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Isoperimetry between exponential and Gaussian

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Abstract

We study the isoperimetric problem for product probability measures with tails between the exponential and the Gaussian regime. In particular we exhibit many examples where coordinate half-spaces are approximate solutions of the isoperimetric problem.

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1 Introduction

This paper establishes infinite dimensional isoperimetric inequalities for a wide class of probability measures. We work in the setting of a Riemannian manifold (M, g). The geodesic distance on M is denoted by d. Furthermore M is equipped with a Borel probability measure μ which is assumed to be absolutely continuous with respect to the volume measure. For $h \geq 0$ the closed h-enlargement of a set $A \subset M$ is

$$A_h := \big\{x \in M;\ d(x, A) \leq h\big\},$$

where $d(x, A) := \inf\{d(x, a);\ a \in A\}$ is $+\infty$ by convention for $A = \emptyset$. We may define the boundary measure, in the sense of μ, of a Borel set A by

$$\mu_s(\partial A) := \liminf_{h \to 0^+} \frac{\mu(A_h \setminus A)}{h}\cdot$$

An isoperimetric inequality is a lower bound on the boundary measure of sets in terms of their measure. Their study is an important topic in geometry, see e.g. (37). Finding sets of given measure and of minimal boundary measure is very difficult. In many cases the only hope is to estimate the isoperimetric function (also called isoperimetric profile) of the metric measured space (M, d, μ), denoted by I_μ

$$I_\mu(a) := \inf\big\{\mu_s(\partial A);\ \mu(A) = a\big\}, \quad a \in [0, 1].$$

For $h > 0$ one may also investigate the best function R_h such that $\mu(A_h) \geq R_h(\mu(A))$ holds for all Borel sets. The two questions are related, and even equivalent in simple situations, see (17). Since the function $\alpha(h) = 1 - R_h(1/2)$ is the so-called concentration function, the isoperimetric problem for probability measures is closely related to the concentration of measure phenomenon. We refer the reader to the book (32) for more details on this topic.

The main probabilistic example where the isoperimetric problem is completely solved is the Euclidean space $(\mathbb{R}^n, |\cdot|)$ with the standard Gaussian measure, denoted γ^n in order to emphasize its product structure

$$d\gamma^n(x) = e^{-|x|^2/2} \frac{dx}{(2\pi)^{n/2}}, \quad x \in \mathbb{R}^n.$$

Sudakov-Tsirel'son (39) and Borell (19) have shown that among sets of prescribed measure, half-spaces have h-enlargements of minimal measure. Setting $G(t) = \gamma((-\infty, t])$, their result reads as follows: for $A \subset \mathbb{R}^n$ set $a = G^{-1}(\gamma^n(A))$, then

$$\gamma^n(A_h) \geq \gamma\big((-\infty, a + h]\big) = G\Big(G^{-1}\big(\gamma^n(A)\big) + h\Big),$$

and letting h go to zero

$$(\gamma^n)_s(\partial A) \geq G'(a) = G'\Big(G^{-1}\big(\gamma^n(A)\big)\Big).$$

These inequalities are best possible, hence $I_{\gamma^n} = G' \circ G^{-1}$ is independent of the dimension n. Such dimension free properties are crucial in the study of large random systems, see e.g. (31; 41). Asking which measures enjoy such a dimension free isoperimetric inequality is therefore a fundamental question. Let us be more specific about the products we are considering: if μ is a probability measure on (M,g), we consider the product μ^n on the product Riemannian manifold M^n where the geodesic distance is the ℓ_2 combination of the distances on the factors. Considering the ℓ_∞ combination is easier and leads to different results, see (12; 16; 7). In the rest of this paper we only consider the ℓ_2 combination.

It can be shown that Gaussian measures are the only symmetric measures on the real line such that for any dimension n, the coordinate half-spaces $\{x \in \mathbb{R}^n;\ x_1 \le t\}$ solve the isoperimetric problem for the corresponding product measure on $\mathbb{R}^n$. See (14; 28; 34) for details and stronger statements. Therefore it is natural to investigate measures on the real line for which half-lines solve the isoperimetric problem, and in any dimension coordinate half-spaces are approximate solutions of the isoperimetric problem for the products, up to a universal factor. More generally, one looks for measures on the line for which there exists $c < 1$ with

$$I_\mu \ge I_{\mu^\infty} \ge c\, I_\mu, \tag{1}$$

where by definition $I_{\mu^\infty} := \inf_{n\ge 1} I_{\mu^n}$. Note that the first inequality is always true. Inequality (1) means that for any n, and $\varepsilon > 0$, among subsets of $\mathbb{R}^n$ with μ^n-measure equal to $a \in (0,1)$ there are sets of the form $A \times \mathbb{R}^{n-1}$ with boundary measure at most $c^{-1} + \varepsilon$ times the minimal boundary measure.

Dimension free isoperimetric inequalities as (1) are very restrictive. Heuristically one can say that they force μ to have a tail behaviour which is intermediate between exponential and Gaussian. More precisely, if $I_{\mu^\infty} \ge cI_\mu$ is bounded from below by a continuous positive function on $(0,1)$, standard arguments imply that the measures μ^n all satisfy a concentration inequality which is independent of n. As observed by Talagrand in (40), this property implies the existence of $\varepsilon > 0$ such that $\int e^{\varepsilon|t|} d\mu(t) < +\infty$, see (16) for more precise results. In particular, the central limit theorem applies to μ. Setting $m = \int x\, d\mu(x)$, it allows to compute the limit of

$$\mu^n\Big(\big\{x \in \mathbb{R}^n;\ \frac{1}{\sqrt{n}}\sum_{i=1}^n (x_i - m) \le t\big\}\Big).$$

Under mild assumptions it follows that for some constant d, $cI_\mu \le I_{\mu^\infty} \le d\, I_\gamma$. Thus the isoperimetric function of μ is at most a multiple of the Gaussian isoperimetric function. In particular if μ is symmetric with a log-concave density, this is known to imply that μ has at least Gaussian tails.

For the symmetric exponential law $d\nu(t) = e^{-|t|}dt/2$, $t \in \mathbb{R}$, Bobkov and Houdré (15) actually showed $I_{\nu^\infty} \ge I_\nu/(2\sqrt{6})$. Their argument uses a functional isoperimetric inequality with the tensorization property. In the earlier paper (40), Talagrand proved a different dimension free isoperimetric inequality for the exponential measure, where the enlargements involve mixtures of ℓ_1 and ℓ_2 balls with different scales (this result does not provide lower bounds on the boundary measure of sets).

In a recent paper (8) we have studied in depth various types of inequalities allowing the precise description of concentration phenomenon and isoperimetric profile for probability measures, in the intermediate regime between exponential and Gaussian. Our approach of the isoperimetric inequality followed the one of Ledoux (30) (which was improved in (4)): we studied the improving properties of the underlying semigroups, but we had to replace Gross hypercontractivity by a notion of Orlicz hyperboundedness, closely related to F-Sobolev inequalities (see Equation (7) in

Section 6 for a definition). This approach yields a dimension free description of the isoperimetric profile for the measures $d\nu_\alpha(t) = e^{-|t|^\alpha} dt$ for $1 \le \alpha \le 2$: there exists a universal constant K such that for all $\alpha \in [1,2]$

$$I_{\nu_\alpha} \ge I_{\nu_\alpha^\infty} \ge \frac{1}{K} I_{\nu_\alpha}.$$

It is plain that the method in (8) allows to deal with more general measures, at the price of rather heavy technicalities.

In this paper, we wish to point out a softer approach to isoperimetric inequalities. It was recently developed by Wang and his coauthors (42; 25; 43) and relies on so called super-Poincaré inequalities. It can be combined with our techniques in order to provide dimension free isoperimetric inequalities for large classes of measures. Among them are the measures on the line with density $e^{-\Phi(|t|)}\, dt/Z$ where $\Phi(0) = 0$, Φ is convex and $\sqrt{\Phi}$ is concave. This is achieved in the first part of the paper: Sections 2–5. The dimension free inequalities are still valid for slight modifications of the above examples. Other approaches and a few examples of perturbation results are developed in the last sections of the article.

Finally, let us present the super-Poincaré inequality as introduced by Wang in order to study the essential spectrum of Markov generators (actually we have found it convenient to exchange the roles of s and $\beta(s)$ in the definition below). We shall say that a probability measure μ on (M, g) satisfies a super-Poincaré inequality, if there exists a nonnegative function β defined on $[1, +\infty[$ such that for all smooth $f : M \to \mathbb{R}$ and all $s \ge 1$,

$$\int f^2 d\mu - s \left(\int |f| d\mu \right)^2 \le \beta(s) \int |\nabla f|^2 d\mu.$$

This family of inequalities is equivalent to the following Nash type inequality: for all smooth f,

$$\int f^2 d\mu \le \left(\int |f| d\mu \right)^2 \Theta \left(\frac{\int |\nabla f|^2 d\mu}{\left(\int |f| d\mu\right)^2} \right),$$

where $\Theta(x) := \inf_{s \ge 1}\{\beta(s)x + s\}$. But it is often easier to work with the first form. Similar inequalities appear in the literature, see (10; 22). Wang discovered that super-Poincaré inequalities imply precise isoperimetric estimates, and are related to Beckner-type inequalities via F-Sobolev inequalities. In fact, Beckner-type inequalities, as developed by Latała-Oleszkiewicz (29) were crucial in deriving dimension-free concentration in our paper (8). In full generality they read as follows: for all smooth f and all $p \in [1, 2)$,

$$\int f^2 d\mu - \left(\int |f|^p d\mu \right)^{\frac{2}{p}} \le T(2-p) \int |\nabla f|^2 d\mu,$$

where $T : (0, 1] \to \mathbb{R}^+$ is a non-decreasing function. Following (9) we could characterize the measures on the line which enjoy this property, and then take advantage of the tensorization property. As the reader noticed, the super-Poincaré and Beckner-type inequalities are formally very similar. It turns out that the tools of (9) apply to both, see for example Lemma 3 below. This remark allows us to present a rather concise proof of the dimension-free isoperimetric inequalities, since the two functional inequalities involved (Beckner type for the tensorization property, and super-Poincaré for its isoperimetric implications) can be studied in one go.

2 A measure-Capacity sufficient condition for super-Poincaré inequality

This section provides a sufficient condition for the super Poincaré inequality to hold, in terms of a comparison between capacity of sets and their measure. This point of view was put forward in (8) in order to give a natural unified presentation of the many functional inequalities appearing in the field.

Given $A \subset \Omega$, the *capacity* $\mathrm{Cap}_\mu(A,\Omega)$, is defined as

$$\begin{aligned}\mathrm{Cap}_\mu(A,\Omega) &= \inf\left\{\int |\nabla f|^2 d\mu;\ f_{|A} \geq 1,\ f_{|\Omega^c} = 0\right\} \\ &= \inf\left\{\int |\nabla f|^2 d\mu;\ \mathbf{1}_A \leq f \leq \mathbf{1}_\Omega\right\},\end{aligned}$$

where the infimum is over locally Lipschitz functions. Recall that Rademacher's theorem (see e.g. (23, 3.1.6)) ensures that such functions are Lebesgue almost everywhere differentiable, hence μ-almost surely differentiable. The latter equality follows from an easy truncation argument, reducing to functions with values in $[0,1]$. Finally we defined in (9) the capacity of A with respect to μ when $\mu(A) < 1/2$ as

$$\mathrm{Cap}_\mu(A) := \inf\{\mathrm{Cap}(A,\Omega);\ A \subset \Omega,\ \mu(\Omega) \leq 1/2\}.$$

Theorem 1. *Assume that for every measurable $A \subset M$ with $\mu(A) < 1/2$, one has*

$$\mathrm{Cap}_\mu(A) \geq \sup_{s\geq 1} \frac{1}{\beta(s)} \left(\frac{\mu(A)}{1+(s-1)\mu(A)}\right).$$

for some function β defined on $[1,+\infty[$.
Then, for every smooth $f: M \to \mathbb{R}$ and every $s \geq 1$ one has

$$\int f^2 d\mu - s\left(\int |f| d\mu\right)^2 \leq 4\beta(s) \int |\nabla f|^2 d\mu.$$

Proof. We use four results that we recall or prove just after this proof. Let $s \geq 1$, $f: M \to \mathbb{R}$ be locally Lipschitz and m a median of the law of f under μ. Define $F_+ = (f-m)_+$ and $F_- = (f-m)_-$. Setting

$$\mathcal{G}_s = \left\{g : M \to [0,1);\ \int (1-g)^{-1} d\mu \leq 1 + \frac{1}{s-1}\right\},$$

it follows from Lemmas 2 and 3 (used with $A = s-1$ and $a = 1/2$) that

$$\begin{aligned}\int f^2 d\mu - s\left(\int |f| d\mu\right)^2 &\leq \int (f-m)^2 d\mu - (s-1)\left(\int |f-m|\, d\mu\right)^2 \\ &\leq \sup\left\{\int (f-m)^2 g\, d\mu; g \in \mathcal{G}_s\right\} \\ &\leq \sup\left\{\int F_+^2 g\, d\mu; g \in \mathcal{G}_s\right\} \\ &\quad + \sup\left\{\int F_-^2 g\, d\mu; g \in \mathcal{G}_s\right\},\end{aligned}$$

where we have used the fact that the supremum of a sum is less than the sum of the suprema. We deal with the first term of the right hand side. By Theorem 5 we have

$$\sup\left\{\int F_+^2 g\, d\mu; g\in\mathcal{G}_s\right\}\leq 4B_s\int|\nabla F_+|^2 d\mu$$

where B_s is the smallest constant so that for all $A\subset M$ with $\mu(A)<1/2$

$$B_s \mathrm{Cap}_\mu(A)\geq \sup\left\{\int \mathbb{1}_A g\, d\mu; g\in\mathcal{G}_s\right\}.$$

On the other hand Lemma 4 insures that

$$\begin{aligned}\sup\left\{\int \mathbb{1}_A g\, d\mu; g\in\mathcal{G}_s\right\} &= \mu(A)\left(1-\left(1+\frac{1}{(s-1)\mu(A)}\right)^{-1}\right)\\ &= \frac{\mu(A)}{1+(s-1)\mu(A)}.\end{aligned}$$

Thus, by our assumption, $B_s\leq\beta(s)$. We proceed in the same way for F_-. Summing up, we arrive at

$$\begin{aligned}\int f^2 d\mu - s\left(\int|f|d\mu\right)^2 &\leq 4\beta(s)(\int|\nabla F_+|^2 d\mu+\int|\nabla F_-|^2 d\mu)\\ &\leq 4\beta(s)\int|\nabla f|^2 d\mu.\end{aligned}$$

In the last bound we used the fact that since f is locally Lipschitz and μ is absolutely continuous, the set $\{f=m\}\cap\{\nabla f\neq 0\}$ is μ-negligible. Indeed $\{f=m\}\cap\{\nabla f\neq 0\}\subset\{x;\ |f-m|$ is not differentiable at $x\}$ has Lebesgue measure zero since $x\mapsto|f(x)-m|$ is locally Lipschitz. □

Lemma 2. *Let (X,P) be a probability space. Then for any function $g\in L^2(P)$, for any $s\geq 1$,*

$$\int g^2 dP - s\left(\int|g|dP\right)^2\leq\int(g-m)^2 dP-(s-1)\left(\int|g-m|dP\right)^2$$

where m is a median of the law of g under P.

Proof. We write

$$\int g^2 dP - s\left(\int|g|dP\right)^2 = \mathbf{Var}_P(|g|)-(s-1)\left(\int|g|dP\right)^2.$$

By the variational definition of the median and the variance respectively, we have $\mathbf{Var}_P(|g|)\leq\mathbf{Var}_P(g)\leq\int(g-m)^2dP$ and $\int|g-m|dP\leq\int|g|dP$. The result follows. □

Lemma 3 ((9))**.** *Let φ be a non-negative integrable function on a probability space (X,P). Let $A\geq 0$ and $a\in(0,1)$, then*

$$\begin{aligned}&\int\varphi\, dP - A\left(\int\varphi^a dP\right)^{\frac{1}{a}}\\ =\ &\sup\left\{\int\varphi g\, dP; g:X\to(-\infty,1)\ \text{and}\ \int(1-g)^{\frac{a}{a-1}}dP\leq A^{\frac{a}{a-1}}\right\}\\ \leq\ &\sup\left\{\int\varphi g\, dP; g:X\to[0,1)\ \text{and}\ \int(1-g)^{\frac{a}{a-1}}dP\leq 1+A^{\frac{a}{a-1}}\right\}.\end{aligned}$$

Note that in (9) it is assumed that $A > 0$. The case $A = 0$ is easy.

Lemma 4 ((9))**.** *Let $a \in (0,1)$. Let Q be a finite positive measure on a space X and let $K > Q(X)$. Let $A \subset X$ be measurable with $Q(A) > 0$. Then*

$$\begin{aligned} & \sup\left\{\int_X \mathbb{1}_A g\, dQ; g: X \to [0,1) \text{ and } \int_X (1-g)^{\frac{a}{a-1}} dQ \le K\right\} \\ = \;& Q(A)\left(1 - \left(1 + \frac{K - Q(X)}{Q(A)}\right)^{\frac{a-1}{a}}\right). \end{aligned}$$

Theorem 5. *Let $\mathcal{G}$ be a family of non-negative Borel functions on M, $\Omega \subset M$ with $\mu(\Omega) \le 1/2$ and for any measurable function f vanishing on Ω^c set*

$$\Phi(f) = \sup_{g \in \mathcal{G}} \int_\Omega fg\, d\mu.$$

Let B denote the smallest constant such that for all $A \subset \Omega$ with $\mu(A) < 1/2$ one has

$$B\, \mathrm{Cap}_\mu(A) \ge \Phi(\mathbb{1}_A).$$

Then for every smooth function $f: M \to \mathbb{R}$ vanishing on Ω^c it holds

$$\Phi(f^2) \le 4B \int |\nabla f|^2 d\mu.$$

Proof. We start with a result of Maz'ja (33), also discussed in (8, Proposition 13): given two absolutely continuous positive measures μ, ν on M, denote by B_ν the smallest constant such that for all $A \subset \Omega$ one has

$$B_\nu \mathrm{Cap}_\mu(A, \Omega) \ge \nu(A).$$

Then for every smooth function $f: M \to \mathbb{R}$ vanishing on Ω^c

$$\int f^2 d\nu \le 4B_\nu \int |\nabla f|^2 d\mu.$$

Following an idea of Bobkov and Götze (13) we apply the previous inequality to the measures $d\nu = g d\mu$ for $g \in \mathcal{G}$. Thus for f as above

$$\Phi(f) = \sup_{g \in \mathcal{G}} \int_\Omega fg\, d\mu \le 4 \sup_{g \in \mathcal{G}} B_{g\, d\mu} \int |\nabla f|^2 d\mu.$$

It remains to check that the constant B is at most $\sup_{g \in \mathcal{G}} B_{g\, d\mu}$. This follows from the definition of Φ and the inequality $\mathrm{Cap}_\mu(A) \le \mathrm{Cap}_\mu(A, \Omega)$. □

Corollary 6. *Assume that $\beta : [1, +\infty) \to \mathbb{R}^+$ is non-increasing and that $s \mapsto s\beta(s)$ is non-decreasing on $[2, +\infty)$. Then, for every $a \in (0, 1/2)$,*

$$\frac{1}{2}\frac{a}{\beta(1/a)} \le \sup_{s \ge 1} \frac{a}{1 + (s-1)a}\frac{1}{\beta(s)} \le 2\frac{a}{\beta(1/a)}. \tag{2}$$

In particular, if for every measurable $A \subset M$ with $\mu(A) < 1/2$, one has

$$\mathrm{Cap}_\mu(A) \geq \frac{\mu(A)}{\beta(1/\mu(A))},$$

then, for every $f : M \to \mathbb{R}$ and every $s \geq 1$ one has

$$\int f^2 d\mu - s\left(\int |f| d\mu\right)^2 \leq 8\beta(s) \int |\nabla f|^2 d\mu.$$

Proof. The choice $s = 1/a$ gives the first inequality in (2). For the second part of (2), we consider two cases:

If $a(s-1) \leq 1/2$ then $s \leq 1 + \frac{1}{2a} \leq \frac{1}{a}$, where we have used $a < 1/2$. Hence, the monotonicity of β yields

$$\frac{a}{1+(s-1)a}\frac{1}{\beta(s)} \leq \frac{a}{\beta(s)} \leq \frac{a}{\beta(1/a)}.$$

If $a(s-1) > 1/2$, note that $a/(1+(s-1)a) \leq 1/s$. Thus by monotonicity of $s \mapsto s\beta(s)$ and since $s \geq 1 + 1/2a = \frac{1+2a}{2a} \geq 2$,

$$\frac{a}{1+(s-1)a}\frac{1}{\beta(s)} \leq \frac{1}{s\beta(s)} \leq \frac{2a}{(1+2a)\beta(1+\frac{1}{2a})} \leq \frac{2a}{\beta(1/a)}.$$

The last step uses the inequality $1 + \frac{1}{2a} \leq \frac{1}{a}$ and the monotonicity of β.

The second part of the Corollary is a direct consequence of Theorem 1 and (2) (replacing β in Theorem 1 by 2β). $\square$

3 Beckner type versus super Poincaré inequality

In this section we use Corollary 6 to derive super Poincaré inequality from Beckner type inequality.

The following criterion was established in (8, Theorem 18 and Lemma 19) in the particular case of $M = \mathbb{R}^n$. As mentioned in the introduction of (8) the extension to Riemannian manifolds is straightforward.

Theorem 7 ((8))**.** *Let $T : [0,1] \to \mathbb{R}^+$ be non-decreasing and such that $x \mapsto T(x)/x$ is non-increasing. Let C be the optimal constant such that for every smooth $f : M \to \mathbb{R}$ one has (Beckner type inequality)*

$$\sup_{p\in(1,2)} \frac{\int f^2 d\mu - \left(\int |f|^p d\mu\right)^{\frac{2}{p}}}{T(2-p)} \leq C \int |\nabla f|^2 d\mu.$$

Then $\frac{1}{6}B(T) \leq C \leq 20B(T)$, where $B(T)$ is the smallest constant so that every $A \subset M$ with $\mu(A) < 1/2$ satisfies

$$B(T)\mathrm{Cap}_\mu(A) \geq \frac{\mu(A)}{T\left(1/\log\left(1+\frac{1}{\mu(A)}\right)\right)}.$$

If $M = \mathbb{R}$, m is a median of μ and ρ_μ is its density, we have more explicitly

$$\frac{1}{6}\max(B_-(T), B_+(T)) \le C \le 20 \max(B_-(T), B_+(T))$$

where

$$\begin{aligned} B_+(T) &= \sup_{x>m} \mu([x,+\infty)) \frac{1}{T\left(1/\log\left(1+\frac{1}{\mu([x,+\infty))}\right)\right)} \int_m^x \frac{1}{\rho_\mu} \\ B_-(T) &= \sup_{x<m} \mu((-\infty,x]) \frac{1}{T\left(1/\log\left(1+\frac{1}{\mu((-\infty,x])}\right)\right)} \int_x^m \frac{1}{\rho_\mu}. \end{aligned}$$

The relations between Beckner-type and super-Poincaré inequalities have been explained by Wang, via F-Sobolev inequalities. Here we give an explicit connection under a natural condition on the rate function T.

Corollary 8 (From Beckner to Super Poincaré)**.** *Let $T : [0,1] \to \mathbb{R}^+$ be non-decreasing and such that $x \mapsto T(x)/x$ is non-increasing. Assume that there exists a constant C such that for every smooth $f : M \to \mathbb{R}$ one has*

$$\sup_{p\in(1,2)} \frac{\int f^2 d\mu - \left(\int |f|^p d\mu\right)^{\frac{2}{p}}}{T(2-p)} \le C \int |\nabla f|^2 d\mu. \tag{3}$$

Define $\beta(s) = T(1/\log(1+s))$ for $s \ge e-1$ and $\beta(s) = T(1)$ for $s \in [1, e-1]$.
Then, every smooth $f : M \to \mathbb{R}$ satisfies for every $s \ge 1$,

$$\int f^2 d\mu - s\left(\int |f| d\mu\right)^2 \le 48 C\beta(s) \int |\nabla f|^2 d\mu.$$

Proof. By Theorem 7, Inequality (3) implies that every $A \subset M$ with $\mu(A) < 1/2$ satisfies

$$6\, C\, \mathrm{Cap}_\mu(A) \ge \frac{\mu(A)}{T\left(1/\log\left(1+\frac{1}{\mu(A)}\right)\right)} = \frac{\mu(A)}{\beta\left(\frac{1}{\mu(A)}\right)}.$$

Since T is non-decreasing, β is non-increasing on $[1,\infty)$. On the other hand, for $s \ge e-1$, we have

$$s\beta(s) = sT(1/\log(1+s)) = \log(1+s) T(1/\log(1+s)) \frac{s}{\log(1+s)}.$$

The map $x \mapsto T(x)/x$ is non-increasing and $s \mapsto \frac{s}{\log(1+s)}$ is non-decreasing. It follows that $s \mapsto s\beta(s)$ is non-decreasing. Corollary 6 therefore applies and yields the claimed inequality. $\square$

A remarkable feature of Beckner type inequalities (3) is the tensorization property: if μ_1 and μ_2 both satisfy (3) with constant C, then so does $\mu_1 \otimes \mu_2$ (29). For this reason inequalities for measures on the real line are inherited by their infinite products. In dimension 1 the criterion given in Theorem 7 allows us to deal with probability measures $d\mu_\Phi(x) = Z_\Phi^{-1} e^{-\Phi(|x|)} dx$ with quite general potentials Φ:

Proposition 9. *Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing convex function with $\Phi(0) = 0$ and consider the probability measure $d\mu_\Phi(x) = Z_\Phi^{-1} e^{-\Phi(|x|)}dx$. Assume that Φ is $\mathcal{C}^2$ on $[\Phi^{-1}(1), +\infty)$ and that $\sqrt{\Phi}$ is concave. Define $T(x) = [1/\Phi' \circ \Phi^{-1}(1/x)]^2$ for $x > 0$ and $\beta(s) = [1/\Phi' \circ \Phi^{-1}(\log(1+s))]^2$ for $s \geq e-1$ and $\beta(s) = [1/\Phi' \circ \Phi^{-1}(1)]^2$ for $s \in [1, e-1]$. Then there exists a constant $C > 0$ such that for any $n \geq 1$, every smooth function $f : \mathbb{R}^n \to \mathbb{R}$ satisfies*

$$\sup_{p\in(1,2)} \frac{\int f^2 d\mu_\Phi^n - \left(\int |f|^p d\mu_\Phi^n\right)^{\frac{2}{p}}}{T(2-p)} \leq C \int |\nabla f|^2 d\mu_\Phi^n.$$

In turn, for any $n \geq 1$, every smooth function $f : \mathbb{R}^n \to \mathbb{R}$ and every $s \geq 1$,

$$\int f^2 d\mu_\phi^n - s\left(\int |f| d\mu_\Phi^n\right)^2 \leq 48C\beta(s) \int |\nabla f|^2 d\mu_\Phi^n.$$

Proof. The proof of the Beckner type inequality comes from (8, proof of Corollary 32): the hypotheses on Φ allow to compute an equivalent of $\mu_\Phi([x, +\infty))$ when x tends to infinity (namely $e^{-\phi}/\phi'$) and thus to bound from above the quantities $B_+(T)$ and $B_-(T)$ of Theorem 7. This yields the Beckner type inequality in dimension 1. Next we use the tensorization property.

The second part follows from Corollary 8 (the hypotheses on Φ ensure that T is non-decreasing and $T(x)/x$ is non-increasing). $\square$

Example 1. A first family of examples is given by the measures $d\mu_p(x) = e^{-|x|^p}dx/(2\Gamma(1+1/p))$, $p \in [1, 2]$. The potential $x \mapsto |x|^p$ fulfills the hypotheses of Proposition 9 with $T_p(x) = \frac{1}{p^2}x^{2(1-\frac{1}{p})}$. Thus, by Proposition 9, for any $n \geq 1$, μ_p^n satisfies a super Poincaré inequality with function $\beta(s) = c_p/\log(1+s)^{2(1-\frac{1}{p})}$ where c_p depends only on p and not on the dimension n.

Note that the corresponding Beckner type inequality

$$\sup_{q\in(1,2)} \frac{\int f^2 d\mu_p^n - \left(\int |f|^q d\mu_p^n\right)^{\frac{2}{q}}}{(2-q)^{2(1-\frac{1}{p})}} \leq \tilde{c}_p \int |\nabla f|^2 d\mu_p^n,$$

goes back to Latała and Oleszkiewicz (29) with a different proof, see also (9).

Example 2. Consider now the larger family of examples given by $d\mu_{p,\alpha}(x) = Z_{p,\alpha}^{-1} e^{-|x|^p(\log(\gamma+|x|))^\alpha}dx$, $p \in [1, 2]$, $\alpha \geq 0$ and $\gamma = e^{\alpha/(2-p)}$. One can see that $\mu_{p,\alpha}^n$ satisfies a super Poincaré inequality with function

$$\beta(s) = \frac{c_{p,\alpha}}{(\log(1+s))^{2(1-\frac{1}{p})}(\log\log(e+s))^{2\alpha/p}}, \qquad s \geq 1.$$

4 Isoperimetric inequalities

In this section we collect results which relate super-Poincaré inequalities with isoperimetry. They follow Ledoux approach of Buser's inequality (30). This method was developed by Bakry-Ledoux (4) and Wang (36; 42), see also (24; 8).

The following result, a particular case of (4, Inequality (4.3)), allows to derive isoperimetric estimates from semi-group bounds.

Theorem 10 ((4; 36)). *Let μ be a probability measure on (M,g) with density e^{-V} with respect to the volume measure. Assume that V is $\mathcal{C}^2$ and such that* $\mathrm{Ricci} + \mathrm{D}^2 V \geq -Rg$ *for some $R \geq 0$. Let $(\mathbf{P}_t)_{t\geq 0}$ be the corresponding semi-group with generator $\mathbf{L} = \Delta - \nabla V \cdot \nabla$. Then, for $t > 0$, every measurable set $A \subset M$ satisfies*

$$\begin{aligned} \frac{\arg\tanh\left(\sqrt{1-e^{-4Rt}}\right)}{2\sqrt{R}}\,\mu_s(\partial A) &\geq \mu(A) - \int (\mathbf{P}_t \mathbb{1}_A)^2 d\mu \\ &= \mu(A^c) - \int (\mathbf{P}_t \mathbb{1}_{A^c})^2 d\mu. \end{aligned}$$

For $R = 0$ the left-hand side term should be understood as its limit $\sqrt{t}\,\mu_s(\partial A)$.

Remark 3. The condition $\mathrm{Ricci} + \mathrm{D}^2 V \geq -Rg$ was introduced by Bakry and Emery (3, Proposition 3). The left hand side term is a natural notion of curvature for manifolds with measures $e^{-V}\,d\mathrm{Vol}$, which takes into account the curvature of the space and the contribution of the potential V.

Proof. We briefly reproduce the line of reasoning of Bakry-Ledoux (4) and its slight improvement given by Röckner-Wang (36). Let f, g be smooth bounded functions. We start with Inequality (4.2) in (4), which describes a regularizing effect of the semigroup:

$$|\nabla \mathbf{P}_s g|^2 \leq \frac{R}{1-e^{-2Rs}} \|g\|_\infty^2.$$

By reversibility, integration by parts and Cauchy-Schwarz inequality, it follows that, for any $t \geq 0$,

$$\begin{aligned} \int g(f - \mathbf{P}_{2t} f) d\mu &= -\int g \left(\int_0^{2t} \mathbf{L}\mathbf{P}_s f ds \right) d\mu = -\int_0^{2t} \int g \mathbf{L}\mathbf{P}_s f d\mu ds \\ &= -\int_0^{2t} \int \mathbf{P}_s g \mathbf{L} f d\mu ds = \int_0^{2t} \int \nabla \mathbf{P}_s g \cdot \nabla f d\mu ds \\ &\leq \int |\nabla f| \int_0^{2t} |\nabla \mathbf{P}_s g| ds d\mu \\ &\leq \int |\nabla f| d\mu \int_0^{2t} \sqrt{\frac{R}{1-e^{-2Rs}}} ds\, \|g\|_\infty \\ &= \frac{1}{\sqrt{R}} \arg\tanh\left(\sqrt{1-e^{-4Rt}}\right) \int |\nabla f| d\mu\, \|g\|_\infty. \end{aligned}$$

This is true for any choice of g so by duality we obtain

$$\int |f - \mathbf{P}_{2t} f|\, d\mu \leq \frac{1}{\sqrt{R}} \arg\tanh\left(\sqrt{1-e^{-4Rt}}\right) \int |\nabla f| d\mu.$$

Applying this to approximations of the characteristic function of the set $A \subset M$ and using the relation $\int \mathbb{1}_A \mathbf{P}_{2t} \mathbb{1}_A\, d\mu = \int (\mathbf{P}_t \mathbb{1}_A)^2 d\mu$ leads to the expected result. □

In order to exploit this result we need the following proposition due to Wang (42). We sketch the proof for completeness.

Proposition 11 ((42))**.** *Let μ be a probability measure on M with density e^{-V} with respect to the volume measure. Assume that V is $\mathcal{C}^2$. Let $(\mathbf{P}_t)_{t\geq 0}$ be the corresponding semi-group with generator $L := \Delta - \nabla V \cdot \nabla$. Then the following are equivalent*
(i) μ satisfies a Super Poincaré inequality: every smooth $f : M \to \mathbb{R}$ satisfies for every $s \geq 1$

$$\int f^2 d\mu - s\left(\int |f| d\mu\right)^2 \leq \beta(s) \int |\nabla f|^2 d\mu.$$

(ii) For every $t \geq 0$, every smooth $f : M \to \mathbb{R}$, and all $s \geq 1$

$$\int (\mathbf{P}_t f)^2 d\mu \leq e^{-\frac{2t}{\beta(s)}} \int f^2 d\mu + s(1 - e^{-\frac{2t}{\beta(s)}}) \left(\int |f| d\mu\right)^2.$$

Proof. (i) follows from (ii) by differentiation at $t = 0$.
On the other hand, if $u(t) = \int (\mathbf{P}_t f)^2 d\mu$, (i) implies that

$$u'(t) = 2\int \mathbf{P}_t f L \mathbf{P}_t f \, d\mu = -2\int |\nabla \mathbf{P}_t f|^2 d\mu \leq -\frac{2}{\beta(s)}\left[u(t) - s\left(\int |f| d\mu\right)^2\right]$$

since $\int |\mathbf{P}_t f| d\mu \leq \int |f| d\mu$. The result follows by integration. □

Theorem 12 ((42))**.** *Let μ be a probability measure on (M, g) with density e^{-V} with respect to the volume measure. Assume that V is $\mathcal{C}^2$ and such that $\mathrm{Ricci} + \mathrm{D}^2 V \geq -Rg$ for some $R \geq 0$. Let $(\mathbf{P}_t)_{t\geq 0}$ be the corresponding semi-group with generator $\Delta - \nabla V \cdot \nabla$. Assume that every smooth $f : M \to \mathbb{R}$ satisfies for every $s \geq 1$*

$$\int f^2 d\mu - s\left(\int |f| d\mu\right)^2 \leq \beta(s) \int |\nabla f|^2 d\mu,$$

with β decreasing. Then there exists a positive number $C(R, \beta(1))$ such that every measurable set $A \subset M$ satisfies

$$\mu_s(\partial A) \geq C(R, \beta(1))\mu(A)(1 - \mu(A)).$$

If $\beta(+\infty) = 0$, any measurable set $A \subset M$ with $p := \min(\mu(A), \mu(A^c)) \leq \min(1/2, 1/(2\beta^{-1}(1/R)))$ satisfies

$$\mu_s(\partial A) \geq \frac{1}{3} \frac{p}{\sqrt{\beta\left(\frac{1}{2p}\right)}}.$$

Proof. From the super-Poincaré inequality and Proposition 11 we have for any smooth $f : M \to \mathbb{R}$ and all $s \geq 1$

$$\int (\mathbf{P}_t f)^2 d\mu \leq e^{-\frac{2t}{\beta(s)}} \int f^2 d\mu + s(1 - e^{-\frac{2t}{\beta(s)}}) \left(\int |f| d\mu\right)^2.$$

Applying this to approximations of characteristic functions we get for any measurable set $A \subset M$,

$$\int (\mathbf{P}_t \mathbb{1}_A)^2 d\mu \leq e^{-\frac{2t}{\beta(s)}} \mu(A) + s(1 - e^{-\frac{2t}{\beta(s)}})\mu(A)^2 \qquad \forall s \geq 1.$$

Hence by Theorem 10, we have for all $t > 0, s \geq 1$,

$$\mu_s(\partial A) \geq \mu(A)(1 - s\mu(A))2\sqrt{R}\frac{1 - e^{-\frac{2t}{\beta(s)}}}{\arg\tanh\left(\sqrt{1 - e^{-4tR}}\right)}. \tag{4}$$

The first isoperimetric inequality is obtained when choosing $s = 1$, $t = \beta(1)$. In fact this is almost exactly the method used by Ledoux to derive Cheeger's inequality from Poincaré inequality when the curvature is bounded from below (30).

For a set A of measure at most 1/2, taking $s = 1/(2\mu(A))$ and $t = \beta(s)/2 = \frac{1}{2}\beta\left(\frac{1}{2\mu(A)}\right)$

$$\begin{aligned}
\mu_s(\partial A) &\geq \mu(A)\frac{\sqrt{R}}{\sqrt{2R\beta\left(\frac{1}{2\mu(A)}\right)}}\frac{(1 - e^{-1})\sqrt{2R\beta\left(\frac{1}{2\mu(A)}\right)}}{\arg\tanh\sqrt{1 - e^{-2R\beta(\frac{1}{2\mu(A)})}}} \\
&\geq \mu(A)\frac{1}{\sqrt{\beta\left(\frac{1}{2\mu(A)}\right)}}\frac{(1 - e^{-1})}{\arg\tanh\sqrt{1 - e^{-2}}} \\
&\geq \frac{1}{3}\frac{\mu(A)}{\sqrt{\beta\left(\frac{1}{2\mu(A)}\right)}},
\end{aligned}$$

where we have used $2R\beta(1/(2\mu(A))) \leq 2$ together with the fact that $x \mapsto (\arg\tanh\sqrt{1 - e^{-x}})/\sqrt{x}$ is increasing, a consequence of the convexity of the function $(\arg\tanh\sqrt{1 - e^{-x}})^2$. For sets with $\mu(A) > 1/2$ we work instead with the expression involving A^c in Theorem 10. □

Combining Theorem 7, the tensorization property of Beckner type-inequalities, Corollary 8 and Theorem 12 allows to derive dimension-free isoperimetric inequalities for the products of large classes of probability measures on the real line. In the next section we focus on log-concave densities.

5 Isoperimetric profile for log-concave measures

Here we apply the previous results to infinite product of the measures: $\mu_\Phi(dx) = Z_\Phi^{-1}\exp\{-\Phi(|x|)\}dx = \varphi(x)dx$, $x \in \mathbb{R}$, with Φ convex and $\sqrt{\Phi}$ concave. The isoperimetric profile of a symmetric log-concave density on the line (with the usual metric) was calculated by Borell (20) (see also Bobkov (11)). He showed that half-lines have minimal boundary among sets of given measure. Since the boundary measure of $(-\infty, x]$ is given by the density of the measure at x, the isoperimetric profile is $I_\Phi(t) = \varphi(H^{-1}(\min(t, 1 - t)) = \varphi(H^{-1}(t))$, $t \in [0, 1]$ where H is the distribution function of μ_Φ. It compares to the function

$$L_\Phi(t) = \min(t, 1 - t)\Phi' \circ \Phi^{-1}\left(\log\frac{1}{\min(t, 1 - t)}\right),$$

where Φ' is the right derivative. More precisely,

Proposition 13. *Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing convex function. Assume that in a neighborhood of $+\infty$, the function Φ is $\mathcal{C}^2$ and $\sqrt{\Phi}$ is concave.*

Let $d\mu_\Phi(x) = Z_\Phi^{-1} e^{-\Phi(|x|)}dx$ be a probability measure with density φ. Let H be the distribution function of μ and $I_\Phi(t) = \varphi(H^{-1}(t))$, $t \in [0,1]$. Then,

$$\lim_{t\to 0} \frac{I_\Phi(t)}{t\Phi' \circ \Phi^{-1}(\log \frac{1}{t})} = 1.$$

Consequently, if $\Phi(0) < \log 2$, L_Φ is defined on $[0,1]$ and there exist constants $k_1, k_2 > 0$ such that for all $t \in [0,1]$,

$$k_1 L_\Phi(t) \le I_\Phi(t) \le k_2 L_\Phi(t).$$

This result appears in (5; 18) in the particular case $\Phi(x) = |x|^p$.

Proof. Since Φ is convex and (strictly) increasing, note that Φ' may vanish only at 0. Under our assumptions on Φ we have $H(y) = \int_{-\infty}^y Z_\Phi^{-1} e^{-\Phi(|x|)}dx \sim Z_\Phi^{-1} e^{-\Phi(|y|)}/\Phi'(|y|)$ when y tends to $-\infty$. Thus using the change of variable $y = H^{-1}(t)$, we get

$$\begin{aligned}\lim_{t\to 0} \frac{I_\Phi(t)}{t\Phi' \circ \Phi^{-1}(\log \frac{1}{t})} &= \lim_{y\to-\infty} \frac{e^{-\Phi(|y|)}}{Z_\Phi H(y)\Phi' \circ \Phi^{-1}(\log \frac{1}{H(y)})} \\ &= \lim_{y\to-\infty} \frac{\Phi'(|y|)}{\Phi' \circ \Phi^{-1}(\log \frac{1}{H(y)})}.\end{aligned}$$

A Taylor expansion of $\Phi' \circ \Phi^{-1}$ between $\log \frac{1}{H(y)}$ and $\Phi(|y|)$ gives

$$\frac{\Phi' \circ \Phi^{-1}(\log \frac{1}{H(y)})}{\Phi'(|y|)} = 1 + \frac{1}{\Phi'(|y|)} \left(\log \frac{1}{H(y)} - \Phi(|y|) \right) \frac{\Phi'' \circ \Phi^{-1}(c_y)}{\Phi' \circ \Phi^{-1}(c_y)}$$

for some $c_y \in [\min(\Phi(|y|), \log \frac{1}{H(y)}), \infty)$.

Since for $y \ll -1$

$$\frac{1}{2}\frac{e^{-\Phi(|y|)}}{Z_\Phi \Phi'(|y|)} \le H(y) \le 2\frac{e^{-\Phi(|y|)}}{Z_\Phi \Phi'(|y|)}$$

we have

$$-\log 2 + \log(Z_\Phi \Phi'(|y|)) \le \log \frac{1}{H(y)} - \Phi(|y|) \le \log 2 + \log(Z_\Phi \Phi'(|y|)). \tag{5}$$

On the other hand, when $\sqrt{\Phi}$ is concave and $\mathcal{C}^2$, $(\sqrt{\Phi})''$ is non positive when it is defined. This leads to $\frac{\Phi''}{\Phi'} \le \frac{\Phi'}{2\Phi}$. Since $(\sqrt{\Phi})'$ is decreasing, it follows that $\Phi'(x) \le c\sqrt{\Phi(x)}$ for x large enough and for some constant $c > 0$. Finally we get $\frac{\Phi''(x)}{\Phi'(x)} \le \frac{c}{\sqrt{\Phi(x)}}$ for x large enough.

All these computations together give

$$\left| \frac{1}{\Phi'(|y|)} \left(\log \frac{1}{H(y)} - \Phi(|y|) \right) \frac{\Phi'' \circ \Phi^{-1}(c_y)}{\Phi' \circ \Phi^{-1}(c_y)} \right| \le \frac{\log 2 + |\log(Z_\Phi \Phi'(|y|))|}{|\Phi'(|y|)|} \frac{c}{\sqrt{c_y}}$$

which goes to 0 as y goes to $-\infty$. This ends the proof. $\square$

The following comparison result will allow us to modify measures without loosing much on their isoperimetric profile. It also shows that even log-concave measures on the real line play a central role.

Theorem 14 ((6; 37))**.** *Let m be a probability measure on $(\mathbb{R}, |.|)$ with even log-concave density. Let μ be a probability measure on (M, g) such that $I_\mu \geq cI_m$. Then for all $n \geq 1$, $I_{\mu^n} \geq cI_{m^n}$.*

Now we show the following infinite dimensional isoperimetric inequality.

Theorem 15. *Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing convex function with $\Phi(0) = 0$ and consider the probability measure $d\mu_\Phi(x) = Z_\Phi^{-1} e^{-\Phi(|x|)} dx$. Assume that Φ is $\mathcal{C}^2$ on $[\Phi^{-1}(1), +\infty)$ and that $\sqrt{\Phi}$ is concave.*

Then there exists a constant $K > 0$ such that for all $t \in [0, 1]$ one has

$$I_{\mu_\Phi^\infty}(t) \geq K L_\Phi(t).$$

Since $I_{\mu_\Phi^\infty}(t) \leq I_{\mu_\Phi}(t) \leq k_2 L_\Phi(t)$, we have, up to constants, the value of the isoperimetric profile of the infinite product.

Proof. For simplicity we assume first that $x \mapsto \Phi(|x|)$ is $\mathcal{C}^2$. We shall explain later how to deal with the general case. Applying Proposition 9 to the measure μ_Φ provides a Beckner-type inequality, with rate function T expressed in terms of Φ. By tensorization the powers of this measure enjoy the same property, which implies a super-Poincaré inequality by Corollary 8. Hence there exists a constant C independent of the dimension n such that for every smooth $f : \mathbb{R}^n \to \mathbb{R}$ one has

$$\int f^2 d\mu_\Phi^n - s \left(\int |f| d\mu_\Phi^n \right)^2 \leq C\beta(s) \int |\nabla f|^2 d\mu_\Phi^n \qquad \forall s \geq 1,$$

where $\beta(s) = [1/\Phi' \circ \Phi^{-1}(\log(1+s))]^2$ for $s \geq e-1$ and $\beta(s) = [1/\Phi' \circ \Phi^{-1}(1)]^2$ for $s \in [1, e-1]$. Next we apply Theorem 12 to the measure μ_Φ^n. Consider first the case $\lim_{x\to\infty} \Phi'(x) = \alpha < +\infty$. The first inequality in Theorem 12 yields

$$I_{\mu_\Phi^n}(t) \geq K_1 \Phi' \circ \Phi^{-1}(1) \min(t, 1-t) \geq K_2 L_\Phi(t),$$

where the constants $K_1, K_2 > 0$ are independent of n and t.

If Φ' tends to infinity, the second part of Theorem 12 allows to conclude that for $t \in [0, 1]$ (note that $\Phi'' \geq 0$ and thus we may take $R = 0$)

$$I_{\mu_\Phi^n}(t) \geq K_3 \min(t, 1-t) \Phi' \circ \Phi^{-1}(\log(1 + \frac{1}{2\min(t, 1-t)})).$$

Next we use elementary inequalities to bound from below $\Phi' \circ \Phi^{-1}(\log(1 + \frac{1}{2\min(t,1-t)}))$ by $\Phi' \circ \Phi^{-1}(\log(\frac{1}{\min(t,1-t)}))$. Their proof is postponed to the next lemma. Using the bound $1 + \frac{1}{2x} \geq (\frac{1}{x})^{\frac{1}{2}}$ for $0 < x \leq 1/2$ we have $\Phi'[\Phi^{-1}(\log(1 + \frac{1}{2x}))] \geq \Phi'[\Phi^{-1}(\log(\frac{1}{x})/2)]$. Then, (i) and (iii) of Lemma 16 ensure that $\Phi'[\Phi^{-1}(\log(1 + \frac{1}{2x}))] \geq \frac{1}{2}\Phi'[\Phi^{-1}(\log(\frac{1}{x}))]$. Thus there exists a constant $K_4 > 0$ such that for any n

$$I_{\mu_\Phi^n}(t) \geq K_4 L_\Phi(t) \qquad \forall t \in [0, 1].$$

This is the expected result in this case.

We now turn to the general case. Assume that Φ is $\mathcal{C}^2$ on $[\Phi^{-1}(1), +\infty)$. Choose an even convex function $\Psi : \mathbb{R}^+ \mapsto \mathbb{R}$ which is $\mathcal{C}^2$, increasing on $[0, +\infty)$ and that coincides with Φ outside an interval $[0, a]$. We also consider the probability measure $d\mu_\Psi(x) = Z_\Psi^{-1} e^{-\Psi(|x|)}$. In the large its density differs from the one of μ_Φ exactly by the multiplicative factor Z_Φ / Z_Ψ. The first statement of Proposition 13 shows that the isoperimetric profiles of μ_Φ and μ_Ψ are equivalent when t tends to 0 or 1. Since they are continuous, there exists constants $c_1, c_2 > 0$ such that

$$c_1 I_{\mu_\Phi} \geq I_{\mu_\Psi} \geq c_2 I_{\mu_\Phi}. \tag{6}$$

The second inequality in the above formula implies that the monotone map $T : \mathbb{R} \to \mathbb{R}$ defined by $T(x) = H_\Psi^{-1} \circ H_\Phi$ is Lipschitz (just compute its derivative). Here H_Φ is the distribution function of μ_Φ. Moreover, by construction the image measure of μ_Φ by T is μ_Ψ. This easily implies that any Sobolev type inequality satisfied by μ_Φ can be transported to μ_Ψ with a change in the constant, see e.g. (31; 5) for more on these methods. As before, applying Proposition 9 to the measure μ_Φ provides a Beckner-type inequality, with rate function T expressed in terms of Φ. For the above reasons it is inherited by μ_Ψ (we could also have used the perturbation results recalled in the last section of the paper). We get by tensorization and Corollary 8 that there exists a constant C' independent of the dimension n such that for every smooth $f : \mathbb{R}^n \to \mathbb{R}$ one has

$$\int f^2 d\mu_\Psi^n - s \left(\int |f| d\mu_\Psi^n \right)^2 \leq C' \beta(s) \int |\nabla f|^2 d\mu_\Psi^n \qquad \forall s \geq 1,$$

where $\beta(s) = [1/\Phi' \circ \Phi^{-1}(\log(1+s))]^2$ for $s \geq e-1$ and $\beta(s) = [1/\Phi' \circ \Phi^{-1}(1)]^2$ for $s \in [1, e-1]$. Following exactly the reasoning of the smooth case, we get that there exists a constant K_4' such that for any n,

$$I_{\mu_\Psi^n}(t) \geq K_4 L_\Phi(t) \qquad \forall t \in [0,1].$$

The first inequality in (6) and Theorem 14 imply that $I_{\mu_{\Phi^n}} \geq \frac{1}{c_1} I_{\mu_{\Psi^n}}$. This achieves the proof. $\square$

Remark 4. The above theorem can be extended in many ways. The regularity assumption and the concavity of $\sqrt{\Phi}$ need only be satisfied in the large. Proving this requires in particular to modify the function T in Proposition 9.

Example 5. The previous theorem applies to the family of measures $d\nu_p(x) = e^{-|x|^p} dx / (2\Gamma(1 + 1/p))$, $p \in [1,2]$. This recovers results in (15; 8).

Example 6. More generally, for $d\mu_{p,\alpha}(x) = Z_{p,\alpha}^{-1} e^{-|x|^p (\log(\gamma + |x|))^\alpha} dx$, $p \in [1,2]$, $\alpha \geq 0$ and $\gamma = e^{2\alpha/(2-p)}$ we get the following isoperimetric inequality: there exists a constant $c_{p,\alpha}$ such that for any dimension n and any Borel set A with $\mu_{p,\alpha}^n(A) \leq 1/2$,

$$(\mu_{p,\alpha}^n)_s(\partial A) \geq c_{p,\alpha} \left(\log\Big(\frac{1}{\mu_{p,\alpha}^n(A)}\Big) \right)^{1-\frac{1}{p}} \left(\log\log\Big(e + \frac{1}{\mu_{p,\alpha}^n(A)}\Big) \right)^{\alpha/p}.$$

Lemma 16. *Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing convex function with $\Phi(0) = 0$. Assume that $\sqrt{\Phi}$ is concave. Then,*

(i) for every $x \geq 0$: $\Phi^{-1}\left(\frac{1}{2}x\right) \geq \frac{1}{2}\Phi^{-1}(x)$;

(*ii*) *for every* $x \geq 0$*:* $\Phi(2x) \leq 4\Phi(x)$*;*

(*iii*) *for every* $x \geq 0$*:* $\Phi'\left(\frac{1}{2}x\right) \geq \frac{1}{2}\Phi'(x)$*.*

Proof. Since Φ is convex, the slope function $(\Phi(x) - \Phi(0))/x = \Phi(x)/x$ is non-decreasing. Comparing the values at x and $2x$ shows that $2\Phi(x) \leq \Phi(2x)$. The claim of (i) follows.

Assertion (ii) is proved along the same line. Since $\sqrt{\Phi}$ is concave and vanishes at 0, the ratio $\sqrt{\Phi(x)}/x$ is non-increasing. Comparing its values at x and $2x$ yields the inequality.

Point (iii) is a direct consequence of (ii). Indeed, since $\sqrt{\Phi}$ is concave, $\Phi'/(2\sqrt{\Phi})$ is non-increasing. Comparing the values at x and $2x$ and using (ii) ensures that

$$\Phi'(2x) \leq \sqrt{\frac{\Phi(2x)}{\Phi(x)}}\Phi'(x) \leq 2\Phi'(x).$$

This completes the proof. $\square$

6 F-Sobolev versus super-Poincaré inequality

We have explained in Section 3 how to get a dimension free super-Poincaré inequality, using the (tensorizable) Beckner inequality and Theorem 1. Another family of tensorizable inequalities is discussed in (8), namely additive ϕ-Sobolev inequalities.

We shall say that μ satisfies a homogeneous F-Sobolev inequality if for all smooth f,

$$\int f^2 F\left(\frac{f^2}{\int f^2 d\mu}\right) d\mu \leq C_F \int |\nabla f|^2 d\mu. \tag{7}$$

Observe that necessarily $F(1) \leq 0$ (for $f = 1$). When $F = \log$ this is the usual tight logarithmic Sobolev inequality. In this case $F(a/b) = F(a) - F(b)$ so that the previous homogeneous inequality can be rewritten in an additive form. In general however this is not the case, so that we have to introduce the additive ϕ-Sobolev inequality, i.e.

$$\int \phi(f^2)d\mu - \phi\left(\int f^2 d\mu\right) \leq C_\phi \int |\nabla f|^2 d\mu, \tag{8}$$

with for example $\phi(x) = xF(x)$. In general, Inequalities (7) and (8) have different features. Note that (7) is an equality for constant f if $F(1) = 0$. We shall say that the inequality is tight in this case, and is defective if $F(1) < 0$. Besides, Inequality (8) is tight by nature. The main advantage of additive inequalities is that they enjoy the tensorization property, see (8) Lemma 12. Both kinds of Sobolev inequalities can be related to measure-capacity inequalities. We shall below complete the picture in (8). The next Lemma shows how to tight a defective homogeneous inequality, in a much more simple way than the extension of Rothaus lemma discussed in (8) Lemma 9 and Theorem 10.

Lemma 17. *Let* $F : (0, +\infty) \to \mathbb{R}$ *be a non-decreasing continuous function such that* $F(x)$ *tends to* $+\infty$ *when* x *goes to* $+\infty$ *and* $xF_-(x)$ *is bounded.*

Assume that μ satisfies the homogeneous F-Sobolev inequality with constant C_F and a Poincaré inequality with constant C_P. Then for all $a > \max(F(2), 0)$ there exits $C_+(a)$ depending on a, F, C_F and C_P such that for all smooth f

$$\int f^2(F-a)_+\left(\frac{f^2}{\int f^2 d\mu}\right) d\mu \leq C_+(a)\int |\nabla f|^2 d\mu\,.$$

Proof. Since F goes to ∞ at ∞, we may find $\rho > 1$ such that $F(2\rho) = a$. Define $\tilde{F}(u) = F(u) - a$ which is thus non-positive on $[0, 2\rho]$ and non-negative on $[2\rho, +\infty[$ since F is non-decreasing. Obviously μ still satisfies an $\tilde{F}$-Sobolev inequality. If $M = \sup_{0\leq u\leq 2\rho}\{-u\tilde{F}(u)\}$, $M < +\infty$ thanks to our hypotheses, so that for a non-negative f such that $\int f^2 d\mu = 1$,

$$\int f^2\tilde{F}_+(f^2)d\mu \leq C_F\int |\nabla f|^2 d\mu + M. \tag{9}$$

Let ψ defined on $\mathbb{R}^+$ as follows : $\psi(u) = 0$ if $u \leq \sqrt{2}$, $\psi(u) = u$ if $u \geq \sqrt{2\rho}$ and $\psi(u) = \sqrt{2\rho}\,(u-\sqrt{2})/(\sqrt{2\rho}-\sqrt{2})$ if $\sqrt{2} \leq u \leq \sqrt{2\rho}$. Since $\psi(f) \leq f$, $\int \psi^2(f)d\mu \leq 1$ so that

$$\begin{aligned}
\int f^2\tilde{F}_+(f^2)d\mu &= \int \psi^2(f)\tilde{F}_+(\psi^2(f))d\mu \\
&\leq \int \psi^2(f)\tilde{F}_+\left(\frac{\psi^2(f)}{\int\psi^2(f)d\mu}\right)d\mu \\
&\leq AC_F\int|\nabla f|^2 d\mu + M\int\psi^2(f)d\mu \\
&\leq AC_F\int|\nabla f|^2 d\mu + M\int_{f^2\geq 2} f^2 d\mu
\end{aligned}$$

where $A = 2\rho/\left((\sqrt{2\rho}-\sqrt{2})\right)^2$. But as shown in (8, Remark 22),

$$\int_{f^2\geq 2} f^2 d\mu \leq 12C_P\int|\nabla f|^2 d\mu \tag{10}$$

(recall that $\int f^2 d\mu = 1$), so that we finally obtain the desired result. □

The previous Lemma is a key to the result below, which we shall use in what follows.

Theorem 18. *Let $d\mu = e^{-V}dx$ a probability measure on $\mathbb{R}^d$, with V a locally bounded potential. Let $F : (0, +\infty) \to \mathbb{R}$ be a non decreasing, concave, $\mathcal{C}^1$ function satisfying for some γ and M*

- *(i) $F(x)$ tends to $+\infty$ when x goes to $+\infty$,*
- *(ii) $xF'(x) \leq \gamma$ for all $x > 0$,*
- *(iii) $F(xy) \leq E + F(x) + F(y)$ for all $x, y > 0$.*

If μ satisfies the homogeneous F-Sobolev inequality (7) *with constant C_F, then μ satisfies an additive ϕ-Sobolev inequality with some constant C_ϕ and $\phi(x) = xF(x)$. Moreover there exists a constant D such that, for all n, the product measure μ^n satisfies a measure-capacity inequality*

$$\mu^n(A)F\left(\frac{1}{\mu^n(A)}\right) \leq D\mathrm{Cap}_{\mu^n}(A), \tag{11}$$

for all A such that $\mu^n(A) \leq 1/2$.

Proof. Since μ has a locally bounded potential V, it follows from the remark after Theorem 3.1 in (36) that it satisfies the following weak Poincaré inequality for some non increasing function $\tau : (0,1/4) \to \mathbb{R}^+$: for every $s \in (0,1/4)$ and every locally Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ it holds

$$\mathbf{Var}_\mu(f) \le \tau(s) \int |\nabla f|^2 d\mu + s\big(\sup(f) - \inf(f)\big)^2.$$

By hypothesis, μ also satisfies a F-Sobolev inequality with F growing to infinity, so (1, Theorem 2.11) ensures that it verifies a Poincaré inequality (actually we also need to check that the function $xF(x)$ is bounded from below; this is a consequence of (ii)).

In turn (see (8, Remark 20)), there exists a constant $D' > 0$ such that for all n and all A with $\mu^n(A) \le 1/2$,

$$\mu^n(A) \le D' \mathrm{Cap}_{\mu^n}(A). \tag{12}$$

For technical reasons, we assume first that $F(8) > 0$. We shall explain in the end how this assumption can be removed. By Lemma 17, μ satisfies an $\tilde{F}$-Sobolev inequality for $\tilde{F} = (F-a)_+$, where a is any number in $(F(2), F(8))$.

According to (8) Theorem 22 and Remark 23, μ will satisfy a measure-capacity inequality as soon as we can find some $x_0 > 2$ such that

(a) $x \mapsto \tilde{F}(x)/x$ is non-increasing on $(x_0, +\infty)$,

(b) there exists some $\lambda > 4$ such that $4\tilde{F}(\lambda x) \le \lambda \tilde{F}(x)$ for $x \ge x_0$.

For large values of x, the derivative of $\tilde{F}(x)/x$ has the sign of $xF'(x) - F(x) + a$. This is non-positive for $x \ge F^{-1}(\gamma + a)$ thanks to (ii) and (i). So Property (a) is valid when $x_0 \ge F^{-1}(\gamma + a)$. For (b) just remark that

$$\tilde{F}(8x) \le E + a + \tilde{F}(8) + \tilde{F}(x) \le 2\tilde{F}(x), \quad \forall x \ge \tilde{F}^{-1}(E + a + F(8)),$$

thanks to (iii). We may choose x_0 as the maximum of the two previous values. As explained in (8) Remark 23, we then have $\mu(A)\tilde{F}(1/\mu(A)) \le K_0 \mathrm{Cap}_\mu(A)$ if $\mu(A) \le 1/x_0$. It follows that

$$\mu(A)\tilde{F}\left(\frac{2}{\mu(A)}\right) \le \mu(A)\tilde{F}\left(\frac{8}{\mu(A)}\right) \le 2\mu(A)\tilde{F}\left(\frac{1}{\mu(A)}\right) \le 2K_0 \mathrm{Cap}_\mu(A).$$

for any A with $\mu(A) \le 1/x_0$. Using Poincaré inequality in the form of Equation (12), we find a constant K_1 such that

$$\mu(A)F\left(\frac{2}{\mu(A)}\right) \le K_1 \mathrm{Cap}_\mu(A),$$

for all A with $\mu(A) \le 1/2$. Theorem 26 in (8) furnishes the additive ϕ-Sobolev inequality (8) with $\phi(x) = xF(x)$.

By the tensorization property of additive ϕ-Sobolev inequalities, the measures μ^n also satisfy (8) (with a constant which does not depend on the dimension n). Consequently μ^n satisfies a homogeneous $\big(F - F(1)\big)$-Sobolev inequality with a dimension-free constant and therefore a homogeneous F-Sobolev inequality (since $F(1) \le 0$). Proceeding exactly as in the beginning of the proof (for μ^n instead of μ) we deduce that

$$\mu^n(A)F\left(\frac{2}{\mu^n(A)}\right) \le D_\phi \mathrm{Cap}_{\mu^n}(A),$$

for some constant D_ϕ (independent on n) and all A with $\mu^n(A) \leq 1/2$. This achieves the proof when $F(8) > 0$.

Finally when $F(8) \leq 0$, we choose $\varepsilon \in (0, F(8) - F(1))$ and define $G := F - F(8) + \varepsilon \geq F$. Note that $G(8) > 0$, $G(1) \leq 0$ and that G also satisfies (i), (ii) and (iii) with possibly worse constants. Hence if we show that μ satisfies a homogeneous G-Sobolev inequality the above reasoning applies and gives the claim of the theorem. Now we show briefly that since μ satisfies a Poincaré inequality, the F-Sobolev inequality may be upgraded to a G-Sobolev inequality. To see this we apply Lemma 17 to get a $(F-1)_+$-Sobolev inequality. The function $(F-1)_+$ is zero before $x_1 := F^{-1}(1) > 8$. Next we add up the latter Sobolev inequality with $(1-F(8)+\varepsilon)$ times the equivalent form of Poincaré inequality given in (10) to get a $G\mathbb{1}_{[x_1,\infty)}$-Sobolev inequality. Finally Lemma 21 in (8) yields the desired G-Sobolev inequality. Indeed this lemma allows any $\mathcal{C}^2$ modification of the function on the interval $[0, x_1]$ provided it vanishes at 1; moreover since G is concave and non-positive at 1 it can be upper bounded by such a function. The proof is complete. $\square$

Remark 7. Part of the previous Theorem is proved in a slightly different form in (35).

We have seen in the proof that under the hypotheses of Theorem 18, μ^n satisfies (12). Thus, in the capacity-measure inequality (11) we may replace F by $1 + F_+$ according to (12), changing the constant D if necessary. As a consequence, using Corollary 6 and Theorem 18 we have

Corollary 19. *Let μ and F as in Theorem 18. Then there exists a constant K such that for all n, for all $f : (\mathbb{R}^d)^n \to \mathbb{R}$ and every $s \geq 1$ one has*

$$\int f^2 d\mu^n - s\left(\int |f| d\mu^n\right)^2 \leq K\beta(s) \int |\nabla f|^2 d\mu^n,$$

with $\beta(s) = 1/(1 + F_+)(s)$.

As the reader readily sees, the previous corollary is not as esthetic as the Beckner type approach for two reasons: first F has to fulfill some hypotheses, second the constant K is not explicit (the main difficulty is to get an estimate on the Poincaré constant from the weak spectral gap property). Nonetheless combined with the results in Section 4, it allows us to obtain isoperimetric inequalities for Boltzmann measures that do not enter the framework of Section 5 (see below).

Finally the results extend to Riemannian manifolds since any probability measure with a locally bounded potential satisfies a local Poincaré inequality, see (36).

7 Further examples.

The main result of this section is Theorem 21. It provides more general examples of measures μ for which the products μ^n satisfy a dimension free isoperimetric inequality. Its main interest is to deal directly with measures μ on $\mathbb{R}^d$.

We start with perturbation results. Let μ be a non-negative measure and $d\nu = e^{-2V} d\mu$ be a probability measure. It is easy to deal with a bounded perturbation V as for the logarithmic Sobolev inequality (27) or the Poincaré inequality: if μ satisfies one of these inequalities with constant C then so does ν with constant at most $Ce^{\mathrm{Osc}(2V)}$, where $\mathrm{Osc}(V) = \sup V - \min V$.

Since Wang (43, Proposition 2.5) proved a similar result for the generalized Beckner inequality, Corollary 8 applies to the perturbed measure ν. When considering unbounded perturbations V, some control on the derivatives seem to be needed. Here is a general result in this direction, extending (8, Section 7.2.).

Lemma 20. *Let μ be a non-negative measure and $d\nu = e^{-2V}d\mu$ be a probability measure on (M, g). Assume that μ satisfies a defective homogeneous L-Sobolev inequality:*

$$\int f^2 L\left(\frac{f^2}{\int f^2 d\mu}\right) d\mu \le C \int |\nabla f|^2 d\mu + C' \int f^2 d\mu \qquad \forall f.$$

Let F be a $\mathcal{C}^1$ function defined on $(0, +\infty)$, satisfying

(i) *$F(x)$ tends to $+\infty$ when x goes to $+\infty$,*

(ii) *There exists $E \in \mathbb{R}$ such that $F(xy) \le E + F(x) + F(y)$ for all $x, y > 0$,*

(iii) *there exists $K \in \mathbb{R}$ such that $xF(x) \le xL(x) + K/\mu(M)$ for all x. If $\mu(M) = +\infty$ we decide that $K = 0$.*

(iv) *$F(e^{2V}) + C(\Delta_\mu V - |\nabla V|^2)$ is bounded from above.*

Then there exists a constant B such that ν satisfies a homogeneous $(F - B)$-Sobolev inequality. Here Δ_μ is an analogue of the Laplace operator for μ with the integration by part property $\int f\, \Delta_\mu g\, d\mu = -\int \nabla f \cdot \nabla g\, d\mu$.

Proof. First of all thanks to (ii),

$$F(g^2) = F(g^2 e^{-2V} e^{2V}) \le E + F(e^{2V}) + F(g^2 e^{-2V}).$$

Hence if $\int g^2 d\nu = 1$ and $f = g\, e^{-V}$ (so that $\int f^2 d\mu = 1$),

$$\begin{aligned}
\int g^2 F(g^2) d\nu &\le E + \int g^2 F(e^{2V}) d\nu + \int f^2 F(f^2) d\mu \\
&\le (E+K) + \int g^2 F(e^{2V}) d\nu + \int f^2 L(f^2) d\mu \\
&\le (E+K+C') + \int g^2 F(e^{2V}) d\nu + C \int |\nabla f|^2 d\mu \\
&\le (E+K+C') + \int g^2 \left(F(e^{2V}) + C(\Delta_\mu V - |\nabla V|^2)\right) d\nu \\
&\qquad + C \int |\nabla g|^2 d\nu
\end{aligned} \tag{13}$$

using (iii), the L-Sobolev inequality for μ and an immediate integration by parts. This gives the expected result by (iv). ∎

Remark 8. If in addition of the hypotheses $(i), (ii), (iii)$, we assume that $x \mapsto xF(x)$ is convex, one can replace Hypothesis (iv) by the weaker assumption

$$(iv') \quad \exists \varepsilon \in (0,1) \text{ such that } \int H\left(\frac{1}{\varepsilon}\left(F(e^{2V}) + C(\Delta_\mu V - |\nabla V|^2)\right)\right) d\nu < +\infty,$$

where H is the convex conjugate of $x \mapsto xF(x)$. Indeed, Young's inequality $xy \leq \varepsilon xF(x) + H(y/\varepsilon)$ allows to bound (13).

Theorem 21. *Let $d\mu = e^{-2V}dx$ be a probability measure on $\mathbb{R}^d$ with V a C^2 potential such that $\mathrm{D}^2V \geq -Rg$ for some $R \geq 0$. Let F be a $\mathcal{C}^1$ function defined on $(0, +\infty)$, satisfying*

(i) $F(x)$ tends to $+\infty$ when x goes to $+\infty$,

(ii) there exists $E \in \mathbb{R}$ such that $F(xy) \leq E + F(x) + F(y)$ for all $x, y > 0$,

(iii) there exists $E' \in \mathbb{R}$ such that $F(x) \leq E' \log_+(x)$ for all x.

(iv) there exists $\gamma > 0$ such that $xF'(x) \leq \gamma$, for all $x > 0$,

(v) there exists $C > 0$ such that $F(e^{2V}) + C(\Delta_\mu V - |\nabla V|^2)$ is bounded from above.

Then there exists $\theta > 0$ such that for all n and all measurable sets $A \subset (\mathbb{R}^d)^n$ with $\mu^n(A) \leq 1/2$

$$\mu_s^n(\partial A) \geq \theta \mu^n(A) \sqrt{1 + F_+\left(\frac{1}{2\mu^n(A)}\right)}.$$

Proof. The Euclidean logarithmic Sobolev inequality (see (22, Theorem 2.2.4)) asserts that for any bounded smooth function $f : \mathbb{R}^d \to \mathbb{R}$ with $\int f^2 dx = 1$ and for every $\eta > 0$,

$$\int f^2 \log_+ f^2 dx \leq 2\eta \int |\nabla f|^2 dx + 2 + \frac{d}{2} \log\left(\frac{1}{\pi\eta}\right).$$

Thus we can apply Lemma 20 with $d\mu = dx$, $L = E' \log_+$ choosing $\eta = C/(2E')$. This leads to a homogeneous $(F - B)$-Sobolev inequality, for some constant $B > 0$. Corollary 19 applies and leads to a super-Poincaré inequality with function $\beta = 1/(1 + F_+)$. Applying Theorem 12 achieves the proof. □

Example 9. Let $1 < \alpha < 2$. Let $V : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^2$ function with $V(x) = |x|^\alpha + \log(1 + |x| \sin^2 x)$ when $|x| \geq \varepsilon > 0$. This potential is an unbounded perturbation of $|x|^\alpha$ and is not convex. Theorem 21 applies to V for $F(u) = \log(1 + u)^{2(1-\frac{1}{\alpha})} - \log(2)^{2(1-\frac{1}{\alpha})}$.

Remark 10. As for the logarithmic Sobolev inequality in (21), the previous result allows us to look at d dimensional spaces from the beginning. Nevertheless, if $d = 1$ it can be compared with the tractable condition one can get for the Beckner type inequality in Section 3. Indeed assume that V' does not vanish near ∞ and that $V''/|V'|^2$ goes to 0 at ∞. Then the Laplace method, see e.g. (2, Corollaire 6.4.2), yields a sufficient condition for $B_+(T)$ and $B_-(T)$ in Theorem 7 to be finite, namely :

$$|V'|^2 T\left(\frac{1}{V + \log(|V'|)}\right) \geq C > 0, \tag{14}$$

near ∞. If $\log(|V'|) \ll V$ near ∞, (14) becomes $|V'|^2 \geq C/T(\frac{1}{V})$ i.e. we have the formal relation $1/T(1/\log u) = F(u)$.

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