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Correlation lengths for random polymer models and for some renewal sequences*

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Abstract

We consider models of directed polymers interacting with a one-dimensional defect line on which random charges are placed. More abstractly, one starts from renewal sequence on $\mathbb{Z}$ and gives a random (site-dependent) reward or penalty to the occurrence of a renewal at any given point of $\mathbb{Z}$. These models are known to undergo a delocalization-localization transition, and the free energy $\textsc{f}$ vanishes when the critical point is approached from the localized region. We prove that the quenched correlation length ξ, defined as the inverse of the rate of exponential decay of the two-point function, does not diverge faster than $1/\textsc{f}$. We prove also an exponentially decaying upper bound for the disorder-averaged two-point function, with a good control of the sub-exponential prefactor. We discuss how, in the particular case where disorder is absent, this result can be seen as a refinement of the classical renewal theorem, for a specific class of renewal sequences.

Key words: Pinning and Wetting Models, Typical and Average Correlation Lengths, Critical Exponents, Renewal Theory, Exponential Convergence Rates.

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1 Introduction and motivations

The present work is motivated by the following two problems:

- *Critical behavior of the correlation lengths for directed polymers with (de-)pinning interactions.* Take a homogeneous Markov chain $\{S_n\}_{n\geq 0}$ on some discrete state space Σ, with $S_0 = 0$ and law $\mathbf{P}$. A trajectory of S is interpreted as the configuration of a directed polymer in the space $\Sigma \times \mathbb{N}$. In typical examples, S is a simple random walk on $\Sigma = \mathbb{Z}^d$ or a simple random walk conditioned to be non-negative on $\Sigma = \mathbb{Z}^+$. Of particular interest is the case where the distribution of the first return time of S to zero, $K(n) := \mathbf{P}(\min\{k > 0 : S_k = 0\} = n)$, decays like a power of n for n large. This holds in particular in the case of the simple random walks mentioned above. We want to model the situation where the polymer gets a reward (or penalty) ω_n each time it touches the line $S \equiv 0$ (which is called *defect line*). In other words, we introduce a polymer-line interaction energy of the form

$$-\sum_{n=1}^{N} \omega_n \mathbf{1}_{\{S_n=0\}},$$

where N will tend to infinity in the thermodynamic limit. The defect line is attractive at points n where $\omega_n > 0$ and repulsive when $\omega_n < 0$. In particular, one is interested in the situation where ω_n are IID quenched random variables. There is a large physics literature (cf. (9, Chapter 1) and references therein) related to this class of models, due to their connection with, e.g., problems of $(1+1)$-dimensional wetting of a disordered wall or with the DNA denaturation transition.

In the *localized phase* where the free energy (defined in next section) is positive and the number of contacts between the polymer and the defect line, $|\{1 \leq n \leq N : S_n = 0\}|$, grows proportionally to N, one knows (11) that the two-point correlation function

$$|\mathbf{P}_{\infty,\omega}(S_{n+k} = 0|S_n = 0) - \mathbf{P}_{\infty,\omega}(S_{n+k} = 0)| \tag{1.1}$$

decays exponentially in k, for every n and for almost every disorder realization. Here, $\mathbf{P}_{\infty,\omega}(.)$ is the Gibbs measure for a given randomness realization and the index ∞ refers to the fact that the thermodynamic limit has been taken. The exponential decay of correlation functions has been applied, for instance, to prove sharp results on the maximal excursions length in the localized phase (11, Theorem 2.5) and bounds on the finite-size correction to the thermodynamic limit of the free energy (11, Theorem 2.8).

The inverse of the rate of decay is identified as a correlation length ξ. A natural question is the relation between ξ and the free energy $\textsc{f}$, in particular in proximity of the delocalization-localization critical point, where the free energy tends to zero (see next section) and the correlation length is expected to tend to infinity. The disorder average of the two-point function (1.1) is also known (11) to decay exponentially with k, possibly with a *different* rate (19).

The important role played by the correlation length, and by its relation with the free energy, in understanding the critical properties of disordered pinning models was emphasized in a recent work by K. Alexander (2).

- *Geometric convergence rates for renewal sequences.* Consider a renewal sequence $\tau := \{\tau_i\}_{i=0,1,2,...}$ of law $\mathbf{P}$ defined as follows: $\tau_0 = 0$, and $\tau_i - \tau_{i-1}$ are IID random variables with values in $\mathbb{N}$ and probability distribution $p(.)$, where $p(n) \geq 0$ and $\sum_{n\in\mathbb{N}} p(n) = 1$. The celebrated renewal theorem (4, Chap. I, Th. 2.2) states that

$$u_n := \mathbf{P}(n \in \tau) \overset{n\to\infty}{\to} u_\infty := \frac{1}{\sum_{n\in\mathbb{N}} np(n)} = \frac{1}{\mathbf{E}(\tau_1)}, \tag{1.2}$$

with the convention that $1/\infty = 0$. It is natural (and quite useful in practice, especially in queuing theory applications) to study the speed of convergence in (1.2). In this respect, it is known (cf. for instance (4, Chapter VII.2), (18)) that, if

$$b := \sup\{s > 0 : \sum_{n\in\mathbb{N}} e^{sn} p(n) < \infty\} > 0, \tag{1.3}$$

then there exist $r > 0$ and $C < \infty$ such that

$$|u_n - u_\infty| \leq Ce^{-rn}. \tag{1.4}$$

However, the relation between b and the largest possible r in Eq. (1.4), call it r_{max}, is not known in general. A lot of effort has been put in investigating this point, and in various special cases, where $p(.)$ satisfies some structural ordering properties, it was proved that $r_{max} \geq b$ (see for instance (5), where power series methods are employed and explicit upper bounds on the prefactor C are given). In even more special cases, for instance when τ_i are the return times of a Markov chain with some stochastic ordering properties, the optimal result $r_{max} = b$ is proved (for details, see (16; 19), which are based on coupling techniques). However, the equality $r_{max} = b$ cannot be expected in general. In particular, if $p(.)$ is a geometric distribution,

$$p(n) = (e^c - 1)e^{-nc}$$

with $c > 0$, then one sees that $u_n = u_\infty$ for every $n \in \mathbb{N}$ so that $r_{max} = \infty$, while $b = c$. On the other hand, if for instance $p(1) = p(2) = 1/2$ and $p(n) = 0$ for $n \geq 3$, then $b = \infty$ while r_{max} is finite. These and other nice counter-examples are discussed in (5).

The two problems are known to be strictly related: indeed, in the homogeneous situation ($\omega_n \equiv$ *const*) the law of the collection $\{n : S_n = 0\}$ of points of polymer-defect contact is given, in the thermodynamic limit, by a renewal process of the type described above, with $p(n)$ proportional to $K(n)e^{-n\textsc{f}}$ (cf., for instance, (9, Chapter 2)). In this case, therefore, the free energy $\textsc{f}$ plays the role of b above.

With respect to the first problem listed above, the main result of this paper is that, in the limit where $\textsc{f}$ tends to zero (i.e., when the parameters of the model are varied in such a way that the critical point is approached from the localized phase), the correlation length ξ is at most of order $1/\textsc{f}$, for almost every disorder realization. An exponentially decaying upper bound, with a "good" control of the sub-exponential prefactor, is derived also for the *disorder average* of the two-point function (1.1), cf. Equation (2.17) of Theorem 2.1 and the discussion in Remark 2.2.

As a corollary we obtain the following result for the second problem above: if the jump law $p(.)$ of the renewal sequence is of the form

$$p(n) = a(b)\frac{L(n)}{n^{\alpha+1}}e^{-bn}, \tag{1.5}$$

with $0 \leq \alpha < \infty$,

$$a(b) = \left(\sum_n L(n) n^{-(\alpha+1)} \exp(-bn) \right)^{-1}$$

and $L(.)$ a slowly varying function (not depending on b), then for b small one has that $r_{max} \gtrsim b$ and $C \lesssim b^{-c}$ for some positive constant c (see Theorem 2.1 and Remarks 2.2, 4.1 below for the precise statements). In particular, this means that $|u_n - u_\infty|$ starts decaying exponentially (with rate at least of order b) as soon as $n \gg 1/b$.

Remark 1.1. After this work was completed, a much sharper result was obtained by G. Giacomin (10) in the homogeneous case: if condition (1.5) holds for some $\alpha > 0$, then for b sufficiently small one has

$$u_n - u_\infty \overset{n\to\infty}{\sim} \frac{a(b)}{(a(b)-1)^2} \frac{L(n)}{n^{\alpha+1}} e^{-bn}. \tag{1.6}$$

The techniques employed in (10) are very different from ours, and do not extend to the situation where disorder is present, i.e., to the study of (1.1) for $\omega \not\equiv const$.

2 Notations and main result

We will define our "directed polymer" model in an abstract way where the Markov chain S mentioned in the introduction does not appear explicitly. In this way the intuitive picture of the Markov chain trajectory as representing a directed polymer configuration is somewhat hidden, but the advantage is that the connection with renewal theory becomes immediate. The link with the polymer model discussed in the introduction is made by identifying the renewal sequence τ below with the set of the return times of the Markov chain S to the site 0.

Let $K(.)$ be a probability distribution on $\mathbb{N} := \{1, 2, \ldots\}$, i.e., $K(n) \geq 0$ for $n \in \mathbb{N}$ and

$$\sum_{n \in \mathbb{N}} K(n) = 1. \tag{2.1}$$

We assume that

$$K(n) = \frac{L(n)}{n^{\alpha+1}} \tag{2.2}$$

for some $0 \leq \alpha < \infty$. Here, $L(.)$ is a slowly varying function, i.e., a positive function $L : \mathbb{R}^+ \ni x \to L(x) \in (0, \infty)$ such that $\lim_{x\to\infty} L(xr)/L(x) = 1$ for every $r > 0$. Given $x \in \mathbb{Z}$, we construct a renewal process $\tau := \{\tau_i\}_{i \in \mathbb{N}\cup\{0\}}$ with law $\mathbf{P}_x$ as follows: $\tau_0 = x$, and $\tau_i - \tau_{i-1}$ are IID integer-valued random variables with law $K(.)$. $\mathbf{P}_x$ can be naturally seen as a law on the set

$$\Omega_x := \{\tau : \tau \subset (\mathbb{Z} \cap [x, \infty)) \text{ and } x \in \tau\}.$$

Note that, thanks to (2.1), τ is a *recurrent* renewal process (possibly, null-recurrent).

Now we modify the law of the renewal by switching on a random interaction as follows. We let $\{\omega_n\}_{n\in\mathbb{Z}}$ be a sequence of IID centered random variables with law $\mathbb{P}$ and $\mathbb{E}\,\omega_0^2 = 1$. For simplicity, we require also ω_n to be bounded. Then, given $h \in \mathbb{R}$, $\beta \geq 0$, $x, y \in \mathbb{Z}$ with $x < y$ and a realization of ω we let

$$\frac{\mathrm{d}\mathbf{P}_{x,y,\omega}}{\mathrm{d}\mathbf{P}_x}(\tau) = \frac{e^{\sum_{n=x+1}^{y}(\beta\omega_n - h)\mathbf{1}_{\{n\in\tau\}}}}{Z_{x,y,\omega}}\mathbf{1}_{\{y\in\tau\}} \tag{2.3}$$

where, of course,

$$Z_{x,y,\omega} = \mathbf{E}_x\left(e^{\sum_{n=x+1}^{y}(\beta\omega_n - h)\mathbf{1}_{\{n\in\tau\}}}\mathbf{1}_{\{y\in\tau\}}\right) \tag{2.4}$$

and $\mathbf{P}_{x,y,\omega}$ is still a law on Ω_x. Note that the normalization condition (2.1) is by no means a restriction: if we had $\Sigma := \sum_{n\in\mathbb{N}} K(n) < 1$, we could perform the replacements $K(.) \to K(.)/\Sigma$, $h \to h - \log\Sigma$ in (2.3) and the measure $\mathbf{P}_{x,y,\omega}$ would be unchanged.

One defines the free energy as

$$\mathrm{F}(\beta, h) = \lim_{N\to\infty}\frac{1}{2N}\log Z_{-N,N,\omega}. \tag{2.5}$$

The convergence holds almost surely and in $L^1(\mathbb{P})$, and $\mathrm{F}(\beta, h)$ is $\mathbb{P}(\mathrm{d}\omega)$-a.s. constant (see (9, Chap. 4) and (3)). It is known that $\mathrm{F}(\beta, h) \geq 0$: to realize this, it is sufficient to observe that

$$\frac{1}{2N}\log Z_{-N,N,\omega} \geq \frac{1}{2N}\log\mathbf{E}_{-N}\left(e^{\sum_{n=-N+1}^{N}(\beta\omega_n - h)\mathbf{1}_{\{n\in\tau\}}}\mathbf{1}_{\{\tau_1 = N\}}\right) \tag{2.6}$$

$$= \frac{\beta\omega_N - h}{2N} + \frac{1}{2N}\log K(2N) \tag{2.7}$$

which tends to zero for $N \to \infty$. One then decomposes the phase diagram into *localized* and *delocalized* regions defined as

$$\mathcal{L} := \{(\beta, h) : \mathrm{F}(\beta, h) > 0\} \tag{2.8}$$

$$\mathcal{D} := \{(\beta, h) : \mathrm{F}(\beta, h) = 0\}, \tag{2.9}$$

separated by the critical line

$$h_c(\beta) := \inf\{h : \mathrm{F}(\beta, h) = 0\}. \tag{2.10}$$

By convexity, the free energy is continuous in β and h and therefore tends to zero when the critical line is approached from the localized region. It is known that typical configurations τ are very different in the two regions. Roughly speaking, if $(\beta, h) \in \mathcal{L}$ then the typical τ has a finite density of points in $\mathbb{N}$, i.e., for N large

$$\frac{1}{N}|\tau \cap \{1, \ldots, N\}| \sim -\partial_h\mathrm{F}(\beta, h) > 0. \tag{2.11}$$

On the other hand, in $\mathcal{D}$ the density tends to zero with N:

$$\frac{1}{N}|\tau \cap \{1, \ldots, N\}| \begin{cases} \leq (\log N)/N & \text{if} \quad h > h_c(\beta) \\ \leq N^{-1/3}\log N & \text{if} \quad h = h_c(\beta) \end{cases} \tag{2.12}$$

(for precise statements see, respectively, (12, Theorem 1.4, part (2)) and (19, Theorem 3.1)).

Another quantity which will play an important role in the following is

$$\mu(\beta, h) = -\lim_{N\to\infty} \frac{1}{2N} \log \mathbb{E} \frac{1}{Z_{-N,N,\omega}}. \tag{2.13}$$

As it is known (cf. (11, Theorem 2.5 and Appendix B)) for $(\beta, h) \in \mathcal{L}$ one has

$$0 < \mu(\beta, h) < \mathrm{F}(\beta, h), \tag{2.14}$$

while $\mathrm{F}(\beta, h) = \mu(\beta, h) = 0$ in $\mathcal{D}$. On the other hand, it is unknown whether the ratio $\mathrm{F}(\beta, h)/\mu(\beta, h)$ remains bounded for $h \to h_c(\beta)$. $\mu(\beta, h)$ is related to the maximal excursion length in the localized phase,

$$\Delta_N := \max_{\substack{0<i<j<N:\\ \{i,\dots,j\}\cap\tau=\emptyset}} |j - i|,$$

in the sense that essentially $\Delta_N \simeq \log N/\mu(\beta, h)$, see (11, Theorem 2.5) (cf. also (1) for a proof of the same fact in a related model, the *heteropolymer at a selective interface*).

As was proven in (11) (but see also (6) for the proof of the almost sure existence of the infinite-volume Gibbs measure for the heteropolymer model in the localized phase), the limit

$$\mathbf{E}_{\infty,\omega}(f) := \lim_{\substack{x\to-\infty\\ y\to\infty}} \mathbf{E}_{x,y,\omega}(f) \tag{2.15}$$

exists, $\mathbb{P}(\mathrm{d}\omega)$–a.s., for every $(\beta, h) \in \mathcal{L}$ and for every bounded local observable f, and is independent of the way the limits $x \to -\infty$, $y \to \infty$ are performed. A bounded local observable is a bounded function $f : \{\tau : \tau \subset \mathbb{Z}\} \to \mathbb{R}$ for which there exists I, finite subset of $\mathbb{Z}$, such that

$$f(\tau_1) = f(\tau_2)$$

whenever $\tau_1 \cap I = \tau_2 \cap I$. The smallest possible I is called support of f. An example of local observable is $|\{\tau \cap I\}|$, the number of points of τ which belong to I. On the other hand, τ_1 is not a local observable.

A useful identity is the following: let $a \in \mathbb{Z}$ and f, g be two local observables, whose supports are contained in $\{\dots, a-2, a-1\}$ and $\{a+1, a+2, \dots\}$, respectively. Then, if $x < a < y$,

$$\mathbf{E}_{x,y,\omega}(f\, g|a \in \tau) = \mathbf{E}_{x,a,\omega}(f)\mathbf{E}_{a,y,\omega}(g). \tag{2.16}$$

In other words, conditioning on the event that a belongs to τ makes the process to the left and to the right of a independent. This is easily checked from the definition (2.3) of the Boltzmann-Gibbs measure and from the IID character of $\tau_i - \tau_{i-1}$ under $\mathbf{P}_x$.

Our first result is an exponentially decaying upper bound on the disorder-averaged two-point correlation function, in the localized phase:

Theorem 2.1. *Let $\epsilon > 0$ and $(\beta, h) \in \mathcal{L}$. There exists $C_1 := C_1(\epsilon, \beta, h) > 0$ such that, for every $k \in \mathbb{N}$,*

$$\mathbb{E}\left|\mathbf{P}_{\infty,\omega}(k \in \tau|0 \in \tau) - \mathbf{P}_{\infty,\omega}(k \in \tau)\right| \le \frac{1}{C_1 \mu(\beta, h)^{1/C_1}} \exp\left(-k\, C_1\, \mu(\beta, h)^{1+\epsilon}\right). \tag{2.17}$$

The constant $C_1(\epsilon, \beta, h)$ does not vanish at the critical line: for every bounded subset $B \subset \mathcal{L}$ one has $\inf_{(\beta,h)\in B} C_1(\epsilon, \beta, h) \ge C_1(B, \epsilon) > 0$.

Remark 2.2. Note that Theorem 2.1 is more than just a bound on the rate of exponential decay of the disorder-averaged two-point correlation. Indeed, thanks to the explicit bound on the prefactor in front of the exponential, Eq. (2.17) says that the exponential decay, with rate at least of order $\mu^{1+\epsilon}$, commences as soon as $k \gg \mu^{-1-\epsilon}|\log \mu|$. This observation reinforces the meaning of Eq. (2.17) as an upper bound on the correlation length of disorder-averaged correlations functions.

It would be possible, via the Borel-Cantelli Lemma, to extract from Eq. (2.17) the almost-sure exponential decay of the disorder-dependent two-point function. However, from (19) one expects the almost-sure exponential decay to be related to $\textsc{f}(\beta,h)$ rather than to $\mu(\beta,h)$. Indeed, we have the following:

Theorem 2.3. *Let $\epsilon > 0$ and $(\beta,h) \in \mathcal{L}$. One has for every $k \in \mathbb{N}$*

$$|\mathbf{P}_{\infty,\omega}(k \in \tau|0 \in \tau) - \mathbf{P}_{\infty,\omega}(k \in \tau)| \le C_2(\omega) \exp\left(-k\, C_1 \textsc{f}(\beta,h)^{1+\epsilon}\right), \tag{2.18}$$

where C_1 is as in Theorem 2.1, while $C_2(\omega) := C_2(\omega,\epsilon,\beta,h)$ is an almost surely finite random variable.

Recalling that $\textsc{f} > \mu$, it is clear that Theorem 2.3 cannot be deduced from Theorem 2.1.

Remark 2.4. It is quite tempting to expect that, in analogy with Theorem 2.1, the (random) prefactor $C_2(\omega)$ is bounded above by

$$\frac{C_5(\omega,\epsilon,\beta,h)}{\textsc{f}(\beta,h)^{C_5(\omega,\epsilon,\beta,h)}},$$

for some random variable C_5 such that, say, $\mathbb{E}C_5(\omega,\epsilon,\beta,h) \le c(B,\epsilon) < \infty$ for (β,h) belonging to a bounded set $B \subset \mathcal{L}$. This would mean that the almost sure exponential decay with decay rate at least of order $\textsc{f}^{1+\epsilon}$ commences as soon as $k \gg n(\omega)\textsc{f}^{-1-\epsilon}|\log \textsc{f}|$, with $n(\omega)$ random but typically of order one even close to the critical point. However, this kind of result seems to be out of reach with the present techniques.

Remark 2.5. As can be extracted from the proof of Theorems 2.1 and 2.3 (see in particular Remark 7.3), if the slowly varying function $L(n)$ in (2.2) tends to a constant for $n \to \infty$, then one can replace the right-hand side of Eqs. (2.17), (2.18) by

$$\frac{1}{C_3(\beta,h)\mu(\beta,h)^{1/C_3(\beta,h)}} \exp\left(-k\, C_3(\beta,h)\, \frac{\mu(\beta,h)}{|\log \mu(\beta,h)|}\right)$$

and

$$C_4(\omega,\beta,h) \exp\left(-k\, C_3(\beta,h) \frac{\textsc{f}(\beta,h)}{|\log \textsc{f}(\beta,h)|}\right)$$

respectively, with $\inf_{(\beta,h)\in B} C_3(\beta,h) \ge C_3(B) > 0$ and C_4 almost surely finite.

Once the exponential decay of the two-point function is proven, it is not difficult to obtain similar results for the correlation between any two given local observables (cf. Remark 5.1 below for some more details):

Theorem 2.6. *Let A and B be two bounded local observables, with supports S_A and S_B, respectively. Assume that S_A is contained in $\mathbb{Z} \cap (-\infty, 0]$ and $S_B \subset \mathbb{Z} \cap [k, \infty)$. Let $(\beta, h) \in \mathcal{L}$, while $\epsilon > 0$. Then,*

$$\mathbb{E}\, |\mathbf{E}_{\infty,\omega}(AB) - \mathbf{E}_{\infty,\omega}(A)\mathbf{E}_{\infty,\omega}(B)| \leq \frac{||A||_\infty ||B||_\infty}{C_1 \mu(\beta,h)^{1/C_1}} \exp\left(-k\, C_1\, \mu(\beta,h)^{1+\epsilon}\right) \tag{2.19}$$

and

$$|\mathbf{E}_{\infty,\omega}(AB) - \mathbf{E}_{\infty,\omega}(A)\mathbf{E}_{\infty,\omega}(B)| \leq ||A||_\infty ||B||_\infty C_2(\omega) \exp\left(-k\, C_1\, \textsc{f}(\beta,h)^{1+\epsilon}\right), \tag{2.20}$$

where C_1 and C_2 are as in Theorems 2.1 and 2.3.

3 Sketch of the idea: auxiliary Markov process and coupling

In this section, we give an informal sketch of the basic ideas underlying the proof of the upper bounds for the two-point function. The actual proof is somewhat involved and takes Sections 4 to 7.

The basic trick is to associate to the renewal probability $K(.)$ a Markov process $\{S_t\}_{t \geq x}$ such that, very roughly speaking, its trajectories are continuous "most of the time" and the random set of integer times $\{t \in \mathbb{Z} \cap [x, \infty) : S_t = 0\}$ has the same distribution as the *discrete* renewal process $\{\tau_i\}_{i \in \mathbb{N} \cup \{0\}}$ associated to $K(.)$, with law $\mathbf{P}_x$. This construction is done in Section 4, where we see that $S_.$ is strictly related to the Bessel process (17) of dimension $2(\alpha + 1)$. Once we have $S_.$, we switch on the interaction

$$-\sum_{n=x+1}^{y} (\beta\omega_n - h)\mathbf{1}_{\{S_n = 0\}}$$

and in the thermodynamic limit $x \to -\infty, y \to \infty$ we obtain a new measure $\hat{\mathbf{P}}_{\infty,\omega}$ on the paths $\{S_t\}_{t \in \mathbb{R}}$. An important point will be that the process $S_.$, under $\hat{\mathbf{P}}_{\infty,\omega}$, is still Markovian, and that the marginal distribution of $\tau := \{t \in \mathbb{Z} : S_t = 0\}$ is just the measure $\mathbf{P}_{\infty,\omega}$ defined in Eq. (2.15). At that point, we take two copies $(S^1_., S^2_.)$ of the process, distributed according to the product measure $\hat{\mathbf{P}}^{\otimes 2}_{\infty,\omega}$, and we define the coupling time $\mathcal{T}(S^1, S^2) = \inf\{t \geq 0 : S^1_t = S^2_t\}$. From the Markov property it follows that

$$|\mathbf{P}_{\infty,\omega}(k \in \tau | 0 \in \tau) - \mathbf{P}_{\infty,\omega}(k \in \tau)| \leq \hat{\mathbf{P}}^{\otimes 2}_{\infty,\omega}(\mathcal{T}(S^1, S^2) > k | S^1_0 = 0). \tag{3.1}$$

Indeed, if the two paths meet before time k, we can let them proceed together from then on and they will either both touch zero at $t = k$, or both will not touch it. Note that at the left-hand side of (3.1) we have just the quantity we wish to bound in Theorems 2.1 and 2.3. Finally, in order to prove Eq. (2.18), we will show in Section 6 that, roughly speaking, in the time interval $[0, k]$ two typical (with respect to $\hat{\mathbf{P}}^{\otimes 2}_{\infty,\omega}$) configurations of the paths $S^1_., S^2_.$ come close to each other at least approximately $k\, \textsc{f}(\beta, h)$ times. The inequality (2.18) then follows by estimating what is the probability that the two (independent!) paths actually succeed in avoiding each other every time they are close: it is rather intuitive that this probability should decrease with k like $\exp(-k\textsc{f}(\beta, h))$. This explains result (2.18) (forget for the moment about ϵ and the constants).

Inequality (2.17) is somewhat less intuitive and we do not try to give a heuristic justification here. The technical difficulties one meets in turning this heuristics into a proof are reflected in the necessity of taking $\epsilon > 0$ in Theorem 2.1.

The most natural question left open by our result is whether lower bounds on the two-point correlation function, complementary to the upper bounds of Eqs. (2.18), (2.17) hold. In Ref. (19) a sharp result was proven in a specific case: if $\mathbf{P}$ is the law of the zeros of the one-dimensional simple random walk conditioned to be non-negative (but that proof works also for the unconditioned simple random walk), then the limit in (2.18) exists for $(\beta, h) \in \mathcal{L}$ and equal exactly $\textsc{f}(\beta, h)$. Similarly, for the disorder-averaged two-point function the analogous limit exists and equals $\mu(\beta, h)$. The simplification that occurs in the situation considered in (19) is that two trajectories of the Markov chain which is naturally associated to $K(.)$, i.e., of the simple random walk, must necessary meet whenever they cross each other. This avoids the construction of the auxiliary Markov chain and makes the coupling argument much more efficient.

Let us emphasize that, in general, it is not even proven that the rate of exponential decay of the (averaged or not) two-point correlation function tends to zero when the critical point is approached (although this is very intuitive, and known for instance in the case considered in (19), as already mentioned).

4 The Markov process

For $\delta \in (2, \infty)$ let $\{\rho_t^{(s)}\}_{t \geq s}$ be the Bessel process of dimension δ and denote its law by $P_\rho^{(s)}$. The Bessel process is actually well defined also for $\delta \leq 2$, but we will not need that here. For the application we have in mind, we choose the initial condition $\rho_s^{(s)} = 1$. For general properties of the Bessel process, we refer to (17, Sections VI.3 and XI.1). This is a diffusion on $\mathbb{R}^+$ with infinitesimal generator

$$\frac{1}{2}\frac{d^2}{dx^2} + \frac{\delta - 1}{2x}\frac{d}{dx}. \tag{4.1}$$

For every real $\delta > 2$, $\rho_\cdot^{(s)}$ is a transient Markov process with continuous trajectories (and, if $\rho_s^{(s)} = 0$ were chosen as initial condition, for δ integer $\rho_\cdot^{(s)}$ would have the same law as the absolute value of the standard Brownian motion in $\mathbb{R}^\delta$ started at the origin at time s). The transition semi-group associated to $\rho_\cdot^{(s)}$, which gives the probability of being in y at time $t_0 + t$ having started at x at time t_0, is known explicitly (17): its density in y with respect to the Lebesgue measure is given, for $t, x > 0$, by

$$p_t^\delta(x, y) := \frac{y}{t}\left(\frac{y}{x}\right)^\nu e^{-(x^2+y^2)/(2t)} I_\nu\left(\frac{xy}{t}\right) \tag{4.2}$$

where $\nu := (\delta/2) - 1$ and $I_\cdot(.)$ is the modified Bessel function of first kind (7, Chapter 7.2.2).

Recall our choice $\rho_s^{(s)} = 1$ and define $T^{(s)} := \inf\{t > s : \rho_t^{(s)} = 1/2\}$. (As will be clear from the proof, the values 1 and 1/2 could be replaced by any a, b with $a > b > 0$.) Then, $0 < P_\rho^{(s)}(T^{(s)} < \infty) < 1$, the upper bound being a consequence of transience. We let also $\{\hat\rho_t^{(s)}\}_{t \geq s}$ with law $\hat P_\rho^{(s)}$ be the process $\rho_\cdot^{(s)}$ conditioned on $T^{(s)} < \infty$. Finally, for $n \in \mathbb{N}$ we set

$\mathcal{K}^{(\delta)}(n) := \hat{P}_\rho^{(0)}(T^{(0)} \in (n-1, n])$ so that

$$\sum_{n\in\mathbb{N}} \mathcal{K}^{(\delta)}(n) = 1. \tag{4.3}$$

One can prove (cf. Appendix A; the proof is an immediate consequence of results in (14) and (13)) that

$$\lim_{n\to\infty} n^{\delta/2}\mathcal{K}^{(\delta)}(n) \in (0, \infty), \tag{4.4}$$

the existence of the limit being part of the statement.

Note that $\hat{\rho}^{(s)}_{\cdot}$ *is not* a Markov process. Indeed, for instance,

$$\begin{aligned} &\hat{P}_\rho^{(0)}(\exists t > 1 : \hat{\rho}_t^{(0)} = 1/2 | \hat{\rho}_1^{(0)} = 2, \exists 0 < s < 1 : \hat{\rho}_s^{(0)} = 1/2) \\ &= P_\rho^{(0)}(\exists t > 1 : \rho_t^{(0)} = 1/2 | \rho_1^{(0)} = 2) < 1 \end{aligned} \tag{4.5}$$

by transience of $\rho^{(0)}_{\cdot}$, while

$$\hat{P}_\rho^{(0)}(\exists t > 1 : \hat{\rho}_t^{(0)} = 1/2 | \hat{\rho}_1^{(0)} = 2, \nexists 0 < s < 1 : \hat{\rho}_s^{(0)} = 1/2) = 1$$

since $T^{(0)} < \infty$ almost surely for $\hat{\rho}^{(0)}_{\cdot}$. However, it is immediately checked that the *stopped* process which equals $\hat{\rho}^{(s)}_t$ for $s \le t < T^{(s)}$ and, say, 0 for $t \ge T^{(s)}$ is again Markovian. This will play a role later.

We choose the parameter of the Bessel process as $\delta = 2(1 + \alpha + \epsilon)$, with $\epsilon > 0$ (this is the same ϵ which appears in the statement of Theorem 2.1). Then, from Eqs. (4.3), (4.4) and (2.2) it is immediate to realize that there exists $p = p(\epsilon)$ with $0 < p < 1$ such that, for every $n \in \mathbb{N}$,

$$K(n) = p\mathcal{K}^{(2(1+\alpha+\epsilon))}(n) + (1 - p)\hat{K}(n) \tag{4.6}$$

where $\hat{K}(n) \ge 0$ and, of course, $\sum_{n\in\mathbb{N}} \hat{K}(n) = 1$. The important point here is the non-negativity of $\hat{K}(n)$, which implies that both $\mathcal{K}^{(2(1+\alpha+\epsilon))}(.)$ and $\hat{K}(.)$ are probabilities on $\mathbb{N}$, to which renewal processes are naturally associated.

Note for later convenience that, as a consequence of (B.2),

$$\frac{\mathcal{K}^{(2(1+\alpha+\epsilon))}(n)}{K(n)} \ge \frac{d_3(\epsilon)}{n^{2\epsilon}}. \tag{4.7}$$

Remark 4.1. Note that, if the slowly varying function $L(n)$ in (2.2) tends to a positive constant for $n \to \infty$, one can choose $\epsilon = 0$ and in that case (4.7) can be improved into

$$\inf_{n\in\mathbb{N}} \frac{\mathcal{K}^{(2(1+\alpha))}(n)}{K(n)} > 0. \tag{4.8}$$

Now, given $x \in \mathbb{Z}$ we construct a continuous-time Markov process $\{S_t^{(x)}\}_{t\ge x} = \{(\phi_t^{(x)}, \psi_t^{(x)})\}_{t\ge x}$, with $\phi_t^{(x)} \ge 0$, $\psi_t^{(x)} \in \{0, 1\}$ and initial condition $S_x^{(x)} = (0, 0)$. The process will satisfy the following two properties:

- Let $t \in \mathbb{Z}$. Conditionally on $\phi_t^{(x)} = 0$, $\{S_u\}_{u>t}$ is independent of $\{S_u\}_{u<t}$.
- Let $t_1 < t_2 \in \mathbb{Z}$. The process $\{S_u\}_{u>t_1}$, conditioned on $\phi_{t_1}^{(x)} = 0$, has the same law as $\{S_u\}_{u>t_2}$ conditioned on $\phi_{t_2}^{(x)} = 0$ and time-shifted to the left of $t_2 - t_1$.

Therefore, we need to construct the trajectories only between two successive integer times where $\phi_t^{(x)} = 0$. The construction proceeds as follows: whenever the condition

$$t \in \mathbb{Z}, \phi_t^{(x)} = 0 \tag{4.9}$$

is realized, we extract (independently of $\{S_u^{(x)}\}_{u \le t}$) a random variable Ψ which takes value 0 with probability $(1-p)$, and 1 with probability p (p being the one which appears in Eq. (4.6)). At that point (see Figure 1):

- If $\Psi = 0$, then we extract a random variable $m \in \mathbb{N}$ with probability law $\hat{K}(.)$ and we let $\phi_u^{(x)} = m + t - u$ for $u \in (t, t+m]$. In the same time interval, we let $\psi_u^{(x)} = \Psi = 0$. At time $t+m$, we are back to condition (4.9) and we start again the procedure with an independent extraction of Ψ.
- If $\Psi = 1$, then we let $\phi_u^{(x)}$ evolve like the process $\hat{\rho}_u^{(t)}$ for $u \in (t, t+T^{(t)})$ where, we recall, $T^{(t)}$ is the (random, but almost surely finite) first time $u > t$ such that $\hat{\rho}_u^{(t)} = 1/2$. In particular, $\phi_{t+}^{(x)} = 1$. Let $\widetilde{T}^{(t)} = \inf\{j \in \mathbb{Z} : j \ge T^{(t)}\}$. Then, we let $\phi_u^{(x)} = 0$ for $u \in [T^{(t)}, \widetilde{T}^{(t)}]$ and $\psi_u^{(x)} = \Psi = 1$ for $u \in (t, \widetilde{T}^{(t)}]$. At time $\widetilde{T}^{(t)}$ we are back to condition (4.9) and we start again with an independent extraction of Ψ.

The process $S_\cdot^{(x)}$ so constructed (whose law will be denoted by $\hat{\mathbf{P}}_x$), satisfies the following properties which are easily checked:

A If $\tau^{(x)} := \{\mathbb{Z} \ni t \ge x : \phi_t^{(x)} = 0\}$, then the marginal distribution of $\tau^{(x)}$ is the law $\mathbf{P}_x$ of Section 2 (the original renewal process associated to $K(.)$ with $\tau_0 = x$). This is obvious from (4.6) and from the construction of $S_\cdot^{(x)}$.

B Let

$$\frac{\mathrm{d}\hat{\mathbf{P}}_{x,y,\omega}}{\mathrm{d}\hat{\mathbf{P}}_x}(S_\cdot^{(x)}) = \frac{e^{\sum_{n=x+1}^{y}(\beta\omega_n - h)\mathbf{1}_{\{n\in\tau^{(x)}\}}}}{Z_{x,y,\omega}} \mathbf{1}_{\{y\in\tau^{(x)}\}}. \tag{4.10}$$

Then, the marginal distribution of $\tau^{(x)}$ is the law $\mathbf{P}_{x,y,\omega}$ introduced in Eq. (2.3).

C For $(\beta, h) \in \mathcal{L}$, the limit $\hat{\mathbf{P}}_{\infty,\omega}(f)$ obtained as $x \to -\infty, y \to \infty$ exists for every bounded local observable f (i.e., bounded function of $\{S_u^{(x)}\}_{u\in I}$, I bounded subset of $\mathbb{R}$.) This is a consequence of the fact that in the localized region τ has a non-zero density in $\mathbb{Z}$ and that the limit exists for functions depending only on τ, as discussed in Section 2. We will call simply $S_\cdot = (\phi_\cdot, \psi_\cdot)$ the limit process obtained as $x \to -\infty, y \to \infty$, and $\tau = \{t \in \mathbb{Z} : \phi_t = 0\}$.

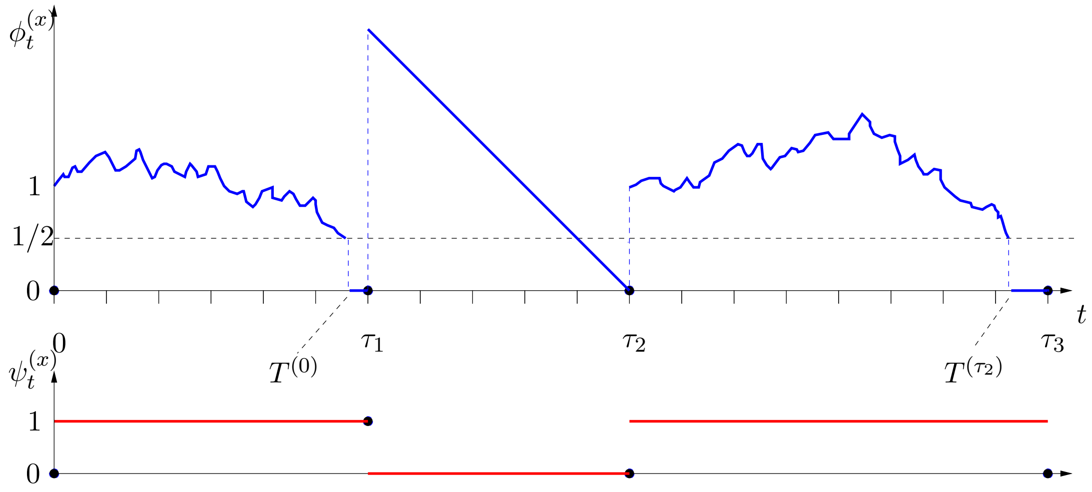

FIGURE 1. An example of trajectory of $S_t^{(x)} = (\phi_t^{(x)}, \psi_t^{(x)})$. In this picture the starting time x equals 0. The top curve represents $\phi_t^{(x)}$, the bottom one $\psi_t^{(x)}$. In this example, $\psi_t^{(x)} = 1$ in $(0, \tau_1]$. At the same time, $\phi_t^{(x)}$ performs a Bessel excursion starting from the value 1, up to the time $T^{(0)}$ when it reaches the value 1/2. Then it equals 0 up to $\tau_1 = \widetilde{T}^{(0)}$. In the time interval $(\tau_1, \tau_2]$, on the other hand, $\psi_t^{(x)}$ equals 0 and $\phi_t^{(x)}$ decreases linearly. In the third time interval, one has again a Bessel excursion for $\phi^{(x)}$ and the value 1 for $\psi^{(x)}$, and so on. The stretches of the trajectory $(\phi_t^{(x)}, \psi_t^{(x)})$ between τ_i and τ_{i+1} are independent.

D The process $S_\cdot$ is Markovian. More precisely: if A is a local event supported on $[u, \infty)$ then

$$\hat{\mathbf{P}}_{\infty,\omega}(A | \{S_t\}_{t \le u}) = \hat{\mathbf{P}}_{\infty,\omega}(A | S_u). \tag{4.11}$$

(This property is easily checked for x, y finite, and then passes to the thermodynamic limit).

E Recall that $\tau = \{t \in \mathbb{Z} : \phi_t = 0\}$ and let $A_{a,b}$ be the event $\{a \in \tau, b \in \tau, \{a+1, \dots, b-1\} \cap \tau = \emptyset\}$, for $a, b \in \mathbb{Z}$ with $x < a < b < y$. Under the law $\hat{\mathbf{P}}_{x,y,\omega}$, conditionally on $A_{a,b}$, the variable $\psi_{a+}(= \psi_u$ for every $u \in (a, b]$, from our construction of $S_\cdot$) is independent of $\{S_t\}_{t \in (-\infty, a) \cup (b, \infty)}$ and is a Bernoulli variable which equals 0 with probability

$$(1-p)\frac{\hat{K}(b-a)}{K(b-a)}$$

and 1 with probability

$$p\frac{\mathcal{K}^{(2(1+\alpha+\epsilon))}(b-a)}{K(b-a)} \ge \frac{d_4(\epsilon)}{(b-a)^{2\epsilon}},$$

where the lower bound follows from (4.7). As for $\{\phi_u\}_{u \in (a,b]}$, conditionally on $A_{a,b}$ it is also independent of $\{S_t\}_{t \in (-\infty, a) \cup (b, \infty)}$. If in addition we condition on $\psi_{a+} = 0$, then $\phi_u = b - u$, while if we condition on $\psi_{a+} = 1$ then $\{\phi_u\}_{u \in (a,b]}$ has the same law as a trajectory of $\rho_u^{(a)}$ *conditioned on* $T^{(a)} \in (b-1, b]$ up to (and excluding) time $T^{(a)}$, and $\phi_u = 0$ in $[T^{(a)}, b]$. This property survives in the limit $x \to -\infty, y \to \infty$.

5 The coupling inequality

Consider two independent copies $S^1_\cdot, S^2_\cdot$ of the process $S_\cdot$, distributed according to the product measure $\hat{\mathbf{P}}^{\otimes 2}_{\infty,\omega}(.)$. As a consequence of property C of Section 4, we can rewrite

$$\mathbf{P}_{\infty,\omega}(k \in \tau | 0 \in \tau) - \mathbf{P}_{\infty,\omega}(k \in \tau) = \hat{\mathbf{E}}^{\otimes 2}_{\infty,\omega}\left(\mathbf{1}_{\{\phi^1_k = 0\}} - \mathbf{1}_{\{\phi^2_k=0\}} \middle| \, \phi^1_0 = 0 \right). \tag{5.1}$$

Given two trajectories of $S_\cdot$, define their *first coupling time after time zero* as

$$\mathcal{T}(S^1, S^2) := \inf\{t \geq 0 : S^1_t = S^2_t\}. \tag{5.2}$$

It is important to remark that we are not requiring $\mathcal{T}(S^1, S^2)$ to be an integer. Then, from the Markov property of S it is clear that the right-hand side of (5.1) equals

$$\hat{\mathbf{E}}^{\otimes 2}_{\infty,\omega}\left(\left(\mathbf{1}_{\{\phi^1_k=0\}} - \mathbf{1}_{\{\phi^2_k=0\}} \right) \mathbf{1}_{\{\mathcal{T}(S^1,S^2)>k\}} \middle| \, \phi^1_0 = 0 \right). \tag{5.3}$$

Therefore, we conclude that

$$|\mathbf{P}_{\infty,\omega}(k \in \tau | 0 \in \tau) - \mathbf{P}_{\infty,\omega}(k \in \tau)| \leq \hat{\mathbf{P}}^{\otimes 2}_{\infty,\omega}\left(\mathcal{T}(S^1, S^2) > k \middle| \, \phi^1_0 = 0 \right). \tag{5.4}$$

Remark 5.1. In analogy with Eqs. (5.1)-(5.4), under the assumptions of Theorem 2.6 on the local observables A, B, one has

$$\begin{aligned} |\mathbf{E}_{\infty,\omega}(AB) - \mathbf{E}_{\infty,\omega}(A)\mathbf{E}_{\infty,\omega}(B)| &= \left| \hat{\mathbf{E}}^{\otimes 2}_{\infty,\omega}\left[(A(\tau^1)B(\tau^1) - A(\tau^1)B(\tau^2))\mathbf{1}_{\{\mathcal{T}(S^1,S^2)\geq k\}} \right] \right| \\ &\leq 2||A||_\infty ||B||_\infty \hat{\mathbf{P}}^{\otimes 2}_{\infty,\omega}\left(\mathcal{T}(S^1, S^2) \geq k \right). \end{aligned} \tag{5.5}$$

The upper bounds of Section 7 on the probability of large coupling times imply therefore Theorem 2.6 (indeed, the proof of Eqs. (7.1) and (7.6) can be easily repeated in absence of the conditioning on the event $\phi^1_0 = 0$.)

To proceed with the proof of Theorems 2.1 and 2.3 we are left with the task of giving upper bounds for the probability that the coupling time is large. This will be done in Section 7, but first we need results on the geometry of the set $\{t \in \mathbb{Z} : \phi_t = 0\} \cap \{1, \ldots, k\}$, for k large and close to the critical line.

6 Estimates on the distribution of returns in a long time interval

Ideas similar to those employed in this section have been already used in Ref. (11) and, more recently, in (2).

To simplify notations, we will from now on set $\underline{v} := (\beta, h)$, $\mu := \mu(\underline{v})$ and $\mathrm{F} := \mathrm{F}(\underline{v})$. Also, in the following whenever a constant $c(\underline{v})$ is such that for every bounded $B \subset \mathcal{L}$ one has $0 < c_-(B) \leq \inf_{\underline{v} \in B} c(\underline{v}) \leq \sup_{\underline{v}\in B} c(\underline{v}) \leq c_+(B) < \infty$, we will say with some abuse of language that it is independent of $\underline{v}$. In particular, this means that $c(\underline{v})$ cannot vanish or diverge when the critical line is approached.

In this section we prove, roughly speaking, that if the interval $\{1, \ldots, k\}$ is large there are sufficiently many points of τ in it, and that these points are rather uniformly distributed. More precisely: take the interval $\{1,\dots,k\}$ and divide it into disjoint blocks $B_\ell := \{(\ell-1)R+1,\dots,\ell R\}$, $\ell=1,\dots,M$ of size

$$R := \frac{c|\log\mu|}{\mu}, \tag{6.1}$$

where c is a large (but independent of $\underline{v}$) positive constant to be chosen later and

$$M = k\frac{\mu}{c|\log\mu|}. \tag{6.2}$$

In order to avoid a plethora of $\lfloor . \rfloor$, we are assuming that R and M are integers. Let η be a positive constant, which will be chosen small (independently of $\underline{v}$) later. Now we want to say that, with probability at least $\simeq (1-\exp(-\mu k))$, a finite fraction of the blocks contain at least a point of τ:

Proposition 6.1. *There exists $c_5 < \infty$ such that*

$$\mathbb{E}\,\mathbf{P}_{\infty,\omega}(\exists I \subset \{1,\dots,M\} : |I| \geq \eta M \text{ and } B_\ell \cap \tau = \emptyset \text{ for every } \ell \in I) \leq c_5 \mu^{-c_5} e^{-k\eta\,\mu/c_5}. \tag{6.3}$$

We will need also an analogous $\mathbb{P}(\,\mathrm{d}\omega)$-almost sure result. However, in this case the strategy has to be modified and $\{1,\dots,k\}$ has to be divided into blocks whose lengths depend on ω: namely, let $i_0(\omega)=0$,

$$i_j(\omega) = \inf\{r > i_{j-1}(\omega) : Z_{i_{j-1}(\omega),i_j(\omega),\omega} \geq \frac{1}{\mathrm{F}^c}\}$$

and $M(\omega) = \sup\{j : i_j(\omega) < k\}$. Again, we define blocks $B^\omega_\ell := \{i_{\ell-1}(\omega)+1,\dots,i_\ell(\omega)\}, \ell = 1,\dots,M(\omega)$, while $B^\omega_{M(\omega)+1} := \{i_{M(\omega)}(\omega)+1,\dots,k\}$. Then, one has:

Proposition 6.2. *There exists a $\mathbb{P}(\,\mathrm{d}\omega)$-almost surely finite random variable $k_0(\omega,\underline{v})$ and a constant $c_6(\underline{v}) > 0$ such that for every $k \geq k_0(\omega,\underline{v})$:*

A

$$M(\omega) \geq k\frac{\mathrm{F}}{2c|\log\mathrm{F}|}. \tag{6.4}$$

B

$$\begin{aligned}&\mathbf{P}_{\infty,\omega}\left(\exists I \subset \{1,\dots,M(\omega)+1\} : |I| \geq \eta M(\omega) \text{ and } B^\omega_\ell \cap \tau = \emptyset \text{ for every } \ell \in I\right)\\ &\leq c_6(\underline{v}) e^{-k\eta\,\mathrm{F}/8}.\end{aligned} \tag{6.5}$$

Proof of Proposition 6.1 Define the event

$$A := \{\exists I \subset \{1,\dots,M\} : |I| \geq \eta M \text{ and } B_\ell \cap \tau = \emptyset \text{ for every } \ell \in I\}.$$

Write

$$\mathbb{E}\,\mathbf{P}_{\infty,\omega}(A) = \sum_{\substack{I\subset\{1,\dots,M\}:\\ |I|\geq\eta M}} \mathbb{E}\,\mathbf{P}_{\infty,\omega}(A_I) \tag{6.6}$$

where A_I is the event

$$A_I := \{B_\ell \cap \tau = \emptyset \text{ for every } \ell \in I\} \cap \{B_\ell \cap \tau \neq \emptyset \text{ for every } \ell \notin I\} \tag{6.7}$$

We can rewrite (in a unique way) $B_I := \cup_{\ell \in I} B_\ell$ as a disjoint union of intervals,

$$B_I = \cup_{r=1}^{m(I)} \{i_r, \ldots, j_r\}, \tag{6.8}$$

with $i_r \geq j_{r-1} + R$. In other words, any two adjacent blocks $B_\ell, B_{\ell+1}$ with $\ell, \ell+1$ belonging to I will be regrouped in the same interval. Of course, $1 \leq m(I) \leq |I|$ if I is not empty. Conditioning on the location x_r of the first point of τ at the left of i_r and on the location y_r of the first point of τ at the right of j_r one has

$$\mathbf{P}_{\infty,\omega}(A_I) \leq e^{m(I)(|h|+\beta\omega_{max})} \sum_{\substack{x_1 \leq i_1 \\ j_1 \leq y_1 \leq (j_1+R)}} \sum_{\substack{(i_{m(I)}-R) \leq x_{m(I)} \leq i_{m(I)} \\ y_{m(I)} \geq j_{m(I)}}} \sum_{\substack{(i_r - R) \leq x_r \leq i_r \\ j_r \leq y_r \leq (j_r+R) \\ 1<r<m(I)}} \prod_{r=1}^{m(I)} \frac{1}{Z_{x_r,y_r,\omega}}. \tag{6.9}$$

(If $m(I) = 1$, the formula is slightly modified in that the sum is only on $x_1 \leq i_1$ and $y_1 \geq j_1$; the estimates which follow hold also in this case). Here we are using the fact that the disorder variables are bounded, say, $|\omega_n| \leq \omega_{max}$. To obtain (6.9) observe that, if $i_r^- := \max\{\tau_i : \tau_i \leq i_r\}$ and $j_r^+ := \min\{\tau_i : \tau_i \geq j_r\}$,

$$\mathbf{P}_{\infty,\omega}(A_I; i_r^- = x_r, j_r^+ = y_r \, \forall r = 1, \ldots, m(I)) \tag{6.10}$$

$$\leq \mathbf{P}_{\infty,\omega}(A_I | i_r^- = x_r, j_r^+ = y_r \, \forall r = 1, \ldots, m(I)) \leq \prod_{r=1}^{m(I)} \frac{K(y_r - x_r)e^{\beta\omega_{y_r} - h}}{Z_{x_r,y_r,\omega}} \tag{6.11}$$

where we used (2.16) in the last step. It is clear that, on the event A_I, $i_r^- \geq i_r - R$ if $r > 1$ (otherwise the block $\{i_r - R, \ldots, i_r - 1\}$ would be contained in B_I, which is not possible due to $i_r \geq j_{r-1} + R$) and similarly $j_r^+ \leq j_r + R$ if $r < m(I)$. Then, (6.9) immediately follows. Note that by the first inequality in (B.3) one can bound $Z_{x_r,y_r,\omega} \geq Z_{x_r,i_r,\omega} Z_{i_r,j_r,\omega} Z_{j_r,y_r,\omega}$. Therefore, using Eqs. (B.1), (B.2) and (B.4), we get that

$$\mathbb{E}\,\mathbf{P}_{\infty,\omega}(A_I) \leq \mu^{-c_7} \prod_{r=1}^{m(I)} c_7 R^{c_7} \mathbb{E} \frac{1}{Z_{i_r,j_r,\omega}} \leq \mu^{-c_7} \prod_{r=1}^{m(I)} c_7 R^{c_7} e^{-\mu(j_r - i_r)} (j_r - i_r)^{c_8} \tag{6.12}$$

for some positive c_7, c_8. The factor μ^{-c_7} comes, through (B.4), from the sum

$$\sum_{x_1 : x_1 \leq i_1} \mathbb{E} \frac{1}{Z_{x_1,i_1,\omega}} \left(= \sum_{y_{m(I)} : y_{m(I)} \geq j_{m(I)}} \mathbb{E} \frac{1}{Z_{j_{m(I)},y_{m(I)},\omega}} \right).$$

Since $m(I) \leq |I|$, one finds then

$$\mathbb{E}\,\mathbf{P}_{\infty,\omega}(A_I) \leq \mu^{-c_7} e^{-|I|(\mu R - c_7 \log R - \log c_7)} e^{c_8 \sum_{r=1}^{m(I)} \log(j_r - i_r)}. \tag{6.13}$$

Now we use Jensen's inequality for the logarithm and the monotonicity of $x \to x \log(1/x)$ for $x > 0$ small to bound

$$e^{c_8 \sum_{r=1}^{m(I)} \log(j_r - i_r)} \leq e^{c_8 |I| \log\left(\frac{k}{|I|}\right)}.$$

From the definition of R one sees then that, for c sufficiently large (independently of $\underline{v}$)

$$\mathbb{E}\,\mathbf{P}_{\infty,\omega}(A_I) \leq c_9 \mu^{-c_7} \exp\left(-\frac{c|I||\log\mu|}{2}\right) e^{c_8|I|\log\left(\frac{k}{|I|}\right)} \tag{6.14}$$

uniformly in I. Finally we can go back to the decomposition (6.6) which, together with elementary combinatorial considerations, gives

$$\begin{aligned}\mathbb{E}\,\mathbf{P}_{\infty,\omega}(A) &\leq c_9\mu^{-c_7} \sum_{j\geq \eta M} \binom{M}{j} e^{-jc|\log\mu|/2} e^{c_8 j\log\left(\frac{c|\log\mu|}{\eta\mu}\right)} \\ &\leq c_{10}\mu^{-c_7} \binom{M}{M/2} e^{-\eta k\mu/4} \leq c_{11}\mu^{-c_7} e^{-\frac{\eta k\mu}{8}}\end{aligned} \tag{6.15}$$

if c is large enough.

Proposition 6.1 $\square$

Proof of Proposition 6.2 Observe first of all that, thanks to (B.3) and to the boundedness of disorder, for every ω and $x < y$

$$\frac{1}{c_{12}} \leq \frac{Z_{x,y,\omega}}{Z_{x,y+1,\omega}} \leq c_{12} \tag{6.16}$$

so that, say,

$$\frac{1}{\textsc{f}^c} \leq Z_{i_j(\omega),i_{j+1}(\omega),\omega} \leq \frac{c}{\textsc{f}^c} \tag{6.17}$$

if c is sufficiently large (the lower bound holds by definition of $i_j(\omega)$, while the upper bound simply says that, since by definition $Z_{i_j(\omega),i_{j+1}(\omega)-1,\omega} < \textsc{f}^{-c}$, then $Z_{i_j(\omega),i_{j+1}(\omega),\omega}$ cannot be much larger than $\textsc{f}^{-c}$). Therefore, denoting (with some abuse of notation) $i_{M(\omega)+1} := k$ and using repeatedly Eq. (B.3), we find

$$Z_{0,k,\omega} \leq \left(\frac{c}{\textsc{f}^c}\right)^{M(\omega)+1} c_1^{M(\omega)} \prod_{r=1}^{M(\omega)+1} (i_j(\omega) - i_{j-1}(\omega))^{c_1} \tag{6.18}$$

and, applying Jensen's inequality to the concave function $x \to \log x$,

$$\frac{1}{k}\log Z_{0,k,\omega} \leq c\frac{M(\omega)+1}{k}|\log \textsc{f}| + (\log c_1 + \log c)\frac{M(\omega)}{k} + c_1\frac{M(\omega)+1}{k}\log\left(\frac{k}{M(\omega)+1}\right). \tag{6.19}$$

Now assume that

$$\frac{M(\omega)+1}{k} \leq \frac{\textsc{f}}{2c|\log \textsc{f}|}. \tag{6.20}$$

Since the function $x \to x\log(1/x)$ is increasing for $x > 0$ small, one deduces from (6.19)

$$\frac{1}{k}\log Z_{0,k,\omega} \leq \frac{3}{4}\textsc{f} \tag{6.21}$$

if c is chosen sufficiently large. But we know that $(1/k)\log Z_{0,k,\omega}$ converges to $\textsc{f}$ almost surely, and therefore the event (6.20) does not happen for k larger than some random but finite $k_0(\omega)$. Equation (6.4) is then proven.

As for (6.5), in view of Lemma B.3 it is sufficient to prove that

$$\mathbf{P}_{\infty,\omega}\left(A;\{0,k+1\}\subset\tau\right)\le c_6(\underline{v})e^{-k\eta\,\mathrm{F}/8} \tag{6.22}$$

for $k \ge k_0(\omega)$, where

$$A^\omega=\{\exists I\subset\{1,\ldots,M(\omega)+1\}:|I|\ge\eta M(\omega)\text{ and }B_\ell^\omega\cap\tau=\emptyset\text{ for every }\ell\in I\}.$$

In analogy with Eqs. (6.7), (6.8) define for $I\subset\{1,\ldots,M(\omega)+1\}$

$$A_I^\omega:=\{B_\ell^\omega\cap\tau=\emptyset\text{ for every }\ell\in I\}\cap\{B_\ell^\omega\cap\tau\neq\emptyset\text{ for every }\ell\notin I\} \tag{6.23}$$

and rewrite $B_I := \cup_{\ell\in I}B_\ell^\omega$ as

$$B_I=\cup_{r=1}^{m(I)}\{i_{x_r}(\omega)+1,\ldots,i_{y_r}(\omega)\}$$

where the indices x_r, y_r are chosen so that $i_{x_r}(\omega)\ge i_{y_{r-1}}(\omega)+2$. Then, with a conditioning argument similar to the one which led to Eq. (6.12), one finds for c sufficiently large

$$\begin{aligned}&\mathbf{P}_{\infty,\omega}(A_I^\omega;\{0,k+1\}\subset\tau)\le\mathbf{P}_{\infty,\omega}(A_I^\omega|\{0,k+1\}\subset\tau)=\mathbf{P}_{0,k+1,\omega}(A_I^\omega)\\&\le\mathrm{F}^{c|I|}\prod_{r=1}^{m(I)}c_{13}[(i_{x_r}(\omega)-i_{x_r-1}(\omega))(i_{y_r+1}(\omega)-i_{y_r}(\omega))]^{c_{13}}\\&\le c_{14}^{|I|}e^{-c|I||\log\mathrm{F}|}\exp\left(c_{14}m(I)\log\left(\frac{k}{m(I)}\right)\right)\le c_{15}(\underline{v})e^{-\frac{c}{2}|I||\log\mathrm{F}|}.\end{aligned} \tag{6.24}$$

In the third inequality we used, once more, Jensen's inequality for the logarithm function and in the fourth one the monotonicity of $x\to x\log(1/x)$ for $x>0$ small, plus Eq. (6.4) and the assumption that $|I|\ge\eta M(\omega)$. Considering all possible sets I of cardinality not smaller than $\eta M(\omega)$, we see that the left-hand side of (6.5) is bounded above by

$$c_{15}(\underline{v})\sum_{j\ge\eta M(\omega)}\binom{M(\omega)+1}{j}e^{-cj|\log\mathrm{F}|/2} \tag{6.25}$$

and recalling (6.4), the desired result Eq. (6.5) holds. Proposition 6.2 □

7 Upper bounds on the probability of large coupling times

Finally, we can go back to the problem of estimating from above the $\hat{\mathbf{P}}^{\otimes 2}_{\infty,\omega}$-probability that the coupling time is larger than k, cf. Section 5. This will conclude the proof of Theorems 2.1, 2.3 and 2.6.

7.1 The average case

We wish first of all to prove that

$$\mathbb{E}\,\hat{\mathbf{P}}^{\otimes 2}_{\infty,\omega}\left(\mathcal{T}(S^1,S^2)>k+1\middle|\,\phi_0^1=0\right)\le\frac{1}{C_1(\epsilon)\mu^{1/C_1(\epsilon)}}e^{-k\,C_1(\epsilon)\mu^{1+\epsilon}}. \tag{7.1}$$

To this purpose observe that, if $\tau^a = \{t \in \mathbb{Z} : \phi_t^a = 0\}$, $a = 1, 2$,

$$\begin{aligned}&\mathbb{E}\,\hat{\mathbf{P}}^{\otimes 2}_{\infty,\omega}\left(\exists I \subset \{1,\dots,M\} : |I| \geq \eta M,\, B_\ell \cap \tau^1 = \emptyset \text{ or } B_\ell \cap \tau^2 = \emptyset \;\forall \ell \in I \middle|\, \phi_0^1 = 0\right)\\ &=: \mathbb{E}\,\hat{\mathbf{P}}^{\otimes 2}_{\infty,\omega}(U|\, \phi_0^1 = 0) \leq 2c_5 \mu^{-c_5} e^{-k\eta\mu/c_5}.\end{aligned} \tag{7.2}$$

This would be an immediate consequence of Proposition 6.1 if the conditioning on $0 \in \tau^1$ were absent. However, the proof of Proposition 6.1 can be repeated exactly in presence of conditioning, i.e., when the measure $\mathbf{P}_{\infty,\omega}(.)$ is replaced by $\mathbf{P}_{0,\infty,\omega}(.) := \lim_{y\to\infty} \mathbf{P}_{0,y,\omega}(.)$ in Eq. (6.3). Therefore,

$$\begin{aligned}\mathbb{E}\,\hat{\mathbf{P}}^{\otimes 2}_{\infty,\omega}\left(\mathcal{T}(S^1,S^2) > k+1 \middle|\, \phi_0^1 = 0\right) \;\leq\;& 2c_5 \mu^{-c_5} e^{-k\eta\mu/c_5}\\ &+ \mathbb{E}\,\hat{\mathbf{P}}^{\otimes 2}_{\infty,\omega}\left(\mathcal{T}(S^1,S^2) > k+1 \middle|\, U^c, \phi_0^1 = 0\right),\end{aligned} \tag{7.3}$$

where U^c is the complementary of the event U. On the other hand, provided that η is chosen sufficiently small (but independent of $\underline{v}$) it is obvious that if the event U^c occurs there exist at least, say, $M/10$ integers $1 < \ell_i < M$ such that $\ell_i > \ell_{i-1} + 2$ and $B_r \cap \tau^a \neq \emptyset$, for every $a \in \{1, 2\}$ and $r \in \{\ell_i - 1, \ell_i, \ell_i + 1\}$. The condition $\ell_i > \ell_{i-1} + 2$ simply guarantees that any two triplets of blocks of the kind $\{B_{\ell_i-1}, B_{\ell_i}, B_{\ell_i+1}\}$ are disjoint for different i, a condition we will need later in this section. We need to introduce the following definition:

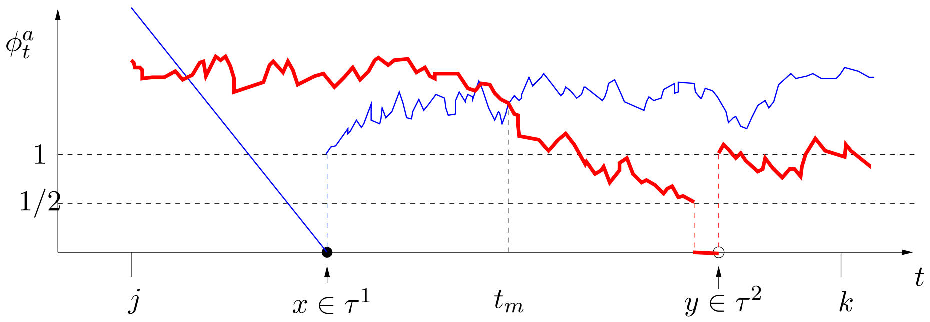

FIGURE 2. An example of goodness. The thin line represents ϕ_t^1 and the thick one represents ϕ_t^2. The important thing is what happens between $x \in \tau^1$ and $y \in \tau^2$. Both paths perform a Bessel excursion in the time interval under consideration, which means that $\psi_t^1 = \psi_t^2 = 1$ there. Since in this example $\phi_x^2 > 1$, there exists necessarily at least a time $t_m \in [x, y]$ where the two paths meet.

Definition 7.1. A configuration of (τ^1, τ^2) is called *good in the interval* $\{j, \dots, k\}$ if there exist $x, y \in \{j, \dots, k\}$, with $x \leq y$, such that the following three conditions are satisfied:

- either $\{x \in \tau^1 \;\text{ and }\; y \in \tau^2\}$ or $\{x \in \tau^2 \;\text{ and }\; y \in \tau^1\}$
- $\{x + 1, \dots, y - 1\} \cap \tau^a = \emptyset$ for $a = 1, 2$
- $\psi_t^a = 1$ for $a = 1, 2$ and $t \in (x, y]$.

Roughly speaking (see Figure 2), this means that (assuming for definiteness $x \in \tau^1$) the point x is overcome by a Bessel excursion of ϕ_t^2 which ends at y, while at x starts a Bessel excursion of ϕ_t^1 which overcomes y and ends at some later time. Such a configuration is called *good* in $\{i, \dots, j\}$ because the paths S_t^1, S_t^2 have a good chance of meeting there, as the next result shows:

Lemma 7.2. *Conditionally on (τ^1,τ^2) being good in the interval $\{j,\dots,k\}$ and on the configuration of $\{S^a_u\}^{a=1,2}_{u\notin[j,k]}$, the $\hat{\mathbf{P}}^{\otimes 2}_{\infty,\omega}$-probability that there exists $t\in[j,k]$ such that $S^1_t=S^2_t$ is bounded below by a positive constant c_0, independent of ω, j, k and of $\{S^a_u\}^{a=1,2}_{u\notin[j,k]}$.*

Lemma 7.2 is proven in Appendix A. Now recall property E, Section 4, of $\hat{\mathbf{P}}_{\infty,\omega}$ and the discussion following Eq. (7.2) above, to conclude that, conditionally on the event U^c, the configuration (τ^1,τ^2) is good in each of the blocks B_{ℓ_i} defined above, with probability at least

$$\left(\frac{d_4(\epsilon)}{R^{2\epsilon}}\right)^2.$$

This holds *independently* of what happens in B_{ℓ_j}, $j\neq i$, thanks to Markovian property (2.16) and the fact that there are points of τ^1 and τ^2 in both B_{ℓ_i-1} and B_{ℓ_i+1}.

Using also Lemma 7.2 one has then that, conditionally on U^c, the $\hat{\mathbf{P}}^{\otimes 2}_{\infty,\omega}$-probability that $\mathcal{T}(S^1,S^2)>k$ does not exceed

$$\left[1-c_0\left(\frac{d_4(\epsilon)}{R^{2\epsilon}}\right)^2\right]^{M/10}. \tag{7.4}$$

Recalling the definitions (6.1) and (6.2) of R and M, one can bound this probability from above with

$$\exp\left(-d_5(\epsilon)k\mu^{1+5\epsilon}/|\log\mu|^2\right). \tag{7.5}$$

Together with Eq. (7.3), this concludes the proof of Eq. (2.17). Theorem 2.1, Eq. (2.17) □

Remark 7.3. A look at Remark 4.1 shows that, if the slowly varying function $L(n)$ in (2.2) tends to a positive constant for $n\to\infty$, the upper bound (7.4) can be improved into $\exp(-\widetilde{d}_5M)$ with $\widetilde{d}_5>0$. From this and Eq. (7.3) the claim of Remark 2.5 follows immediately.

7.2 The almost-sure case

Let us finally prove that, almost surely,

$$\hat{\mathbf{P}}^{\otimes 2}_{\infty,\omega}\left(\mathcal{T}(S^1,S^2)>k+1\,\middle|\,\phi^1_0=0\right)\le C_2(\omega)e^{-C_1\mathrm{F}^{1+\epsilon}}. \tag{7.6}$$

The proof is quite similar to that of the average case, so we will be a bit sketchy. Define (with the notations of Section 6) the event

$$W(\omega):=\left\{\exists I\subset\{1,\dots,M(\omega)\}:|I|\ge\eta k\frac{\mathrm{F}}{2c|\log\mathrm{F}|},B^\omega_\ell\cap\tau^1=\emptyset\text{ or }B^\omega_\ell\cap\tau^2=\emptyset\;\forall\ell\in I\right\} \tag{7.7}$$

so that

$$\begin{aligned}\hat{\mathbf{P}}^{\otimes 2}_{\infty,\omega}\left(\mathcal{T}(S^1,S^2)>k+1\,\middle|\,\phi^1_0=0\right)\;&\le\;2c_6(\underline{v})e^{-k\eta\mathrm{F}/16}\\&\quad+\hat{\mathbf{P}}^{\otimes 2}_{\infty,\omega}\left(\mathcal{T}(S^1,S^2)>k+1\,\middle|\,W(\omega)^c,\phi^1_0=0\right)\end{aligned} \tag{7.8}$$

$\mathbb{P}(\,d\omega)$-almost surely, for $k > k_0(\omega)$. If the event $W(\omega)^c$ occurs, one can find $G(\omega) \geq k_{\mathrm{F}}/(20c|\log \mathrm{F}|)$ integers $1 < \ell_i < M(\omega)$ such that $\ell_i > \ell_{i-1} + 2$ and $B^\omega_r \cap \tau^a \neq \emptyset$, for every $a \in \{1,2\}$ and $r \in \{\ell_i - 1, \ell_i, \ell_i + 1\}$. (τ^1, τ^2) is good in each of the blocks $B^\omega_{\ell_j}$ with probability at least

$$\left(\frac{d_4(\epsilon)}{(i_{\ell_j}(\omega) - i_{\ell_j - 1}(\omega))^{2\epsilon}}\right)^2.$$

Therefore, conditionally on $W(\omega)^c$, the $\hat{\mathbf{P}}^{\otimes 2}_{\infty,\omega}$-probability that $\mathcal{T}(S^1, S^2) > k$ does not exceed

$$\begin{aligned}
&\prod_{j=1}^{G(\omega)} \left[1 - c_0 \left(\frac{d_4(\epsilon)}{(i_{\ell_j}(\omega) - i_{\ell_j - 1}(\omega))^{2\epsilon}}\right)^2\right] \leq \exp\left(-d_6(\epsilon) \sum_{j=1}^{G(\omega)} (i_{\ell_j}(\omega) - i_{\ell_j - 1}(\omega))^{-4\epsilon}\right) \quad (7.9)\\
&\leq \exp\left[-d_6(\epsilon) G(\omega) \left(\frac{G(\omega)}{\sum_{j=1}^{G(\omega)} (i_{\ell_j}(\omega) - i_{\ell_j - 1}(\omega))}\right)^{4\epsilon}\right] \leq \exp\left[-d_6(\epsilon) k \left(\frac{G(\omega)}{k}\right)^{1+4\epsilon}\right],
\end{aligned}$$

where we used Jensen's inequality for the convex function $x \to x^{-4\epsilon}$. The lower bound $G(\omega) \geq k_{\mathrm{F}}/(20c|\log \mathrm{F}|)$, together with Eq. (7.8) are then enough to obtain the desired estimate (7.6).

Theorem 2.1, Eq. (2.18) □

A Some technical facts on Bessel processes

Take $0 < b < a < \infty$ and consider a Bessel process $\rho^{(0)}_t, t \geq 0$ of dimension δ starting from $\rho^{(0)}_0 = a$ at time 0. Let $T_{a,b}$ be the first hitting time of b, i.e., $T_{a,b} = \inf\{t \geq 0 : \rho^{(0)}_t = b\}$. Then, it follows from (14, Theorem 3.1) plus (13, Theorem 2.5) that, conditionally on $T_{a,b} < \infty$, the density of the probability distribution of $T_{a,b}$ with respect to the Lebesgue measure on $\mathbb{R}^+$ is proportional to

$$p(t) := \int_0^\infty B(z) e^{-tz/2} dz := \int_0^\infty \frac{J_\nu(b\sqrt{z}) Y_\nu(a\sqrt{z}) - J_\nu(a\sqrt{z}) Y_\nu(b\sqrt{z})}{J^2_\nu(b\sqrt{z}) + Y^2_\nu(b\sqrt{z})} e^{-tz/2} dz \quad (A.1)$$

where $J_\nu(z)$ and $Y_\nu(z)$ are Bessel function of the first and second kind, respectively (7, Chapter 7.2.1), and $\nu = (\delta/2) - 1$. From (7, Chap. 7.2.1, Eqs. (3)-(4)) one deduces that $B(z) z^{-\nu} \to c_\nu(a,b)$ for $z \to 0^+$, where $c_\nu(a,b)$ is a finite and positive constant whose precise value is not needed for our purposes. Therefore, the Abelian Theorem (20, Chapter 5, Corollary 1a) gives

$$p(t) t^{\nu+1} = p(t) t^{\delta/2} \overset{t \to +\infty}{\longrightarrow} \frac{c_\nu(a,b) \Gamma(\nu+2) 2^{\nu+1}}{\nu+1}. \quad (A.2)$$

From Eq. (A.2), the asymptotic behavior (4.4) immediately follows taking $a = 1, b = 1/2$ (of course, any other values $0 < b < a < \infty$ would be equally good).

A.1 Proof of Lemma 7.2

Let x, y be any pair of sites which satisfies the conditions required by Definition 7.1. Assume for definiteness that $x \in \tau^1$ and $y \in \tau^2$. We assume also that $x < y$, otherwise the lemma is trivial. For technical reasons, it is also convenient to treat apart the case $x = y-1$. In this case, the lemma follows immediately from (B.3). Indeed, from this is easily deduced in particular that, conditionally on $y \in \tau^2$, the probability that also $y-1 \in \tau^2$ is greater than some positive constant, independent of ω.

As for the more difficult case where $x < y-1$, it is clear that there exists $x \le t \le y$ such that $\phi^1_t = \phi^2_t$ whenever $\phi^2_x \ge 1$ (we assume that $x \notin \tau^2$, otherwise the existence of t such that $\phi^1_t = \phi^2_t$ is trivial). This follows (see also Figure 2) from the observation that $\phi^1_{x^+} = 1, \phi^1_y \ge 1/2$ and that there exists $y-1 < s \le y$ with $\phi^2_s = 1/2$, together with the fact that the trajectories of the Bessel process are continuous almost surely. Therefore, the Lemma follows if we can prove that the probability that $\phi^2_x \ge 1$ is bounded below by a positive constant. This is the content of (A.4) below.

In order to state (A.4), we need to introduce the Bessel Bridge process of dimension δ (17, Chapter XI.3). Given $u \ge 0$ and $a, v > 0$, the Bessel Bridge is a continuous process $\{X_t\}_{t\in[0,a]}$ (whose law is denoted by $P^{a,\delta}_{u,v}$) which starts from u at time 0, ends at v at time a and such that, given $0 < s_1 < \ldots < s_k < a$, the law of $(X_{s_1}, \ldots, X_{s_k})$ has density

$$p^\delta_{s_1}(u, x_1)p^\delta_{s_2-s_1}(x_1, x_2)\ldots p^\delta_{a-s_k}(x_k, v)/p^\delta_a(u, v). \tag{A.3}$$

Then, what we need is

$$\inf_{u,v\ge 1/2} P^{2,\delta}_{u,v}(X_1 \ge 1 | X_s > 1/2\ \forall\ s \in [0, 2]) > 0. \tag{A.4}$$

Of course, u, v correspond to the values $\phi^2_{x-1}, \phi^2_{x+1}$, respectively. It is immediate to realize that (A.4) concludes the proof of Lemma 7.2.

Inequality (A.4) is easily proven: indeed, via FKG inequalities (8) (15) one has (see details below)

$$P^{2,\delta}_{u,v}(X_1 \ge 1 | X_s > 1/2\ \forall\ s \in [0, 2]) \ge P^{2,\delta}_{u,v}(X_1 \ge 1). \tag{A.5}$$

Using formula (A.3), the right-hand side of (A.5) equals

$$\frac{\int_1^\infty p^\delta_1(u, w)p^\delta_1(w, v)\, \mathrm{d}w}{p^\delta_2(u, v)} = \frac{e^{-(u^2+v^2)/4}}{I_\nu(uv/2)} \int_1^\infty w\, e^{-w^2} I_\nu(uw)I_\nu(vw)\, \mathrm{d}w. \tag{A.6}$$

Since $I_\nu(w) > 0$ for $w > 0$ and

$$\lim_{w\to\infty} e^{-w}\sqrt{w}I_\nu(w) \in (0, \infty) \tag{A.7}$$

(this can be extracted from (7, Chap. 7.13.1, Eq. (5); cf. Chap. 7.2.6 for the definition of the Hankel symbol (ν, m))), one has

$$0 < c^-_\nu := \inf_{z\ge 1/2} e^{-z}\sqrt{z}I_\nu(z) \le \sup_{z\ge 1/2} e^{-z}\sqrt{z}I_\nu(z) =: c^+_\nu < \infty. \tag{A.8}$$

Therefore, the left-hand side of (A.5) is bounded below by

$$\frac{(c^-_\nu)^2}{\sqrt{2}c^+_\nu} \int_1^\infty e^{-(w-\frac{u+v}{2})^2}\, \mathrm{d}w, \tag{A.9}$$

which tends to a positive constant if $u \to +\infty$ or $v \to +\infty$ (or both), thus yielding Eq. (A.4).

Finally, we show how (A.5) follows from the FKG inequalities. Due to the continuity of the trajectories of the Bessel Bridge, the probability in the left-hand side of (A.4) equals

$$\lim_{n\to\infty} P^{2,\gamma}_{u,v}(X_1 \geq 1 | X_{i/n} > 1/2,\ i = 1, \dots, 2n-1). \tag{A.10}$$

Let $p(x_1, \dots, x_{2n-1})$ be the probability density of $(X_{1/n}, \dots, X_{(2n-1)/n})$. Given $\underline{x}^a := (x^a_1, \dots, x^a_{2n-1})$, $x^a_j > 0$, $a = 1, 2$, define $\underline{x}^1 \vee \underline{x}^2 := ((x^1_1 \vee x^2_1), \dots, (x^1_{2n-1} \vee x^2_{2n-1}))$ and analogously $\underline{x}^1 \wedge \underline{x}^2$. Then, from the continuity and Markov property of the Bessel Bridge process (17, Chapter XI.3) it is clear that $p(\underline{x}^1 \vee \underline{x}^2)p(\underline{x}^1 \wedge \underline{x}^2) \geq p(\underline{x}^1)p(\underline{x}^2)$. This is just the FKG inequality, which implies in particular that the probability in (A.10), for any given n, is not smaller than $P^{2,\delta}_{u,v}(X_1 \geq 1)$.

Lemma 7.2 $\square$

B Technical estimates on $Z_{x,y,\omega}$ and $\mathbf{P}_{\infty,\omega}$

In this section we collect some technical estimates, which in very similar form have been already used in the previous literature. Let us notice at first that, for every $x < y$ and uniformly in ω,

$$Z_{x,y,\omega} \geq e^{\beta\omega_y - h} K(y-x). \tag{B.1}$$

Also, Eq. (2.2) and the property of slow variation imply that for every $\epsilon > 0$ there exist positive constants $d_1(\epsilon), d_2(\epsilon)$ such that, for every $n \in \mathbb{N}$,

$$\frac{d_1(\epsilon)}{n^{1+\alpha+\epsilon}} \leq K(n) \leq \frac{d_2(\epsilon)}{n^{1+\alpha-\epsilon}}. \tag{B.2}$$

In Lemma A.1 of (11) it was proven that there exists c_1, which in the case of bounded disorder can be chosen independent of ω, such that for every $x < z < y$

$$Z_{x,z,\omega} Z_{z,y,\omega} \leq Z_{x,y,\omega} \leq c_1((z-x) \wedge (y-z))^{c_1} Z_{x,z,\omega} Z_{z,y,\omega}. \tag{B.3}$$

As it was shown in (11, Proposition 2.7), this immediately implies that there exists $c_1' > 0$ such that, for every $y > x$,

$$\left| \frac{1}{|y-x|} \mathbb{E} \log Z_{x,y,\omega} - \mathrm{F}(\underline{v}) \right| \leq c_1' \frac{\log|y-x|}{|y-x|}.$$

Similarly, one can see that

$$\left| -\frac{1}{|y-x|} \log \mathbb{E} \frac{1}{Z_{x,y,\omega}} - \mu(\underline{v}) \right| \leq c_1' \frac{\log|y-x|}{|y-x|} : \tag{B.4}$$

this follows immediately observing that (B.3) implies

$$\frac{1}{c_1(2N)^{c_1}} \left(\mathbb{E} \frac{1}{Z_{-N,N,\omega}} \right)^2 \leq \mathbb{E} \frac{1}{Z_{-2N,2N,\omega}} \leq \left(\mathbb{E} \frac{1}{Z_{-N,N,\omega}} \right)^2. \tag{B.5}$$

A minor modification of the proof of (11, Lemma A.1) gives also

Lemma B.1. *Let A be a local event supported in $\{1,\dots,a\}$. Then,*

$$\mathbf{P}_{\infty,\omega}(A;\{-k,\dots,0\}\cap\tau\neq\emptyset)\leq c_1(ak)^{c_1}\mathbf{P}_{\infty,\omega}(A;\{0,a+1\}\in\tau). \tag{B.6}$$

We will also need the following result, which follows from (11, Lemma 3.1):

Lemma B.2. *For every $\underline{v}\in\mathcal{L}$ there exist positive constants $c_2(\omega;\underline{v}),c_3(\underline{v})$ (with $c_2(\omega;\underline{v})$ finite $\mathbb{P}(\,\mathrm{d}\omega)-$almost surely) such that, for every $k\in\mathbb{N}$,*

$$\mathbf{P}_{\infty,\omega}(\tau\cap\{-k^2,\dots,0\}=\emptyset)\leq c_2(\omega;\underline{v})e^{-c_3(\underline{v})k^2}. \tag{B.7}$$

As a consequence of Lemmas B.2 and B.1, we have finally

Lemma B.3. *Let A be a local event supported in $\{1,\dots,k\}$. Then,*

$$\mathbf{P}_{\infty,\omega}(A)\leq c_1k^{c_1}\mathbf{P}_{\infty,\omega}(A;\{0,k+1\}\subset\tau)+c_2(\omega;\underline{v})e^{-c_3(\underline{v})k^2}. \tag{B.8}$$

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