a_{n-1} \end{pmatrix}.$$

Now, we note that A^j represents multiplication by x^j for $j \geq 1$. Therefore,

$$(e_0, A^j e_0) = (f_0, x^j f_0) = \int f_0 x^j f_0 \mu(dx) = \int x^j \mu(dx)$$

for each $j \geq 1$. So by the first part of the argument, $m_j(\mu) = m_j(\mu_A)$ for $j = 1, 2, \ldots$ where $m_j(\nu)$ stands the j-th moment of the measure ν. As we already know that μ is supported only on n points, we have $\mu_A = \mu$.

Also we now claim that $b_j \neq 0$ for $0 \leq j \leq n-2$. If not,

$$Af_j = b_{j-1} f_{j-1} + a_j f_j + b_j f_{j+1} = b_{j-1} f_{j-1} + a_j f_j.$$

Then, we have

$$\text{span}\{f_0, Af_0, Af_1, \ldots, Af_j\} \subseteq | : \text{span}\{f_0, f_1, \ldots, f_j\}.$$

But since $\text{span}\{f_0, f_1, \ldots, f_j\} = \text{span}\{1, x, \ldots, x^j\}$ and A is a linear operator, we have

$$\text{span}\{Af_0, Af_1, \ldots, Af_j\} = \text{span}\{A1, Ax, \ldots, Ax^j\} = \text{span}\ \{x, x^2, \ldots, x^{j+1}\}.$$

$$\begin{aligned} \text{Therefore, } \text{span}\{f_0, f_1, \ldots, f_{j+1}\} &= \text{span}\{1, x, \ldots, x^{j+1}\} \\ &= \text{span}\{f_0, Af_0, Af_1, \ldots, Af_j\} \\ &\subseteq \text{span}\{f_0, f_1, \ldots, f_j\}. \end{aligned}$$

This contradicts the orthogonality of the vectors $\{f_0, f_1, \ldots, f_{j+1}\}$. $\square$

Now we are in a position to complete the proof of the key lemma. We assume that μ is supported on n points. We now need to construct another measure $\overline{\mu}$ such that first $(2n-1)$ moments of both μ and $\overline{\mu}$ are same.

Consider the tridiagonal matrix A as in Lemma 3.3 such that $\mu = \mu_A$. We construct another tridiagonal matrix B of order $(n+1) \times (n+1)$ in the following way. Choose $b_{n-1}, a_n \in \mathbb{R}$ such that $b_{n-1} \neq 0$ and consider the matrix

$$B = \begin{pmatrix} A & g_n \\ g_n' & a_n \end{pmatrix}$$

where $g_n' = (\overbrace{0\ 0\ \ldots\ 0}^{n-1}\ \ b_{n-1})$ and finally set $\overline{\mu} = \mu_B$.

Now, we claim that the first diagonal entries of A^j and B^j are same for $j = 1, 2, \ldots, 2n-1$ and prove it for $j = (2n-1)$ only. For other values of j, the same argument works. We have,

$$B^{2n-1}(1,1) = \sum b_{1,i_1} b_{i_1,i_2} \ldots b_{i_{2n-1},1}$$

where $b_{i,j}$ is the (i,j)-th entry of the matrix B and the sum extends over all such possible combinations with each $b_{i,j}$ non-zero.

Now, for each such possible combination, we can assign a path of $(2n-1)$ steps, on the set of numbers $\{1, 2, \ldots, n+1\}$ in the following way: for the combination b_{1,i_1}, b_{i_1,i_2}, $\ldots b_{i_{2n-2},1}$ we assign the path $1 \to i_1 \to i_2 \to i_3 \ldots \to i_{2n-2} \to 1$ (the first step is $1 \to i_1$, the second is $i_1 \to i_2$ and so on).

Now since the given matrix is tridiagonal, $b_{i,j}$ is non-zero only when $|i-j| \leq 1$. In other words, the path can move from a point i to another point j which is at most at a distance 1 from i (i.e., to $i+1$, $i-1$ or i). So, a path starting from 1, will take at least n steps to reach $n+1$ and will take at least another n more steps to come back to 1. Therefore, any path of $(2n-1)$ steps starting from 1 cannot reach $n+1$. This means that no term of the type $b_{i,n+1}$ or $b_{n+1,i}$ can be present in the above sum. This proves that the first diagonal entry of the matrix A^{2n-1} is same as that of B^{2n-1}.

Similarly, considering the first diagonal entry of A^j and B^j, for any $j = 1, 2, \ldots, 2n-1$, we observe that

$$\int x^j \mu_B(dx) = (e_0, B^j e_0) = (e_0, A^j e_0) = \int x^j \mu_A(dx)$$

for $j = 1, 2 \dots, 2n-1$.

Finally, to conclude that $\overline{\mu} = \mu_B \neq \mu_A = \mu$, we show that μ_B and μ_A differ in their $2n$-th moments, that is, $B^{2n}(1,1) \neq A^{2n}(1,1)$. For this, we simply repeat the argument of the previous paragraphs to observe that

$$B^{2n}(1,1) = \sum b_{1,i_1} b_{i_1,i_2} \cdots b_{i_{2n-1},1}$$

so that the only way any of the indices i_j can equal $n+1$ is through $i_1 = 2$, $i_2 = 3$, $\dots$, $i_n = n+1$, $i_{n+1} = n$, $\dots$, $i_{2n-1} = 2$. Thus, $B^{2n}(1,1)$ is free of a_n, and also the only term in it that involves b_{n-1} equals $\prod_{i=0}^{n-1} b_i^2$. In other words,

$$B^{2n}(1,1) = \prod_{i=0}^{n-1} b_i^2 + \sum_{1 \leq i_j \leq n} b_{1,i_1} b_{i_1,i_2} \cdots b_{i_{2n-1},1} = \prod_{i=0}^{n-1} b_i^2 + A^{2n}(1,1) \neq A^{2n}(1,1).$$

This completes the proof. □

Remark 3.1 The choice of the matrix B from the matrix A as in the above proof can be made by the application of a fixed continuous function $\mathbb{R}^{2n-1} \to \mathbb{R}^{2n+1}$ by putting 1, for example, for both the numbers a_n and b_{n-1}.

The pair of lemmas 3.2 and 3.3 establish a one-to-one correspondence between the set of Borel probability measures $\mathbb{P}_n$ supported on n points and $\mathcal{A}_n$, a certain set of symmetric tridiagonal matrices of order n; namely, those which arise as described in the proof of Lemma 3.3. A natural question then is, whether this correspondence has any topological significance. Clearly, $\mathcal{A}_n$, looked at as a subset of $\mathbb{R}^{2n-1}$, inherits the relative topology, and we can equip $\mathbb{P}_n$ with the relative topology inherited from the topology of weak convergence. The following lemma will yield the bicontinuity of this bijection. We shall require this result for the non-homogeneous case.

Lemma 3.4 *If A_l, $l \geq 1$ and A are in $\mathcal{A}_n$, then μ_{A_l} converges weakly to μ_A if and only if A_l converges to A as $l \to \infty$.*

Proof : Let us write, for simplicity, μ for μ_A and μ_l for μ_{A_l}, $l \geq 1$. The "if" part of the statement is easily derived from the fact that the convergence of A_l to A implies the convergence of each moment of μ_l to the corresponding moment of μ. Since μ, being supported only on a finite set of points, is determined by its moments, this gives us the required weak convergence.

For the "only if" part, we claim first that all moments of μ_l converge to the corresponding moments of μ. This is because all μ_l must be supported inside one compact set (taking union of closed intervals around the points in the support of μ will suffice).

As a result, the entries of the matrix A_l, which are but rational, hence continuous, functions of the moments of μ_l, will also converge to the respective functions of the moments of μ, or in other words, the corresponding entries of A. □

Remark 3.2 If the tridiagonal matrices A and B are as in Remark 3.1, the foregoing lemma establishes the fact that the resulting measure μ_B is also obtained from μ_A by the application of a continuous function from $\mathbb{P}_n$ into $\mathbb{P}_{n+1}$, where both sides inherit the topology of weak convergence.

4 Proofs of The Theorems

We first recall from section 2 the necessary and sufficient condition for non-emptiness of $\{P_j : 1 \le j \le k\}$: the moment functions $t \mapsto \mathbf{E}X_t^j$, be polynomials $\forall\, 1 \le j \le k$. In order to interpret our problem in terms of the Lévy measure and Kolmogorov measure, we need to consider what are known as the *cumulants* of X_t for $t \ge 0$.

The cumulant of order $j \ge 1$ of a random variable is defined as i^{-j} times the j-th derivative of the logarithm of its characteristic function at 0 if it exists (see Gnedenko [G], page 253). Therefore the equivalence of the existence of cumulants and moments of each order follows. In particular, the first cumulant equals the mean and the second, the variance. Further, one can easily show that the cumulant of every order j is a polynomial in the moments of order 1 through j; and vice versa. Thus, an equivalent formulation to the condition for existence of time-space harmonic polynomials given earlier in Proposition 2.1 is that each of the first k cumulants of X_t exist and also be polynomials in t. These, seen as functions of t, are called the cumulant functions of the process and denoted $c_j(t)$, $j \ge 1$, in the sequel. Of course, as mentioned earlier, this condition is trivially satisfied by Lévy processes X since any cumulant of X_t is just the corresponding cumulant of X_1 multiplied by t.

The importance of the function m and the Kolmogorov measure K for an additive process arise in this context. It turns out that $m(t)$ is the mean, hence the first cumulant of X_t, and for $j \ge 2$, the j-th cumulant of X_t exists if and only if the integral $\int u^{j-2} K([0,t] \otimes du)$ is finite and in this case they are equal. In other words, for $t \ge 0$,

$$c_1(t) = m(t) \quad \text{and} \quad c_j(t) = \int u^{j-2} K([0,t] \otimes du). \tag{4.7}$$

In the case when all the cumulants exist this follows easily from (2.3) since

$$\sum_{j=1}^{\infty} c_j(t) \frac{(i\alpha)^j}{j!} \;=\; \log \mathbf{E}(e^{i\alpha X_t}) \;=\; \sum_{j=2}^{\infty} \frac{(i\alpha)^j}{j!} \int u^{j-2} K([0,t] \otimes du).$$

Of course, for Lévy processes X the quatities in the right hand sides of (4.7) reduce to $t \cdot m$ and $t \int u^{j-2} \eta(du)$, both clearly polynomials.

Proof of Theorem 2.1 : First suppose that X is a Lévy process with Kolmogorov measure η_X and $\mathrm{Card}(\eta_X) = n$. We have to exhibit a k such that whenever for some Lévy process Y we have, $\mathcal{P}_j(Y) = \mathcal{P}_j(X)$ for all $1 \le j \le k$, then $Y \stackrel{d}{=} X$. We show this for $k = 2n+2$.

For a Lévy process Z ($Z = X$ or Y), the mean of Z_1 and the Kolmogorov measure are denoted by m_Z and η_Z respectively. It is clear that we have to show $m_Y = m_X$ and $\eta_Y = \eta_X$.

From the equation (4.7) and the comment following it, we have

$$m_X = m_Y \quad \text{and} \quad \int_{\mathbb{R}} u^j \eta_X(du) = \int_{\mathbb{R}} u^j \eta_Y(du) \quad \text{for } j = 0, 1, \ldots, 2n. \tag{4.8}$$

Since, η_X is supported on n points, we now apply Lemma 3.1 to conclude that $\eta_X = \eta_Y$.

For the converse part, we assume that η_X is not finitely supported. For any fixed $k \geq 1$, we have to contruct a Lévy process Y such that $\mathcal{P}_j(Y) = \mathcal{P}_j(X)$ for all $1 \leq j \leq k$. From the foregoing discussion, it is clear that this is tantamount to obtaining a finite measure $\eta \neq \eta_X$ such that $\int u^j \eta_X(du) = \int u^j \eta(du)$ for $j = 0, 1, \ldots, k-2$, and setting $m_Y = m_X$ and $\eta_Y = \eta$. Since η_X is not finitely supported, the first assertion of Lemma 3.1 guarantees the existence of such a measure η and this completes the proof of the theorem. $\square$

Proof of Theorem 2.2 : In view of the proof of Theorem 2.1, it is enough to show that if for a Lévy process X, $\text{Card}(\eta_X) = n$, then $\{\mathcal{P}_j : 1 \leq j \leq 2n+1\}$ does not determine the law of X; that is, there exists another Lévy process Y such that $\mathcal{P}_j(X) = \mathcal{P}_j(Y)$ for $1 \leq j \leq 2n+1$ such that the laws of X and Y are not the same. As we have noted in the proof of Theorem 2.1, it is equivalent to constructing another measure $\overline{\eta}$ such that $\overline{\eta} \neq \eta_X$ but $\int u^j \eta_X(du) = \int u^j \overline{\eta}(du)$ for $j = 0, 1, \ldots, 2n-1$ and setting $m_Y = m_X$ and $\eta_Y = \overline{\eta}$. But this is achieved by the second part of Lemma 3.1. $\square$

To carry out the same program in the non-homogeneous setup, we first need an important result (Lemma 4.1). But first let us briefly touch on a heuristic justification for expecting it to be true. If for a substantial set of points t, $\kappa(t,\cdot)$ were supported on more than n points, then like in the homogeneous case, it could not be determined uniquely from the first $2n$, leave alone n, of its moments. However, unlike in the homogeneous case, to define the law of another process in $\mathcal{C}$, one has to handle not just the distribution of one single random variable, but that of the whole process; that is, define either a different mean function or a different Kolmogorov measure, or equivalently, derivative measure $\overline{\kappa}$ which would keep the first n moment functionss of the process intact. The crux of the matter lies naturally in constructing $\overline{\kappa}$ while retaining its measurability with respect to its first argument. This requires a variant of a certain result of Descriptive Set Theory, known as Novikov's Selection Theorem, stated for example in Kechris ([K], page 220, Theorem 28.8).

Theorem 4.1 *Suppose U is a Standard Borel space and V a Polish space and $B \subseteq U \times V$ a Borel set whose projection onto U is the whole of U, that is, $\Pi_U(B) = U$. Suppose further that the sections B_x of B, $x \in U$, are all compact. Then there is a Borel measurable function $h : U \to V$ whose graph is contained in B, that is, $h(x) \in B_x \, \forall \, x \in U$.*

From now on, an additive process will always mean an element of the class $\mathcal{C}$ defined earlier.

Lemma 4.1 *Suppose X is a process in the class $\mathcal{C}$ whose distribution in $\mathcal{C}$ is uniquely determined by the moment functions $\mathbf{E}X_t^j$, $1 \leq j \leq k$ where $k > 1$. Then for any version of κ, the set $T \subseteq [0,\infty)$ defined by $T = \{t \geq 0 : \text{Card}(\kappa(t,\cdot)) > k\}$ is Borel and has zero lebesgue measure.*

Proof: Let us start with any version of κ, and construct the set T as in the statement. Towards the first assertion, define, for each $n \geq 1$ and $j \in \mathbb{Z}$, the Borel sets

$$I_{j,n} = \left\{ t : \kappa(t, (\frac{j}{2^n}, \frac{j+1}{2^n}]) > 0 \right\}.$$

For each $n \geq 1$, the possibly $+\infty$-valued function f_n on $[0, \infty)$ defined as

$$f_n(t) = \sum_{j=-\infty}^{\infty} 1_{I_{j,n}}(t)$$

is measurable. Clearly, for each t, $f_n(t)$ is increasing in n, hence $f = \lim_n f_n(t)$ exits and is measurable also. (Observe $f(t) = \infty$ whenever $f_n(t) = \infty$ for some n). Finally, we just have to note that $T = \{t : f(t) > k\}$, which implies that T is Borel. In fact, $f(t)$ equals exactly $\mathrm{Card}(\kappa(t, \cdot))$ in case the latter is finite. The first statement is thus established.

For the second statement, suppose if possible that $\ell(T) > 0$. Now note that the hypothesis of the lemma implies that X has finite moments of orders at least upto k. Therefore, for each t, $\int_0^t \int_{\mathbb{R}} |u|^j \kappa(s, du) ds < \infty$, $0 \leq j \leq k-2$. In particular, the set

$$I = \bigcup_{j=0}^{k-2} \left\{ t : \int |u|^j \kappa(t, du) = +\infty \right\}$$

is a Borel set of zero lebesgue measure. Set $\tilde{T} = T \cap I^c$. Then clearly, $\ell(\tilde{T}) = \ell(T) > 0$. We now apply Theorem 4.1 to produce a contradiction.

Choose for U the set $\tilde{T}$, and for V, the following σ-compact Banach space:

$$V = \left\{ \sum_{i=1}^{k} c_i g_i : c_i \in \mathbb{R}, 1 \leq i \leq k \right\},$$

where $g_1, g_2, \ldots, g_k$ are non-vanishing continuous functions on $\mathbb{R} \to \mathbb{R}$, each bounded by 1, with the property that for any set $E \subset \mathbb{R}$ of cardinality at least k, the functions $g_1 1_E$, $g_2 1_E$, $\ldots$, $g_k 1_E$ are linearly independent. For $f = \sum_{i=1}^k c_i g_i \in V$, define $\|f\| = \left(\sum_{i=1}^k |c_i|^2\right)^{1/2}$ so that V is isometrically isomorphic to $\mathbb{R}^k$.

To define the set B, we introduce a definition and a notation: for $t \in \tilde{T}$, define the linear map $\Lambda_t : V \to \mathbb{R}^{k-1}$ as $\Lambda_t(f) = \left(\int_{\mathbb{R}} f(u) u^j \kappa(t, du)\right)_{0 \leq j \leq k-2}$. Define now $B = \{(t, f) \in \tilde{T} \times V : \Lambda_t f = 0,\ \frac{1}{2} \leq \|f\| \leq 1\}$. To show both that B is Borel and that its sections B_t are compact in V, we use the following

Lemma 4.2 *The map $(t, f) \mapsto \Lambda_t f$ as a function on $\tilde{T} \times V$ is measurable in the first argument t keeping the second f fixed, and continuous in the latter fixing the former. In particular, it is jointly measurable on $\tilde{T} \times V$.*

The proof of this lemma involves only routine applications of standard results in measure theory and we omit it.

Obviously, as a result, B is the intersection of two Borel sets in the product $\tilde{T} \times V$, therefore Borel itself. Next, for every fixed $t \in \tilde{T}$, $B_t = \{f : \Lambda_t f = 0\} \cap \{f : \frac{1}{2} \leq \|f\| \leq 1\}$, is the intersection of a closed set (being the inverse image of a closed set under a continuous map) and a compact set, hence is itself compact.

It remains to show that $\Pi_{\tilde{T}}(B) = \tilde{T}$, or equivalently, that B_t is nonempty for all $t \in \tilde{T}$. Fix a $t \in \tilde{T}$. Denote the support of $\kappa(t, \cdot)$ by S_t. By definition of $\tilde{T}$, S_t contains at least k points. Consider now the vector space $V_t = \{f 1_{S_t} : f \in V\}$. It is not difficult to show, using $|S_t| \geq k$, that V_t has dimension at least k.

We now define, for $t \in \tilde{T}$, a linear map $\zeta_t : V_t \to \mathbb{R}^{k-1}$, as $\zeta_t(v) = \Lambda_t f$ where $v = f 1_{S_t}$, $f \in V$. This map is well-defined, since if for some $g \in V$, $v = g 1_{S_t}$ also, then $\Lambda_t g = \Lambda_t f$. Further, ζ_t necessarily has a nontrivial kernel, its range being of strictly smaller dimension than its domain. Choosing any nonzero element $v = f 1_{S_t}$ in the kernel and scaling it appropriately so that $\frac{1}{2} \leq \|f\| \leq 1$, we get $f \in B_t$. Thus $B_t \neq \emptyset \; \forall\, t \in \tilde{T}$.

Applying Theorem 4.1 therefore, we get a 'measurable selection' $h : \tilde{T} \to V$ such that $h(t) \in B_t \; \forall\, t \in \tilde{T}$. For $t \in \tilde{T}$ and $u \in \mathbb{R}$, let us define $h(t, u)$ as $h(t)$ evaluated at u and write $h(t, u) = \sum_{i=1}^{k} c_i(t) g_i(u)$. Now, each of the maps $\sum_{i=1}^{k} c_i g_i \mapsto c_i$, $1 \leq i \leq k$ is continuous, therefore measurable. Thus the measurability of h implies that $\forall\, u \in \mathbb{R}$, the map $t \mapsto h(t, u)$ is measurable. But for every t, $h(t) \in V$ is known to be continuous, therefore $h : \tilde{T} \times \mathbb{R} \to \mathbb{R}$ is jointly measurable.

Define now, using this function h, the new derivative measure $\overline{\kappa}$ and Kolmogorov measure $\overline{K}$ as

$$\begin{aligned} \overline{\kappa}(t, du) &= \kappa(t, du) 1_{\tilde{T}^c}(t) + (1 + h(t, u)) \kappa(t, du) 1_{\tilde{T}}(t), \quad \text{and} \\ \overline{K}(A) &= \int_A \overline{\kappa}(t, du) dt \qquad \text{for Borel } A \subseteq [0, \infty) \times \mathbb{R}. \end{aligned}$$

The fact that $\forall\, t \in \tilde{T}$, $|h(t, u)| \leq 1$ for $\kappa(t, \cdot)$-a.e. u, ensures that $\overline{K}$ is a measure. This is the Kolmogorov measure of our candidate for an additive process Y. We retain for Y the same mean function as that of X. Then, for all $s \geq 0$ and $0 \leq j \leq k - 2$,

$$\begin{aligned} \int u^j \overline{K}([0, s] \otimes du) &= \int_0^s dt \int u^j \overline{\kappa}(t, du) \\ &= \int_{[0,s] \cap \tilde{T}^c} dt \int u^j \kappa(t, du) + \int_{[0,s] \cap \tilde{T}} dt \int u^j (1 + h(t, u)) \kappa(t, du) \\ &= \int_{[0,s] \cap \tilde{T}^c} dt \int u^j \kappa(t, du) + \int_{[0,s] \cap \tilde{T}} dt \int u^j \kappa(t, du) \\ &\quad + \int_{[0,s] \cap \tilde{T}} dt \int u^j h(t, u) \kappa(t, du) \\ &= \int_0^s dt \int u^j \kappa(t, du) \qquad \text{(by construction of } h) \\ &= \int u^j L([0, s] \otimes du); \end{aligned}$$

that is, Y has the same cumulants and consequently, the same moments, as X upto order k. However, $\overline{K} \neq K$, which can be seen as follows. First of all, since $\ell(\tilde{T}) > 0$, there is a $t_0 > 0$, such that $\ell(\tilde{T} \cap [0, t_0]) > 0$. Denoting $\tilde{T_0} = \tilde{T} \cap [0, t_0]$, consider the Borel set $H \subseteq [0, t_0] \times \mathbb{R}$ defined as $H = \{(t, u) : t \in \tilde{T_0}, h(t, u) > 0\}$. Since $H \subseteq [0, t_0] \times \mathbb{R}$, clearly $K(H) < \infty$. On the other hand, since for each $t \in \tilde{T_0}$, $\|h(t, \cdot)\| \geq 1/2$ and $\int h(t, u) \kappa(t, du) = 0$, we must have

$\kappa(t, \{u : h(t,u) > 0\}) > 0$. For, if $h(t, \cdot) \equiv 0$ on S_t, then using the linear independence of the g_i's, we get that $c_i = 0$ for every $1 \le i \le k$ which would contradict $\|h(t,\cdot)\| > 1/2$. This, along with $\ell(\tilde{T_0}) > 0$, implies that

$$\overline{K}(H) \;=\; K(H) + \int_{\tilde{T_0}} dt \int_{\{u:h(t,u)>0\}} h(t,u)\kappa(t,du) \;>\; K(H).$$

Thus $Y \overset{d}{\neq} X$, contradicting the hypothesis. $\square$

Proof of Theorem 2.3 : The proof of part (a) is quite similar to the corresponding part in the homogeneous case. If κ is indeed of the given form, then $\{\mathcal{P}_j : 1 \le j \le 2n+2\}$ determines the law of X. First of all, the two conditions imply that for all $1 \le j \le 2n+2$, the j-th cumulant of X_t is given by

$$c_j(t) \;=\; \int u^{j-2} K([0,t]\otimes du) \;=\; \int_0^t \int u^{j-2}\kappa(s,du) \;=\; \int_0^t \sum_{i=1}^n p_i(t)\{x_i(t)\}^{j-2},$$

a polynomial in t. This ensures that $\mathcal{P}_j(X) \neq \emptyset$ for $1 \le j \le 2n+2$.

If now Y is another additive process of the class $\mathcal{C}$ such that $\mathcal{P}_j(Y) = \mathcal{P}_j(X)$ for $1 \le j \le 2n+2$, then for every $t \ge 0$, the cumulants of order upto $2n+2$ of Y_t agree with those of X_t. Denote the mean function, Kolmogorov measure and derivative measure for Y by $\overline{m}$, $\overline{K}$ and $\overline{\kappa}$ respectively. Then by the relation (4.7), $\overline{m}(t) = c_1(t) = m(t)$, and for all $0 \le j \le 2n$ and $t \ge 0$,

$$\int_0^t \int u^j \overline{\kappa}(s,du) \;=\; \int u^j \overline{K}([0,t]\otimes du) \;=\; \int u^j K([0,t]\otimes du) \;=\; \int_0^t \int u^j \kappa(s,du).$$

It follows that for almost all $t \ge 0$, $\int u^j \overline{\kappa}(t,du) = \int u^j \kappa(t,du)$, $0 \le j \le 2n$, and consequently that

$$\int \prod_{j=1}^n (u - x_j(t))^2\, \overline{\kappa}(t,du) = \int \prod_{j=1}^n (u - x_j(t))^2\, \kappa(t,du) = 0.$$

By the same argument as in the proof of the 'if' part of Theorem 2.1, this proves $\overline{\kappa}(t,\cdot) = \kappa(t,\cdot)$ for almost all t; and therefore $\overline{K} = K$, implying $Y \overset{d}{=} X$. Thus $\{\mathcal{P}_j : 1 \le j \le 2n+2\}$ characterises X.

Let us now prove part (b). The hypothesis now entails the existence of a positive integer k such that $\{\mathcal{P}_j(X), 1 \le j \le k\}$, determines the law of X. We may thus apply Lemma 4.1. Consider any version of κ and the set T as defined in its proof. For $t \in T$, redefine $\kappa(t,\cdot)$ as zero measure. The resulting transition function still remains a version of κ. Now recall, for $t \ge 0$, the notation S_t for the support of $\kappa(t,\cdot)$. By our construction, $|S_t| \le k \;\forall\, t \ge 0$. Let us partition $[0,\infty)$ by the cardinality of S_t, that is, let

$$T_j = \{t \ge 0 : |S_t| = j\}, \qquad 1 \le j \le k.$$

By the same argument used to prove T (of Lemma 4.1) is Borel, one can conclude that so is each T_j. Notice that $T = (\cup_{j=1}^k T_j)^c$.

For $t \in T_j$, order the elements of S_t as $x_1(t) < x_2(t) < \ldots < x_j(t)$, and denote the $\kappa(t,\cdot)$-masses at these points by $p_1(t), p_2(t), \ldots, p_j(t)$ respectively.

Also, for $j+1 \le i \le k$, let $x_i(t) = x_{i-1}(t) + 1$ and $p_i(t) = 0$. For $t \in T = (\cup_{j=1}^k T_j)^c$, set $x_i(t) \equiv y_i$ and $p_i(t) \equiv 1$, $1 \le i \le k$. With these notations, it follows that

$$\kappa(t,\cdot) = \sum_{i=1}^{k} p_i(t)\delta_{x_i(t)}.$$

We now need to prove that the real-valued functions $x_i(t)$, and the non-negative functions $p_i(t)$, are all measurable. It suffices to show the measurability of these functions only on $\cup_{j=1}^k T_j$. First we deal with the x_i's. It is enough to show that each x_i is measurable on each T_j, and that too, only for $i \le j$.

Fix $j \ge 1$. By the definition of support of a measure, $S_t = \text{supp}\,(\kappa(t,\cdot))$ is closed, and, $x_1(t) = \inf S_t$ for every $t \in T_j$. Therefore,

$$\{t \in T_j : x_1(t) \ge a\} = \bigcap_{\substack{q\in\mathbb{Q}\\ q<a}} \{t \in T_j : \kappa(t,(-\infty,q)) = 0\}.$$

This means that the function $x_1 : T_j \to \mathbb{R}$ is measurable. Next, if $j \ge 2$, then

$$\begin{aligned}&\{t \in T_j : x_2(t) \ge a\}\\ =\;& \{t \in T_j : x_1(t) \ge a\} \bigcup \Big\{\{t \in T_j : x_1(t) < a\} \cap \{t \in T_j : x_2(t) \ge a\}\Big\},\end{aligned}$$

and the second set can be written as

$$\{t \in T_j : x_1(t) < a\} \cap \bigcap_{\substack{q\in\mathbb{Q}\\ q<a}} \{\{t \in T_j : x_1(t) \ge q\} \cup \{t \in T_j : \kappa(t,(q,a)) = 0\}\}.$$

Thus x_2 is measurable on T_j. In similar fashion, it can be shown that the rest of the functions $x_i(t)$, $1 \le i \le j$, are each measurable on T_j. Now the task remains to show that the p_i's are also measurable. But observe that

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ x_1(t) & x_2(t) & \dots & x_k(t) \\ x_1^2(t) & x_2^2(t) & \dots & x_k^2(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{k-1}(t) & x_2^{k-1}(t) & \dots & x_k^{k-1}(t) \end{pmatrix} \begin{pmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \\ \vdots \\ p_k(t) \end{pmatrix} = \begin{pmatrix} \int \kappa(t,du) \\ \int u\kappa(t,du) \\ \int u^2\kappa(t,du) \\ \vdots \\ \int u^{k-1}\kappa(t,du) \end{pmatrix}$$

The matrix D on the left is nonsingular, being a Vandermonde matrix, and since each of its elements is a measurable function of t, so is each element of its inverse. Each element of the vector v on the right is also measurable, by approximating the functions u^j by simple functions. It follows that the elements p_i of $D^{-1}v$ are also measurable.

Finally, for every j, $0 \le j \le k-2$,

$$\sum_{j=1}^{k} p_i(t)\{x_i(t)\}^j = \int u^j\,\kappa(t,du) = \frac{\mathrm{d}}{\mathrm{dt}} \int u^j K([0,t] \otimes du) = c'_{j+2}(t),$$

where the second equality holds for lebesgue-almost every t. The left hand side is therefore a polynomial almost everywhere. $\square$

The proof of Theorem 2.4 requires the following lemma whose proof we only sketch. Consider the class $\mathbb{P}$ of all Borel probability measures on the real line, with the σ-field $\Sigma := \sigma < e_A : A \in \mathcal{B}(\mathbb{R}) >$ generated by the evaluation maps $\{e_A\}$, defined as $e_A(P) = P(A)$. Denote the topology of weak convergence on $\mathbb{P}$ by $\mathcal{T}$.

Lemma 4.3 *The Borel σ-field $\sigma(\mathcal{T})$ on $\mathbb{P}$ equals Σ.*

Proof : We know from Billingsley ([B2], pages 236-237) that the class

$$\mathcal{U} = \{\{Q : Q(F_i) < P(F_i) + \epsilon, 1 \leq i \leq n\} : P \in \mathbb{P}, \epsilon > 0, n \geq 1, F_1, \ldots, F_n \subseteq \mathbb{R} \text{ closed}\}$$

forms a base for $\mathcal{T}$. Since $\mathcal{T}$ is second countable, it is enough to show that every $B \in \mathcal{U}$ is in Σ to conclude that $\sigma(\mathcal{T}) \subseteq \Sigma$. But this is trivially seen to be true.

On the other hand, that $\sigma(\mathcal{T}) \supseteq \Sigma$ follows from Dynkin's π-λ Theorem (see [B1], page 37) since closed subsets of $\mathbb{R}$ form a λ-class, the class $\mathcal{A} = \{A \text{ such that } e_A : \mathbb{P} \to [0,1] \text{ is measurable}\}$ is a π-class and e_F is easily seen to be a measurable map on $\mathbb{P}$ for closed $F \subseteq \mathbb{R}$. $\square$

Proof of Theorem 2.4 : Here too, since the proof of Theorem 2.3 reveals that $\{P_j : 1 \leq j \leq 2n+2\}$ determines the law of X in $\mathcal{C}$, it is enough to show that $\{P_j : 1 \leq j \leq 2n+1\}$ does not.

The hypothesis of the theorem says that the set $T_n := \{t \geq 0 : \text{Card}(\kappa(t,\cdot)) = n\}$ has positive lebesgue measure. Clearly, for $t \in T_n$, $\kappa(t,\cdot)$ is a finite positive measure, hence it can be normalised into a Borel probability measure $p\,(t,\cdot)$ on the line; namely, $p\,(t,E) = \frac{\kappa(t,E)}{\kappa(t,\mathbb{R})}$, $E \in \mathcal{B}(\mathbb{R})$. Further, for every Borel E, the map $p\,(\cdot,E) : T_n \to [0,1]$ is Borel measurable, by the similar property enjoyed by κ. This implies, by Lemma 4.3, that the map $p : T_n \to \mathbb{P}$ is a Borel measurable map. Moreover, p is actually a measurable map from T_n into $\mathbb{P}_n$. That $\mathbb{P}_n$ is a Borel subset of $\mathbb{P}$ is guaranteed by Lemma 4.3 again, along the same lines as the proof of the measurability of the set T in the proof of Lemma 4.1.

Now, for $t \in T_n$, we replace $\kappa(t,\cdot)$ by a different derivative measure while retaining measurability in the sense described before, to obtain a new additive process. First, for the measure $p\,(t,\cdot)$ construct the $n \times n$ symmetric tridiagonal matrix A as in Lemma 3.2, and call it $A(t)$. Corresponding to this matrix, choose the symmetric tridiagonal matrix B of order $(n+1) \times (n+1)$ as in the Remark 3.1, and call it $B(t)$. Clearly, our objective will be achieved if we can show that the resulting probability measure $\mu_{B(t)}$ satisfies the condition of measurability in t, in the sense already described. Consider now the composition of the maps $p\,(t,\cdot) \mapsto A(t) \mapsto B(t) \mapsto \overline{p}\,(t,\cdot) := \mu_{B(t)}$ on $\mathbb{P}_n \to \mathbb{P}_{n+1}$. The first and the last are continuous by Lemma 3.4, while the middle one is so by Remark 3.2. Therefore the composition is itself continuous; hence Borel measurable. By Lemma 4.3, it follows that for each $E \in \mathcal{B}(\mathbb{R})$, $t \mapsto \overline{p}\,(t,E)$ is a measurable function $T_n \to [0,1]$. This implies that the new transition measure $\overline{\kappa}$ defined as

$$\overline{\kappa}(t,E) := \kappa(t,\mathbb{R})\,\overline{p}\,(t,E)$$

is a valid derivative measure. Define now a new additive process Y as before with this new derivative measure $\overline{\kappa}$. It follows that $\mathcal{P}_k(Y) = \mathcal{P}_k(X)$ for $1 \leq k \leq 2n+1$. That Y has a distinct

Kolmogorov measure from X so that $Y \overset{d}{\neq} X$ is an easy consequence of the fact that $(\mathcal{B})\mathbb{R}$ is countably generated and is left to the reader. $\square$

Remark. Given the form of the derivative measure, it is natural to speculate what can be said about the exact nature of the functions x_i and p_i, $1 \leq i \leq k$. We exhibit some possible forms in the case $n = 2$ in the section devoted to examples.

5 Examples

The following are some examples of additive processes, a few of which are fpd and some which are not. The first three types are in the general setup. More non-homogeneous examples can easily be constructed by minor modifications.

- **2-polynomially determined processes**. The only 2-polynomially determined additive processes are those which are deterministic, that is, processes X for which X_t equals with probability 1 a polynomial $P(t)$ with $P(0) = 0$. Clearly, then, $\beta(t) = m(t) = P(t)$, $\sigma^2(t) \equiv 0$ and $L = K = 0$. The two time-space harmonic polynomials characterising such a process are $P_1(t,x) = x - P(t)$ and $P_2(t,x) = (x - P(t))^2$.

- **4-polynomially determined processes**. Such processes are clearly determined by three functions m, p and x on $[0,\infty)$ with $m(0) = 0$ and $p \geq 0$ such that $m(t)$, $p(t)$, $x(t)p(t)$ and $x^2(t)p(t)$ are polynomials, where m stands for the mean function, and a version of the derivative measure is given by $p(t)\delta_{x(t)}$.

- **6-polynomially determined processes**. For these, clearly, the value of n as in Theorem 2.4 is 2. Thus, such processes are determined by an arbitrary polynomial $m(\cdot)$ with $m(0) = 0$ as mean function, and the functions x_1, x_2, p_1 and p_2. Some possible forms for these functions are

  1. $x_1(t)$, $x_2(t)$, $p_1(t) \geq 0$ and $p_2(t) \geq 0$ are polynomials,
  2. $x_1(t) = a(t) + \sqrt{b(t)}$, $x_2(t) = a(t) - \sqrt{b(t)}$, $p_1(t) = c(t) + d(t)\sqrt{b(t)}$ and $p_2(t) = c(t) - d(t)\sqrt{b(t)}$, where a, b, c and d are polynomials so chosen that $c \pm d\sqrt{b}$ are both non-negative on $[0,\infty)$.
  3. $x_1(t) = a(t)b(t)$, $x_2(t) = c(t)b(t)$, $p_1(t) = \frac{d(t)}{b(t)}$, and $p_2(t) = \frac{e(t)}{b(t)}$. Here, a, $b > 0$, c, $d \geq 0$ and $e \geq 0$ are polynomials such that $b|(d+e)$.

  Although in each of these examples, the underlying process admits time-space harmonic polynomials of each degree in the space variable, that is, $\mathcal{P}_k \neq \emptyset \ \forall\, k \geq 1$, one can easily obtain examples of fpd additive processes for which this property is violated. That is, there are additive processes for which only finitely many $\mathcal{P}_k$'s are non-empty and serve to determine its law ([S1], page 78).

- **Standard Brownian motion.** For Brownian motion, it is well-known that the Lévy measure $m(du) \equiv 0$, and hence the Kolmogorov measure is $\eta = \delta_0$. Thus the first four Hermite polynomials
$$H_1(t,x) = x,\ H_2(t,x) = x^2 - t,$$

$$H_3(t,x) = x^3 - 3tx,\ H_4(t,x) = x^4 - 6tx^2 + 4t^2,$$

determine it uniquely among Lévy processes. An example of another Lévy process for which the first three time-space harmonic polynomials agree, is specified by the following:

$$m = 0 \quad \text{and} \quad \eta = \frac{1}{2}(\delta_{-1} + \delta_1).$$

We may note here that Brownian motion with a constant drift is also 4-polynomially determined.

- **Poisson Process.** In this case, $m = \lambda$ and $\eta = \lambda\delta_1$. Here too, the first four Poisson-Charlier polynomials

$$C_1(t,x) = x - \lambda t, \quad C_2(t,x) = (x - \lambda t)^2 - \lambda t,$$

$$C_3(t,x) = (x - \lambda t)^3 - 3\lambda t(x - \lambda t) - \lambda t,$$

$$C_4(t,x) = (x - \lambda t)^4 - 6\lambda t(x - \lambda t)^2 - 4\lambda t(x - \lambda t) + 3(\lambda t)^2 - \lambda t,$$

determine it. Another Lévy process with the first three matching is given by

$$m = \lambda \quad \text{and} \quad \eta = \frac{\lambda}{2}(\delta_0 + \delta_2).$$

- **Gamma Process.** This is a counterexample in contrast to the earlier two. Here $m = \alpha/\lambda$ and

$$\eta(du) = \alpha\, u\, e^{-\lambda u} du, \qquad u \geq 0$$

which is clearly not finitely supported. Therefore the Gamma process is not fpd.

- Finally, we present an example of a Lévy process which is not even *infinitely* polynomially determined, leave alone being fpd. Take $m = 0$ and as η, any measure that is not determined by its moments, as for instance that in Feller ([F], page 224)

$$\eta(du) = e^{\sqrt[4]{u}}(1 - \alpha \sin \sqrt[4]{u})\, du,\ u \geq 0, \quad \text{for some } 0 < \alpha < 1.$$

Obviously, since this η is not finitely supported, the underlying process has no chance of being fpd. The non-existence of the m.g.f. makes this construction possible.

Acknowledgements. We are grateful to Professors S. M. Srivastava, A. Goswami and B. V. R. Bhatt and an anonymous referee for their helpful comments and suggestions.

References

[B1] P. Billingsley. *Probability and Measure.* John Wiley, New York, 1986.

[B2] P. Billingsley. *Convergence of Probability Measures.* John Wiley, London, 1968.

[D] J. L. Doob. *Stochastic Processes.* John Wiley & Sons, New York, 1952.

[F] W. Feller. *An Introduction to Probability Theory and Its Applications.* Wiley Eastern University Edition, 1988.

[G] B. V. Gnedenko. *The Theory of Probability.* Chelsea, New York, 1963.

[GK] B. V. Gnedenko & A. N. Kolmogorov. *Limit Distributions for Sums of Independent Random Variables.* Addison-Wesley Publications, Cambridge, 1954.

[I] K. Ito. *Lectures on Stochastic Processes.* Tata Institute of Fundamental Research Lecture Notes. Narosa, New Delhi, 1984.

[K] A. S. Kechris. *Classical Descriptive Set Theory.* Springer-Verlag, Graduate Texts in Mathematics 156, 1995.

[MRR] P. McGill, B. Rajeev and B. V. Rao. Extending Lévy's characterisation of Brownian motion. *Seminaire de Prob. XXIII*, Lecture Notes in Mathematics # 1321, pp. 163 – 165, Springer-Verlag, 1988.

[S1] Arindam Sengupta. *Time-space Harmonic Polynomials for Stochastic Processes.* Ph. D. Thesis (unpublished), Indian Statistical Institute, Calcutta, 1998.

[S2] Arindam Sengupta. Time-space Harmonic Polynomials for Continuous-time Processes and an Extension. *Journal of Theoretical Probability,* **13**, pp. 951 – 976, 2000.

[W] S. Watanabe. On discontinuous additive functionals and Lévy measures of a Markov process. *Japanese J. Math.*, **34**, pp. 53 – 70, 1964.