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# Avoiding-probabilities for Brownian snakes and super-Brownian motion

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**Abstract:** We investigate the asymptotic behaviour of the probability that a normalized $d$-dimensional Brownian snake (for instance when the life-time process is an excursion of height 1) avoids 0 when starting at distance $\varepsilon$ from the origin. In particular we show that when $\varepsilon$ tends to 0, this probability respectively behaves (up to multiplicative constants) like $\varepsilon^4$, $\varepsilon^{2\sqrt{2}}$ and $\varepsilon^{(\sqrt{17}-1)/2}$, when $d = 1$, $d = 2$ and $d = 3$. Analogous results are derived for super-Brownian motion started from $\delta_x$ (conditioned to survive until some time) when the modulus of $x$ tends to 0.

**Key words:** Brownian snakes, superprocesses, non-linear differential equations

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# Avoiding-probabilities for Brownian snakes and super-Brownian motion

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## 1 Introduction

The Brownian snake is a path-valued Markov process that has been shown to be a powerful tool to study probabilistically some non-linear partial differential equations (see e.g. Le Gall [13, 14, 15], Dhersin-Le Gall [8]). It has also been studied on its own right (see e.g. [24], [25]), and because of its intimate connection with super-Brownian motion, it can be used to study certain branching particle systems and superprocesses (see for instance Le Gall-Perkins [18], Serlet [21, 22]).

Our first purpose here is to study the asymptotical probability (when $\varepsilon \to 0+$) that a $d$-dimensional Brownian snake started at distance $\varepsilon$ from the origin does not hit 0, for instance when the life-time process is conditioned on hitting 1. This has a rather natural interpretation in terms of critical branching particule systems that we shall briefly point out at the end of this introduction. Recall that if $B$ denotes a linear Brownian motion started from $\varepsilon$ under the probability measure $P_\varepsilon$, then the reflection principle immediately shows that

$$P_\varepsilon(0 \notin B[0,1]) \sim \sqrt{\frac{2}{\pi}}\varepsilon, \text{ when } \varepsilon \to 0. \tag{1}$$

We shall for instance see (see Theorem 1 below) that the analogous quantity for a conditioned one-dimensional Brownian snake behaves (up to a multiplicative constant) like $\varepsilon^4$ when $\varepsilon \to 0$. Note that the type of conditioning (conditioned to hit a fixed point, to survive after time $t$ etc.) will only change the multiplicative constant, but not the power of $\varepsilon$. We will also use these to study analogous problems for conditioned $d$-dimensional super-Brownian motion.

Let us first briefly recall the definition of the $d$-dimensional Brownian snake (see e.g. Le Gall [13, 14] for a detailed construction and some properties of this process): It is a Markov process $W = ((W_s, \zeta_s), s \in [0, \sigma])$ where $W_s$ takes values in the set of stopped continuous functions

$$\mathcal{F} = \{w : [0, T(w)] \to \mathbb{R}^d, \ w \text{ is continuous}\}$$

and its life-time is $T(W_s) = \zeta_s$. Conditionally on the life-time process $(\zeta_s, s \leq \sigma)$, the process $(W_s, s \geq 0)$ is a time-inhomogeneous Markov process in $\mathcal{F}$, and its transition kernels are described by the two following properties: For any $s < s' < \sigma$,

- $W_s(t) = W_{s'}(t)$ for all $t \leq \inf_{[s,s']} \zeta$.
- $W_{s'}(\inf_{[s,s']} \zeta + t) - W_{s'}(\inf_{[s,s']} \zeta)$ for $t \in [0, \zeta_{s'} - \inf_{[s,s']} \zeta]$ is a Brownian motion in $\mathbb{R}^d$ started from 0 and independent of $W_s$.

Loosely speaking, $W_s$ is a Brownian path in $\mathbb{R}^d$ with random lifetime $\zeta_s$. When $\zeta$ decreases, the extremity of the path is erased and when $\zeta$ increases, the path is extended by adding an independent Brownian motion.

We shall essentially focus on the cases where the life-time process $\zeta = (\zeta_s, s \in [0, \sigma])$ is a Brownian excursion away from 0. When $\zeta$ is reflected Brownian motion, then the above description defines the Brownian snake, which is a Markov process for which the trajectory started from $x$ and with zero life-time is regular. We will denote the associated (infinite) excursion measure $\mathbb{N}_x^{[d]}$. We choose $\mathbb{N}_x^{[d]}$ in such a way that $\zeta$ is a standard Brownian excursion defined under the Itô measure. We shall also consider the case where $\zeta$ is a Brownian excursion renormalized by its height (i.e. $\sup_{[0,\sigma]} \zeta = 1$); in this case, the 'normalized' Brownian snake started from $x$ is defined under a probability measure that we shall denote $\tilde{\mathbb{N}}_x^{[d]}$.

Some large deviation results for the Brownian snake normalized by the length of the excursion have been recently derived by Dembo-Zeitouni [6], Serlet [24].

The range $\mathcal{R}$ of the Brownian snake $W$ is defined as follows:

$$\mathcal{R} = \{W_s(\zeta_s), \ s \in [0, \sigma]\} = \{W_s(t), \ s \in [0, \sigma], \ t \in [0, \zeta_s]\}.$$

It is the set of points visited by the snake. Recall that points are not polar (i.e. a fixed point has a striclty positive probability/measure to be in $\mathcal{R}$) only for $d < 4$ (see e.g. Dawson-Iscoe-Perkins [4]).

Here is the statement of our main results for the Brownian snake:

**Theorem 1** *Define*

$$\begin{aligned} \alpha(1) &= 4, \\ \alpha(2) &= 2\sqrt{2} \\ \alpha(3) &= \frac{\sqrt{17} - 1}{2}. \end{aligned}$$

*Then for $d = 1, 2, 3$:*
*(i) There exists constants $k_1(d) \in (0, \infty)$ such that*

$$\tilde{\mathbb{N}}_\varepsilon^{[d]}(0 \notin \mathcal{R}) \sim k_1(d)\ \varepsilon^{\alpha(d)} \text{ when } \varepsilon \to 0+ .$$

*(ii) There exist constants $k_2(d) \in (0, \infty)$, such that for all fixed $X > 0$,*

$$\mathbb{N}_\varepsilon^{[d]}(0 \notin \mathcal{R} \text{ and } \mathcal{R} \cap \{z \in \mathbb{R}^d,\ \|z\| = X\} \neq \emptyset) \sim \frac{k_2(d)}{X^{\alpha(d)+2}} \varepsilon^{\alpha(d)} \text{ when } \varepsilon \to 0+ .$$

*For $d = 1$, one has*

$$k_2(1) = \frac{\Gamma(1/3)^{18}}{1792\pi^6} \approx 29.315. \tag{2}$$

The two statements (i) and (ii) are similar, but their proofs differ significantly. The proof of (ii) relies on analytical arguments (Sections 4 and 5.1), whereas that of (i) uses the (probabilistic) decomposition of the snake via its longest path (Section 3 and 5.2).

We will then use these results to study similar quantities for $d$-dimensional super-Brownian motion. For references on superprocesses, see e.g. the monographs and lecture notes by Dawson [3], Dawson-Perkins [5] and Dynkin [10] and the references therein. Let $(\mu_t)_{t \geq 0}$ denote a super-Brownian motion with branching mechanism $2u^2$ (just as in Le Gall [17], Theorem 2.3) started from the finite positive measure $\nu$ on $\mathbb{R}^d$ under the probability measure $P_\nu$. Let $|\mu_1|$ denote the total mass of $\mu_1$ and the closure $\mathcal{S}$ of the union over all $t > 0$ of the supports of $\mu_t$. Then the following Theorem will turn out to be a simple consequence of Theorem 1:

**Theorem 2** *Define $\alpha(d)$ and $k_1(d)$ as in Theorem 1. Then for $d = 1$, 2 and 3, one has:*

$$P_{\delta_\varepsilon}(0 \notin \mathcal{S} \text{ and } |\mu_1| > 0) \sim k_3(d)\ \varepsilon^{\alpha(d)}\ \exp\left\{-\frac{4-d}{2\varepsilon^2}\right\}, \text{ when } \varepsilon \to 0 \tag{3}$$

*and where $k_3(d) = k_1(d)/(1 + \alpha(d)/2)$.*

This paper is organized as follows: The next section is devoted to some preliminaries (we put down some notation and recall some relevant facts). For the sake of clarity, we will then first focus on the one-dimensional case. In Section 3, we use probabilistic arguments to derive Theorem 1-(i) for $d = 1$, and in Section 4, we derive Theorem 1-(ii) for $d = 1$ using the Laurent series expansion of elliptic functions. In section 5, we study the multi-dimensional problem, and more generally, the case where the Brownian snake is replaced by a snake, where the underlying spatial Markov process is a Bessel process of dimension $\delta < 4$ (we will somewhat improperly call this a Bessel snake). In Section 6, we use these results to derive Theorem 2. Finally, we conclude with some remarks in Section 7.

**An informal motivation:** Consider a critical branching particule system that can be modelized in the long-time scale by super-Brownian motion (for instance, each particule moves randomly without drift, and when it dies, it gives birth to exactly two offsprings with probability $1/2$, and it has no offsprings with propability $1/2$). Take a living particule at time $T$ ($T$ is very large compared to the mean life-time of one particule), and look at its ancester $X$ at time 0 (this corresponds very roughly speaking to condition $X$ on having at least one living descendant at time $T$). Let $\Pi$ be a hyperplane at distance $\varepsilon$ of the position of $X$ at time 0 ($\varepsilon$ is small, but large compared to the mean displacement of $X$ during its life-time). Then, the probability that NO descendant of $X$ before time $T$ has crossed the hyperplane $\Pi$ decays up to a multiplicative constant like $\varepsilon^4$ when $\varepsilon \to 0$.

In view of possible applications, we will also (in Section 6.2) consider the following problem: Suppose that we have to implant a large number of particules (we can choose how many) at distance $\varepsilon$ of an 'infected' hyperplane $\Pi$. We want to maximize the probability that the system survives up to time $T$, and that no particule hits the hyperplane $\Pi$ before $T$ (i.e. the system does not die out and does not get infected by the hyperplane $\Pi$). It turns out that for this problem, the optimal probability decays like a constant times $\varepsilon^6$, when $\varepsilon \to 0$.

# 2 Preliminaries, Notation

Throughout this paper:

- $B$ will denote a one-dimensional Brownian motion started from $x$ under the probability measure $P_x$.

- When $\delta > 0$, $R$ will denote a $\delta$-dimensional Bessel process started from $x$ under the probability measure $P_x^{(\delta)}$; the filtration associated to the process $(R_t)_{t\geq 0}$ is denoted by $(\mathcal{F}_t)_{t\geq 0}$. Let us recall that Bessel processes hit zero with positive probability only when their dimension is strictly smaller than 2.

  The following result (Yor [27], Proposition 1) shall be very useful in this paper:

  **Proposition 1 (Yor)** *Suppose that $\Phi_t$ is a positive $\mathcal{F}_t$-measurable random variable and that $\lambda \geq 0$ and $\mu > -1$; then for $x > 0$, one has:*

$$E_x^{(2+2\mu)}\left[\Phi_t \exp\left\{-\frac{\lambda^2}{2}\int_0^t \frac{ds}{(R_s)^2}\right\}\right] = x^{\nu-\mu} E_x^{(2+2\nu)}\left[(R_t)^{-\nu+\mu}\Phi_t\right], \quad (4)$$

  *where $\nu^2 = \lambda^2 + \mu^2$.*

- $n$ will denote the Itô measure on positive Brownian excursions $(\zeta(s), s \leq \sigma)$. The set of all positive excursions will be denoted $\mathcal{E}$. The length of the excursion $\zeta$ will be denoted $\sigma(\zeta)$ and its height will be denoted $H(\zeta) = \sup_{s \leq \sigma} \zeta(s)$. Recall that

$$n(H(\zeta) > x) = \frac{1}{2x}. \tag{5}$$

- We also put

$$n^{>h} = n1_{\{H(\zeta)>h\}} \text{ and } n^{<h} = n1_{\{H(\zeta)<h\}}.$$

  $n^{=h}$ will denote the law of Brownian excursions renormalized by their height (so that $n^{=h}$ is a probability measure and $n^{=h}(H(\zeta) = h) = 1$). For convenience, we put $n^{=1} = \tilde{n}$.

- $W = ((W_s, \zeta_s), s \leq \sigma)$ will now denote a one-dimensional Brownian snake associated to the excursion $\zeta$. When $\zeta$ is defined under the measure $n$ (respectively $n^{<h}$, $n^{=h}$, $n^{>h}$ and $\tilde{n}$) then the snake $W$ is started from $x$ under the measure $\mathbb{N}_x$ (respectively $\mathbb{N}_x^{<h}$, $\mathbb{N}_x^{=h}$, $\mathbb{N}_x^{>h}$ and $\tilde{\mathbb{N}}_x$). Throughout the paper, we put

$$f(\varepsilon) = \tilde{\mathbb{N}}_\varepsilon(0 \notin \mathcal{R}).$$

- The following result is a direct consequence in terms of the Brownian snake of a result of Dawson, Iscoe and Perkins ([4], Theorem 3.3):

$$\mathbb{N}_0^{<h}(1 \in \mathcal{R}) \leq \frac{c}{h^{1/2}} \exp(-a/h), \tag{6}$$

  for all sufficiently small $h$ ($c$ and $a$ are fixed positive constants).

  Theorem 3.3 in [4] states in particular that in dimension $d$, for all small enough $t$,

$$P_{\delta_0}(\exists s \leq t, \mu_s(\{x \in \mathbb{R}^d,\ \|x\| \geq 1\}) > 0) \leq \frac{c'}{t^{d/2}} \exp(-a/t)$$

  where $a$ and $c'$ are fixed positive constants (the constant $a$ instead of $1/2$ is because the super-Brownian motion $\mu$ has branching mechanism $2u^2$ and not $u^2$; we do not need the exact value of $a$ here). This implies that for $d = 1$,

$$P_{\delta_0}(1 \in \mathcal{S} \text{ and } \mu_t = 0) \leq \frac{c'}{t^{1/2}} \exp(-a/t).$$

  But on the other hand, the Poissonian representation of super-Brownian motion (see e.g. Le Gall [17], Theorem 2.3) and the exponential formula (see e.g. Revuz-Yor [20], Chapter XII, (1.12)) imply that

$$\begin{aligned} P_{\delta_0}(1 \in \mathcal{S} \text{ and } \mu_t = 0) &= 1 - P_{\delta_0}(1 \notin \mathcal{S} \text{ or } \mu_t \neq 0) \\ &= 1 - \exp\left\{-\int_0^1 \mathbb{N}_0(H(\zeta) < t \text{ and } 1 \in \mathcal{R})du\right\} \end{aligned}$$

Comining the last two statements implies (6). For exact related asymptotics, see e.g. Dhersin [7].

- When $\delta > 0$, $W^{(\delta)} = (W^{(\delta)}, s \leq \sigma)$ will denote a $\delta$-dimensional Bessel snake. This process is defined exactly as the one-dimensional Brownian snake, except that the spatial motion is this time a $\delta$-dimensional Bessel process (absorbed or reflected at 0; this does not make any difference for the quantities that we shall consider here). For convenience, we shall also denote its range by $\mathcal{R}$. When the excursion $\zeta$ is defined under the measure $n$ (respectively $\tilde{n}$), then the Bessel snake is defined under the measure $\mathbb{N}^{(\delta)}$ (resp. $\tilde{\mathbb{N}}^{(\delta)}$). We will put

$$f^{(\delta)}(\varepsilon) = \tilde{\mathbb{N}}_\varepsilon^{(\delta)}(0 \notin \mathcal{R}).$$

- When $A$ and $A'$ are two topological spaces, $\mathcal{C}(A, A')$ will denote the set of continuous functions of $A$ into $A'$.

# 3 Probabilistic approach in one dimension

## 3.1 Decomposition of the normalized snake

We are first going to decompose the normalized Brownian snake (defined under $\tilde{\mathbb{N}}_x$) with respect to its longest path. Our goal in this part is to derive the following statement:

**Proposition 2** *For all $\varepsilon > 0$,*

$$\begin{aligned} f(\varepsilon) &= P_\varepsilon(0 \notin B[0,1]) \\ &\quad \times E_\varepsilon\left(\exp\left\{-2\int_0^1 dt \int_0^{1-t} \frac{dh}{h^2}(1 - f(B_t/\sqrt{h}))\right\} \,\middle|\, 0 \notin B[0,1]\right). \end{aligned}$$

Suppose that $\zeta$ is defined under the probability measure $\tilde{n}$ and $W$ under $\tilde{\mathbb{N}}_x$. Define

$$T = \inf\{s \geq 0,\ \zeta_s = 1\}.$$

It is well-known (see e.g. Revuz-Yor [20]) that under $\tilde{n}$, the two processes $(\zeta_s, s \leq T)$ and $(\zeta_{\sigma-s}, s \leq \sigma - T)$ are two independent three-dimensional Bessel processes started from 0 and stopped at their respective hitting times of 1. Combining this with the definition of the Brownian snake shows that under $\tilde{\mathbb{N}}_x$ and conditionally on $\{W_T = \gamma\}$, the two processes $W^+ = (W_s, s \leq T)$ and $W^- = (W_{\sigma-s}, s \leq \sigma - T)$ are independent and that their conditional laws are identical. Define the reversed process:

$$\overleftarrow{\zeta}_s = 1 - \zeta_{T-s}$$

for $s \in [0,T]$. It is well-known (see e.g. Proposition (4.8), Revuz-Yor [20], chapter VII) that the law of $(\overleftarrow{\zeta}_s, s \le T)$ is again that of a three-dimensional Bessel process stopped at its hitting time of 1. We now put

$$\overleftarrow{\zeta}^*_s = \sup_{u<s} \overleftarrow{\zeta}_u$$

and we define, for $u \in [0,1]$,

$$\alpha_u = \inf\{s > 0,\ \overleftarrow{\zeta}_s = u\}$$

and

$$\beta_u = \inf\{s > \alpha_u,\ \overleftarrow{\zeta}_s = u\}.$$

$[\alpha_u, \beta_u]_{u\in[0,1]}$ are the excursion intervals of the process $\overleftarrow{\zeta}^*_s - \overleftarrow{\zeta}_s$ above level 0. Let $\zeta^u$ denote the excursion of that process in the interval $[\alpha_u, \beta_u]$ (i.e. $\zeta^u$ is defined on the time-interval $[0, \beta_u - \alpha_u]$).

**Lemma 1** *The random measure $\sum_{u\in[0,1]} \delta_{(u,\zeta^u)}$ is a Poisson measure on $[0,1]\times \mathcal{E}$ of intensity $2dt \times n^{<t}(de)$.*

*Proof-* Recall that for all $u_0 > 0$, the law of $(\overleftarrow{\zeta}_{s+\alpha_{u_0}}, s \le T - \alpha_{u_0})$ is that of a linear Brownian motion started from $u_0$ and conditioned to hit 1 before 0, and that it is independent of $(\overleftarrow{\zeta}_s, s \le \alpha_{u_0})$.

Suppose now that $B$ is a linear Brownian motion started from $u_0$, and define

$$B^*_s = \sup_{u<s} B_u,$$

and the excursion intervals $[\alpha_u, \beta_u]$ of $B^* - B$ above zero for $u \in [u_0, 1]$ exactly as above. Let $B^u$ denote the associated excursion. Then (as $B^* - B$ is reflecting Brownian motion, see e.g. Revuz-Yor [20]), the measure $\sum_{u\in[u_0,1]} \delta_{(u,B^u)}$ is a Poisson point process of intensity $2dt \times n(de)$. Note that $B$ hits 1 before 0 if and only if

$$H(B^u) < u$$

for all $u \ge u_0$. Hence, the measure

$$\sum_{u\in[u_0,1]} \delta_{(u,\zeta^u)}$$

is a Poisson measure on $[u_0, 1] \times \mathcal{E}$ of intensity $2dt \times n^{<t}(de)$. As $(\overleftarrow{\zeta}_s, s \le \alpha_{u_0})$ and $(\overleftarrow{\zeta}_s, s \ge \alpha_{u_0})$ are independent, the lemma follows, letting $u_0 \to 0+$. $\square$

We are now going to state the following counterpart of this lemma for the Brownian snake: Note that for all $s \in [\alpha_u, \beta_u]$, the path $W_{T-s}$ coincides with

$W_T$ on the interval $[0, 1-u]$. Let us define, for all $u \in [0,1]$, and for all $s \in [0, \beta_u - \alpha_u]$,

$$W_s^u(t) = W_{T-\alpha_u-s}(1-u+t) \text{ for } t \in [0, \zeta_{T-\alpha_u-s} - (1-u)].$$

Note that all the paths $W_s^u$ are started from $W_T(1-u)$. In the sequel $W^u$ will denote the process $(W_s^u,\ s \in [0, \beta_u - \alpha_u])$.

**Lemma 2** *Under the probability measure $\tilde{\mathbb{N}}_x$ and conditionaly on $\{W_T = \gamma\}$, the random measure $\sum_{u\in[0,1]} \delta_{(u,W^u)}$ is a Poisson measure on $[0,1] \times \mathcal{C}(\mathbb{R}_+, \mathcal{F})$ with intensity $2dt\mathbb{N}^{<t}_{\gamma(1-t)}(d\kappa)$.*

*Proof-* The proof goes along similar lines than that of Proposition 2.5 in Le Gall [14]. This type of result can also be found in Serlet [23]. Let $\Theta^{\zeta^u}_{\gamma(1-u)}(d\kappa)$ denote the law of $W^u$ under $\tilde{\mathbb{N}}_x$ and conditional on $\{W_T = \gamma\}$ and $\zeta^u$. Let $F(t,\kappa)$ denote a positive measurable functional on the space $[0,1] \times \mathcal{C}(\mathbb{R}_+, \mathcal{F})$. Note that for all $u \neq u'$, conditionaly on $\zeta^u$ and $\zeta^{u'}$, $W^u$ and $W^{u'}$ are independent, because of the Markov property of the Brownian snake. Hence,

$$\begin{aligned}
&\tilde{\mathbb{N}}_x \left( \exp\left\{ -\sum_{u\in[0,1]} F(u, W^u) \right\} \,\middle|\, W_T = \gamma \right) \\
&= \tilde{n} \left( \prod_{u\in[0,1]} \int \Theta^{\zeta^u}_{\gamma(1-u)}(d\kappa) e^{-F(u,\kappa)} \right).
\end{aligned}$$

Using Lemma 1 and the exponential formula (see [20], Chapter XII, (1.12)), we then get

$$\begin{aligned}
&\tilde{\mathbb{N}}_x \left( \exp\left\{ -\sum_{u\in[0,1]} F(u, W^u) \right\} \,\middle|\, W_T = \gamma \right) \\
&= \exp\left\{ -\int_0^1 2du\ n^{<u} \left[ 1 - \int \Theta^{\zeta}_{\gamma(1-u)}(d\kappa) e^{-F(u,\kappa)} \right] \right\} \\
&= \exp\left\{ -2\int_0^1 du\ n^{<u} \left[ \int \Theta^{\zeta}_{\gamma(1-u)}(d\kappa) \left( 1 - e^{-F(u,\kappa)} \right) \right] \right\} \\
&= \exp\left\{ -2\int_0^1 du\ \mathbb{N}^{<u}_{\gamma(1-u)} \left( 1 - e^{-F(u,W^u)} \right) \right\}.
\end{aligned}$$

The lemma follows. $\square$

In particular,

$$\tilde{\mathbb{N}}_x(\exists u \in [0,1],\ 0 \in W^u \mid W_T = \gamma)$$

$$
\begin{aligned}
&= \exp\left\{-2\int_0^1 du \int_0^u \frac{dh}{2h^2}\left(1-\mathbb{N}^{=h}_{\gamma(1-u)}(0\in\mathcal{R})\right)\right\}\\
&= \exp\left\{-\int_0^1 du \int_0^{1-u} \frac{dh}{h^2}(1-f(B_u/\sqrt{h}))\right\}
\end{aligned}
$$

Combining this with the fact that $W_T$ is a linear Brownian motion started from $x$ and that, conditionally on the path $W_T$, the two processes $W^+$ and $W^-$ are independent and have the same law, one gets

$$
\begin{aligned}
f(\varepsilon) &= E_\varepsilon\left(1_{\{0\notin B_{[0,1]}\}}\,\mathbb{N}_\varepsilon\left(\exists u\in[0,1],\ 0\in W^u \mid W_T(\cdot)=B(\min(\cdot,1))\right)^2\right)\\
&= P_\varepsilon(0\notin B[0,1])\times\\
&\qquad E_\varepsilon\left(\exp\left\{-2\int_0^1 du\int_0^{1-u}\frac{dh}{h^2}(1-f(B_u/\sqrt{h}))\right\}\ \middle|\ 0\notin B[0,1]\right).
\end{aligned}
$$

### 3.2 Application

We are now going to derive Theorem 1-(i) for $d=1$ without giving the explicit exact value of $\alpha$. The law of $(B_t, t\le 1)$ with $B_0=\varepsilon$ conditioned by the event $\{0\notin B[0,1]\}$ is that of a Brownian meander of length 1, started from $\varepsilon$. It is well-known (see e.g. Biane-Yor [2], Imhof [12] or Yor [26], formula (3.5)) that its density with respect to the law of a three-dimensional Bessel process $(R_t, t\in[0,1])$ started from $R_0=\varepsilon$ (defined under the probability measure $P_\varepsilon^{(3)}$) is equal to $1/R_1$ up to the renormalizing constant $E_\varepsilon^{(3)}((R_1)^{-1})^{-1}$. Hence, Proposition 2 can be rewritten as:

$$
\begin{aligned}
f(\varepsilon) &= \frac{P_\varepsilon(0\notin B[0,1])}{E_\varepsilon^{(3)}((R_1)^{-1})}E_\varepsilon^{(3)}\left(\frac{1}{R_1}\exp\left\{-2\int_0^1 dt\int_0^{1-t}\frac{dh}{h^2}(1-f(R_t/\sqrt{h}))\right\}\right)\\
&= \frac{P_\varepsilon(0\notin B[0,1])}{E_\varepsilon^{(3)}((R_1)^{-1})}\\
&\qquad\times E_\varepsilon^{(3)}\left(\frac{1}{R_1}\exp\left\{-4\int_0^1\frac{dt}{R_t^2}\int_{R_t/\sqrt{1-t}}^\infty v(1-f(v))dv\right\}\right). \qquad (7)
\end{aligned}
$$

Note that (6) implies easily that $\int_0^\infty v(1-f(v))dv<\infty$. Note also that

$$
\lim_{\varepsilon\to 0+} E_\varepsilon^{(3)}((R_1)^{-1}) = E_0^{(3)}((R_1)^{-1}) = \sqrt{\frac{2}{\pi}}
$$

(see for instance Yor [26] formula (3.4)). Hence, combining this with (1), one gets

$$
f(\varepsilon)\sim\varepsilon g(\varepsilon) \text{ when } \varepsilon\to 0+
$$

where

$$g(\varepsilon) := E_\varepsilon^{(3)} \left( \frac{1}{R_1} \exp\left\{ -4 \int_0^1 \frac{dt}{R_t^2} \int_{R_t/\sqrt{1-t}}^\infty v(1-f(v))dv \right\} \right).$$

Let us put

$$\lambda = \left( 8 \int_0^\infty v(1-f(v))dv \right)^{1/2} \tag{8}$$

and define the function

$$G(x) = 4 \int_0^x v(1-f(v))dv.$$

We now rewrite $g(\varepsilon)$ as follows:

$$g(\varepsilon) = E_\varepsilon^{(3)} \left( \frac{\exp\{\int_0^1 R_t^{-2} G(R_t/\sqrt{1-t})dt\}}{R_1} \exp\left\{ -\frac{\lambda^2}{2} \int_0^1 \frac{dt}{R_t^2} \right\} \right).$$

We can now apply directly Proposition 1 (for $\mu = 1/2$) so that

$$g(\varepsilon) = \varepsilon^{\nu-1/2} E_\varepsilon^{(2+2\nu)} \left( \frac{\exp\{\int_0^1 R_t^{-2} G(R_t/\sqrt{1-t})dt\}}{(R_1)^{\nu+1/2}} \right),$$

where

$$\nu = \left( \frac{1}{4} + \lambda^2 \right)^{1/2}. \tag{9}$$

We now need the following result:

**Lemma 3** *One has*

$$\begin{aligned} &\lim_{\varepsilon \to 0+} E_\varepsilon^{(2+2\nu)} \left( \frac{\exp\{\int_0^1 R_t^{-2} G(R_t/\sqrt{1-t})dt\}}{(R_1)^{\nu+1/2}} \right) \\ &= \quad E_0^{(2+2\nu)} \left( \frac{\exp\{\int_0^1 R_t^{-2} G(R_t/\sqrt{1-t})dt\}}{(R_1)^{\nu+1/2}} \right) < \infty. \end{aligned}$$

This lemma combined with the above shows that

$$f(\varepsilon) \sim k_1 \varepsilon^\gamma \text{ when } \varepsilon \to 0+, \tag{10}$$

where

$$k_1 = E_0^{(2+2\nu)} \left( \frac{\exp\{\int_0^1 R_t^{-2} G(R_t/\sqrt{1-t})dt\}}{(R_1)^{\nu+1/2}} \right) \tag{11}$$

and

$$\gamma = \nu + 1/2. \tag{12}$$

*Proof of the Lemma-* It suffices to apply a dominated convergence argument. Note that $G(x) \leq \lambda^2/2$ for all $x$, so that $G$ is bounded. Note also that $G(x) \leq 4\int_0^x vdv = 2x^2$. Let $\rho_{1/2}^{\varepsilon}(.)$ denote the density of $R_{1/2}$ under the probability measure $P_\varepsilon^{(2+2\nu)}$. Then

$$\begin{aligned}\int_0^1 R_t^{-2} G(R_t/\sqrt{1-t})dt &\leq \frac{\lambda^2}{2}\int_{1/2}^1 R_t^{-2}dt + \int_0^{1/2} R_t^{-2} G(\sqrt{2}R_t)dt \\ &\leq \frac{\lambda^2}{2}\int_{1/2}^1 R_t^{-2} + 2.\end{aligned}$$

Hence (here $\varepsilon$ can take the value 0),

$$\begin{aligned} h_\varepsilon^\nu &:= E_\varepsilon^{(2+2\nu)}\left(\frac{\exp\{\int_0^1 R_t^{-2}G(R_t/\sqrt{1-t})dt\}}{(R_1)^{\nu+1/2}}\right) \\ &\leq e^2 E_\varepsilon^{(2+2\nu)}\left(\frac{\exp\{\frac{\lambda^2}{2}\int_{1/2}^1 R_t^{-2}dt\}}{(R_1)^{\nu+1/2}}\right) \\ &= e^2\int_0^\infty d\rho_{1/2}^\varepsilon(a) E_a^{(2+2\nu)}\left(\frac{\exp\{\frac{\lambda^2}{2}\int_0^{1/2} R_t^{-2}dt\}}{(R_{1/2})^{\nu+1/2}}\right).\end{aligned}$$

Using Proposition 1 once again, we get:

$$\begin{aligned} h_\varepsilon^\nu &\leq e^2\int_0^\infty d\rho_{1/2}^\varepsilon(a) E_a^{(3)}\left(\frac{(R_{1/2})^{\nu-1/2}}{a^{\nu-1/2}(R_{1/2})^{\nu+1/2}}\right) \\ &= e^2\int_0^\infty d\rho_{1/2}^\varepsilon(a) a^{1/2-\nu} E_a^{(3)}((R_{1/2})^{-1}).\end{aligned}$$

Note that a simple combination of the strong Markov and the scaling property for the Bessel processes yields

$$E_a^{(3)}((R_{1/2})^{-1}) \leq E_0^{(3)}((R_{1/2})^{-1}) < \infty$$

so that

$$h_\varepsilon^\nu \leq e^2 E_0^{(3)}((R_{1/2})^{-1}) E_\varepsilon^{(2+2\nu)}((R_{1/2})^{1/2-\nu}).$$

Note that similarly (as $\nu \geq 1/2$)

$$E_\varepsilon^{(2+2\nu)}((R_{1/2})^{1/2-\nu}) \leq E_0^{(2+2\nu)}((R_{1/2})^{1/2-\nu}) < \infty$$

so that $h_\varepsilon^\nu$ is bounded by a constant uniformly in $\varepsilon$, which completes the proof of the lemma. □

### 3.3 Explicit value of the exponent

It now remains to compute $\int_0^\infty v(1-f(v))dv$ to complete the proof of Theorem 1-(i) for $d=1$. Note that the scaling property and (5) imply that

$$\begin{aligned}\int_0^\infty v(1-f(v))dv &= \int_0^\infty v\mathbb{N}_v^{=1}(0 \in \mathcal{R})\ dv \\ &= \int_0^\infty v\mathbb{N}_1^{=1/v^2}(0 \in \mathcal{R})\ dv \\ &= \int_0^\infty \frac{du}{2u^2}\mathbb{N}_1^{=u}(0 \in \mathcal{R}) \\ &= \mathbb{N}_1(0 \in \mathcal{R})\end{aligned}$$

Define $u(x) = \mathbb{N}_x(0 \in \mathcal{R})$. The scaling property implies that

$$u(x) = \frac{\mathbb{N}_1(0 \in \mathcal{R})}{x^2},$$

and it is also well-known (see e.g. Le Gall [17]) that $u$ solves the equation

$$u'' = 4u^2$$

in $(0,\infty)$. This implies immediately that

$$N_1(0 \in \mathcal{R}) = \frac{3}{2}. \tag{13}$$

Hence combining this with (12), (8) and (9) yields

$$\gamma = \frac{1}{2} + \sqrt{(1+48)/4} = 4,$$

and the proof of Theorem 1-(i) for $d=1$ is complete.

## 4 Analytical approach in one dimension

In this section, we are going to estimate the asymptotic behaviour of

$$\mathbb{N}_x\,(0 \notin \mathcal{R} \text{ and } X \in \mathcal{R})$$

when $x \to 0+$ and $X > 0$ is fixed.

Throughout this section $X > 0$ is fixed. Let us now define for $x \in (0, X)$,

$$u(x) = \mathbb{N}_x(0 \in \mathcal{R} \text{ or } X \in \mathcal{R}).$$

Note that by symmetry, $u(x) = u(X-x)$ for all $x \in (0, X)$; the scaling property of the Brownian snake implies that

$$u(x) \geq \mathbb{N}_x(0 \in \mathcal{R}) = x^{-2}\mathbb{N}_1(0 \in \mathcal{R}). \tag{14}$$

Hence

$$\lim_{x\to 0+} u(x) = \lim_{x\to X-} u(x) = +\infty.$$

It is in fact well-known that $u$ is the only positive solution to the equation

$$u''(x) = \Delta u(x) = 4u^2(x) \quad \text{for } x \in (0, X) \tag{15}$$

with boundary conditions $u(0) = u(X) = +\infty$ (see e.g. Le Gall [15]). In other words, integrating this equation once,

$$u'^2(x) = \frac{8}{3}u^3(x) - c, \tag{16}$$

where $c$ is a constant (depending on $X$) that we now determine using the boundary conditions: Symmetry considerations imply that $u'(X/2) = 0$; hence, $c = 8u(X/2)^3/3$ and

$$\frac{X}{2} = \int_0^{X/2} dx = \int_{u(X/2)}^{\infty} \frac{dv}{\sqrt{\frac{8}{3}(v^3 - u(X/2)^3)}}.$$

Hence

$$\begin{aligned}\sqrt{\frac{2}{3}}X &= \int_{u(X/2)}^{\infty} \frac{dv}{\sqrt{v^3 - u(X/2)^3}} \\ &= \frac{1}{\sqrt{u(X/2)}} \int_1^{\infty} \frac{dh}{\sqrt{h^3 - 1}}.\end{aligned}$$

Let us from now on put

$$I = \int_1^{\infty} \frac{dh}{\sqrt{h^3 - 1}}.$$

Then

$$u(X/2) = \frac{3I^2}{2X^2} \text{ and } c = \frac{9I^6}{X^6}. \tag{17}$$

If we put $v = 2u/3$, we have

$$v'^2 = 4v^3 - \frac{4c}{9} = 4v^3 - \frac{4I^6}{X^6}.$$

$v$ is therefore a Weierstrass function. It can be extended into its Laurent series in the neighbourhood of 0 as follows (see e.g. Abramowitz-Stegun [1], Chapter 18)

$$v(x) = \frac{1}{x^2} + \sum_{p=1}^{\infty} c_p \frac{I^{6p}}{X^{6p}} x^{6p-2},$$

where $(c_p)$ is the sequence of rational numbers defined by induction as follows: $c_1 = 1/7$ and for all $p \geq 2$,

$$c_p = \frac{1}{(6p+1)(p-1)} \sum_{j=1}^{j=p-1} c_j c_{p-j}.$$

In particular,

$$u(x) - \frac{3}{2x^2} \sim \frac{3I^6}{14X^6} x^4 \text{ when } x \to 0+. \tag{18}$$

On the other hand, recall that $\mathbb{N}_x(0 \in \mathcal{R}) = x^{-2}\mathbb{N}_1(0 \in \mathcal{R}) 3/(2x^2)$, (cf. (13)) so that

$$\begin{aligned} \mathbb{N}_x(0 \notin \mathcal{R} \text{ and } X \in \mathcal{R}) &= \mathbb{N}_x(0 \in \mathcal{R} \text{ or } X \in \mathcal{R}) - \mathbb{N}_x(0 \in \mathcal{R}) \\ &= \sum_{p=1}^{\infty} c_p \frac{I^{6p}}{X^{6p}} x^{6p-2}, \end{aligned} \tag{19}$$

which proves Part (ii) of the Theorem with

$$k_2 = \frac{3I^6}{14X^6}. \tag{20}$$

$I$ can be expressed in terms of $\Gamma(1/3)$, noticing that in fact

$$I = \frac{1}{3} \int_0^1 dy y^{-5/6}(1-y)^{-1/2} = \frac{\Gamma(1/6)\Gamma(1/2)}{3\Gamma(2/3)}$$

and using the duplication formulae (e.g. 6.1.18 and 6.1.19 in Abramowitz-Stegun [1]), which imply that

$$I = \frac{\Gamma(1/3)^3}{\pi 2^{4/3}}.$$

Plugging the result into (20) gives (2).

**Remark-** It is not surprising that we obtain the same exponent as in Theorem 1-(i). There is indeed a direct way to show that the two quantities are logarythmically equivalent:

$$\log \mathbb{N}_\varepsilon(0 \notin \mathcal{R} \text{ and } 1 \in \mathcal{R}) \sim \log \tilde{\mathbb{N}}_\varepsilon(0 \notin \mathcal{R}) \text{ when } \varepsilon \to 0;$$

however, Theorem 1-(i) (respectively (ii)) does not imply directly (ii) (respectively (i)).

## 5 'Bessel snakes' and dimensions 2 and 3

We are now going to turn our attention towards the multi-dimensional problem (i.e. cases $d = 2$ and $d = 3$ in Theorem 1), and its generalization for 'Bessel snakes'. Suppose that $\delta < 4$, that $W^\delta$ is a 'Bessel snake' as defined in section 2, defined under the measures $\mathbb{N}_x^{(\delta)}$ or $\tilde{\mathbb{N}}_x^{(\delta)}$. Bessel processes have the same scaling property than Brownian motion so that we will be able to adapt the proofs as we did give in the $\delta = 1$ case (i.e. the one-dimensional Brownian case). The following theorem clearly contains Theorem 1:

**Theorem 3** *Define for all $\delta < 4$,*

$$\alpha(\delta) = \frac{2 - \delta + \sqrt{\delta^2 - 20\delta + 68}}{2}.$$

*Then:*
*(i) There exists a constant $k_1(\delta) \in (0, \infty)$ such that*

$$\tilde{\mathbb{N}}_\varepsilon^{(\delta)}(0 \notin \mathcal{R}) \sim k_1 \varepsilon^{\alpha(\delta)} \textit{ when } \varepsilon \to 0+.$$

*(ii) There exists a constant $k_2$, such that for all fixed $X > 0$,*

$$\mathbb{N}_\varepsilon^{(\delta)}(0 \notin \mathcal{R} \textit{ and } X \in \mathcal{R}) \sim k_2 \frac{\varepsilon^{\alpha(\delta)}}{X^{\alpha(\delta)+2}} \textit{ when } \varepsilon \to 0+.$$

As the proof goes along similar lines than that of Theorem 1 for $d = 1$, we will only focus on the aspects where they differ significantly.

### 5.1 The analytical approach

Suppose now that $\delta < 4$ and $X > 0$ are fixed.

**Lemma 4** *The function $u(x) = \mathbb{N}_x^{(\delta)}(0 \in \mathcal{R}$ or $X \in \mathcal{R})$ is the maximal solution of the equation*

$$u'' + \frac{\delta - 1}{x} u' = 4u^2 \textit{ in } (0, X). \tag{21}$$

*Proof of the lemma-* One has to be a little bit careful because $1/x$ is not bounded in the neighbourhood of 0, so that this is not a subcase of the results stated for nicer diffusions (we can not use the maximum principle for the solutions defined on a larger interval). However, there is no real difficulty in adapting the proofs to this particular case: Note that for integer $\delta$, the lemma can be deduced from the known results for $\delta$-dimensional Brownian motion in a ball.

Theorem 3.2 in [17] implies readily that $u$ defined in the Lemma is a solution of (21). It is very easy to check directly that $\lim_{X-} u = +\infty$ but it is not immediate to see for which values of $\delta$, one has $\lim_{0+} u = +\infty$, because 0 can be polar for Bessel processes. Define now for all sufficiently large $n$,

$$u_n(x) = \mathbb{N}_x^{(\delta)}(1/n \in \mathcal{R} \text{ or } X - 1/n \in \mathcal{R}).$$

Similarly, one has

$$u_n'' + \frac{\delta - 1}{x} u_n' = 4u^2 \text{ in } (n^{-1}, X - n^{-1}), \tag{22}$$

and it is easy to check that $\lim_{x \to (X-(1/n))-} u_n(x) = \lim_{x \to (1/n)+} u_n(x) = \infty$. Moreover, the maximum principle (see e.g. Dynkin [9], appendix) shows that $u_n$ is in fact the maximal solution of the differential equation (22) in $(1/n, X - 1/n)$. It is also easy to check (using the compactness of $\mathcal{R}$) that for all $x \in (0, X)$,

$$u(x) = \lim_{n \to \infty} u_n(x).$$

Combining the results above yields readily that $u$ is the maximal solution of (21) in $(0, X)$. $\square$

The scaling property shows again that

$$\mathbb{N}_x^{(\delta)}(0 \in \mathcal{R}) = \frac{1}{x^2} \mathbb{N}_1^{(\delta)}(0 \in \mathcal{R}). \tag{23}$$

Let $c_0 = \mathbb{N}_1^{(\delta)}(0 \in \mathcal{R})$. It is then very easy to check (because $v(x) = c_0 x^{-2}$ is also a solution of $v'' + (\delta - 1)v'/x = 4v^2$), that

$$c_0 = \frac{4 - \delta}{2}. \tag{24}$$

By analogy with the case $\delta = 1$, we are looking for solution of (21) of the type

$$u(x) = \frac{1}{x^2} \sum_{k=0}^{\infty} a_k x^{\beta k},$$

for some $\beta > 0$. Formally, that would imply that

$$\sum_{k=0}^{\infty} (\beta k - 2)(\beta k + \delta - 4) a_k x^{\beta k - 4} = 4 \sum_{k=0}^{\infty} \left( \sum_{j=0}^{k} a_j a_{k-j} \right) x^{\beta k - 4}.$$

Identifying the terms of these series yields that

$$a_0 = 2 - \frac{\delta}{2},$$

$$a_1 \left( (\beta - 2)(\beta + \delta - 4) - 8a_0 \right) = 0, \tag{25}$$

and for $k \geq 2$,

$$a_k\left((\beta k-2)(\beta k+\delta-4)-4(4-\delta)\right)=\sum_{j=1}^{k-1} a_j a_{k-j}. \tag{26}$$

Conversely, let us now define

$$\beta=\frac{6-\delta+\sqrt{\delta^2-20\delta+68}}{2} \tag{27}$$

so that (25) is satisfied for any $a_1$ (if $a_0=2-\delta/2$). We now define by induction the sequence $(b_k)$, such that $b_0=2-\delta/2$, $b_1=1$, and for any $k \geq 2$,

$$b_k\left(\beta(6-\delta)(k^2-k)+2(4-\delta)(k^2-1)\right)=\sum_{j=1}^{k-1} b_j b_{k-j} \tag{28}$$

(this definition parallels Equation (26) with $\beta$ given as in (27)).

**Lemma 5** *For all $k \geq 0$, $b_k>0$. The radius of convergence $R$ of the series $\sum_{k \geq 1} b_k x^k$ is strictly positive and finite. Moreover, one has $\sum_{k \geq 1} b_k R^k=+\infty$.*

*Proof of the lemma-* The sequence $(b_k)$ is well-defined, and it is immediate to check using (28) that $b_k \geq 0$ for all $k \geq 1$. (28) also implies that there exists $0<m<M<\infty$ such that for all $k \geq 2$,

$$m k^{-2} \sum_{j=1}^{k-1} b_j b_{k-j} \leq b_k \leq M k^{-2} \sum_{j=1}^{k-1} b_j b_{k-j}.$$

As

$$\lim_{k \to \infty} \frac{1}{k^3} \sum_{j=1}^{k-1} j(k-j)=\int_0^1 t(1-t) dt=\frac{1}{6},$$

there also exist $0<m'<M'<\infty$, such that for all $k \geq 2$,

$$m' \leq \frac{1}{k^3} \sum_{j=1}^{k-1} j(k-j) \leq M'.$$

Hence, an easy induction shows that for all $k \geq 1$,

$$k(mm')^{k-1} \leq b_k \leq k(MM')^{k-1},$$

and this implies that the radius of convergence $R$ of $\sum b_k x^k$ is strictly positive and finite. Let us now define

$$w(y)=\sum_{k \geq 0} b_k y^k$$

so that $w$ solves the equation

$$\beta^2 y^2 w'' + \beta(\delta + \beta - 6) y w' + (8 - 2\delta) w = 4w^2 \tag{29}$$

on $(0, R)$. Suppose now that $w(R) = \sum_{k \geq 0} b_k R^k < \infty$. Then, as

$$\left( y^{(\delta+\beta-6)/\beta} w' \right)' = y^{(\delta-\beta-6)/\beta} \left( \frac{4w^2 - (8 - 2\delta) w}{\beta^2} \right)$$

on $[R/2, R)$, $w'$ and $w''$ are also uniformly bounded on $[R/2, R)$, and therefore, $w'(R)$ is well-defined. Cauchy's theorem, then implies the existence of a planar neighbourhood $V \subset \mathbb{C}$ of the point $R$ and the existence and unicity of a holomorphic solution $\tilde{w}$ of equation (29) in $V$, such that $\tilde{w}(R) = w(R)$ and $\tilde{w}'(R) = w'(R)$. The unicity ensures that $w = \tilde{w}$ on $V \cap \{z \in \mathbb{C},\ |z| < R\}$, and as both $w$ and $\tilde{w}$ are holomorphic, $w$ can be extended analytically onto $\{z \in \mathbb{C},\ |z| < R\} \cup V$. This contradicts the definition of $R$ and the fact that for all $k \geq 1$, $b_k > 0$. The lemma is proved. □

**Lemma 6** *For all $x \in (0, X)$,*

$$u(x) = \frac{1}{x^2} \sum_{k=0}^{\infty} b_k \left( \frac{R x^\beta}{X^\beta} \right)^k = \frac{1}{x^2} w \left( \frac{x^\beta R}{X^\beta} \right).$$

*Proof of the lemma*- Define, for all $Y > 0$,

$$u^Y(x) = \frac{1}{x^2} \sum_{k=0}^{\infty} b_k \left( \frac{R x^\beta}{Y^\beta} \right)^k = \frac{1}{x^2} w \left( \frac{x^\beta R}{Y^\beta} \right).$$

The radius of convergence of this series is $Y$; it is straightforward to check that $u^Y$ is a solution of (21) on $(0, Y)$ and that $u^Y(0+) = u^Y(Y-) = +\infty$. We can not apply directly the maximum priciple as we cannot compare $u$ and $u^X$ at the neighbourhood of 0. We therefore go back to the function

$$r(x) = \mathbb{N}_x^{(\delta)}(0 \notin \mathcal{R} \text{ and } X \in \mathcal{R}).$$

Using (23) and (24), one has

$$r(x) = u(x) - \mathbb{N}_x^{(\delta)}(0 \in \mathcal{R}) = u(x) - \frac{c_0}{x^2}$$

on $(0, X)$. Hence, one gets immediately that

$$r'' + \frac{\delta - 1}{x} r' = 4r(r + 2c_0 x^{-2}) \tag{30}$$

on $(0, X)$. Note also that the definition of $r$ ensures that

$$\lim_{0+} r = 0,\ \lim_{X-} r = +\infty \text{ and } r \geq 0.$$

Define also, for all $Y > 0$,

$$r^Y(x) = u^Y(x) - \frac{c_0}{x^2} = \sum_{k=1}^{\infty} b_k \frac{R^k x^{\beta k - 2}}{Y^{\beta k}}.$$

Note that $\beta > 2$ (because $\delta < 4$), so that $\lim_{0+} r^Y = 0$. As a consequence of the results for $u^Y$, $r^Y \geq 0$, $\lim_{Y-} r^Y = +\infty$, and $r^Y$ satisfies the same differential equation as $r$ (i.e. (30)) on the time-interval $(0, Y)$. Take a sequence $Y_n$, such that $Y_n \leq Y$ and $\lim_{n\to\infty} Y_n = X$. We can now apply the maximum principle for this new equation in $(0, Y_n)$, so that

$$r \leq r^{Y_n} \text{ in } (0, Y_n).$$

Clearly, for all $x \in (0, X)$,

$$\lim_{n\to\infty} r^{Y_n}(x) = \lim_{n\to\infty} \frac{X^2}{Y_n^2} r^X(xX/Y_n) = r^X(x).$$

Hence, $r \leq r^X$ on $(0, X)$, and eventually,

$$u(x) \leq u^X(x).$$

Combining this with Lemma 4 finishes the proof of this lemma. □

*Proof of Theorem 3(ii)*- The previous lemma shows that

$$\mathbb{N}_x^{(\delta)}(0 \notin \mathcal{R} \text{ and } X \in \mathcal{R}) = r(x) = \sum_{k=1}^{\infty} \frac{b_k R^k}{X^{\beta k}} x^{\beta k - 2}. \tag{31}$$

In particular,

$$\mathbb{N}_x^{(\delta)}(0 \notin \mathcal{R} \text{ and } X \in \mathcal{R}) \sim \frac{R}{X^\beta} x^{\beta - 2} \text{ when } x \to 0+, \tag{32}$$

which proves part (ii) of Theorem 3.

## 5.2 The probabilistic approach

### 5.2.1 The case $\delta \geq 2$

In the case $\delta \geq 2$, the path of one Bessel process of dimension $\delta$ never reaches 0. The decomposition of the snake with respect to its longest branch is therefore even simpler than in the Brownian case, as there is no need to condition by the event $\{0 \notin R[0,1]\}$. Hence, one immediately gets

$$\begin{aligned}&\tilde{N}_x^{(\delta)}(0 \notin \mathcal{R}) \\ = \;& E_\varepsilon^{(\delta)}\left(\exp\left\{\frac{-\lambda^2}{2}\int_0^1 \frac{dt}{R_t^2}\right\}\exp\left\{\int_0^1 \frac{G^{(\delta)}(R_t/\sqrt{1-t})dt}{R_t^2}\right\}\right),\end{aligned}$$

where

$$G^{(\delta)}(x) = 4\int_0^x v\tilde{N}_v^{(\delta)}(0 \in \mathcal{R})dv$$

and

$$\lambda = (2(G^{(\delta)}(\infty))^{1/2}.$$

Using (4), and if we put $\mu(\delta) = -1 + \delta/2$ and $\nu = \sqrt{\lambda^2 + \mu^2}$, we get

$$\tilde{N}_x^{(\delta)}(0 \notin \mathcal{R}) = \varepsilon^{\nu-\mu} E_\varepsilon^{(2+2\nu)}\left(\frac{1}{R_1^{\nu-\mu}} \exp\left\{\int_0^1 \frac{G^{(\delta)}(R_t/\sqrt{1-t})dt}{R_t^2}\right\}\right).$$

Using exactly the same arguments than for Lemma 3, one can show the convergence of

$$E_\varepsilon^{(2+2\nu)}\left(\frac{1}{R_1^{\nu-\mu}} \exp\left\{\int_0^1 \frac{G^{(\delta)}(R_t/\sqrt{1-t})dt}{R_t^2}\right\}\right)$$

towards a finite constant $k_1$, when $\varepsilon \to 0$, so that

$$\tilde{N}_x^{(\delta)}(0 \notin \mathcal{R}) \sim k_1 \varepsilon^{\nu-\mu}, \text{ when } \varepsilon \to 0+.$$

### 5.2.2 The case $\delta < 2$

This time, a Bessel process of dimension $\delta$ hits zero with strictly positive probability. The proof now follows exactly the same lines as in Section 3. We now have to use the following fact (which follows for instance from formula (3.5) in Yor [26], paragraph 3.6): The conditional law of $(R_t, t \in [0,1])$ under $P_\varepsilon^{(\delta)}$ conditioned by the event $\{0 \notin R[0,1]\}$ is that of a Bessel meander of dimension $\delta$, and is proportional (up to a constant term) to

$$\frac{1}{R_1^{2-\delta}} P_\varepsilon^{(4-\delta)}.$$

Also, recall that

$$P_\varepsilon^{(\delta)}(0 \notin R[0,1]) \sim k(\delta)\varepsilon^{2-\delta} \text{ when } \varepsilon \to 0+.$$

Hence,

$$\tilde{N}_x^{(\delta)}(0 \notin \mathcal{R}) \sim$$

$$k'(\delta)\varepsilon^{2-\delta} E_\varepsilon^{(4-\delta)}\left(\frac{1}{R_1^{2-\delta}} \exp\left\{\frac{-\lambda^2}{2}\int_0^1 \frac{dt}{R_t^2}\right\} \exp\left\{\int_0^1 \frac{G^{(\delta)}(R_t/\sqrt{1-t})dt}{R_t^2}\right\}\right)$$

when $\varepsilon \to 0$, with the same notation than above. Note that $\mu(4-\delta) = -\mu(\delta)$ (as functions of $\delta$), so that one eventually gets

$$\tilde{N}_x^{(\delta)}(0 \notin \mathcal{R}) \sim k'''(\delta)\varepsilon^{\nu+\mu-2\mu} \text{ when } \varepsilon \to 0+.$$

### 5.2.3 Identification

It now remains to show that

$$\int_0^\infty v(1-\tilde{\mathbb{N}}_v^{(\delta)}(0\notin\mathcal{R}))dv = \frac{4-\delta}{2}$$

to complete the proof of Theorem 3. This can be done exactly as in Section 3.3, and is safely left to the reader.

# 6 Super-Brownian motion

## 6.1 Proof of Theorem 2

We now turn our attention towards super-Brownian motion. We shall use Theorem 1 and the Poissonian representation of super-Brownian motion in terms of the Brownian snake (see e.g. Le Gall [17], Theorem 2.3 or Dynkin-Kuznetsov [11]) to derive Theorem 2. Let $(\mu_t)_{t\geq 0}$ denote a super-Brownian motion started from the finite positive measure $\nu$ on $\mathbb{R}^d$ under the probability measure $P_\nu$. Throughout this section, to avoid complications, we shall always omit the superscript $d$ (the spatial dimension), which is fixed ($d=1$, 2 or 3).

Fix $\lambda\geq 0$. Consider a Brownian snake $W$ started from $\varepsilon$ (this is a point in $\mathbb{R}^d$ at distance $\varepsilon$ from the origin) with life-time process a reflected Brownian motion $\zeta$, killed when its local time at level 0 exceeds $\lambda$. Define $(W^u,\zeta^u)_{u\in[0,\lambda]}$, the corresponding excursions of $W$ away from the path started at $\varepsilon$ with zero life-time. Let $H(\zeta^u)$ denote the height of the excursion $\zeta^u$. We now construct the super-Brownian motion $\mu_t$ under the probability measure $P_{\lambda\delta_\varepsilon}$ using the Brownian snake $W$ as in Le Gall [17], Theorem 2.3. Then, one has:

$$\{|\mu_1|>0\}=\{\exists u\leq\lambda,\ H(\zeta^u)>1\}.$$

Hence, as the excursion process is Markovian, shows that

$$\begin{aligned}&P_{\lambda\delta_\varepsilon}(0\notin\mathcal{S}\text{ and }|\mu_1|>0)\\ &\quad= P_{\lambda\delta_\varepsilon}(\exists u\leq\lambda,\ H(\zeta^u)>1\text{ and }W^u\text{ does not hit }0)\times P_{\lambda\delta_\varepsilon}(0\notin\mathcal{S}).\end{aligned}$$

The exponential formula (see e.g. (1.12) in [20], chapter XII) implies that

$$P_{\lambda\delta_\varepsilon}(0\notin\mathcal{S})=\exp\left\{-\int_0^\lambda \mathbb{N}_\varepsilon(0\in\mathcal{R})du\right\}.$$

Using (24) and (23), one gets,

$$P_{\lambda\delta_\varepsilon}(0\notin\mathcal{S})=\exp\left\{-\frac{\lambda(4-d)}{2\varepsilon^2}\right\}.$$

On the other hand, the exponential formula also implies that

$$P_{\lambda\delta_\varepsilon}(\exists u \le \lambda,\ H(\zeta^u) > 1 \text{ and } W^u \text{ does not hit } 0) = 1 - \exp(\lambda \mathbb{N}_\varepsilon^{>1}(0 \notin \mathcal{R})).$$

Hence

$$\begin{aligned} & P_{\lambda\delta_\varepsilon}(0 \notin \mathcal{S} \text{ and } |\mu_1| > 0) \\ = \ & \exp\left\{\frac{\lambda(4-d)}{2\varepsilon^2}\right\}\left\{1 - \exp\{-\lambda \mathbb{N}_\varepsilon^{>1}(0 \notin \mathcal{R})\}\right\} \end{aligned} \tag{33}$$

Exactly as in the one-dimensional case, Theorem 1 then implies immediately that

$$\mathbb{N}_\varepsilon^{>1}(0 \notin \mathcal{R}) \sim \frac{k_1(d)\varepsilon^{\alpha(d)}}{1+\alpha/2} \text{ when } \varepsilon \to 0.$$

Combining this with (33) when $\lambda = 1$ implies immediately Theorem 2.

### 6.2 An optimization problem

We are now going to derive the following result that we announced at the end of the introduction:

**Proposition 3** *One has*

$$\sup_{\lambda \ge 0} P_{\lambda\delta_\varepsilon}(0 \notin \mathcal{S} \textit{ and } |\mu_1| > 0) \sim \frac{k_3(d)}{4-d}\varepsilon^{2+\alpha(d)}, \textit{ when } \varepsilon \to 0. \tag{34}$$

*Moreover,* $\sup_{\lambda \ge 0} P_{\lambda\delta_\varepsilon}(0 \notin \mathcal{S} \textit{ and } |\mu_1| > 0)$ *is obtained when*

$$\lambda = \lambda^*(\varepsilon) = \frac{1}{\mathbb{N}_\varepsilon^{>1}(0 \notin \mathcal{R})} \log\left(1 + \frac{2\varepsilon^2 \mathbb{N}_\varepsilon^{>1}(0 \notin \mathcal{R})}{4-d}\right) \sim \frac{2\varepsilon^2}{4-d} \textit{ when } \varepsilon \to 0. \tag{35}$$

*Proof-* (35) is an immediate consequance of (33). The first statement follows immediately.

## 7 Other related problems

### 7.1 Non-intersection between two ranges

It is very easy to use Theorem 1 to investigate the probability that the ranges of two independent one-dimensional Brownian snakes do not intersect. Let us give an example for Brownian snakes renormalized by their height: Suppose now that $\mathcal{R}$ and $\mathcal{R}'$ denote the ranges of two independent Brownian snakes, renormalized by their height (the height of the excursions is equal to 1) and respectively started from 0 and $\varepsilon$ under the probability measure $\overline{N}_\varepsilon$. Then:

**Proposition 4**

$$\overline{N}_\varepsilon(\mathcal{R}\cap\mathcal{R}'=\emptyset)\sim\frac{k_1^2}{70}\varepsilon^8 \ \ when\ \varepsilon\to 0+\,.$$

*Proof-* Fix the integer $p\geq 2$ for a while. Let us put

$$\mathcal{R}_*=\sup\mathcal{R}\ \text{and}\ \mathcal{R}'_{\#}=\inf\mathcal{R}'.$$

It is easy to notice that

$$\begin{aligned}
\overline{N}_\varepsilon(\mathcal{R}\cap\mathcal{R}'=\emptyset) &= \sum_{j=1}^{p}\overline{N}_\varepsilon\left(\mathcal{R}\cap\mathcal{R}'=\emptyset\ \text{and}\ \mathcal{R}_*\in\left[\frac{(j-1)\varepsilon}{p},\frac{j\varepsilon}{p}\right]\right)\\
&\leq \sum_{j=1}^{p}\overline{N}_\varepsilon\left(\mathcal{R}_*\in\left[\frac{(j-1)\varepsilon}{p},\frac{j\varepsilon}{p}\right]\ \text{and}\ \mathcal{R}'_{\#}\geq\frac{(j-1)\varepsilon}{p}\right)\\
&\leq \sum_{j=1}^{p}\left(f\left(\frac{j\varepsilon}{p}\right)-f\left(\frac{(j-1)\varepsilon}{p}\right)\right)f\left(\frac{(p-j+1)\varepsilon}{p}\right).
\end{aligned}$$

Using the asymptotic expansion of $f$, we get

$$\limsup_{\varepsilon\to 0+}\varepsilon^{-8}\overline{N}_\varepsilon(\mathcal{R}\cap\mathcal{R}'=\emptyset)\leq (k_1)^2\sum_{j=1}^{p}\frac{j^4-(j-1)^4}{p^8}(p-j+1)^4,$$

and hence (letting $p\to\infty$), one gets immediately (as the above is a Riemann sum)

$$\limsup_{\varepsilon\to 0+}\varepsilon^{-8}\overline{N}_\varepsilon(\mathcal{R}\cap\mathcal{R}'=\emptyset)\leq 4(k_1)^2\int_0^1 x^3(1-x)^4dx=\frac{(k_1)^2}{70}.$$

Similarly, one gets

$$\liminf_{\varepsilon\to 0+}\varepsilon^{-8}\overline{N}_\varepsilon(\mathcal{R}\cap\mathcal{R}'=\emptyset)\geq (k_1)^2\sum_{j=1}^{p}\frac{j^4-(j-1)^4}{p^8}(p-j)^4,$$

and therefore,

$$\liminf_{\varepsilon\to 0+}\varepsilon^{-8}\overline{N}_\varepsilon(\mathcal{R}\cap\mathcal{R}'=\emptyset)\geq\frac{(k_1)^2}{70}$$

which completes the proof.

In higher dimensions, this problem seems to be more complicated. Of course one gets lower and upper bounds for non-intersection probabilities between ranges like

$$c_1\varepsilon^8\leq\overline{N}_\varepsilon^{(d)}(\mathcal{R}\cap\mathcal{R}'=\emptyset)\leq c_2\varepsilon^{2\alpha(d)},$$

with obvious notation, but this is not very informative (especially when $d=4$, 5, 6 or 7!).

### 7.2 Other remarks

One could also have derived results analogous to Theorem 1 for Brownian snakes renormalized by the length (and not the height) of its genealogical excursion. This can be for instance derived from Theorem 1-(i), using the fact that for all $M > 1$,

$$n^{=1}(\sigma(\zeta) > M) \leq a e^{-bM}$$

for some well-chosen $a$ and $b$, and the scaling property.

If $W$ now denotes a Brownian snake in $\mathbb{R}^d$, and $H$ a smooth $d'$-dimensional manifold in $\mathbb{R}^d$ (with $d' = d-1,\ d-2$ or $d-3$). The above results can be easily adapted to show that the asymptotic behaviour of the probability that $W$, started at distance $\varepsilon$ from $H$ and conditionned to survive after a certain time decays also like $\varepsilon^{\alpha(d-d')}$, when $\varepsilon \to 0+$.

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