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Degenerate stochastic differential equations arising from catalytic branching networks

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Abstract

We establish existence and uniqueness for the martingale problem associated with a system of degenerate SDE's representing a catalytic branching network. For example, in the hypercyclic case:

$$dX_t^{(i)} = b_i(X_t)dt + \sqrt{2\gamma_i(X_t)X_t^{(i+1)}X_t^{(i)}}dB_t^i,\ X_t^{(i)} \geq 0,\ i = 1, \ldots, d,$$

where $X^{(d+1)} \equiv X^{(1)}$, existence and uniqueness is proved when γ and b are continuous on the positive orthant, γ is strictly positive, and $b_i > 0$ on $\{x_i = 0\}$. The special case $d = 2$, $b_i = \theta_i - x_i$ is required in work of [DGHSS] on mean fields limits of block averages for 2-type branching models on a hierarchical group. The proofs make use of some new methods, including Cotlar's lemma to establish asymptotic orthogonality of the derivatives of an associated semigroup at different times, and a refined integration by parts technique from [DP1]. As a by-product of the proof we obtain the strong Feller property of the associated resolvent.

Key words: stochastic differential equations, perturbations, resolvents, Cotlar's lemma

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1. Introduction. In this paper we establish well-posedness of the martingale problem for certain degenerate second order elliptic operators. The class of operators we consider arises from models of catalytic branching networks including catalytic branching, mutually catalytic branching and hypercyclic catalytic branching systems (see [DF] for a survey of these systems). For example, the hypercyclic catalytic branching model is a diffusion on $\mathbb{R}_+^d$, $d \geq 2$, solving the following system of stochastic differential equations:

$$dX_t^{(i)} = (\theta_i - X_t^{(i)})dt + \sqrt{2\gamma_i(X_t)X_t^{(i+1)}X_t^{(i)}}dB_t^i,\ i = 1, \dots, d. \tag{1.1}$$

Here $X(t) = (X_t^{(1)}, \dots, X_t^{(d)})$, addition of the superscripts is done cyclically so that $X_t^{(d+1)} = X_t^{(1)}$, $\theta_i > 0$, and $\gamma_i > 0$.

Uniqueness results of this type are proved in [DP1] under Hölder continuity hypotheses on the coefficients. Our main result here is to show the uniqueness continues to hold if this is weakened to continuity. One motivation for this problem is that for $d = 2$, (1.1) arises in [DGHSS] as the mean field limit of the block averages of a system of SDE's on a hierarchical group. The system of SDEs models two types of individuals interacting through migration between sites and at each site through interactive branching, depending on the masses of the types at that particular site. The branching coefficients γ_i of the resulting equation for the block averages arise from averaging the original branching coefficients at a large time (reflecting the slower time scale of the block averages) and so are given in terms of the equilibrium distribution of the original equation. The authors of [DGHSS] introduce a renormalization map which gives the branching coefficients γ_i of the block averages in terms of the previous SDE. They wish to iterate this map to study higher order block averages. Continuity is preserved by this map on the interior of $\mathbb{R}_+^d$, and is conjectured to be preserved at the boundary (see Conjecture 2.7 of [DGHSS]). It is not known whether Hölder continuity is preserved (in the interior and on the boundary), which is why the results of [DP1] are not strong enough to carry out this program. The weakened hypothesis also leads to some new methods.

The proofs in this paper are substantially simpler in the two-dimensional setting required for [DGHSS] (see Section 8 below) but as higher dimensional analogues of their results are among the "future challenges" stated there, we thought the higher-dimensional results worth pursuing.

Further motivation for the study of such branching catalytic networks comes from [ES] where a corresponding system of ODEs was proposed as a macromolecular precursor to early forms of life. There also have been a number of mathematical works on mutually catalytic branching ((1.1) with $d = 2$ and γ_i constant) in spatial settings where a special duality argument ([M], [DP2]) allows a more detailed analysis, and even in spatial analogues of (1.1) for general d, but now with much more restricted results due in part to the lack of any uniqueness result ([DFX], [FX]). See the introduction of [DP1] for more background material on the model.

Earlier work in [ABBP] and [BP] show uniqueness in the martingale problem for the operator $\mathcal{A}^{(b,\gamma)}$ on $C^2(\mathbb{R}_+^d)$ defined by

$$\mathcal{A}^{(b,\gamma)}f(x) = \sum_{i=1}^{d} \left(b_i(x)\frac{\partial f}{\partial x_i} + \gamma_i(x) x_i \frac{\partial^2 f}{\partial x_i^2} \right), \quad x \in \mathbb{R}_+^d.$$

Here b_i, γ_i $i = 1, \ldots, d$ are continuous functions on $\mathbb{R}_+^d$, with $b_i(x) \geq 0$ if $x_i = 0$, and also satisfy some additional regularity or non-degeneracy condition. If $b_i(x) = \sum_j x_j q_{ji}$ for some $d \times d$ Q-matrix (q_{ji}), then such diffusions arise as limit points of rescaled systems of critical branching Markov chains in which (q_{ji}) governs the spatial motions of particles and $\gamma_i(x)$ is the branching rate at site i in population $x = (x_1, \ldots, x_d)$. In both [ABBP] and [BP] a Stroock-Varadhan perturbation approach was used in which one views the generator in question as a perturbation of an independent collection of squared Bessel processes. The perturbation argument, however, is carried out on different Banach spaces; in [ABBP] it was an appropriate L^2 space, while in [BP] it was a weighted Hölder space. In this work we again proceed by such a perturbation argument on an appropriate L^2 space (as in [ABBP]) but the methods of [ABBP] or [BP] will not apply to systems such as (1.1) because now the branching rates γ_i may be zero and so the process from which we are perturbing will be more involved. In fact the appropriate class of processes was introduced in [DP1]. So it would appear that a combination of the ideas of [ABBP] and [DP1] is needed, but we will see that we will in fact have to significantly extend the integration by parts formulae of [DP1] and an invoke a new analytic ingredient, Cotlar's Lemma, to carry out the proof. Admittedly the choice of operator from which one perturbs and Banach space in which to carry out the perturbation is a bit of an art at present, but we feel the set of methods introduced to date may also handle a number of other degenerate diffusions including higher order multiplicative catalysts.

We will formulate our results in terms of catalytic branching networks in which the catalytic reactions are given by a finite directed graph $(V, \mathcal{E})$ with vertex set $V = \{1, \ldots, d\}$ and edge set $\mathcal{E} = \{e_1, \ldots, e_k\}$. This will include (1.1) and all of the two-dimensional systems arising in [DGHSS]. As in [DP1] we assume throughout:

Hypothesis 1.1. $(i, i) \notin \mathcal{E}$ for all $i \in V$ and each vertex is the second element of at most one edge. □

The restrictive second part of this hypothesis has been removed by Kliem [K] in the Hölder continuous setting of [DP1]. It is of course no restriction if $|V| = 2$ (as in [DGHSS]), and holds in the cyclic setting of (1.1).

Vertices denote types and an edge $(i,j) \in \mathcal{E}$ indicates that type i catalyzes the type j branching. Let C denote the set of vertices (catalysts) which appear as the first element of an edge and R denote the set of vertices that appear as the second element (reactants). Let $c : R \to C$ be such that for $j \in R$, c_j denotes the unique $i \in C$ such that $(i,j) \in \mathcal{E}$, and for $i \in C$, let $R_i = \{j : (i,j) \in \mathcal{E}\}$.

Here are the hypotheses on the coefficients:

Hypothesis 1.2. For $i \in V$,

$$\gamma_i : \mathbb{R}_+^d \to (0,\infty), \quad b_i : \mathbb{R}_+^d \to \mathbb{R},$$

are continuous such that $|b_i(x)| \le c(1+|x|)$ on $\mathbb{R}_+^d$, and $b_i(x) > 0$ if $x_i = 0$. □

The positivity condition on $b_i|_{x_i=0}$ is needed to ensure the solutions remain in the first orthant.

If $D \subset \mathbb{R}^d$, $C_b^2(D)$ denotes the space of twice continuously differentiable bounded functions on D whose first and second order partial derivatives are also bounded. For $f \in C_b^2(\mathbb{R}_+^d)$, and with the above interpretations, the generators we study are

$$\mathcal{A}f(x) = \mathcal{A}^{(b,\gamma)}f(x) = \sum_{j\in R} \gamma_j(x) x_{c_j} x_j f_{jj}(x) + \sum_{j\notin R} \gamma_j(x) x_j f_{jj}(x) + \sum_{j \in V} b_j(x) f_j(x).$$

(Here and elsewhere we use f_i and f_{ij} for the first and second partial derivatives of f.)

Definition. Let $\Omega = C(\mathbb{R}_+, \mathbb{R}_+^d)$, the continuous functions from $\mathbb{R}_+$ to $\mathbb{R}_+^d$. Let $X_t(\omega) = \omega(t)$ for $\omega \in \Omega$, and let $(\mathcal{F}_t)$ be the canonical right continuous filtration generated by X. If ν is a probability on $\mathbb{R}_+^d$, a probability $\mathbb{P}$ on Ω solves the martingale problem $MP(\mathcal{A},\nu)$ if under $\mathbb{P}$, the law of X_0 is ν and for all $f \in C_b^2(\mathbb{R}_+^d)$,

$$M_f(t) = f(X_t) - f(X_0) - \int_0^t \mathcal{A}f(X_s)\,ds$$

is a local martingale under $\mathbb{P}$.

A natural state space for our martingale problem is

$$S = \Big\{ x \in \mathbb{R}_+^d : \prod_{(i,j)\in\mathcal{E}} (x_i + x_j) > 0 \Big\}.$$

The following result is Lemma 5 of [DP1] – the Hölder continuity assumed there plays no role in the proof.

Lemma 1.3. *If $\mathbb{P}$ is a solution of $MP(\mathcal{A},\nu)$, where ν is a probability on $\mathbb{R}_+^d$, then $X_t \in S$ for all $t > 0$ $\mathbb{P}$-a.s.*

Here is our main result.

Theorem 1.4. *Assume Hypotheses 1.1 and 1.2 hold. Then for any probability ν on S, there is exactly one solution to MP$(\mathcal{A}, \nu)$.*

The cases required in Theorem 2.2 of [DGHSS] are the three possible directed graphs for $V = \{1, 2\}$:

(i) $\mathcal{E} = \emptyset$;

(ii) $\mathcal{E} = \{(2, 1)\}$ or $\mathcal{E} = \{(1, 2)\}$;

(iii) $\mathcal{E} = \{(1, 2), (2, 1)\}$.

The state space here is $S = \mathbb{R}^2 - \{(0, 0)\}$. In addition, [DGHSS] takes $b_i(x) = \theta_i - x_i$ for $\theta_i \geq 0$. As discussed in Remark 1 of [DGHSS], weak uniqueness is trivial if either θ_i is 0, as that coordinate becomes absorbed at 0, so we may assume $\theta_i > 0$. In this case Hypotheses 1.1 and 1.2 hold, and Theorem 2.2, stated in [DGHSS] (the present paper is cited for a proof), is immediate from Theorem 1.4 above. See Section 8 below for further discussion about our proof and how it simplifies in this two-dimensional setting. In fact, in Case (i) the result holds for any ν on all of $\mathbb{R}^2_+$ (as again noted in Theorem 2.2 of [DGHSS]) by Theorem A of [BP].

Our proof of Theorem 1.4 actually proves a stronger result. We do not require that the γ_i be continuous, but only that their oscillation not be too large. More precisely, we prove that there exists $\varepsilon_0 > 0$ such that if (1.2) below holds, then there is exactly one solution of $MP(\mathcal{A}, \nu)$. The condition needed is

For each $i = 1, \ldots, d$ and each $x \in \mathbb{R}^d_+$ there exists a neighborhood N_x such that

$$\operatorname{Osc}_{N_x} \gamma_i < \varepsilon_0, \tag{1.2}$$

where $\operatorname{Osc}_A f = \sup_A f - \inf_A f$.

As was mentioned above, the new analytic tool we use is Cotlar's lemma, Lemma 2.13, which is also at the heart of the famous T1 theorem of harmonic analysis. For a simple application of how Cotlar's lemma can be used, see [Fe], pp. 103–104.

We consider certain operators T_t (defined below in (2.18)) and show that

$$\|T_t\|_2 \leq c/t. \tag{1.3}$$

We require L^2 bounds on $\int_0^\infty e^{-\lambda t} T_t \, dt$, and (1.3) is not sufficient to give these. This is where Cotlar's lemma comes in: we prove L^2 bounds on $T_t T_s^*$ and $T_t^* T_s$, and these together with Cotlar's lemma yield the desired bounds on $\int_0^t e^{-\lambda t} T_t \, dt$. The use of Cotlar's lemma to operators arising from a decomposition of the time axis is perhaps noteworthy. In all other applications of Cotlar's lemma that we are aware of, the corresponding operators arise from a decomposition of the space variable. The L^2 bounds on $T_t T_s^*$ and $T_t^* T_s$ are the hardest and lengthiest parts of the paper. At the heart of these bounds is an integration by parts formula which refines a result used in [DP1] (see the proof of Proposition 17 there) and is discussed in the next section.

Theorem 1.4 leaves open the question of uniqueness in law starting at points in the complement of S, that is at points where the reactant and catalyst are both zero. For such starting points the generator from which we are perturbing will be too degenerate to have a resolvent with the required smoothing properties. In the simple two-dimensional case, the question amounts to showing uniqueness in law starting from the origin for the martingale problem associated with

$$\mathcal{A}f(x) = \sum_{j=1}^{2} \gamma_j(x)x_1x_2f_{jj}(x) + b_j(x)f_j(x),$$

or

$$\mathcal{A}'f(x) = \gamma_1(x)x_1f_{11}(x) + \gamma_2(x)x_1x_2f_{22}(x) + \sum_{j=1}^{2} b_j(x)f_j(x).$$

If γ_j and b_j (as in Hypothesis 1.2) are Lipschitz we can use different methods to prove pathwise uniqueness until the process hits the axes, necessarily at a strictly positive time. Hence uniqueness in law now follows from Theorem 1.4. The question of uniqueness starting in the complement of S, assuming only continuity of the coefficients or in higher dimensions, remains open.

In Section 3 we give a proof of Theorem 1.4. The proofs of all the hard steps, are, however, deferred to later sections. A brief outline of the rest of the paper is given at the end of Section 2.

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2. Structure of the proof.

We first reduce Theorem 1.4 to a local uniqueness result (Theorem 2.1 below). Many details are suppressed as this argument is a minor modification of the proof of Theorem 4 in [DP1]. By the localization argument in Section 6.6 of [SV] it suffices to fix $x^0 \in S$ and show that for some $r_0 = r_0(x^0) > 0$, there are coefficients which agree with γ_i, b_i on $B(x^0, r_0)$, the open ball of radius r_0 centered at x^0, and for which the associated martingale problem has a unique solution for all initial distributions. Following [DP1], let $Z = \{i \in V : x_i^0 = 0\}$, $N_1 = \cup_{i \in Z \cap C} R_i$, $\bar{N}_1 = N_1 \cup (Z \cap C)$, and $N_2 = V - \bar{N}_1$. Note that $N_1 \cap Z = \emptyset$ because $x^0 \in S$. Define

$$\tilde{\gamma}_j(x) = \begin{cases} x_j\gamma_j(x) & \text{if } j \in N_1; \\ x_{c_j}\gamma_j(x) & \text{if } j \in (Z \cap C) \cup (N_2 \cap R); \\ \gamma_j(x) & \text{if } j \in N_2 \cap R^c, \end{cases}$$

and note that $\gamma_j^0 \equiv \tilde{\gamma}_j(x^0) > 0$ for all j because $x^0 \in S$. We may now write

$$\mathcal{A}^{b,\gamma} f(x) = \sum_{i \in Z \cap C} \Big[\sum_{j \in R_i} \tilde{\gamma}_j(x) x_i f_{jj}(x) \Big] + \tilde{\gamma}_i(x) x_i f_{ii}(x) + \sum_{j \in N_2} \tilde{\gamma}_j(x) x_j f_{jj}(x) + \sum_{j \in V} b_j(x) f_j(x).$$

Let $\delta = \delta(x^0) = \min_{i \in Z} b_i(x^0) > 0$ (set it equal to 1 if Z is empty), and define

$$\tilde{b}_j(x) = \begin{cases} b_j(x) & \text{if } j \in N_1; \\ b_j(x) \vee \frac{\delta}{2} & \text{if } j \notin N_1, \end{cases}$$

and let $b_j^0 = \tilde{b}_j(x^0)$, so that $b_j^0 > 0$ for $j \notin N_1$. Although $b_j(x^0) \le 0$ is possible for $j \in N_2 \cap Z^c$ (and so $\tilde{b}_j$ may differ from b_j here), a simple Girsanov argument will allow us to assume that $b_j(x^0) \ge \delta$ for $j \in N_2 \cap Z^c$ (see the proof below) and so $\tilde{b}_j = b_j$ near x^0. With this reduction we see that by Hypothesis 1.2 and the choice of δ, $\tilde{b}_j(x) = b_j(x)$ for x near x^0. By changing $\tilde{b}$ and $\tilde{\gamma}$ outside a small ball centered at x^0 we may assume $\tilde{\gamma}_j > 0$ for all j, $\tilde{b}_j > 0$ for $j \notin N_1$, $\tilde{\gamma}_j, \tilde{b}_j$ are all bounded continuous and constant outside a compact set, and

$$\varepsilon_0 \equiv \sum_{j=1}^{d} \big(\|\tilde{\gamma} - \gamma_j^0\|_\infty + \|\tilde{b}_j - b_j^0\|_\infty \big) \tag{2.1}$$

is small. For these modified coefficients introduce

$$\tilde{\mathcal{A}} f(x) = \sum_{i \in Z \cap C} \Big[\sum_{j \in R_i} \tilde{\gamma}_j(x) x_i f_{jj}(x) \Big] + \tilde{\gamma}_i(x) x_i f_{ii}(x) + \sum_{j \in N_2} \tilde{\gamma}_j(x) x_j f_{jj}(x) + \sum_{j \in V} \tilde{b}_j(x) f_j(x), \tag{2.2}$$

and also define a constant coefficient operator

$$\begin{aligned} \mathcal{A}^0 f(x) &= \sum_{i \in Z \cap C} \Big[\sum_{j \in R_i} \gamma_j^0 x_i f_{jj}(x) + b_j^0 f_j(x) \Big] + \gamma_i^0 x_i f_{ii}(x) + b_i^0 f_i(x) \\ &\quad + \sum_{j \in N_2} \gamma_j^0 x_j f_{jj}(x) + b_j^0 f_j(x) \\ &\equiv \sum_{i \in Z \cap C} \mathcal{A}_i^1 + \sum_{j \in N_2} \mathcal{A}_j^2. \end{aligned} \tag{2.3}$$

As $b_j^0 \le 0$ and $\tilde{b}_j|_{x_j=0} \le 0$ is possible for $j \in N_1$ (recall we have modified $\tilde{b}_j$), the natural state space for the above generators is the larger

$$S^0 \equiv S(x^0) = \{x \in \mathbb{R}^d : x_j \ge 0 \text{ for all } j \notin N_1\}.$$

When modifying $\tilde{\gamma}_j$ and $\tilde{b}_j$ it is easy to extend them to this larger space, still ensuring all of the above properties of $\tilde{b}_j$ and $\tilde{\gamma}_j$. If ν_0 is a probability on S^0, a solution to the martingale problem $MP(\tilde{\mathcal{A}}, \nu_0)$ is a probability $\mathbb{P}$ on $C(\mathbb{R}_+, S^0)$ satisfying the obvious analogue of the definition given for $MP(\mathcal{A}, \nu)$. As we have $\mathcal{A}f(x) = \tilde{\mathcal{A}}f(x)$ for x near x^0, the localization in [SV] shows that Theorem 1.4 follows from:

Theorem 2.1. *Assume $\tilde{\gamma}_j : S(x^0) \to (0, \infty)$, $\tilde{b}_j : S(x^0) \to \mathbb{R}$ are bounded continuous and constant outside a compact set with $\tilde{b}_j > 0$ for $j \notin N_1$. For $j \leq d$, let $\gamma_j^0 > 0$, $b_j^0 \in \mathbb{R}$, $b_j^0 > 0$ if $j \notin N_1$, and*

$$M_0 = \max_{j \leq d}(\gamma_j^0, (\gamma_j^0)^{-1}, |b_j^0|) \vee \max_{j \notin N_1}(b_j^0)^{-1}. \tag{2.4}$$

There is an $\varepsilon_1(M_0) > 0$ so that if $\varepsilon_0 \leq \varepsilon_1(M_0)$, then for any probability ν on $S(x^0)$, there is a unique solution to $MP(\tilde{\mathcal{A}}, \nu)$.

Proof of reduction of Theorem 1.4 to Theorem 2.1. This proceeds as in the proof of Theorem 4 in [DP1]. The only change is that in Theorem 2.1 we are now assuming $\tilde{b}_j > 0$ and $b_j^0 > 0$ for all $j \notin N_1$, not just $\tilde{b}_j \geq 0$ on $\{x_j = 0\}$ for $j \notin N_1$ and $b_j^0 > 0$ for $j \in Z \cap (R \cup C)$ with $b_j^0 \geq 0$ for other values of $j \notin N_1$. If $b_j(x^0) > 0$ for all $j \in N_2$, then the proof of Theorem 4 in [DP1] in Case 1 applies without change. The strict positivity is needed (unlike [DP1]) to utilize Theorem 2.1. We therefore need only modify the argument in Case 2 of the proof of Theorem 4 in [DP1] so that it applies if $b_j(x^0) \leq 0$ for some $j \in N_2$. This means $x_j^0 > 0$ by our (stronger) Hypothesis 1.2 and the Girsanov argument given there now allows us to locally modify b_j so that $b_j(x^0) > 0$. The rest of the argument now goes through as before. $\square$

Turning to the proof of Theorem 2.1, existence is proved as in Theorem 1.1 of [ABBP]–instead of the comparison argument given there, one can use Tanaka's formula and (2.4) to see that solutions must remain in $S(x^0)$.

We focus on uniqueness from here on.

The operator $\mathcal{A}_j^2$ is the generator of a Feller branching diffusion with immigration. We denote its semigroup by Q_t^j. It will be easy to give an explicit representation for the semigroup P_t^i associated with $\mathcal{A}_i^1$ (see (3.2) below). An elementary argument shows that the martingale problem associated with $\mathcal{A}^0$ is well-posed and the associated diffusion has semigroup

$$P_t = \prod_{i \in Z \cap C} P_t^i \prod_{j \in N_2} Q_t^j, \tag{2.5}$$

and resolvent $R_\lambda = \int e^{-\lambda t} P_t \, dt$. Define a reference measure μ on S^0 by

$$\mu(dx) = \prod_{i \in Z \cap C} \Big[\prod_{j \in R_i} dx_j\Big] x_i^{b_i^0/\gamma_i^0 - 1} dx_i \times \prod_{j \in N_2} x_j^{b_j^0/\gamma_j^0 - 1} dx_j = \prod_{i \in Z \cap C} \mu_i \prod_{j \in N_2} \mu_j.$$

The norm on $L^2 \equiv L^2(S^0,\mu)$ is denoted by $\|\cdot\|_2$.

The key analytic bound we will need to carry out the Stroock-Varadhan perturbation analysis is the following:

Proposition 2.2. *There is a dense subspace $\mathcal{D}_0 \subset L^2$ and a $K(M_0) > 0$ such that $R_\lambda : \mathcal{D}_0 \to C_b^2(S^0)$ for all $\lambda > 0$ and*

$$\Big[\sum_{i\in Z\cap C}\Big[\sum_{j\in R_i}\|x_i(R_\lambda f)_{jj}\|_2\Big] + \|x_i(R_\lambda f)_{ii}\|_2\Big] + \Big[\sum_{j\in N_2}\|x_j(R_\lambda f)_{jj}\|_2\Big] + \Big[\sum_{j\in V}\|(R_\lambda f)_j\|_2\Big]$$
$$\le K\|f\|_2 \quad \text{for all } f\in\mathcal{D}_0 \text{ and } \lambda\ge 1. \tag{2.6}$$

Here are the other two ingredients needed to complete the proof of Theorem 2.1.

Proposition 2.3. *Let $\mathbb{P}$ be a solution of $MP(\tilde{\mathcal{A}},\nu)$ where $d\nu = \rho\, d\mu$ for some $\rho \in L^2$ with compact support and set $S_\lambda f = \mathbb{E}_{\mathbb{P}}\Big(\int_0^\infty e^{-\lambda t} f(X_t)\,dt\Big)$. If*

$$\varepsilon_0 \le (2K(M_0))^{-1} \wedge (48dM_0^5)^{-1}, \tag{2.7}$$

then for all $\lambda \ge 1$,

$$\|S_\lambda\| := \sup\{|S_\lambda f| : \|f\|_2 \le 1\} \le \frac{2\|\rho\|_2}{\lambda} < \infty.$$

Proposition 2.4. *Assume $\{\mathbb{P}^x : x\in S^0\}$ is a collection of probabilities on $C(\mathbb{R}_+, S^0)$ such that:*
(i) For each $x \in S^0$, $\mathbb{P}^x$ is a solution of $MP(\tilde{\mathcal{A}},\delta_x)$.
(ii) $(\mathbb{P}^x, X_t)$ is a Borel strong Markov process.
Then for any bounded measurable function f on S^0 and any $\lambda > 0$,

$$S_\lambda f(x) = \mathbb{E}^x\Big(\int_0^\infty e^{-\lambda t} f(X_t)\,dt\Big)$$

is a continuous function in $x \in S^0$.

Remark 2.5. Our proof of Proposition 2.4 will also show the strong Feller property of the resolvent for solutions to the original MP$(\mathcal{A},\nu)$ in Theorem 1.4–see Remark 6.2.

Assuming Propositions 2.2–2.4 the proof of Theorem 2.1 is then standard and quite similar to the proof of Proposition 2.1 in Section 7 of [ABBP]. Unlike [ABBP] the state space here is not compact, so we present the proof for completeness.

Proof of Theorem 2.1. Let $\mathbb{Q}_k, k = 1,2$, be solutions to $MP(\tilde{\mathcal{A}},\nu)$ where ν is as in Proposition 2.3 and define $S_\lambda^k f = \mathbb{E}_k\Big(\int e^{-\lambda t} f(X_t)\,dt\Big)$, where $\mathbb{E}_k$ denotes expectation with respect to $\mathbb{Q}_k$. Let $f \in C_b^2(S^0)$. The martingale problem shows that there is a local martingale M^f satisfying

$$f(X_t) = f(X_0) + M^f(t) + \int_0^t \tilde{\mathcal{A}} f(X_s)\, ds. \tag{2.8}$$

Note that for $t > 0$,

$$\begin{aligned}\mathbb{E}_k(\sup_{s\le t}|M^f(s)|) &\le 2\|f\|_\infty + \int_0^t \mathbb{E}_k(|\tilde{\mathcal{A}} f(X_s)|)\, ds \\ &\le 2\|f\|_\infty + c\int_0^t \mathbb{E}_k\Big(\sum_{j\notin N_1} X_s^j + 1\Big)\, ds < \infty,\end{aligned}$$

where the finiteness follows by considering the associated SDE for X^j and using the boundedness of $\tilde{b}_j$. This shows that M^f is a martingale under $\mathbb{Q}_k$. Let $g \in \mathcal{D}_0$. Multiply (2.8) by $\lambda e^{-\lambda t}$ integrate over t, take expectations (just as in (7.3) of [ABBP]), and set $f = R_\lambda g \in C_b^2$ to derive

$$S_\lambda^k g = \int R_\lambda g\, d\nu + S_\lambda^k((\tilde{\mathcal{A}} - \mathcal{A}^0)R_\lambda g).$$

Taking the difference of this equation when $k = 1, 2$, we obtain

$$|(S_\lambda^1 - S_\lambda^2)g| \le \|S_1^\lambda - S_\lambda^2\|\ \|(\tilde{\mathcal{A}} - \mathcal{A}^0)R_\lambda g\|_2 \le \|S_\lambda^1 - S_\lambda^2\|\varepsilon_0 K(M_0)\|g\|_2,$$

where we have used the definition of ε_0 (in (2.1)) and Proposition 2.2. Set $\varepsilon_1(M_0) = (2K(M_0))^{-1}$ to conclude $\|S_\lambda^1 - S_\lambda^2\| \le \frac{1}{2}\|S_\lambda^1 - S_\lambda^2\|$. Proposition 2.3 implies the above terms are finite for $\lambda \ge 1$ and so we have

$$\|S_\lambda^1 - S_\lambda^2\| = 0 \text{ for all } \lambda \ge 1. \tag{2.9}$$

To prove uniqueness we first use Krylov selection (Theorem 12.2.4 of [SV]) to see that it suffices to consider Borel strong Markov processes $((\mathbb{Q}_k^x)_{x\in S^0}, X_t)$, $k = 1, 2$, where $\mathbb{Q}_k^x$ solves $MP(\tilde{\mathcal{A}}, \delta_x)$, and to show that $\mathbb{Q}_1^x = \mathbb{Q}_2^x$ for all $x \in S^0$ (see the argument in the proof of Proposition 2.1 of [ABBP], but the situation here is a bit simpler as there is no killing). If S_λ^k are the resolvent operators associated with $(\mathbb{Q}_k^x, x \in S^0)$, then (2.9) implies that

$$\int S_\lambda^1 f(x)\rho(x)d\mu(x) = \int S_\lambda^2 f(x)\rho(x)d\mu(x)$$
$$\text{for all } f \in L^2, \text{compactly supported } \rho \in L^2, \text{ and } \lambda \ge 1.$$

For f and λ as above this implies $S_\lambda^1 f(x) = S_\lambda^2 f(x)$ for Lebesgue a.e. x and so for all x by Proposition 2.4. From this one deduces $\mathbb{Q}_1^x = \mathbb{Q}_2^x$ for all x (e.g., see Theorem VI.3.2 of [B98]). $\square$

It remains to prove Propositions 2.2–2.4. Propositions 2.3 and 2.4 follow along the lines of Propositions 2.3 and 2.4, respectively, of [ABBP], and are proved in Sections 5 and 6, respectively. There are some additional complications in the present setting. Most of the work, however, will go into the proof of Proposition 2.2 where a different approach than those in [ABBP] or [DP1] is followed. In [DP1] a canonical measure formula (Proposition 14 of that work) is used to represent and bound derivatives of the semigroups $P_t^i f(x)$ in (2.5) (see Lemma 3.8 below). This approach will be refined (see, e.g., Lemmas 3.11 and 7.1 below) to give good estimates on the derivatives of the the actual transition densities using an integration by parts formula. The formula will convert spatial derivatives on the semigroup or density into differences involving Poisson random variables which can be used to represent the process with semigroup P_t from which we are perturbing. The construction is described in Lemma 3.4 below. The integration by parts formula underlies the proof of Lemma 7.1 and is explicitly stated in the simpler setting of first order derivatives in Proposition 8.1.

In [ABBP] we differentiate an explicit eigenfunction expansion for the resolvent of a killed squared Bessel process to get an asymptotically orthogonal expansion. We have less explicit information about the semigroup P_t of $\mathcal{A}^0$ and so instead use Cotlar's Lemma (Lemma 2.13 below), to get a different asymptotically orthogonal expansion for the derivatives of the resolvent R_λ–see the proof of Proposition 2.2 later in this section.

Notation 2.6. Set $\underline{d} = |Z \cap C| + |N_2| = |N_1^c| \le d$. Here $|\cdot|$ denotes cardinality.

Convention 2.7. All constants appearing in statements of results concerning the semigroup P_t and its associated process may depend on d and the constants $\{b_j^0, \gamma_j^0 : j \le d\}$, but, if M_0 is as in (2.4), these constants will be uniformly bounded for $M_0 \le M$ for any $M > 0$.

We state an easy result on transition densities which will be proved in Section 3.

Proposition 2.8. *The semigroup $(P_t, t \ge 0)$, has a jointly continuous transition density $p_t : S^0 \times S^0 \to [0, \infty)$, $t > 0$. This density, $p_t(x, y)$ is C^3 on S^0 in each variable (x or y) separately, and satisfies the following:*
(a) $p_t(y, x) = \hat{p}_t(x, y)$, where $\hat{p}_t$ is the transition density associated with $\hat{\mathcal{A}}^0$ with parameters $\hat{\gamma}^0 = \gamma^0$ and

$$\hat{b}_j^0 = \begin{cases} -b_j^0 & \text{if } j \in N_1 \\ b_j^0 & \text{otherwise.} \end{cases}$$

In particular

$$\int p_t(x, y)\mu(dy) = \int p_t(x, y)\mu(dx) = 1. \tag{2.10}$$

(b) If D_x^n is any nth order partial differential operator in $x \in S^0$ and $0 \le n \le 3$, then

$$\sup_x |D_x^n p_t(x,y)| \le c_{2.8} t^{-n-(d-\underline{d})-\sum_{i\notin N_1} b_i^0/\gamma_i^0} \prod_{j\in N_2} [1+(y_j/t)^{1/2}] \quad \textit{for all } y \in S^0, \quad (2.11)$$

and

$$\sup_y |D_y^n p_t(x,y)| \le c_{2.8} t^{-n-(d-\underline{d})-\sum_{i\notin N_1} b_i^0/\gamma_i^0} \prod_{j\in N_2} [1+(x_j/t)^{1/2}] \quad \textit{for all } x \in S^0. \quad (2.12)$$

(c) If $0 \le n \le 3$,

$$\sup_x \int |D_y^n p_t(x,y)| d\mu(y) \le c_{2.8} t^{-n}. \quad (2.13)$$

(d) For all bounded Borel $f : S^0 \to \mathbb{R}$, $P_t f \in C_b^2(S^0)$, and for $n \le 2$ and D_x^n as in (b),

$$D_x^n P_t f(x) = \int D_x^n p_t(x,y) f(y) d\mu(y) \quad (2.14)$$

and

$$\|D_x^n P_t f\|_\infty \le c_{2.8} t^{-n} \|f\|_\infty. \quad (2.15)$$

Notation 2.9. Throughout $\tilde{D}_x$ will denote one of the following first or second order differential operators:

$$D_{x_j},\ j \le d,\ x_i D^2_{x_j x_j},\ i \in Z \cap C, j \in R_i, \text{ or } x_j D^2_{x_j x_j}, j \notin N_1.$$

A deeper result is the following bound which sharpens Proposition 2.8.

Proposition 2.10. *For $\tilde{D}_x$ as above and all $t > 0$,*

$$\sup_x \int |\tilde{D}_x p_t(x,y)| \mu(dy) \le c_{2.10} t^{-1}. \quad (2.16)$$

$$\sup_y \int |\tilde{D}_x p_t(x,y)| \mu(dx) \le c_{2.10} t^{-1}. \quad (2.17)$$

This is proved in Section 4 below. The case $\tilde{D}_x = x_j D^2_{x_j x_j}$ for $j \in Z \cap C$ will be the most delicate.

For $\tilde{D}$ as in Notation 2.9 and $t > 0$, define an integral operator $T_t = T_t(\tilde{D})$ by

$$T_t f(x) = \int \tilde{D}_x p_t(x,y) f(y) \mu(dy), \text{ for } f : S^0 \to \mathbb{R} \text{ for which the integral exists.} \quad (2.18)$$

By (d) above T_t is a bounded operator on L^∞, but we will study these operators on $L^2(S^0, \mu)$.

Lemma 2.11. *Assume $K : S^0 \times S^0 \to \mathbb{R}$ is a measurable kernel on S^0.*
(a) If

$$\Big\| \int |K(\cdot,y)|\mu(dy)\Big\|_\infty \le c_1 \text{ and } \Big\| \int |K(x,\cdot)|\mu(dx)\Big\|_\infty \le c_2,$$

then $Kf(x) = \int K(x,y)f(y)\mu(dy)$ is a bounded operator on L^2 with norm $\|K\| \le \sqrt{c_1c_2}$.

(b) If

$$|K|^2 := \Big\| \int\int |K(x,y')|\,|K(x,\cdot)|\mu(dy')\mu(dx)\Big\|_\infty < \infty,$$

then $Kf(\cdot) = \int K(\cdot,y)f(y)\mu(dy)$ is bounded on $L^2(\mu)$ and its norm satisfies $\|K\| \le |K|$.

Proof. (a) is well known; see [B95], Theorem IV.5.1, for example, for a proof.
(b) Let $K^*K(\cdot,\cdot)$ denote the integral kernel associated with the operator K^*K. The hypothesis implies that

$$\Big\| \int |K^*K(y,\cdot)|\,\mu(dy)\Big\|_\infty \le |K|^2.$$

By (a) and the fact that K^*K is symmetric, we have $\|K\|^2 = \|K^*K\| \le |K|^2$. □

Corollary 2.12. *(a) For $f \in L^2(\mu)$ and $t, \lambda > 0$, $\|P_tf\|_2 \le \|f\|_2$ and $\|R_\lambda f\|_2 \le \lambda^{-1}\|f\|_2$.*
(b) If $g \in C_b^2(S^0) \cap L^2(\mu)$ and $\mathcal{A}^0 g \in L^2(\mu)$, then $t \to P_tg$ is continuous in $L^2(\mu)$.

Proof. (a) This is immediate from Lemma 2.11(a) and (2.10).
(b) By $(MP(\mathcal{A}^0,\nu))$, if $0 \le s < t$, then

$$\begin{aligned}\|P_tg - P_sg\|_2 &= \Big\| \int_s^t \mathcal{A}^0 P_r g\, dr\Big\|_2 \\ &\le \int_s^t \|P_r\mathcal{A}^0 g\|_2 dr \\ &\le (t-s)\|\mathcal{A}^0 g\|_2.\end{aligned}$$

We have used (a) in the last line. □

Proposition 2.10 allows us to apply Lemma 2.11 to T_t and conclude

$$T_t \text{ is a bounded operator on } L^2 \text{ with norm } \|T_t\| \le c_{2.10}t^{-1} \tag{2.19}$$

Unfortunately this is not integrable near $t = 0$ and so we can not integrate this bound to prove Proposition 2.2. We must take advantage of some cancellation in the integral over t and this is where we use Cotlar's Lemma:

Lemma 2.13 (Cotlar's Lemma). *Assume $\{U_j : j \in \mathbb{Z}_+\}$ are bounded operators on $L^2(\mu)$ and $\{a(j) : j \in \mathbb{Z}\}$ are non-negative real numbers such that*

$$\|U_j U_k^*\| \vee \|U_j^* U_k\| \le a(j-k)^2 \quad \textit{all } j,k. \tag{2.20}$$

Then

$$\Big\|\sum_{j=0}^{N} U_j\Big\| \le A := \sum_{j=-\infty}^{\infty} a(j) \quad \textit{for all } N.$$

Proof. See, e.g., Lemma XI.4.1 in [T]. □

The subspace $\mathcal{D}_0$ in Proposition 2.2 will be

$$\mathcal{D}_0 = \{P_{2^{-j}} g : j \in \mathbb{N}, g \in C_b^2(S^0) \cap L^2(\mu), \mathcal{A}^0 g \in L^2(\mu)\}. \tag{2.21}$$

As we can take $g \in C^2$ with compact support, denseness of $\mathcal{D}_0$ in L^2 follows from Corollary 2.12(b). To see that $\mathcal{D}_0$ is a subspace, let $P_{2^{-j_i}} g_i \in \mathcal{D}_0$ for $i = 1,2$ with $j_2 \ge j_1$. If $\tilde{g}_1 = P_{2^{-j_1} - 2^{-j_2}} g_1$, then $\tilde{g}_1$ is in L^2 by Corollary 2.12 (a) and also in $C_b^2(S^0)$ by Proposition 2.8(d). In addition,

$$\|\mathcal{A}^0 \tilde{g}_1\|_2 = \|P_{2^{-j_1} - 2^{-j_2}} \mathcal{A}^0 g_1\|_2 \le \|\mathcal{A}^0 g_1\|_2 < \infty,$$

where we have used Corollary 2.12(a) again. Hence $P_{2^{-j_1}} g_1 = P_{2^{-j_2}} \tilde{g}_1$ where $\tilde{g}_1$ satisfies the same conditions as g_1. Therefore

$$P_{2^{-j_1}} g_1 + P_{2^{-j_2}} g_2 = P_{2^{-j_2}}(\tilde{g}_1 + g_2) \in \mathcal{D}_0.$$

We show below how Cotlar's Lemma easily reduces Proposition 2.2 to the following result.

Proposition 2.14. *There is an $\eta > 0$ and $c_{2.14}$ so that if $\tilde{D}_x$ is any of the operators in Notation 2.9, then*

$$\begin{aligned} &\|T_s^* T_t f\|_2 \le c_{2.14} s^{-1-\eta/2} t^{-1+\eta/2} \|f\|_2 \textit{ and} \\ &\|T_s T_t^* f\|_2 \le c_{2.14} s^{-1-\eta/2} t^{-1+\eta/2} \|f\|_2 \\ &\qquad\qquad \textit{for any } 0 < t \le s \le 2, \textit{ and any bounded Borel } f \in L^2(\mu). \end{aligned} \tag{2.22}$$

Assuming this result, we can now give the

Proof of Proposition 2.2. Fix a choice of $\tilde{D}_x$ (recall Notation 2.9), let $\lambda \ge 1$, and for $k \in \mathbb{Z}_+$, define

$$U_k = U_k(\tilde{D}_x) = \int_{2^{-k}}^{2^{-k+1}} e^{-\lambda s} T_s \, ds.$$

By (2.19), U_k is bounded operator on L^2. Moreover if $k > j$ then

$$\begin{aligned}
\|U_j^* U_k f\|_2 &= \Big\| \int_{2^{-j}}^{2^{-j+1}} \Big[\int_{2^{-k}}^{2^{-k+1}} e^{-\lambda(s+t)} T_s^* T_t f\, dt\Big]\, ds \Big\|_2 \\
&\le \int_{2^{-j}}^{2^{-j+1}} \Big[\int_{2^{-k}}^{2^{-k+1}} c_{2.14} s^{-1-\eta/2} t^{-1+\eta/2}\, dt\Big]\, ds \|f\|_2 \\
&\le c_{2.14} 2^{-(\eta/2)(k-j)} \|f\|_2.
\end{aligned}$$

If $k = j$ a similar calculation where the contributions to the integral from $\{s \ge t\}$ and $\{t \ge s\}$ are evaluated separately shows

$$\|U_j^* U_j f\|_2 \le c_{2.14} \|f\|_2.$$

Cotlar's Lemma therefore shows that

$$\Big\| \sum_{j=0}^{N} U_j \Big\| \le \sqrt{c_{2.14}}\, 2(1 - 2^{-\eta/4})^{-1} := C(\eta) \text{ for all } N. \tag{2.23}$$

Now let $f = P_{2^{-N}} g \in \mathcal{D}_0$ where g is as in the definition of $\mathcal{D}_0$, and for $M \in \mathbb{N}$ set $h = h_M = P_{2^{-N}(1-2^{-M})} g$. Then

$$\begin{aligned}
\tilde{D}_x R_\lambda f &= \tilde{D}_x \int_0^\infty e^{-\lambda t} P_{t+2^{-M-N}} h\, dt \\
&= \exp(\lambda 2^{-M-N}) \Big[\tilde{D}_x \Big[\int_{2^{-N-M}}^{2} e^{-\lambda u} P_u h\, du\Big] + \tilde{D}_x \Big[\int_2^\infty e^{-\lambda u} P_u h\, du\Big]\Big] \\
&= \exp(\lambda 2^{-M-N}) \Big[\sum_{j=0}^{M+N} U_j h + \sum_{k=1}^{\infty} e^{-\lambda k} \int_{k+1}^{k+2} e^{-\lambda(u-k)} T_{u-k}(P_k h)\, du \Big].
\end{aligned}$$

In the last line the bound (2.15) allows us to differentiate through the t integral and (2.14) allows us to differentiate through the $\mu(dy)$ integral and conclude $\tilde{D}_x P_u h = T_u h$. A change of variables in the above now gives

$$\tilde{D}_x R_\lambda f = \exp(\lambda 2^{-M-N}) \Big[\sum_{j=0}^{M+N} U_j h + \sum_{k=1}^{\infty} e^{-\lambda k} U_0(P_k h) \Big].$$

So (2.23) shows that

$$\begin{aligned}
\|\tilde{D}_x R_\lambda f\|_2 &\le \exp(\lambda 2^{-M-N}) C(\eta) \Big[\|h_M\|_2 + \sum_{k=1}^{\infty} e^{-\lambda k} \|P_k h_M\|_2 \Big] \\
&\le \exp(\lambda 2^{-M-N}) C(\eta) (1 - e^{-\lambda})^{-1} \|h_M\|_2.
\end{aligned} \tag{2.24}$$

Corollary 2.12(b) shows that $\|h_M\|_2 = \|P_{2^{-N}-2^{-N-M}}g\|_2 \to \|f\|_2$ as $M \to \infty$. Now let $M \to \infty$ in (2.24) to conclude

$$\|\tilde{D}_x R_\lambda f\|_2 \le C(\eta)(1-e^{-\lambda})^{-1}\|f\|_2,$$

and the result follows. □

For Proposition 2.14, an easy calculation shows that for $0 < s \le t$,

$$T_s^* T_t f(x) = \int K_{s,t}^{(1)}(x,y)f(y)d\mu(y) \text{ and } T_s T_t^* f(x) = \int K_{s,t}^{(2)}(x,y)f(y)d\mu(y), \tag{2.25}$$

where

$$K_{s,t}^{(1)}(y,z) = \int \tilde{D}_x p_s(x,y)\tilde{D}_x p_t(x,z)d\mu(x), \tag{2.26}$$

and

$$K_{s,t}^{(2)}(x,y) = \int \tilde{D}_x p_s(x,z)\tilde{D}_y p_t(y,z)d\mu(z). \tag{2.27}$$

Lemma 2.11(b) shows that (2.22) follows from

$$\sup_y \int\int |K_{s,t}^{(i)}(x,y')|\,|K_{s,t}^{(i)}(x,y)|\,d\mu(y')\,d\mu(x) \le c_{2.14}s^{-2-\eta}t^{-2+\eta}$$
$$\text{for all } 0 < t \le s \le 2 \text{ and } i = 1,2. \tag{2.28}$$

This calculation will reduce fairly easily to the case N_2 empty and $Z \cap C$ a singleton (see the proof of Proposition 2.14 at the end of Section 4 below). Here there are essentially 4 distinct choices of $\tilde{D}_x$, making our task one of bounding 8 different 4-fold integrals involving first and second derivatives of the transition density $p_t(x,y)$. Fairly explicit formulae (see (4.7)–(4.9)) are available for all the derivatives except those involving the unique index j in $Z \cap C$, and as a result Proposition 2.14 is easy to prove for all derivatives but those with respect to j (Proposition 4.3). Even here the first order derivatives are easily handled, leaving $\tilde{D}_x = x_j D_{x_j x_j}$. This is the reason for most of the rather long calculations in Section 7. In the special case $d = 2$, of paramount importance to [DGHSS], one can avoid this case using the identity $\mathcal{A}^0 R_\lambda f = \lambda R_\lambda f - f$, as is discussed in Section 8.

We give a brief outline of the rest of the paper. Section 3 studies the transition density associated with the resolvent in Proposition 2.2 for the key special case when $Z \cap C$ is a singleton and $N_2 = \emptyset$. This includes the canonical measure formulae for these densities (Lemmas 3.4 and 3.11) and the proof of Proposition 2.8. In addition some important formulae for Feller branching processes with immigration, conditional on their value at time t, are proved (see Lemmas 3.2, 3.14 and Corollary 3.15). In Section 4, the proofs of Propositions 2.14 and 2.10 are reduced to a series of technical bounds on the derivatives of the transition densities (Lemmas 4.5, 4.6 and 4.7). Most of the work here is in the setting of the key special case considered in Section 3, and then at the end of Section 4 we show how the general case of Proposition 2.14 follows fairly easily thanks to the product structure in (2.5). Propositions 2.3 and 2.4 are proved in Sections 5 and 6, respectively. Lemmas 4.5–4.7 are finally proved in Section 7, thus completing the proof of Theorem 2.1 (and 1.4). The key inequality in Section 7 is Lemma 7.1 which comes from the integration by parts identity for the dominant term (see Proposition 8.1 for a simple special case). In Section 8 we describe how all of this becomes considerably simpler in the 2-dimensional setting required in [DGHSS].

3. The basic semigroups. Unless otherwise indicated, in this section we work with the generator in (2.3) where $Z \cap C = \{d\}$ and $N_2 = \emptyset$. Taking $d = m+1$ to help distinguish this special setting, this means we work with the generator

$$\mathcal{A}^1 = \Big[\sum_{j=1}^m b_j^0 \frac{\partial}{\partial x_j} + \gamma_j^0 x_{m+1} \frac{\partial^2}{\partial x_j^2}\Big] + b_{m+1}^0 \frac{\partial}{\partial x_{m+1}} + \gamma_{m+1}^0 x_{m+1} \frac{\partial^2}{\partial x_{m+1}^2},$$

with semigroup P_t on the state space $S_m = \mathbb{R}^m \times \mathbb{R}_+$ ($m \in \mathbb{N}$). Here we write $b = b_{m+1}^0$, $\gamma = \gamma_{m+1}^0$ and assume

$$\gamma_j^0 > 0,\ b_j^0 \in \mathbb{R} \quad \text{for } j \le m, \qquad \text{and} \qquad \gamma > 0, b > 0.$$

Our Convention 2.7 on constants therefore means:

Convention 3.1. Constants appearing in statements of results may depend on m and $\{b_j^0, \gamma_j^0 :\ j \le m+1\}$. If

$$M_0 = M_0(\gamma^0, b^0) := \max_{i \le m+1} (\gamma_i^0 \vee (\gamma_i^0)^{-1} \vee |b_i^0|) \vee (b_{m+1}^0)^{-1} < \infty,$$

then these constants will be uniformly bounded for $M_0 \le M$ for any fixed $M > 0$.

Note that $M_0 \ge 1$.

It is easy to see that the martingale problem $MP(\mathcal{A}^1, \nu)$ is well-posed for any initial law ν on S_m. In fact, we now give an explicit formula for P_t. Let $X_t = (X_t^{(1)}, \ldots, X_t^{(m+1)})$ be a solution to this martingale problem. By considering the associated SDE, we see that $X^{(m+1)}$ is a Feller branching diffusion (with immigration) with generator

$$\mathcal{A}_0' = b\frac{d}{dx} + \gamma x \frac{d^2}{dx^2}, \tag{3.1}$$

and is independent of the driving Brownian motions of the first m coordinates. Let $\mathbb{P}_{x_{m+1}}$ be the law of $X^{(m+1)}$ starting at x_{m+1} on $C(\mathbb{R}_+, \mathbb{R}_+)$. By conditioning on X^{m+1} we see that the first m coordinates are then a time-inhomogeneous Brownian motion. Therefore if $I_t = \int_0^t X_s^{(m+1)}\,ds$ and $p_t(z) = (2\pi t)^{-1/2} e^{-z^2/2t}$, then (see (20) in [DP1])

$$P_t f(x_1, \ldots, x_{m+1}) = \mathbb{E}_{x_{m+1}}\Big[\int f(y_1, \ldots, y_m, X_t^{(m+1)}) \prod_{j=1}^m p_{2\gamma_j^0 I_t}(y_j - x_j - b_j^0 t)\,dy_j\Big]. \quad (3.2)$$

If $x = (x_1, \ldots, x_{m+1}) = (x^{(m)}, x_{m+1}) \in S_m$, let

$$\mu(dx) = x_{m+1}^{\frac{b}{\gamma}-1} dx = dx^{(m)} \mu_{m+1}(dx_{m+1}).$$

Recall (see, e.g., (2.2) of [BP]) that $X^{(m+1)}$ has a symmetric density $q_t = q_t^{b,\gamma}(x,y)$ $(x, y \geq 0)$ with respect to $\mu_{m+1}(dy)$, given by

$$q_t^{b,\gamma}(x,y) = (\gamma t)^{-b/\gamma} \exp\Big\{\frac{-x-y}{\gamma t}\Big\}\Big[\sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+b/\gamma)}\Big(\frac{x}{\gamma t}\Big)^m \Big(\frac{y}{\gamma t}\Big)^m\Big], \quad (3.3)$$

and associated semigroup $Q_t = Q_t^{b,\gamma}$. Let

$$\bar{r}_t(x_{m+1}, y_{m+1}, dw) = \mathbb{P}_{x_{m+1}}\Big(\int_0^t X_s^{(m+1)}\,ds \in dw | X_t^{(m+1)} = y_{m+1}\Big),$$

or more precisely a version of this collection of probability laws which is symmetric in (x_{m+1}, y_{m+1}) and such that $(x_{m+1}, y_{m+1}) \to \bar{r}_t(x_{m+1}, y_{m+1}, dw)$ is a jointly continuous map with respect to the weak topology on the space of probability measures. The existence of such a version follows from Section XI.3 of [RY]. Indeed, Corollary 3.3 of the above states that if $\gamma = 2$, then

$$\begin{aligned} L(\lambda, x, y) &:= \int \exp\Big\{\frac{-\lambda^2}{2} w\Big\} \bar{r}_1(x, y, dw) \\ &= \begin{cases} \frac{\lambda}{\sinh\lambda} \exp\Big\{\Big(\frac{x+y}{2}\Big)(1 - \lambda\coth\lambda)\Big\} I_\nu\Big(\frac{\lambda\sqrt{xy}}{\sinh\lambda}\Big) / I_\nu(\sqrt{xy}) & \text{if } xy > 0; \\ \Big(\frac{\lambda}{\sinh\lambda}\Big)^{b/2} \exp\Big\{\Big(\frac{x+y}{2}\Big)(1 - \lambda\coth\lambda)\Big\} & \text{if } xy = 0. \end{cases} \end{aligned} \quad (3.4)$$

Here $\nu = \frac{b}{2} - 1$ and $I_\nu(z) = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{1}{\Gamma(m+\nu+1)} \Big(\frac{z}{2}\Big)^{2m+\nu}$ is the modified Bessel function of the first kind of index $\nu > -1$. The continuity and symmetry of L in (x, y) gives the required continuous and symmetric version of $\bar{r}_1(x_{m+1}, y_{m+1})$. A scaling argument (see the proof of Lemma 3.2 below) gives the required version of $\bar{r}_t$ for general $\gamma > 0$.

Now define $r_t(x_{m+1}, y_{m+1}, dw) = q_t^{b,\gamma}(x_{m+1}, y_{m+1}) \bar{r}_t(x_{m+1}, y_{m+1}, dw)$, so that

$$(x_{m+1}, y_{m+1}) \to r_t(x_{m+1}, y_{m+1}, dw) \text{ is symmetric and weakly continuous} \quad (3.5)$$

and

$$\int\int \psi(y_{m+1},w)r_t(x_{m+1},y_{m+1},dw)\mu_{m+1}(dy_{m+1})$$
$$= \mathbb{E}_{x_{m+1}}(\psi(X_t^{(m+1)},I_t)) \text{ for all } x_{m+1}\geq 0 \text{ and Borel } \psi:\mathbb{R}_+^2\to\mathbb{R}_+. \tag{3.6}$$

(Weakly continuous means continuity with respect to the weak topology on the space of probability measures.) Combine (3.6) and (3.2) to conclude that X has a transition density with respect to $\mu(dy)$ given by

$$\begin{aligned} p_t(x,y) &= \int_0^\infty \prod_{j=1}^m p_{2\gamma_j^0 w}(y_j - x_j - b_j^0 t) r_t(x_{m+1},y_{m+1},dw) \\ &= p_t(x^{(m)},x_{m+1},y^{(m)},y_{m+1}) = p_t(x^{(m)}-y^{(m)},x_{m+1},0,y_{m+1}) \\ &\equiv p_t^0(x^{(m)}-y^{(m)},x_{m+1},y_{m+1}); \end{aligned} \tag{3.7}$$

here we write $x = (x^{(m)},x_{m+1})$ and similarly for y, we write 0 for $0^{(m)}$, and we use translation invariance to get the third equality. In particular we see that conditional on the last component at time t, the first m components are translation invariant. Moreover if we set $b^0=(b_1^0,\dots,b_m^0)\in\mathbb{R}^m$ and write $p_t^{b^0}(x,y)$ for $p_t(x,y)$, then (3.5) implies

$$p_t^{b^0}(x,y) = p_t^{-b^0}(y,x) \text{ for all } x,y\in S_m. \tag{3.8}$$

The next result is a refinement of Lemma 7(b) of [DP1].

Lemma 3.2. *For any $p>0$ there is a $c_{3.2}(p)$ such that for all $x,y\geq 0$ and $t>0$,*

$$\begin{aligned} \mathbb{E}_x\Big(\Big(\int_0^t X_s^{(m+1)}ds\Big)^{-p}|X_t^{(m+1)}=y\Big) &\equiv \int w^{-p}\bar{r}_t(x,y,dw) \\ &\leq c_{3.2}(x+y+t)^{-p}t^{-p}. \end{aligned}$$

Proof. Assume first $\gamma=2$, $t=1$ so that we may use (3.4) to conclude (recall $\nu=\frac{b}{2}-1$)

$$\frac{L(\lambda,x,y)}{L(\lambda,x,0)} = \begin{cases} \left(\frac{\lambda}{\sinh\lambda}\right)^{-\nu} \frac{I_\nu\left(\frac{\lambda}{\sinh\lambda}\sqrt{xy}\right)}{I_\nu(\sqrt{xy})} \exp\{\frac{y}{2}(1-\lambda\coth\lambda)\} & \text{if } x>0; \\ \exp\{\frac{y}{2}(1-\lambda\coth\lambda)\} & \text{if } x=0. \end{cases}$$

A bit of calculus shows

$$\lambda\coth\lambda\geq 1,\ \alpha(\lambda) := \frac{\lambda}{\sinh\lambda}\in[0,1] \text{ for all } \lambda\geq 0, \tag{3.9}$$

with the first inequality being strict if $\lambda > 0$. The above series expansion shows that $I_\nu(\alpha z) \le \alpha^\nu I_\nu(z)$ for all $z \ge 0,\ \alpha \in [0,1]$, and so using (3.9) in the above ratio bound, we get

$$L(\lambda, x, y) \le L(\lambda, x, 0) \text{ for all } \lambda, x, y \ge 0. \tag{3.10}$$

We have

$$\begin{aligned}
\int w^{-p}\bar{r}_1(x,y,dw) &= 2p\int_0^\infty \bar{r}_1(x,y,[0,u^2])u^{-2p-1}du \\
&\le 2p\sqrt{e}\int_0^\infty L(u^{-1},x,y)u^{-2p-1}du \\
&\le 2p\sqrt{e}\int_0^\infty L(u^{-1},x,0)u^{-2p-1}du \quad \text{(by (3.10))} \\
&= c_p\int_0^\infty \Big(\frac{u^{-1}}{\sinh u^{-1}}\Big)^{b/2}\exp\{\frac{x}{2}(1-u^{-1}\coth u^{-1})\}u^{-2p-1}du \\
&= c_p\int_0^\infty \Big(\frac{w}{\sinh w}\Big)^{b/2}\exp\{\frac{-x}{2}(w\coth w-1)\}w^{2p-1}dw \\
&\le c_p\Big[\int_0^1 \exp\{-cxw^2\}w^{2p-1}dw \\
&\qquad + \int_1^\infty \exp\{-cxw - wb/2\}2^{b/2}w^{b/2+2p-1}dw\Big],
\end{aligned}$$

where in the last line $c > 0$ and we have used (3.9), $\inf_{0\le w\le 1}\frac{w\coth w-1}{w^2} = c_1 > 0$, and $\inf_{w\ge 1}\frac{w\coth w-1}{w} = c_2 > 0$. For $x \le 1$ we may bound the above by (recall Convention 3.1)

$$c_p\Big[\int_0^1 w^{2p-1}\,dw + \int_1^\infty (e^{-w}2w)^{b/2}w^{2p-1}\,dw\Big] \le c_1(p),$$

and for $x \ge 1$ we may, using $(2we^{-w})^{b/2} \le 1$ for $w \ge 1$, bound it by

$$c_p\Big[\int_0^1 \exp\{-cxw^2\}w^{2p-1}dw + \int_1^\infty e^{-cxw}w^{2p-1}dw\Big] \le c_2(p)x^{-p}.$$

These bounds show that $\int w^{-p}\bar{r}_1(x,y,dw) \le c(p)(1+x)^{-p}$ and so by symmetry in x and y we get

$$\int w^{-p}\bar{r}_1(x,y,dw) \le c(p)(1+x+y)^{-p} \text{ for all } x, y \ge 0.$$

For general γ and t, $\hat{X}_s = \frac{2}{t\gamma}X_{ts}$ is as above with $\hat{\gamma} = 2$ and $\hat{b} = \frac{2b}{\gamma}$. We have $\int_0^t X_s ds = \left(\frac{t^2\gamma}{2}\right)\int_0^1 \hat{X}_u\,du$, and so, using the above case,

$$\begin{aligned}
\int w^{-p}\bar{r}_t^{b,\gamma}(x,y,dw) &= \Big(\frac{t^2\gamma}{2}\Big)^{-p}\int w^{-p}\bar{r}_1^{\hat{b},2}\Big(\frac{2x}{t\gamma},\frac{2y}{t\gamma},dw\Big) \\
&\le c(p)t^{-p}\Big(\frac{t\gamma}{2}\Big)^{-p}\Big(1+\frac{2x}{t\gamma}+\frac{2y}{t\gamma}\Big)^{-p} \\
&\le c(p)t^{-p}(t+x+y)^{-p}.
\end{aligned}$$

$\square$

We observe that there exist c_1, c_2 (recall Convention 3.1 is in force) such that

$$c_1 m^{b/\gamma-1} m! \le \Gamma(m+b/\gamma) \le c_2 m^{b/\gamma-1} m! \text{ for all } m \in \mathbb{N}. \tag{3.11}$$

To see this, suppose first $b/\gamma \equiv r \ge 1$ and use Jensen's inequality to obtain

$$\frac{\Gamma(m+r)}{m!} = \int x^r x^{m-1} e^{-x} \frac{dx}{\Gamma(m)} m^{-1} \ge \Big(\int x \cdot x^{m-1} e^{-x} \frac{dx}{\Gamma(m)}\Big)^r m^{-1} = m^{r-1}.$$

Next suppose $r \in [1,2]$ and again use Jensen's inequality to see that

$$\begin{aligned}\frac{\Gamma(m+r)}{m!} &= \int x^{r-1} x^m e^{-x} \frac{dx}{\Gamma(m+1)} \\ &\le \Big(\int x \cdot x^m e^{-x} \frac{dx}{\Gamma(m+1)}\Big)^{r-1} = (m+1)^{r-1} \le 2m^{r-1}.\end{aligned}$$

These two inequalities imply (3.11) by using the identity $\Gamma(m+r+1) = (m+r)\Gamma(m+r)$ a finite number of times.

Lemma 3.3. *There is a $c_{3.3}$ so that for all $t > 0$, $x_{m+1}, y_{m+1} \ge 0$:*
(a) $q_t^{b,\gamma}(x_{m+1}, y_{m+1}) \le c_{3.3}[t^{-b/\gamma} + 1_{(b/\gamma<1/2)}(x_{m+1} \wedge y_{m+1})^{1/2-b/\gamma} t^{-1/2}]$.

(b) For all $t > 0$, $(x, y) \to p_t(x, y)$ is continuous on S_m^2 and

$$\sup_{x,y\in S_m} p_t(x, y) \le c_{3.3} t^{-m-b/\gamma}.$$

(c) $\mathbb{E}_{x_{m+1}}(\exp(-\lambda X_t^{(m+1)})) = (1 + \lambda\gamma t)^{-b/\gamma} \exp(-x_{m+1}\lambda/(1 + \lambda\gamma t))$ for all $\lambda > -(\gamma t)^{-1}$.
(d) If $0 < p < b/\gamma$ then

$$\mathbb{E}_{x_{m+1}}((X_t^{(m+1)})^{-p}) \le c_{3.3}\Big(\frac{p}{(b/\gamma) - p} + 1\Big)(x_{m+1} + t)^{-p}.$$

(e) $\mathbb{E}_{x_{m+1}}((X_t^{(m+1)})^2) \le c_{3.3}(x_{m+1} + t)^2$.
(f) For any $p > 0$, $\mathbb{E}_{x_{m+1}}\Big(\Big(\int_0^t X_s^{(m+1)} ds\Big)^{-p}\Big) \le c_{3.2}(p) t^{-p}(t + x_{m+1})^{-p}$.
(g) $\sup_{y\ge0}\Big|\int_0^\infty (x - y) x^{b/\gamma} D_x^2 q_t(x, y) dx\Big| \le c_{3.3}$.

Proof. (a) If $q(x, y) = e^{-x-y} \sum_{m=0}^\infty \frac{(xy)^m}{m!\Gamma(m+b/\gamma)}$, then $q_t(x, y) = (\gamma t)^{-b/\gamma} q(x/\gamma t, y/\gamma t)$ and it suffices to show

$$q(x, y) \le c_2(1 + 1_{(b/\gamma<1/2)}(x \wedge y)^{1/2-b/\gamma}) \text{ for all } x, y \ge 0. \tag{3.12}$$

By (3.11) and Stirling's formula we have

$$\begin{aligned}
q(x,y) &\le c_1^{-1} e^{-x-y} \sum_{m=1}^{\infty} \frac{(2m)!}{m!m!} 2^{-2m} m^{1-b/\gamma} \frac{(2\sqrt{xy})^{2m}}{(2m)!} + e^{-x-y}\Gamma(b/\gamma)^{-1} \\
&\le c\Big[1 + e^{-x-y+2\sqrt{xy}} \sum_{m=1}^{\infty} m^{1/2-b/\gamma} \frac{(2\sqrt{xy})^m}{m!} e^{-2\sqrt{xy}}\Big] \\
&\le c\Big[1 + e^{-(\sqrt{x}-\sqrt{y})^2} (2\sqrt{xy}+1)^{1/2-b/\gamma}\Big],
\end{aligned}$$

where in the last line we used an elementary Poisson expectation calculation (for $b/\gamma \ge 1/2$ see Lemma 3.3 of [BP]). If $b/\gamma \ge 1/2$, the above is bounded and (3.12) is immediate. Assume now that $p = 1/2 - b/\gamma > 0$ and $x \ge y$. Then the above is at most

$$\begin{aligned}
c(1 + e^{-(\sqrt{x}-\sqrt{y})^2}(\sqrt{x}\sqrt{y})^p) &\le c(1 + \sqrt{y}^p(\sqrt{y}^p + (\sqrt{x}-\sqrt{y})^p)e^{-(\sqrt{x}-\sqrt{y})^2}) \\
&\le c(1 + y^p + y^{p/2}) \le c(1+y^p).
\end{aligned}$$

This proves (3.12) and hence (a).

(b) The continuity follows easily from (3.7), the continuity of $q_t(\cdot,\cdot)$, the weak continuity of $\bar{r}_t(\cdot,\cdot,dw)$, Lemma 3.2 and dominated convergence. Using Lemma 3.2 and (a) in (3.7), we obtain (recall Convention 3.1)

$$\begin{aligned}
p_t(x,y) &\le c(t + x_{m+1} + y_{m+1})^{-m/2} q_t^{b,\gamma}(x_{m+1}, y_{m+1}) t^{-m/2} \\
&\le c[t^{-m/2} t^{-b/\gamma} t^{-m/2} + 1_{(b/\gamma<1/2)}(x_{m+1}+y_{m+1})^{b/\gamma-1/2} t^{-m/2+1/2-b/\gamma} \\
&\qquad\qquad \times (x_{m+1}\wedge y_{m+1})^{1/2-b/\gamma} t^{-1/2} t^{-m/2}] \\
&\le ct^{-m-b/\gamma}.
\end{aligned}$$

(c) This is well-known, and is easily derived from (3.3).

(d) The expectation we need to bound equals

$$\begin{aligned}
p&\int_0^\infty v^{-1-p}\mathbb{P}_{x_{m+1}}(X_t^{(m+1)} < v)\,dv \\
&\le p\int_0^{x_{m+1}\vee t} v^{-1-p} e\mathbb{E}_{x_{m+1}}(e^{-X_t^{(m+1)}/v})\,dv + p\int_{x_{m+1}\vee t}^\infty v^{-1-p}\,dv \\
&= ep\int_0^{x_{m+1}\vee t} v^{-1-p+b/\gamma}(v+\gamma t)^{-b/\gamma} e^{-x_{m+1}/(v+\gamma t)}\,dv + (x_{m+1}\vee t)^{-p}
\end{aligned}$$

We have used (c) in the last line. Set $c_r = \sup_{x\ge 0} x^r e^{-x}$ and $M_1 = M_0^2$. If $x_{m+1} \ge t$, use

$$(v+\gamma t)^{-b/\gamma} e^{-x_{m+1}/(v+\gamma t)} \le c_{b/\gamma} x_{m+1}^{-b/\gamma} \le (c_{M_1}+1) x_{m+1}^{-b/\gamma}$$

to bound the above expression for $\mathbb{E}_{x_{m+1}}((X^{(m+1)})^{-p})$ by

$$\begin{aligned}(c_{M_1}+1)epx_{m+1}^{-b/\gamma}\int_0^{x_{m+1}} v^{-1-p+b/\gamma}\,dv + x_{m+1}^{-p}\\ \leq \Big(\frac{(c_{M_1}+1)ep}{b/\gamma-p}+1\Big)x_{m+1}^{-p}.\end{aligned}$$

On the other hand if $x_{m+1}<t$, the above expression for $\mathbb{E}_{x_{m+1}}((X^{(m+1)})^{-p})$is trivially at most

$$ep\int_0^t v^{-1-p+b/\gamma}\gamma^{-b/\gamma}t^{-b/\gamma}\,dv + t^{-p} = \Big(\frac{ep\gamma^{-b/\gamma}}{b/\gamma-p}+1\Big)t^{-p}.$$

The result follows from these two bounds.
(e) This is standard (e.g., see Lemma 7(a) of [DP1]).
(f) Multiply the bound in Lemma 3.2 by $q_t(x,y)y^{b/\gamma-1}$ and integrate over y.
(g) Let $b'=\gamma/b$. Since $q_t(x,y)=t^{-b/\gamma}q_1(x/t,y/t)$, a simple change of variables shows the quantity we need to bound is

$$\begin{aligned}&\sup_{y\geq 0}\Big|\int_0^\infty (x-\gamma y)x^{b'}D_x^2q_1(x,\gamma y)dx\Big|\\ &\quad=\sup_{y\geq 0}\Big|\sum_{m=0}^\infty e^{-y}\frac{y^m}{m!}\int_0^\infty (x-y)x^{b'}D_x^2(e^{-x}x^m)\,dx/\Gamma(m+b')\Big|.\end{aligned}$$

Carrying out the differentiation and resulting Gamma integrals, we see the absolute value of the above summation equals

$$\begin{aligned}&\Big|\sum_{m=0}^\infty e^{-y}\frac{y^m}{m!}\Big[(m+1+b')(m+b')-2m(m+b')+m(m-1)\\ &\qquad -y(m+b')+2my-ym(1-\frac{b'}{m+b'-1})1_{(m\geq 1)}\Big]\\ &\leq \sum_{m=0}^\infty e^{-y}\frac{y^m}{m!}\Big[b'(1+b')+yb'|-1+1_{(m\geq 1)}m(m+b'-1)^{-1}|\Big]\\ &\leq b'(1+b')+b'\sum_{m=1}^\infty e^{-y}\frac{y^{m+1}}{(m+1)!}|1-b'|\frac{m+1}{m+b'-1}+e^{-y}yb'\leq c_{3.3}.\end{aligned}$$

□

Now let $\{\mathbb{P}_x^0 : x\geq 0\}$ denote the laws of the Feller branching process with generator $\mathcal{L}^0 f(x)=\gamma xf''(x)$. If $\omega\in C(\mathbb{R}_+,\mathbb{R}_+)$ let $\zeta(\omega)=\inf\{t>0:\omega(t)=0\}$. There is a unique σ-finite measure $\mathbb{N}_0$ on

$$C_{ex}=\{\omega\in C(\mathbb{R}_+,\mathbb{R}_+):\omega(0)=0,\ \zeta(\omega)>0,\ \omega(t)=0\ \forall t\geq\zeta(\omega)\}$$

such that for each $h > 0$, if Ξ^h is a Poisson point process on C_{ex} with intensity $h\mathbb{N}_0$, then

$$X = \int_{C_{ex}} \nu \, \Xi^h(d\nu) \text{ has law } \mathbb{P}^0_h; \tag{3.13}$$

see, e.g., Theorem II.7.3 of [P] which can be projected down to the above situation by considering the total mass function. Moreover for each $t > 0$ we have (Theorems II.7.2(iii) and II.7.3(b) of [P])

$$\mathbb{N}_0(\{\nu : \nu_t > 0\}) = (\gamma t)^{-1} \tag{3.14}$$

and so we may define a probability on C_{ex} by

$$\mathbb{P}^*_t(A) = \frac{\mathbb{N}_0(\{\nu \in A : \nu_t > 0\})}{\mathbb{N}_0(\{\nu : \nu_t > 0)\}}. \tag{3.15}$$

The above references in [P] also give the well-known

$$\mathbb{P}^*_t(\nu_t > x) = e^{-x/\gamma t}, \tag{3.16}$$

and so this together with (3.14) implies

$$\int_{C_{ex}} \nu_t \, d\mathbb{N}_0(\nu) = 1. \tag{3.17}$$

The representation (3.13) leads to the following decomposition of $X^{(m+1)}$ from Lemma 10 of [DP1]. As it is consistent with the above notation, we will use $X^{(m+1)}$ to denote a Feller branching diffusion (with immigration) starting at x_{m+1} and with generator given by (3.1), under the law $\mathbb{P}_{x_{m+1}}$.

Lemma 3.4. *Let $0 \le \rho \le 1$.*
(a) We may assume

$$X^{(m+1)} = X'_0 + X_1, \tag{3.18}$$

where X'_0 is a diffusion with generator $\mathcal{A}'_0$ as in (3.1), starting at ρx_{m+1}, X_1 is a diffusion with generator $\gamma x f''(x)$ starting at $(1-\rho)x_{m+1} \ge 0$, and X'_0, X_1 are independent. In addition, we may assume

$$X_1(t) = \int_{C_{ex}} \nu_t \, \Xi(d\nu) = \sum_{j=1}^{N_\rho(t)} e_j(t), \tag{3.19}$$

where Ξ is independent of X'_0 and is a Poisson point process on C_{ex} with intensity $(1-\rho)x_{m+1}\mathbb{N}_0$, $\{e_j, j \in \mathbb{N}\}$ is an i.i.d. sequence with common law $\mathbb{P}^*_t$, and $N_\rho(t) = \Xi(\{\nu :$

$\nu_t > 0\}$) is a Poisson random variable (independent of the $\{e_j\}$) with mean $\frac{(1-\rho)x_{m+1}}{t\gamma^0_{m+1}}$.
(b) We also have

$$\begin{aligned}\int_0^t X_1(s)ds &= \int_{C_{ex}} \int_0^t \nu_s \, ds \, 1_{(\nu_t \neq 0)} \Xi(d\nu) + \int_{C_{ex}} \int_0^t \nu_s \, ds \, 1_{(\nu_t = 0)} \Xi(d\nu) \\ &\equiv \sum_{j=1}^{N_\rho(t)} r_j(t) + I_1(t), \end{aligned} \tag{3.20}$$

$$\int_0^t X_s^{(m+1)} ds = \sum_{j=1}^{N_\rho(t)} r_j(t) + I_2(t), \tag{3.21}$$

where $r_j(t) = \int_0^t e_j(s)ds$ and $I_2(t) = I_1(t) + \int_0^t X_0'(s)ds$.

Remark 3.5. A double application of the decomposition in Lemma 3.4(a), first with general ρ and then $\rho = 0$ shows we may write

$$X_t^{(m+1)} = \tilde{X}_0'(t) + \sum_{j=1}^{N_0'(t)} e_j^2(t) + \sum_{j=1}^{N_\rho(t)} e_j^1(t), \tag{3.22}$$

where $\tilde{X}_0'$ is as in Lemma 3.4(a) with $\rho = 0$, $\{e_j^1(t), e_k^2(t), j, k\}$ are independent exponential variables with mean $(1/\gamma t)$, $N_0'(t)$, $N_\rho(t)$ are independent Poisson random variables with means $\rho x_{m+1}/\gamma t$ and $(1-\rho)x_{m+1}/\gamma t$, respectively, and $(\tilde{X}_0'(t), \{e_j^1(t), e_k^2(t)\}, N_0'(t), N_\rho(t))$ are jointly independent. The group of two sums of exponentials in (3.22) may correspond to $X_1(t)$ in (3.18) and (3.19), and so we may use this as the decomposition in Lemma 3.4(a) with $\rho = 0$. Therefore we may take N_0 to be $N_0' + N_\rho$, and hence may couple these decompositions so that

$$N_\rho \leq N_0. \tag{3.23}$$

The decomposition in Lemma 3.4 also gives a finer interpretation of the series expansion (3.3) for $q_t^{b,\gamma}(x, y)$, as we now show. Note that the decomposition (from (3.18),(3.19)),

$$X^{(m+1)}(t) = X_0'(t) + \sum_{j=1}^{N_\rho(t)} e_j(t),$$

where X_0', $N_\rho(t)$ and $\{e_j(t)\}$ satisfy the distributional assumptions in (a), uniquely determines the joint law of $(X_t^{(m+1)}, N_\rho(t))$. This can be seen by conditioning on $N_\rho(t) = n$. Both this result and method are used in the following.

Lemma 3.6. *Assume $\phi : \mathbb{R}_+ \times \mathbb{Z}_+ \to \mathbb{R}_+$ is Borel measurable. If $N_\rho(t)$ is as in Lemma 3.4, then*

(a)

$$\mathbb{E}_x(\phi(X_t^{(m+1)}, N_0(t))) = \int_0^\infty \sum_{n=0}^\infty \phi(y,n)(n!\Gamma(n+b/\gamma))^{-1} \\ \times \Big(\frac{x}{\gamma t}\Big)^n \Big(\frac{y}{\gamma t}\Big)^n \exp\Big\{-\frac{x}{\gamma t} - \frac{y}{\gamma t}\Big\}(\gamma t)^{-b/\gamma} d\mu_{m+1}(y).$$

(b)

$$\mathbb{E}_x(\phi(X_t^{(m+1)}, N_{1/2}(t))) = \int_0^\infty \sum_{n=0}^\infty \sum_{k=0}^n \binom{n}{k} 2^{-n} \phi(y,k)(n!\Gamma(n+b/\gamma))^{-1} \\ \times \Big(\frac{x}{\gamma t}\Big)^n \Big(\frac{y}{\gamma t}\Big)^n \exp\Big\{-\frac{x}{\gamma t} - \frac{y}{\gamma t}\Big\}(\gamma t)^{-b/\gamma} d\mu_{m+1}(y).$$

Proof. (a) Set $x = 0$ in (3.3) to see that $X_0'(t)$ has Lebesgue density

$$\exp\{\frac{-y}{\gamma t}\}(y/\gamma t)^{b/\gamma - 1}(\gamma t)^{-1}\Gamma(b/\gamma)^{-1},$$

that is, has a gamma distribution with parameters $(b/\gamma, \gamma t)$. It follows from Lemma 3.4(a), (3.16) and the joint independence of $(\{e_j(t)\}, X_0'(t), N_0(t))$ that, conditional on $N_0(t) = n$, $X_t^{(m+1)}$ has a gamma distribution with parameter $(n + b/\gamma, \gamma t)$. This gives (a).
(b) Apply (3.22) with $\rho = 1/2$ to see that the decomposition in (3.18) for $\rho = 1/2$ is given by (3.22) with $X_0'(t) = \tilde{X}_0'(t) + \sum_{j=1}^{N_0'(t)} e_j^2(t)$ and $e_j(t) = e_j^1(t)$. As in (a), conditional on $(N_0'(t), N_{1/2}(t)) = (j,k)$, $X_t^{(m+1)}$ has a gamma distribution with parameters $(j + k + b/\gamma, \gamma t)$. A short calculation (with $n = j + k$) now gives (b). □

Notation 3.7 Let D^n denote any nth order partial differential operator on S_m and let $D^n_{x_i}$ denote the nth partial derivative with respect to x_i.

If $X \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\nu^i \in C_{ex}$, $G : \mathbb{R}_+^2 \to \mathbb{R}$ and $t \geq 0$, let

$$\Delta_t G(X, \nu^1) = \Delta_t^1 G(X, \nu^1) = G\Big(\int_0^t X_s + \nu_s^1\, ds, X_t + \nu_t^1\Big) - G\Big(\int_0^t X_s\, ds, X_t\Big),$$

$$\begin{aligned}\Delta_t^2 G(X, \nu^1, \nu^2) =& G\Big(\int_0^t X_s + \nu_s^1 + \nu_s^2\, ds, X_t + \nu_t^1 + \nu_t^2\Big) - G\Big(\int_0^t X_s + \nu_s^1\, ds, X_t + \nu_t^1\Big) \\ &- G\Big(\int_0^t X_s + \nu_s^2\, ds, X_t + \nu_t^2\Big) + G\Big(\int_0^t X_s\, ds, X_t\Big) \\ =& \Delta_t G(X + \nu^1, \nu^2) - \Delta_t G(X, \nu^2),\end{aligned}$$

$$\begin{aligned}\Delta_t^3 G(X,\nu^1,\nu^2,\nu^3) =& G\Big(\int_0^t X_s+\nu_s^1+\nu_s^2+\nu_s^3\,ds, X_t+\nu_t^1+\nu_t^2+\nu_t^3\Big)\\ &-\sum_{i=1}^3 G\Big(\int_0^t X_s+\nu_s^1+\nu_s^2+\nu_s^3-\nu_s^i\,ds, X_t+\nu_t^1+\nu_t^2+\nu_t^3-\nu_t^i\Big)\\ &+\sum_{i=1}^3 G\Big(\int_0^t X_s+\nu_s^i, X_t+\nu_t^i\Big)-G\Big(\int_0^t X_s\,ds, X_t\Big).\end{aligned}$$

Lemma 3.8. *If $f : S_m \to \mathbb{R}$ is a bounded Borel function and $t > 0$, then $P_t f \in C_b^3(S_m)$ and for $n \le 3$*

$$\|D^n P_t f\|_\infty \le c_{3.8}\|f\|_\infty t^{-n}. \tag{3.24}$$

Moreover if $f \in C_b(S_m)$, then for $n \le 3$,

$$D^n_{x_{m+1}} P_t f(x) = \mathbb{E}_{x_{m+1}}\Big[\int \Delta_t^n(G_{t,x^{(m)}}f)(X,\nu^1,\ldots,\nu^n)\prod_{j=1}^n d\mathbb{N}_0((\nu^j))\Big], \tag{3.25}$$

where for $x^{(m)} \in \mathbb{R}^m$

$$G_{t,x^{(m)}}f(I,X) = \int_{\mathbb{R}^m} f(z_1,\ldots,z_m,X)\prod_{j=1}^m p_{2\gamma_j^0 I}(z_j - x_j - b_j^0 t)dz_j. \tag{3.26}$$

Proof. Proposition 14 and Remark 15 of [DP1] show that $P_t f \in C_b^2(S_m)$ and give (3.24) and (3.25) for $n \le 2$. The proof there shows how to derive the $n = 2$ case from the $n = 1$ case and similar reasoning, albeit with more terms to check, allows one to derive the $n = 3$ case from the $n = 2$ case. □

Recall that Q_t is the semigroup of $X^{(m+1)}$, the squared Bessel diffusion with transition density given by (3.3).

Corollary 3.9. *If $g : \mathbb{R}_+ \to \mathbb{R}$ is a bounded Borel function and $t > 0$, then $Q_t g \in C_b^3(\mathbb{R}_+)$ and for $n \le 3$,*

$$\|D^n Q_t g\|_\infty \le c_{3.8}\|g\|_\infty t^{-n}. \tag{3.27}$$

Proof. Apply Lemma 3.8 to $f(y) = g(y_{m+1})$. □

Notation 3.10. For $t, \delta > 0$, $x^{(m)} \in \mathbb{R}^m$, $y \in S_m$, $1 \le j \le m$, $I, X \ge 0$, define

$$G^\delta_{t,x^{(m)},y}(I,X) = \prod_{i=1}^m p_{\delta+2\gamma_i^0 I}(x_i - y_i + b_i^0 t)q_\delta^{b,\gamma}(y_{m+1},X).$$

Note that in the above notations, $G = G_{t,x^{(m)}}f$ and $G = G^\delta_{t,x^{(m)},y}$ are real-valued functions on $\mathbb{R}_+^2$ and so, according to the notation above, for such a G, $\Delta_t^n G$ will be a ral-valued function of $(X,\nu^1,\ldots,\nu^n) \in C(\mathbb{R}_+,\mathbb{R}_+)\times C_{ex}^n$, $n = 1,2,3$.

Lemma 3.11. *(a) For each $t > 0$ and $y \in S_m$, the functions $x \to p_t(x, y)$ and $x \to p_t(y, x)$ are in $C_b^3(S_m)$, and if D_x^n denotes any nth order partial differential operator in the x variable, then*

$$|D_y^n p_t(x, y)| + |D_x^n p_t(x, y)| \leq c_{3.11} t^{-n-m-b/\gamma} \text{ for all } x, y \in S_m \text{ and } 0 \leq n \leq 3. \tag{3.28}$$

(b) $\sup_x \int |D_x^n p_t(x, y)| \mu(dy) \leq c_{3.8} t^{-n}$ *for all $t > 0$ and $0 \leq n \leq 3$.*

(c) *For $n \leq 3$ and $t > 0$, $y \to D_x^n p_t(x, y)$ and $x \to D_y^n p_t(x, y)$ are in $C_b(S_m)$.*

(d) *For $n = 1, 2, 3$,*

$$D^n_{x_{m+1}} p_t(x, y) = \lim_{\delta \downarrow 0} \mathbb{E}_{x_{m+1}} \Big(\int \Delta_t^n G^\delta_{t, x^{(m)}, y}(X, \nu^1, \ldots, \nu^n) \prod_{i=1}^n d\mathbb{N}_0(\nu^i) \Big).$$

Proof. (a) By the Chapman-Kolmogorov equations, $p_t(x, y) = \mathbb{E}_x(p_{t/2}(X_{t/2}, y))$. Both the required regularity and (3.28) now follow for $x \to p_t(x, y)$ from Lemma 3.3(b) and Lemma 3.8 with $f(x) = p_{t/2}(x, y)$. By (3.8) it follows for $y \to p_t(x, y)$.
(b) For $n = 1, 2, 3$, $N \in \mathbb{N}$ and $x \in S_m$, let $f(y) = \text{sgn}\,(D_x^n p_t(x, y)) 1_{(|y| \leq N)}$. Then

$$\begin{aligned} \int |D_x^n p_t(x, y)| 1_{(|y| \leq N)} d\mu(y) &= \Big| \int D_x^n p_t(x, y) f(y) d\mu(y) \Big| \\ &= |D_x^n P_t f(x)|, \end{aligned}$$

where the last line follows by dominated convergence, the uniform bound in (3.28) and the fact that f has compact support. An application of (3.24) implies

$$\int |D_x^n p_t(x, y)| 1_{(|y| \leq N)} d\mu(y) \leq c_{3.8} t^{-n},$$

and the result follows upon letting $N \to \infty$.
(c)

$$\begin{aligned} D_x^n p_t(x, y) &= D_x^n \int p_{t/2}(x, z) p_{t/2}(z, y) d\mu(z) \\ &= \int D_x^n p_{t/2}(x, z) p_{t/2}(z, y) d\mu(z) \end{aligned} \tag{3.29}$$

by dominated convergence and the uniform bounds in (a). The integrability of $D_x^n p_{t/2}(x, z)$ with respect to $\mu(dz)$ (from (b)) and the fact that $p_{t/2}(z, \cdot) \in C_b(S_m)$ allow us to deduce the continuity of (3.29) in y from dominated convergence. Now use symmetry, i.e., (3.8), to complete the proof.

(d) If $y, z \in S_m$, $\delta > 0$, let

$$f^{y,\delta}(z) = \prod_{i=1}^{m} p_\delta(z_i - y_i) q_\delta(y_{m+1}, z_{m+1}).$$

f is bounded and continuous in z by Lemma 3.3(a) (with a bound depending on y, δ). Let $n \leq 3$. The uniform bounds in (a) and integrability of $f^{y,\delta}$ allow us to apply dominated convergence to differentiate through the integral and conclude

$$\begin{aligned} D^n_{x_{m+1}} P_t f^{y,\delta}(x) = & \int\int D^n_{x_{m+1}} p_t(x,z) f^{y,\delta}(z)\, d\mu(z) \\ & \to D^n_{x_{m+1}} p_t(x,y) \text{ as } \delta \downarrow 0. \end{aligned} \tag{3.30}$$

In the last line we have used (c). Now note that if $I, X \geq 0$, then from (3.26) and Chapman-Kolmogorov,

$$\begin{aligned} G_{t,x^{(m)}} f^{y,\delta}(I,X) = & \int_{\mathbb{R}^m} f^{y,\delta}(z,X) \prod_{j=1}^{m} p_{2\gamma_j^0 I}(z_j - x_j - b_j^0 t) dz_j \\ = & q_\delta(y_{m+1}, X) \prod_{j=1}^{m} p_{\delta + 2\gamma_j^0 I}(y_j - x_j - b_j^0 t) = G^\delta_{t,x^{(m)},y}(I,X). \end{aligned} \tag{3.31}$$

Use this in (3.25) to conclude that

$$D^n_{x_{m+1}} P_t f^{y,\delta}(x) = \mathbb{E}_{x_{m+1}}\Big(\int\int \Delta^n_t G^\delta_{t,x^{(m)},y}(X, \nu^1, \dots, \nu^n) \prod_{i=1}^{n} d\mathbb{N}_0(\nu^i)\Big). \tag{3.32}$$

Combine (3.32) and (3.30) to derive (d). □

Lemma 3.12. *(a) For each $t > 0$ and $y \in \mathbb{R}_+$, the functions $x \to q_t(x,y)$ and $x \to q_t(y,x)$ are in $C_b^3(\mathbb{R}_+)$, and*

$$|D^n_y q_t(x,y)| \leq c_{3.12} t^{-n-b/\gamma}[1 + \sqrt{y/t}] \text{ for all } x, y \geq 0 \text{ and } 0 \leq n \leq 3. \tag{3.33}$$

(b) $\sup_{x\geq 0} \int |D^n_x q_t(x,y)| \mu_{m+1}(dy) \leq c_{3.12} t^{-n}$ for all $t > 0$ and $0 \leq n \leq 3$.

Proof. This is a minor modification of the proofs of Lemma 3.11 (a),(b). Use the bound (from Lemma 3.3(a))

$$\begin{aligned} q_{t/2}(\cdot, y) & \leq c_{3.3}(t^{-b/\gamma} + 1_{(b/\gamma<1/2)} y^{1/2-b/\gamma} t^{-1/2}) \\ & \leq c_{3.3} t^{-b/\gamma}(2 + \sqrt{y/t}) \end{aligned}$$

in place of Lemma 3.3(b), and (3.27) in place of (3.24), in the above argument. Of course this is much easier and can also be derived by direct calculation from our series expansion for q_t. □

Expectation under $\mathbb{P}^*_t$ is denoted by $\mathbb{E}^*_t$ and we let $e(s), s \geq 0$, denote the canonical excursion process under this probability.

Lemma 3.13. *If $0 < s \le t$, then*

$$\mathbb{E}_t^*(e(s)|e(t)) = (s/t)((s/t)e(t) + 2\gamma(t-s)) \le e(t) + 2\gamma s, \quad \mathbb{P}_t^* - \text{ a.s.}$$

Proof. Let $\mathbb{P}_h^0$ denote the law of the diffusion with generator $\gamma x \frac{d^2}{dx^2}$ starting at h. If $f \in C_b(\mathbb{R}_+)$ has compact support, then Proposition 9 of [DP1] and (3.14) show that

$$\begin{aligned}\mathbb{E}_t^*(e(s)f(e(t))) &= \lim_{h\downarrow 0} h^{-1}\mathbb{E}_h^0(X_s f(X_t)1_{(X_t>0)})\gamma t \\ &= \lim_{h\downarrow 0} \frac{t}{s}\int\int \frac{\gamma s q_s^{0,\gamma}(h,y)}{h} y q_{t-s}^{0,\gamma}(y,x) f(x)\, dy\, dx. \qquad (3.34)\end{aligned}$$

Here we extend the notation in (3.3) by letting $q_s^{0,\gamma}$ denote the absolutely continuous part of the transition kernel for $\mathbb{E}^0$ (it also has an atom at 0). We have also extended the convergence in [DP1] slightly as the functional $e(s)f(e(t))$ is not bounded but this extension is justified by a uniform integrability argument–the approximating functionals are L^2 bounded. By (2.4) of [BP]

$$\begin{aligned}\frac{\gamma s}{h} q_s^{0,\gamma}(h,y) &= \exp\Big\{\frac{-h-y}{\gamma s}\Big\}\sum_{m=0}^{\infty}\frac{1}{(m+1)!}\Big(\frac{h}{\gamma s}\Big)^m \frac{1}{m!}\Big(\frac{y}{\gamma s}\Big)^m (\gamma s)^{-1} \\ &\to \exp(-y/\gamma s)(\gamma s)^{-1} \quad \text{ as } h \downarrow 0,\end{aligned}$$

and also $\frac{\gamma s}{h} q_s^{0,\gamma}(h,y) \le (\gamma s)^{-1}$. Dominated convergence allows us to take the limit in (3.34) through the integral and deduce that

$$\mathbb{E}_t^*(e(s)f(e(t))) = \frac{t}{s}\int\int \exp(-y/\gamma s)(\gamma s)^{-1} y q_{t-s}^{0,\gamma}(y,x)\, dy f(x)\, dx.$$

By (3.16) we conclude that

$$\mathbb{E}_t^*(e(s)|e(t)=x) = (t/s)^2 e^{x/\gamma t}\int e^{-y/\gamma s} y q_{t-s}^{0,\gamma}(y,x)\, dy.$$

By inserting the above series expansion for $q_{t-s}^{0,\gamma}(y,x)$ and calculating the resulting gamma integrals, the result follows. $\square$

Recall the modified Bessel function, I_ν, introduced prior to Lemma 3.2.

Lemma 3.14. *Let $\nu = b/\gamma - 1$, and $\kappa_\nu(z) = \frac{I_\nu'}{I_\nu}(z)z + 1$ for $z \ge 0$, where $\kappa_\nu(0) \equiv \lim_{z\downarrow 0}\kappa_\nu(z) = \nu + 1$. Then*

$$\begin{aligned}\mathbb{E}_{z_{m+1}}\Big(\int_0^t X_s^{(m+1)}\, ds \Big| X_t^{(m+1)} = y\Big) &= \kappa_\nu\Big(\frac{2\sqrt{z_{m+1}y}}{t\gamma}\Big)\frac{t^2\gamma}{6} + \frac{(z_{m+1}+y)t}{3} \\ &\le c_{3.14}[t^2 + t(z_{m+1}+y)].\end{aligned}$$

Proof. Write X for $X^{(m+1)}$ and z for z_{m+1}. The scaling argument used in the proof of Lemma 3.2 allows us to assume $t = 1$, $\hat{\gamma} = 2$ and $\hat{b} = 2b/\gamma$. Dominated convergence implies that

$$\mathbb{E}_z\Big(\int_0^1 X_s\, ds\Big|X_1 = y) = \lim_{\lambda\to 0+} -\lambda^{-1}\frac{d}{d\lambda}\mathbb{E}_z\Big(\exp\Big((-\lambda^2/2)\int_0^1 X_s\, ds\Big)\Big|X_1 = y\Big).$$

The right side can be calculated explicitly from the two formulae in (3.4) and after some calculus we arrive at

$$\mathbb{E}_z\Big(\int_0^1 X_s\, ds\Big|X_1 = y\Big) = \frac{\kappa_\nu(\sqrt{zy})}{3} + \frac{(z+y)}{3},$$

where $\nu = \hat{b}/2 - 1 = b/\gamma - 1$. This gives the required equality.

To obtain the bound (recall Convention 3.1) it suffices to show

$$\kappa_\nu(z) \le c(\varepsilon)(1+z) \text{ for all } z \ge 0 \text{ and } \nu + 1 \in [\varepsilon, \varepsilon^{-1}].$$

Set $\alpha = \nu + 1$ and recall that

$$I_\nu(z) = (z/2)^\nu \sum_{n=0}^\infty \frac{1}{n!\Gamma(n+\alpha)}(z/2)^{2n},$$

and

$$zI'_\nu(z) = 2(z/2)^\nu \sum_{n=1}^\infty \frac{1}{(n-1)!\Gamma(n+\alpha)}(z/2)^{2n} + \nu I_\nu(z).$$

Taking ratios of the above and setting $w = (z/2)^2$, we see that it suffices to show

$$\sum_{n=1}^\infty \frac{w^n}{(n-1)!\Gamma(n+\alpha)} \le c_0(\varepsilon)\Big[\sum_{n=0}^\infty \frac{w^{n+1/2}}{n!\Gamma(n+\alpha)}\Big] \text{ for all } w \ge 0, \varepsilon \le \alpha \le \varepsilon^{-1}. \tag{3.35}$$

We claim in fact that

$$\frac{w^n}{(n-1)!\Gamma(n+\alpha)} \le \frac{1}{2}(\alpha^{-1/2}\vee 1)\Big[\frac{w^{n-1/2}}{(n-1)!\Gamma(n-1+\alpha)} + \frac{w^{n+1/2}}{n!\Gamma(n+\alpha)}\Big]$$
$$\text{for all } n \ge 1, w \ge 0, \alpha \in [\varepsilon, \varepsilon^{-1}]. \tag{3.36}$$

Assuming this, (3.35) will follow with $c_0(\varepsilon) = \varepsilon^{-1/2}$ by summing (3.36) over $n \ge 1$. The proof of (3.36) is an elementary application of the quadratic formula once factors of $w^{n-1/2}$ are canceled. $\square$

Corollary 3.15. *If $N_0(t)$ is as in Lemma 3.4 then*

$$\mathbb{E}_{z_{m+1}}\Big(\int_0^t X_s^{(m+1)}\, ds\Big|X_t^{(m+1)}, N_0(t)\Big) \le c_{3.15}[t^2(1+N_0(t)) + t(X_t^{(m+1)} + z_{m+1})].$$

Proof. We write X for $X^{(m+1)}$ and z for z_{m+1}. Recall the decomposition (3.21):

$$\int_0^t X_s\,ds = \int_0^t X_0'(s)\,ds + \int\int_0^t \nu_s ds 1_{(\nu_t=0)}\Xi(d\nu) + \sum_{i=1}^{N_0(t)} \int_0^t e_i(s)ds,$$

where the second integral is independent of $(\{e_j(t), j \in \mathbb{N}\}, N_0(t) = \int 1_{(\nu_t>0)}\Xi(d\nu), X_0')$ by elementary properties of Poisson point processes and the independence of X_0' and Ξ. Therefore

$$\begin{aligned}
&\mathbb{E}_z\Big(\int_0^t X_s ds \Big| X_0'(t), N_0(t), \{e_j(t), j \in \mathbb{N}\}\Big) \\
&= \mathbb{E}_z\Big(\int_0^t X_0'(s)ds\Big|X_0'(t)\Big) + \mathbb{E}_z\Big(\int\int_0^t \nu_s ds 1_{(\nu_t=0)}\Xi(d\nu)\Big) \\
&\qquad\qquad + \sum_{i=1}^{N_0(t)} \mathbb{E}_z\Big(\int_0^t e_i(s)ds\Big|X_0'(t), N_0(t), \{e_j(t), j \in \mathbb{N}\}\Big) \\
&= \mathbb{E}_z\Big(\int_0^t X_0'(s)ds\Big|X_0'(t)\Big) + \mathbb{E}_z\Big(\int\int_0^t \nu_s ds 1(\nu_t=0)\Xi(d\nu)\Big) \\
&\qquad\qquad\qquad + \sum_{i=1}^{N_0(t)} \mathbb{E}_z\Big(\int_0^t e_i(s)ds|e_i(t)\Big).
\end{aligned}$$

In the last line we have used the independence of X_0' and Ξ, of $N_0(t)$ and $\{e_j, j \in \mathbb{N}\}$, and the joint independence of the $\{e_j\}$ (see Lemma 3.4). Now use Lemmas 3.13 and 3.14 to bound the last and first terms, respectively, and note the second term is bounded by the mean of $\int_0^t X_1(s)\,ds$, where X_1 is as in (3.19). This bounds the above by

$$\begin{aligned}
&c_{3.14}[t^2 + tz + tX_0'(t)] + \mathbb{E}_z\Big(\int_0^t X_1(s)ds\Big) + \sum_{i=1}^{N_0(t)} (e_i(t)t + \gamma t^2) \\
&\quad \le c[t^2 + tX_t + zt] + \gamma N_0(t)t^2.
\end{aligned}$$

Condition the above on $\sigma(N_0(t), X_t)$ to complete the proof. □

We now return to the general setting of Propositions 2.2 and 2.8.

Proof of Proposition 2.8. From (2.5) we may write

$$p_t(x,y) = \prod_{i\in Z\cap C} p_t^i(x_{(i)}, y_{(i)}) \prod_{j\in N_2} q_t^j(x_j, y_j),$$

where p_t^i are the transition densities from Lemma 3.11 and q_t^j are the transition densities from Lemma 3.12. The joint continuity and smoothness in each variable is immediate from these properties for each factor (from Lemma 3.11 and (3.3)). (a) is also immediate from (3.8). The first part of (b) is also clear from the above factorization and the upper bounds in Lemmas 3.11(a) and 3.12(a). The second part of (b) is then immediate from (a). (c) also follows from Lemmas 3.11(b) and 3.12(b) and a short calculation.

(d) is an exercise in differentiating through the integral, but, as we will be doing a lot of this in the future, we outline the proof here and refer to this argument for such manipulations hereafter. Let f be a bounded Borel function on S^0 and $0 \leq n$. If $I(x)$ denotes the right-hand side of (2.14), then $I(x)$, is finite by (c) and also continuous in x. To see the latter, choose a unit vector e_j, set $x'_i = x_i$ if $i \neq j$ and x'_j variable, and note that for $h > 0$,

$$\begin{aligned}|I(x+he_j) - I(x)| &\leq \int \int_{x_j}^{x_j+h} |D_{x_j} D_x^n p_t(x', y)| dx'_j f(y) \mu(dy) \\ &\leq \|f\|_\infty c_{2.8} t^{-n-1} h.\end{aligned}$$

We have used (c) in the above.

Let $f_N(y) = f(y) 1_{(|y| \leq N)}$. By the integrability in (c), the left-hand side of (2.14) equals

$$\lim_{N\to\infty} \int D_x^n p_t(x,y) f_N(y) d\mu(y) = \lim_{N\to\infty} D_x^n \int p_t(x,y) f_N(y) d\mu(y),$$

where the differentiation through the integral over a compact set is justified by the bounds in (b) and dominated convergence. The bound in (c) shows this convergence is uniformly bounded (in x). For definiteness assume $n = 2$ and $D_x^2 = D_{x_i x_j}^2$ for $i \neq j$. By the above convergence and dominated convergence we get

$$\begin{aligned}\int_0^{x'_j} \int_0^{x'_i} \int D_x^2 p_t(x,y) f(y) d\mu(y) dx_i dx_j &= \lim_{N\to\infty} \int_0^{x'_j} \int_0^{x'_i} D_x^2 \int p_t(x,y) f_N(y) d\mu(y) dx_i dx_j \\ &= \int [p_t(x,y)|_{x_i=0}^{x_i=x'_i}|_{x_j=0}^{x_j=x'_j}] f(y) d\mu(y)\end{aligned}$$

Now differentiate both sides with respect x'_j and then x'_i and use the Fundamental Theorem of Calculus and the continuity of $\int D_x^2 p_t(x,y) f(y) d\mu(y)$, noted above, to obtain (2.14). This shows $P_t f \in C_b^2$ as continuity in x was established above. Finally (2.15) is now immediate from (2.13) and (2.14). □

4. Proofs of Propositions 2.14 and 2.10.

By Lemma 2.11(b), Proposition 2.14 will follow from (2.28). We restate this latter inequality explicitly. Recall that $\tilde{D}_x$ is one of the first or second order partial differential operators listed in Notation 2.9. (2.28) then becomes

$$\sup_y \int \Big[\int \Big| \int \tilde{D}_z p_s(z,x) \tilde{D}_z p_t(z,y') \mu(dz) \Big| \mu(dy') \Big| \int \tilde{D}_z p_s(z,x) \tilde{D}_z p_t(z,y) \mu(dz) \Big| \Big] \mu(dx)$$
$$\leq c_{2.14} s^{-2-\eta} t^{-2+\eta} \text{ for all } 0 < t \leq s \leq 2, \tag{4.1}$$

and

$$\sup_y \int \Big[\int \Big| \int \tilde{D}_x p_s(x,z) \tilde{D}_{y'} p_t(y',z) \mu(dz) \Big| \mu(dy') \Big| \int \tilde{D}_x p_s(x,z) \tilde{D}_y p_t(y,z) \mu(dz) \Big| \Big] \mu(dx)$$
$$\leq c_{2.14} s^{-2-\eta} t^{-2+\eta} \text{ for all } 0 < t \leq s \leq 2, \tag{4.2}$$

We have stated these conditions with $s \leq 2$ for other potential uses; in our case we will verify (2.28) for all $0 < t \leq s$. Recall also our Convention 3.1 for constants applies to $c_{2.14}$ and η.

Until otherwise indicated, we continue to work in the setting of the last section and use the notation introduced there. In particular, Convention 3.1 will be in force and the differential operators in Notation 2.9 are

$$\tilde{D}_x = D_{x_i},\ i \leq m+1, \text{ or } \tilde{D}_x = x_{m+1} D^2_{x_i},\ i \leq m+1. \tag{4.3}$$

In [DP1] a number of bounds were obtained on the derivatives of the semigroup $P_t f$; (3.24) in the last section was one such bound. Propositions 16 and 17 of [DP1] state there exists a $c_{4.1}$ such that for all $\tilde{D}_x$ as in Notation 2.9, $t > 0$ and bounded Borel function f,

$$\sup_{x \in S_m} |\tilde{D}_x P_t f(x)| \leq c_{4.1} t^{-1} \|f\|_\infty. \tag{4.4}$$

Although these results are stated for $m = 1$ in [DP1], the same argument works in $m+1$ dimensions (see, for example, Proposition 20 of [DP1]). As a simple consequence of this result we get:

Lemma 4.1. *For all $t > 0$, and all $\tilde{D}_x$ as in (4.3)*

$$\sup_{x \in S_m} \int |\tilde{D}_x p_t(x,y)| \mu(dy) \leq c_{4.1} t^{-1}, \tag{4.5}$$

and

$$\sup_{x_{m+1} \geq 0} \int [x_{m+1} |D^2_{x_{m+1}} q_t(x_{m+1}, y)| + |D_{x_{m+1}} q_t(x_{m+1}, y)|]\, \mu_{m+1}(dy) \leq c_{4.1} t^{-1}. \tag{4.6}$$

Proof. Apply (4.4) and (2.14) to $f(y) = \operatorname{sgn}(\tilde{D}_x p_t(x,y))$ to obtain (4.5), and to $f(y) = \operatorname{sgn}(\tilde{D}_x q_t(x_{m+1}, y_{m+1}))$ to obtain (4.6). □

One of the ingredients we will need is a bound like (4.5) but with the integral on the left with respect to x instead of y. For derivatives with respect to $x_i, i \le m$, this is straightforward as we now show.

By differentiating through the integral in (3.7) we find for $i \le m$,

$$D_{x_i} p_t(x,y) = \int_0^\infty \frac{(y_i - x_i - b_i^0 t)}{2\gamma_i^0 w} \prod_{j=1}^m p_{2\gamma_j^o w}(y_j - x_j - b_j^0 t) r_t(x_{m+1}, y_{m+1}, dw), \qquad (4.7)$$

$$\begin{aligned} D_{x_i}^2 p_t(x,y) = \int_0^\infty &\Big[\frac{(y_i - x_i - b_i^0 t)^2}{2\gamma_i^0 w} - 1\Big](2\gamma_i^0 w)^{-1} \\ &\times \prod_{j=1}^m p_{2\gamma_j^0 w}(y_j - x_j - b_j^0 t) r_t(x_{m+1}, y_{m+1}, dw), \end{aligned} \qquad (4.8)$$

and

$$\begin{aligned} D_{x_i}^3 p_t(x,y) = \int_0^\infty &\Big[\frac{(y_i - x_i - b_i^0 t)^3}{(2\gamma_i^0 w)} - 3(y_i - x_i - b_i^0 t)\Big](2\gamma_0^i w)^{-2} \\ &\times \prod_{j=1}^m p_{2\gamma_j^0 w}(y_j - x_j - b_j^0 t) r_t(x_{m+1}, y_{m+1}, dw). \end{aligned} \qquad (4.9)$$

Integration through the integral is justified by the bounds in Lemma 3.2 and dominated convergence.

Lemma 4.2. *For all $t > 0$, and $i \le m$,*

(a) $\int |D_{z_i} p_t(z,y)| dz^{(m)} \le c_{4.2} t^{-1/2} (t + z_{m+1} + y_{m+1})^{-1/2} q_t(y_{m+1}, z_{m+1})$.
(b) For all $0 \le p \le 2$, $\int z_{m+1}^p |D_{z_i} p_t(z,y)| \mu(dz) \le c_{4.2} t^{-1/2} (t + y_{m+1})^{p-1/2}$.
(c) For $\tilde{D}_z = z_{m+1} D_{z_i}^2$ or D_{z_i},

$$\sup_y \int |\tilde{D}_z p_t(z,y)| \mu(dz) \le c_{4.2} t^{-1}, \qquad (4.10)$$

and

$$\sup_z \int |\tilde{D}_z p_t(z,y)| \mu(dy) \le c_{4.2} t^{-1}. \qquad (4.11)$$

Proof. (a) By (4.7) the integral in (a) is

$$\begin{aligned} &\int \Big| \int_0^\infty \Big[\frac{(y_i - z_i - b_i^0 t)}{2\gamma_i^0 w}\Big] \prod_{j=1}^m p_{2\gamma_j^0 w}(y_j - z_j - b_j^0 t) r_t(z_{m+1}, y_{m+1}, dw) \Big| dz^{(m)} \\ &\le c \int_0^\infty w^{-1/2} r_t(y_{m+1}, z_{m+1}, dw) \\ &\le c t^{-1/2} (t + z_{m+1} + y_{m+1})^{-1/2} q_t(y_{m+1}, z_{m+1}). \end{aligned}$$

where in the first inequality we used the symmetry of r_t (recall (3.5)) and in the last inequality we have used Lemma 3.2.

(b) Integrate the inequality in (a) to bound the integral in (b) by

$$\begin{aligned}&ct^{-1/2}\int z_{m+1}^{p}(t+z_{m+1}+y_{m+1})^{-1/2}q_t(y_{m+1},z_{m+1})\mu_{m+1}(dz_{m+1})\\&\leq ct^{-1/2}(t+y_{m+1})^{-1/2}\mathbb{E}_{y_{m+1}}((X_t^{(m+1)})^p)\\&\leq ct^{-1/2}(t+y_m)^{p-1/2}.\end{aligned}$$

In the last line we used Lemma 3.3(e).

(c) For $\tilde{D}_z = D_{z_i}$, (4.10) follows from (b) upon taking $p=0$. The other cases are similarly proved, now using (4.8) for the second order derivatives. ((4.11) is also immediate from Lemma 4.1.) $\square$

Consider first (2.28) for $\tilde{D}_x = D_{x_i}$ or $x_{m+1}D_{x_i}^2$ for some $i \leq m$.

Proposition 4.3. *If $\tilde{D}_x = D_{x_i}$ or $x_{m+1}D_{x_i}^2$ for some $i \leq m$, then (2.28) holds with $\eta = 1/2$.*

Proof. Consider (4.1) for $\tilde{D}_x = x_{m+1}D_{x_i}^2$. We may as well take $i=1$. Assume $0<t\leq s$ and let

$$J=\int\Big|\int \tilde{D}_z p_s(z,x)\tilde{D}_z p_t(z,y')\mu(dz)\Big|\mu(dy').$$

Then

$$\begin{aligned}J\leq&\int\Big|\int z_{m+1}D_{y_1'}^2 p_s(y_1',z_2,\dots,z_{m+1},x)z_{m+1}D_{z_1}^2 p_t(z,y')\mu(dz)\Big|\mu(dy')\\&+\int\Big|\int z_{m+1}[D_{y_1'}^2 p_s(y_1',z_2,\dots,z_{m+1},x)-D_{z_1}^2 p_s(z,x)]z_{m+1}D_{z_1}^2 p_t(z,y')\mu(dz)\Big|\mu(dy')\\\equiv&J_1+J_2.\end{aligned}$$

To evaluate J_1 do the dz_1 integral first and use (4.8) to see

$$\int z_{m+1}D_{z_1}^2 p_t(z,y')\,dz_1=0,$$

and so $J_1=0$. (Lemma 3.2 handles integrability issues.)

Let J_2' and J_2'' denote the contribution to the integral defining J_2 from $\{z_1\leq y_1'\}$ and $\{z_1\geq y_1'\}$, respectively. Then by (4.8),

$$\begin{aligned}J_2'\leq\int\int z_{m+1}^2\Big|\int\int 1_{(z_1\leq z'\leq y_1')}D_{z'}^3 p_s(z',z_2,\dots,z_{m+1},x)\\\times\Big(\int_0^\infty\Big[\frac{(y_1'-z_1-b_1^0t)^2}{2\gamma_1^0 w}-1\Big](2\gamma_1^0 w)^{-1}\prod_{j=1}^m p_{2\gamma_j^0 w}(y_j'-z_j-b_j^0 t)\Big)\\r_t(z_{m+1},y_{m+1}',dw)dz^{(m)}dz'\Big|\mu_{m+1}(dz_{m+1})\mu(dy').\end{aligned}$$

Do the z_1 integral first in the above and if $X = \frac{-y_1'+z'+b_1^0 t}{\sqrt{2\gamma_1^0 w}}$, note the absolute value of this integral is

$$\begin{aligned}
&\Big|\int_{-\infty}^{z'} \Big[\frac{(y_1'-z_1-b_1^0 t)^2}{2\gamma_1^0 w} - 1\Big] p_{2\gamma_1^0 w}(y_1'-z_1-b_1^0 t)\, dz_1\Big| \\
&\quad = \Big|\int_{-\infty}^{X} [v^2-1] p_1(v)\, dv\Big| \\
&\quad \le c|X| e^{-X^2/2} \\
&\quad \le c|y_1'-z'-b_1^0 t| p_{2\gamma_1^0 w}(y_1'-z'-b_1^0 t), \qquad (4.12)
\end{aligned}$$

where the first inequality follows by an elementary calculation–consider $|X|\ge 1$ and $|X|<1$ separately, note that $\int_0^\infty (v^2-1)p_1(v)\, dv = 0$, and in the last case use

$$\Big|\int_{-\infty}^{X} p_1(v)[v^2-1]dv\Big| = \int_0^{|X|} p_1(v)[1-v^2]dv.$$

Take the absolute value inside the remaining integrals, then integrate over $dy_2' \ldots dy_m'$, and use (4.9) to express the third order derivative. This and (4.12) lead to

$$\begin{aligned}
J_2' \le \int\int z_{m+1}^2 \Big\{ \int\int 1_{(z'\le y_1')} \Big[\int_0^\infty \Big[\frac{|x_1-z'-b_1^0 s|^3}{(2\gamma_1^0 w')} + |x_1-z'-b_1^0 s|\Big] (2\gamma_1^0 w')^{-2} \\
\times p_{2\gamma_1^0 w'}(x_1-z'-b_1^0 s) \prod_{j=2}^m p_{2\gamma_j^0 w'}(x_j-z_j-b_1^0 s) dz_j r_s(z_{m+1}, x_{m+1}, dw')\Big] \\
\times \int_0^\infty c\frac{|-y_1'+z'+b_1^0 t|}{2\gamma_1^0 w} p_{2\gamma_1^0 w}(y_1'-z'-b_1^0 t) r_t(z_{m+1}, y_{m+1}', dw) dy_1' dz' \Big\} \\
\times \mu_{m+1}(dz_{m+1})\mu_{m+1}(dy_{m+1}').
\end{aligned}$$

Do the trivial integral over $dz_2 \ldots dz_m$ and then consider the $dy_1'\, dz'$ integral of the resulting integrand. If $z'' = z'-x_1+b_1^0 s$ and $y_1'' = y_1'-x_1-b_1^0(t-s)$, this integral equals

$$\begin{aligned}
&\int\int 1_{(z''\le y_1''+b_1^0 t)} \Big[\frac{|z''|^3}{(2\gamma_1^0 w')} + |z''|\Big] (2\gamma_1^0 w')^{-2} p_{2\gamma_1^0 w'}(z'') \frac{|y_1''-z''|}{2\gamma_1^0 w} \\
&\qquad \times p_{2\gamma_1^0 w}(y_1''-z'') dy_1''\, dz'' \\
&\quad \le c(2\gamma_1^0 w')^{-3/2} (2\gamma_1^0 w)^{-1/2}.
\end{aligned}$$

Use this in the above bound on J_2' and the symmetry of $r_s(z_{m+1}, x_{m+1}, \cdot)$ in (z_{m+1}, x_{m+1})

(recall (3.5)), and conclude

$$
\begin{aligned}
J_2' &\le c \int\int\int\int z_{m+1}^2 (w')^{-3/2} w^{-1/2} r_s(x_{m+1}, z_{m+1}, dw') r_t(z_{m+1}, y'_{m+1}, dw) \\
&\qquad\qquad \times \mu_{m+1}(dy'_{m+1}) \mu_{m+1}(dz_{m+1}) \\
&= c \int\int z_{m+1}^2 (w')^{-3/2} \mathbb{E}_{z_{m+1}} \Big(\Big(\int_0^t X_r^{(m+1)} dr \Big)^{-1/2} \Big) r_s(x_{m+1}, z_{m+1}, dw') \mu_{m+1}(dz_{m+1}) \\
&\le c \int\int (w')^{-3/2} z_{m+1}^2 t^{-1/2} (t + z_{m+1})^{-1/2} r_s(x_{m+1}, z_{m+1}, dw') \mu_{m+1}(dz_{m+1}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 3.3(f))} \\
&\le c t^{-1/2} \mathbb{E}_{x_{m+1}} \Big(\Big(X_s^{(m+1)} \Big)^{3/2} \Big(\int_0^s X_r^{(m+1)} dr \Big)^{-3/2} \Big) \qquad \text{(recall (3.6))} \\
&\le t^{-1/2} \Big(\mathbb{E}_{x_{m+1}} \Big(X_s^{(m+1)} \Big)^2 \Big)^{3/4} \Big(\mathbb{E}_{x_{m+1}} \Big(\int_0^s X_r^{(m+1)} dr \Big)^{-6} \Big)^{1/4}.
\end{aligned}
$$

Another application of Lemma 3.3(e,f) now shows

$$
J_2' \le c t^{-1/2} (x_{m+1} + s)^{3/2} (x_{m+1} + s)^{-6/4} s^{-6/4} = c t^{-1/2} s^{-3/2}.
$$

Symmetry (switching z_1 and y_1' in the integral defining J_2' amounts to switching the sign of b_1^0) gives the same bound on J_2'' and hence for J. This implies that the left-hand side of (4.1) is at most

$$
\begin{aligned}
&c t^{-1/2} s^{-3/2} \sup_y \int\int z_{m+1}^2 |D_{z_1}^2 p_s(z,x)| |D_{z_1}^2 p_t(z,y)| \mu(dx) \mu(dz) \\
&\quad \le c t^{-1/2} s^{-5/2} \sup_y \int z_{m+1} |D_{z_1}^2 p_t(z,y)| \mu(dz) \text{ (by Lemma 4.1)} \\
&\quad \le c t^{-3/2} s^{-5/2},
\end{aligned}
$$

where (4.10) is used in the last line.

This completes the proof of (4.1) with $\eta = 1/2$ for $\tilde{D}_x = x_{m+1} D_{x_1}^2$. The proof for D_{x_1} is similar and a bit easier. Finally, very similar arguments (the powers change a bit in the last part of the bound on J_2') will verify (4.2) for these operators. $\square$

Recall the notation $\hat{p}_t$ from Proposition 2.8.

Lemma 4.4. *If D_x^n and D_y^k are nth and kth order partial differential operators in x and y, respectively, then for all $t > 0$, $k, n \le 2$, $D_y^k D_x^n p_t(x,y)$ exists, is bounded, is continuous in each variable separately, and equals*

$$
\int D_y^k \hat{p}_{t/2}(y,z) D_x^n p_{t/2}(x,z) \, \mu(dz).
$$

For $k \leq 1$, $D_y^k D_x^n p_t(x,y)$ is jointly continuous.

Proof. From (3.29) we have

$$D_x^n p_t(x,y) = \int D_x^n p_{t/2}(x,z)\hat{p}_{t/2}(y,z)\mu(dz).$$

Apply (2.14), with $f(z) = D_x^n p_{t/2}(x,z)$ and $\hat{p}_{t/2}$ in place of p_t, to differentiate with respect to y through the integral and derive the above identity. Uniform boundedness in (x,y), and continuity in each variable separately follows from the boundedness in Lemma 3.11(a), the L^1-boundedness in Lemma 3.11(b) and dominated convergence. If $k =\leq 1$ the uniform boundedness of the above derivatives implies continuity in y uniformly in x and hence joint continuity. □

We now turn to the verification of (2.28) and Proposition 2.10 for $\tilde{D}_x = D_{x_{m+1}}$ or $x_{m+1}D^2_{x_{m+1}}$. The argument here seems to be much harder (at least in the second order case) and so we will reduce its proof to three technical bounds whose proofs are deferred to Section 7. Not surprisingly these proofs will rely on the representations in Lemmas 3.4 and 3.11 as well as the other explicit expressions obtained in Section 3 such as Lemmas 3.6 and 3.14.

Lemma 4.5. *There is a $c_{4.5}$ such that for all $t > 0$:*
(a) If $-(2M_0^2)^{-1} \leq p \leq 1/2$ then for all $y \in S_m$,

$$\int z_{m+1}^p |D_{z_{m+1}} p_t(z,y)|\mu(dz) \leq c_{4.5} t^{-1/2}(t + y_{m+1})^{p-1/2}, \tag{4.13}$$

and for all $z \in S_m$,

$$\int y_{m+1}^p |D_{z_{m+1}} p_t(z,y)|\mu(dy) \leq c_{4.5} t^{-1/2}(t + z_{m+1})^{p-1/2}. \tag{4.14}$$

(b) If $0 \leq q \leq 2$ and $-(2M_0^2)^{-1} \leq p \leq 2$, then for all $j \leq m$ and all $y \in S_m$

$$\int |y_j - z_j|^q z_{m+1}^p |D^2_{z_{m+1}} p_t(z,y)|\mu(dz) \leq c_{4.5} t^{q/2-1}(t + y_{m+1})^{p+q/2-1}. \tag{4.15}$$

(c) If $0 \leq q \leq 2$, then for all $j \leq m$ and $z \in S_m$,

$$\int |y_j - z_j|^q z_{m+1} |D^2_{z_{m+1}} p_t(z,y)|\mu(dy) \leq c_{4.5}[t^{q-1} + t^{q/2-1} z_{m+1}^{q/2}]. \tag{4.16}$$

(d)

$$\sup_y \int z_{m+1}^{3/2} |D^3_{z_{m+1}} p_t(z,y)|\mu(dz) \leq c_{4.5} t^{-3/2}. \tag{4.17}$$

(e) If $0 \le p \le 1/2$, then for all $(y^{(m)}, z_{m+1}) \in S_m$,

$$\int z_{m+1} y_{m+1}^p |D_{y_{m+1}} D^2_{z_{m+1}} p_t(z,y)| dz^{(m)} \mu_{m+1}(dy_{m+1}) \le c_{4.5} t^{p-2}, \tag{4.18}$$

and for all $j \le m$,

$$\int z_{m+1} y_{m+1}^p |D_{z_j} D^2_{z_{m+1}} p_t(z,y)| dz^{(m)} \mu_{m+1}(dy_{m+1}) \le c_{4.5} t^{p-2}. \tag{4.19}$$

(f) If $0 \le p \le 3/2$, then for all $j \le m$,

$$\sup_y \int z_{m+1}^p |D_{z_j} D^2_{z_{m+1}} p_t(z,y)| \mu(dz) \le c_{4.5} t^{p-3}. \tag{4.20}$$

Lemma 4.6. *There is a $c_{4.6}$ such that for all $0 < t \le s$,*

$$t^{b/\gamma} \int_0^{\gamma t} \Big[\int |D_{z_{m+1}} p_s(z,y)| dz^{(m)} \Big] dz_{m+1} \le c_{4.6} t/s, \tag{4.21}$$

and

$$t^{b/\gamma} \int_0^{\gamma t} \Big[\int |D^3_{z_{m+1}} p_s(z,y)| dz^{(m)} \Big] dz_{m+1} \le c_{4.6} t s^{-3} \le c_{4.6} s^{-2} \tag{4.22}$$

Lemma 4.7. *There is a $c_{4.7}$ such that if $1 \le p \le 2$, then for all $t > 0$, $w > 0$ and $y^{(m)} \in \mathbb{R}^m$,*

$$\begin{aligned} \int\int &(1_{(y_{m+1} \le w \le z_{m+1})} + 1_{(z_{m+1} \le w \le y_{m+1})}) z_{m+1}^p |D^2_{z_{m+1}} p_t(z,y)| \mu(dz) \mu_{m+1}(dy_{m+1}) \\ &\le c_{4.7} [1_{(w \le \gamma t)} t^{p-2+b/\gamma} + 1_{(w > \gamma t)} t^{-1/2} w^{p-3/2+b/\gamma}]. \end{aligned} \tag{4.23}$$

Assuming these results we now verify (2.28) and Proposition 2.10 for $\tilde{D}_x = D_{x_{m+1}}$ or $x_{m+1} D^2_{x_{m+1}}$. The analogue of Proposition 2.10 is immediate.

Proposition 4.8. *For $\tilde{D}_x$ as in (4.3), and all $t > 0$,*

$$\sup_x \int |\tilde{D}_x p_t(x,y)| \mu(dy) \le c_{4.8} t^{-1}. \tag{4.24}$$

$$\sup_y \int |\tilde{D}_x p_t(x,y)| \mu(dx) \le c_{4.8} t^{-1}. \tag{4.25}$$

Proof. This is immediate from Lemma 4.1, Lemma 4.2(c), Lemma 4.5(a) (with $p = 0$) and Lemma 4.5(b) (with $q = 0$ and $p = 1$). □

Proposition 4.9. *(2.28) holds for $\tilde{D}_x = D_{x_{m+1}}$ and $\eta = (2M_0^2)^{-1}$.*

Proof. Define η as above and fix $0 < t \le s \le 2$. We will verify (4.1) even with the absolute values taken inside all the integrals. By (4.14) with $p = 0$,

$$\int |D_{z_{m+1}} p_t(z, y')|\mu(dy') \le c_{4.5} t^{-1/2}(t + z_{m+1})^{-1/2} \le c_{4.5} t^{-1+\eta} z_{m+1}^{-\eta}.$$

In the last inequality we used $\eta \le 1/2$ (recall $M_0 \ge 1$). Therefore the left side of (4.1) (even with absolute values inside all the integrals) is at most

$$\begin{aligned}
&\sup_y c_{4.5} \int \Big[\int |D_{z_{m+1}} p_s(z, x)| z_{m+1}^{-\eta} \mu(dz) t^{-1+\eta} \\
&\qquad\qquad \times \Big[\int |D_{z'_{m+1}} p_s(z', x)|\, |D_{z'_{m+1}} p_t(z', y)| \mu(dz') \Big]\Big] \mu(dx) \\
&\le c_{4.5}^2 s^{-1-\eta} t^{-1+\eta} \sup_y \int \Big[\int |D_{z'_{m+1}} p_s(z', x)| \mu(dx) \Big] |D_{z'_{m+1}} p_t(z', y)| \mu(dz'), \quad (4.26)
\end{aligned}$$

where we have used (4.13) with $p = -\eta$ in the last line. Now apply (4.14) to the above integral in x and then (4.13) to the integral in z', both with $p = 0$, and conclude that (4.26) is at most

$$cs^{-2-\eta} t^{-2+\eta},$$

as required. The derivation of (4.2) (with absolute values inside the integral) is almost the same. One starts with (4.13) with $p = 0$ to bound the integral in y' as above, and then uses (4.14) with $p = -\eta$ to bound the resulting integral in z. $\square$

It remains to verify (2.28) for $\tilde{D}_x = x_{m+1} D^2_{x_{m+1}}$. This is the hard part of the proof and we will not be able to take the absolute values inside the integrals in (2.28).

Lemma 4.10. *For $j = 1, 2$, $\int |D^j_{z_{m+1}} p_t(z, y)| dz^{(m)} < \infty$ for all $z_{m+1} > 0$ and $y \in S_m$, and*

$$\int D^j_{z_{m+1}} p_t(z, y) dz^{(m)} = D^j_{z_{m+1}} \int p_t(z, y) dz^{(m)} = D^j_{z_{m+1}} q_t(z_{m+1}, y_{m+1})$$

for all $z_{m+1} > 0$ and $y \in S_m$. (4.27)

Proof. We give a proof as this differentiation is a bit delicate, and the result is used on a number of occasions. Set $j = 2$ as $j = 1$ is slightly easier. Fix $y \in S_m$ and $t > 0$. By Lemma 4.5(b,d),

$$\int |D^2_{z_{m+1}} p_t(z, y)| + 1_{(z_{m+1} > \varepsilon)} |D^3_{z_{m+1}} p_t(z, y)| \mu(dz) < \infty \text{ for all } \varepsilon > 0. \quad (4.28)$$

We claim

$$z_{m+1} \to \int D^2_{z_{m+1}} p_t(z,y)dz^{(m)} \equiv F(z_{m+1}) \text{ is continuous on } \{z_{m+1} > 0\}. \tag{4.29}$$

Note that $F(z_{m+1}) < \infty$ for almost all $z_{m+1} > 0$ by (4.28). The Fundamental Theorem of Calculus and Lemma 3.11(a) imply that if $z'_{m+1} > z_{m+1} > 0$, then

$$\begin{aligned}&\int |D^2_{z_{m+1}} p_t(z^{(m)}, z'_{m+1}, y) - D^2_{z_{m+1}} p_t(z^{(m)}, z_{m+1}, y)|\, dz^{(m)} \\ &\quad\le \int \int_{z_{m+1}}^{z'_{m+1}} |D^3_w p_t(z^{(m)}, w, y)| dw\, dz^{(m)} \to 0 \text{ as } z'_{m+1} \to z_{m+1} \text{ or } z_{m+1} \to z'_{m+1},\end{aligned}$$

where dominated convergence and (4.28) are used to show the convergence to 0. This allows us to first conclude that

$$\int |D^2_{z_{m+1}} p_t(z,y)| dz^{(m)} < \infty \text{ for all } z_{m+1} > 0, \tag{4.30}$$

and in particular, $F(z_{m+1})$ is finite for all $z_{m+1} > 0$, and also that F is continuous.

The differentiation through the integral now proceeds as in the proof of Proposition 2.8(d) given in Section 3 (using the Fundamental Theorem of Calculus). The last equality follows from (3.7) and the definition of r_t. □

Lemma 4.11. *There is a $c_{4.11}$ so that for $\tilde{D}_z \equiv \tilde{D}_{z_{m+1}} = D_{z_{m+1}}$ or $z_{m+1} D^2_{z_{m+1}}$, all $t > 0$ and all $y' \in S_m$,*

$$\Big|\int \tilde{D}_z p_t(z, y')\mu(dz)\Big| \le c_{4.11}(t + y'_{m+1})^{-1} \tag{4.31}$$

Proof. Use (4.27) to see that

$$\int \tilde{D}_z p_t(z,y')\mu(dz) = \int \tilde{D}_{z_{m+1}} q_t(z_{m+1}, y'_{m+1})\mu_{m+1}(dz_{m+1}).$$

Changing variables, we must show

$$\Big|\int \tilde{D}_z q_t(z,y) z^{b/\gamma - 1} dz\Big| \le c_{4.10}(t+y)^{-1}. \tag{4.32}$$

The arguments are the same for either choice of $\tilde{D}_z$ so let us take $\tilde{D}_z = D_z$ for which the algebra is slightly easier. Let $w = y/\gamma t$ and $x = z/\gamma t$. By differentiating the power series (3.3) the left side of (4.32) is then

$$\begin{aligned}&\Big|\int_0^\infty (\gamma t)^{-1} e^{-w} \sum_{m=0}^\infty \frac{w^m}{m!\Gamma(m+b/\gamma)} e^{-x}[m x^{m-1} - x^m] x^{b/\gamma - 1} dx\Big| \\ &\quad= \Big|(\gamma t)^{-1}\Big(\sum_{m=1}^\infty e^{-w} \frac{w^m}{m!\Gamma(m+b/\gamma)}[m\Gamma(m-1+b/\gamma) - \Gamma(m+b/\gamma)]\Big) - (\gamma t)^{-1} e^{-w}\Big| \\ &\quad= \Big|(\gamma t)^{-1}\Big(\sum_{m=1}^\infty e^{-w} \frac{w^m}{m!} \frac{1-b/\gamma}{m+b/\gamma-1}\Big) - (\gamma t)^{-1} e^{-w}\Big| \\ &\quad\le c_1(\gamma t)^{-1}[\mathbb{E}\,(N(w)^{-1} 1_{(N(w)\ge 1)}) + e^{-w}],\end{aligned}$$

where $N(w)$ is a Poisson random variable with mean w, c_1 satisfies Convention 3.1, and we have used $(n+b/\gamma-1)^{-1} \leq cn^{-1}$ for all $n \in \mathbb{N}$. An elementary calculation (e.g. Lemma 3.3 of [BP]) bounds the above by

$$c_2 t^{-1}(1 \wedge w^{-1} + e^{-w}) \leq 2c_2 t^{-1}[1 \wedge w^{-1}] \leq c_3(t+y)^{-1}.$$

(4.32) follows. $\square$

Proposition 4.12. *(2.28) holds for $\tilde{D}_x = x_{m+1} D^2_{x_{m+1}}$ and $\eta = 1/2$.*

Proof. Consider general $0 < \eta < 1$ for now. (4.1) and (4.2) (and hence (2.28)) will follow from

$$\sup_x \int \Big| \int \tilde{D}_z p_s(z,x) \tilde{D}_z p_t(z,y') \mu(dz) \Big| \mu(dy') \leq c_{4.11} s^{-1-\eta} t^{-1+\eta} \text{ for } 0 < t \leq s \leq 2, \quad (4.33)$$

and

$$\sup_x \int \Big| \int \tilde{D}_x p_s(x,z) \tilde{D}_{y'} p_t(y',z) \mu(dz) \Big| \mu(dy') \leq c_{4.11} s^{-1-\eta} t^{-1+\eta} \text{ for } 0 < t \leq s \leq 2. \quad (4.34)$$

To see (4.1), multiply both sides of (4.33) by

$$\int \int |\tilde{D}_z p_s(z,x)|\, |\tilde{D}_z p_t(z,y)| \mu(dz) \mu(dx).$$

After taking a supremum over y, the resulting left-hand side is an upper bound for the left-hand side of (4.1). For the resulting right-hand side, use (4.24) to first bound the integral in x, uniformly in z, by $c_{4.8} s^{-1}$ and (4.25) to then bound the integral in z, uniformly in y by $c_{4.8} t^{-1}$. This gives (4.1) and similar reasoning derives (4.2) from (4.34).

Next use Lemma 4.11 to see that

$$\begin{aligned} &\int \Big| \int \tilde{D}_{y'} p_s(y',x) \tilde{D}_z p_t(z,y') \mu(dz) \Big| \mu(dy') \\ &\leq \int |\tilde{D}_{y'} p_s(y',x)| c_{4.11} (t + y'_{m+1})^{-1} \mu(dy') \\ &\leq \int |D^2_{y'_{m+1}} p_s(y',x)| \mu(dy') \leq c s^{-2} \leq c s^{-1-\eta} t^{-1+\eta}. \end{aligned} \quad (4.35)$$

In the next to last inequality we have used Lemma 4.5(b) with $q = p = 0$. Therefore the triangle inequality shows that (4.33) (with perhaps a different constant) will follow from

$$\int \Big| \int (\tilde{D}_z p_s(z,x) - \tilde{D}_{y'} p_s(y',x)) \tilde{D}_z p_t(z,y') \mu(dz) \Big| \mu(dy') \leq c_{4.12} s^{-1-\eta} t^{-1+\eta}$$

$$\text{for } 0 < t \leq s \leq 2. \quad (4.36)$$

The analogous reduction for (4.34) is easier as (2.14) with $f \equiv 1$ implies

$$\int \tilde{D}_x p_s(x, y') \tilde{D}_{y'} p_t(y', z) \mu(dz) = 0.$$

Use this in place of (4.35) and again apply the triangle inequality to see that (4.34) will follow from

$$\int \left| \int (\tilde{D}_x p_s(x, z) - \tilde{D}_x p_s(x, y')) \tilde{D}_{y'} p_t(y', z) \mu(dz) \right| \mu(dy') \le c_{4.12} s^{-1-\eta} t^{-1+\eta} \qquad \text{for } 0 < t \le s \le 2. \tag{4.37}$$

Having reduced our problem to (4.36) and (4.37), we consider (4.36) first and take $\eta = 1/2$ for the rest of the proof. The left-hand side of (4.36) is bounded by

$$\begin{aligned}
&\int \left| \int (z_{m+1}(D^2_{z_{m+1}} p_s(z, x) - D^2_{y'_{m+1}} p_s(z^{(m)}, y'_{m+1}, x)) z_{m+1} D^2_{z_{m+1}} p_t(z, y') \mu(dz) \right| \mu(dy') \\
&\quad + \int \left| \int (z_{m+1} D^2_{y'_{m+1}} p_s(z^{(m)}, y'_{m+1}, x) - y'_{m+1} D^2_{y'_{m+1}} p_s(y', x)) \right. \\
&\qquad\qquad \left. \times z_{m+1} D^2_{z_{m+1}} p_t(z, y') \mu(dz) \right| \mu(dy') \\
&\quad := T_{a,1} + T_{a,2}.
\end{aligned} \tag{4.38}$$

Use the Fundamental Theorem of Calculus (recall Proposition 2.8 for the required regularity) to see that

$$\begin{aligned}
T_{a,1} \le \int \int \int &(1_{(z_{m+1} < w < y'_{m+1})} + 1_{(y'_{m+1} < w < z_{m+1})}) z_{m+1} |D^3_w p_s(z^{(m)}, w, x)| \\
&\times z_{m+1} |D^2_{z_{m+1}} p_t(z, y')| \, dw \, \mu(dz) \mu(dy').)
\end{aligned}$$

Now recall from (3.7) that $p_t(z, y) = p^0_t(z^{(m)} - y^{(m)}, z_{m+1}, y_{m+1})$. First do the $\mu(dy') \, \mu_{m+1}(dz_{m+1})$ integrals and change variables to $y'' = z^{(m)} - y'^{(m)}$ in this integral to see that

$$\begin{aligned}
T_{a,1} &\le \int \int \int \int \int (1_{(z_{m+1} < w < y'_{m+1})} + 1_{(y'_{m+1} < w < z_{m+1})}) \\
&\qquad \times z^2_{m+1} |D^2_{z_{m+1}} p^0_t(y'', z_{m+1}, y'_{m+1})| dy'' \mu_{m+1}(dz_{m+1}) \mu(dy_{m+1}) \\
&\qquad \times |D^3_w p_s(z^{(m)}, w, x)| dz^{(m)} dw \\
&\le c_{4.7} \int_{\gamma t}^{\infty} \int t^{-1/2} w^{3/2} |D^3_w p_s(z^{(m)}, w, x)| dz^{(m)} \, w^{b/\gamma - 1} dw \\
&\qquad + c_{4.7} \int_0^{\gamma t} \int t^{b/\gamma} |D^3_w p_s(z^{(m)}, w, x)| dz^{(m)} dw.
\end{aligned} \tag{4.39}$$

In the last line we have used Lemma 4.7 with $p = 2$. Now use Lemma 4.5(d) to bound the first term by $ct^{-1/2}s^{-3/2}$, and use (4.22) to bound the second term by $cs^{-2} \le ct^{-1/2}s^{-3/2}$. We have proved

$$T_{a,1} \le c_1 t^{-1/2} s^{-3/2}. \tag{4.40}$$

Note that

$$\begin{aligned} T_{a,2} \le & \int \Big| \int (z_{m+1}(D^2_{y'_{m+1}} p_s(z^{(m)}, y'_{m+1}, x) - D^2_{y'_{m+1}} p_s(y', x)) \\ & \qquad \times z_{m+1} D^2_{z_{m+1}} p_t(z, y') \mu(dz) \Big| \mu(dy') \\ & + \int \Big| \int (z_{m+1} - y'_{m+1}) D^2_{y'_{m+1}} p_s(y', x) z_{m+1} D^2_{z_{m+1}} p_t(z, y') \mu(dz) \Big| \mu(dy') \\ := & T_{a,3} + T_{a,4}. \end{aligned} \tag{4.41}$$

By Lemma 4.10,

$$\int D^2_{z_{m+1}} p_t(z, y') dz^{(m)} = D^2_{z_{m+1}} q_t(z_{m+1}, y'_{m+1}),$$

and so using Lemma 3.3(g) we have

$$\begin{aligned} T_{a,4} = & \int \Big| \int (z_{m+1} - y'_{m+1}) z^{b/\gamma}_{m+1} D^2_{z_{m+1}} q_t(z_{m+1}, y'_{m+1}) dz_{m+1} \Big| |D^2_{y'_{m+1}} p_s(y', x))| \mu(dy') \\ \le & c_{3.3} \int |D^2_{y'_{m+1}} p_s(y', x))| \mu(dy') \le c_4 s^{-2} \le c_4 t^{-1/2} s^{-3/2}, \end{aligned} \tag{4.42}$$

where we have used Lemma 4.5(b) in the last line with $p = q = 0$.

For $T_{a,3}$ use the Fundamental Theorem of Calculus to write

$$\begin{aligned} T_{a,3} = & \int \Big| \int \Big[\int_0^1 \sum_{j=1}^m (z_j - y'_j) D_j D^2_{y'_{m+1}} p^0_s(y'^{(m)} - x^{(m)} + r(z^{(m)} - y'^{(m)}), y'_{m+1}, x_{m+1})\, dr \Big] \\ & \times z^2_{m+1} D^2_{z_{m+1}} p^0_t(z^{(m)} - y'^{(m)}, z_{m+1}, y'_{m+1}) dz^{(m)} \mu_{m+1}(dz_{m+1}) \Big| dy'^{(m)} \mu_{m+1}(dy'_{m+1}). \end{aligned}$$

Now take the absolute values inside the integrals and summation, do the integral in r last, and for each r carry out the linear change of variables for the other ($2m$-dimensional) Lebesgue integrals: $(u, w) = (z^{(m)} - y'^{(m)}, y'^{(m)} - x^{(m)} + r(z^{(m)} - y'^{(m)}))$ (noting that $|dz^{(m)} dy'^{(m)}| \le 2^m |du\, dw|$ for all $0 \le r \le 1$. This shows

$$\begin{aligned} T_{a,3} \le & c \sum_{j=1}^m \int \int \Big[\int \int |u_j| z^2_{m+1} |D^2_{z_{m+1}} p^0_t(u, z_{m+1}, y'_{m+1})| du\, \mu_{m+1}(dz_{m+1}) \Big] \\ & \times |D_{w_j} D^2_{y'_{m+1}} p^0_s(w, y'_{m+1}, x)|\, dw\, \mu_{m+1}(dy'_{m+1}) \\ \le & c \sum_{j=1}^m t^{-1/2} \int \int (t^{3/2} + (y'_{m+1})^{3/2}) |D_{w_j} D^2_{y'_{m+1}} p^0_s(w, y'_{m+1}, x_{m+1})|\, dw\, \mu_{m+1}(dy'_{m+1}). \end{aligned} \tag{4.43}$$

For the last inequality we have used Lemma 4.5(b) with $q = 1$ and $p = 2$. Now use Lemma 4.5(f) with $p = 0$ (for the $t^{3/2}$ term) and then with $p = 3/2$ (for the $(y'_{m+1})^{3/2}$ term) to conclude that

$$T_{a,3} \leq c[ts^{-3} + t^{-1/2}s^{-3/2}] \leq c_3 t^{-1/2}s^{-3/2} \tag{4.44}$$

Combining (4.40), (4.42) and (4.44) now gives (4.36) (with $\eta = 1/2$).

The left side of (4.37) is at most

$$\begin{aligned}
&\int\Big|\int x_{m+1}(D^2_{x_{m+1}}p_s(x,z) - D^2_{x_{m+1}}p_s(x,z^{(m)},y'_{m+1}))y'_{m+1}D^2_{y'_{m+1}}p_t(y',z)\mu(dz)\Big|\mu(dy')\\
&\quad+\int\Big|\int x_{m+1}(D^2_{x_{m+1}}p_s(x,z^{(m)},y'_{m+1}) - D^2_{x_{m+1}}p_s(x,y'))y'_{m+1}D^2_{y'_{m+1}}p_t(y',z)\mu(dz)\Big|\mu(dy')\\
&\quad:= T_{b,1} + T_{b,2}.
\end{aligned} \tag{4.45}$$

The Fundamental Theorem of Calculus gives (Lemma 4.4 gives the required regularity)

$$\begin{aligned}
T_{b,1} \leq \int\int\int\int\int &(1_{(z_{m+1}<w<y'_{m+1})} + 1_{(y'_{m+1}<w<z_{m+1})})x_{m+1}\\
&\times |D_w D^2_{x_{m+1}}p_s(x,z^{(m)},w)|y'_{m+1}|D^2_{y'_{m+1}}p_t(y',z)|\\
&dw\, dz^{(m)}\, dy'^{(m)}\mu_{m+1}(dz_{m+1})\mu_{m+1}(dy'_{m+1}).
\end{aligned}$$

Re-express p_t in terms of p^0_t and set $y'' = y'^{(m)} - z^{(m)}$ to conclude

$$\begin{aligned}
T_{b,1} \leq \int\int\int &(1_{(z_{m+1}<w<y'_{m+1})} + 1_{(y'_{m+1}<w<z_{m+1})})\\
&\times y'_{m+1}|D^2_{y'_{m+1}}p^0_t(y'',y'_{m+1},z_{m+1}))|\,dy''\mu_{m+1}(dy'_{m+1})\mu_{m+1}(dz_{m+1})\\
&\times x_{m+1}|D_w D^2_{x_{m+1}}p_s(x,z^{(m)},w)|dw\,dz^{(m)}\\
\leq c_{4.7}\int_{\gamma t}^{\infty}\int &t^{-1/2}w^{1/2}w^{b/\gamma-1}x_{m+1}|D_w D^2_{x_{m+1}}p_s(x,z^{(m)},w)|dz^{(m)}dw\\
+ c_{4.7}\int_0^{\gamma t}\int &t^{b/\gamma-1}x_{m+1}|D_w D^2_{x_{m+1}}p_s(x,z^{(m)},w)|dz^{(m)}dw.
\end{aligned} \tag{4.46}$$

In the last line we used Lemma 4.7 with $p = 1$. Now use (4.18) with $p = 1/2$ to bound the first term by $c_{4.5}c_{4.7}t^{-1/2}s^{-3/2}$. By Lemma 4.4, the second term in (4.46) is at most

$$\begin{aligned}
&ct^{-1}\int\Big[\int_0^{\gamma t}\int t^{b/\gamma}|D_w\hat{p}_{s/2}(z^{(m)},w,z')|dz^{(m)}dw\Big]\,x_{m+1}|D^2_{x_{m+1}}p_{s/2}(x,z')|\mu(dz')\\
&\leq ct^{-1}(t/s)s^{-1} \leq cs^{-2} \leq ct^{-1/2}s^{-3/2},
\end{aligned}$$

where we have used (4.21) then (4.5). (We are applying (4.21) to $\hat{p}_{s/2}$.) We have shown

$$T_{b,1} \le ct^{-1/2}s^{-3/2}. \tag{4.47}$$

For $T_{b,2}$, an argument similar to that leading to (4.43) bounds $T_{b,2}$ above by

$$\begin{aligned}
&c\sum_{j=1}^{m}\int\int\Big[\int\int |u_j| y'_{m+1}|D^2_{y'_{m+1}}p^0_t(u, y'_{m+1}, z_{m+1})|du\, \mu_{m+1}(dz_{m+1})\Big]\\
&\qquad\times x_{m+1}|D_{w_j}D^2_{x_{m+1}}p^0_s(w, x_{m+1}, y'_{m+1})|dw\, \mu_{m+1}(dy'_{m+1})\\
&\le c\int\int c_{4.5}[1+t^{-1/2}(y'_{m+1})^{1/2}]x_{m+1}|D_{w_j}D^2_{x_{m+1}}p^0_s(w, x_{m+1}, y'_{m+1})|dw\, \mu_{m+1}(dy'_{m+1}).
\end{aligned}$$

In the last line we have used the identity $p^0_t(u, y'_{m+1}, z_{m+1}) = p_t(0, y'_{m+1}, -u, z_{m+1})$ and then Lemma 4.5(c) with $q = 1$. Finally use (4.19) with $p = 0$ and $p = 1/2$ to bound the above by $c(s^{-2} + t^{-1/2}s^{-3/2}) \le ct^{-1/2}s^{-3/2}$. Use this and (4.47) in (4.45) to complete the proof of (4.37). □

Having obtained (2.28) and Proposition 2.10 for the special case N_2 null and $Z\cap C = \{d\}$, we now turn to the general case. In the rest of this section we work in the general setting of Propositions 2.2 and 2.14.

Proof of Proposition 2.14. We need to establish (2.28) (thanks to Lemma 2.11(b)), and first do this for the special case when our transition density is $q_t = q^{b,\gamma}_t$, that is $Z\cap C$ empty and N_2 a singleton. (See the beginning of Section 2 for the definition of N_2.) Let p_t be the transition density considered above with $m = 1$. Recall from (3.7) that

$$\int p_t(x, z)dz_1 = \int p_t(x, z)dx_1 = q_t(x_2, z_2). \tag{4.48}$$

Let $\tilde{D}_{y_2} = D_{y_2}$ or $y_2D_{y_2y_2}$. We claim that we can differentiate through the above integrals and so

$$\int \tilde{D}_{x_2}p_t(x, z)dz_1 = \tilde{D}_{x_2}q_t(x_2, z_2) \text{ for almost all } z_2 > 0 \text{ and all } x, \tag{4.49}$$

and

$$\int \tilde{D}_{x_2}p_t(x, z)dx_1 = \tilde{D}_{x_2}q_t(x_2, z_2) \text{ for all } x_2 > 0 \text{ and } z. \tag{4.50}$$

Lemma 4.10 implies (4.50). The proof of (4.49) uses $\int |\tilde{D}_{x_2}p_t(x, z)|dz_1 < \infty$ for a.a. $z_2 > 0$ by Lemma 3.11(b), and then proceeds using the Fundamental Theorem of Calculus as in the proof of Proposition 2.8(d) in Section 3. (The stronger version of (4.49) also holds but this result will suffice.)

Consider first (4.2) for q_t. Let $0 < t \le s \le 2$. By (4.49) and (4.50), we have for all $x_2, y_2 > 0$,

$$
\begin{aligned}
&\Big|\int \tilde{D}_{x_2} q_s(x_2, z_2)\tilde{D}_{y_2} q_t(y_2, z_2)\mu_2(dz_2)\Big| \\
&\quad = \Big|\int \Big[\int \tilde{D}_{x_2} p_s(x, z)dx_1\Big]\Big[\int \tilde{D}_{y_2} p_t(y, z)dz_1\Big]\mu_2(dz_2)\Big| \\
&\quad \le \int \Big|\int \tilde{D}_{x_2} p_s(x, z)\tilde{D}_{y_2} p_t(y, z)\mu(dz)\Big| dx_1. \qquad (4.51)
\end{aligned}
$$

Similarly, for all $x_2 > 0$,

$$
\begin{aligned}
&\int \Big| \int \tilde{D}_{x_2} q_s(x_2, z_2)\tilde{D}_{y'_2} q_t(y'_2, z_2)\mu_2(dz_2)\Big|\mu_2(dy'_2) \\
&\quad = \inf_{x_1} \int \Big|\int \Big[\int \tilde{D}_{x_2} p_s(x, z)dz_1\Big]\Big[\int \tilde{D}_{y'_2} p_t(y', z)dy'_1\Big]\mu_2(dz_2)\Big|\mu_2(dy'_2) \\
&\quad \le \inf_{x_1} \int \Big|\int \tilde{D}_{x_2} p_s(x, z)\tilde{D}_{y'_2} p_t(y', z)\mu(dz)\Big|\mu(dy'). \qquad (4.52)
\end{aligned}
$$

Integrability issues are handled by Lemma 4.10 and Proposition 4.8. The infimum in the second line can be omitted as the expression following does not depend on x_1. Multiply (4.52) and (4.51) and integrate with respect to $\mu_2(dx_2)$ to see that for any $y_2 > 0$,

$$
\begin{aligned}
&\int \Big|\int \tilde{D}_{x_2} q_s(x_2, z_2)\tilde{D}_{y'_2} q_t(y'_2, z_2)\mu_2(dz_2)\Big| \\
&\qquad \times \Big|\int \tilde{D}_{x_2} q_s(x_2, z_2)\tilde{D}_{y_2} q_t(y_2, z_2)\mu_2(dz_2)\Big|\mu_2(dy'_2)\mu_2(dx_2) \\
&\le \int \Big|\int \tilde{D}_{x_2} p_s(x, z)\tilde{D}_{y'_2} p_t(y', z)\mu(dz)\Big|\Big|\int \tilde{D}_{x_2} p_s(x, z)\tilde{D}_{y_2} p_t(y, z)\mu(dz)\Big|\mu(dy')\mu(dx) \\
&\le cs^{-2-\eta}t^{-2+\eta},
\end{aligned}
$$

the last by Proposition 4.12. This gives (4.2) for q_t. A similar argument works for (4.1).

Next we consider (2.28) in the general case. Write $x = ((x_{(i)})_{i\in Z\cap C}, (x_j)_{j\in N_2})$ so that (from (2.5))

$$
p_t(x, y) = \prod_{j\in Z\cap C} p_t^j(x_{(j)}, y_{(j)}) \prod_{j\in N_2} q_t^j(x_j, y_j). \qquad (4.53)
$$

For $j \in (Z\cap C)\cup N_2$, let $x_{\hat{j}}$ denote x but with $x_{(j)}$ (if $j \in Z\cap C$) or x_j (if $j \in N_2$) omitted, $\mu_{\hat{j}} = \prod_{i\neq j} \mu_i$, and let $p_t^{\hat{j}}(x_{\hat{j}}, y_{\hat{j}})$ denote the above product of transition densities but with the jth factor (which may be a p_t^j or a q_t^j) omitted. Consider (4.2) and let $\tilde{D}_x \equiv \tilde{D}_{x_{(j)}}$ be one of the differential operators in Notation 2.9 acting on the variable $j' \in \{j\}\cup R_j$ for some $j \in Z\cap C$. (The case $j' = j \in N_2$ is considered below.) In this case (4.53) shows that the left-hand side of (4.2) equals

$$\sup_y \int \Big[\int \Big| \int \tilde{D}_{x_{(j)}} p_s^j(x_{(j)}, z_{(j)}) \tilde{D}_{y'_{(j)}} p_t^j(y'_{(j)}, z_{(j)}) p_s^{\hat{j}}(x_{\hat{j}}, z_{\hat{j}}) p_t^{\hat{j}}(y'_{\hat{j}}, z_{\hat{j}}) \\ \mu_j(dz_{(j)}) \mu_{\hat{j}}(dz_{\hat{j}}) \Big| \mu_j(dy'_{(j)}) \mu_{\hat{j}}(dy'_{\hat{j}}) \\ \times \Big| \int \tilde{D}_{x_{(j)}} p_s^j(x_{(j)}, z_{(j)}) \tilde{D}_{y_{(j)}} p_t^j(y_{(j)}, z_{(j)}) p_s^{\hat{j}}(x_{\hat{j}}, z_{\hat{j}}) p_t^{\hat{j}}(y_{\hat{j}}, z_{\hat{j}}) \\ \mu_j(dz_{(j)}) \mu_{\hat{j}}(dz_{\hat{j}}) \Big| \Big] \mu_j(dx_{(j)}) \mu_{\hat{j}}(d\hat{x}_j).$$

Take the absolute values inside the two $\mu_{\hat{j}}(dz_{\hat{j}})$ integrals (giving an upper bound) and pull the $p^{\hat{j}}$ terms out of the $\mu_j(dz_{(j)})$ integrals. Now we can integrate the $p^{\hat{j}}$ integrals using (3.8) by first integrating over $y'_{\hat{j}}$, then the first $z_{\hat{j}}$ integral, then the $x_{\hat{j}}$ integral, and finally the second $z_{\hat{j}}$ integral. This shows that the left-hand side of (4.2) is at most

$$\sup_{y_{(j)}} \int \int \Big| \int \tilde{D}_{x_{(j)}} p_s^j(x_{(j)}, z_{(j)}) \tilde{D}_{y'_{(j)}} p_t^j(y'_{(j)}, z_{(j)}) \mu_j(dz_{(j)}) \Big| \mu_j(dy'_{(j)}) \\ \times \Big| \int \tilde{D}_{x_{(j)}} p_s^j(x_{(j)}, z_{(j)}) \tilde{D}_{y_{(j)}} p_t^j(y_{(j)}, z_{(j)}) \mu_j(dz_{(j)}) \Big| \mu_{(j)}(dx_{(j)}) \\ \leq c s^{-2-\eta} t^{-2+\eta}.$$

In the last line we used (4.2) for p^j (i.e., Proposition 4.12). For $j' = j \in N_2$ we would use (4.2) for q_t, which was established above. This completes the proof of (4.2) and the proof for (4.1) is similar. □

Proof of Proposition 2.10. By Proposition 4.8 the required result holds for each p_t^j factor and then using (4.49) and (4.50), one easily verifies it for q_t (as was done implicitly in the previous proof). The general case now follows easily from the product structure (4.53) and a short calculation which is much simpler than that given above. □

5. Proof of Proposition 2.3.

Assume $\mathbb{P}$ is a solution of $M(\tilde{\mathcal{A}}, \nu)$ where $\tilde{\mathcal{A}}$ is as in (2.2) and $d\nu = \rho\, d\mu$ is as in Proposition 2.3, and assume (2.7) throughout this section. Without loss of generality we may work on a probability space carrying a d-dimensional Brownian motion B and realize $\mathbb{P}$ as the law of a solution X of

$$X_t^i = X_0^i + \int_0^t \sqrt{2\tilde{\gamma}_i(X_s) X_s^i}\, dB_s^i + \int_0^t \tilde{b}_i(X_s)\, ds, \quad i \notin N_1,$$
$$X_t^j = X_0^j + \int_0^t \sqrt{2\tilde{\gamma}_j(X_s) X_s^i}\, dB_s^j + \int_0^t \tilde{b}_j(X_s)\, ds, \quad j \in R_i, i \in Z \cap C.$$

Set $[s]_n = (([ns]-1)/n) \vee 0$, define $\tilde{\gamma}_j(X_s) \equiv \gamma^0_j$, $\tilde{b}_j(X_s) \equiv b^0_j$ if $s < 0$, and consider the unique solution X^n to

$$
\begin{aligned}
X^{n,i}_t &= X^i_0 + \int_0^t \sqrt{2\tilde{\gamma}_i(X_{[s]_n})X^{n,i}_s}\, dB^i_s + \int_0^t \tilde{b}_i(X_{[s]_n})\, ds, \quad i \notin N_1, \\
X^{n,j}_t &= X^j_0 + \int_0^t \sqrt{2\tilde{\gamma}_j(X_{[s]_n})X^{n,i}_s}\, dB^j_s + \int_0^t \tilde{b}_j(X_{[s]_n})\, ds, \quad j \in R_i, i \in Z \cap C.
\end{aligned} \tag{5.1}
$$

Note that

for $k \geq 0$, on $[\frac{k}{n}, \frac{k+1}{n}]$ and conditional on $\mathcal{F}_{k/n}$, X^n has generator $\mathcal{A}^0$ but with γ^0, b^0 replaced with the (random) $\gamma^k \equiv \tilde{\gamma}(X_{(k-1)/n})$, $b^k \equiv \tilde{b}(X_{(k-1)/n})$. (5.2)

We see in particular that pathwise uniqueness of X^n follows from the classical Yamada-Watanabe theorem. An easy stochastic calculus argument, using Burkholder's inequalities and the boundedness of $\tilde{\gamma}$, $\tilde{b}$ and X_0, shows that

$$
\mathbb{E}\left((X^{n,i}_t)^p\right) \leq c_p(1+t^p) \text{ for all } t \geq 0 \text{ and } p \in \mathbb{N}. \tag{5.3}
$$

Here c_p may depend on the aforementioned bounds and is independent of n (although we will not need the latter).

By making only minor modifications in the proof of Lemma 5.1 in [ABBP] we have:

Lemma 5.1. *For any $T > 0$, $\sup_{t \leq T} \|X^n_t - X_t\| \to 0$ in probability as $n \to \infty$.*

For $k \in \mathbb{Z}_+$, let

$$
\mu_k(dx) = \prod_{i \in Z \cap C} \Big(\prod_{j \in R_i} dx_j\Big) x_i^{b^k_i/\gamma^k_i - 1} dx_i \times \prod_{j \in N_2} x_j^{b^k_j/\gamma^k_j - 1} dx_j,
$$

and let $p^k_t(x,y)$ denote the (random) transition density with respect to μ_k of the diffusion described in (5.2) operating on the interval $[k/n, (k+1)/n]$. Proposition 2.8(b) with $n = 0$ implies that

$$
p^k_t(x,y) \leq c_{5.1} t^{-c_{5.1}} \prod_{j \in N_2} (t^{1/2} + y_j^{1/2}) \leq c_{5.1} t^{-c_{5.1}} \prod_{j \in N_2} (1 + y_j^{1/2}), \text{ for } x, y \in S^0, 0 < t \leq 1, \tag{5.4}
$$

where as usual $c_{5.1}$ may depend on M_0, d but not on k. We are also using (2.7) here to bound b^k_i/γ^k_i.

Let $S^n_\lambda f = \mathbb{E}\left(\int_0^\infty e^{-\lambda t} f(X^n_t)\, dt\right)$ and define $\|S^n_\lambda\| = \sup\{|S^n_\lambda f| : \|f\|_2 \leq 1\}$, where as usual the L^2 norm refers to the fixed measure μ.

Lemma 5.2. *If (2.7) holds, then* $\|S^n_\lambda\| < \infty$ *for all* $\lambda > 0$, $n \in \mathbb{N}$.

Proof. It suffices to consider $|S^n_\lambda f|$ for non-negative $f \in L^2(\mu)$. Let

$$\delta = \sup_{i \notin N_1, x} \Big| \frac{\tilde{b}^i(x)}{\tilde{\gamma}^i(x)} - \frac{b^0_i}{\gamma^0_i} \Big|. \tag{5.5}$$

A bit of algebra using (2.7) shows that

$$\delta \le \frac{\varepsilon_0 M_0^2}{M_0^{-1} - \varepsilon_0} \le 2\varepsilon_0 M_0^3. \tag{5.6}$$

Let $\mathbb{E}^{\,k}_x$ denote expectation starting at x with respect to the law of the diffusion with (random) transition density p^k, and let R^k_λ and $r^k_\lambda(x,y)$ denote the corresponding resolvent and resolvent density with respect to μ_k. Then (5.2) shows that

$$\begin{aligned} S^n_\lambda f &= \sum_{k=0}^\infty e^{-\lambda k/n} \mathbb{E}\left(\mathbb{E}^{\,k}_{X^n_{k/n}} \Big(\int_0^{1/n} e^{-\lambda t} f(X_t)\, dt \Big) \right) \\ &\le \int R^0_\lambda f(x) \rho(x) d\mu(x) + \sum_{k=1}^\infty e^{-\lambda k/n} \mathbb{E}\left(\mathbb{E}\left(R^k_\lambda f(X^n_{k/n}) | \mathcal{F}_{(k-1)/n} \right)\right) \\ &\le \|R^0_\lambda f\|_2 \|\rho\|_2 + \sum_1^\infty e^{-\lambda k/n} \mathbb{E}\Big(\int R^k_\lambda f(x) p^{k-1}_{1/n}(X^n_{(k-1)/n}, x) \mu_{k-1}(dx) \Big) \\ &\le \lambda^{-1} \|\rho\|_2 \|f\|_2 + \sum_1^\infty e^{-\lambda k/n} \mathbb{E}\left(I^n_k \right), \end{aligned} \tag{5.7}$$

where

$$I^n_k = \int\int r^k_\lambda(x,y) p^{k-1}_{1/n}(X^n_{(k-1)/n}, x) \mu_{k-1}(dx) f(y) \prod_{i \notin N_1} (y_i^\delta + y_i^{-\delta}) \mu(dy)$$

and we have used Corollary 2.12(a) to see $\|R^0_\lambda f\|_2 \le \lambda^{-1} \|f\|_2$.

As usual we suppress dependence on d and M_0 in our constants (which may change from line to line) but will record dependence on n. Use (5.4) and (5.5) to see that

$$\begin{aligned} &\int r^k_\lambda(x,y) p^{k-1}_{1/n}(X^n_{(k-1)/n}, x) \mu_{k-1}(dx) \\ &\le c_n \int r^k_\lambda(x,y) \prod_{j \in N_2} (1 + \sqrt{x_j}) \prod_{i \notin N_1} (x_i^{2\delta} + x_i^{-2\delta}) \mu_k(dx) \\ &\le c_n \int r^k_\lambda(x,y) \prod_{i \notin N_1} (x_i^{-2\delta} + x_i^{1/2 + 2\delta}) \mu_k(dx). \end{aligned} \tag{5.8}$$

Let $\hat{\mathbb{E}}_x^k$ denote expectation with respect to the diffusion with transition kernel $\hat{p}_t^k(x,y) = p_t^k(y,x)$ (as in Proposition 2.8(a)).Then (5.8) is bounded by

$$\begin{aligned}
&c_n \int_0^\infty e^{-\lambda s} \hat{\mathbb{E}}_y^k\Big(\prod_{i \notin N_1} ((X_s^i)^{-2\delta} + (X_s^i)^{1/2+2\delta})\Big)\, ds \\
&= c_n \int_0^\infty e^{-\lambda s} \prod_{i \notin N_1} \hat{\mathbb{E}}_y^k\Big((X_s^i)^{-2\delta} + (X_s^i)^{1/2+2\delta})\Big)\, ds \text{ (by (2.5))} \\
&\le c_n \int_0^\infty e^{-\lambda s} \prod_{i \notin N_1} \Big(s^{-2\delta} + [\hat{\mathbb{E}}_y^k(X_s^i)]^{1/2+2\delta}\Big)\, ds.
\end{aligned}$$

In the next to last line we have used the conditional independence of $\{X^i : i \notin N_1\}$ (recall (2.5)). In the last line we have used (5.6) and (2.7) to see that $1/2 + 2\delta \le 1$, and also used Lemma 3.3(d). We can apply this last result because for all $i \notin N_1$ and all $k \in \mathbb{Z}_+$,

$$\begin{aligned}
b_i^k/\gamma_i^k - 2\delta &\ge \frac{M_0^{-1} - \varepsilon_0}{M_0 + \varepsilon_0} - 2\delta \\
&\ge (4M_0^2)^{-1} - 4\varepsilon_0 M_0^3 \ge (8M_0^2)^{-1},
\end{aligned} \tag{5.9}$$

thanks to (5.6) and (2.7). For $i \notin N_1$ we have $\mathbb{E}_y^k(X_s^i) \le y_i + M_0 s$, and by (2.7) and (5.6) we have $2d\delta \le 1/12$. Therefore if $f(s) = s^{-2\delta} + (M_0 s)^{1/2+2\delta}$ (clearly $f \ge 1$) we may now bound (5.8) by

$$\begin{aligned}
&c_n \int_0^\infty e^{-\lambda s} \prod_{i \notin N_1} (f(s) + y_i^{1/2+2\delta})\, ds \\
&\le c_n \int_0^\infty e^{-\lambda s} f(s)^{|N_1^c|} \prod_{i \notin N_1} (1 + (y_i^{1/2+2\delta}/f(s)))\, ds \\
&\le c_n \int_0^\infty e^{-\lambda s} f(s)^{|N_1^c|}\, ds \prod_{i \notin N_1} (1 + y_i^{1/2+2\delta}) \le c_{n,\lambda} \prod_{i \notin N_1} (1 + y_i^{1/2+2\delta}).
\end{aligned}$$

In the definition of I_k^n use Hölder's inequality and then this bound on (5.8) on one of the resulting squared factors to see that

$$\begin{aligned}
I_k^n &\le c\|f\|_2 \Big[\int \Big[\int r_\lambda^k(x,y) p_{1/n}^{k-1}(X_{(k-1)/n}^n, x)\mu_{k-1}(dx)\Big]^2 \prod_{i \notin N_1} (y_i^{2\delta} + y_i^{-2\delta})\mu(dy)\Big]^{1/2} \\
&\le c_{n,\lambda}\|f\|_2 \Big[\int \int r_\lambda^k(x,y) p_{1/n}^{k-1}(X_{(k-1)/n}^n, x) \prod_{i \notin N_1} (1 + y_i^{1/2+2\delta}) \\
&\qquad \times \prod_{i \notin N_1} (y_i^{2\delta} + y_i^{-2\delta})\mu(dy)\mu_{k-1}(dx)\Big]^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&\leq c_{n,\lambda}\|f\|_2\Big[\int p_{1/n}^{k-1}(X^n_{(k-1)/n},x)\int r^k_\lambda(x,y)\prod_{i\notin N_1}(1+y_i^{1/2+2\delta})\\
&\qquad\times\prod_{i\notin N_1}(y_i^{3\delta}+y_i^{-3\delta})\mu_k(dy)\mu_{k-1}(dx)\Big]^{1/2}\\
&\leq c_{n,\lambda}\|f\|_2\Big[\int p_{1/n}^{k-1}(X^n_{(k-1)/n},x)J_{k,\lambda}(x)\mu_{k-1}(dx)\Big]^{1/2}. \qquad (5.10)
\end{aligned}
$$

Here

$$
\begin{aligned}
J_{k,\lambda}(x) &= \int r^k_\lambda(x,y)\prod_{i\notin N_1}(y_i^{1/2+5\delta}+y_i^{-3\delta})\mu_k(dy)\\
&= \int_0^\infty e^{-\lambda s}\mathbb{E}^{\,k}_x\Big(\prod_{i\notin N_1}((X^i_s)^{1/2+5\delta}+(X^i_s)^{-3\delta})\Big)\,ds\\
&\leq c\int_0^\infty e^{-\lambda s}\prod_{i\notin N_1}\mathbb{E}^{\,k}_x\Big(X^i_s+(X^i_s)^{-3\delta}\Big)\,ds\\
&\leq c\int_0^\infty e^{-\lambda s}\prod_{i\notin N_1}[x_i+M_0s+s^{-3\delta}]\,ds.
\end{aligned}
$$

In the next to last line we have again used the independence of $X^i, i\notin N_1$ under $\mathbb{E}^{\,k}_x$, and the bound $1/2+5\delta\leq 1$ which follows from (5.6) and (2.7). In the last line we have again used Lemma 3.3(d) whose applicability can again be checked as in (5.9). An elementary calculation on the above bound, again using (5.6) and (2.7) to see that $3\delta d\leq 1/8$, now shows that

$$
J_{k,\lambda}(x)\leq c_\lambda\prod_{i\notin N_1}(1+x_i),
$$

and therefore by (5.10),

$$
\begin{aligned}
I^n_k &\leq c_{n,\lambda}\|f\|_2\Big[\int p_{1/n}^{k-1}(X^n_{(k-1)/n},x)\prod_{i\notin N_1}(1+x_i)\mu_{k-1}(dx)\Big]^{1/2}\\
&\leq c_{n,\lambda}\|f\|_2\prod_{i\notin N_1}\Big(1+X^{n,i}_{(k-1)/n}+\frac{M_0}{n}\Big).
\end{aligned}
$$

Take expectations in the above and use some elementary inequalities to conclude that

$$
\begin{aligned}
\mathbb{E}\,(I^n_k) &\leq c_{n,\lambda}\|f\|_2(1+M_0/n)^d\,\mathbb{E}\,\Big(\prod_{i\notin N_1}(1+X^{n,i}_{(k-1)/n})\Big)\\
&\leq c_{n,\lambda}\|f\|_2\sum_{i\notin N_1}\mathbb{E}\,((1+X^{n,i}_{(k-1)/n})^d)\\
&\leq c_{n,\lambda}\|f\|_2(1+(k/n)^d),
\end{aligned}
$$

the last by (5.3). Put this bound into (5.7) to complete the proof. □

Remark 5.3. The above argument is considerably longer than its counterpart (Lemma 5.3) in [ABBP]. This is in part due to the non-compact state space in the above leading to the unboundedness of the densities on this state space (recall the bound (5.4)). More significantly, it is also because the argument in [ABBP] is incomplete. In (5.4) of [ABBP] the norm on f actually depends on k and is not the norm on the canonical L^2 space. The argument above, however, will also give a correct proof of Lemma 5.3 of [ABBP]–in fact the compact state space there leads to considerable simplification.

Proof of Proposition 2.3. This now proceeds by making only minor changes in the proof of Proposition 2.3 in Section 5 of [ABBP]. One uses the above Lemmas 5.1, 5.2 and Proposition 2.2. We only point out the (trivial) changes required. For $f \in C_b^2(S^0)$ one uses Itô's Lemma and (5.1) to obtain the semimartingale decomposition of $f(X_t^n)$. The local martingale part is a martingale as in the proof of Theorem 2.1 in Section 2 (use (5.3)). Corollary 2.12(a) is used, instead of the eigenfunction expansion in [ABBP], to conclude that the constant coefficient resolvent R_λ has bound λ^{-1} as an operator on L^2. The rest of the proof proceeds as in [ABBP] where the bound $\varepsilon_0 \leq (2K(M_0))^{-1}$ is used to get the final bound, first on $S_\lambda^n(|f|)$, and then on $S_\lambda(|f|)$ by Fatou's lemma. □

6. Proof of Proposition 2.4.

Let $(\mathbb{P}^x, X_t)$ $(x \in S^0)$ be as in the statement of Proposition 2.4. Throughout this section, for any Borel set A we let $T_A = T_A(X) = \inf\{t : X_t \in A\}$ and $\tau_A = \tau_A(X) = \inf\{t : X_t \notin A\}$, be the first entrance and exit times, respectively, and let $|A|$ denote the Lebesgue measure of A. We say a function h is harmonic in $D = B(x,r) \cap S^0$ if h is bounded on $\overline{D}$ and $h(X_{t\wedge\tau_D})$ is a right continuous martingale with respect to $\mathbb{P}_y$ for each y.

The key step in the proof of Proposition 2.4 is the following.

Proposition 6.1. *Let $z \in S^0$. There exist positive constants $r, c_{6.1}$ and α, depending on z, such that if h is harmonic in $B(z,r) \cap S^0$, then*

$$|h(x) - h(z)| \leq c_{6.1}\Big(\frac{|x-z|}{r}\Big)^\alpha \Big(\sup_{B(z,r)\cap S^0} |h|\Big), \qquad x \in B(z,r/2) \cap S^0. \tag{6.1}$$

Proof. By relabeling the axes we may assume that $S^0 = \{x \in \mathbb{R}^d : x_i \geq 0 \text{ for } i > J_0\}$. If z is in the interior of S^0, the result is easy, because the generator is locally uniformly elliptic, and (6.1) follows by the first paragraph of the proof of Theorem 6.4 of [ABBP]. So suppose $z \in \partial S^0$. Then $J_0 < d$ and we may assume, again by reordering the axes, that there is a $K \in \{J_0, \dots, d-1\}$ so that $z_i = 0$ for all $i > K$ and $z_i > 0$ if $J_0 < i \le K$. Assume (set $\min \emptyset = 1$)

$$0 < r < \min_{J_0 < i \le K} \frac{z_i}{2}. \tag{6.2}$$

Since our result only depends on the values of h in $B(x,r) \cap S^0$, we may change the diffusion and drift coefficients of the generator of X outside $\overline{B(z,r)} \cap S^0$ as we please. By changing the coefficients in this way and again relabeling the axes if necessary, we may suppose that our generator is

$$\begin{aligned} \tilde{\mathcal{A}} f(x) = \sum_{i=1}^{J} \sigma_i(x) f_{ii}(x) &+ \sum_{i=J+1}^{K} \sigma_i(x) x_{a(i)} f_{ii}(x) \\ &+ \sum_{i=K+1}^{d} \sigma_i(x) x_i f_{ii}(x) + \sum_{i=1}^{d} b_i(x) f_i(x), \end{aligned} \tag{6.3}$$

where $J \le K$, $a(i) \in \{K+1, \dots, d\}$ for $i = J+1, \dots, K$, each σ_i is continuous and bounded above and below by positive constants, each b_i is continuous and bounded, and each b_i for $i > K$ is bounded below by a positive constant. We have extended our coefficients to the possibly larger space $S^1 = \{x \in \mathbb{R}^d : x_i \ge 0 \text{ for all } i > K\}$ as this is the natural state space for $\tilde{\mathcal{A}}$. As $B(z,r) \cap S^0 = B(z,r) \cap S^1$ (by (6.2)) this will not affect the harmonic functions we are dealing with. For $0 \le \delta < 1$ let

$$\begin{aligned} Q_n(\delta) &= \prod_{i=1}^{J} [z_i - 2^{-n/2}, z_i + 2^{-n/2}] \times \prod_{i=J+1}^{K} [z_i - 2^{-n}, z_i + 2^{-n}] \times \prod_{i=K+1}^{d} [\delta 2^{-n}, 2^{-n}] \\ R_n(\delta) &= \prod_{i=1}^{J} [z_i - \tfrac{3}{2} \cdot 2^{-n/2}, z_i + \tfrac{3}{2} \cdot 2^{-n/2}] \times \prod_{i=J+1}^{K} [z_i - \tfrac{3}{2} \cdot 2^{-n}, z_i + \tfrac{3}{2} \cdot 2^{-n}] \\ &\qquad \times \prod_{i=K+1}^{d} [\delta 2^{-n}, 2^{-n}]. \end{aligned}$$

Take $n \ge 1$ large enough so that $Q_n(0) \subset B(z, r/2) \cap S^1$.

We will first show there exist $c_2, \delta > 0$ independent of n such that

$$\mathbb{P}_x(T_{R_{n+1}(\delta)} < \tau_{Q_n(0)}) \ge c_2, \qquad x \in Q_{n+1}(0). \tag{6.4}$$

We may assume there exist independent one-dimensional Brownian motions B_t^i such that

$$X_t^i = X_0^i + M_t^i + A_t^i,$$

where $dA_t^i = b_i(X_t)\,dt$ and

$$\begin{aligned} dM_t^i &= (2\sigma_i(X_t))^{1/2}\, dB_t^i, \qquad i \le J, \\ dM_t^i &= (2\sigma_i(X_t) X_t^{a(i)})^{1/2}\, dB_t^i, \qquad J+1 \le i \le K, \\ dM_t^i &= (2\sigma_i(X_t) X_t^i)^{1/2}\, dB_t^i, \qquad K+1 \le i \le d. \end{aligned}$$

Since the b_i and σ_i are bounded, there exists t_0 small such that for all $x \in Q_{n+1}(0)$

$$\mathbb{P}_x\big(\sup_{s\le t_0 2^{-n}} |M_s^i| > \tfrac14 \cdot 2^{-n/2}\big) \le \tfrac{1}{8d}, \qquad i \le J,$$
$$\mathbb{P}_x\big(\sup_{s\le t_0 2^{-n}} |M_s^i| > \tfrac14 \cdot 2^{-n}\big) \le \tfrac{1}{8d}, \qquad J+1 \le i \le d,$$

and

$$\sup_{s\le t_0 2^{-n}} |A_s^i| \le \tfrac14 \cdot 2^{-n}, \qquad 1 \le i \le d.$$

The first and last bounds are trivial, and the second inequality is easily proved by first noting

$$\sup_{x\in Q_{n+1}(0)} \mathbb{E}_x\Big(\sum_{i=1}^d \int_0^{t_0 2^{-n}} X_s^i\, ds\Big) \le c 2^{-2n},$$

and then using the Dubins-Schwarz Theorem and Markov's inequality. Hence

$$\sup_{x\in Q_{n+1}(0)} \mathbb{P}_x(\tau_{R_{n+1}(0)} < t_0 2^{-n}) \le \tfrac14. \tag{6.5}$$

By Lemma 6.2 of [ABBP] there exists δ such that if U is uniformly distributed on $[t_0/2, t_0]$, then

$$\sup_{x\in S^1} \mathbb{P}(X_U^i \le \delta) \le 1/4d, \qquad K+1 \le i \le d.$$

Scaling shows that

$$\sup_{x\in S^1} \mathbb{P}_x(X^i_{U2^{-n}} \le \delta 2^{-n}) \le 1/4d, \qquad K+1 \le i \le d.$$

Therefore by (6.5), for any $x \in Q_{n+1}(0)$ with $\mathbb{P}^x$-probability at least $1/2$, $X_{U2^{-n}} \in R_{n+1}(\delta)$. Since $R_{n+1}(0) \subset Q_n(0)$ and $U2^{-n} \le t_0 2^{-n}$, this and (6.5) proves (6.4).

Take δ even smaller if necessary so that $|Q_n(0) - Q_n(\delta)| < \frac14 |Q_n(0)|$ and $\delta \le \sigma_i \le \delta^{-1}$, $|b_i| \le \delta^{-1}$ for all i. Next we show that if $G \subset Q_n(0)$ and $|G| \ge |Q_n(0)|/3$, then there is a $c_3(\delta) > 0$, independent of n, so that

$$\mathbb{P}_x(T_G < \tau_{Q_n(\delta/2)}) \ge c_3, \qquad x \in R_{n+1}(\delta). \tag{6.6}$$

Let

$$Y_t^i = \begin{cases} 2^n X^i_{2^{-n}t}, & i > J, \\ 2^{n/2} X^i_{2^{-n}t}, & i \le J. \end{cases}$$

It is straightforward (cf. [ABBP], Proof of Theorem 6.4) to see that for $t \le \tau_{Q_0(\delta/2)}$, $Y_0 \in Q_0(\delta/2)$, Y_t solves

$$dY_t^i = \widehat{\sigma}_i(Y_t)\, d\widehat{B}_t^i + \widehat{b}_i(Y_t)\, dt,$$

where the $\widehat{B}^i$ are independent one-dimensional Brownian motions, the $\widehat{b}_i$ are bounded above, and the $\widehat{\sigma}_i$ are bounded above and below by positive constants depending on δ but not n. (6.6) now follows by Proposition 6.1 of [ABBP] (with a minor change to account for the fact that $|G \cap Q_n(\delta/2)|/|Q_n(\delta/2)|$ is greater than $\frac{1}{12}$ rather than $\frac{1}{2}$).

Combining (6.4) and (6.6) and using the strong Markov property, we see that if $c_4 = c_2 c_3$, then

$$\mathbb{P}_x(T_G < \tau_{Q_n(0)}) \geq c_4 > 0, \qquad x \in Q_{n+1}(0).$$

Suppose h is harmonic on Q_{n_0} for some n_0. Our conclusion will follow by setting $\alpha = \log(1/\rho)/\log 2$ if we show there exists $\rho < 1$ such that

$$\operatorname{Osc}_{Q_{n+1}(0)} h \leq \rho \operatorname{Osc}_{Q_n(0)} h, \qquad n \geq n_0, \tag{6.7}$$

where $\operatorname{Osc}_A h = \sup_A h - \inf_A h$. Take $n \geq n_0$ and by looking at $c_5 h + c_6$, we may suppose $\sup_{Q_n(0)} h = 1$ and $\inf_{Q_n(0)} h = 0$. By looking at $1-h$ if necessary, we may suppose $|G| \geq \frac{1}{2}|Q_n(0)|$, where $G = \{x \in Q_n(0) : h(x) \geq 1/2\}$. By Doob's optional stopping theorem

$$h(x) \geq \mathbb{E}_x[h(X_{T_G}); T_G < \tau_{Q_n(0)}] \geq \tfrac{1}{2}\mathbb{P}_x(T_G < \tau_{Q_n(0)}) \geq c_4/2, \qquad x \in Q_{n+1}(0).$$

Hence $\operatorname{Osc}_{Q_{n+1}(0)} h \leq 1 - c_4/2$, and (6.7) follows with $\rho = 1 - c_4/2$. $\square$

Proof of Proposition 2.4. We can now proceed as in the proof of Theorem 6.4 of [ABBP]. To obtain the analogue of (6.14) in [ABBP], we note from (2.2) that if $x \in \partial S^0$, at least one coordinate can be bounded below by a squared Bessel process with positive drift starting at zero.

Remark 6.2. Essentially the same argument shows that if for each $x \in S$, $\mathbb{P}^x$ is a solution of $\mathrm{MP}(\mathcal{A}, \delta_x)$ as in Theorem 1.4 (it will be Borel strong Markov by Theorem 1.4), then the resolvent S_λ maps bounded Borel measurable functions to continuous functions. After localizing the problem, one is left with a generator in the same form as (6.3) and so the proof proceeds as above. $\square$

7. Proofs of Lemmas 4.5, 4.6 and 4.7.

We work in the setting and with the notation from Sections 3 and 4. Recall, in particular, the Poisson random variables $N_\rho(t)$ from Lemma 3.4.

Lemma 7.1. *There is a $c_{7.1}$ such that for all $0 \leq q \leq 2$, $1 \leq j \leq m$, $y \in S_m$, $0 < t$, and $z' = z_{m+1}/\gamma t > 0$:*

(a) If for $x \geq 0$ and $n \in \mathbb{Z}_+$,

$$\psi_1(z', n) \equiv (z'+1)^{q/2-1}\Big[1_{(n\leq 1)} + 1_{(n=1)}{z'}^{-1} + 1_{(n\geq 2)}{z'}^{-2}\Big(n + \Big(n - \frac{z'}{2}\Big)^2\Big)\Big],$$

and

$$\psi_2(z', x, n) \equiv 1_{(n\le 2)}(1+z'^{-n})(1+z'^{q/2}+x^{q/2})$$
$$+ 1_{(n\ge 3)}z'^{-3}(|n-z'|^3+3n|n-z'|+n)(z'^{q/2}+n^{q/2}+x^{q/2}),$$

then

$$\int |y_j - z_j|^q |D^3_{z_{m+1}} p_t(z,y)| dz^{(m)} \tag{7.1}$$
$$\le c_{7.1} t^{q-3} \liminf_{\delta\to 0} \mathbb{E}_{z_{m+1}}(q_\delta(y_{m+1}, X_t^{(m+1)})[\psi_1(z', N_{1/2}(t)) + \psi_2(z', X_t^{(m+1)}/t, N_0(t))]).$$

(b) If for x, n as in (a),

$$\phi(z', x, n) \equiv 1_{(n\le 1)}(1+x^{q/2}+(z')^{q/2}) + 1_{(n=1)}(z')^{-1}(1+(z')^{q/2}+x^{q/2})$$
$$+ 1_{(n\ge 2)}(z')^{-2}(n+(n-z')^2)((z')^{q/2}+n^{q/2}+x^{q/2}),$$

then

$$\int |y_j - z_j|^q |D^2_{z_{m+1}} p_t(z,y)| dz^{(m)}$$
$$\le c_{7.1} t^{q-2} \liminf_{\delta\to 0} \mathbb{E}_{z_{m+1}}(q_\delta(y_{m+1}, X_t^{(m+1)})\phi(z', X_t^{(m+1)}/t, N_0(t))). \tag{7.2}$$

(c)

$$\int |D_{z_{m+1}} p_t(z,y)| dz^{(m)}$$
$$\le c_{7.1} t^{-1} \liminf_{\delta\to 0} \mathbb{E}_{z_{m+1}}\Big(q_\delta(y_{m+1}, X_t^{(m+1)})\Big((1+z')^{-1} + \frac{|N_0 - z'|}{z'}\Big)\Big). \tag{7.3}$$

In addition for all $z \in S_m$,

$$\int |D_{z_{m+1}} p_t(z,y)| dy^{(m)}$$
$$\le c_{7.1} t^{-1} \liminf_{\delta\to 0} \mathbb{E}_{z_{m+1}}\Big(q_\delta(y_{m+1}, X_t^{(m+1)})\Big((1+z')^{-1} + \frac{|N_0 - z'|}{z'}\Big)\Big). \tag{7.4}$$

Proof. The proof of (a) is lengthy and the reader may want to first take a look at the simpler proof of (c) given in Section 8.

(a) By Lemma 3.11(d), Fatou's lemma and symmetry we have

$$\int |y_j - z_j|^q |D^3_{z_{m+1}} p_t(z,y)| dz^{(m)}$$
$$\le \liminf_{\delta\to 0} \int |y_j - z_j|^q \Big| \mathbb{E}_{z_{m+1}}\Big(\Delta^3_t G^\delta_{t,z^{(m)},y}(X, \nu^1, \nu^2, \nu^3)[1_{(\nu^i_t = 0 \text{ for } i=1,2,3)}$$
$$+ 3\cdot 1_{(\nu^1_t>0, \nu^2_t=\nu^3_t=0)} + 3\cdot 1_{(\nu^1_t>0,\nu^2_t>0,\nu^3_t=0)} + 1_{(\nu^i_t>0 \text{ for } i=1,2,3)}]$$
$$\times \prod_{i+1}^{3} \mathbb{N}_0(d\nu^i)\Big)\Big| dz^{(m)}$$
$$:= \liminf_{\delta\to 0} E^\delta_1 + 3E^\delta_2 + 3E^\delta_3 + E^\delta_4. \tag{7.5}$$

Consider E_1^δ first. An explicit differentiation shows

$$|D_u^k p_u(w)| \le c p_{2u}(w) u^{-k} \text{ for } k = 1, 2, 3, \tag{7.6}$$

which implies

$$|D_I^k G^\delta_{t,z^{(m)},y}(I, X)| \le c_2 q_\delta(y_{m+1}, X) I^{-k} \prod_{i+1}^{m} p_{2\delta+4\gamma_i^0 I}(z_i - y_i + b_i^0 t), \text{ for } k = 1, 2, 3. \tag{7.7}$$

Use this (with $k = 3$) with the Fundamental Theorem of Calculus to see that on $\{\nu_t^i > 0 \text{ for } i = 1, 2, 3\}$,

$$\begin{aligned}
&\int |y_j - z_j|^q |\Delta_t^3 G^\delta_{t,z^{(m)},y}(X, \nu^1, \nu^2, \nu^3)| dz^{(m)} \\
&\le c_2 \int\int |y_j - z_j|^q 1_{(u_k \le \int_0^t \nu_s^k ds, k=1,2,3)} I_t^{-3} q_\delta(y_{m+1}, X_t^{(m+1)}) \\
&\qquad\qquad \times \prod_{i=1}^{m} [p_{2\delta+4\gamma_i^0(I_t + \sum_{k=1}^3 u_k)}(z_i - y_i + b_i^0 t) dz_i] \prod_{k=1}^{3} du_k \\
&\le c I_t^{-3} q_\delta(y_{m+1}, X_t^{(m+1)}) \Big(t^q + \Big(\delta + I_t + \sum_{k=1}^{3} \int_0^t \nu_s^k ds\Big)^{q/2}\Big) \prod_{k=1}^{3} \int_0^t \nu_s^k ds.
\end{aligned} \tag{7.8}$$

(3.14) and (3.16) imply

$$\int \nu_s^p \, \mathbb{N}_0(d\nu) = (\gamma s)^{p-1} \Gamma(p+1) \text{ for } p \ge 0, \tag{7.9}$$

and so by Jensen

$$\begin{aligned}
\int \Big(\int_0^t \nu_s ds\Big)^{q/2+1} \mathbb{N}_0(d\nu) &\le t^{q/2+1} \int\int \nu_s^{q/2+1} \mathbb{N}_0(d\nu) \frac{ds}{t} \\
&= \Gamma(2 + q/2) t^{q/2} \int_0^t (\gamma s)^{q/2} ds \le c t^{q+1}.
\end{aligned} \tag{7.10}$$

This bound, (7.9) with $p = 1$, and (7.8), together with the expression for E_1^δ, shows that

$$\begin{aligned}
E_1^\delta &\le c \mathbb{E}_{z_{m+1}}(I_t^{-3} q_\delta(y_{m+1}, X_t^{(m+1)})(t^q + \delta^{q/2} + I_t^{q/2}) t^3) \\
&\le c \mathbb{E}_{z_{m+1}}\Big(q_\delta(y_{m+1}, X_t^{(m+1)})[(t^{q+3} + \delta^{q/2} t^3) \mathbb{E}_{z_{m+1}}(I_t^{-3} | X_t^{(m+1)}) \\
&\qquad\qquad + t^3 \mathbb{E}_{z_{m+1}}(I_t^{q/2-3} | X_t^{(m+1)})]\Big) \\
&\le c \mathbb{E}_{z_{m+1}}\Big(q_\delta(y_{m+1}, X_t^{(m+1)})\Big)[(t^q + \delta^{q/2})(z_{m+1} + t)^{-3} + t^{q/2}(t + z_{m+1})^{q/2-3}] \\
&\le c t^{q-3} \mathbb{E}_{z_{m+1}}\Big(q_\delta(y_{m+1}, X_t^{(m+1)})\Big)(1 + (\delta^{1/2}/t)^q)(1 + z')^{q/2-3},
\end{aligned} \tag{7.11}$$

where Lemma 3.2 is used in the next to last inequality.

Let us jump ahead to E_4^δ which will be the dominant (and most interesting) term. We use the decomposition and notation from Lemma 3.4 with $\rho = 0$. Let $S_n = \sum_{i=1}^n e_i(t)$, $R_n = \sum_{i=1}^n r_i(t)$, $p_k(z') = e^{-z'}\frac{(z')^k}{k!} = P(N_0(t) = k)$, and $N^{(k)} = N(N-1)\dots(N-k+1)$. What follows is an integration by parts formula on function space. Recalling that $\mathbb{N}_0(\nu_t > 0) = (\gamma t)^{-1}$ from (3.14), we have from Lemma 3.4 and the exponential law of μ_t under P_t^* (recall (3.16)) that

$$
\begin{aligned}
&\Big|\mathbb{E}_{z_{m+1}}\Big(\int \Delta_t^3 G^\delta_{t,z^{m)},y}(X,\nu^1,\nu^2,\nu^3)\prod_{i=1}^3(1_{(\nu_t^i>0)}\mathbb{N}_0(d\nu^i))\Big)\Big| \\
&= (\gamma t)^{-3}\Big|\mathbb{E}_{z_{m+1}}\Big(G^\delta_{t,z^{(m)},y}(R_{N_0+3}+I_2(t), S_{N_0+3}+X_0'(t)) \\
&\qquad - 3G^\delta_{t,z^{(m)},y}(R_{N_0+2}+I_2(t), S_{N_0+2}+X_0'(t)) \\
&\qquad + 3G^\delta_{t,z^{(m)},y}(R_{N_0+1}+I_2(t), S_{N_0+1}+X_0'(t)) \\
&\qquad - G^\delta_{t,z^{(m)},y}(R_{N_0}+I_2(t), S_{N_0}+X_0'(t))\Big)\Big| \\
&= (\gamma t)^{-3}\Big|\sum_{n=0}^\infty (p_{n-3}(z') - 3p_{n-2}(z') + 3p_{n-1}(z') - p_n(z')) \\
&\qquad \times \mathbb{E}_{z_{m+1}}(G^\delta_{t,z^{(m)},y}(R_n + I_2(t), S_n + X_0'(t)))\Big| \\
&= (\gamma t)^{-3}\Big|\sum_{n=0}^\infty p_n(z')(z')^{-3}[n^{(3)} - 3n^{(2)}z' + 3n(z')^2 - (z')^3] \\
&\qquad \times \mathbb{E}_{z_{m+1}}(G^\delta_{t,z^{(m)},y}(R_n + I_2(t), S_n + X_0'(t)))\Big| \\
&= z_{m+1}^{-3}\Big|\mathbb{E}_{z_{m+1}}\Big([N_0^{(3)} - 3N_0^{(2)}z' + 3N_0(z')^2 - (z')^3]G^\delta_{t,z^{(m)},y}(I_t, X_t^{(m+1)})\Big)\Big|. \qquad (7.12)
\end{aligned}
$$

In the last line we have again used Lemma 3.4 to reconstruct $(I_t, X_t^{(m+1)})$.

We also have

$$
\begin{aligned}
&\int |y_j - z_j|^q G^\delta_{t,z^{(m)},y}(I_t, X_t^{(m+1)})dz^{(m)} \\
&= q_\delta(y_{m+1}, X_t^{(m+1)})\int |y_j - z_j|^q p_{\delta+2\gamma_j^0 I_t}(z_j - y_j + b_j^0 t)dz_j \\
&\le cq_\delta(y_{m+1}, X_t^{(m+1)})[t^q + \delta^{q/2} + I_t^{q/2}]. \qquad (7.13)
\end{aligned}
$$

Combine (7.12) and (7.13) to derive

$$
\begin{aligned}
E_4^\delta \le cz_{m+1}^{-3}\mathbb{E}_{z_{m+1}}\Big(&|N_0^{(3)} - 3N_0^{(2)}z' + 3N_0(z')^2 - (z')^3| \\
&\times q_\delta(y_{m+1}, X_t^{(m+1)})[t^q + \delta^{q/2} + I_t^{q/2}]\Big). \qquad (7.14)
\end{aligned}
$$

Apply Jensen's inequality (as $q/2 \le 1$) in Corollary 3.15 to see that

$$\mathbb{E}_{z_{m+1}}(I_t^{q/2}|N_0, X_t^{(m+1)}) \le c[t^q + t^q N_0^{q/2} + t^{q/2}(X_t^{(m+1)})^{q/2} + t^{q/2} z_{m+1}^{q/2}], \tag{7.15}$$

and so

$$\begin{aligned} E_4^\delta \le ct^{q-3}z'^{-3}\mathbb{E}_{z_{m+1}}\Big(&|N_0^{(3)} - 3N_0^{(2)}z' + 3N_0(z')^2 - (z')^3|q_\delta(y_{m+1}X_t^{(m+1)}) \\ &\times [1 + (\delta^{1/2}/t)^q + (X_t^{(m+1)}/t)^{q/2} + (z')^{q/2} + N_0^{q/2}]\Big) \end{aligned} \tag{7.16}$$

Next consider E_2^δ. Not surprisingly the argument is a combination of the ideas used to bound E_1^δ and E_4^δ. Define

$$\begin{aligned} H^\delta_{t,z^{(m)},y}(I, X, \nu^2, \nu^3) =& G^\delta_{t,z^{(m)},y}\Big(I + \int_0^t \nu_s^2 + \nu_s^3\, ds, X\Big) - G^\delta_{t,z^{(m)},y}\Big(I + \int_0^t \nu_s^2\, ds, X\Big) \\ &- G^\delta_{t,z^{(m)},y}\Big(I + \int_0^t \nu_s^3\, ds, X\Big) + G^\delta_{t,z^{(m)},y}(I, X). \end{aligned}$$

Now apply the decomposition in Lemma 3.4 with $\rho = 1/2$ so that $N_{1/2}(t)$ is Poisson with mean $z'/2$. Arguing as in the derivation of (7.12), but now with a simpler first order summation by parts (which we leave for the reader), we obtain

$$\begin{aligned} &\Big|\mathbb{E}_{z_{m+1}}\Big(\int \Delta_t^3 G^\delta_{t,z^{(m)},y}(X, \nu^1, \nu^2, \nu^3) 1_{(\nu_t^1>0, \nu_t^2=\nu_t^3=0)} \prod_{k=1}^3 \mathbb{N}_0(d\nu^k)\Big)\Big| \\ &= (\gamma t)^{-1}\Big|\mathbb{E}_{z_{m+1}}\Big(\int\int H^\delta_{t,z^{(m)},y}\Big(\int_0^t X_s^{(m+1)} + \nu_s^1 ds, X_t^{(m+1)} + \nu_t^1, \nu^2, \nu^3\Big) \\ &\qquad - H^\delta_{t,z^{(m)},y}\Big(\int_0^t X_s^{(m+1)} ds, X_t^{(m+1)}, \nu^2, \nu^3\Big) P_t^*(d\nu^1) 1_{(\nu_t^2=\nu_t^3=0)} \prod_{k=2}^3 \mathbb{N}_0(d\nu^k)\Big)\Big| \\ &= (\gamma t)^{-1}\Big|\mathbb{E}_{z_{m+1}}\Big(\int\int H^\delta_{t,z^{(m)},y}(I_2(t) + R_{N_{1/2}+1}, X_0'(t) + S_{N_{1/2}+1}, \nu^2, \nu^3) \\ &\qquad - H^\delta_{t,z^{(m)},y}(I_2(t) + R_{N_{1/2}}, X_0'(t) + S_{N_{1/2}}, \nu^2, \nu^3) \prod_{k=2}^3 \mathbb{N}_0(d\nu^k)\Big)\Big| \\ &= \frac{2}{z_{m+1}}\Big|\int \mathbb{E}_{z_{m+1}}\Big((N_{1/2} - z'/2) H^\delta_{t,z^{(m)},y}(I(t), X_t^{(m+1)}, \nu^2, \nu^3)\Big) \prod_{k=2}^3 \mathbb{N}_0(d\nu^k)\Big|. \end{aligned} \tag{7.17}$$

Now use (7.7) (with $k = 2$) and argue as in (7.8) to see that

$$\begin{aligned} &\int\int |y_j - z_j|^q |H^\delta_{t,z^{(m)},y}(I, X, \nu^2, \nu^3)| dz^{(m)} \prod_{k=2}^3 \mathbb{N}_0(d\nu^k) \\ &\le cI^{-2} q_\delta(y_{m+1}X) \int (t^q + (\delta + I + \sum_2^3 \int_0^t \nu_s^k ds)^{q/2}) \prod_{k=2}^3 \Big[\int_0^t \nu_s^k ds \mathbb{N}_0(d\nu^k)\Big] \\ &\le cI^{-2} q_\delta(y_{m+1}, X)[t^q + \delta^{q/2} + I^{q/2}] t^2, \end{aligned} \tag{7.18}$$

where the last line uses (7.10).

Take the absolute values inside the inside the integral in (7.17), multiply by $|y_j - z_j|^q$, integrate with respect to $z^{(m)}$, and use the above bound to conclude that

$$E_2^\delta \le c z_{m+1}^{-1} t^2 \mathbb{E}_{z_{m+1}}\Big(|N_{1/2} - z'/2| q_\delta(y_{m+1}, X_t^{(m+1)}) \\ \times \Big[(t^q + \delta^{q/2}) I(t)^{-2} + I(t)^{q/2-2}\Big]\Big). \tag{7.19}$$

If $r \ge 0$, the independence of X_0' from $(N_{1/2}, \{e_j\})$ and Lemma 3.2, applied to X_0', imply that

$$\begin{aligned}
&\mathbb{E}_{z_{m+1}}(I(t)^{-r}|X_t^{(m+1)}, N_{1/2})\\
&\quad\le \mathbb{E}\Big(\mathbb{E}\Big(\Big(\int_0^t X_0'(s)ds\Big)^{-r}|X_0'(t), N_{1/2}, \{e_j\}\Big)\Big|X_t^{(m+1)}, N_{1/2}\Big)\\
&\quad= \mathbb{E}\Big(\mathbb{E}\Big(\Big(\int_0^t X_0'(s)ds\Big)^{-r}\Big|X_0'(t)\Big)\Big|X_t^{(m+1)}, N_{1/2}\Big)\\
&\quad\le c(t + z_{m+1}/2)^{-r} t^{-r}.
\end{aligned} \tag{7.20}$$

The last line is where it is convenient that $\rho = 1/2 > 0$.

Use (7.20) in (7.19) with $r = 2$ and $2 - q/2$. After a bit of algebra this leads to

$$E_2^\delta \le c t^{q-3} (z')^{-1} \mathbb{E}_{z_{m+1}}(|N_{1/2} - z'/2| q_\delta(y_{m+1}, X_t^{(m+1)})) \\ \times \Big[\Big(\frac{\sqrt{\delta}}{t}\Big)^q (1+z')^{-2} + (1+z')^{q/2-2}\Big]. \tag{7.21}$$

The argument for E_3^δ is similar to the above. One works with

$$\tilde{H}^\delta_{t,z^{(m)},y}(I, X, \nu^3) = G^\delta_{t,z^{(m)},y}\Big(I + \int_0^t \nu_s^3 ds, X\Big) - G^\delta_{t,z^{(m)},y}(I, X).$$

The required third order difference of $G^\delta_{t,z^{(m)},y}$ on $\{\nu_t^1 > 0, \nu_t^2 > 0, \nu_t^3 = 0\}$ is now a second order difference of $\tilde{H}^\delta_{t,z^{(m)},y}$. Minor modifications of the derivation of (7.21) lead to

$$E_3^\delta \le c t^{q-3} (z')^{-2} \mathbb{E}_{z_{m+1}}(|N_{1/2}^{(2)} - N_{1/2} z' + (z'/2)^2| q_\delta(y_{m+1}, X_t^{(m+1)})) \\ \times [(\sqrt{\delta}/t)^q (1+z')^{-1} + (1+z')^{q/2-1}]. \tag{7.22}$$

The above bounds in E_i^δ $i = 1, \ldots 4$ may be used in (7.5) and after the terms involving $\sqrt{\delta}/t$ are neglected (for $q = 0$ these terms are bounded by their neighbours, and for $q > 0$, if they do not approach 0, the right side below must be infinite) we find

$$\begin{aligned}
&\int |y_j - z_j|^q |D^2_{z_{m+1}} p_t(z, y)| dz^{(m)}\\
&\quad\le c t^{q-3} \liminf_{\delta\to 0} \mathbb{E}_{z_{m+1}}\Big(q_\delta(y_{m+1}, X_t^{(m+1)})\Big[(1+z')^{q/2-1} \\
&\qquad\times \Big((1+z')^{-2} + |N_{1/2} - z'/2|(1+z')^{-1}(z')^{-1} + |N_{1/2}^{(2)} - N_{1/2} z' + (z'/2)^2|(z')^{-2}\Big)\\
&\qquad + |N_0^{(3)} - 3N_0^{(2)} z' + 3N_0 (z')^2 - (z')^3|(z')^{-3}[1 + (X_t^{(m+1)}/t)^{q/2} + (z')^{q/2} + N_0^{q/2}]\Big]\Big).
\end{aligned} \tag{7.23}$$

The required bound follows from the above by a bit of algebra but as the reader may be fatigued at this point we point out the way. Trivial considerations show it suffices to show the following inequalities for $n_0, n_{1/2} \in \mathbb{Z}_+$ and $z' \geq 0$:

$$|n_0^{(3)} - 3n_0^{(2)} z' - 3n_0(z')^2 - (z')^3|(z')^{-3}$$
$$\leq c[1_{(n_0 \leq 2)}(1 + (z')^{-n_0}) + 1_{(n_0 \geq 3)}(z')^{-3}(|n_0 - z'|^3 + 3n_0|n_0 - z'| + n_0), \quad (7.24)$$

and

$$[(1 + z')^{-2} + |n_{1/2} - z'/2|(1 + z')^{-1}(z')^{-1} + |n_{1/2}^{(2)} - n_{1/2}z' + (z'/2)^2|(z')^{-2}$$
$$\leq c[1_{(n_{1/2} \leq 1)} + 1_{(n_{1/2}=1)}z'^{-1} + 1_{(n_{1/2} \geq 2)}z'^{-2}(n_{1/2} + (n_{1/2} - \frac{z'}{2})^2)]. \quad (7.25)$$

(7.24) is easy. (7.25) reduces fairly directly to showing that for $n_{1/2} \geq 2$,

$$(1 + z')^{-1} \leq c(n_{1/2} + (n_{1/2} - z'/2)^2)(z')^{-2}.$$

If $z' \leq 1$ this is trivial and for $z' > 1$ consider $n_{1/2} \leq z'/4$ and $n_{1/2} > z'/4$ separately. This completes the proof of (a).

(b) The proof of this second order version of (a) is very similar to, but simpler than that of (a). One now only has a second order difference and three E_i^δ terms to consider. In fact we will not actually need $q > 0$ in (a) but included it so that the reader will not complain about the missing details in the proof of (b) (where $q > 0$ has been used in Proposition 4.12). We do comment on the lack of $N_{1/2}$ in this bound.

An argument similar to that leading to (7.23) shows that $\int |z_j - y_j|^q |D^2_{z_{m+1}} p_t(z, y)| dz^{(m)}$ is bounded by

$$ct^{q-2} \liminf_{\delta \to 0} \mathbb{E}_{z_{m+1}} \Big(q_\delta(y_{m+1}, X_t^{(m+1)}) \Big[(1 + z')^{-1+q/2} \Big((1 + z')^{-1} + |N_{1/2} - z'/2|(z')^{-1} \Big)$$
$$+ |N_0^{(2)} - 2N_0 z' + (z')^2|(z')^{-2}(1 + (X_t^{(m+1)}/t)^{q/2} + (z')^{q/2} + N_0^{q/2}) \Big] \Big)$$
$$\equiv ct^{q-2} \liminf_{\delta \to 0} \mathbb{E}_{z_{m+1}} \Big(q_\delta(y_{m+1}, X_t^{(m+1)}) [T_{1/2} + T_0] \Big).$$

It is easy to check that

$$T_0 \leq c\phi(z', X_t^{(m+1)}/t, N_0),$$

and, using $N_{1/2} \leq N_0$ from (3.23), that

$$T_{1/2} \leq 2(1 + z')^{-1+q/2}(1 + N_0(z')^{-1} + 1) \equiv \bar{T}_{1/2}.$$

Hence to prove (b), it remains to verify

$$\bar{T}_{1/2} \leq c\phi(z', X_t^{(m+1)}/t, N_0).$$

Trivial considerations reduce this to showing that $(1+z')^{-1+q/2} \le c\phi(z', X_t^{(m+1)}/t, N_0)$ for $N_0 \ge 2$. This is easily verified by considering $N_0 < z'/2$ and $N_0 \ge z'/2$ separately.

(c) Note that (7.3) is the first order version of (a) and (b) with $q = 0$. The proof is substantially simpler, but, as it plays the pivotal role in the proof for the important 2-dimensional case, we give the proof in Section 8. (7.4) then follows immediately since the spatial homogeneity in the first m variables, (3.7), implies

$$p_t(z, y) = p_t(-y^{(m)}, z_{m+1}, -z^{(m)}, y_{m+1}). \tag{7.26}$$

$\square$

Proof of Lemma 4.5(b). Let J denote the integral to be bounded in the statement of (b), and $p_n(w) = e^{-w}w^n/n!$ be the Poisson probabilities. Let Γ_n be a Gamma random variable with density

$$g_n(x) = x^{n+b/\gamma-1}e^{-x}\Gamma(n+b/\gamma)^{-1}, \tag{7.27}$$

and recall $z' = z_{m+1}/\gamma t$. By integrating the bound from Lemma 7.1(b) in z_{m+1} (using Fatou's Lemma) we see that

$$J \le c_{7.1}t^{q-2}\liminf_{\delta\to 0}\int z_{m+1}^p \mathbb{E}_{z_{m+1}}\Big(q_\delta(y_{m+1}, X_t^{(m+1)})\phi(z', X_t^{(m+1)}/t, N_0)\Big)\, z_{m+1}^{b/\gamma-1}dz_{m+1}.$$

Our formula for the joint distribution of $(X_t^{(m+1)}, N_0)$ (Lemma 3.6(a)) allows us to evaluate the above and after changing variables and the order of integration we see that if $y' = y/\gamma t$, then

$$\begin{aligned}
J &\le c_{7.1}t^{q-2+p}\liminf_{\delta\to 0}\int (q_\delta(y_{m+1}, y)\sum_{n=0}^{\infty}p_n(y') \\
&\quad\times\Big[\int_0^\infty g_n(z')[(1_{(n\le 1)} + 1_{(n=1)}(z')^{-1})(1+(y')^{q/2}+(z')^{q/2}) \\
&\qquad + 1_{(n\ge 2)}(z')^{-2}[(n-z')^2+n][(y')^{q/2}+(z')^{q/2}+n^{q/2}](z')^p dz'\Big]y^{b/\gamma-1}dy \\
&= c_{7.1}t^{q-2+p}\liminf_{\delta\to 0}\int q_\delta(y_{m+1}, y)\sum_{n=0}^{\infty}p_n(y') \\
&\quad\times\Big[1_{(n\le 1)}\mathbb{E}\left((1+(y')^{q/2}+\Gamma_n^{q/2})\Gamma_n^p\right) + 1_{(n=1)}\mathbb{E}\left((1+(y')^{q/2}+\Gamma_n^{q/2})\Gamma_n^{-1+p}\right) \\
&\qquad + 1_{(n\ge 2)}\mathbb{E}\left(((n-\Gamma_n)^2+n)\Gamma_n^{-2+p}((y')^{q/2}+\Gamma_n^{q/2}+n^{q/2})\right)\Big]y^{b/\gamma-1}dy. \quad (7.28)
\end{aligned}$$

There is a constant c_0 (as in Convention 3.1) so that

$$\mathbb{E}\left(\Gamma_n^r\right) \le c_0(n\vee 1)^r \text{ for all } |r| \le 4 \text{ and } n \in \mathbb{Z}_+ \text{ satisfying } r+n \ge \frac{-3}{4M_0^2}. \tag{7.29}$$

Indeed the above expectation is $\Gamma(n+b/\gamma+r)/\Gamma(n+b/\gamma)$, where

$$r+n+b/\gamma \geq \frac{-3}{4M_0^2}+M_0^{-2}=(2M_0)^{-2}.$$

The result now follows by an elementary, and easily proved, property of the Gamma function.

Assume now the slightly stronger condition

$$|r| \leq 3, n \in \mathbb{Z}_+ \text{ and } r+n \geq \frac{-1}{2M_0^2}. \tag{7.30}$$

Then $\Gamma_n = \Gamma_0 + S_n$, where S_n is a sum of n i.i.d. mean one exponential random variables. If s and s' are Hölder dual exponents, where s is taken close enough to 1 so that the conditions of (7.29) remain valid with rs in place of r, then

$$\begin{aligned} \mathbb{E}\left((\Gamma_n-n)^2\Gamma_n^r\right) &\leq \mathbb{E}\left((\Gamma_n-n)^{2s'}\right)^{1/s'}\mathbb{E}\left(\Gamma_n^{sr}\right)^{1/s} \\ &\leq cn(n\vee 1)^r, \end{aligned} \tag{7.31}$$

where we have used an elementary martingale estimate for $|S_n - n|$ and (7.29). Here c again is as in Convention 3.1.

We now use (7.31) and (7.29) to bound the Gamma expectations in (7.28). It is easy to check that our bounds on p and q imply the powers we will be bounding satisfy (7.30). This leads to

$$\begin{aligned} J &\leq ct^{q-2+p}\liminf_{\delta\to 0}\int q_\delta(y_{m+1},y)\sum_{n+0}^{\infty}p_n(y') \\ &\qquad\times\left[1_{(n\leq 1)}(1+(y')^{q/2})+1_{(n\geq 2)}n^{-1+p}((y')^{q/2}+n^{q/2})\right]y^{b/\gamma-1}\,dy \\ &\leq ct^{q-2+p}\liminf_{\delta\to 0}\int q_\delta(y_{m+1},y)\Big[e^{-y'}(1+y')(1+(y')^{q/2}) \\ &\qquad+\mathbb{E}\left(1_{(N(y')\geq 2)}((y')^{q/2}N(y')^{p-1}+N(y')^{q/2-1+p})\right)\Big]y^{b/\gamma-1}dy. \end{aligned} \tag{7.32}$$

In the last line $N(y')$ is a Poisson random variable with mean y'. Well-known properties of the Poisson distribution show that for a universal constant c_2

$$\mathbb{E}\left(N(y')^r 1_{(N(y')\geq 2)}\right) \leq h_r(y') \equiv c_2(1+y')^r \text{ for all } y' \geq 0, |r| \leq 2. \tag{7.33}$$

For negative values of r see Lemma 4.3(a) of [BP] where the constant depends on r but the argument there easily shows for r bounded one gets a uniform constant. If

$$h(y') = e^{-y'}(1+y')(1+(y')^{q/2}) + h_{q/2-1+p}(y') + y'^{q/2}h_{p-1}(y'),$$

then clearly

$$h(y') \le c_3(1+y')^{q/2-1+p}.$$

As all of the powers appearing in (7.32) satisfy the bounds in (7.33), we may use (7.33) to bound the left-hand side of (7.32) and arrive at

$$\begin{aligned}
J &\le ct^{q-2+p} \liminf_{\delta\to 0} \mathbb{E}_{y_{m+1}}(h(X_\delta^{(m+1)}/\gamma t)) \\
&\le ct^{q-2+p} \liminf_{\delta\to 0} \mathbb{E}_{y_{m+1}}((1+(X_\delta^{(m+1)}/\gamma t)^{q/2-1+p}) \\
&= ct^{q-2+p}(1+y')^{q/2-1+p} \qquad \text{(Dominated Convergence)} \\
&= ct^{q/2-1}(t+y_{m+1})^{q/2-1+p}.
\end{aligned}$$

□

Proof of Lemma 4.5(c). The spatial homogeneity (7.26) shows the integral to be bounded equals

$$\int |y_j - (-z_j)|^q z_{m+1} |D^2_{z_{m+1}} p_t(y^{(m)}, z_{m+1}, -z^{(m)}, y_{m+1})| dy^{(m)} \mu_{m+1}(dy_{m+1}).$$

This shows we can again use the upper bound in Lemma 7.1(b) to bound the integral over $y^{(m)}$ in the above. One then must integrate the resulting bound in y_{m+1} instead of z_{m+1}. This actually greatly simplifies the calculation just given as one can integrate y_{m+1} at the beginning and hence the q_δ term conveniently disappears (see the proof of (4.14) below). For example, if we neglect the insignificant $n \le 1$ contribution to ϕ in Lemma 7.1, the resulting integral is bounded by

$$ct^{q-1}(z')^{-1}\mathbb{E}\,(1_{(N_0\ge 2)}(N_0 + (N_0 - z')^2)((z')^{q/2} + N_0^{q/2} + (X_t^{(m+1)}/t)^{q/2})).$$

This can be bounded using elementary estimates of the Poisson and Hölder's inequality, the latter being much simpler than invoking Lemma 3.6. We omit the details. □

Proof of Lemma 4.5(a). (4.13) is the first order version of Lemma 4.5 (b) and we omit the proof which is much simpler. (4.14) is a bit different from (c). Integrate (7.4) over y_{m+1} to see that

$$\begin{aligned}
&\int y_{m+1}^p |D_{z_{m+1}} p_t(z,y)| \mu(dy) \\
&\quad \le c_{7.1} t^{-1} \liminf_{\delta\to 0} \int y_{m+1}^p \mathbb{E}_{z_{m+1}}\Big(q_\delta(y_{m+1}, X_t^{(m+1)})[(1+z')^{-1} + |N_0 - z'|/z']\Big)\mu_{m+1}(dy_{m+1}) \\
&\quad = c_{7.1} t^{-1} \liminf_{\delta\to 0} \mathbb{E}_{z_{m+1}}\Big([(1+z')^{-1} + |N_0 - z'|/z']\mathbb{E}_{X_t^{(m+1)}}((X_\delta^{(m+1)})^p)\Big).
\end{aligned}$$

Now use the moment bounds in Lemma 3.3(d,e) to bound the above by

$$ct^{-1}\mathbb{E}_{z_{m+1}}\Big([(1+z')^{-1}+|N_0-z'|/z'](X_t^{(m+1)})^p\Big). \tag{7.34}$$

The first term is trivially bounded by the required expression using Lemma 3.3 again. Using the joint density formula (Lemma 3.6), the Gamma power bounds (7.29), and arguing as in the proof of (b) above, the term in (7.34) involving N_0 is at most

$$ct^{p-1}(z')^{-1}\mathbb{E}\left(|N-z'|(N\vee 1)^p\right), \tag{7.35}$$

where $N=N(z')$ is a Poisson random variable with mean z'. We have

$$\mathbb{E}\left(|N-z'|N^p 1_{(N>0)}\right)\le c_0(z'\wedge(z')^{1/2+p}) \text{ for all } z'>0 \text{ and } -1\le p\le 1/2. \tag{7.36}$$

For $p\le 0$ Lemma 3.3 of [BP] shows this (the uniformity for bounded p is again clear). For $1/2\ge p>0$ use Cauchy-Schwarz to prove (7.36). Separating out the contribution from $N=0$, we see from (7.36) that (7.35) is at most

$$ct^{p-1}(z')^{-1}[e^{-z'}z'+(z'\wedge(z')^{1/2+p})]\le ct^{p-1}(e^{-z'}+1\wedge(z')^{p-1/2})\le ct^{p-1}(z'+1)^{p-1/2}.$$

The result follows. $\square$

Proof of Lemma 4.5(f). By the spatial homogeneity in the first m variables (7.26) we may use Lemma 4.4 to conclude

$$|D_{z_j}D^2_{z_{m+1}}p_t(z,y)|=\Big|\int D_{z_j}p_{t/2}(x,-z^{(m)},y_{m+1})D^2_{z_{m+1}}p_{t/2}(-y^{(m)},z_{m+1},x)\mu(dx)\Big|.$$

Therefore

$$\begin{aligned}&\int z^p_{m+1}|D_{z_j}D^2_{z_{m+1}}p_t(z,y)|\mu(dz)\\ &\quad\le\int\int\Big[\int|D_{z_j}p_{t/2}(x,-z^{(m)},y_{m+1})|dz^{(m)}\Big]\\ &\qquad\qquad\times z^p_{m+1}|D^2_{z_{m+1}}p_{t/2}(-y^{(m)},z_{m+1},x)|\mu(dx)\mu_{m+1}(dz_{m+1}).\end{aligned}$$

Use Lemma 4.2(a) to bound the first integral in square brackets and so bound the above by

$$\begin{aligned}&ct^{-1/2}\int z^p_{m+1}|D^2_{z_{m+1}}p_{t/2}(-y^{(m)},z_{m+1},x)|dx^{(m)}\,\mu_{m+1}(dz_{m+1})\\ &\qquad\times(t+x_{m+1}+y_{m+1})^{-1/2}q_{t/2}(x_{m+1},y_{m+1})\mu_{m+1}(dx_{m+1}).\end{aligned}$$

The spatial homogeneity (7.26) implies

$$p_{t/2}(-y^{(m)}, z_{m+1}, x) = p_{t/2}(-x^{(m)}, z_{m+1}, y^{(m)}, x_{m+1}),$$

and so we conclude from the above that

$$\begin{aligned}
&\int z_{m+1}^p |D_{z_j} D^2_{z_{m+1}} p_t(z,y)|\mu(dz) \\
&\quad \le ct^{-1/2} \int \Big[\int z_{m+1}^p |D^2_{z_{m+1}} p_{t/2}(-x^{(m)}, z_{m+1}, y^{(m)}, x_{m+1})| dx^{(m)} \mu_{m+1}(dz_{m+1})\Big] \\
&\qquad\qquad\qquad\qquad \times (t + x_{m+1} + y_{m+1})^{-1/2} q_{t/2}(x_{m+1}, y_{m+1}) \mu_{m+1}(dx_{m+1}) \\
&\quad \le ct^{-3/2} \int (t + x_{m+1})^{p-1} (t + x_{m+1} + y_{m+1})^{-1/2} q_{t/2}(x_{m+1}, y_{m+1}) \mu_{m+1}(dx_{m+1}) \\
&\quad \le ct^{-3/2} \mathbb{E}_{y_{m+1}} \Big((t + X_t^{(m+1)})^{p-3/2}\Big) \le ct^{p-3}.
\end{aligned}$$

We have used Lemma 4.5(b) with $q = 0$ in the next to last inequality and $p \le 3/2$ in the last line. $\square$

Proof of Lemma 4.5(e). For (4.18), use Lemma 4.4 and the spatial homogeneity (7.26) to bound the left-hand side of (4.18) by

$$\begin{aligned}
&\int \int \Big[\int y_{m+1}^p |D_{y_{m+1}} \hat{p}_{t/2}(-x^{(m)}, y_{m+1}, -y^{(m)}, x_{m+1})| \\
&\qquad\qquad \times z_{m+1} |D^2_{z_{m+1}} p_{t/2}(0, z_{m+1}, x^{(m)} - z^{(m)}, x_{m+1})| \mu(dx)\Big] dz^{(m)} \mu_{m+1}(dy_{m+1}).
\end{aligned}$$

Use the substitution (for $z^{(m)}$) $w = x^{(m)} - z^{(m)}$ and do the $dx^{(m)}\mu_{m+1}(dy_{m+1})$ integral first, using (4.13) to bound this integral by $c_{4.5}t^{p-1}$ (as $p \le 1/2$). Now use (4.5) to bound the remaining $dw\mu_{m+1}(dx_{m+1})$ integral by $c_{4.5}t^{-1}$.

The derivation of (4.19) is almost the same as above. One uses Lemma 4.2(b) now to bound the first integral.

Proof of Lemma 4.5(d). The approach is similar to that in (b) as we integrate the bound in Lemma 7.1(a). There is some simplification now even with the higher derivative as $q = 0$. We use the notation from that proof, so that g_n is the Gamma density in (7.27), $p_n(w)$ are the Poisson probabilities with mean w, and Γ_n is a random variable with density g_n. Also let B_n be a Binomial $(n, 1/2)$ random variable independent of Γ_n. To ease the transition to Lemma 4.6 we replace t with s in this calculation. We also keep the notation $z' = z_{m+1}/\gamma s$, $y' = y/\gamma s$. If

$$\phi_1(z', k) = (z' + 1)^{-1}[1_{(k \le 1)}(1 + (z')^{-k}) + 1_{(k \ge 2)}(z')^{-2}[k + (k - z'/2)^2]],$$

and

$$\phi_2(z',k) = 1_{(k\le 2)}(1+(z')^{-k}) + 1_{(k\ge 3)}(z')^{-3}[|k-z'|^3 + 3k|k-z'| + k],$$

then by Lemma 7.1(a) and Fatou's lemma, the integral we need to bound is at most

$$\begin{aligned}
&\liminf_{\delta\to 0} cs^{-3}\Big(\int \mathbb{E}_{z_{m+1}}\Big(q_\delta(y_{m+1}, X_s^{(m+1)})\phi_1(z', N_{1/2})\Big) z_{m+1}^{3/2}\mu_{m+1}(dz_{m+1}) \\
&\qquad\qquad + \int \mathbb{E}_{z_{m+1}}\Big(q_\delta(y_{m+1}, X_s^{(m+1)})\phi_2(z', N_0)\Big) z_{m+1}^{3/2}\mu_{m+1}(dz_{m+1})\Big) \\
&:= \liminf_{\delta\to 0} cs^{-3}[J_1(\delta) + J_2(\delta)]. \qquad (7.37)
\end{aligned}$$

By Lemma 3.6(b),

$$\begin{aligned}
J_1(\delta) &\le cs^{3/2}\int_0^\infty q_\delta(y_{m+1}, y)\sum_{n=0}^\infty p_n(y')\int_0^\infty \sum_{k=0}^n \binom{n}{k} 2^{-n} g_n(z')\phi_1(z',k)(z')^{3/2}dz' y^{b/\gamma-1}dy \\
&= cs^{3/2}\int_0^\infty q_\delta(y_{m+1}, y)\sum_{n=0}^\infty p_n(y')\mathbb{E}\,(\phi_1(\Gamma_n, B_n)\Gamma_n^{3/2})y^{b/\gamma-1}dy. \qquad (7.38)
\end{aligned}$$

The moment bounds (7.29) (the conditions there will be trivially satisfied now) give us

$$\begin{aligned}
\mathbb{E}\,(\phi_1(\Gamma_n, B_n)\Gamma_n^{3/2}) &\le c\Big(P(B_n = 0)\mathbb{E}\,(\Gamma_n^{3/2}) + P(B_n = 1)\mathbb{E}\,(\Gamma_n^{1/2} + \Gamma_n^{3/2}) \\
&\qquad + \mathbb{E}\,(1_{(B_n\ge 2)}B_n)\mathbb{E}\,(\Gamma_n^{-3/2}) + \mathbb{E}\,(1_{(B_n\ge 2)}(B_n - \Gamma_n/2)^2\Gamma_n^{-3/2})\Big) \\
&\le c\Big(2^{-n}(1+n)(n^{1/2} + n^{3/2}) + 1_{(n\ge 2)}n^{-1/2} \\
&\qquad + 1_{(n\ge 2)}\mathbb{E}\,(\mathbb{E}\,((B_n - \Gamma_n/2)^2|\Gamma_n)\Gamma_n^{-3/2})\Big). \qquad (7.39)
\end{aligned}$$

The conditional expectation in the last term is $[(\Gamma_n - n)^2 + \Gamma_n]/4$. Therefore we may now use (7.31) and also (7.29) (as $n \ge 2$ and $r = -1/2$ or $-3/2$ the conditions there are satisfied) to see that for $n \ge 2$

$$\mathbb{E}\,(\mathbb{E}\,((B_n - \Gamma_n/2)^2|\Gamma_n)\Gamma_n^{-3/2}) \le (\mathbb{E}\,((\Gamma_n - n)^2\Gamma_n^{-3/2}) + \mathbb{E}\,(\Gamma_n^{-1/2}))/4 \le cn^{-1/2}.$$

Insert this bound into (7.39) and conclude that

$$\mathbb{E}\,(\phi_1(\Gamma_n, B_n)\Gamma_n^{3/2}) \le c(2^{-n}(1+n^{5/2}) + 1_{(n\ge 2)}n^{-1/2}) \le c.$$

Therefore we can sum over n and integrate over y in (7.38) and obtain

$$J_1(\delta) \le c_1 s^{3/2}, \qquad (7.40)$$

where as always c_1 satisfies Convention 3.1.

Lemma 3.6(a) and the argument leading to (7.38) shows that

$$J_2(\delta) \le cs^{3/2} \int_0^\infty q_\delta(y_{m+1}, y) \sum_{n=0}^\infty p_n(y') \mathbb{E}\left(\phi_2(\Gamma_n, n)\Gamma_n^{3/2}\right) y^{b/\gamma - 1} dy. \tag{7.41}$$

We have

$$\begin{aligned}
&\mathbb{E}\left(\phi_2(\Gamma_n, n)\Gamma_n^{3/2}\right) \\
&\quad = 1_{(n\le 2)}\mathbb{E}\left((1+\Gamma_n^{-n})\Gamma_n^{3/2}\right) + 1_{(n\ge 3)}\mathbb{E}\left(\Gamma_n^{-3/2}(|n-\Gamma_n|^3 + 3n|n-\Gamma_n| + n)\right).
\end{aligned}$$

Some simple Gamma distribution calculations like those in the proof of (b), and which the reader can easily provide (recall Convention 3.1), show that the above is bounded by a constant depending only on M_0. As before by using this bound in (7.41) and integrating out n and y we arrive at

$$J_2(\delta) \le c_2 s^{3/2}. \tag{7.42}$$

Insert the above bounds on $J_i(\delta)$ into (7.37) to complete the proof. □

Proof of Lemma 4.6. Consider (4.22). The functions ϕ_1 and ϕ_2 are as in the previous argument. Argue just as in the derivation of (7.37) to bound the left-hand side of (4.22) by

$$\begin{aligned}
&\liminf_{\delta\to 0} cs^{-3}\Big[t^{b/\gamma}\int_0^{\gamma t} \mathbb{E}_{z_{m+1}}\Big(q_\delta(y_{m+1}, X_s^{(m+1)})\phi_1(z', N_{1/2})\Big) dz_{m+1} \\
&\qquad\qquad + t^{b/\gamma}\int_0^{\gamma t} \mathbb{E}_{z_{m+1}}\Big(q_\delta(y_{m+1}, X_s^{(m+1)})\phi_2(z', N_0)\Big) dz_{m+1}\Big] \\
&:= \liminf_{\delta\to 0} cs^{-3}[K_1(\delta) + K_2(\delta)].
\end{aligned} \tag{7.43}$$

Note we are integrating with respect to z_{m+1} and not $\mu_{m+1}(dz_{m+1})$ as in the previous calculation. Lemma 3.6(b) implies that

$$\begin{aligned}
K_1(\delta) &\le cs \int_0^\infty q_\delta(y_{m+1}, y) \sum_{n=0}^\infty p_n(y') \\
&\qquad \times (t/s)^{b/\gamma} \int_0^{t/s} \mathbb{E}\left(\phi_1(z', B_n)\right) e^{-z'} \frac{(z')^n}{\Gamma(n+b/\gamma)} dz' y^{b/\gamma-1} dy \\
&\le cs \int_0^\infty q_\delta(y_{m+1}, y) \sum_{n=0}^\infty p_n(y') \int_0^{t/s} \mathbb{E}\left(\phi_1(z', B_n)\right) \frac{(z')^n}{(n+1)^2} dz' y^{b/\gamma-1} dy.
\end{aligned} \tag{7.44}$$

We have bounded $\Gamma(n+b/\gamma)^{-1}$ in a rather crude manner in the last line.

For $0 < z' \le t/s \le 1$ we have

$$\begin{aligned}
&\mathbb{E}\,(\phi_1(z',B_n))(z')^n(n+1)^{-2}\\
&\quad\le c\Big[\mathbb{P}(B_n=0)+\mathbb{P}(B_n=1)(1+(z')^{-1})\\
&\qquad\quad + 1_{(n\ge 2)}(z'+1)^{-1}(z')^{-2}\mathbb{E}\,(B_n+(B_n-z'/2)^2)\Big](z')^n(n+1)^{-2}\\
&\quad\le c\Big[2^{-n}((z')^n+n(1+(z')^{n-1})+1_{(n\ge 2)}(z')^{n-2}(n+(n-z')^2)(n+1)^{-2}\Big]\le c.
\end{aligned}$$

This, together with (7.44), shows that

$$K_1(\delta)\le ct.$$

Next use Lemma 3.6(a) to see that

$$\begin{aligned}
K_2(\delta) &\le cs\int_0^\infty q_\delta(y_{m+1},y)\sum_{n=0}^\infty p_n(y')\\
&\qquad\qquad \times (t/s)^{b/\gamma}\int_0^{t/s}\phi_2(z',n)e^{-z'}\frac{(z')^n}{\Gamma(n+b/\gamma)}dz'y^{b/\gamma-1}dy\\
&\le cs\int_0^\infty q_\delta(y_{m+1},y)\sum_{n=0}^\infty p_n(y')\int_0^{t/s}\phi_2(z',n)\frac{(z')^n}{(n+1)^3}dz'y^{b/\gamma-1}dy.
\end{aligned}$$

As above, an elementary argument shows that for $0 < z' \le 1$, $\phi_2(z',n)(z')^n(n+1)^{-3}$ is uniformly bounded in n, z' and also (b,γ) as in Convention 3.1. Hence, we may infer

$$K_2(\delta)\le ct.$$

Put the bounds on $K_i(\delta)$ into (7.43) to complete the proof of (4.22).

We omit the proof of (4.21) which is the first order analogue of (4.22) and is considerably easier. □

We need a probability estimate for Lemma 4.7. As usual $X^{(m+1)}$ is the Feller branching diffusion with generator (3.1).

Lemma 7.2. *(a)* $\mathbb{P}_z(X_t^{(m+1)}\ge w)\le (w/z)^{b/2\gamma}\exp\Big\{\frac{-(\sqrt{z}-\sqrt{w})^2}{\gamma t}\Big\}$ *for all* $w > z \ge 0$.

(b) $\mathbb{P}_z(X_t^{(m+1)}\le w)\le (w/z)^{b/2\gamma}\exp\Big\{\frac{-(\sqrt{z}-\sqrt{w})^2}{\gamma t}\Big\}$ for all $0\le w\le z$.

Proof. This is a simple estimate using the Laplace transform in Lemma 3.3(c). Write X_t for $X_t^{(m+1)}$. If $-(\gamma t)^{-1} < \lambda \le 0$, then

$$\mathbb{P}_z(X_t\ge w)\le e^{\lambda w}\mathbb{E}_z(e^{-\lambda X_t}) = e^{\lambda w}(1+\lambda t\gamma)^{-b/\gamma}\exp\Big\{\frac{-z\lambda}{1+\lambda\gamma t}\Big\}.$$

If $\lambda \geq 0$, then

$$\mathbb{P}_z(X_t \leq w) \leq e^{\lambda w}\mathbb{E}_z(e^{-\lambda X_t}) = e^{\lambda w}(1+\lambda t\gamma)^{-b/\gamma}\exp\Big\{\frac{-z\lambda}{1+\lambda\gamma t}\Big\}.$$

Now set $\lambda = \frac{\sqrt{z/w}-1}{\gamma t}$ in both cases. This is in $(-(\gamma t)^{-1}, 0)$ if $0 \leq z < w$ and in $[0,\infty)$ if $0 \leq z \geq w$. A bit of algebra gives the bounds. $\square$

Proof of Lemma 4.7. We will again integrate the bound in Lemma 7.2(b) over z_{m+1} but as $q = 0$ we will use the function

$$\psi(z', n) = 1_{(n\leq 1)} + 1_{(n=1)}(z')^{-1} + 1_{(n\geq 2)}(z')^{-2}[(n-z')^2 + n].$$

We then have from Lemma 7.2(b), that for each $z' > 0$,

$$\begin{aligned}
&J(z_{m+1})\\
&\quad \equiv \int\int (1_{(y_{m+1}\leq w\leq z_{m+1})} + 1_{(z_{m+1}\leq w\leq y_{m+1})}) z_{m+1}^p\\
&\qquad\qquad\qquad \times |D^2_{z_{m+1}} p_t(z,y)| dz^{(m)} \mu_{m+1}(dy_{m+1})\\
&\quad \leq ct^{-2} z_{m+1}^p \liminf_{\delta\to 0} \mathbb{E}_{z_{m+1}}\Big(\psi(z', N_0)\int q_\delta(y_{m+1}, X_t^{(m+1)})(1_{(y_{m+1}\leq w\leq z_{m+1})}\\
&\qquad\qquad\qquad + 1_{(z_{m+1}\leq w\leq y_{m+1})})\mu_{m+1}(dy_{m+1})\\
&\quad = ct^{-2} z_{m+1}^p \mathbb{E}_{z_{m+1}}\Big(\psi(z', N_0)[1_{(w\leq z_{m+1})}1_{(X_t^{(m+1)}\leq w)}\\
&\qquad\qquad\qquad + 1_{(w\geq z_{m+1})}1_{(X_t^{(m+1)}\geq w)}]\Big)\\
&\quad \leq ct^{-2} z_{m+1}^p \mathbb{E}_{z_{m+1}}(\psi(z', N_0)^2)^{1/2}[1_{(w\leq z_{m+1})}\mathbb{P}_{z_{m+1}}(X_t^{(m+1)} \leq w)^{1/2}\\
&\qquad\qquad\qquad + 1_{(w\geq z_{m+1})}\mathbb{P}_{z_{m+1}}(X_t^{(m+1)} \geq w)^{1/2}].
\end{aligned} \tag{7.45}$$

In the third line we have used the a.s. and, hence weak, convergence of $X_\delta^{(m+1)}$ to $X_0^{(m+1)}$ as $\delta \to 0$ and the fact that $X_t^{(m+1)} \neq w$ a.s.

We have

$$\begin{aligned}
\mathbb{E}(\psi(z', N_0)^2) \leq c\Big(&(1+z')e^{-z'} + (z')^{-1}e^{-z'}\\
&+ (z')^{-4}[\mathbb{E}(1_{(N_0\geq 2)}(N_0 - z')^4) + \mathbb{E}(1_{(N_0\geq 2)}N_0^2)]\Big).
\end{aligned}$$

An elementary calculation (consider small z' and large z' separately) shows that the term in square brackets is at most $c(z')^2$. Therefore we deduce that

$$\mathbb{E}(\psi(z', N_0)^2)^{1/2} \leq c(z')^{-1}. \tag{7.46}$$

If we set $w' = w/\gamma t$, then this, together with (7.45) and Lemma 7.2 allows us to conclude that

$$\begin{aligned}
&\int J(z_{m+1})\mu_{m+1}(dz_{m+1}) \\
&\leq ct^{-2}\int z_{m+1}^{p-1+b/\gamma}(z')^{-1}(w/z_{m+1})^{b/4\gamma}\exp\Big(\frac{-(\sqrt{z_{m+1}}-\sqrt{w})^2}{2\gamma t}\Big)dz_{m+1} \\
&= ct^{p-2+b/\gamma}(w')^{b/4\gamma}\int (z')^{p-2+3b/4\gamma}\exp\Big(\frac{-(\sqrt{z'}-\sqrt{w'})^2}{2}\Big)dz' \\
&\equiv ct^{p-2+b/\gamma}(w')^{b/4\gamma}K_{p-2+3b/4\gamma}(w').
\end{aligned} \tag{7.47}$$

A simple calculation using the obvious substitution $x = (\sqrt{z'}-\sqrt{w'})^2$ shows that for any $\varepsilon > 0$ there is a $c_0(\varepsilon)$ such that

$$K_r(w') \leq c_0[1_{(w'\leq 1)} + (w')^{r+1/2}1_{(w'>1)}] \text{ for all } w' \geq 0 \text{ and } -1+\varepsilon \leq r \leq \varepsilon^{-1}.$$

Our bounds on p and Convention 3.1 imply that $r = p-2+3b/4\gamma \in [-1+3(4M_0^2)^{-1}, M_0^2]$. Therefore the left-hand side of (7.47) is at most

$$\begin{aligned}
&ct^{p-2+b/\gamma}(w')^{b/4\gamma}(1_{(w'\leq 1)} + 1_{(w'>1)}(w')^{p-3/2+3b/4\gamma}) \\
&\leq c(1_{(w\leq\gamma t)}t^{p-2+b/\gamma} + 1_{(w>\gamma t)}t^{-1/2}w^{p-3/2+b/\gamma}).
\end{aligned}$$

□

8. A Remark on the Two-dimensional Case.

As has already been noted, the proof of Proposition 2.2 (by far the most challenging step) simplifies substantially if $d = 2$. As this is the case required in [DGHSS], we now describe this simplification in a bit more detail.

Recall the three cases (i)–(iii) for $d = 2$ listed following Theorem 1.4. As noted there, the case $\mathcal{E} = \emptyset$ is covered by Theorem A of [BP] (with $d = 2$) without removing $(0,0)$ from the state space, so we will focus mainly on the other two cases here (but see the last paragraph below). In these cases the localization in Theorem 2.1 reduces the problem to the study of the martingale problem for a perturbation of the constant coefficient operator

$$\mathcal{A}^0 = \sum_{i=1}^{2} b_i^0 D_{x_i} + \gamma_i^0 x_2 D_{x_i}^2, \tag{8.1}$$

with resolvent R_λ and semigroup P_t. Our job is to establish Proposition 2.2 for this resolvent.

For $f \in \mathcal{D}_0$ ($\mathcal{D}_0$ as in (2.21)), we have

$$\|\mathcal{A}^0 R_\lambda f\|_2 = \|\lambda R_\lambda f - f\|_2 \le 2\|f\|_2, \tag{8.2}$$

the latter by Corollary 2.12. We may therefore remove the term $x_2(R_\lambda f)_{22}$ from the summation in (2.6) because L^2-boundedness of this term will follow from the other three and (8.2). Recall the required boundedness of any of the terms was reduced to (2.28) by Cotlar's Lemma. (Proposition 2.10 was only used to ensure the operator T_t was bounded on L^2 so that Cotlar's Lemma may be employed in the proof of Proposition 2.2. This boundedness, as well as the bound of ct^{-1}, is also implied by (2.28) with $s = t$ and the elementary Lemma 2.11(b). So we only need consider (2.28).) For the two derivatives involving x_1, (2.28) was fairly easily checked in Lemma 4.3 thanks to the "explicit" formulae (4.7) and (4.8), and the bound in Lemma 3.3(f).

It remains only to check (2.28) for D_{x_2}. This was done in Proposition 4.9, using only Lemma 4.5(a) and in fact only used (4.14) for $p = 0$ and (4.13) for $p \le 0$. These proofs in turn were fairly simple consequences of part (c) of the key Lemma 7.1. (Admittedly the proof of (4.13) was omitted, being much simpler than that of (4.15).) As (7.4) was a trivial consequence of (7.3) (recall (7.26)), we have essentially reduced the two-dimensional case to the proof of (7.3). To justify our earlier statements, that this really is much simpler than that of (7.1), we give the proof. At the risk of slightly lengthening the argument we will take this opportunity to explicitly write an integration by parts formula which was implicit (and hidden) in the more complicated setting of Lemma 7.1. Recall that $m = 1$ (the proofs below are the same for general m), $I_t = \int_0^t X_s^{(2)}\, ds$, $\gamma = \gamma_2^0$, and (see (3.26))

$$G_{t,z_1} f(I, X) = \int f(x_1, X) p_{2\gamma_1^0 I}(x_1 - z_1 - b_1^0 t)\, dx_1.$$

$N_0 = N_0(t)$ is the Poisson variable in Lemma 3.4.

Proposition 8.1 (Integration by Parts Formula). *If $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is bounded and Borel, then*

$$D_{z_2} P_t f(z) = (\gamma t)^{-1} \mathbb{E}_{z_2}\Big(\frac{(N_0 - (z_2/\gamma t))}{z_2/\gamma t} G_{t,z_1} f(I_t, X_t^{(2)})\Big) + E_1(t, z, f), \tag{8.3}$$

where E_1 is given by (8.6) below and satisfies

$$\begin{aligned}|E_1(t, z, f)| \le \mathbb{E}_{z_2}\Big(\int\int\int &|f(x_1, X_t^{(2)})| 4 I_t^{-1} p_{4\gamma_1^0(u+I_t)}(x_1 - z_1 - b_1^0 t) \\ &\times 1_{(u \le \int_0^t \nu_s ds)}\, dx_1\, du\, d\mathbb{N}_0(\nu)\Big).\end{aligned} \tag{8.4}$$

Proof. By (3.25)

$$\begin{aligned}
D_{z_2}P_tf(z) &= \mathbb{E}_{z_2}\Big(\int[G_{t,z_1}f\Big(\int X_s^{(2)}+\nu_s ds, X_t^{(2)}\Big) - G_{t,z_1}f\Big(\int X_s^{(2)}, X_t^{(2)}\Big)]1_{(\nu_t=0)}\mathbb{N}_0(d\nu)\Big) \\
&\quad + \mathbb{E}_{z_2}\Big(\int[G_{t,z_1}f\Big(\int X_s^{(2)}+\nu_s ds, X_t^{(2)}+\nu_t\Big) \\
&\qquad\qquad - G_{t,z_1}f\Big(\int X_s^{(2)}ds, X_t^{(2)}\Big)]1_{(\nu_t>0)}\mathbb{N}_0(d\nu)\Big) \\
&\equiv E_1(t,z,f)+E_2(t,z,f).
\end{aligned} \tag{8.5}$$

Now use

$$\frac{\partial p_t}{\partial t}(z) = (z^2t^{-1}-1)(2t)^{-1}p_t(z),$$

and (by some calculus)

$$\Big|\frac{\partial p_t}{\partial t}(z)\Big| \le \frac{4}{t}p_{2t}(z),$$

together with the Fundamental Theorem of Calculus, to obtain

$$\begin{aligned}
E_1(t,z,f) =& \mathbb{E}_{z_2}\Big(\int\int\int f(x_1,X_t^{(2)})\Big[\frac{(x_1-z_1-b_1^0t)^2}{2\gamma_1^0(u+I_t)}-1\Big](2(u+I_t))^{-1} \\
&\times p_{2\gamma_1^0(u+I_t)}(x_1-z_1-b_1^0t)1_{(u\le\int_0^t \nu_s ds)}dx_1\,du1_{(\nu_t=0)}d\mathbb{N}_0(\nu)\Big),
\end{aligned} \tag{8.6}$$

and

$$\begin{aligned}
|E_1(t,z,f)| \le \mathbb{E}_{z_2}\Big(&\int\int\int|f(x_1,X_t^{(2)})|4(u+I_t)^{-1}p_{4\gamma_1^0(u+I_t)}(x_1-z_1-b_1^0t) \\
&\times 1_{(u\le\int_0^t \nu_s ds)}dx_1\,du\,d\mathbb{N}_0(\nu).
\end{aligned}$$

The latter inequality gives (8.4).

For E_2 we use the decomposition in Lemma 3.4 with $\rho=0$. S_n and R_n are the sum of the first n of the e_i and r_i, respectively, and we continue to write $p_n(w)=e^{-w}w^n/n!$ and $z'=z_2/\gamma t$. Then Lemma 3.4 allows us to write

$$\begin{aligned}
E_2(t,z,f) &= \mathbb{N}_0(\nu_t>0)\mathbb{E}_{z_2}\Big(G_{t,z_1}f(I_2(t)+R_{N_0+1},X_0'(t)+S_{N_0+1}) \\
&\qquad\qquad - G_{t,z_1}f(I_2(t)+R_{N_0},X_0'(t)+S_{N_0})\Big) \\
&= (\gamma t)^{-1}\sum_{n=0}^{\infty}(p_{n-1}(z')-p_n(z'))\mathbb{E}_{z_2}(G_{t,z_1}f(I_2(t)+R_n,X_0'(t)+S_n)) \\
&= (\gamma t)^{-1}\sum_{n=0}^{\infty}p_n(z')(n-z')(z')^{-1}\mathbb{E}_{z_2}(G_{t,z_1}f(I_2(t)+R_n,X_0'(t)+S_n)) \\
&= (\gamma t)^{-1}\mathbb{E}_{z_2}((N_0-z')(z')^{-1}G_{t,z_1}f(I_t,X_t^{(2)})).
\end{aligned} \tag{8.7}$$

In the last line we have again used Lemma 3.4 to reconstruct $X^{(2)}$. (8.7) and (8.5) complete the proof. $\square$

Remark 8.2. Since $|G_{t,z_1}f| \le \|f\|_\infty$, the above implies the sup norm bound

$$\begin{aligned}
&|D_{z_2}P_tf(z)| \\
&\le (\gamma t)^{-1}\mathbb{E}_{z_2}\Big(\frac{|N_0 - z'|}{z'}\Big)\|f\|_\infty + 4\|f\|_\infty \mathbb{E}_{z_2}(I_t^{-1})\int\int_0^t \nu_s\, ds\, d\mathbb{N}_0(\nu)\Big) \\
&\le (\gamma t)^{-1}2\|f\|_\infty + 4t^{-1}\|f\|_\infty.
\end{aligned}$$

We have used Lemma 3.3(f) and (3.17) in the last. This gives a derivation of (4.4). More importantly we can use the above to derive an L^1 bound which will allow us to take $f = \delta_y$. Recall that $q_t(x,y)$ is the transition density of $X^{(2)}$ with respect to $y^{b/\gamma-1}dy$.

Corollary 8.3. *If $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is bounded and Borel, then*

$$\begin{aligned}
&\int |D_{z_2}P_tf(z)|dz_1 \\
&\qquad \le c_{8.3}t^{-1}\mathbb{E}_{z_2}\Big(\int |f(z_1, X_t^{(2)})|dz_1\Big[\frac{|N_0 - z_2/\gamma t|}{z_2/\gamma t} + \Big(\frac{z_2}{\gamma t}+1\Big)^{-1}\Big]\Big).
\end{aligned}$$

Proof. Note first that $\int |G_{t,z_1}f(I,X)|dz_1 \le \int |f(x_1,X)|dx_1$, and then integrate over z_1 in Proposition 8.1 to see that the above integral is at most ($z' = z_2/\gamma t$ as usual)

$$\begin{aligned}
&(\gamma t)^{-1}\mathbb{E}_{z_2}\Big(\frac{|N_0 - z'|}{z'}\int |f(x_1, X_t^{(2)})|dx_1\Big) \\
&\qquad + \mathbb{E}_{z_2}\Big(\int\int |f(x_1,X_t^{(2)})|dx_1 4I_t^{-1}\Big)\Big(\int\int_0^t \nu_s\, ds\, d\mathbb{N}_0(\nu)\Big). \qquad (8.8)
\end{aligned}$$

Use Lemma 3.3(f) and (3.17) again to bound the last term by

$$4c_{3.2}(t+z_2)^{-1}\mathbb{E}_{z_2}\Big(\int |f(x_1, X_t^{(2)})|dx_1\Big).$$

Use this in (8.8) to derive the required bound. $\square$

Proof of (7.3). Let $f^{y,\delta}(z_1,z_2) = p_\delta(z_1 - y_1)q_\delta(y_2,z_2)$ (bounded in z by Lemma 3.3(a)). Then (3.30) shows that $\lim_{\delta\to 0} D_{z_2}P_tf^{y,\delta}(z) = D_{z_2}p_t(z,y)$. Apply Fatou's Lemma and Corollary 8.3 to conclude

$$\begin{aligned}
&\int |D_{z_2}p_t(z,y)|dz_1 \\
&\quad \le \liminf_{\delta\to 0} c_{7.3}t^{-1}\mathbb{E}_{z_2}\Big(\int |f^{y,\delta}(z_1,X_t^{(2)})|dz_1\Big[\frac{|N_0 - z_2/\gamma t|}{z_2/\gamma t} + \Big(\frac{z_2}{\gamma t}+1\Big)^{-1}\Big]\Big) \\
&\quad \le \liminf_{\delta\to 0} c_{7.3}t^{-1}\mathbb{E}_{z_2}\Big(q_\delta(y_2, X_t^{(2)})\Big[\frac{|N_0 - z_2/\gamma t|}{z_2/\gamma t} + \Big(\frac{z_2}{\gamma t}+1\Big)^{-1}\Big]\Big).
\end{aligned}$$

The required result follows. $\square$

If we wanted to include the case $\mathcal{E} = \emptyset$ to make the above "short proof" self-contained, then we need to consider Proposition 2.2 and hence (4.1) and (4.2) for the case

$$\mathcal{A}'^0 = \sum_{i=1}^{2} b_i^0 D_{x_i} + \gamma_i^0 x_i D_{x_i}^2.$$

The associated semigroup $P_t = \prod_{j=1}^2 Q_t^j$ is a product of one-dimensional Feller branching (with immigration) semigroups with transition densities given by (3.3). As in the the last part of the proof of Proposition 2.14 at the end of Section 4, (4.1) and (4.2) reduce easily to checking (4.1) and (4.2) for each one dimensional Q_t^i. In the first part of the proof of Proposition 2.14 (in Section 4) we saw that these easily followed for each differential operator by projecting down the corresponding result for $\mathcal{A}^0$ (as in (8.1)) to the second coordinate. This was checked in the "short" proof above for the the first order operators. It therefore only remains to check (4.1) and (4.2) for $\tilde{D}_x = xD_x^2$ and q_t in place of p_t. As in the proof of Proposition 4.12, we must verify (4.33), (4.34), (4.24), and (4.25) for this operator and one-dimensional density. These, however, can be done by direct calculation using the series expansion (3.3)–the arguments are much simpler and involve direct summation by parts with Poisson probabilities and elementary Poisson bounds.

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