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HAUSDORFF DIMENSION OF CUT POINTS FOR BROWNIAN MOTION

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Abstact: Let B be a Brownian motion in $\mathbf{R}^d$, $d = 2, 3$. A time $t \in [0, 1]$ is called a cut time for $B[0, 1]$ if $B[0, t) \cap B(t, 1] = \emptyset$. We show that the Hausdorff dimension of the set of cut times equals $1 - \zeta$, where $\zeta = \zeta_d$ is the intersection exponent. The theorem, combined with known estimates on ζ_3, shows that the percolation dimension of Brownian motion (the minimal Hausdorff dimension of a subpath of a Brownian path) is strictly greater than one in $\mathbf{R}^3$.

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Hausdorff Dimension of Cut Points for Brownian Motion

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Abstract

Let B be a Brownian motion in $\mathbf{R}^d$, $d = 2, 3$. A time $t \in [0, 1]$ is called a cut time for $B[0, 1]$ if $B[0, t) \cap B(t, 1] = \emptyset$. We show that the Hausdorff dimension of the set of cut times equals $1 - \zeta$, where $\zeta = \zeta_d$ is the intersection exponent. The theorem, combined with known estimates on ζ_3, shows that the percolation dimension of Brownian motion (the minimal Hausdorff dimension of a subpath of a Brownian path) is strictly greater than one in $\mathbf{R}^3$.

1 Introduction

Let $B_t = B(t)$ denote a Brownian moton taking values in $\mathbf{R}^d, (d = 2, 3)$. A time $t \in [0, 1]$ is called a *cut time* for $B[0, 1]$ if

$$B[0, t) \cap B(t, 1] = \emptyset.$$

We call $B(t)$ a *cut point* for $B[0, 1]$ if t is a cut time. Let L denote the set of cut times for $B[0, 1]$. Dvoretkty, Erdős, and Kakutani [8] showed that for each $t \in (0, 1)$,

$$\mathbf{P}\{t \in L\} = 0.$$

However, Burdzy [1] has shown that nontrivial cut points exists, i.e., with probability one $L \cap (0, 1) \neq \emptyset$. In this paper we give another proof of the existence of cut points and compute the Hausdorff dimension of L in terms of a particular exponent, called the *intersection exponent.*

The intersection exponent is defined as follows. Let B^1, B^2 be independent Brownian motions in $\mathbf{R}^d$ $(d < 4)$ starting at distinct points x, y with $|x| = |y| = 1$. For each $n \geq 1$, let

$$T_n^i = \inf\{t : |B^i(t)| = n\}.$$

Then the intersection exponent is the number $\zeta = \zeta_d$ such that as $n \to \infty$,

$$\mathbf{P}\{B^1[0, T_n^1] \cap B[0, T_n^2] = \emptyset\} \approx n^{-2\zeta}, \tag{1}$$

[1] Research supported by the National Science Foundation and NSERC

where $\approx$ denotes that the logarithms of the two sides are asymptotic as $n \to \infty$. It is not difficult [12, Chapter 5] to show that such a ζ exists. This exponent is also the intersection exponent for random walks with mean zero and finite variance [3, 6]. Sometimes the exponent $\xi = 2\zeta$ is called the intersection exponent. The exact value of ζ is not known. The best rigorous bounds are [4]

$$\frac{1}{2} + \frac{1}{8\pi} \leq \zeta < \frac{3}{4}, \quad d = 2,$$

$$\frac{1}{4} \leq \zeta < \frac{1}{2}, \quad d = 3.$$

Duplantier and Kwon [7] have conjectured using nonrigorous conformal field theory that $\zeta = 5/8$ in two dimensions. This is consistent with Monte Carlo simulations [5, 16] and simulations suggest that $.28 \leq \zeta \leq .29$ in three dimensions. The purpose of this paper is to prove the following theorem. We let $\dim_h$ denote Hausdorff dimension.

Theorem 1.1 *If B is a Brownian motion in $\mathbf{R}^d$, $d = 2, 3$ and L is the set of cut times of $B[0,1]$, then with probability one,*

$$\dim_h(L) = 1 - \zeta,$$

where $\zeta = \zeta_d$ is the intersection exponent.

Although the theorem is stated for dimensions 2 and 3, it is actually true in all dimensions. If $d \geq 4$ [8] the probability on the left hand side of (1) equals 1 for all n. Hence we can say that the intersection exponent $\zeta_d = 0$ for $d \geq 4$. But, with probability one, $L = [0,1]$ for $d \geq 4$, and hence $\dim_h(L) = 1$. If $d = 1$, cut times must be points of increase or points of decrease. It is known [9] that with probability one, one dimensional Brownian motion has no points of increase. Hence $L = \emptyset$. But, it is easy to check that the intersection exponent, as defined in (1) equals one for $d = 1$.

Define the set of global cut times to be

$$L^G = \{t : B[0,t) \cap B(t,\infty) = \emptyset\}.$$

In two dimensions, with probability one, $L^G = \{0\}$. In three dimensions, there are nontrivial global cut times. As part of the proof of Theorem 1.1 we prove that with probability one

$$\dim_h(L^G) = 1 - \zeta, \quad d = 3.$$

The set of local cut times can be defined by

$$L^{\text{loc}} = \{t \in [0,1] : \text{there exists } \epsilon > 0 \text{ with } B[t-\epsilon, t) \cap B(t, t+\epsilon] = \emptyset\}.$$

Then

$$L^{\text{loc}} = \bigcup_{n=1}^{\infty} L_n,$$

where

$$L^n = \{t \in [0,1] : B[t - \frac{1}{n}, t) \cap B(t, t + \frac{1}{n}] = \emptyset\}.$$

It follows from Theorem 1.1 that $\dim_h(L^n) = 1 - \zeta$ for each n. Hence, with probability one,

$$\dim_h(L^{\mathrm{loc}}) = 1 - \zeta.$$

Let

$$B(L) = \{B(t) : t \in L\}$$

denote the set of cut points on $B[0,1]$. It is well known [11, 17] that Brownian motion doubles the Hausdorff dimension of sets in $[0,1]$. Therefore, it follows from the theorem that with probability one,

$$\dim_h[B(L)] = 2(1-\zeta).$$

As in [2], we can define the *percolation dimension* of Brownian motion to be the infimum of all numbers a such that there exists a continuous curve $\gamma : [0,1] \to \mathbf{R}^d$ with $\gamma(0) = 0, \gamma(1) = B(1)$, $\gamma[0,1] \subset B[0,1]$, and such that the Hausdorff dimension of $\gamma[0,1]$ equals a. It is easy to check that for any such γ,

$$B(L) \subset \gamma[0,1].$$

Using the fact that $\zeta_3 < 1/2$, we have the following corollary.

Corollary 1.2 *The percolation dimension of Brownian motion is greater than or equal to* $2(1-\zeta)$. *In particular, the percolation dimension of three dimensional Brownian motion is strictly greater than* 1.

The main techinical tool in proving Theorem 1.1 is an improvement of the estimate (1). As before, let B^1, B^2 be independent Brownian motions in $\mathbf{R}^d$ and let

$$T_n^i = \inf\{t : |B^i(t)| = n\}.$$

Let

$$a_n = \sup_{|x|,|y|=1} \mathbf{P}^{x,y}\{B^1[0,T_n^1] \cap B^2[0,T_n^2] = \emptyset\},$$

where $\mathbf{P}^{x,y}$ denotes probabilities assuming $B^1(0) = x, B^2(0) = y$. Let

$$b_n = \mathbf{P}^{0,0}\{B^1[1,n] \cap B^2[0,n] = \emptyset\}.$$

We prove in Section 3 that there exist constants $0 < c_3 < c_4 < \infty$ such that

$$c_3 n^{-2\zeta} \le a_n \le c_4 n^{-2\zeta}, \tag{2}$$

$$c_3 n^{-\zeta} \le b_n \le c_4 n^{-\zeta} \tag{3}$$

A somewhat different proof of (2) for $d = 2$ can be found in [15]; this proof generalizes to the case of multiple intersections. However, since we need (2) for $d = 3$ as well as $d = 2$, we include a complete proof in this paper. A similar argument has been used in [14] to relate the Hausdorff dimension of the frontier or outer boundary of planar Brownian motion to a "disconnection exponent." The analogous result for simple random walk can be found in [13].

For $1 \le k \le 2^n$, we let $A(k,n)$ be the event

$$A(k,n) = \{B[0, \frac{k-1}{2^n}] \cap B[\frac{k}{2^n}, 1] = \emptyset, \}$$

and let

$$J_n = \#\{k : \frac{1}{4}2^n \leq k \leq \frac{3}{4}2^n, A(k,n) \text{ holds }\}.$$

It follows from (3) that

$$\mathbf{E}(J_n) \geq c_1(2^n)^{1-\zeta}.$$

(Here and throughout this paper we use $c, c_1, c_2, \ldots$ for arbitrary positive constants. The values of c, c_1, c_2 may vary from place to place, but the values of $c_3, c_4, \ldots$ will not change.) We can also use (3) to estimate higher moments. In particular,

$$\mathbf{E}(J_n^2) \leq c_2(2^n)^{2(1-\zeta)}.$$

Hence, by a standard argument, we get that $J_n \geq c(2^n)^{1-\zeta}$ with some positive probability, independent of n. This gives a good indication that the Hausdorff dimension of L should be $1-\zeta$, and with this bound standard techniques can be applied to establish the result.

In the next section we will give the proof of Theorem 1.1, saving the proofs of the key estimates for the final two sections. In Section 3, we prove the estimates (2) and (3) and in the final section we establish the bound on $\mathbf{E}(J_n^2)$.

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2 Proof of Main Theorem

In this section we prove Theorem 1.1, delaying the proofs of some of the estimates to the next sections. We will not need to know the exact value of ζ, but we will use the fact that

$$\zeta \in [\frac{1}{4}, \frac{3}{4}].$$

The upper bound on the Hausdorff dimension is fairly straightforward using (3). Let

$$K_n = \#\{k : 1 \leq k \leq 2^n : A(k,n) \text{ holds }\},$$

where, as before,

$$A(k,n) = \{B[0, \frac{k-1}{2^n}] \cap B[\frac{k}{2^n}, 1] = \emptyset\}.$$

By (3),

$$\mathbf{E}(K_n) \leq c_5(2^n)^{1-\zeta}.$$

If $\epsilon > 0$, Markov's inequality gives

$$\mathbf{P}\{K_n \geq (2^n)^{1-\zeta+\epsilon}\} \leq c_5(2^n)^{-\epsilon},$$

and hence by the Borel-Cantelli Lemma,

$$\mathbf{P}\{K_n \geq (2^n)^{1-\zeta+\epsilon} \text{ i. o.}\} = 0.$$

But if $K_n \leq (2^n)^{1-\zeta+\epsilon}$ for all sufficiently large n, L can be covered by $(2^n)^{1-\zeta+\epsilon}$ intervals of length $(2^n)^{-1}$. By standard arguments, this implies $\dim_h(L) \leq 1-\zeta+\epsilon$. Since this holds with probability one for all $\epsilon > 0$,

$$\mathbf{P}\{\dim_h(L) \leq 1-\zeta\} = 1.$$

The lower bound is the difficult result. We will need the following standard criterion for estimating the Hausdorff dimension of a subset of $[0,1]$ (see [10] for relevant facts about Hausdorff dimension).

Lemma 2.1 *[10, Theorem 4.13] Suppose $X \subset [0,1]$ is a closed set and let μ be a positive measure supported on X with $\mu(X) > 0$. Let the β-energy, $I_\beta(\mu)$, be defined by*

$$I_\beta(\mu) = \int_0^1 \int_0^1 |s-t|^{-\beta}\, d\mu(s)\, d\mu(t).$$

If $I_\beta(\mu) < \infty$, then

$$\dim_h(X) \geq \beta.$$

We will start by showing that there is a positive probability that $\dim_h(L) = 1-\zeta$. More precisely, we prove the following. Let $\mathcal{B}(x,r)$ denote the open ball of radius r in $\mathbf{R}^d$.

Proposition 2.2 *There exist a $c_6 > 0$ such that*

$$\mathbf{P}\{\dim_h(L \cap [\frac{1}{4}, \frac{3}{4}]) = 1-\zeta; B[0, \frac{1}{4}] \subset \mathcal{B}(0, \frac{1}{2}); |B(1)| \geq 1\} \geq c_6.$$

Consider J_n and $A(k,n)$ as defined in the previous section. Then by (3), there exsits a $c > 0$ such that

$$\mathbf{E}(J_n) \geq c(2^n)^{1-\zeta}.$$

If we let

$$\tilde{A}(k,n) = A(k,n) \cap \{B[0, \frac{1}{4}] \subset \mathcal{B}(0, \frac{1}{2}); |B(1)| \geq 1\},$$

and

$$\tilde{J}_n = \#\{k : \frac{1}{4}2^n < k \leq \frac{3}{4}2^n, \tilde{A}(k,n) \text{ holds }\},$$

then a similar argument (see Lemma 3.17) shows that there is a constant $c_7 > 0$ such that

$$\mathbf{E}(\tilde{J}_n) \geq c_7(2^n)^{1-\zeta}. \tag{4}$$

We will need estimates on higher moments of J_n. We prove the following in Section 4.

Lemma 2.3 *There exists a $c_8 < \infty$ such that if $2^{n-2} \leq j \leq k \leq 3 \cdot 2^{n-2}$,*

$$\mathbf{P}[A(j,n) \cap A(k,n)] \leq c_8(2^n)^{-\zeta}(k-j+1)^{-\zeta}, \tag{5}$$

and hence

$$\mathbf{E}(J_n^2) \leq c(2^n)^{2(1-\zeta)}. \tag{6}$$

Standard arguments now can be used to show that (4) and (6) imply that there exists a $c_9 > 0$ such that for each n,

$$\mathbf{P}\{\tilde{J}_n \geq c_9(2^n)^{1-\zeta}\} \geq c_9, \tag{7}$$

and hence

$$\mathbf{P}\{\tilde{J}_n \geq c_9(2^n)^{1-\zeta} \text{ i.o.}\} \geq c_9. \tag{8}$$

Let μ_n be the (random) measure whose density, with respect to Lebesgue measure, is $(2^n)^\zeta$ on each interval $[(k-1)2^{-n}, k2^{-n}]$ with $2^{n-2} < k \leq 3 \cdot 2^{n-2}$ such that $\tilde{A}(k,n)$ holds and assigns measure zero elsewhere. It is easy to check that $\mathrm{supp}(\mu_{n+1}) \subset \mathrm{supp}(\mu_n)$ and, with probability one,

$$\bigcap_{n=1}^{\infty} \mathrm{supp}(\mu_n) \subset L \cap [\frac{1}{4}, \frac{3}{4}].$$

Also, μ_n is the zero measure on the complement of the event

$$\{B[0, \frac{1}{4}] \subset \mathcal{B}(0, \frac{1}{2}), |B(1)| \geq 1\}.$$

By (8), with probability at least c_9, we can find a subsequence μ_{n_i} such that

$$\mu_{n_i}([0,1]) \geq c_9. \tag{9}$$

This shows that $L \cap [\frac{1}{4}, \frac{3}{4}]$ is nonempty with positive probability.

Let $\beta = 1 - \zeta - \epsilon$ with $\epsilon > 0$ and let $I_\beta(\mu_n)$ denote the β-energy of μ_n as described in Lemma 2.1. Then, by (5),

$$\begin{aligned}
\mathbf{E}[I_\beta(\mu_n)] &= \sum 2^{2\zeta n}[\int_{(j-1)2^{-n}}^{j2^{-n}} \int_{(k-1)2^{-n}}^{k2^{-n}} (s-t)^{-\beta}\, ds\, dt]\mathbf{P}[A(j,n) \cap A(k,n)] \\
&\leq u_\beta \sum 2^{2\zeta n}[2^{(\beta-2)n}(|k-j|+1)^{-\beta}]2^{-n\zeta}(|k-j|+1)^{-\zeta} \\
&\leq u_\beta \sum 2^{-n}2^{-\epsilon n}(|k-j|+1)^{\epsilon-1} \\
&\leq u_\beta.
\end{aligned}$$

Here the sums are over all $2^{n-2} < j, k \leq 3 \cdot 2^{n-2}$ and u_β is a positive constant, depending on β, whose value may change from line to line. In particular,

$$\mathbf{P}\{I_\beta(\mu_n) \geq 2u_\beta/c_9\} \leq \frac{1}{2}c_9.$$

Therefore, using (8),

$$\mathbf{P}\{\mu_n([\frac{1}{4}, \frac{3}{4}]) \geq c_9;\ I_\beta(\mu_n) \leq 2u_\beta/c_9 \text{ i.o.}\} \geq \frac{1}{2}c_9.$$

On the event above, let μ be any weak limit of the μ_n. Then it is easy to verify that μ is supported on $L \cap [1/4, 3/4]$; $\mu(L) \geq c_9$; and $I_\beta(\mu) \leq 2u_\beta/c_9$. By Lemma 2.1, we can conclude that

$$\mathbf{P}\{\dim_h(L \cap [\frac{1}{4}, \frac{3}{4}]) \geq 1 - \zeta - \epsilon; B[0, \frac{1}{4}] \subset \mathcal{B}(0, \frac{1}{2}), |B(1)| \geq 1\} \geq \frac{c_9}{2}.$$

Since this holds for every $\epsilon > 0$,

$$\mathbf{P}\{\dim_h(L \cap [\frac{1}{4}, \frac{3}{4}]) \geq 1 - \zeta; B[0, \frac{1}{4}] \subset \mathcal{B}(0, \frac{1}{2}), |B(1)| \geq 1\} \geq \frac{c_9}{2}.$$

This gives Proposition 2.2.

If $d = 3$, let L^G be the set of global cut times,

$$L^G = \{t : B[0,t) \cap B(t,\infty) = \emptyset\}.$$

It follows immediately from Proposition 2.2 and the transience of Brownian motion, that

$$\mathbf{P}\{\dim_h(L^G \cap [0,1]) = 1 - \zeta\} > 0.$$

However, scaling tells us that

$$\mathbf{P}\{\dim_h(L^G \cap [0,t]) = 1 - \zeta\},$$

is independent of t. It is not difficult to see that the only way this happens is if

$$\mathbf{P}\{\dim_h(L^G \cap [0,1]) = 1 - \zeta\} = 1.$$

We now finish the proof of Theorem 1.1 for $d = 2$. Assume $d = 2$ and let $L_t = L \cap [0,t]$ and for $t \leq s \leq 1$,

$$L_t(s) = \{r : B[0,r) \cap B(r,s] = \emptyset\} \cap [0,t].$$

Note that $L_t = L_t(1)$. Let $W_t(s)$ be the event

$$W_t(s) = \{\dim_h(L_t(s)) = 1 - \zeta\},$$

and U_s the event

$$U_s = \bigcap_{n=1}^{\infty} W_{1/n}(s).$$

For any $t < s < 1$,

$$W_t(s) \setminus W_t(1) \subset \{B[0,t] \cap B[s,1] = \emptyset\}.$$

For fixed $s > 0$, the probability of the event on the right goes to zero as $t \to 0$. Hence for every $s > 0$,

$$\mathbf{P}(U_s \setminus U_1) = 0,$$

i.e., the sets agree up to an event of probability zero. If

$$U = \bigcup_{n=1}^{\infty} U_{1/n},$$

then it easy easy to see from the Blumenthal 0-1 Law that $\mathbf{P}(U)$ must be 0 or 1. Since $\mathbf{P}(U) = \mathbf{P}(U_1)$, U_1 satisfies a 0-1 Law, and hence it suffices to prove that $\mathbf{P}(U_1) > 0$.

Let

$$V_n = \{\dim_h(L \cap [\frac{1}{4}2^{-n}, \frac{3}{4}2^{-n}]) = 1 - \zeta\}.$$

By Proposition 2.2 and a standard estimate, there is a $c_{10} > 0$ such that

$$\mathbf{P}(V_n) \geq \frac{c_{10}}{n}.$$

However, the straightforward estimate

$$\mathbf{P}\{B[0,1] \cap B[3,n] = \emptyset\} \leq \frac{c}{\ln n},$$

allows us to conclude, for $m < n-3$,

$$\begin{aligned}\mathbf{P}(V_n \cap V_m) &\leq \mathbf{P}\{B[0,\frac{1}{4}2^{-n}] \cap [\frac{3}{4}2^{-n},\frac{1}{8}2^{-m}] = \emptyset, \\ &\qquad B[\frac{1}{8}2^{-m},\frac{1}{4}2^{-m}] \cap B[\frac{3}{4}2^{-m},1] = \emptyset\} \\ &\leq \frac{c}{m(n-m)}.\end{aligned}$$

Let

$$Z_n = \sum_{j=1}^{n} I(V_j),$$

where again I denotes indicator function. The estimates above allow us to conclude that there are constants c_{11} and c_{12} such that

$$\mathbf{E}(Z_n) \geq c_{11} \ln n, \quad \mathbf{E}(Z_n^2) \leq c_{12}(\ln n)^2,$$

so we may conclude again using standard arguments that

$$\mathbf{P}(U_1) \geq \mathbf{P}\{V_n \text{ i.o}\} > 0.$$

This finishes the proof of the theorem.

3 Estimate for Intersection Probabilities

The goal of this section is to proof the estimates (2) and (3). Let B^1, B^2 be independent Brownian motions in $\mathbf{R}^d$, $d = 2,3$. Let

$$T_n^i = \inf\{t : |B^i(t)| = n\},$$

and let A_n be the event

$$A_n = \{B^1[0,T_n^1] \cap B^2[0,T_n^2] = \emptyset\}.$$

Let

$$a_n = \sup_{|x|,|y|=1} \mathbf{P}^{x,y}(A_n),$$

where $\mathbf{P}^{x,y}$ denotes probabilities assuming $B^1(0) = x, B^2(0) = y$. If $d = 2$, it can be shown [6] that the supremum is taken on when $|x-y| = 2$, but we will not use that fact here. Scaling and rotational invariance imply that

$$a_{nm} \leq a_n a_m,$$

and hence by standard arguments, using the subadditivity of $\log a_{2^n}$, there exists a $\zeta > 0$ such that

$$a_n \approx n^{-2\zeta},$$

and, in fact, $a_n \geq n^{-2\zeta}$ for all n. The exact value of ζ is not known; for this section, it will suffice to know the bounds

$$\zeta_2 \in (\frac{1}{2},\frac{3}{4}), \quad \zeta_3 \in [\frac{1}{4},\frac{1}{2}).$$

The lower bound in (2) follows from submultiplicativity. To get a bound in the other direction, it suffices to show that there is a constant $c > 0$ such that for all n, m,

$$a_{nm} \geq c a_n a_m. \tag{10}$$

(One can check this by noting that (10) implies that $b_n = \log a_{2^n} + \log c$ is superadditive.) As before, we let $\mathcal{B}(x, r)$ denote the open ball of radius r about x in $\mathbf{R}^d$.

Lemma 3.1 *There exists a $c_{13} > 0$ such that for all n,*

$$a_{n+1} \geq c_{13} a_n.$$

Proof. Let V_n be the event

$$V_n = \{B^1[0, T_n^1] \cap \mathcal{B}(B^2(T_n^2), 5) = \emptyset, B^2[0, T_n^2] \cap \mathcal{B}(B^1(T_n^1), 5) = \emptyset\}.$$

¿From the Harnack principle and rotational symmetry of Brownian motion, we can see that for any $|x|, |y| \leq n/2$,

$$\mathbf{P}^{x,y}(V_n^c) \leq c n^{1-d}.$$

Hence, if $|x|, |y| = 1$,

$$\mathbf{P}^{x,y}(V_n^c \cap A_n) \leq \mathbf{P}^{x,y}(A_{n/2}) \mathbf{P}^{x,y}(V_n^c \mid A_{n/2}) \leq c n^{-2}$$

(the last inequality follows for $d = 2$, since $\zeta_2 > 1/2$). Since $\zeta < 1$, this implies for all n sufficiently large,

$$\sup_{|x|,|y|=1} \mathbf{P}^{x,y}(A_n \cap V_n) \geq \frac{1}{2} a_n.$$

It is easy to see that a Brownian motion starting on the sphere of radius n has a positive probability (independent of n) of reaching the sphere of radius $n + 1$ while remaining within distance 2 of its starting point. Hence there is a constant $c > 0$ such that for all $|x|, |y| = 1$,

$$\mathbf{P}^{x,y}(A_{n+1} \mid A_n \cap V_n) \geq c. \quad \Box$$

Lemma 3.2 *There exists a $c_{14} > 0$ such that for all n and all $|x|, |y| = 1$,*

$$\mathbf{P}^{x,y}(A_n) \leq c_{14} |x - y|^{1/8} a_n.$$

Proof. For any x, y with $|x| = 1, |x - y| \leq 1$, let

$$\tau^i = \inf\{t : |B^i(t) - x| = 1\}.$$

Since $\zeta \geq 1/4$, there exists a $c > 0$ such that

$$\mathbf{P}^{x,y}\{B^1[0, \tau^1] \cap B^2[0, \tau^2] = \emptyset\} \leq c|x - y|^{1/8}.$$

Hence by the strong Markov property,

$$\begin{aligned} \mathbf{P}^{x,y}(A_n) &\leq \mathbf{P}^{x,y}\{B^1[0, \tau^1] \cap B^2[0, \tau^2] = \emptyset\} \\ &\qquad \mathbf{P}^{x,y}(B^1[\tau^1, T_n^1] \cap B^2[\tau^2, T_n^2] = \emptyset \mid B^1[0, \tau^1] \cap B^2[0, \tau^2] = \emptyset\} \\ &\leq c|x - y|^{1/8} a_{n-1}. \end{aligned}$$

The result then follows from Lemma 3.1. $\Box$

Lemma 3.3 *For every $\epsilon > 0$, let*

$$U_n^i = U_n^i(\epsilon) = \{B^i[0, T_n^i] \cap \mathcal{B}(0,1) \subset \mathcal{B}(B^i(0), \epsilon)\},$$

There exists a constant $c_{15} > 0$ such that for every $\epsilon > 0$

$$\sup_{|x|,|y|=1} \mathbf{P}^{x,y}(A_n \cap U_n^1) \geq c_{15}\epsilon a_n.$$

Proof. Without loss of generality we assume $\epsilon \leq 1/10, n \geq 2$. By the previous lemma, there exists a $\delta > 0$ such that if $|x|, |y| \leq 1, |x-y| \leq \delta$,

$$\mathbf{P}^{x,y}(A_n) \leq \frac{1}{2} a_n.$$

Fix such a $\delta > 0$. Fix n and choose some $|x|, |y| = 1$ that maximize $\mathbf{P}^{x,y}(A_n)$. (It is easy to see that $f(x,y) = \mathbf{P}^{x,y}(A_n)$ is continuous and hence the maximum is obtained.) For $\epsilon > 0$ let

$$\rho = \rho_\epsilon = \inf\{t : B^1(t) \in \mathcal{B}(0,1) \setminus \mathcal{B}(B^1(0), \epsilon)\},$$

$$\sigma = \sigma_\epsilon = \inf\{t \geq \rho : |B^1(t)| = 1 - \epsilon\},$$

$$\tau = \tau_\epsilon = \inf\{t \geq \rho : |B^1(t)| = 1\}.$$

Note that there is an $r > 0$ (independent of $\epsilon < 1/10$) such that

$$\mathbf{P}^{x,y}\{\sigma < \tau \mid \rho < T_n^1\} \geq r.$$

(This can be verified by noting that

$$\mathbf{P}\{|B(\rho)| \leq 1 - \frac{\epsilon}{2}; \rho < T_n^1\} \geq c,$$

and

$$\mathbf{P}\{\sigma < \tau \mid |B(\rho)| \leq 1 - \frac{\epsilon}{2}, \rho < T_n^1\} \geq c.)$$

If a Brownian motion is on the sphere of radius $1-\epsilon$, there is a probability of at least $c\epsilon$ that it reaches the sphere of radius $1/2$ before leaving the ball of radius 1. By the Harnack principle, a Brownian motion starting on the sphere of radius $1/2$ has a positive probabilty, independent of the starting point, of hitting the unit sphere first within distance δ of y (the probability depends on δ, but we have fixed δ). Hence

$$\mathbf{P}^{x,y}\{|B(\tau) - y| < \delta \mid \rho < T_n^1\} \geq c\epsilon.$$

Using the strong Markov property, we conclude

$$\mathbf{P}^{x,y}(A_n \mid \rho < T_n^1, \sigma < \tau) \leq (1 - c\epsilon)a_n + c\epsilon(a_n/2).$$

But,

$$\mathbf{P}^{x,y}(A_n \mid \rho < T_n^1, \sigma \geq \tau) \leq a_n,$$

and hence

$$\mathbf{P}^{x,y}(A_n; \rho < T_n^1) \leq \mathbf{P}^{x,y}(A_n \mid \rho < T_n^1) \leq (1 - c_{15}\epsilon)a_n,$$

for appropriately chosen $c_{15} > 0$. Therefore,

$$\mathbf{P}^{x,y}(A_n; \rho > T_n^1) \geq c_{15} \epsilon a_n. \quad \Box$$

For any n, ϵ, consider the x, y with $|x| = |y| = 1$ that maximize

$$\mathbf{P}^{x,y}(U_n^1 \cap A_n).$$

It is intuitively obvious that the x, y that maximize this quantity are not very close to each other. However, it takes some effort to prove this. Let $|x| = 1$ and let

$$Y_n = \sup_{|y|=1} \mathbf{P}^{x,y}\{B^1[0, T_n^1] \cap B^2[0, T_n^2] = \emptyset \mid B^1[0, T_n^1]\}.$$

Lemma 3.4 *For every $M < \infty$ there exists a $\delta > 0$ and a constant $k < \infty$ such that*

$$\mathbf{P}\{Y_n \geq n^{-\delta}\} \leq k n^{-M}.$$

Proof. We only sketch the proof. We will consider $Y^n = Y_{2^n}$. For $j \leq n$, let

$$R^j = \inf_{|z| \leq 2^{j-1}} \mathbf{P}^z\{B^2[0, T_{2^j}^1] \cap B^1[T_{2^{j-1}}^1, T_{2^j}^1] \neq \emptyset\},$$

where $\mathbf{P}^z$ indicates that $B^2(0) = z$ and R^j is considered as a function of $B^1[0, T_{2^n}^1]$. It is easy to show that

$$\mathbf{P}\{R^j = 0\} = 0,$$

and hence for any $\epsilon > 0$ there is a $\gamma > 0$ with

$$\mathbf{P}\{R^j \leq \gamma\} < \epsilon.$$

Since $R^2, \ldots, R^n$ are independent, identically distributed, standard large deviation estimates for binomial random variables allow us to choose γ and k so that

$$\mathbf{P}\{R^j \leq \gamma \text{ for at least } n/2 \text{ values of } j \leq n\} \leq k(2^{n+1})^{-M}.$$

But if $R^j \geq \gamma$ for at least $n/2$ values, then

$$Y^n \leq (1-\gamma)^{n/2} \leq (2^n)^{-\delta},$$

for appropriately chosen δ. $\Box$

Lemma 3.5 *There exist a $c_{16} > 0$ and a $c_{17} < \infty$ such that if $U_n = U_n^1(\epsilon)$ is defined as in Lemma 3.3,*

$$\sup_{|x|=1, |y-x| \leq 10\epsilon} \mathbf{P}^{x,y}(A_n \cap U_n^1) \leq c_{17} \epsilon^{1+c_{16}} a_n.$$

Proof. Let
$$\tau^i = \inf\{t : |B^i(t) - B^1(0)| = 1\},$$
and let $\tilde{U} = \tilde{U}(\epsilon)$ be the event
$$\tilde{U} = \{B^1[0,\tau^1] \cap \mathcal{B}(0,1) \subset \mathcal{B}(B^1(0),\epsilon)\}.$$
A standard estimate gives for $|x| = 1$,
$$c_1\epsilon \leq \mathbf{P}^x(\tilde{U}) \leq c_2\epsilon.$$
Lemma 3.4 can be used to see that there is a $c_{16} > 0$ and a $c < \infty$ such that for $|y - x| \leq 10\epsilon$,
$$\mathbf{P}^{x,y}\{B^1[0,\tau^1] \cap B^2[0,\tau^2] = \emptyset \mid \tilde{U}\} \leq c\epsilon^{c_{16}},$$
and hence
$$\mathbf{P}^{x,y}(\tilde{U}; B^1[0,\tau^1] \cap B^2[0,\tau^2] = \emptyset) \leq c\epsilon^{1+c_{16}}.$$
By using the strong Markov property as in Lemma 3.2, we get the result. $\square$

Lemma 3.6 *Let U_n^i be as defined in Lemma 3.3. Then there exist an $\epsilon > 0$ and a $c_{18} > 0$ such that*
$$\sup_{|x|,|y|=1,|x-y|\geq 10\epsilon} \mathbf{P}^{x,y}(A_n \cap U_n^1 \cap U_n^2) \geq c_{18}a_n.$$

Proof. Fix n and choose $|x|, |y| = 1$ that maximize $\mathbf{P}^{x,y}(A_n \cap U_n^1)$. By Lemmas 3.3 and 3.5 we can find an ϵ so that $|x - y| \geq 10\epsilon$ for all n. The proof then proceeds as in Lemma 3.3. $\square$

Now fix ϵ, c_{18} as in Lemma 3.6. For any $\lambda < \epsilon$, let
$$W_\lambda^i = \{B^i[0,T_n^i] \cap \mathcal{B}(B^{3-i}(0),\lambda) = \emptyset\}.$$
By Lemma 3.2 and the strong Markov property,
$$\sup_{|x|,|y|=1} P^{x,y}(A_n \cap (W_\lambda^i)^c) \leq \sup_{|x-y|\leq\lambda} P^{x,y}(A_n) \leq c\lambda^{1/8}a_n.$$
Hence we can choose $\lambda > 0$ so that
$$\sup_{|x|,|y|=1,|x-y|\geq 10\epsilon} \mathbf{P}^{x,y}[A_n \cap U_n^1 \cap U_n^2 \cap W_\lambda^1 \cap W_\lambda^2] \geq \frac{1}{2}c_{18}a_n.$$
We have therefore proved the following lemma.

Lemma 3.7 *There exist positive constants $c_{20}, c_{21}, c_{22} < 1/10$, such that the following is true. For any $|x| = 1$, let*
$$\Gamma^i(x) = \Gamma_n^i(x, c_{20}) = \{B^i[0,T_n^i] \cap \mathcal{B}(x,c_{20}) = \emptyset\},$$
$$\Lambda^i(x) = \Lambda_n^i(x, c_{21}) = \{B^i[0,T_n^i] \cap \mathcal{B}(0,1) \subset \mathcal{B}(x,c_{21})\}.$$
Then for every n, there exist $|x|, |y| = 1$ with $|x - y| \geq 10c_{20}$ such that
$$\mathbf{P}^{x,y}[A_n \cap \Gamma^1(y) \cap \Gamma^2(x) \cap \Lambda^1(x) \cap \Lambda^2(y)] \geq c_{22}a_n.$$

It is easy to verify, using standard estimates for the Poisson kernel of the ball, that there is a $c_{23} > 0$ such that for $|w - x| \le c_{20}/2, |z - y| \le c_{20}/2$,

$$\mathbf{P}^{w,z}[A_n \cap \Gamma^1(y) \cap \Gamma^2(x) \cap \Lambda^1(x) \cap \Lambda^2(y)] \ge c_{23} a_n,$$

where x, y are chosen as in Lemma 3.7

Corollary 3.8 *There is a constant $c_{24} > 0$ such that for all n,*

$$a_{2n} \ge c_{24} a_n.$$

Proof. Let c_{20}, c_{21}, c_{22} be as above. It is easy to verify that there is a $p > 0$ such that if $|x|, |y| = 1$, $|x - y| \ge 10c_{20}$; e_1 is the unit vector whose first component equals 1; and B^1 and B^2 are independent Brownian motions starting at $e_1/2$ and $-e_1/2$ respectively, then

$$\mathbf{P}\{B^1[0, T_1^1] \cap B^2[0, T_1^2] = \emptyset; B^1[0, T_1^1] \cap \mathcal{B}(y, c_{21}) = \emptyset; B^2[0, T_1^2] \cap \mathcal{B}(x, c_{21}) = \emptyset;$$
$$|B^1(T_1^1) - x| \le c_{20}/2; |B^2(T_1^2) - y| \le c_{20}/2\} \ge p.$$

Then by combining this path with the paths mentioned above and scaling by a factor of 2 we see that

$$\mathbf{P}^{e_1, -e_1}(A_{2n}) \ge p c_{23} a_n. \ \Box$$

The choice of $e_1, -e_1$ in the proof above was arbitrary. In fact, the same idea can be used to prove the following.

Corollary 3.9 *For every $\delta > 0$ there is an $\epsilon > 0$ such that if $|x|, |y| = 1, |x - y| \ge \delta$,*

$$\mathbf{P}^{x,y}(A_n) \ge \mathbf{P}^{x,y}(A_{2n}) \ge \epsilon a_n.$$

We omit the proof of the next easy lemma.

Lemma 3.10 *For every $\delta > 0$ there is an $\epsilon > 0$ such that for $|x|, |y| = 1$,*

$$\mathbf{P}^{x,y}\{B^i[0, T_2^i] \cap \mathcal{B}(B(T_2^{3-i}), \epsilon) = \emptyset, i = 1, 2\} \ge 1 - \delta.$$

Let

$$J_n = J_n(\epsilon) = \{B^1[0, T_n^1] \cap \mathcal{B}(B(T_n^2), \epsilon n) \ne \emptyset \text{ or } B^2[0, T_n^2] \cap \mathcal{B}(B(T_n^1), \epsilon n) \ne \emptyset\}.$$

Then it follows from Corollary 3.8, Lemma 3.10, and the strong Markov property that for every $\delta > 0$, there is an $\epsilon > 0$ such that

$$\mathbf{P}(J_n \cap A_n) \le \mathbf{P}(J_n \cap A_{n/2}) \le \delta a_{n/2} \le c\delta a_n.$$

We have therefore proved the following.

Corollary 3.11 *There exists a $c_{25} > 0$ and a $c_{26} < \infty$ such that if $J_n = J_n(c_{25})$ is defined as above,*

$$\mathbf{P}(A_n \cap J_n^c) \ge c_{26} a_n.$$

This corollary tells us that Brownian paths that do not intersect have a good chance of being reasonably far apart. Once we know that they can be reasonably far apart we can attach many different configurations to the ends of the two walks and still get an event with probability greater than a constant times a_n. As an example, we state the following corollary whose proof we omit.

Corollary 3.12 *Let z^1 denote the first coordinate of $z \in \mathbf{R}^d$ and divide the ball of radius n into three sets:*

$$\mathcal{B}(0,n) = \mathcal{B}_+(n) \cup \mathcal{B}_-(n) \cup \mathcal{B}_0(n),$$

where

$$\mathcal{B}_+(n) = \{z \in \mathcal{B}(0,n) : z^1 \geq \frac{n}{8}\},$$

$$\mathcal{B}_-(n) = \{z \in \mathcal{B}(0,n) : z^1 \leq -\frac{n}{8}\},$$

$$\mathcal{B}_0(n) = \mathcal{B}(0,n) \setminus (\mathcal{B}_+(n) \cup \mathcal{B}_-(n)).$$

Let

$$V_n^1 = \{B^1[0,T_n^1] \subset \mathcal{B}_-(n) \cup \mathcal{B}(0,\frac{n}{2})\},$$

$$V_n^2 = \{B^2[0,T_n^2] \subset \mathcal{B}_+(n) \cup \mathcal{B}(0,\frac{n}{2})\}.$$

Then there is a $c_{27} > 0$ such that

$$\sup_{|x|,|y|=1} \mathbf{P}^{x,y}(A_n \cap V_n^1 \cap V_n^2) \geq c_{27} a_n.$$

Lemma 3.7, Corollary 3.9, and Corollary 3.12 can now be combined to give (2).

Corollary 3.13 *There exists a $c > 0$ such that for all n, m,*

$$a_{nm} \geq c_{28} a_n a_m,$$

and hence there is a constant $c_4 < \infty$ such that

$$n^{-2\zeta} \leq a_n \leq c_4 n^{-2\zeta}.$$

Moreover, for every $\delta > 0$ there is an $\epsilon > 0$ such that if $|x|, |y| = 1, |x-y| \geq \delta$,

$$\mathbf{P}^{x,y}(A_n) \geq \epsilon n^{-2\zeta}.$$

We will now prove the estimate for fixed times, (3). We note that there is a constant $\beta > 0$ and a $c_{28} < \infty$ such that for $|x|, |y| \leq n/2, k > 0$,

$$\mathbf{P}^{x,y}\{\min(T_n^1, T_n^2) \leq kn^2\} \leq c_{28} e^{-\beta/k}. \tag{11}$$

$$\mathbf{P}^{x,y}\{\max(T_n^1, T_n^2) \geq kn^2\} \leq c_{28} e^{-\beta k}, \tag{12}$$

(By scaling, it suffices to show this for $n = 1$. The first inequality can be derived from

$$\begin{aligned} \mathbf{P}^x\{T_1^1 \leq k\} &\leq \mathbf{P}^0\{\sup_{t \leq k} |B^1(t)| \geq \frac{1}{2}\} \\ &\leq 2\mathbf{P}^0\{|B^1(k)| \geq \frac{1}{2}\} = 2\mathbf{P}^0\{|B^1(1)| \geq \frac{1}{2\sqrt{k}}\}, \end{aligned}$$

and the second by iterating

$$\mathbf{P}^x\{T_1^1 \geq k+1 \mid T_1^1 \geq k\} \leq \mathbf{P}^0\{|B(1)| \leq 2\} < 1. \ )$$

Lemma 3.14 *There exist constants c_{29}, c_{30} such that for every n and every $|x|, |y| = 1$, $a > 0$,*

$$\mathbf{P}^{x,y}(A_n; \min(T_n^1, T_n^2) \leq an^2) \leq c_{29} e^{-c_{30}/a} n^{-2\zeta}.$$

$$\mathbf{P}^{x,y}(A_n; \max(T_n^1, T_n^2) \geq an^2) \leq c_{29} e^{-c_{30}a} n^{-2\zeta}.$$

Proof. The first inequality is easy, since

$$\begin{aligned} \mathbf{P}^{x,y}(A_n; \min(T_n^1, T_n^2) \leq an^2) &\leq \mathbf{P}^{x,y}(A_{n/2}) \mathbf{P}^{x,y}\{\min(T_n^1, T_n^2) \leq an^2 \mid A_{n/2}\} \\ &\leq cn^{-2\zeta} \sup_{|w|,|z|=n/2} P^{w,z}\{\min(T_n^1, T_n^2) \leq an^2\}, \end{aligned}$$

and the second term can be estimated using (11).

We will prove the second inequality for $m = 2^n$. Choose $|x| = |y| = 1$ and write $\mathbf{P}$ for $\mathbf{P}^{x,y}$. Assume $T_m = T_m^1 \geq a(2^n)^2$. Then there must be at least one $j = 1, \ldots, n$ with

$$T_{2^j} - T_{2^{j-1}} \geq (\frac{1}{2})^{n-j-1} a(2^n)^2 = \frac{a}{2}(2^j)^2 2^{n-j}.$$

By considering the three intervals $[0, T_{2^{j-1}}], [T_{2^{j-1}}, T_{2^j}], [T_{2^j}, T_{2^n}]$ separately we see, using (12), that

$$\mathbf{P}\{T_{2^j} - T_{2^{j-1}} \geq (2^j)^2 \frac{a}{2} 2^{n-j}; A_{2^n}\} \leq$$

$$[\sup_{|z|,|w|=1} \mathbf{P}^{z,w}(A_{2^{j-1}})][\sup_{|z|=2^{j-1}} \mathbf{P}^z\{T_{2^j} \geq (2^j)^2 \frac{a}{2} 2^{n-j}\}][\sup_{|z|,|w|=2^j} \mathbf{P}^{z,w}(A_{2^n})],$$

and hence,

$$\begin{aligned} \mathbf{P}\{T_{2^j} - T_{2^{j-1}} \geq (2^j)^2 \frac{a}{2} 2^{n-j}; A_{2^n}\} &\leq c(2^n)^{-2\zeta} \sup_{|z|=2^{j-1}} \mathbf{P}^z\{T_{2^j} \geq \frac{a}{2}(2^j)^2 2^{n-j}\} \\ &\leq c(2^n)^{-2\zeta} \exp\{-ac_{30} 2^{n-j}\}, \end{aligned}$$

for appropriately chosen c_{30}. Hence if $a \geq 1$ (it suffices to prove the lemma for $a \geq 1$),

$$\begin{aligned} \mathbf{P}\{T_{2^n} \geq a(2^n)^2; A_{2^n}\} &\leq \sum_{j=1}^{n} \mathbf{P}\{T_{2^j} - T_{2^{j-1}} \geq (2^j)^2 \frac{a}{2} 2^{n-j}; A_{2^n}\} \\ &\leq c \sum_{j=1}^{n-1} (2^n)^{-2\zeta} \exp\{-ac_{30} 2^{n-j}\} \\ &\leq c(2^n)^{-2\zeta} e^{-c_{30}a}. \end{aligned}$$

A similar argument holds for $T_{2^n}^2$. $\square$

Proposition 3.15 *There exist constants c_{31}, c_{32} such that, for all $n \geq 1$,*

$$\begin{aligned} c_{31} n^{-2\zeta} &\leq \sup_{|x|,|y|=1} \mathbf{P}^{x,y}\{B^1[0, n^2] \cap B^2[0, n^2] = \emptyset\} \\ &\leq \sup_{|x|,|y|=1} \mathbf{P}^{x,y}\{B^1[0, \min(T_n^1, n^2)] \cap B^2[0, \min(T_n^2, n^2)] = \emptyset\} \\ &\leq c_{32} n^{-2\zeta} \end{aligned}$$

Proof. The second inequality is trivial so we will only consider the first and third inequalities. Let $a > 0$. By Corollary 3.13 and Lemma 3.14,

$$\begin{aligned}
&\sup_{|x|,|y|=1} \mathbf{P}^{x,y}\{B^1[0,n^2] \cap B^2[0,n^2] = \emptyset\} \\
&\geq \sup_{|x|,|y|=1} [\mathbf{P}^{x,y}\{B^1[0,T^1_{an}] \cap B^2[0,T^2_{an}] = \emptyset\} \\
&\qquad -\mathbf{P}^{x,y}\{B^1[0,T^1 an] \cap B^2[0,T^2 an] = \emptyset; \min(T^1_{an},T^2_{an}) \leq n^2\}] \\
&\geq (an)^{-2\zeta} - c_{29}(an)^{-2\zeta}e^{-c_{30}a^2}.
\end{aligned}$$

If a is chosen sufficiently large, the last expression is greater than $(an)^{-2\zeta}/2$. This gives the first inequality.

It suffices to prove the third inequality for $n = 2^k$. For any $|x|,|y| = 1$,

$$\mathbf{P}^{x,y}\{B^1[0,\min(n^2,T^1_n)] \cap B^2[0,\min(n^2,T^2_n)] = \emptyset\} \leq$$
$$\mathbf{P}^{x,y}(A_n) + \sum_{j=1}^{k} \mathbf{P}^{x,y}(A_{2^{j-1}}; \max(T^1_{2^j},T^2_{2^j}) \geq n^2).$$

But, (12) and Lemma 3.14 allow us to conclude that there exist c and β such that

$$\mathbf{P}^{x,y}(A_{2^{j-1}}; \max(T^1_{2^j},T^2_{2^j}) \geq n^2\} \leq$$
$$c(2^{j-1})^{-2\zeta}\exp\{-\beta 2^{n-j}\} = c(2^n)^{-2\zeta}(2^{n-j})^{2\zeta}\exp\{-2^{n-j}\beta\}.$$

By summing over j, we get the lemma. $\square$.

The following can be proved similarly. We omit the proof.

Proposition 3.16 *There exist constants c_{33}, c_{34} such that if B^1 and B^2 are independent Brownian motions starting at the origin, $n \geq 1$,*

$$\begin{aligned}
c_{33}n^{-2\zeta} &\leq \mathbf{P}\{B^1[1,n^2] \cap B^2[0,n^2] = \emptyset\} \\
&\leq \mathbf{P}\{B^1[1,\min(n^2,T^1_n)] \cap B^2[0,\min(n^2,T^2_n)] = \emptyset\} \\
&\leq c_{34}n^{-2\zeta}.
\end{aligned}$$

Note that we have now proved the estimate (3). There is one slight improvement on this estimate that we will need. Let $a > 0$ and consider the event $A_{an} \cap V^1_{an} \cap V^2_{an}$ as descibed in Corollary 3.12. By the corollary and Lemma 3.14, we can see that we can choose a sufficiently small so that there is a $\lambda = \lambda(a) > 0$ such that for $|x|,|y| = 1$, $|x - y| \geq 1$,

$$\mathbf{P}^{x,y}(A_{an} \cap V^1_{an} \cap V^2_{an}; \max(T^1_{an},T^2_{an}) \geq n^2/10) \geq \lambda n^{-2\zeta}.$$

By appropriately extending the paths we can conclude the following.

Lemma 3.17 . *Let B^1, B^2 be independent Brownian motions starting at the origin. Define events Q^1_n, Q^2_n, Q^3_n by*

$$Q^1_n = \{B^1[0,20n^2] \subset \mathcal{B}(0,\frac{n^2}{10})\},$$

$$Q_n^2 = \{B^2[\frac{n^2}{4}, 20n^2] \cap \mathcal{B}(0, n^2) = \emptyset\},$$

$$Q_n^3 = \{B^1[1, 20n^2] \cap B^2[0, 20n^2] = \emptyset\}.$$

Then there exists a constant c_{35} such that

$$\mathbf{P}(Q_n^1 \cap Q_n^2 \cap Q_n^3) \geq c_{35} n^{-2\zeta}.$$

4 Moment Bound

In this section we prove Lemma 2.3. Let

$$D(k, n) = \{B[0, \frac{k-1}{n}] \cap B[\frac{k}{n}, 1] = \emptyset\},$$

and

$$G_n = \sum_{k=1}^{n} I(D(k, n)).$$

Note that $A(k, n) = D(k, 2^n)$ and $J_n \leq G_{2^n}$, where $A(k, n)$ and J_n are as defined in Sections 1 and 2. We will prove the following lemma. If $0 \leq i_1 \leq i_2 \leq \cdots \leq i_r \leq n$, let

$$f_{r,n}(i_1, \ldots, i_r) = [(i_1 + 1)(i_2 - i_1 + 1) \cdots (i_r - i_{r-1} + 1)(n - i_r + 1)]^{-\zeta}.$$

Lemma 4.1 *There exists a constant $c < \infty$ such that for all $0 \leq i \leq j \leq n$,*

$$\mathbf{P}[D(i, n) \cap D(j, n)] \leq c n^{\zeta} f_{2,n}(i, j). \tag{13}$$

It follows immediately from Lemma 4.1 that

$$\mathbf{E}(G_n^2) \leq c n^{\zeta} \sum_{0 \leq i \leq j \leq n} f_{2,n}(i, j).$$

But it is a straightforward exercise to show that the right hand side is bounded by

$$c n^{2(1-\zeta)}.$$

Hence to prove Lemma 2.3 it suffices to prove Lemma 4.1.

To prove Lemma 4.1 we will assume for ease that $j - i$ is even (a very easy modification is needed for $j - i$ odd) and we let $k = (j - i)/2$. Up to symmetry there are three cases to consider: Case I, $i \leq n - j \leq k$; Case II, $i \leq k \leq n - j$; and Case III, $k \leq i \leq n - j$.

For Case I,

$$\mathbf{P}[D(i, n) \cap D(j, n)] \leq$$
$$\mathbf{P}\{B[0, \frac{i-1}{n}] \cap B[\frac{i}{n}, \frac{i+k}{n}] = \emptyset, B[\frac{j-k}{n}, \frac{j}{n}] \cap B[\frac{j+1}{n}, 1] = \emptyset\}. \tag{14}$$

By independence and Proposition 3.15, this probability is bounded by a constant times $(i+1)^{-\zeta}(n - j + 1)^{-\zeta}$. Since $k \geq n/6$, we easily get the result.

Case II is similar except that we use Proposition 3.15 to conclude that the probability in (14) is bounded by a constant times $(i+1)^{-\zeta}(k+1)^{-\zeta}$. Since $n-j \geq n/6$, we get the result.

To handle Case III, let

$$V = V_{i,j,n} = \{B[\frac{i-k}{n}, \frac{i}{n}] \cap B[\frac{i+1}{n}, \frac{i+k}{n}] = \emptyset, B[\frac{j-k}{n}, \frac{j}{n}] \cap B[\frac{j+1}{n}, \frac{j+k}{n}] = \emptyset\},$$

$$U = U_{i,j,n} = \{B[0, \frac{i-k}{n}] \cap B[\frac{j+k}{n}, 1] = \emptyset\}.$$

Then,

$$\mathbf{P}[D(i,n) \cap D(j,n)] \leq \mathbf{P}(V \cap U).$$

For any integer $a \geq 0$, let

$$V(a) = V_{i,j,n}(a) = V \cap \{\sqrt{\frac{ak}{n}} \leq |B(\frac{j+k}{n}) - B(\frac{i-k}{n})| < \sqrt{\frac{(a+1)k}{n}}\}.$$

By arguments similar to those in Lemma 3.14 one can see that there is a $\beta < \infty$ such that

$$\mathbf{P}[V(a)] \leq ck^{-2\zeta}e^{-a\beta}.$$

By Proposition 3.15 and Brownian scaling,

$$\mathbf{P}[U \mid V(a)] \leq c[\frac{(a+1)(k+1)}{i+1}]^{\zeta}.$$

By summing over a and noting that $n-j \geq n/6$, we get the result. This proves Lemma 4.1. We comment that similar arguments can be used to prove estimates for higher moments.

Lemma 4.2 *For every positive integer r there exists $\alpha_r < \infty$ and such that if $k > 0$, and*

$$1 \leq i_1 \leq i_2 \leq \cdots \leq i_r \leq n,$$

then

$$\mathbf{P}[D(i_1,n) \cap \cdots \cap D(i_r,n)] \leq \alpha_r n^{\zeta} f_{r,n}(i_1,\ldots,i_r). \tag{15}$$

By summing (15) we conclude

$$\mathbf{E}(J_n^r) \leq \alpha_r n^{r(1-\zeta)},$$

for appropriately chosen α_r.

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