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Lyapunov exponents for the one-dimensional parabolic Anderson model with drift

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Abstract

We consider the solution u to the one-dimensional parabolic Anderson model with homogeneous initial condition $u(0,\cdot) \equiv 1$, arbitrary drift and a time-independent potential bounded from above. Under ergodicity and independence conditions we derive representations for both the quenched Lyapunov exponent and, more importantly, the p-th annealed Lyapunov exponents for *all* $p \in (0,\infty)$.
These results enable us to prove the heuristically plausible fact that the p-th annealed Lyapunov exponent converges to the quenched Lyapunov exponent as $p \downarrow 0$. Furthermore, we show that u is p-intermittent for p large enough.
As a byproduct, we compute the optimal quenched speed of the random walk appearing in the Feynman-Kac representation of u under the corresponding Gibbs measure; related results for the discrete time case have been derived by [GdH92] and [Flu07]. In our context, depending on the negativity of the potential, a phase transition from zero speed to positive speed appears as the drift parameter or diffusion constant increase, respectively.

Key words: Parabolic Anderson model, Lyapunov exponents, intermittency, large deviations.

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1 Introduction

1.1 Model and notation

We consider the one-dimensional parabolic Anderson model with arbitrary drift and homogeneous initial condition, i.e. the Cauchy problem

$$\left.\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \kappa\Delta_h u(t,x) + \xi(x)u(t,x), && (t,x)\in\mathbb{R}_+\times\mathbb{Z}, \\ u(0,x) &= 1, && x\in\mathbb{Z}, \end{aligned}\right\} \tag{1.1}$$

where κ is a positive diffusion constant, $h \in (0,1]$ an arbitrary drift and Δ_h denotes the discrete Laplace operator with drift h given by

$$\Delta_h u(t,x) := \frac{1+h}{2}(u(t,x+1)-u(t,x)) + \frac{1-h}{2}(u(t,x-1)-u(t,x)).$$

Here and in the following, $(\xi(x))_{x\in\mathbb{Z}} \in \Sigma := (-\infty,0]^{\mathbb{Z}}$ is a non-constant ergodic potential bounded from above. The distribution of ξ will be denoted Prob and the corresponding expectation $\langle\cdot\rangle$. In our context ergodicity is understood with respect to the left-shift θ acting on $\zeta \in \Sigma$ via $\theta((\zeta(x))_{x\in\mathbb{Z}}) := (\zeta(x+1))_{x\in\mathbb{Z}}$. Without further loss of generality we will assume

$$\operatorname{ess\,sup}\xi(0) = 0. \tag{1.2}$$

Note that the case $\operatorname{ess\,sup}\xi(0) = c$ reduces to (1.2) by the transformation $u \mapsto e^{ct}u$. We can now write

$$\mathrm{Prob} \in \mathscr{M}_1^e(\Sigma). \tag{1.3}$$

Here, $\mathscr{M}_1(E)$ denotes the space of probability measures on a topological space E, and if we have a shift operator defined on E (such as θ for $E = \Sigma$), then by $\mathscr{M}_1^s(E)$ and $\mathscr{M}_1^e(E)$ we denote the spaces of shift-invariant and ergodic probability measures on E, respectively. If not mentioned otherwise, we will always assume the measures to be defined on the corresponding Borel σ-algebra and the spaces of measures to be endowed with the topology of weak convergence. We denote $\Sigma_b := [b,0]^{\mathbb{Z}}$ and $\Sigma_b^+ := [b,0]^{\mathbb{N}_0}$ for $b \in (-\infty,0)$. Since the potential plays the role of a (random) medium, we likewise refer to ξ as the medium.

Examples motivating the study of (1.1) reach from chemical kinetics (cf. [GM90] and [CM94]) to evolution theory (see [EEEF84]). In particular, we may associate to (1.1) the following branching particle system: At time $t = 0$ at each site $x \in \mathbb{Z}$ there starts a particle moving independently from all others according to a continuous-time random walk with generator $\kappa\Delta_{-h}$. It is killed with rate ξ^- and splits into two with rate ξ^+. Each descendant moves independently from all other particles according to the same law as its ancestor. The expected number of particles at time t and site x given the medium ξ solves equation (1.1).

1.2 Motivation

Our central interest is in the quenched and p-th annealed Lyapunov exponents, which if they exist, are given by

$$\lambda_0 := \lim_{t\to\infty}\frac{1}{t}\log u(t,0) \quad \text{a.s.} \tag{1.4}$$

and

$$\lambda_p := \lim_{t\to\infty} \frac{1}{t} \log\langle u(t,0)^p\rangle^{1/p}, \quad p \in (0,\infty), \tag{1.5}$$

respectively (cf. Theorems 2.1 and 2.7). By means of these Lyapunov exponents we will then investigate the occurrence of intermittency of the solution u, which heuristically means that u is irregular and exhibits pronounced spatial peaks. The motivation for this is as follows. In equation (1.1) two competing effects are present. On the one hand, the operator $\kappa\Delta_h$ induces a diffusion (in combination with a drift) which tends to smooth the solution. On the other hand, the influence of the random potential ξ favours the spatial inhomogeneity of the solution u. The presence of intermittency is therefore evidence that the effect of the potential dominates that of the diffusion.

Corresponding problems have been investigated in the zero-drift case (see [BK01a]) and it is therefore natural to ask if similar effects occur for the model with drift.

A standard procedure in the study of intermittency is to investigate the exponential growth of moments.[1] While the following definition is a straightforward generalisation of the corresponding definition in [ZMRS88] or [GdH06], it is more restrictive than the definition given in the zero-drift case of [BK01a] which exhibits itself through more refined second order asymptotics only.

Definition 1.1. For $p \in (0,\infty)$, the solution u to (1.1) is called *p-intermittent* if $\lambda_{p+\varepsilon} > \lambda_p$ for all $\varepsilon > 0$ sufficiently small.

Remark 1.2. Note that $\lambda_{p+\varepsilon} \geq \lambda_p$ is always fulfilled due to Jensen's inequality. Furthermore, it will turn out in Proposition 2.11 (a) that p-intermittency implies $\lambda_{p+\varepsilon} > \lambda_p$ for all $\varepsilon > 0$.

If u is p-intermittent, then Chebyshev's inequality yields

$$\mathrm{Prob}(u(t,0) > e^{\alpha t}) \leq e^{-\alpha p t}\langle u(t,0)^p\rangle \asymp e^{(-\alpha+\lambda_p)pt} \to 0$$

for $\alpha \in (\lambda_p, \lambda_{p+\varepsilon})$ and, at the same time,

$$\langle u(t,0)^{p+\varepsilon}\mathbb{1}_{u(t,0)\leq e^{\alpha t}}\rangle \leq e^{\alpha(p+\varepsilon)t} = o(\langle u(t,0)^{p+\varepsilon}\rangle)$$

as $t \to \infty$, which again implies

$$\langle u(t,0)^{p+\varepsilon}\mathbb{1}_{u(t,0)>e^{\alpha t}}\rangle \sim \langle u(t,0)^{p+\varepsilon}\rangle.$$

In particular, setting $\Gamma(t) := \{x \in \mathbb{Z} : u(t,x) > e^{\alpha t}\}$ and considering large centered intervals I_t, we get

$$|I_t|^{-1}\sum_{x\in I_t} u(t,x)^{p+\varepsilon} \approx |I_t|^{-1}\sum_{x\in I_t\cap\Gamma(t)} u(t,x)^{p+\varepsilon}$$

due to Birkhoff's ergodic theorem. This justifies the interpretation that for large times the solution u develops (relatively) higher and higher peaks on fewer and fewer islands. For further reading, see [GK05] and [GM90].

Motivated by the definition of intermittency, the main goal of this article is to find closed formulae in particular for the p-th annealed Lyapunov exponent for all $p \in (0,\infty)$, cf. Theorem 2.7. As a first step towards this aim we compute the quenched Lyapunov exponent (see Theorem 2.1). On the one

[1] An explicit geometric characterisation is more difficult and beyond the scope of this article. For the zero-drift case, see [GKM07].

hand the auxiliary results leading to Theorem 2.1 can be employed to obtain results on the optimal speed of the random walk under the random potential ξ (see Corollary 2.3 and Proposition 2.5), while on the other hand the techniques used in its proof prepare the ground for the computation of the annealed Lyapunov exponents thereafter. Having these formulae at hand, we then investigate the occurence of intermittency (see Proposition 2.9).

In systems such as the one we are considering one formally derives

$$\lim_{p\downarrow 0}\frac{1}{p}\log\langle u(t,0)^p\rangle = \Big[\frac{d}{dp}\log\langle u(t,0)^p\rangle\Big]\Big|_{p=0} = \frac{\langle(\log u(t,0))u(t,0)^p\rangle}{\langle u(t,0)^p\rangle}\Big|_{p=0} = \langle\log u(t,0)\rangle. \tag{1.6}$$

Although one still has to scrutinise whether the interchange of limits and integration is valid, it is believed due to (1.6) that the p-th annealed Lyapunov exponent converges to the quenched Lyapunov exponent as $p\downarrow 0$. Since we are able compute the p-th annealed Lyapunov exponent for all $p\in(0,\infty)$, we can prove this conjecture, cf. Theorem 2.10.

In order to formulate our results, we introduce some more notation. Let $Y=(Y_t)_{t\in\mathbb{R}_+}$ be a continuous-time random walk on $\mathbb{Z}$ with generator $\kappa\Delta_{-h}$. By $\mathbb{P}_x$ we denote the underlying probability measure with $\mathbb{P}_x(Y_0=x)=1$ and we write $\mathbb{E}_x$ for the expectation with respect to $\mathbb{P}_x$. Let T_n be the first hitting time of $n\in\mathbb{Z}$ by Y and define for $\beta\in\mathbb{R}$,

$$L^+(\beta) := \Big\langle\Big(\log\mathbb{E}_1\exp\Big\{\int_0^{T_0}(\xi(Y_s)+\beta)\,ds\Big\}\Big)^+\Big\rangle,$$

$$L^-(\beta) := \Big\langle\Big(\log\mathbb{E}_1\exp\Big\{\int_0^{T_0}(\xi(Y_s)+\beta)\,ds\Big\}\Big)^-\Big\rangle$$

as well as

$$L(\beta) := L^+(\beta)-L^-(\beta) \tag{1.7}$$

if this expression is well-defined, i.e. if at least one of the two terms on the right-hand side is finite. We denote β_{cr} the critical value such that $L^+(\beta)=\infty$ for $\beta>\beta_{cr}$ and $L^+(\beta)<\infty$ for $\beta<\beta_{cr}$. With this notation, we observe that $L(\beta)$ is well-defined for all $\beta\in(-\infty,\beta_{cr})$ at least. By constraining the random walk Y to stay at site x with $\xi(x)\approx 0$ (cf. (1.2)), it can be easily shown that

$$\beta_{cr}\in[0,\kappa]. \tag{1.8}$$

2 Main results

2.1 Quenched regime

We start by considering the quenched Lyapunov exponent λ_0 and note that even the existence of the limit on the right-hand side of (1.4) is not immediately obvious. We will impose that either the random field ξ is ergodic and bounded, i.e.

$$\mathrm{Prob}\in\mathscr{M}_1^e(\Sigma_b) \tag{2.1}$$

for some $b<0$, or that ξ consists of i.i.d. random variables, i.e.

$$\mathrm{Prob}=\eta^{\mathbb{Z}} \tag{2.2}$$

for some law $\eta\in\mathscr{M}_1((-\infty,0])$. We then have the following result.

Theorem 2.1. *Assume (1.2) and either (2.1) or (2.2). Then the quenched Lyapunov exponent λ_0 exists a.s. and is non-random. Furthermore, λ_0 equals the zero of $\beta \mapsto L(-\beta)$ in $(-\beta_{cr}, 0)$ or, if such a zero does not exist, it equals $-\beta_{cr}$.*

Remark 2.2. As β_{cr} plays a crucial role here and in the following, we note in anticipation of Lemma 5.5 that

$$\beta_{cr} = \kappa(1 - \sqrt{1-h^2})$$

holds if (2.2) is fulfilled. In particular, we observe that β_{cr} is independent of the very choice of the potential in this case.

For an outline of the proof of Theorem 2.1 and in order to understand the corollary below, we remark that the unique bounded non-negative solution to (1.1) is given by the Feynman-Kac formula

$$\begin{aligned} u(t,x) = \mathbb{E}_0 \exp\Big\{ \int_0^t \xi(X_s)\, ds \Big\} &= \sum_{n\in\mathbb{Z}} \mathbb{E}_0 \exp\Big\{ \int_0^t \xi(X_s)\, ds \Big\} \mathbb{1}_{X_t=n} \\ &= \sum_{n\in\mathbb{Z}} \mathbb{E}_n \exp\Big\{ \int_0^t \xi(Y_s)\, ds \Big\} \mathbb{1}_{Y_t=0}. \end{aligned} \tag{2.3}$$

Here, in analogy to Y we denote by $X = (X_t)_{t\in\mathbb{R}_+}$ a continuous-time random walk on $\mathbb{Z}$ with generator $\kappa\Delta_h$. Note that X and Y may be regarded as time reversals of each other. Departing from (2.3), the strong Markov property supplies us with

$$\begin{aligned} &\mathbb{E}_n \exp\Big\{ \int_0^t \xi(Y_s)\, ds \Big\} \mathbb{1}_{Y_t=0} \\ &= \mathbb{E}_n \Big(\exp\Big\{ \int_0^{T_0} \xi(Y_s)\, ds \Big\} \mathbb{1}_{T_0\le t} \Big(\mathbb{E}_0 \exp\Big\{ \int_0^{t-r} \xi(Y_s)\, ds \Big\} \mathbb{1}_{Y_{t-r}=0} \Big)_{r=T_0} \Big). \end{aligned} \tag{2.4}$$

The advantage of considering the time reversal of (2.3) is now apparent: For a fixed realisation of the medium, the term $\mathbb{E}_0 \exp\{\int_0^{t-r} \xi(Y_s)\, ds\} \mathbb{1}_{Y_{t-r}=0}$ sees the same part of the medium, independent of which $n \in \mathbb{Z}$ the random walk Y is starting from.

The proof of Theorem 2.1 now roughly proceeds as follows. Considering (2.3) and (2.4), the idea is that the main contributions should stem from summands $n \approx \alpha^* t$, i.e.

$$u(t,0) \asymp \sum_{n\approx\alpha^* t} \mathbb{E}_n \exp\Big\{ \int_0^t \xi(Y_s)\, ds \Big\} \mathbb{1}_{Y_t=0}. \tag{2.5}$$

Here, $\alpha^* \ge 0$ denotes the optimal speed of the random walk X within the random medium, cf. Corollary 2.3 below. To show the desired behaviour we use large deviations for $(T_0/n)_{n\in\mathbb{N}}$ under the perturbed measure suggested by (2.4) which in combination with further estimates yield the variational formula

$$\lambda_0 = \sup_{\alpha\in[0,\gamma]} \inf_{\beta<\beta_{cr}} (-\beta + \alpha L(\beta)), \tag{2.6}$$

for γ large enough, cf. Corollary 4.5. In this formula, the infimum over β optimises the behaviour of T_0/n while the supremum over α optimises the speed of the random walk. As with respect to the competition between the two factors $\exp\{\int_0^{T_0} \xi(Y_s)\, ds\} \mathbb{1}_{T_0\le t}$ and $(\mathbb{E}_0 \exp\{\int_0^{t-r} \xi(Y_s)\, ds\} \mathbb{1}_{Y_{t-r}=0})_{r=T_0}$

appearing in (2.4), one can show that it makes sense for the random walk to linger in the bulk and only hit 0 shortly before time t, i.e. $T_0 \approx t$ (indeed, this is implied by the reasoning about the supremum directly after (4.3)). With (2.6) at hand, it is an easy task to complete the proof of Theorem 2.1. See section 4 for further details.

As a byproduct we obtain the following corollary on the optimal speed of the random walk X in the random potential ξ, i.e. under the Gibbs measure

$$\mathbb{P}_t^{\xi}(\cdot) := \frac{\mathbb{E}_0 \exp\{\int_0^t \xi(X_s)\,ds\}\mathbb{1}_{X_t\in\cdot}}{\mathbb{E}_0 \exp\{\int_0^t \xi(X_s)\,ds\}}$$

on $\mathbb{R}$. In particular, we say that the random walk X in the random potential ξ has speed α^* if $X_t/t \to \alpha^*$ in $\mathbb{P}_t^{\xi}$ probability as $t \to \infty$.

Corollary 2.3. *Let the assumptions of Theorem 2.1 be fulfilled.*

(a) *If* $\lim_{\beta\uparrow\beta_{cr}} L(\beta) > 0$, *then for all* $\varepsilon > 0$,

$$\lambda_0 > \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}_0 \exp\left\{ \int_0^t \xi(X_s)\,ds \right\} \mathbb{1}_{X_t \notin t(\alpha^*-\varepsilon,\alpha^*+\varepsilon)}$$

with

$$\alpha^* := (L'(-\lambda_0))^{-1} = \left\langle \frac{\mathbb{E}_1 T_0 \exp\{\int_0^{T_0} (\xi(Y_s)-\lambda_0)\,ds\}}{\mathbb{E}_1 \exp\{\int_0^{T_0} (\xi(Y_s)-\lambda_0)\,ds\}} \right\rangle^{-1} \in (0,\infty).$$

(b) *If* $\lim_{\beta\uparrow\beta_{cr}} L(\beta) = 0$, *then for* $m \in [0, (\lim_{\beta\uparrow\beta_{cr}} L'(\beta))^{-1}]$ *and all* $\varepsilon > 0$,

$$\lambda_0 = \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}_0 \exp\left\{ \int_0^t \xi(X_s)\,ds \right\} \mathbb{1}_{X_t \in t(m-\varepsilon,m+\varepsilon)},$$

while

$$\lambda_0 > \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}_0 \exp\left\{ \int_0^t \xi(X_s)\,ds \right\} \mathbb{1}_{X_t \notin t[-\varepsilon,(\lim_{\beta\uparrow\beta_{cr}} L'(\beta))^{-1}+\varepsilon]}.$$

(c) *If* $\lim_{\beta\uparrow\beta_{cr}} L(\beta) < 0$, *then for all* $\varepsilon > 0$,

$$\lambda_0 > \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}_0 \exp\left\{ \int_0^t \xi(X_s)\,ds \right\} \mathbb{1}_{X_t \notin t(-\varepsilon,\varepsilon)}.$$

Remark 2.4. (a) The existence of L' under the assumptions of part (b) from above will be shown in Lemma 3.5 below. Since L is increasing and convex on $(-\infty,\beta_{cr})$, the limit $\lim_{\beta\uparrow\beta_{cr}} L'(\beta) > 0$ then exists.

(b) Part (b) of the corollary can be viewed as a transition between the cases (a) and (c), which correspond to the positive and zero-speed regimes, respectively. Note that the result of (c) can be considered a screening effect, where the random walk is prevented from moving with positive speed due to the distribution of ξ putting much mass on very negative values, cf. also [BK01b].

(c) Inspecting the proof of this corollary, one may observe that continuing the corresponding ideas we would obtain a large deviations principle for the position of the random walk under the above Gibbs measure. However, since our emphasis is rather on Lyapunov exponents and intermittency, we will not carry out the necessary modifications.

(d) In this context it is worth mentioning that in [GdH92] the authors consider a discrete time branching random walk with drift in random environment. They derive a variational formula for the optimal speed which on the one hand is more involved than the one we obtain in the above corollary, but on the other hand provides information on the optimal path behaviour of the random walker as well.

Furthermore, also in the discrete time context, in [Flu07] large deviations principles for a random walk with drift in random potential have been derived using the Laplace-Varadhan method applied to earlier results of [Zer98]. However, it has to be emphasised that in the model treated there the influence of the drift of the random walk is essentially different from our situation; in fact, it only appears in terms of the *end point* of the random walker at a given time, while in our model the influence of the drift is also via the *length* of the random walk path.

Similarly to corresponding results in [GdH92], in the context of Corollary 2.3 it is interesting to investigate the dependence of the speed on h. For this purpose, for the rest of this subsection we write L_h to denote the function we previously denoted L.

Proposition 2.5. *Assume (1.2) and (2.2).*

(a) For $h > 0$ small enough the assumptions of Corollary 2.3 (c) are fulfilled and thus a.s. the random walk in the random potential ξ has speed 0.

(b) If $\lim_{\beta\uparrow\kappa} L_1(\beta) > 0$, then for $h \le 1$ large enough the assumptions of Corollary 2.3 (a) are fulfilled and thus a.s. the random walk in the random potential ξ has speed α^*.

If $\lim_{\beta\uparrow\kappa} L_1(\beta) < 0$, then for $h \le 1$ large enough the assumptions of Corollary 2.3 (c) are fulfilled and thus a.s. the random walk in the random potential ξ has speed 0.

Note here that it may happen that the random walk has speed 0 for h arbitrarily close to 1 which is different from the results of [GdH92].

In addition to the behaviour proven in Proposition 2.5, we conjecture that the speed of the random walk in the random potential ξ is nondecreasing as a function of the drift parameter $h \in (0,1]$.

Since the proof of Proposition 2.5 requires results developed later on, it is postponed to section 7.

2.1.1 Further properties of the quenched Lyapunov exponent

Writing $\lambda_0(\kappa)$ to denote the dependence of λ_0 on κ, we get the following properties for the quenched Lyapunov exponent.

Proposition 2.6. *Let the assumptions of Theorem 2.1 be fulfilled.*

(a) The function $(0,\infty) \ni \kappa \mapsto \lambda_0(\kappa)$ is convex and nonincreasing.

(b) $\lim_{\kappa\downarrow 0}\lambda_0(\kappa)=0$.

(c) *The limits* $\lim_{\kappa\to\infty}\kappa^{-1}\lambda_0(\kappa)$ *and* $\lim_{\kappa\downarrow 0}\kappa^{-1}\lambda_0(\kappa)$ *exist and are given by*

$$\lim_{x\downarrow 0}\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}_0\exp\left\{x\int_0^t\xi(X_s)\,ds\right\}=0 \tag{2.7}$$

and

$$\lim_{x\to\infty}\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}_0\exp\left\{x\int_0^t\xi(X_s)\,ds\right\}\in[-1,0), \tag{2.8}$$

respectively, where X *is generated by* Δ_h.

2.2 Annealed regime

In order to avoid technical difficulties, we always assume that

$$\mathrm{Prob}=\eta^{\mathbb{Z}} \tag{2.9}$$

for some $\eta\in\mathscr{M}_1([b,0])$ and $b\in(-\infty,0)$ in the annealed case. We are interested in the existence of the annealed Lyapunov exponents λ_p for all $p>0$ and will derive specific formulae for them. The proof will use process level large deviations applied to the random medium ξ. In order to be able to formulate our result, we have to introduce some further notation. For $\zeta\in\Sigma_b$ we denote by $R_n(\zeta)$ the restriction of the empirical measure $n^{-1}\sum_{k=0}^{n-1}\delta_{\theta^k\circ\zeta}\in\mathscr{M}_1(\Sigma_b)$ to $\mathscr{M}_1(\Sigma_b^+)$.[2] Using assumption (2.9) we get that the uniformity condition (U) in section 6.3 of [DZ98] is satisfied for $(\xi(x))_{x\in\mathbb{Z}}$. Hence, Corollaries 6.5.15 and 6.5.17 of the same reference provide us with a full process level large deviations principle for the random sequence of empirical measures $(R_n\circ\xi)_{n\in\mathbb{N}}$ on scale n with rate function given by

$$\mathscr{I}(\nu):=\begin{cases} H(\nu_1^*|\nu_0^*\otimes\eta), & \text{if } \nu \text{ is shift-invariant,}\\ \infty, & \text{otherwise,}\end{cases} \tag{2.10}$$

for $\nu\in\mathscr{M}_1(\Sigma_b^+)$. In this expression, H denotes relative entropy and writing π_k for the projection mapping from $\mathbb{R}^{\mathbb{N}_0}$ to $\mathbb{R}^k$ given by $(x_n)_{n\in\mathbb{N}_0}\mapsto(x_0,\dots,x_{k-1})$, measures ν_i^* are defined as follows: For $i\in\{0,1\}$ and shift-invariant $\nu\in\mathscr{M}_1(\Sigma_b^+)$, we denote by ν_i^* the unique probability measure on $[b,0]^{\mathbb{Z}\cap(-\infty,i]}$ such that, for each $k\in\mathbb{N}$ and each Borel set $\Gamma\subseteq[b,0]^k$,

$$\nu_i^*(\{(\dots,x_{i-k+1},\dots,x_i):(x_{i-k+1},\dots,x_i)\in\Gamma\})=\nu\circ\pi_k^{-1}(\Gamma).$$

Note that ν_i^* is well-defined due to the shift-invariance of ν. Furthermore, set

$$L(\beta,\nu):=\int_{\Sigma_b^+}\log\mathbb{E}_1\exp\left\{\int_0^{T_0}(\zeta(Y_s)+\beta)\,ds\right\}\nu(d\zeta)$$

[2]If clear from the context, we will interpret elements ν of $\mathscr{M}_1(\Sigma_b)$ as elements of $\mathscr{M}_1(\Sigma_b^+)$ without further mentioning by considering $\nu\circ\pi_+^{-1}$ instead, where $\pi_+:(x_n)_{n\in\mathbb{Z}}\mapsto(x_n)_{n\in\mathbb{N}_0}$. In the same fashion, we consider elements of Σ_b as elements of Σ_b^+.

for all $\beta \in \mathbb{R}$ and $\nu \in \mathcal{M}_1(\Sigma_b^+)$. In particular, we have $L(\beta) = L(\beta, \mathrm{Prob})$. Employing the notation

$$L_p^{sup}(\beta) := \sup_{\nu \in \mathcal{M}_1^s(\Sigma_b^+)} \Big(L(\beta, \nu) - \frac{\mathcal{I}(\nu)}{p} \Big), \quad \beta \in \mathbb{R}, \tag{2.11}$$

we are ready to formulate our main result for the annealed setting.

Theorem 2.7. *Assume (1.2) and (2.9). Then, for each $p \in (0, \infty)$, the p-th annealed Lyapunov exponent λ_p exists. Furthermore, λ_p equals the zero of $\beta \mapsto L_p^{sup}(-\beta)$ in $(-\beta_{cr}, 0)$ or, if such a zero does not exist, it equals $-\beta_{cr}$.*

Remark 2.8. (a) In fact, L_p^{sup} has at most one zero as it is strictly increasing, cf. Lemma 5.11 below.

(b) Recall that as (2.9) implies (2.2), we once again infer $\beta_{cr} = \kappa(1 - \sqrt{1 - h^2})$, cf. Remark 2.2.

With respect to the proof of this theorem, it turns out that the asymptotics of the p-th moment $\langle u(t, 0)^p \rangle$ is the same as the quenched behaviour of $u(t, 0)^p$ but under a different distribution of the environment ξ. This will be made precise by the use of the aforementioned process level large deviations for $R_n \circ \xi$.

The term L_p^{sup} defined in (2.11) and appearing in the above characterisation of λ_p admits a convenient interpretation as follows. On the one hand, distributions ν of our random medium which provide us with high values of $L(\beta, \nu)$ can play an important role in attaining the supremum in the right-hand side of (2.11). On the other hand, we have to pay a price for obtaining such (rare) distributions, which is given by $\mathcal{I}(\nu)/p$. As is heuristically intuitive and evident from formula (2.11), this price in relation to the gain obtained by high values of $L(\beta, \nu)$ becomes smaller as p gets larger. Note that, heuristically, $L(\cdot, \nu)$ corresponds to the function L characterising the quenched Lyapunov exponent for a potential distributed according to ν.

As mentioned before, we are interested in the intermittency of u for which we have the following result:

Proposition 2.9. *Let the assumptions of Theorem 2.7 be fulfilled. Then for $p > 0$ large enough, the solution u to (1.1) is p-intermittent.*

Furthermore, as mentioned previously, one expects the p-th annealed Lyapunov exponent λ_p to converge to the quenched Lyapunov exponent λ_0 as $p \downarrow 0$.

Theorem 2.10. *Let the assumptions of Theorem 2.7 be fulfilled. Then*

$$\lim_{p \downarrow 0} \lambda_p = \lambda_0.$$

2.2.1 Further properties of the annealed Lyapunov exponent

In addition to the previous results we have the following properties of the annealed Lyapunov exponents.

Proposition 2.11. *Let the assumptions of Theorem 2.7 be fulfilled.*

(a) *The function $p \mapsto \lambda_p$ is nondecreasing in $p \in [0,\infty)$.*

(b) *The function $p \mapsto p\lambda_p$ is convex in $p \in (0,\infty)$.*

(c) *For any $p \in [0,\infty)$, $\kappa \mapsto \lambda_p(\kappa)$ is convex in $\kappa \in (0,\infty)$.*

(d) *If u is p-intermittent for some $p \in (0,\infty)$, it is q-intermittent for all $q > p$ as well.*

2.3 Related work

The parabolic Anderson model without drift and i.i.d. or Gaussian potential is well-understood, see the survey [GK05] as well as the references therein. As a common feature, these treatments take advantage of the self-adjointness of the random Hamiltonian $\kappa\Delta + \xi$ which allows for a spectral theory approach to the respective problems. In our setting, however, the (random) operators $\kappa\Delta_h + \xi$ are not self-adjoint, whence we do not have the common functional calculus at our disposal. As hinted at earlier, we therefore retreat to a large deviations principle for the sequence T_0/n. Heuristically, another difference caused by the drift is that the drift term of the Laplace operator makes it harder for the random walk X appearing in the Feynman-Kac representation (2.3) of the solution to stay at islands of values of ξ close to its supremum 0.

Our model without drift has been dealt with in [BK01a] (not restricted to one dimension) and [BK01b]. Here the authors found formulae for the quenched and p-th annealed Lyapunov exponents for all $p \in (0,\infty)$ using a spectral theory approach; as mentioned before, in their investigations they relied on a weaker notion of intermittency. Furthermore, they investigated the so-called screening effect that can appear in dimension one.

A situation similar in spirit to ours has been examined in the seminal article [GdH92] motivated from the point of view of population dynamics. The model treated there is a discrete-time branching model in random environment with drift and corresponds to the case of a bounded i.i.d. potential.

Their aim is the investigation of the quenched and first annealed Lyaupunov exponents and their starting point is similar to ours in the sense that they look for the speed of optimal trajectories of the random walker, which in our situation corresponds to α^* of (2.5). Due to the discrete time nature of their model, it is then possible to single out the dependence of the drift term just in terms of large deviations for what in our model corresponds to Y_n/n under the perturbed measure. In contrast, the dependence of the drift term in our model appears in the function L which is less explicit.

Continuing their investigation they obtain a representation of the quenched and annealed Lyapunov exponents by the use of large deviation principles for the speed as well as for a functional of a empirical pair distribution of a certain Markov process; the latter is important due to its close connection to local times of the random walk. As a consequence, the resulting variational formulae characterising the Lyapunov exponents are on the one hand more involved than the ones we obtain; on the other hand, their evaluation gives rise to a deeper understanding of the path behaviour of optimal trajectories of the random walk.

While the authors obtain a more explicit dependence of the results on the drift parameter h than we do, an advantage of our approach is that we may compute the p-th annealed Lyapunov exponents for *all* $p \in (0,\infty)$ and characterise them in a simpler way.

Also in the context of discrete time, it is well worth mentioning recent results of [Flu07]. Departing from a different model, the author computed the quenched and first annealed Lyapunov exponents and obtains large deviations for discrete-time random walks with drift under the influence of a random potential in arbitrary dimensions. Using the Laplace-Varadhan method, he derives the result from a large deviations principle established in [Zer98] for the case without drift. Note also that the moment generating function appearing in the derivation of the large deviations principle of [Zer98] is quite similar in spirit to our function Λ.

However, coming back to [Flu07], it is not clear how to apply the corresponding techniques to our situation. Firstly, as pointed out in [Zer98], this large deviations principle does not carry over to the continuous-time case automatically, which also involves large deviations on the number of jumps leading to quantitatively different results. In particular, the way the drift term is taken into account is substantially different to the situation we are dealing with: While in the model of [Flu07] the drift only enters through the end point of the random walk at a certain time, in our situation the influence of the drift also depends on the number of jumps the random walk has performed up to a certain time; again, this is an effect that cannot appear in the context of discrete time.

Secondly, and more importantly, it is not clear how to adapt the methods of [Zer98] and [Flu07] to obtain λ_p for general $p \in (0, \infty)$, which is the main focus of this paper.

2.4 Outline

Section 3 contains auxiliary results both for the quenched and annealed context. The proofs of Theorem 2.1 and Corollary 2.3 will be carried out in section 4. In section 5 we prove some results needed for the proof of Theorem 2.7. The latter is then the subject of section 6, while section 7 contains the proofs of the result on the transition from zero to positive speed (Proposition 2.5) as well as the proofs of the results of subsections 2.1.1 and 2.2.1. Furthermore, the intermittency and continuity results, Proposition 2.9 and Theorem 2.10, are proven in this section.

While the results we gave in section 2 are valid for arbitrary $h \in (0, 1]$, the corresponding proofs in sections 3 to 6 contain steps which a priori hold true for $h \in (0, 1)$ only. Section 8 deals with the adaptations necessary to obtain their validity for $h = 1$ also. Finally, in section 8 we will also give a more convenient representation for λ_p with $p \in \mathbb{N}$, see Proposition 8.2.

3 Auxiliary results

In this section we prove auxiliary results which will primarily facilitate the proof of the quenched results given in section 2, but will also play a role when deriving the annealed results.

All of the results hereafter implicitly assume (1.2) and (1.3) mentioned in subsection 1.1.

The main results of this section are Proposition 3.1, which controls the aforementioned term, and the large deviations principle of Theorem 3.8, which helps to control the remaining part of the right-hand side in (2.4). The remaining statements of this section are of a more technical nature.

The following result is motivated in spirit by section VII.6 in [Fre85]. Note that we exclude the case of absolute drift $h = 1$.

Proposition 3.1. *(a) For $h \in (0, 1)$ and $x, y \in \mathbb{Z}$, the finite limit*

$$c^* := \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}_x \exp\left\{ \int_0^t \xi(Y_s)\, ds \right\} \mathbb{1}_{Y_t = y} \tag{3.1}$$

exists a.s., equals

$$\sup_{t\in(0,\infty)} \frac{1}{t}\log \mathbb{E}_0 \exp\Big\{\int_0^t \xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=0} \tag{3.2}$$

as well as

$$\lim_{t\to\infty} \frac{1}{t}\log \mathbb{E}_0 \exp\Big\{\int_0^t \xi(Y_s)\,ds\Big\}\mathbb{1}_{T_M>t,Y_t=0} \tag{3.3}$$

for all $M \in \mathbb{Z}\backslash\{0\}$, and is non-random. Furthermore, $c^* \le -\beta_{cr}$. If either (2.1) or (2.2) hold true, then

$$c^* = -\beta_{cr}. \tag{3.4}$$

(b) *For $\beta > -c^*$, we have*

$$\mathbb{E}_1 \exp\Big\{\int_0^{T_0} (\xi(Y_s)+\beta)\,ds\Big\} = \infty \quad a.s. \tag{3.5}$$

If either (2.1) or (2.2) hold true, then for each $\beta < \beta_{cr}$ there exists a non-random constant $C_\beta < \infty$ such that

$$\mathbb{E}_1 \exp\Big\{\int_0^{T_0} (\xi(Y_s)+\beta)\,ds\Big\} \le C_\beta \quad a.s. \tag{3.6}$$

Remark 3.2. Identity (3.4) links the two expectations in (2.4), one involving fixed time, the other a hitting time. As such, it will prove useful for the simplification of the variational problems arising in sections 4 and 6.

Proof. We start with the proof of (a) and split it into four steps.

(i) We first show that for all $x, y \in \mathbb{Z}$, the limit in (3.1) exists and equals the expression in (3.2).

For $t \ge 0$ and $x, y \in \mathbb{Z}$, define

$$p_{x,y}(t) := \log \mathbb{E}_x \exp\Big\{\int_0^t \xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=y}.$$

Using the Markov property, we observe that $p_{0,0}$ is super-additive. Therefore, the limit c^* of $p_{0,0}(t)/t$ as $t \to \infty$ exists and

$$c^* = \sup_{t\in(0,\infty)} p_{0,0}(t)/t \in (-\infty, 0]. \tag{3.7}$$

For $x, y \in \mathbb{Z}$, the Markov property applied at times 1 and $t+1$ yields

$$\begin{aligned} &\mathbb{E}_x \exp\Big\{\int_0^{t+2} \xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_{t+2}=y} \\ &\quad\ge \mathbb{E}_x\Big(\mathbb{1}_{Y_s\in\{0\wedge x,\dots,0\vee x\}\,\forall s\in[0,1],Y_1=0} \exp\Big\{\int_0^{t+2} \xi(Y_s)\,ds\Big\} \\ &\qquad \times \mathbb{1}_{Y_{t+1}=0,Y_s\in\{0\wedge y,\dots,0\vee y\}\forall s\in[t+1,t+2],Y_{t+2}=y}\Big) \\ &\quad\ge c_{x,y}\mathbb{E}_0 \exp\Big\{\int_0^t \xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=0}, \end{aligned} \tag{3.8}$$

where $c_{x,y}$ is an a.s. positive random variable given by

$$\begin{aligned} c_{x,y} := & \min_{k\in\{0\wedge x,\dots,0\vee x\}} e^{\xi(k)} \times \mathbb{P}_x(Y_s \in \{0\wedge x,\dots,0\vee x\}\,\forall s\in[0,1], Y_1 = 0) \\ & \times \min_{k\in\{0\wedge y,\dots,0\vee y\}} e^{\xi(k)} \times \mathbb{P}_0(Y_s \in \{0\wedge y,\dots,0\vee y\}\,\forall s\in[0,1], Y_1 = y). \end{aligned} \tag{3.9}$$

Similarly,

$$\mathbb{E}_0 \exp\Big\{\int_0^{t+2} \xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_{t+2}=0} \geq c_{y,x}\mathbb{E}_x \exp\Big\{\int_0^t \xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=y}. \tag{3.10}$$

Now, combining (3.7) to (3.10) we conclude that $\lim_{t\to\infty} p_{x,y}(t)/t$ exists and equals (3.2).

(ii) We next show that c^* is non-random and (3.3) holds.

Naming the dependence of c^* on the realisation explicitly, we obtain

$$c^* = c^*(\xi) = c^*(\theta\circ\xi)$$

by the use of (i). Thus, c^* is non-random by Birkhoff's ergodic theorem.

In order to derive (3.3), observe that for $M\in\mathbb{N}$ the function

$$p_M(t) := \log \mathbb{E}_0 \exp\Big\{\int_0^t \xi(Y_s)\,ds\Big\}\mathbb{1}_{T_{-M}>t, Y_t=0}$$

is super-additive. Hence $c_M^* := \lim_{t\to\infty} p_M(t)/t$ is well-defined and equals $\sup_{t\in(0,\infty)} p_M(t)/t$. Obviously, $p_M(t)$ is nondecreasing in M and $p_M(t)\leq p_{0,0}(t)$, whence

$$c_M^* \leq c^* \tag{3.11}$$

for all $M\in\mathbb{N}$. On the other hand, since $c_M^*\geq p_M(t)/t$ and $p_M(t)\uparrow p_{0,0}(t)$ as $M\to\infty$, we get $\lim_{M\to\infty} c_M^* \geq p_{0,0}(t)/t$ for all t and, consequently, $\lim_{M\to\infty} c_M^*\geq c^*$. Together with (3.11) it follows that

$$\lim_{M\to\infty} c_M^* = c^*. \tag{3.12}$$

Similarly to the previous step we compute

$$\begin{aligned} & \mathbb{E}_0 \exp\Big\{\int_0^{t+2} \xi(Y_s)\,ds\Big\}\mathbb{1}_{T_{-M}>t+2, Y_{t+2}=0} \\ & \quad \geq c_{1,1}\mathbb{E}_1 \exp\Big\{\int_0^t \xi(Y_s)\,ds\Big\}\mathbb{1}_{T_{-M}>t, Y_t=1} \\ & \quad = c_{1,1}\mathbb{E}_0 \exp\Big\{\int_0^t (\theta\circ\xi)(Y_s)\,ds\Big\}\mathbb{1}_{T_{-(M+1)}>t, Y_t=0}. \end{aligned}$$

Taking logarithms, dividing both sides by t and letting t tend to infinity, we obtain $c_M^*(\xi)\geq c_{M+1}^*(\theta\circ\xi)$. Iterating this procedure and using the monotonicity of c_M^* in M, we obtain

$$c_M^*(\xi) \geq \frac{1}{n}\sum_{j=1}^{k-1} c_{M+j}^*(\theta^j\circ\xi) + \frac{1}{n}\sum_{j=k}^{n} c_{M+k}^*(\theta^j\circ\xi)$$

for all $n \in \mathbb{N}$ and $k \le n$. Birkhoff's ergodic theorem now yields $c_M^*(\xi) \ge \langle c_{M+k}^*(\xi)\rangle$ a.s. for all $k \in \mathbb{N}$. Because we clearly have $c_M^*(\xi) \le c_{M+k}^*(\xi)$, this implies that c_M^* is constant a.s. and independent of M. Due to (3.12) this gives $c_M^* = c^*$ a.s. for all $M \in \mathbb{N}$, and thus c^* equals (3.3) for all $M \in -\mathbb{N}$. By a similar derivation as above we find that c^* equals (3.3) for all $M \in \mathbb{Z}\backslash\{0\}$.

(iii) The next step is to prove that $\beta_{cr} \le -c^*$.

Given $t, \varepsilon > 0$, we apply the Markov property at time t to obtain

$$
\begin{aligned}
&\mathbb{E}_0 \exp\Big\{\int_0^{T_{-1}} (\xi(Y_s) - c^* + \varepsilon)\, ds\Big\} \\
&\quad \ge \mathbb{E}_0 \exp\Big\{\int_0^{T_{-1}} (\xi(Y_s) - c^* + \varepsilon)\, ds\Big\} \mathbb{1}_{T_{-1}>t, Y_t=0} \\
&\quad = \mathbb{E}_0 \exp\Big\{\int_0^{t} (\xi(Y_s) - c^* + \varepsilon)\, ds\Big\} \mathbb{1}_{T_{-1}>t, Y_t=0} \\
&\qquad \times \mathbb{E}_0 \exp\Big\{\int_0^{T_{-1}} (\xi(Y_s) - c^* + \varepsilon)\, ds\Big\}.
\end{aligned}
$$

The second factor on the right-hand side is positive a.s. while, as we infer from (3.3) for $M = 1$, the first one is logarithmically equivalent to $e^{t\varepsilon}$. Thus, we deduce

$$
\mathbb{E}_0 \exp\Big\{\int_0^{T_{-1}} (\xi(Y_s) - c^* + \varepsilon)\, ds\Big\} = \infty \quad \text{a.s.,} \tag{3.13}
$$

and, using the shift invariance of ξ, we get

$$
\mathbb{E}_1 \exp\Big\{\int_0^{T_0} (\xi(Y_s) - c^* + \varepsilon)\, ds\Big\} = \infty \quad \text{a.s.} \tag{3.14}
$$

In particular, this implies $L^+(-c^* + \varepsilon) = \infty$ and thus $\beta_{cr} \le -c^*$.

(iv) This part consists of showing that $\beta_{cr} \ge -c^*$ if either (2.1) or (2.2) is fulfilled.

Note that the shift invariance of ξ yields

$$
L^+(\beta) = \Big\langle \Big(\log \mathbb{E}_0 \exp\Big\{\int_0^{T_{-1}} (\xi(Y_s) + \beta)\, ds\Big\}\Big)^+ \Big\rangle.
$$

Using (3.7) as well as (3.10) and taking into account that $c^* \le 0$, we get

$$
\mathbb{E}_0 \exp\Big\{\int_0^{t} \xi(Y_s)\, ds\Big\} \mathbb{1}_{Y_t=-1} \le e^{c^* t}/c_{-1,0} \tag{3.15}
$$

for all $t \in (0,\infty)$. Consequently, we compute for $n \in \mathbb{N}$ and $\varepsilon > 0$:

$$
\begin{aligned}
&\mathbb{E}_0 \exp\Big\{\int_0^{T_{-1}} (\xi(Y_s) - c^* - \varepsilon)\, ds\Big\} \mathbb{1}_{T_{-1}\in(n-1,n], Y_s=-1\, \forall s\in[T_{-1},n]} \\
&\quad \le \mathbb{E}_0 \exp\Big\{\int_0^{n} \xi(Y_s)\, ds\Big\} \mathbb{1}_{Y_n=-1} \exp\{-\xi(-1)\} \exp\{-c^* n - \varepsilon(n-1)\} \\
&\quad \le \exp\{-\varepsilon(n-1) - \xi(-1)\}/c_{-1,0} \quad \text{a.s.,}
\end{aligned} \tag{3.16}
$$

where we have used $c^* \le 0$ to deduce the first inequality and (3.15) to obtain the last one. Analogously, the strong Markov property at time T_{-1} supplies us with the lower bound

$$\begin{aligned} &\mathbb{E}_0 \exp\Big\{\int_0^{T_{-1}} (\xi(Y_s) - c^* - \varepsilon)\, ds\Big\} \mathbb{1}_{T_{-1}\in(n-1,n], Y_s=-1\, \forall s\in[T_{-1},n]} \\ &\quad \ge \mathbb{E}_0 \exp\Big\{\int_0^{T_{-1}} (\xi(Y_s) - c^* - \varepsilon)\, ds\Big\} \mathbb{1}_{T_{-1}\in(n-1,n]} \mathbb{P}_{-1}(Y_s = -1\, \forall s \in [0,1]). \end{aligned} \tag{3.17}$$

Since $\mathbb{P}_0(T_{-1} < \infty) = 1$, combining (3.16) with (3.17) and summing over $n \in \mathbb{N}$, we get

$$\mathbb{E}_0 \exp\Big\{\int_0^{T_{-1}} (\xi(Y_s) - c^* - \varepsilon)\, ds\Big\} \le C \sum_{n\in\mathbb{N}} \exp\{-\varepsilon n\} < \infty \quad \text{a.s.,} \tag{3.18}$$

where

$$C := \big(\mathbb{P}_{-1}(Y_s = -1\, \forall s \in [0,1])\big)^{-1} \exp\{\varepsilon - \xi(-1)\}/c_{-1,0}. \tag{3.19}$$

We now distinguish cases and first assume (2.1). In this case, $c_{-1,0}$ can be bounded from below by some constant $\underline{c}_{-1,0} > 0$ a.s., whence C can be bounded from above by the non-random constant

$$\overline{C} := \big(\mathbb{P}_{-1}(Y_s = -1\, \forall s \in [0,1])\big)^{-1} \exp\{\varepsilon - b\}/\underline{c}_{-1,0}. \tag{3.20}$$

In particular, using (3.18) this implies $L^+(-c^* - \varepsilon) < \infty$, whence we deduce $\beta_{cr} \ge -c^*$.

To treat the second case assume (2.2). Due to (1.2) and (2.2), we infer $\mathrm{Prob}(\xi(-1), \xi(0) \ge b) > 0$ for any $b \in (-\infty, 0)$; fix one such b. On $\{\xi(-1), \xi(0) \ge b\}$, as before, C may be bounded from above by the corresponding non-random constant $\overline{C}$ of (3.20) and therefore

$$\mathbb{E}_0 \exp\Big\{\int_0^{T_{-1}} (\xi(Y_s) - c^* - \varepsilon)\, ds\Big\} \le \overline{C} \sum_{n\in\mathbb{N}} \exp\{-\varepsilon n\} < \infty \tag{3.21}$$

$\mathrm{Prob}(\cdot | \xi(-1), \xi(0) \ge b)$-a.s. Since the left-hand side of (3.21) does not depend on the actual realisation of $\xi(-1)$, (3.21) even holds $\mathrm{Prob}(\cdot | \xi(0) \ge b)$-a.s. But the left-hand side of (3.21) is increasing in $\xi(0)$, whence (3.21) holds Prob-a.s. This finishes the proof of part (a).

It remains to prove (b). The first part was already established in (3.14). Under assumption (2.1), the upper bound is a consequence of (3.18) with C replaced by $\overline{C}$ of (3.20); otherwise, if (2.2) is fulfilled, the upper bound follows from the last conclusion in the proof of part (a) (iv). $\square$

Proposition 3.1 enables us to control the asymptotics of the second expectation on the right of (2.4). To deal with the first expression, we define for $n \in \mathbb{N}$ and $\zeta \in \Sigma^+$ the probability measures

$$\mathbb{P}_n^\zeta(A) := (Z_n^\zeta)^{-1} \mathbb{E}_n \exp\Big\{\int_0^{T_0} \zeta(Y_s)\, ds\Big\} \mathbb{1}_A$$

with $A \in \mathscr{F}$ and the normalising constant

$$Z_n^\zeta := \mathbb{E}_n \exp\Big\{\int_0^{T_0} \zeta(Y_s)\, ds\Big\}.$$

The expectation with respect to $\mathbb{P}_n^\zeta$ will be denoted $\mathbb{E}_n^\zeta$. By considering $\mathbb{P}_n^\xi \circ (T_0/n)^{-1}$, we obtain a random sequence of probability measures on $\mathbb{R}_+$ for which we aim to prove a large deviations principle (see Theorem 3.8 below). As common in the context of large deviations, we define the moment generating function

$$\Lambda(\beta) := \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}_n^\xi \exp\{\beta T_0\} = \langle \log \mathbb{E}_1^\xi \exp\{\beta T_0\}\rangle, \quad \beta \in \mathbb{R},$$

where the last equality stems from Birkhoff's ergodic theorem. Note that

$$\Lambda(\beta) = L(\beta) - L(0) \tag{3.22}$$

whenever the right-hand side is well-defined.

The following lemma tells us that the critical value β_{cr} of L^+ also applies to Λ and is positive.

Lemma 3.3. *Assume $h \in (0,1)$. Then*

(a) $\Lambda(\beta) < \infty$ for $\beta < \beta_{cr}$, while $\Lambda(\beta) = \infty$ for $\beta > \beta_{cr}$;

(b) β_{cr} is positive.

Remark 3.4. Note that for $h = 1$ we can explicitly compute $\beta_{cr} = \kappa$ as well as $c^* = \xi(0) - \kappa$. In particular, $h = 1$ is the only case in which c^* is random, cf. Proposition 3.1 (a).

Proof. (a) Since $Z_1^\xi \le 1$, we get

$$\Big(\log \mathbb{E}_1 \exp\Big\{\int_0^{T_0} (\xi(Y_s) + \beta)\, ds\Big\}\Big)^+ \le \log \mathbb{E}_1^\xi \exp\{\beta T_0\} \tag{3.23}$$

for $\beta \ge 0$. Consequently, since $\beta_{cr} \ge 0$, it is evident that $\Lambda(\beta) = \infty$ for $\beta > \beta_{cr}$. For the remaining part of the statement, we estimate with $\beta \in [0, \kappa)$:

$$\begin{aligned}
\mathbb{E}_1^\xi \exp\{\beta T_0\} &\le \frac{\mathbb{E}_1 \exp\{\int_0^{T_0} (\xi(Y_s) + \beta)\, ds\} \mathbb{1}_{T_0 \le T_2}}{\mathbb{E}_1 \exp\{\int_0^{T_0} \xi(Y_s)\, ds\} \mathbb{1}_{T_0 \le T_2}} \\
&\quad + \frac{\mathbb{E}_1 \exp\{\int_0^{T_0} (\xi(Y_s) + \beta)\, ds\} \mathbb{1}_{T_2 \le T_0}}{\mathbb{E}_1 \exp\{\int_0^{T_0} \xi(Y_s)\, ds\} \mathbb{1}_{T_0 \le T_2}} \\
&= \frac{\frac{1+h}{2} \frac{\kappa}{\kappa - \xi(1) - \beta} + \frac{1-h}{2} \frac{\kappa}{\kappa - \xi(1) - \beta} \mathbb{E}_2 \exp\{\int_0^{T_0} (\xi(Y_s) + \beta)\, ds\}}{\frac{1+h}{2} \frac{\kappa}{\kappa - \xi(1)}} \\
&\le \frac{\kappa - \xi(1)}{\kappa - \xi(1) - \beta} \Big(1 + \mathbb{E}_2 \exp\Big\{\int_0^{T_0} (\xi(Y_s) + \beta)\, ds\Big\}\Big) \\
&\le \frac{\kappa}{\kappa - \beta} \Big(1 + \mathbb{E}_1 \exp\Big\{\int_0^{T_0} ((\theta \circ \xi)(Y_s) + \beta)\, ds\Big\} \\
&\quad \times \mathbb{E}_1 \exp\Big\{\int_0^{T_0} (\xi(Y_s) + \beta)\, ds\Big\}\Big) \quad \text{a.s.}
\end{aligned} \tag{3.24}$$

Taking logarithms on both sides and using the inequality $\log(1+xy) \le \log 2 + \log^+ x + \log^+ y$ for $x, y > 0$, we arrive at

$$\log \mathbb{E}_1^\xi \exp\{\beta T_0\} \le \log \frac{\kappa}{\kappa - \beta} + \log 2 + \Big(\log \mathbb{E}_1 \exp\Big\{ \int_0^{T_0} ((\theta \circ \xi)(Y_s) + \beta)\, ds \Big\}\Big)^+ \\ + \Big(\log \mathbb{E}_1 \exp\Big\{ \int_0^{T_0} (\xi(Y_s) + \beta)\, ds \Big\}\Big)^+. \tag{3.25}$$

We observe that for $\beta < \beta_{cr} (\le \kappa$ due to (1.8)) the right-hand side is integrable with respect to Prob and hence so is the left-hand side; thus, $\Lambda(\beta) < \infty$ for $\beta < \beta_{cr}$.

(b) It is sufficient to prove $\beta_{cr} > 0$ for the vanishing potential $\xi(x) \equiv 0$. But in this case we have $\beta_{cr} = -c^*$ due to part (a) of Proposition 3.1. Therefore, using the definition of c^*, the proof reduces to a standard large deviations bound and will be omitted.

□

We next prove the following properties of L defined by (1.7). Recall that L is well-defined for $\beta < \beta_{cr}$.

Lemma 3.5. *(a) If $L(\beta) > -\infty$ for some $\beta \in (-\infty, \beta_{cr})$, then the same is true for all $\beta \in (-\infty, \beta_{cr})$.*

(b) If the function L is finite on $(-\infty, \beta_{cr})$, then it is continuously differentiable on this interval. Its derivative is given by

$$L'(\beta) = \Big\langle \frac{\mathbb{E}_1 T_0 \exp\{\int_0^{T_0} (\xi(Y_s) + \beta)\, ds\}}{\mathbb{E}_1 \exp\{\int_0^{T_0} (\xi(Y_s) + \beta)\, ds\}} \Big\rangle. \tag{3.26}$$

(c) If the assumptions of (b) apply, then

$$\lim_{\beta \to -\infty} L'(\beta) = 0.$$

Remark 3.6. When L is finite, this lemma yields that $\Lambda'(\beta)$ is also given by the expression in (3.26) (cf. (3.22)).

Proof. (a) Assume $L(\beta) > -\infty$ for some $\beta \in (-\infty, \beta_{cr})$. Due to the monotonocity of L, it suffices to show $L(\beta - c) > -\infty$ for all $c > 0$. We apply a reverse Hölder inequality for $q < 0 < r < 1$ with $q^{-1} + r^{-1} = 1$ to obtain

$$\begin{aligned}
&\mathbb{E}_1 \exp\Big\{ \int_0^{T_0} (\xi(Y_s) + \beta - c)\, ds \Big\} \\
&= \mathbb{E}_1 \exp\Big\{ \int_0^{T_0} (\xi(Y_s) + \beta)/r\, ds \Big\} \exp\Big\{ \int_0^{T_0} ((\xi(Y_s) + \beta)/q - c)\, ds \Big\} \\
&\ge \Big(\mathbb{E}_1 \exp\Big\{ r \int_0^{T_0} (\xi(Y_s) + \beta)/r\, ds \Big\}\Big)^{\frac{1}{r}} \Big(\mathbb{E}_1 \exp\Big\{ q \int_0^{T_0} ((\xi(Y_s) + \beta)/q - c)\, ds \Big\}\Big)^{\frac{1}{q}} \\
&= \Big(\mathbb{E}_1 \exp\Big\{ \int_0^{T_0} (\xi(Y_s) + \beta)\, ds \Big\}\Big)^{\frac{1}{r}} \Big(\mathbb{E}_1 \exp\Big\{ \int_0^{T_0} (\xi(Y_s) + \beta - qc)\, ds \Big\}\Big)^{\frac{1}{q}}.
\end{aligned}$$

Using the definition of L, we obtain

$$L(\beta - c) \geq \frac{1}{r}L(\beta) + \frac{1}{q}L(\beta - qc)$$

and for $|q| > 0$ small enough such that $\beta - qc < \beta_{cr}$, the second summand is finite. The first summand is finite by assumption, whence the claim follows.

(b) The proof is standard and uses assertion (3.6) of Proposition 3.1 in the case $h \in (0,1)$. The details are left to the reader.

(c) Due to (b) it is sufficient to show that the integrand converges to 0 pointwise and then apply dominated convergence[3] to infer the desired result.

For this purpose fix a realisation of the medium and observe $\mathbb{P}_1^{\xi} \circ T_0^{-1} \ll \lambda$ with λ denoting the Lebesgue measure on $\mathbb{R}_+$. Then the random density

$$f := \frac{d\mathbb{P}_1^{\xi} \circ T_0^{-1}}{d\lambda}$$

is well-defined. It follows that

$$\mathbb{E}_1^{\xi} T_0 \exp\{\beta T_0\} = \int_{\mathbb{R}_+} x \exp\{\beta x\} f(x)\, dx$$

and splitting this integral we compute for $\varepsilon > 0$ and $\beta < 0$:

$$\begin{aligned} \int_0^{\varepsilon} x \exp\{\beta x\} f(x)\, dx \geq c \int_0^{\varepsilon} x \exp\{\beta x\}\, dx &= c\Big(\frac{1}{\beta} x \exp\{\beta x\}|_{x=0}^{\varepsilon} - \frac{1}{\beta^2}(\exp\{\beta\varepsilon\} - 1)\Big) \\ &= c\Big(\frac{\varepsilon}{\beta} \exp\{\beta\varepsilon\} + \frac{1}{\beta^2} - \frac{1}{\beta^2}\exp\{\beta\varepsilon\}\Big), \end{aligned} \tag{3.27}$$

where $c > 0$ is chosen such that $f \geq c$ holds $\lambda_{[0,\varepsilon]}$-a.s. Similarly, the remaining part is estimated by

$$\int_{\varepsilon}^{\infty} x \exp\{\beta x\} f(x)\, dx \leq \varepsilon \exp\{\beta\varepsilon\} \int_{\varepsilon}^{\infty} f(x)\, dx \leq \varepsilon \exp\{\beta\varepsilon\}$$

for $\beta < -\varepsilon^{-1}$. Thus, for each $\varepsilon > 0$ we can choose β small enough such that $\mathbb{E}_1^{\xi} T_0 \exp\{\beta T_0\} \mathbb{1}_{T_0 \geq \varepsilon} \leq \mathbb{E}_1^{\xi} T_0 \exp\{\beta T_0\} \mathbb{1}_{T_0 \leq \varepsilon}$ whence it follows for such β that

$$\mathbb{E}_1^{\xi} T_0 \exp\{\beta T_0\} \leq 2\varepsilon \mathbb{E}_1^{\xi} \exp\{\beta T_0\}.$$

This proves that the above integrand converges to 0 a.s. for $\beta \to -\infty$ and the result follows. □

In contrast to L, the function Λ may never take the value $-\infty$, as is seen in the following lemma.

Lemma 3.7. $\Lambda(\beta) > -\infty$ *for all* $\beta \in (-\infty, \beta_{cr})$.

[3] Indeed, dominated convergence is applicable since the integrand is increasing in β as one can check by considering its derivative, and it is integrable for $\beta = 0$, cf. e.g. (3.24), Lemma 3.3 (b) and Proposition 3.1 (b).

Proof. Due to monotonicity, it is sufficient to show $\Lambda(\beta) > -\infty$ for all $\beta \in (-\infty, 0)$. For this purpose we choose such β and estimate

$$\frac{\mathbb{E}_1 \exp\{\int_0^{T_0}(\xi(Y_s)+\beta)\,ds\}}{\mathbb{E}_1 \exp\{\int_0^{T_0}\xi(Y_s)\,ds\}} \geq \frac{\mathbb{E}_1 \exp\{\int_0^{T_0}(\xi(Y_s)+\beta)\,ds\}\mathbb{1}_{T_0\leq T_2}}{\mathbb{E}_1 \exp\{\int_0^{T_0}\xi(Y_s)\,ds\}\mathbb{1}_{T_0\leq T_2}(\mathbb{P}_1(T_0 \leq T_2))^{-1}}$$
$$= \frac{1+h}{2}\frac{\kappa-\xi(1)}{\kappa-\xi(1)-\beta}.$$

Taking logarithms and expectations, we see that $\Lambda(\beta) > -\infty$ for all $\beta \in (-\infty, 0)$. □

We now have the necessary tools available to tackle the desired large deviations principle. Let Λ^* denote the Fenchel-Legendre transform of Λ given by

$$\Lambda^*(\alpha) := \sup_{\beta\in\mathbb{R}}(\beta\alpha - \Lambda(\beta)) = \sup_{\beta<\beta_{cr}}(\beta\alpha - \Lambda(\beta)), \quad \alpha \in \mathbb{R},$$

where the second equality is due to Lemma 3.3 (a). Furthermore, for $M > 0$, $n \in \mathbb{N}$ and $\zeta \in \Sigma^+$ define

$$\mathbb{P}^{\zeta}_{M,n} := \mathbb{P}^{\zeta}_n(\cdot|\{\tau_k \leq M \ \forall k \in \{1,\dots,n\}\}) \tag{3.28}$$

where

$$\tau_k := T_{k-1} - T_k, \quad k \in \mathbb{N}. \tag{3.29}$$

The corresponding expectation is denoted by $\mathbb{E}^{\zeta}_{M,n}$.

Theorem 3.8. *For almost all realisations of ξ, the sequence of probability measures $(\mathbb{P}^{\xi}_n \circ (T_0/n)^{-1})_{n\in\mathbb{N}}$ on $\mathbb{R}_+$ satisfies a large deviations principle on scale n with deterministic, convex good rate function Λ^*.*

Proof. Being the supremum of affine functions, Λ^* is lower semi-continuous and convex. Furthermore, since $\Lambda(0) = 0$, it follows that $\Lambda^*(u) \geq 0$ for all $u \in \mathbb{R}$. Choosing $\beta \in (0, \beta_{cr})$, which is possible due to Lemma 3.3 (b), we find that for any $M \geq 0$ the set

$$\{\alpha \in \mathbb{R} : \beta\alpha - \Lambda(\beta) \leq M\} \cap \{\alpha \in \mathbb{R} : -\beta\alpha - \Lambda(-\beta) \leq M\}$$

is compact and, in particular, Λ^* has compact level sets; thus, Λ^* is a good convex rate function.

The upper large deviations bound for closed sets is a direct consequence of the Gärtner-Ellis theorem (cf. Theorem 2.3.6 in [DZ98]).

To prove the lower large deviations bound for open sets, we cannot directly apply the Gärtner-Ellis theorem since the steepness assumption (cf. Definition 2.3.5 (c) in [DZ98]) is possibly not fulfilled. Indeed, if $h = 1$ it may occur that $\lim_{\beta\uparrow\beta_{cr}} |\Lambda'(\beta)| < \infty$ since in this case $\beta_{cr} = \kappa$,

$$\Lambda'(\beta) = \Big\langle \frac{\mathbb{E}^{\xi}_1 T_0 \exp\{\beta T_0\}}{\mathbb{E}^{\xi}_1 \exp\{\beta T_0\}} \Big\rangle = \Big\langle \frac{1}{\kappa - \beta - \xi(0)} \Big\rangle \tag{3.30}$$

and Λ is steep if and only if $-1/\xi(0)$ is not integrable.

To circumvent this problem, we retreat to the measures $\mathbb{P}^{\xi}_{M,n}$ and for the corresponding logarithmic moment generating function we write

$$\Lambda_M(\beta) := \lim_{n\to\infty}\frac{1}{n}\log \mathbb{E}^{\xi}_{M,n}\exp\{\beta T_0\} = \langle \log \mathbb{E}^{\xi}_{M,1}\exp\{\beta T_0\}\rangle, \quad \beta \in \mathbb{R},$$

where the equality is due to Birkhoff's ergodic theorem. Using dominated convergence, one checks that Λ_M is essentially smooth (cf. Definition 2.3.5 in [DZ98]). We may therefore apply the Gärtner-Ellis theorem to the sequence $(\mathbb{P}^{\xi}_{M,n} \circ (T_0/n)^{-1})_{n\in\mathbb{N}}$ to obtain for any $G \subseteq \mathbb{R}_+$ open and $x \in G$ the estimate

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}^{\xi}_{M,n} \circ (T_0/n)^{-1}(G) \geq -\Lambda^*_M(x), \tag{3.31}$$

where $\Lambda^*_M(\alpha) := \sup_{\beta\in\mathbb{R}}(\beta\alpha - \Lambda_M(\beta))$ denotes the Fenchel-Legendre transform for $\alpha \in \mathbb{R}$.

In order to use this result for our original problem, we recall that a sequence of functions $(f_n)_{n\in\mathbb{N}}$ from $\mathbb{R}$ to $\overline{\mathbb{R}}$ *epi-converges* to a function $f : \mathbb{R} \to \overline{\mathbb{R}}$ at $x_0 \in \mathbb{R}$ if and only if

$$\liminf_{n\to\infty} f_n(x_n) \geq f(x_0)$$

for all sequences $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ converging to x_0 and

$$\limsup_{n\to\infty} f_n(x_n) \leq f(x_0)$$

for some sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ converging to x_0.

Using the facts that

(a) Λ is continuous on $(-\infty, \beta_{cr})$,

(b) $\Lambda_M \to \Lambda$ pointwise as $M \to \infty$,

(c) Λ_M is a monotone function and the sequence $(\Lambda_M)_{M\in\mathbb{N}}$ is monotone when restricted either to $(-\infty, 0]$ or $[0, \infty)$,

we deduce that Λ_M epi-converges towards Λ as $M \to \infty$. Therefore, since we note that Λ (cf. Lemma 3.7) and the $(\Lambda_M)_{M\in\mathbb{N}}$ are proper, lower semi-continuous and convex functions, we conclude using Theorem 11.34 in [RW98] that Λ^*_M epi-converges towards Λ^* as $M \to \infty$ along $\mathbb{N}$. Choosing G and x as above we therefore find a sequence $(x_M)_{M\in\mathbb{N}} \subset G$ with $\lim_{M\to\infty} \Lambda^*_M(x_M) = \Lambda^*(x)$. Employing (3.31) we thus obtain

$$\limsup_{M\to\infty} \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}^{\xi}_{M,n}(T_0/n)^{-1}(G) \geq -\Lambda^*(x),$$

which in combination with

$$\mathbb{P}^{\xi}_n \circ (T_0/n)^{-1}(G) \geq \mathbb{P}^{\xi}_{M,n}(T_0/n)^{-1}(G) \cdot \mathbb{P}^{\xi}_M(\tau_k \leq M \; \forall k \in \{1, \dots, n\})$$

yields

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}^{\xi}_n \circ (T_0/n)^{-1}(G) \geq -\Lambda^*(x).$$

This finishes the proof of the lower bound.

□

4 Proofs for the quenched regime

The outline of this section is as follows. We will use the large deviations principle of Theorem 3.8 to derive a variational formula for the lower logarithmic bound of u (cf. Lemma 4.1). In combination with further estimates, the large deviations principle will also prove valuable in establishing a similar estimate for the upper bound, see Lemma 4.2. Combining Lemmas 4.1 and 4.2 we obtain Corollary 4.5 and can then complete the proof of Theorem 2.1. Both the lower and upper bounds will essentially depend on the fact that (2.3) can be used to obtain (2.5). This result will then be shown explicitly to yield the proof of Corollary 2.3. Here, $\asymp$ means exponential equivalence.

We start with the proof of the lower bound.

Lemma 4.1. *Let $I \subset [0,\infty)$ be an interval. Then for almost all realisations of ξ,*

$$\liminf_{t\to\infty} \frac{1}{t} \log \sum_{n\in tI} \mathbb{E}_n \exp\left\{ \int_0^t \xi(Y_s)\, ds \right\} \mathbb{1}_{Y_t=0} \geq \sup_{\alpha \in \mathring{I} \cup (I\cap\{0\})} \inf_{\beta<\beta_{cr}} (-\beta + \alpha L(\beta)). \tag{4.1}$$

Proof. For $\delta > 0$ and $\alpha \in \mathring{I}$ we obtain using (2.4):

$$\begin{aligned}
&\mathbb{E}_{\lfloor \alpha t\rfloor} \exp\left\{ \int_0^t \xi(Y_s)\, ds \right\} \mathbb{1}_{Y_t=0} \\
&\geq \mathbb{E}_{\lfloor \alpha t\rfloor} \Bigg(\exp\left\{ \int_0^{T_0} \xi(Y_s)\, ds \right\} \mathbb{1}_{\frac{T_0}{\lfloor \alpha t\rfloor} \leq \frac{1}{\alpha}} \left(\mathbb{E}_0 \exp\left\{ \int_0^{t-r} \xi(Y_s)\, ds \right\} \mathbb{1}_{Y_{t-r}=0} \right)_{r=T_0} \Bigg) \\
&\geq \sup_{m\in(0,1/\alpha)} \mathbb{E}_{\lfloor \alpha t\rfloor} \Bigg(\exp\left\{ \int_0^{T_0} \xi(Y_s)\, ds \right\} \mathbb{1}_{\frac{T_0}{\lfloor \alpha t\rfloor} \in (0,1/\alpha)\cap(m-\delta,m+\delta)} \\
&\qquad \times \left(\mathbb{E}_0 \exp\left\{ \int_0^{t-r} \xi(Y_s)\, ds \right\} \mathbb{1}_{Y_{t-r}=0} \right)_{r=T_0} \Bigg).
\end{aligned}$$

Applying Proposition 3.1 (a) to the inner expectation and Theorem 3.8 to the remaining part of the right-hand side, we have

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}_{\lfloor \alpha t\rfloor} \exp\left\{ \int_0^t \xi(Y_s)\, ds \right\} \mathbb{1}_{Y_t=0} \\
&\quad \geq \sup_{m\in(0,1/\alpha)} \Bigg(\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}_{\lfloor \alpha t\rfloor} \exp\left\{ \int_0^{T_0} \xi(Y_s)\, ds \right\} \mathbb{1}_{\frac{T_0}{\lfloor \alpha t\rfloor} \in (0,1/\alpha)\cap(m-\delta,m+\delta)} \\
&\qquad + \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}_0 \exp\left\{ \int_0^{t(1-\alpha(m-\delta))} \xi(Y_s)\, ds \right\} \mathbb{1}_{Y_{t(1-\alpha(m-\delta))}=0} \Bigg) \\
&\quad \geq \sup_{m\in(0,1/\alpha)} \Big(\alpha \big(- \inf_{x\in(0,1/\alpha)\cap(m-\delta,m+\delta)} \Lambda^*(x) + L(0) \big) + (1-\alpha(m-\delta))c^* \Big).
\end{aligned} \tag{4.2}$$

Note here that for a potential unbounded from below, $L(0) = -\infty$ is possible; nevertheless, observe that in this case also $L(\beta) = -\infty$ for all $\beta < \beta_{cr}$, see Lemma 3.5 (a). Therefore, the following computations hold true even if $L(0) = -\infty$. The lower semi-continuity of Λ^* supplies us with

$\lim_{\delta\downarrow 0}\inf_{x\in(0,1/\alpha)\cap(m-\delta,m+\delta)}\Lambda^*(x)=\Lambda^*(m)$. Hence, taking $\delta\downarrow 0$ in (4.2) yields

$$\begin{aligned}\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}_{\lfloor\alpha t\rfloor}\exp\Big\{\int_0^t\xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=0}\\ &\geq\sup_{m\in(0,1/\alpha)}\big(\alpha(-\Lambda^*(m)+L(0))+(1-\alpha m)c^*\big)\\ &=c^*+\alpha\sup_{m\in(0,1/\alpha)}\inf_{\beta<\beta_{cr}}(m(\beta_{cr}-\beta)+L(\beta)),\end{aligned}\tag{4.3}$$

where we used (3.22) and (3.4) to obtain the equality. Thus, the supremum in m is taken for $m=1/\alpha$. Hence,

$$\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}_{\lfloor\alpha t\rfloor}\exp\Big\{\int_0^t\xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=0}\geq\inf_{\beta<\beta_{cr}}(-\beta+\alpha L(\beta)).\tag{4.4}$$

Now for the case $\alpha=0$ we observe

$$\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}_0\exp\Big\{\int_0^t\xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=0}=-\beta_{cr}$$

due to Proposition 3.1 (a). Since $\inf_{\beta<\beta_{cr}}(-\beta+\alpha L(\beta))$ evaluates to $-\beta_{cr}$ for $\alpha=0$, in combination with (4.4) this finishes the proof of (4.1). $\square$

Next, we turn to the upper bound which is slightly more involved.

Lemma 4.2. *Let $I\subset[0,\infty)$ be a compact interval. Then for almost all realisations of ξ,*

$$\limsup_{t\to\infty}\frac{1}{t}\log\sum_{n\in tI}\mathbb{E}_n\exp\Big\{\int_0^t\xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=0}\leq\sup_{\alpha\in I}\inf_{\beta<\beta_{cr}}(-\beta+\alpha L(\beta)).\tag{4.5}$$

Proof. (i) First, assume that $\inf I>0$ and write $I=[\varepsilon,\gamma]$. Then for $\delta>0$ we choose numbers $(\alpha_k^\delta)_{k=1}^n$ such that $\varepsilon=\alpha_1^\delta<\alpha_2^\delta<\cdots<\alpha_n^\delta=\gamma$ and $\max_{k=1,\dots,n-1}(\alpha_{k+1}^\delta-\alpha_k^\delta)<\delta$. Then

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t}\log\sum_{n\in tI}\mathbb{E}_n\exp\Big\{\int_0^t\xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=0}\\ &\quad=\max_{k=1,\dots,n-1}\limsup_{t\to\infty}\frac{1}{t}\log\sum_{n\in t[\alpha_k^\delta,\alpha_{k+1}^\delta]}\mathbb{E}_n\exp\Big\{\int_0^t\xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=0}.\end{aligned}$$

Using (2.4) and Proposition 3.1 (a) we get

$$\begin{aligned}&\sum_{n\in t[\alpha_k^\delta,\alpha_{k+1}^\delta]}\mathbb{E}_n\exp\Big\{\int_0^t\xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=0}\\ &=\sum_{n\in t[\alpha_k^\delta,\alpha_{k+1}^\delta]}\mathbb{E}_n\Big(\exp\Big\{\int_0^{T_0}\xi(Y_s)\,ds\Big\}\mathbb{1}_{T_0\leq t}\Big(\mathbb{E}_0\exp\Big\{\int_0^{t-r}\xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_{t-r}=0}\Big)_{r=T_0}\Big)\qquad(4.6)\\ &\leq\sum_{n\in t[\alpha_k^\delta,\alpha_{k+1}^\delta]}\mathbb{E}_n\exp\Big\{\int_0^{T_0}\xi(Y_s)\,ds\Big\}\mathbb{1}_{\frac{T_0}{n}\leq\frac{1}{\alpha_k^\delta}}\exp\{c^*(t-T_0)\},\end{aligned}$$

which by (3.4) and the exponential Chebyshev inequality can be bounded from above by

$$\begin{aligned}&\inf_{\beta>0}\sum_{n\in t[\alpha_k^\delta,\alpha_{k+1}^\delta]}\mathbb{E}_n\exp\Big\{\int_0^{T_0}(\xi(Y_s)-c^*-\beta)\,ds\Big\}\exp\{c^*t\}\exp\{\beta n/\alpha_k^\delta\}\\&\le\exp\{c^*t\}\inf_{\beta<\beta_{cr}}\sum_{n\in t[\alpha_k^\delta,\alpha_{k+1}^\delta]}\mathbb{E}_n\exp\Big\{\int_0^{T_0}(\xi(Y_s)+\beta)\,ds\Big\}\exp\{(-\beta-c^*)n/\alpha_k^\delta\}.\end{aligned}\tag{4.7}$$

Therefore, combining (4.6) and (4.7) we arrive at

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t}\log\sum_{n\in t[\alpha_k^\delta,\alpha_{k+1}^\delta]}\mathbb{E}_n\exp\Big\{\int_0^t\xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=0}\\&\le c^*\Big(1-\frac{\alpha_{j_k}^\delta}{\alpha_k^\delta}\Big)+\alpha_{j_k}^\delta\inf_{\beta<\beta_{cr}}(-\beta/\alpha_k^\delta+(\Lambda(\beta)+L(0))),\end{aligned}\tag{4.8}$$

where $j_k=k$ if the summands on the right-hand side of (4.7) have nonpositive exponential rates in n for some $\beta>0$ and $j_k=k+1$ otherwise. Now if $L(0)=-\infty$, then obviously the right-hand side of (4.8) equals $-\infty$ and (4.5) holds true. Therefore, we assume $L(0)>-\infty$ from now on, which due to Lemma 3.5 (a) implies $L(\beta)>-\infty$ for all $\beta\in(-\infty,\beta_{cr})$. By (3.22), the right-hand side of (4.8) evaluates to

$$c^*\Big(1-\frac{\alpha_{j_k}^\delta}{\alpha_k^\delta}\Big)+\frac{\alpha_{j_k}^\delta}{\alpha_k^\delta}\inf_{\beta<\beta_{cr}}(-\beta+\alpha_k^\delta L(\beta)).\tag{4.9}$$

Using Lemma 3.5 (c) one can show that the family of functions indexed by δ, which are defined piecewise constant by the right-hand side of (4.9) for $\alpha\in[\alpha_k^\delta,\alpha_{k+1}^\delta)$, $k\in\{1,\dots,n-2\}$, and $\alpha\in[\alpha_{n-1}^\delta,\alpha_n^\delta]$, converges uniformly in $\alpha\in[\varepsilon,\gamma]$ to

$$\inf_{\beta<\beta_{cr}}(-\beta+\alpha L(\beta))\tag{4.10}$$

as $\delta\downarrow 0$. Taking $\delta\downarrow 0$ and the supremum over $\alpha\in[\varepsilon,\gamma]$, we therefore obtain from the previous relations:

$$\limsup_{t\to\infty}\frac{1}{t}\log\sum_{n\in t[\varepsilon,\gamma]}\mathbb{E}_n\exp\Big\{\int_0^t\xi(Y_s)ds\Big\}\mathbb{1}_{Y_t=0}\le\sup_{\alpha\in[\varepsilon,\gamma]}\inf_{\beta<\beta_{cr}}(-\beta+\alpha L(\beta)).\tag{4.11}$$

(ii) It remains to consider the case that $\inf I=0$. Then we either find $\varepsilon>0$ such that

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t}\log\sum_{n\in tI}\mathbb{E}_n\exp\Big\{\int_0^t\xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=0}\\&=\limsup_{t\to\infty}\frac{1}{t}\log\sum_{n\in t(I\cap[\varepsilon,\infty))}\mathbb{E}_n\exp\Big\{\int_0^t\xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=0},\end{aligned}\tag{4.12}$$

in which case the problem reduces to the previous case and (4.5) holds true in particular. Otherwise, for each $\varepsilon > 0$ we have "$>$" in (4.12) instead of "$=$". We would then find a function $\varphi : [0,\infty) \to \mathbb{N}_0$ such that $\varphi(t)/t \to 0$ as $t \to \infty$ and which satisfies the first equality in

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t}\log\sum_{n\in tI}\mathbb{E}_n\exp\Big\{\int_0^t \xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=0}\\
&\quad=\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}_{\varphi(t)}\exp\Big\{\int_0^t \xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=0}\\
&\quad=\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}_{\varphi(t)}\Big(\exp\Big\{\int_0^{T_0} \xi(Y_s)\,ds\Big\}\mathbb{1}_{T_0\le t}\\
&\qquad\times\Big(\mathbb{E}_0\exp\Big\{\int_0^{t-r} \xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_{t-r}=0}\Big)_{r=T_0}\Big)\\
&\quad\le\sup_{\alpha\in[\delta,1-\delta]}\Big(\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}_{\varphi(t)}\Big(\exp\Big\{\int_0^{T_0} \xi(Y_s)\,ds\Big\}\mathbb{1}_{T_0\in t[\alpha-\delta,\alpha+\delta]}\Big)\\
&\qquad\times\sup_{r\in t[\alpha-\delta,\alpha+\delta]}\mathbb{E}_0\exp\Big\{\int_0^{t-r} \xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_{t-r}=0}\Big)
\end{aligned}\tag{4.13}$$

with $\delta > 0$ small. The exponential Chebyshev inequality for $\beta \in (0,\beta_{cr})$ supplies us with

$$\begin{aligned}
&\mathbb{E}_{\varphi(t)}\exp\Big\{\int_0^{T_0} \xi(Y_s)\,ds\Big\}\mathbb{1}_{T_0\ge(\alpha-\delta)t}\\
&\qquad\le\mathbb{E}_{\varphi(t)}\exp\Big\{\int_0^{T_0} (\xi(Y_s)+\beta)\,ds\Big\}\exp\{-\beta(\alpha-\delta)t\}.
\end{aligned}\tag{4.14}$$

Taking $\delta\downarrow 0$ and $\beta\uparrow\beta_{cr}$ in this inequality, we deduce

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}_{\varphi(t)}\exp\Big\{\int_0^{T_0} \xi(Y_s)\,ds\Big\}\mathbb{1}_{T_0\ge(\alpha-\delta)t}\le-\alpha\beta_{cr}.$$

Now taking $\delta\downarrow 0$ in (4.13) we obtain in combination with Proposition 3.1 (a):

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t}\log\sum_{n\in tI}\mathbb{E}_n\exp\Big\{\int_0^t \xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=0}\\
&\qquad\le\sup_{\alpha\in[0,1]}(-\alpha\beta_{cr}+(1-\alpha)c^*)=-\beta_{cr}.
\end{aligned}\tag{4.15}$$

But $-\beta_{cr}$ is just the result when replacing $\sup_{\alpha\in I}$ by $\alpha = 0$ in (4.5). This finishes the proof of the lemma. ☐

The next result establishes the intuitively plausible fact that only summands in the direction of the drift are relevant on an exponential scale.

Lemma 4.3. *For all $\delta \geq 0$ and $\gamma \geq \delta$ we have*

$$\limsup_{t\to\infty}\frac{1}{t}\log\sum_{n\in t[-\gamma,-\delta]}\mathbb{E}_n\exp\Big\{\int_0^t\xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=0}$$
$$\leq \delta\log\frac{1-h}{1+h}+\liminf_{t\to\infty}\frac{1}{t}\log\sum_{n\in t[\delta,\gamma]}\mathbb{E}_n\exp\Big\{\int_0^t\xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=0}\quad a.s. \tag{4.16}$$

Proof. First we observe that the function

$$(0,\infty)\ni\alpha\mapsto\inf_{\beta<\beta_{cr}}(-\beta+\alpha L(\beta))$$

is either constant $-\infty$ (if and only if $L\equiv-\infty$ on $(-\infty,\beta_{cr})$, cf. Lemma 3.5 (a)) or continuous, since it is concave. In combination with the proofs of Lemmas 4.1 and 4.2 we therefore infer the existence of $\alpha_+\in[\delta,\gamma]$ such that

$$\lim_{t\to\infty}\frac{1}{t}\log\sum_{n\in t[\delta,\gamma]}\mathbb{E}_n\exp\Big\{\int_0^t\xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=0}=\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}_{\lfloor\alpha_+t\rfloor}\exp\Big\{\int_0^t\xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=0}$$
$$=\sup_{\alpha\in[\delta,\gamma]}\inf_{\beta<\beta_{cr}}(-\beta+\alpha L(\beta))\quad\text{a.s.}$$

Employing similar arguments, one may show that the analogues of Lemmas 4.1 and 4.2 for $I\subset(-\infty,0]$ also hold and we infer the existence of $\alpha_-\in[-\gamma,-\delta]$ such that

$$\lim_{t\to\infty}\frac{1}{t}\log\sum_{n\in t[-\gamma,-\delta]}\mathbb{E}_n\exp\Big\{\int_0^t\xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=0}=\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}_{\lfloor\alpha_-t\rfloor}\exp\Big\{\int_0^t\xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=0} \tag{4.17}$$

exists and is deterministic also. Next we observe that for $n\in\mathbb{N}$

$$\begin{aligned}\mathbb{E}_{-n}\exp\Big\{\int_0^t\xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=0}&=\mathbb{E}_0\exp\Big\{\int_0^t\xi(X_s)\,ds\Big\}\mathbb{1}_{X_t=-n}\\&=\Big(\frac{1-h}{1+h}\Big)^n\mathbb{E}_0\exp\Big\{\int_0^t\xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=-n}\\&\overset{d}{=}\Big(\frac{1-h}{1+h}\Big)^n\mathbb{E}_n\exp\Big\{\int_0^t\xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=0},\end{aligned} \tag{4.18}$$

where the first equality follows from time reversal, the second by comparing the transition probabilities of X and Y, and the last equality follows from the shift invariance of ξ.

Employing (4.18) in combination with (4.17) we conclude

$$\begin{aligned}
\lim_{t\to\infty}\frac{1}{t}\log & \sum_{n\in t[-\gamma,-\delta]} \mathbb{E}_n \exp\Big\{\int_0^t \xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=0} \\
&= \lim_{t\to\infty}\frac{1}{t}\log \mathbb{E}_{\lfloor\alpha_- t\rfloor} \exp\Big\{\int_0^t \xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=0} \\
&= |\alpha_-|\frac{1-h}{1+h} + \lim_{t\to\infty}\frac{1}{t}\log \mathbb{E}_{-\lfloor\alpha_- t\rfloor} \exp\Big\{\int_0^t \xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=0} \\
&\le |\alpha_-|\frac{1-h}{1+h} + \lim_{t\to\infty}\frac{1}{t}\log \sum_{n\in t[\delta,\gamma]} \mathbb{E}_n \exp\Big\{\int_0^t \xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=0}.
\end{aligned}$$

Indeed, to justify the last equality note that the limits on both sides exist and are constant a.s.; (4.18) then yields the equality in question. This finishes the proof. □

Lemma 4.4. *We have*

$$\limsup_{t\to\infty}\frac{1}{t}\log \sum_{n\notin t[-\gamma,\gamma]} \mathbb{E}_n \exp\Big\{\int_0^t \xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=0} \to -\infty \tag{4.19}$$

as $\gamma\to\infty$.

Proof. Note that by the use of Stirling's formula we obtain for $\gamma > \kappa e$:

$$\begin{aligned}
\sum_{n\ge\gamma t} \mathbb{E}_n \exp\Big\{\int_0^t \xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=0} &\le \sum_{n\ge\gamma t}\mathbb{P}_n(T_0\le t) \le \sum_{n\ge\gamma t}\sum_{k\ge n} e^{-\kappa t}\frac{(\kappa t)^k}{k!} \\
&\le e^{-\kappa t}\sum_{n\ge\gamma t}\sum_{k\ge n}\Big(\frac{\kappa t e}{\gamma t}\Big)^k \\
&= Ce^{-\kappa t}\sum_{n\ge\gamma t}\Big(\frac{\kappa e}{\gamma}\Big)^n = Ce^{-\kappa t}\Big(\frac{\kappa e}{\gamma}\Big)^{\lfloor\gamma t\rfloor}
\end{aligned}$$

where C is a generic constant depending on κ and γ, swallowing all sums appearing in the geometric series. Since an analogous result is valid for

$$\sum_{n\le-\gamma t} \mathbb{E}_n \exp\Big\{\int_0^t \xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=0},$$

we infer that (4.19) holds as $\gamma\to\infty$. □

Corollary 4.5. *The quenched Lyapunov exponent* λ_0 *exists and is given by*

$$\lambda_0 = \sup_{\alpha\in[0,\gamma]}\inf_{\beta<\beta_{cr}}(-\beta+\alpha L(\beta)) \tag{4.20}$$

for all $\gamma>0$ *large enough.*

Proof. We take advantage of (2.3) to split for $\gamma > 0$:

$$
\begin{aligned}
u(t,0) \leq & \sum_{n \notin t[-\gamma,\gamma]} \mathbb{E}_n \exp\left\{\int_0^t \xi(Y_s)\,ds\right\} \mathbb{1}_{Y_t=n} \\
& + \sum_{n \in [0,\gamma t]} \mathbb{E}_n \exp\left\{\int_0^t \xi(Y_s)\,ds\right\} \mathbb{1}_{Y_t=n} \\
& + \sum_{n \in [-\gamma t,0]} \mathbb{E}_n \exp\left\{\int_0^t \xi(Y_s)\,ds\right\} \mathbb{1}_{Y_t=n}.
\end{aligned} \tag{4.21}
$$

Lemma 4.3 yields that the third summand is logarithmically negligible when compared to the second. Since the first summand can be made arbitrarily small for γ large, according to Lemma 4.4, we obtain in combination with Lemma 4.2:

$$
\limsup_{t\to\infty} \frac{1}{t} \log u(t,0) \leq \sup_{\alpha \in [0,\gamma]} \inf_{\beta < \beta_{cr}} (-\beta + \alpha L(\beta))
$$

for γ large enough.

With respect to the lower bound, Lemma 4.1 in combination with (2.3) supplies us with

$$
\liminf_{t\to\infty} \frac{1}{t} \log u(t,0) \geq \sup_{\alpha \geq 0} \inf_{\beta < \beta_{cr}} (-\beta + \alpha L(\beta)).
$$

Combining these two estimates we infer the existence of λ_0 and the variational formula (4.20). $\square$

We are now ready to prove the results of subsection 2.1.

Proof of Theorem 2.1. Corollary 4.5 supplies us with the existence of λ_0 and the variational formula (4.20). If L does not have a zero in $(0,\beta_{cr})$, then we have $L(\beta) < 0$ for all $\beta < \beta_{cr}$. Thus, the supremum in α is taken in $\alpha = 0$ and the right-hand side of (4.20) evaluates to $-\beta_{cr}$. If L does have a zero in $(0,\beta_{cr})$, then inspecting (4.20) and differentiating with respect to β, we observe that the supremum over α is a maximum taken in $\alpha = (L'(\beta_z))^{-1}$, with β_z denoting the zero of L in $(0,\beta_{cr})$. Consequently, we deduce that λ_0 equals $-\beta_z$, which finishes the proof. $\square$

Proof of Corollary 2.3. (a) Note that L has a zero in $(0,\beta_{cr})$ by assumption and thus Theorem 2.1 implies $-\lambda_0 < \beta_{cr}$. Therefore, by Lemma 3.5 (b) we may deduce $(L'(-\lambda_0))^{-1} \in (0,\infty)$.

Using the time reversal of (2.3) and Lemma 4.4, it suffices to show

$$
\lambda_0 > \limsup_{t\to\infty} \frac{1}{t} \log \sum_{n \in t([-\gamma,\gamma] \setminus (\alpha^*-\varepsilon,\alpha^*+\varepsilon))} \mathbb{E}_n \exp\left\{\int_0^t \xi(Y_s)\,ds\right\} \mathbb{1}_{Y_t=0} \tag{4.22}
$$

for γ large enough. First, observe that due to Lemma 4.2 we have

$$
\begin{aligned}
\limsup_{t\to\infty} \frac{1}{t} \log & \sum_{n \in t([0,\gamma] \setminus (\alpha^*-\varepsilon,\alpha^*+\varepsilon))} \mathbb{E}_n \exp\left\{\int_0^t \xi(Y_s)\,ds\right\} \mathbb{1}_{Y_t=0} \\
& \leq \sup_{\alpha \in [0,\gamma] \setminus (\alpha^*-\varepsilon,\alpha^*+\varepsilon)} \inf_{\beta < \beta_{cr}} (-\beta + \alpha L(\beta)).
\end{aligned} \tag{4.23}
$$

Differentiating the expression

$$-\beta + \alpha L(\beta) \tag{4.24}$$

with respect to β we obtain

$$-1 + \alpha \Big\langle \frac{\mathbb{E}_1^{\xi} T_0 \exp\{\beta T_0\}}{\mathbb{E}_1^{\xi} \exp\{\beta T_0\}} \Big\rangle, \tag{4.25}$$

cf. Lemma 3.5 (b). Now (4.25) as a function of β is continuous at $-\lambda_0$ and inserting $\beta = -\lambda_0$ as well as $\alpha = \alpha^*$, the term in (4.25) evaluates to 0. Therefore, for $\varepsilon \in (0, \alpha^*)$ there exists $\delta > 0$ such that for all α with $|\alpha - \alpha^*| \geq \varepsilon$ and β with $|\beta - (-\lambda_0)| < \delta$, the derivative (4.25) is bounded away from 0. Since, according to Theorem 2.1, setting $\beta = -\lambda_0$ in (4.24) evaluates to λ_0 independently of the value of α, this boundedness yields

$$\inf_{\beta < \beta_{cr}} (-\beta + \alpha L(\beta)) \leq \lambda_0 - \delta^*$$

for some $\delta^* > 0$ and all $\alpha \notin (\alpha^* - \varepsilon, \alpha^* + \varepsilon)$. Consequently, we get

$$\sup_{\alpha \in [0,\gamma] \setminus (\alpha^* - \varepsilon, \alpha^* + \varepsilon)} \inf_{\beta < \beta_{cr}} (-\beta + \alpha L(\beta)) \leq \lambda_0 - \delta^* < \lambda_0.$$

Therefore, using (4.23) we have

$$\limsup_{t \to \infty} \frac{1}{t} \log \sum_{n \in t([0,\gamma] \setminus (\alpha^* - \varepsilon, \alpha^* + \varepsilon))} \mathbb{E}_n \exp\Big\{ \int_0^t \xi(Y_s)\, ds \Big\} \mathbb{1}_{Y_t = 0} < \lambda_0.$$

Combining this estimate with Lemma 4.3 thus yields

$$\limsup_{t \to \infty} \frac{1}{t} \log \sum_{n \in t([-\gamma,0] \setminus (-\alpha^* - \varepsilon, -\alpha^* + \varepsilon))} \mathbb{E}_n \exp\Big\{ \int_0^{T_0} \xi(Y_s)\, ds \Big\} \mathbb{1}_{Y_t = 0} < \lambda_0. \tag{4.26}$$

Furthermore, the same lemma supplies us with the first inequality in

$$\begin{aligned} &\limsup_{t \to \infty} \frac{1}{t} \log \sum_{n \in t(-\alpha^* - \varepsilon, -\alpha^* + \varepsilon)} \mathbb{E}_n \exp\Big\{ \int_0^{T_0} \xi(Y_s)\, ds \Big\} \mathbb{1}_{Y_t = 0} \\ &\qquad \leq (\alpha^* - \varepsilon) \log \frac{1-h}{1+h} + \sup_{\alpha \in [0,\gamma] \setminus (\alpha^* - \varepsilon, \alpha^* + \varepsilon)} \inf_{\beta < \beta_{cr}} (-\beta + \alpha L(\beta)) < \lambda_0. \end{aligned} \tag{4.27}$$

Combining (4.23), (4.26) and (4.27) gives (4.22) and hence finishes the proof of part (a).

(b) To show the equality, observe that the assumption $\lim_{\beta \uparrow \beta_{cr}} L(\beta) = 0$ implies that L has no zero in $(-\beta_{cr}, 0)$ and hence $\lambda_0 = -\beta_{cr}$ due to Theorem 2.1. Now choose $m \in [0, (\lim_{\beta \uparrow \beta_{cr}} L'(\beta))^{-1}]$ and $\varepsilon \in (0, m)$. Employing time reversal and Lemma 4.1 we arrive at

$$\begin{aligned} &\liminf_{t \to \infty} \frac{1}{t} \log \mathbb{E}_0 \exp\Big\{ \int_0^t \xi(X_s)\, ds \Big\} \mathbb{1}_{X_t \in t(m - \varepsilon, m + \varepsilon)} \\ &\qquad = \liminf_{t \to \infty} \frac{1}{t} \log \sum_{n \in t(m - \varepsilon, m + \varepsilon)} \mathbb{E}_n \exp\Big\{ \int_0^t \xi(Y_s)\, ds \Big\} \mathbb{1}_{Y_t = 0} \\ &\qquad \geq \sup_{\alpha \in (m - \varepsilon, m + \varepsilon)} \inf_{\beta < \beta_{cr}} (-\beta + \alpha L(\beta)) = \inf_{\beta < \beta_{cr}} (-\beta + (m - \varepsilon) L(\beta)), \end{aligned} \tag{4.28}$$

where the last equality follows since $L(\beta) < 0$ for $\beta < \beta_{cr}$. Differentiating the inner term of the right-hand side with respect to β yields $-1 + (m - \varepsilon)L'(\beta)$ which due to our choice of m is smaller than 0 for all $\beta < \beta_{cr}$. Thus, (4.28) evaluates to $-\beta_{cr}$, and this proves the desired equality.

To prove the inequality, observe that Lemmas 4.2, 4.3 and 4.4 yield

$$\limsup_{t\to\infty} \frac{1}{t} \log \sum_{n \le -t\varepsilon} \mathbb{E}_n \exp\Big\{ \int_0^t \xi(Y_s)\, ds \Big\} \mathbb{1}_{Y_t=0} \le \varepsilon \log \frac{1-h}{1+h} + \sup_{\alpha \ge \varepsilon} \inf_{\beta < \beta_{cr}} (-\beta + \alpha L(\beta)).$$

Using Corollary 4.5 we have that the right-hand side is strictly smaller than λ_0.

For the remaining summands, Lemma 4.2 in combination with Lemma 4.4 yields

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{t} \log \sum_{n \ge t((\lim_{\beta\uparrow\beta_{cr}} L'(\beta))^{-1}+\varepsilon)} \mathbb{E}_n \exp\Big\{ \int_0^t \xi(Y_s)\, ds \Big\} \mathbb{1}_{Y_t=0} \\ &\qquad \le \sup_{\alpha \ge (\lim_{\beta\uparrow\beta_{cr}} L'(\beta))^{-1}+\varepsilon} \inf_{\beta < \beta_{cr}} (-\beta + \alpha L(\beta)), \end{aligned} \tag{4.29}$$

and the derivative $-1 + \alpha L'(\beta)$ of the inner term on the right-hand side with respect to β is positive and bounded away from 0 for all $\beta < \beta_{cr}$ large enough and all $\alpha \ge (\lim_{\beta\uparrow\beta_{cr}} L'(\beta))^{-1} + \varepsilon$. Thus, we conclude that the right-hand side of (4.29) is strictly smaller than $\lambda_0 = -\beta_{cr}$, which finishes the proof.

(c) Using time reversal we get for $\gamma > \varepsilon$:

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}_0 \exp\Big\{ \int_0^t \xi(X_s)\, ds \Big\} \mathbb{1}_{X_t \notin t(-\varepsilon,\varepsilon)} \\ &\quad = \limsup_{t\to\infty} \frac{1}{t} \log \sum_{n \notin t(-\varepsilon,\varepsilon)} \mathbb{E}_n \exp\Big\{ \int_0^t \xi(Y_s)\, ds \Big\} \mathbb{1}_{Y_t=0} \\ &\quad = \limsup_{t\to\infty} \frac{1}{t} \log \sum_{n \notin t(-\gamma,\gamma)} \mathbb{E}_n \exp\Big\{ \int_0^t \xi(Y_s)\, ds \Big\} \mathbb{1}_{Y_t=0} \\ &\qquad \vee \limsup_{t\to\infty} \frac{1}{t} \log \sum_{n \in t((-\gamma,\gamma)\setminus(-\varepsilon,\varepsilon))} \mathbb{E}_n \exp\Big\{ \int_0^t \xi(Y_s)\, ds \Big\} \mathbb{1}_{Y_t=0} \end{aligned}$$

According to Lemma 4.4, the first term on the right-hand side tends to $-\infty$ as $\gamma \to \infty$, while Lemma 4.3 combined with Lemma 4.2 implies that the second can be estimated from above by

$$\sup_{\alpha \in [\varepsilon,\gamma]} \inf_{\beta < \beta_{cr}} (-\beta + \alpha L(\beta)) = \inf_{\beta < \beta_{cr}} (-\beta + \varepsilon L(\beta)),$$

where the equality follows since $L(\beta) < 0$ for all $\beta < \beta_{cr}$ by assumption. Thus, this expression evaluates to $-\beta_{cr} + \varepsilon \lim_{\beta\uparrow\beta_{cr}} L(\beta) < -\beta_{cr} = \lambda_0$, and the statement follows. $\square$

5 Auxiliary results particular to the annealed regime

This section contains mainly technical results, which will be employed in the proof of Theorem 2.7. The results given here are in parts generalisations of corresponding results for a finite state space given in section $IX.2$ and $A.9$ of [Ell85].

Lemma 5.1. *Let $n \in \mathbb{N}$, $\rho \in \mathscr{M}_1(\mathbb{R})$ and $\nu \in \mathscr{M}_1^s(\mathbb{R}^{\mathbb{N}_0})$. Then*

$$\sum_{i=1}^{n} H(\pi_i \nu | \pi_{i-1} \nu \otimes \rho) = H(\pi_n \nu | \rho^n). \tag{5.1}$$

Proof. The result is a consequence of the decomposition of relative entropy given for example in Theorem D.13, [DZ98]. Indeed, from this theorem it follows that

$$H(\pi_n \nu | \rho^n) = H(\pi_{n-1} \nu | \rho^{n-1}) + \int_{\mathbb{R}^{n-1}} H\big(\pi_n \nu^{(x_1,\dots,x_{n-1})} | (\rho^n)^{(x_1,\dots,x_{n-1})}\big) \pi_{n-1} \nu(d(x_1,\dots,x_{n-1})) \tag{5.2}$$

where for a measure μ on $\mathscr{B}(\mathbb{R}^n)$ the regular conditional probability distribution of μ given π_{n-1} is denoted by $\mathbb{R}^{n-1} \ni (x_1,\dots,x_{n-1}) \mapsto \mu^{(x_1,\dots,x_{n-1})} \in \mathscr{M}_1(\mathbb{R}^n)$. Thus, to establish (5.1) it suffices to show

$$H(\pi_n \nu | \pi_{n-1} \nu \otimes \rho) = \int_{\mathbb{R}^{n-1}} H\big(\pi_n \nu^{(x_1,\dots,x_{n-1})} | (\rho^n)^{(x_1,\dots,x_{n-1})}\big) \pi_{n-1} \nu(d(x_1,\dots,x_{n-1})) \tag{5.3}$$

for all $n \in \mathbb{N} \backslash \{1\}$. But applying the quoted theorem to the left-hand side of the previous equation we obtain

$$H(\pi_n \nu | \pi_{n-1} \nu \otimes \rho) = \int_{\mathbb{R}^{n-1}} H\big(\pi_n \nu^{(x_1,\dots,x_{n-1})} | (\pi_{n-1} \nu \otimes \rho)^{(x_1,\dots,x_{n-1})}\big) \pi_{n-1} \nu(d(x_1,\dots,x_{n-1})),$$

and since $(\pi_{n-1} \nu \otimes \rho)^{(x_1,\dots,x_{n-1})} = \delta_{(x_1,\dots,x_{n-1})} \otimes \rho = (\rho^n)^{(x_1,\dots,x_{n-1})}$, (5.3) follows. □

Proposition 5.2. *The function $\mathscr{I}_{|\mathscr{M}_1^s(\Sigma_b^+)}$ is affine and for $\nu \in \mathscr{M}_1^s(\Sigma_b^+)$ we have*

$$H(\pi_n \nu | \pi_{n-1} \nu \otimes \eta) \uparrow \mathscr{I}(\nu) \tag{5.4}$$

as $n \to \infty$.

Proof. We know (cf. Lemma 6.5.16, Corollary 6.5.17 and the preceding discussion in [DZ98]) that for $\nu \in \mathscr{M}_1^s(\Sigma_b^+)$ the value of $\mathscr{I}(\nu)$ is given as the limit of the nondecreasing sequence $H(\pi_n \nu | \pi_{n-1} \nu \otimes \eta)$ of relative entropies.

To show that $\mathscr{I}$ restricted to $\mathscr{M}_1^s(\Sigma_b^+)$ is affine, let $\beta \in (0,1)$ and $\mu, \nu \in \mathscr{M}_1^s(\Sigma_b^+)$. We distinguish cases:

(i) Assume $\mathscr{I}(\nu), \mathscr{I}(\mu) < \infty$. Then (5.4) applies and using Lemma 5.1 we deduce $\pi_n \nu \ll \eta^n$ and $\pi_n \mu \ll \eta^n$ for all $n \in \mathbb{N}$. The convexity of relative entropy yields

$$\begin{aligned}
&\beta H(\pi_n \nu | \eta^n) + (1-\beta) H(\pi_n \mu | \eta^n) \\
&\qquad \geq H(\beta \pi_n \nu + (1-\beta) \pi_n \mu | \eta^n) \\
&\qquad \geq \int_{[b,0]^n} \Big(\beta \frac{d\pi_n \nu}{d\eta^n} \log\big(\beta \frac{d\pi_n \nu}{d\eta^n}\big) + (1-\beta) \frac{d\pi_n \mu}{d\eta^n} \log\big((1-\beta) \frac{d\pi_n \mu}{d\eta^n}\big)\Big) d\eta^n \\
&\qquad = \beta H(\pi_n \nu | \eta^n) + \beta \log \beta + (1-\beta) H(\pi_n \mu | \eta^n) + (1-\beta) \log(1-\beta).
\end{aligned} \tag{5.5}$$

Dividing by n and taking $n \to \infty$ we obtain in combination with Lemma 5.1 and (5.4) that

$$\mathscr{I}(\beta\nu + (1-\beta)\mu) = \beta\mathscr{I}(\nu) + (1-\beta)\mathscr{I}(\mu). \tag{5.6}$$

(ii) It remains to consider the case where at least one of the terms $\mathscr{I}(\mu), \mathscr{I}(\nu)$ equals infinity. In this case we want to have $\mathscr{I}(\beta\nu + (1-\beta)\mu) = \infty$, and in consideration of (5.4) the only nontrivial situation can occur if we have $H(\pi_n(\beta\nu+(1-\beta)\mu)|\pi_{n-1}(\beta\nu+(1-\beta)\mu)\otimes\eta) < \infty$ for all $n \in \mathbb{N}$. Then $\pi_n(\beta\nu + (1-\beta)\mu) \ll \pi_{n-1}(\beta\nu + (1-\beta)\mu) \otimes \eta$ and iteratively we deduce $\pi_n(\beta\nu + (1-\beta)\mu) \ll \eta^n$ and thus $\pi_n\nu \ll \eta^n$ as well as $\pi_n\mu \ll \eta^n$ for all $n \in \mathbb{N}$. The same reasoning as in (5.5) and (5.6) then yields the desired result.

□

Corollary 5.3. *The only zero of $\mathscr{I}$ is given by $\eta^{\mathbb{N}_0}$.*

Proof. Proposition 5.2 in combination with Lemma 5.1 shows that for ν such that $\mathscr{I}(\nu)$ is finite, $\mathscr{I}(\nu)$ is given as the limit of the nondecreasing sequence $(H(\pi_n\nu|\eta^n)/n)_{n\in\mathbb{N}}$. Now since the only zero of $H(\cdot|\eta^n)$ is given by η^n, we have $H(\pi_n\nu|\eta^n) = 0$ for all $n \in \mathbb{N}$ if and only if $\pi_n\nu = \eta^n$ for all $n \in \mathbb{N}$. This, however, is equivalent to $\nu = \eta^{\mathbb{N}_0}$ by Kolmogorov's consistency theorem, which finishes the proof. □

The next lemma is standard.

Lemma 5.4. *The set of extremal points of $\mathscr{M}_1^s(\Sigma_b^+)$ is given by $\mathscr{M}_1^e(\Sigma_b^+)$.*

Proof. The proof proceeds analogously to Theorem A.9.10 of [Ell85] and is omitted here. □

The following result is closely connected to Proposition 3.1 (b) and shows that the critical value β_{cr} also applies to the constant zero-potential. It is crucial for proving the finiteness of L_p^{sup} on $(-\infty, \beta_{cr})$ (cf. Lemma 5.11) and as such in the transition from the variational formula of Corollary 6.6 to the representation of the annealed Lyapunov exponents given in Theorem 2.7.

Lemma 5.5. *Assume (2.2) and $h \in (0,1)$. Then*

$$\mathbb{E}_1 \exp\{\beta T_0\} = \infty$$

for all $\beta > \beta_{cr}$, while

$$\mathbb{E}_1 \exp\{\beta_{cr} T_0\} \le 2\sqrt{\frac{1+h}{1-h}} < \infty.$$

In particular,

$$\beta_{cr} = \kappa(1 - \sqrt{1-h^2}),$$

and thus β_{cr} is independent of the very choice of the potential ξ.

Proof. The first equality follows from the definition of β_{cr}. To prove the inequality, we start with showing that $\mathbb{E}_1 \exp\{\beta T_0\}$ is finite for all $\beta < \beta_{cr}$. For this purpose choose such β. We now assume

$$\mathbb{E}_1 \exp\{\beta T_0\} = \infty \tag{5.7}$$

and lead this assumption to a contradiction. Indeed, setting $\varepsilon := \beta_{cr} - \beta > 0$, due to Proposition 3.1 (b) there exists a finite constant $C_{\beta+\varepsilon/2}$ such that

$$\mathbb{E}_1 \exp\Big\{ \int_0^{T_0} (\xi(Y_s) + \beta + \varepsilon/2)\, ds \Big\} \le C_{\beta+\varepsilon/2} \quad \text{a.s.} \tag{5.8}$$

But

$$\begin{aligned} &\mathbb{1}_{\xi(m) \ge -\varepsilon/2\, \forall m \in \{1,\dots,n\}} \mathbb{E}_1 \exp\Big\{ \int_0^{T_0} (\xi(Y_s) + \beta + \varepsilon/2)\, ds \Big\} \mathbb{1}_{Y_s \in \{1,\dots,n\}\, \forall s \in [0,T_0)} \\ &\qquad \ge \mathbb{1}_{\xi(m) \ge -\varepsilon/2\, \forall m \in \{1,\dots,n\}} \mathbb{E}_1 \exp\{\beta T_0\} \mathbb{1}_{Y_s \in \{1,\dots,n\}\, \forall s \in [0,T_0)}. \end{aligned} \tag{5.9}$$

With (5.7), we deduce

$$\mathbb{E}_1 \exp\{\beta T_0\} \mathbb{1}_{Y_s \in \{1,\dots,n\}\, \forall s \in [0,T_0)} \to \infty \tag{5.10}$$

as $n \to \infty$; furthermore, due to (1.2) and (2.2), $(\{\xi(m) \ge -\varepsilon/2\, \forall m \in \{1,\dots,n\}\})_n$ is a decreasing sequence of sets with positive probability each, and therefore, (5.10) in combination with (5.9) yields a contradiction to the a.s. boundedness given in (5.8). Hence, (5.7) cannot hold true. To finish the proof we decompose for $\beta < \beta_{cr}$:

$$\begin{aligned} \mathbb{E}_1 \exp\{\beta T_0\} &= \frac{1+h}{2} \frac{\kappa}{\kappa - \beta} + \frac{1-h}{2} \frac{\kappa}{\kappa - \beta} \mathbb{E}_2 \exp\{\beta T_0\} \\ &\ge \frac{(1-h)\kappa}{2(\kappa - \beta)} (\mathbb{E}_1 \exp\{\beta T_0\})^2. \end{aligned}$$

Consequently,

$$\mathbb{E}_1 \exp\{\beta T_0\} \le \frac{2(\kappa - \beta)}{\kappa(1-h)}$$

and monotone convergence yields $\mathbb{E}_1 \exp\{\beta_{cr} T_0\} \le \frac{2(\kappa - \beta_{cr})}{\kappa(1-h)} = 2\sqrt{\frac{1+h}{1-h}} < \infty$. Here, the equality follows using the formula for β_{cr} given in last statement of this lemma.
To prove this presentation of β_{cr}, expand

$$\mathbb{E}_1 \exp\{\beta T_0\} = \frac{\kappa}{\kappa - \beta} \Big(\frac{1+h}{2} + \frac{1-h}{2} \mathbb{E}_1 \exp\{\beta T_0\} \Big)$$

and investigate the solvability of this equation in $\mathbb{E}_1 \exp\{\beta T_0\}$ in dependence of β. ☐

Lemma 5.6. *For fixed $\beta \in (-\infty, \beta_{cr})$,*

(a) there exist constants $0 < c < C < \infty$ such that

$$\mathbb{E}_1 \exp\Big\{ \int_0^{T_0} (\zeta(Y_s) + \beta)\, ds \Big\} \in [c, C]$$

for all $\zeta \in \Sigma_b^+$.

(b) the mapping

$$\Sigma_b^+ \ni \zeta \mapsto \log \mathbb{E}_1 \exp\Big\{ \int_0^{T_0} (\zeta(Y_s) + \beta)\, ds \Big\}$$

is continuous.

Proof. (a) For any $\zeta \in \Sigma_b^+$ we have

$$\mathbb{E}_1 \exp\left\{\int_0^{T_0} (\zeta(Y_s)+\beta)\,ds\right\} \geq \mathbb{E}_1 \exp\{(b+\beta)T_0\} =: c > 0.$$

The upper bound follows from Lemma 5.5.

(b) This follows using part (a) and dominated convergence. □

Corollary 5.7. *For fixed $\beta \in (-\infty, \beta_{cr})$, the mapping*

$$\mathscr{M}_1(\Sigma_b^+) \ni \nu \mapsto L(\beta, \cdot)$$

is continuous and bounded.

Proof. (a) This follows directly from the previous lemma. □

For technical reasons we will need the following two lemmas in the proof of the lower annealed bound; they can be considered refinements of the corresponding results in the quenched case (cf. Lemma 3.5 (b) and (c)).

Lemma 5.8. *For fixed $\nu \in \mathscr{M}_1(\Sigma_b^+)$, the mapping*

$$(-\infty, \beta_{cr}) \ni \beta \mapsto L(\beta, \nu)$$

is continuously differentiable with derivative

$$\frac{\partial L}{\partial \beta}(\beta, \nu) = \int_{\Sigma_b^+} \frac{\mathbb{E}_1 T_0 \exp\{\int_0^{T_0} (\zeta(Y_s)+\beta)\,ds\}}{\mathbb{E}_1 \exp\{\int_0^{T_0} (\zeta(Y_s)+\beta)\,ds\}} \nu(d\zeta). \tag{5.11}$$

Proof. The proof proceeds in analogy to the proof of Lemma 3.5 (b) and takes advantage of Lemma 5.5. □

With respect to the following lemma, recall (3.28) for the definition of $\mathbb{E}_{M,1}^{\zeta}$.

Lemma 5.9. *(a) For arbitrary $y \in (0,\infty)$, $\nu \in \mathscr{M}_1(\Sigma_b^+)$ and large enough $M \in (0,\infty)$, there exists $\beta_M(y) \in \mathbb{R}$ such that*

$$y = \int_{\Sigma_b^+} \frac{\mathbb{E}_{M,1}^{\zeta} T_0 \exp\{\beta_M(y) T_0\}}{\mathbb{E}_{M,1}^{\zeta} \exp\{\beta_M(y) T_0\}} \nu(d\zeta).$$

(b) For all $b \in (-\infty, 0)$,

$$\lim_{\beta \to -\infty} \sup_{\nu \in \mathscr{M}^1(\Sigma_b^+)} \frac{\partial L}{\partial \beta}(\beta, \nu) = 0.$$

Proof. (a) It suffices to show the assertions that

$$\int_{\Sigma_b^+} \frac{\mathbb{E}_{M,1}^{\zeta} T_0 \exp\{\beta T_0\}}{\mathbb{E}_{M,1}^{\zeta} \exp\{\beta T_0\}} \nu(d\zeta) \to \infty \tag{5.12}$$

for β large enough and $M \to \infty$ as well as that its integrand tends to 0 a.s. for M fixed as $\beta \to -\infty$. The result then follows from the continuity of this integrand and the Intermediate Value theorem in combination with dominated convergence.

The second of these assertions follows as in the proof of part (c) of Lemma 3.5, replacing $\mathbb{P}_1^{\xi}$ by $\mathbb{P}_{M,1}^{\zeta}$. For the first assertion, let β be large enough such that $\mathbb{E}_1^{\zeta} \exp\{\beta T_0\} = \infty$ on a set of positive measure. It follows for such β that the above integrand tends to infinity as $M \to \infty$ on the corresponding set, from which we infer (5.12) for $M \to \infty$.

(b) For this purpose, due to Lemma 5.8, it suffices to show that

$$\frac{\partial L}{\partial \beta}(\beta, \delta_{\zeta}) \to 0$$

uniformly in $\zeta \in \Sigma_b^+$ as $\beta \to -\infty$. As in the above we obtain the estimate (3.27), but now $c > 0$ can be chosen not to depend on $\zeta \in \Sigma_b^+$ due to the uniform boundedness of ζ. Proceeding as in part (a), the claim follows. □

The next lemma states that the supremum in the definition of L_p^{sup} is actually a maximum.

Lemma 5.10. *For $\beta \in (-\infty, \beta_{cr})$, there exists $\nu \in \mathscr{M}_1^s(\Sigma_b^+)$ such that $L_p^{sup}(\beta) = L(\beta, \nu) - \mathscr{I}(\nu)/p$.*

Proof. Fix $\beta \in (-\infty, \beta_{cr})$. Since $\mathscr{M}_1^s(\Sigma_b^+)$ endowed with the weak topology is a compact metric space, we find a converging sequence $(\nu_n)_{n\in\mathbb{N}_0} \subset \mathscr{M}_1^s(\Sigma_b^+)$ such that $L(\beta, \nu_n) - \mathscr{I}(\nu_n)/p \to L_p^{sup}(\beta)$ for $n \to \infty$. As $L(\beta, \cdot)$ is continuous (Corollary 5.7) and $\mathscr{I}$ is lower semi-continuous, we deduce $L(\beta, \nu) - \mathscr{I}(\nu)/p = L_p^{sup}(\beta)$ for $\nu := \lim_{n\to\infty} \nu_n \in \mathscr{M}_1^s(\Sigma_b^+)$. □

In order to deduce the representation for λ_p given in Theorem 2.7, we will need the following lemma.

Lemma 5.11. *The function $\beta \mapsto L_p^{sup}(\beta)$ is finite, strictly increasing, convex and continuous on $(-\infty, \beta_{cr})$.*

Proof. Lemma 5.5 implies that L_p^{sup} is finite on $(-\infty, \beta_{cr})$. With respect to the strict monotonicty, choose $\nu \in \mathscr{M}_1^s(\Sigma_b^+)$ such that $L_p^{sup}(\beta) = L(\beta, \nu) - \mathscr{I}(\nu)/p$, which is possible due to Lemma 5.10. The fact that $L(\cdot, \nu)$ is strictly increasing and $L_p^{sup} \geq L(\cdot, \nu) - \mathscr{I}(\nu)/p$ now imply that L_p^{sup} is strictly increasing.

The convexity follows since $L(\cdot, \nu)$ is convex and thus L_p^{sup} as a supremum of convex functions is convex; continuity is implied by convexity. □

6 Proofs for the annealed regime

The aim of this section is to prove Theorem 2.7. Similarly to the quenched case we derive upper and lower bounds for $t^{-1}\log\langle u(t,0)^p\rangle^{1/p}$ as $t\to\infty$ (cf. Lemmas 6.1 and 6.3). The additional techniques needed here are Varadhan's Lemma (see proof of Lemma 6.1) as well as an exponential change of measure (in the proof of Lemma 6.3), both applied to the sequence $(R_n\circ\xi)_{n\in\mathbb{N}}$ of empirical measures. Further estimates similar to the quenched regime (Lemmas 6.4 and 6.5) lead to a variational formula for λ_p given in Corollary 6.6. Results on the properties of L_p^{sup} (Lemmas 5.10 and 5.11) then complete of the proof of Theorem 2.7.

As in Theorem 2.7, we assume (1.2) and (2.9) for the rest of this section. Notice that since the potential is bounded, L is well-defined on $\mathbb{R}$.

Lemma 6.1. *Let $I\subset[0,\infty)$ be a compact interval and $p\in(0,\infty)$. Then*

$$\limsup_{t\to\infty}\frac{1}{t}\log\Big\langle\Big(\sum_{n\in tI}\mathbb{E}_n\exp\Big\{\int_0^t\xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=0}\Big)^p\Big\rangle^{\frac{1}{p}}\le\sup_{\alpha\in I}\inf_{\beta<\beta_{cr}}(-\beta+\alpha L_p^{sup}(\beta)).\tag{6.1}$$

Proof. (i) With the same notations as in the quenched case we first assume $I=[\varepsilon,\gamma]$ with $\varepsilon>0$ and deduce using the exponential Chebyshev inequality:

$$\begin{aligned}
&\Big\langle\Big(\sum_{n\in t[\alpha_k^\delta,\alpha_{k+1}^\delta]}\mathbb{E}_n\exp\Big\{\int_0^t\xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=0}\Big)^p\Big\rangle &(6.2)\\
&\le\Big\langle\Big(\sum_{n\in t[\alpha_k^\delta,\alpha_{k+1}^\delta]}\mathbb{E}_n\exp\Big\{\int_0^{T_0}\xi(Y_s)\,ds\Big\}\mathbb{1}_{\frac{T_0}{n}\le\frac{1}{\alpha_k^\delta}}\exp\{c^*(t-T_0)\}\Big)^p\Big\rangle\\
&\le\inf_{\beta>0}\Big\langle\Big(\sum_{n\in t[\alpha_k^\delta,\alpha_{k+1}^\delta]}\mathbb{E}_n\exp\Big\{\int_0^{T_0}(\xi(Y_s)-\beta)\,ds\Big\}\exp\{-\beta_{cr}(t-T_0)\}\Big)^p\Big\rangle\exp\Big\{\frac{\beta pn}{\alpha_k^\delta}\Big\}\\
&\le\inf_{\beta>0}\exp\Big\{pt\Big(\frac{\beta\alpha_{k+1}^\delta}{\alpha_k^\delta}-\beta_{cr}\Big)\Big\}\Big\langle\Big(\sum_{n\in t[\alpha_k^\delta,\alpha_{k+1}^\delta]}\exp\big\{nL(-\beta+\beta_{cr},R_n\circ\xi)\big\}\Big)^p\Big\rangle, &(6.3)
\end{aligned}$$

where to obtain the penultimate line we used (3.4). Recall that at the beginning of subsection 2.2, R_n was defined as the empirical measure of a shifted sequence. Consequently, we conclude

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t}\log\Big\langle\Big(\sum_{n\in t[\alpha_k^\delta,\alpha_{k+1}^\delta]}\mathbb{E}_n\exp\Big\{\int_0^t\xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=0}\Big)^p\Big\rangle\\
&\le\inf_{\beta>0}\Big[p\Big(\frac{\beta\alpha_{k+1}^\delta}{\alpha_k^\delta}-\beta_{cr}\Big)+\limsup_{t\to\infty}\frac{1}{t}\log\big\langle\exp\big\{\alpha_{j_k}^\delta ptL(-\beta+\beta_{cr},R_{\lfloor\alpha_{j_k}^\delta t\rfloor}\circ\xi)\big\}\big\rangle\Big] &(6.4)
\end{aligned}$$

with $j_k=k+1$ if the second summand in (6.4) is positive in that case and $j_k=k$ otherwise (note that this decision depends on β but not on the choice of j_k). Corollary 5.7 tells us that the conditions concerning $L(\beta,\cdot)$ with respect to the upper bound of Varadhan's lemma (Lemma 4.3.6 in [DZ98]) are fulfilled. Thus, bearing in mind the large deviations principle for $R_n \circ \xi$ given in Corollary 6.5.15 of [DZ98] with rate function $\mathscr{I}$ (cf. (2.10)), we can estimate the right-hand side of (6.4) by

$$\inf_{\beta > -\beta_{cr}} \Big[\Big(p \frac{(\beta + \beta_{cr}) \alpha^{\delta}_{k+1}}{\alpha^{\delta}_{k}} - \beta_{cr} \Big) + \alpha^{\delta}_{j_k} \sup_{\nu \in \mathscr{M}^s_1(\Sigma^+_b)} \big(pL(-\beta, \nu) - \mathscr{I}(\nu) \big) \Big].$$

Since $\inf I > 0$ by assumption, the ratios $\alpha^{\delta}_{k+1}/\alpha^{\delta}_{k}$ are bounded from above and similarly to the quenched case (proof of Lemma 4.2) we obtain in combination with the previous and taking $\delta \downarrow 0$:

$$\limsup_{t\to\infty} \frac{1}{t} \log \Big\langle \Big(\sum_{n \in tI} \mathbb{E}_n \exp\Big\{ \int_0^t \xi(Y_s)\, ds \Big\} \mathbb{1}_{Y_t = 0} \Big)^p \Big\rangle^{\frac{1}{p}} \le \sup_{\alpha \in I} \inf_{\beta < \beta_{cr}} (-\beta + \alpha L^{sup}_p(\beta)).$$

(ii) Now assume $\inf I = 0$. We proceed similarly to the quenched part (cf. proof of Lemma 4.2) and use Proposition 3.1 (b) to estimate the first factor on the right-hand side of (4.14) uniformly by a constant. $\square$

In order to prove the lower bound, we need the following technical lemma for which we recall the definition of τ_k as $T_{k-1} - T_k$ in (3.29).

Lemma 6.2. *For $\beta \in \mathbb{R}$, $M > 0$, $\zeta \in \Sigma^+_b$ and $n \in \mathbb{N}$ denote*

$$\mathbb{P}^{\zeta}_{n,\beta,M}(A) := \frac{\mathbb{E}_n \exp\{ \int_0^{T_0} (\zeta(Y_s) + \beta)\, ds \} \mathbb{1}_{\tau_k \le M\ \forall k \in \{1,\dots,n\}} \mathbb{1}_A}{\mathbb{E}_n \exp\{ \int_0^{T_0} (\zeta(Y_s) + \beta)\, ds \} \mathbb{1}_{\tau_k \le M\ \forall k \in \{1,\dots,n\}}}$$

with $A \in \mathscr{F}$. Then, for $\varepsilon > 0$ and $\nu \in \mathscr{M}^s_1(\Sigma^+_b)$ there exist $\delta > 0$ and a neighbourhood $U(\nu)$ of ν such that for all $\zeta \in \Sigma^+_b$,

$$\mathbb{P}^{\zeta}_{n,\beta,M}(T_0/n \notin (y - \varepsilon, y + \varepsilon)) \mathbb{1}_{R_n(\zeta) \in U(\nu)} \le 2 \exp\{-n\delta\} \mathbb{1}_{R_n(\zeta) \in U(\nu)}$$

holds for all $n \in \mathbb{N}$, where

$$y := \int_{\Sigma^+_b} \mathbb{E}^{\zeta}_{n,\beta,M}(T_0)\, d\nu.$$

Proof. Define the function

$$G_M(\beta) : \Sigma^+_b \ni \zeta \mapsto \log \mathbb{E}_1 \exp\Big\{ \int_0^{T_0} (\zeta(Y_s) + \beta)\, ds \Big\} \mathbb{1}_{T_0 \le M}.$$

Using the exponential Chebyshev inequality for $\alpha \ge 0$, we compute with $\delta > 0$

$$\begin{aligned} &\mathbb{P}^{\zeta}_{n,\beta,M}(T_0/n \ge y + \varepsilon) \mathbb{1}_{R_n(\zeta) \in U(\nu)} \le \mathbb{E}^{\zeta}_{n,\beta,M} \exp\{\alpha T_0 - n\alpha(y + \varepsilon)\} \mathbb{1}_{R_n(\zeta) \in U(\nu)} \\ &\le \exp\{-n\alpha(y+\varepsilon)\} \exp\Big\{ n \Big(\int_{\Sigma^+_b} (G_M(\beta + \alpha) - G_M(\beta))\, d\nu + \delta \Big) \Big\} \mathbb{1}_{R_n(\zeta) \in U(\nu)} \end{aligned} \tag{6.5}$$

for some neighbourhood $U(\nu)$ of ν (depending on α also) such that

$$\Big| \int_{\Sigma^+_b} (G_M(\beta + \alpha) - G_M(\beta))\, d\nu - \int_{\Sigma^+_b} (G_M(\beta + \alpha) - G_M(\beta))\, d\mu \Big| \le \delta$$

holds for all $\mu \in U(\nu)$. Writing

$$g(\alpha) := -\alpha(y+\varepsilon) + \int_{\Sigma_b^+} (G_M(\beta+\alpha) - G_M(\beta))\, d\nu,$$

the right-hand side of (6.5) equals $\exp\{n(g(\alpha)+\delta)\}\mathbb{1}_{R_n(\zeta)\in U(\nu)}$. We observe $g(0)=0$ and $g'(0) = -y-\varepsilon+y<0$. Hence, there exists $\underline{\alpha}>0$ such that $g(\underline{\alpha})<0$. Setting $\delta := -g(\underline{\alpha})/2$ and refining $U(\nu)$ such that

$$\Big| \int_{\Sigma_b^+} G_M(\underline{\alpha})\, d\nu - \int_{\Sigma_b^+} G_M(\underline{\alpha})\, d\mu \Big| < \delta$$

holds for all $\mu \in U(\nu)$, we deduce from (6.5) with $\alpha = \underline{\alpha}$ that

$$\mathbb{P}^{\zeta}_{n,\beta,M}(T_0/n \geq y+\varepsilon)\mathbb{1}_{R_n(\zeta)\in U(\nu)} \leq \exp\{-n\delta\}\mathbb{1}_{R_n(\zeta)\in U(\nu)}.$$

In complete analogy we obtain

$$\mathbb{P}^{\zeta}_{n,\beta,M}(T_0/n \leq y-\varepsilon)\mathbb{1}_{R_n(\zeta)\in U(\nu)} \leq \exp\{-n\delta\}\mathbb{1}_{R_n(\zeta)\in U(\nu)},$$

where possibly $\delta>0$ is even smaller and $U(\nu)$ even more refined; the result then follows. □

We can now proceed to prove the lower annealed bound.

Lemma 6.3. *Let $I \subset [0,\infty)$ be an interval and $p \in (0,\infty)$. Then*

$$\begin{aligned} \liminf_{t\to\infty} \frac{1}{t} \log \Big\langle \Big(\sum_{n\in tI} \mathbb{E}_n \exp\Big\{ \int_0^t \xi(Y_s)\, ds \Big\} \mathbb{1}_{Y_t=0} \Big)^p \Big\rangle^{\frac{1}{p}} \\ \geq \sup_{\alpha \in \mathring{I} \cup (I\cap\{0\})} \inf_{\beta<\beta_{cr}} (-\beta + \alpha L_p^{sup}(\beta)). \end{aligned} \tag{6.6}$$

Proof. Observe that for $\alpha \in \mathring{I}$, $y \in (0,1/\alpha)$ and $\varepsilon>0$ small enough we have due to the independence of the medium

$$\Big\langle \mathbb{E}_{\lfloor \alpha t\rfloor} \exp\Big\{ \int_0^{T_0} \xi(Y_s)\, ds \Big\} \mathbb{1}_{Y_t=0} \Big)^p \Big\rangle \tag{6.7}$$

$$\begin{aligned} &\geq \Big\langle \Big(\mathbb{E}_{\lfloor \alpha t\rfloor} \exp\Big\{ \int_0^{T_0} \xi(Y_s)\, ds \Big\} \mathbb{1}_{\frac{T_0}{\lfloor \alpha t\rfloor} \in (y-\varepsilon,y+\varepsilon)} \Big(\mathbb{E}_0 \exp\Big\{ \int_0^{t-r} \xi(Y_s)\, ds \Big\} \mathbb{1}_{Y_{t-r}=0} \Big)_{r=T_0} \Big)^p \Big\rangle \\ &\geq \Big\langle \Big(\mathbb{E}_{\lfloor \alpha t\rfloor} \exp\Big\{ \int_0^{T_0} \xi(Y_s)\, ds \Big\} \mathbb{1}_{\frac{T_0}{\lfloor \alpha t\rfloor} \in (y-\varepsilon,y+\varepsilon)} \Big)^p \Big\rangle \\ &\quad \times \Big\langle \min_{r\in(y-\varepsilon,y+\varepsilon)} \Big(\mathbb{E}_0 \exp\Big\{ \int_0^{t(1-\alpha r)} \xi(Y_s)\, ds \Big\} \mathbb{1}_{Y_{t(1-\alpha r)}=0,\, T_1>t(1-\alpha r)} \Big)^p \Big\rangle. \end{aligned} \tag{6.8}$$

(i) To deal with the second of the factors on the right-hand side we observe

$$\frac{1}{t}\log \mathbb{E}_0 \exp\Big\{ \int_0^t \xi(Y_s)\, ds \Big\} \mathbb{1}_{Y_t=0,\, T_1>t} \geq b + \frac{1}{t}\log \mathbb{P}_0(Y_s = 0\ \forall s \in [0,t]) = b-\kappa \tag{6.9}$$

a.s. Hence, Jensen's inequality and a modified version of Fatou's lemma (taking advantage of (6.9)) apply to yield the inequality in

$$\begin{aligned}\liminf_{t\to\infty}\frac{1}{t}\log\Big\langle\Big(\mathbb{E}_0\exp\Big\{\int_0^{t(1-\alpha y)}\xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_{t(1-\alpha y)}=0,\,T_1>t(1-\alpha y)}\Big)^p\Big\rangle\\ \geq p\Big\langle\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}_0\exp\Big\{\int_0^{t(1-\alpha y)}\xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_{t(1-\alpha y)}=0,\,T_1>t(1-\alpha y)}\Big\rangle\\ =(1-\alpha y)pc^*.\end{aligned}\tag{6.10}$$

To obtain the equality, we used the fact that c^* equals (3.3) for $M=1$, cf. Proposition 3.1 (a).

(ii) With respect to the first factor on the right-hand side of (6.8), we aim to perform an exponential change of measure and introduce for $\beta\in\mathbb{R}$, $n\in\mathbb{N}$ as well as $M>0$ large enough (in order to apply Lemma 5.9 (a)) the function

$$G_M^{(n)}(\beta):\Sigma_b^+\ni\zeta\mapsto\log\mathbb{E}_n\exp\Big\{\int_0^{T_0}(\zeta(Y_s)+\beta)\,ds\Big\}\mathbb{1}_{\tau_k\leq M\,\forall k\in\{1,\dots,n\}}.$$

Choosing $\nu\in\mathscr{M}_1^s(\Sigma_b^+)$ such that

$$\nu=\varepsilon\,\mathrm{Prob}+(1-\varepsilon)\nu_\varepsilon\tag{6.11}$$

for some $\varepsilon>0$ and $\nu_\varepsilon\in\mathscr{M}_1^s(\Sigma_b^+)$, we fix $\beta_M\in\mathbb{R}$ such that

$$\int_{\Sigma_b^+}\frac{\mathbb{E}_1T_0\exp\{\int_0^{T_0}(\zeta(Y_s)+\beta_M)\,ds\}\mathbb{1}_{T_0\leq M}}{\mathbb{E}_1\exp\{\int_0^{T_0}(\zeta(Y_s)+\beta_M)\,ds\}\mathbb{1}_{T_0\leq M}}\,\nu(d\zeta)=y,\tag{6.12}$$

which is possible due to part (a) of Lemma 5.9. Since for β fixed, $G_M(\beta):=G_M^{(1)}(\beta)$ is bounded as a function on Σ_b^+, we find for each $\delta>0$ a neighbourhood $U(\nu)$ of ν in $\mathscr{M}_1(\Sigma_b^+)$ such that

$$\Big|\int_{\Sigma_b^+}G_M(\beta_M)\,d\nu-\int_{\Sigma_b^+}G_M(\beta_M)\,d\mu\Big|\leq\delta/2\tag{6.13}$$

for all $\mu\in U(\nu)$. We obtain

$$\begin{aligned}&\Big\langle\Big(\mathbb{E}_n\exp\Big\{\int_0^{T_0}\xi(Y_s)\,ds\Big\}\mathbb{1}_{\frac{T_0}{n}\in(y-\varepsilon,y+\varepsilon)}\Big)^p\Big\rangle\\ &=\Big\langle\Big(\mathbb{E}_n\exp\Big\{\int_0^{T_0}\xi(Y_s)\,ds+\beta_MT_0-G_M^{(n)}(\beta_M)-\beta_MT_0+G_M^{(n)}(\beta_M)\Big\}\times\mathbb{1}_{\frac{T_0}{n}\in(y-\varepsilon,y+\varepsilon)}\Big)^p\Big\rangle\\ &\geq\Big\langle\Big(\frac{\mathbb{E}_n\exp\{\int_0^{T_0}(\xi(Y_s)+\beta_M)\,ds\}\mathbb{1}_{\tau_k\leq M\,\forall k\in\{1,\dots,n\}}\mathbb{1}_{\frac{T_0}{n}\in(y-\varepsilon,y+\varepsilon)}}{\mathbb{E}_n\exp\{\int_0^{T_0}(\xi(Y_s)+\beta_M)\,ds\}\mathbb{1}_{\tau_k\leq M\,\forall k\in\{1,\dots,n\}}}\Big)^p\\ &\quad\times\exp\{pG_M^{(n)}(\beta_M)\}\mathbb{1}_{R_n\circ\xi\in U(\nu)}\Big\rangle\exp\{-n\beta_Mp(y\pm\varepsilon)\}\\ &\geq\exp\{-n\beta_Mp(y\pm\varepsilon)\}\exp\Big\{pn\Big(\int_{\Sigma_b^+}G_M(\beta_M)\,d\nu-\delta/2\Big)\Big\}\\ &\quad\times\big\langle\big(1-\mathbb{P}^\zeta_{n,\beta,M}(T_0/n\notin(y-\varepsilon,y+\varepsilon))\big)^p\mathbb{1}_{R_n\circ\xi\in U(\nu)}\big\rangle,\end{aligned}\tag{6.14}$$

where $y \pm \varepsilon$ is supposed to denote $y + \varepsilon$ if $\beta_M > 0$ and $y - \varepsilon$ otherwise.

Therefore, choosing $\delta > 0$ and $U(\nu)$ according to Lemma 6.2 we infer that $\mathbb{P}^{\zeta}_{n,\beta,M}(T_0/n \notin (y-\varepsilon, y+\varepsilon))\mathbb{1}_{R_n \circ \xi \in U(\nu)}$ decays exponentially in n. Thus, in combination with the large deviations principle for $R_n \circ \xi$ given in Corollary 6.5.15 of [DZ98] we obtain

$$\liminf_{n\to\infty} \frac{1}{n} \log \left\langle \left(\mathbb{P}^{\zeta}_{n,\beta,M}(T_0/n \in (y-\varepsilon, y+\varepsilon))\right)^p \mathbb{1}_{R_n \circ \xi \in U(\nu)} \right\rangle \geq -\mathscr{I}(\nu).$$

Continuing (6.14) we get taking $\varepsilon \downarrow 0$ on the right-hand side

$$\begin{aligned} \liminf_{n\to\infty} \frac{1}{n} \log \left\langle \Big(\mathbb{E}_n \exp\Big\{ \int_0^{T_0} \xi(Y_s)\, ds \Big\} \mathbb{1}_{\frac{T_0}{n} \in (y-\varepsilon, y+\varepsilon)} \Big)^p \right\rangle \\ \geq -\beta_M p y + p\Big(\int_{\Sigma_b^+} G_M(\beta_M) d\nu - \delta/2 \Big) - \mathscr{I}(\nu). \end{aligned} \tag{6.15}$$

We observe that $M \mapsto G_M(\beta)$ is nondecreasing, whence the sets

$$\Big\{ \beta \in \mathbb{R} : -\beta y + \int_{\Sigma_b^+} G_M(\beta)\, d\nu \leq \lim_{M\to\infty} \inf_{\beta \in \mathbb{R}} \Big(-\beta y + \int_{\Sigma_b^+} G_M(\beta)\, d\nu \Big) \Big\}$$

are nonincreasing in M. Furthermore, they are non-empty since differentiation of

$$\beta \mapsto -\beta y + \int_{\Sigma_b^+} G_M(\beta)\, d\nu \tag{6.16}$$

with respect to β yields that this function takes its infimum in $\beta = \beta_M$ (cf. (6.12)). From this in combination with the strict convexity of the function in (6.16) we infer the boundedness of the above sets. Furthermore, these sets are closed since the map in (6.16) is continuous. We therefore conclude that the intersection over all $M > 0$ large enough of these sets contains some $\beta_\nu \in (-\infty, \beta_{cr}]$ (the fact that $\beta_\nu \leq \beta_{cr}$ follows from (6.11)) and in combination with (6.15) we deduce

$$\begin{aligned} \liminf_{n\to\infty} \frac{1}{n} \log \left\langle \Big(\mathbb{E}_n \exp\Big\{ \int_0^{T_0} \xi(Y_s)\, ds \Big\} \mathbb{1}_{\frac{T_0}{n} \in (y-\varepsilon, y+\varepsilon)} \Big)^p \right\rangle \\ \geq -\beta_\nu p y + p(L(\beta_\nu, \nu) - \delta/2) - \mathscr{I}(\nu). \end{aligned} \tag{6.17}$$

We now write $\mathscr{S} := \{\nu \in \mathscr{M}_1^s(\Sigma_b^+) \cap \{\mathscr{I} < \infty\}$ such that (6.11) holds$\}$. Taking $\delta \downarrow 0$, (6.8), (6.10) and (6.17) supply us with

$$\begin{aligned} \liminf_{t\to\infty} \frac{1}{t} \log \left\langle \Big(\mathbb{E}_{\lfloor \alpha t \rfloor} \exp\Big\{ \int_0^{t} \xi(Y_s)\, ds \Big\} \mathbb{1}_{Y_t = 0} \Big)^p \right\rangle^{\frac{1}{p}} \\ \geq \alpha \sup_{\nu \in \mathscr{S}} \big(-\beta_\nu y + L(\beta_\nu, \nu) - \mathscr{I}(\nu)/p \big) + (1 - \alpha(y-\varepsilon))c^* \\ \geq \alpha \sup_{\nu \in \mathscr{S}} \inf_{c \leq \beta \leq \beta_{cr}} \big(-\beta y + L(\beta, \nu) - \mathscr{I}(\nu)/p \big) + (1 - \alpha(y-\varepsilon))c^*, \end{aligned} \tag{6.18}$$

where the last line holds for $c \in (-\infty, 0)$ small enough due to Lemma 5.9 (b).

Now observe that $f : [c, \beta_{cr}] \times \mathscr{S} \ni (\beta, \nu) \mapsto -\beta y + L(\beta, \nu) - \mathscr{I}(\nu)/p$ is a real-valued function. In addition, the function $f(\beta, \cdot)$ is upper semi-continuous and concave on $\mathscr{S}$ for any $\beta \in [c, \beta_{cr}]$, while

$f(\cdot,\nu)$ is lower semi-continuous and convex on $[c,\beta_{cr}]$ for any $\nu\in\mathscr{S}$. Since furthermore $[c,\beta_{cr}]$ is compact and convex while $\mathscr{S}$ is convex, we may apply Sion's minimax theorem (cf. e.g. [Kom88]). This then yields that the first summand of the last line of (6.18) equals

$$\alpha \inf_{c\le\beta\le\beta_{cr}} \sup_{\nu\in\mathscr{S}} \big(-\beta y + L(\beta,\nu) - \mathscr{I}(\nu)/p\big) = \alpha \inf_{c\le\beta\le\beta_{cr}} \sup_{\nu\in\mathscr{M}_1^s(\Sigma_b^+)} \big(-\beta y + L(\beta,\nu) - \mathscr{I}(\nu)/p\big);$$

equality holds since for $\beta\in[c,\beta_{cr}]$ and $\nu\in\mathscr{M}_1^s(\Sigma_b^+)$ we find a sequence $(\nu_n)_{n\in\mathbb{N}}\subset\mathscr{S}$ with $L(\beta,\nu_n)-\mathscr{I}(\nu_n)/p \to L(\beta,\nu)-\mathscr{I}(\nu)/p$ as $n\to\infty$ (indeed, choose $\nu_n := (\text{Prob}+(n-1)\nu)/n$ and use Lemma 5.2).

Taking $\varepsilon\downarrow 0$ in (6.18) and the suprema in y and α we therefore get

$$\begin{aligned}
&\liminf_{t\to\infty}\frac{1}{t}\log\Big\langle \sum_{n\in tI}\mathbb{E}_n \exp\Big\{\int_0^t \xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=0}\Big)^p\Big\rangle^{\frac{1}{p}}\\
&\quad\ge \sup_{\alpha\in\overset{\circ}{I}}\ \sup_{y\in(0,1/\alpha)}\Big(\alpha \inf_{\beta\le\beta_{cr}}\ \sup_{\nu\in\mathscr{M}_1^s(\Sigma_b^+)}\big(-\beta y + L(\beta,\nu)-\mathscr{I}(\nu)/p\big) + (1-\alpha y)c^*\Big)\\
&\quad\ge \sup_{\alpha\in\overset{\circ}{I}}\ \inf_{\beta<\beta_{cr}} (-\beta + \alpha L_p^{sup}(\beta)).
\end{aligned}$$

For the case $0\in I$ it remains to estimate

$$\liminf_{t\to\infty}\frac{1}{t}\log\Big\langle\Big(\mathbb{E}_0 \exp\Big\{\int_0^t \xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=0}\Big)^p\Big\rangle^{\frac{1}{p}}.$$

Analogously as for the second factor of (6.8) we obtain

$$\liminf_{t\to\infty}\frac{1}{t}\log\Big\langle\Big(\mathbb{E}_0 \exp\Big\{\int_0^t \xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=0}\Big)^p\Big\rangle^{\frac{1}{p}} \ge c^* = -\beta_{cr}.$$

This finishes the proof.

□

Similarly to the quenched case, we have the following two results.

Lemma 6.4. *For all $\delta\ge 0$ and $\gamma\ge\delta$ we have*

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t}\log\Big\langle\Big(\sum_{n\in t[-\gamma,-\delta]}\mathbb{E}_n \exp\Big\{\int_0^t \xi(Y_s)ds\Big\}\mathbb{1}_{Y_t=0}\Big)^p\Big\rangle^{\frac{1}{p}}\\
&\quad\le \delta\log\frac{1-h}{1+h} + \liminf_{t\to\infty}\frac{1}{t}\log\Big\langle\Big(\sum_{n\in t[\delta,\gamma]}\mathbb{E}_n \exp\Big\{\int_0^t \xi(Y_s)ds\Big\}\mathbb{1}_{Y_t=0}\Big)^p\Big\rangle. \qquad (6.19)
\end{aligned}$$

Proof. The proof proceeds similarly to the proof of Lemma 4.3 and is omitted here. □

Lemma 6.5. *We have*

$$\limsup_{t\to\infty}\frac{1}{t}\log\Big\langle\Big(\sum_{n\notin t[-\gamma,\gamma]}\mathbb{E}_n \exp\Big\{\int_0^t \xi(Y_s)\,ds\Big\}\mathbb{1}_{Y_t=0}\Big)^p\Big\rangle \to -\infty \qquad (6.20)$$

as $\gamma\to\infty$.

Proof. The proof is similar to that of Lemma 4.4 and is omitted here. □

We are now ready to prove the existence of λ_p and give a variational formula.

Corollary 6.6. *For $p \in (0,\infty)$, the annealed Lyapunov exponent λ_p exists and is given by*

$$\lambda_p = \sup_{\alpha\in[0,\gamma]} \inf_{\beta<\beta_{cr}} (-\beta + \alpha L_p^{sup}(\beta)) \tag{6.21}$$

for all $\gamma > 0$ large enough.

Proof. Using Lemmas 6.1, 6.3, 6.4 and 6.5 one may proceed similarly to the quenched case. □

Proof of Theorem 2.7. In order to derive the representation of Theorem 2.7, we distinguish two cases: First assume that L_p^{sup} does not have a zero in $(0,\beta_{cr})$. From the fact that L_p^{sup} is continuous and increasing (cf. Lemma 5.11), we infer that $L_p^{sup}(\beta) < 0$ for all $\beta \in (-\infty,\beta_{cr})$, whence taking $\alpha \downarrow 0$ in Corollary 6.6 yields $\lambda_p = -\beta_{cr}$.

Otherwise, if such a zero exists, the properties of L_p^{sup} derived in Lemma 5.11 first imply the uniqueness of such a zero and then, in an analogous way to the proof of Theorem 2.1 and in combination with (6.21), that λ_p equals the zero of $L_p^{sup}(-\cdot)$. □

7 Further results

While in sections 4 and 6 we derived the existence of the corresponding Lyapunov exponents and gave formulae for them, we now concentrate on the proofs of the remaining results.

7.1 Quenched regime

We start with proving the result on the transition from zero to positive speed of the random walk in random potential, Proposition 2.5.

Proof of Proposition 2.5. (a) In order to deal with the explicit dependence of the respective quantities on h, we use the notation $(Y_t^h)_{t\in\mathbb{R}_+}$ for a continuous-time random with generator $\kappa\Delta_h$ as well as $\mathscr{Z}_n$ for the set of discrete time simple random walk paths on $\mathbb{Z}$ starting in 1 and hitting 0 for the first time at time $2n+1$. Furthermore, $J(Y)$ denotes the number of jumps of the process Y to the right before it hits 0 and we may and do assume $h < 1$. For $\beta \le \beta_{cr}^h := \beta_{cr} = \kappa(1-\sqrt{1-h^2})$ we then have

$$\begin{aligned}
L_h(\beta) &= \Big\langle \log \mathbb{E}_1 \exp\Big\{ \int_0^{T_0} (\xi(Y_s^h)+\beta)\,ds \Big\} \Big\rangle \\
&= \Big\langle \log \sum_{n\in\mathbb{N}_0} \sum_{Z\in\mathscr{Z}_n} \frac{1+h}{2}\Big(\frac{1-h^2}{4}\Big)^n \prod_{k=0}^{2n} \frac{\kappa}{\kappa-\xi(Z_k)-\beta} \Big\rangle \qquad (7.1) \\
&= \log(1+h) + \Big\langle \log \mathbb{E}_1 \exp\Big\{ \int_0^{T_0} (\xi(Y_s^0)+\beta)\,ds + J(Y^0)\log(1-h^2) \Big\} \Big\rangle.
\end{aligned}$$

Now in order to prove (a), according to Corollary 2.3 it is enough to show

$$\limsup_{h\downarrow 0} L_h(\beta_{cr}^h) < 0. \tag{7.2}$$

By direct inspection or Lemma 5.5 we get that a.s. the expression

$$\begin{aligned}
&\log(1+h) + \log \mathbb{E}_1 \exp\Big\{ \int_0^{T_0} (\xi(Y_s^0) + \beta_{cr}^h)\, ds + J(Y^0)\log(1-h^2) \Big\} \\
&\qquad = \log \sum_{n\in\mathbb{N}_0} \sum_{z\in\mathscr{Z}_n} \frac{\sqrt{1+h}}{2^{2n+1}\sqrt{1-h}} \prod_{k=0}^{2n} \frac{\kappa\sqrt{1-h^2}}{\kappa\sqrt{1-h^2} - \xi(Z_k)}
\end{aligned} \tag{7.3}$$

is bounded from above uniformly in $h \in [0, 1/2)$. Considering the right-hand side of (7.3) one observes that for $h \downarrow 0$ it converges to (7.3) evaluated at $h = 0$. Since furthermore $L_0(\beta_{cr}^0) < 0$, dominated convergence yields (7.2).

(b) Departing from (7.1) we write

$$L_h(\beta_{cr}^h) = \Big\langle \log \sum_{n\in\mathbb{N}_0} \sum_{Z\in\mathscr{Z}_n} \frac{2^{-(2n+1)}\kappa\sqrt{1+h}}{\kappa\sqrt{1-h} - \xi(1)} \prod_{k=1}^{2n} \frac{\kappa\sqrt{1-h^2}}{\kappa\sqrt{1-h^2} - \xi(Z_k)} \Big\rangle.$$

Now if $L_1(\beta) \to \infty$ for $\beta \uparrow \beta_{cr}^1 = \kappa$, then $L_h(\beta_{cr}^h) \to \infty$ for $h \uparrow 1$ as well. Otherwise, if $\lim_{\beta\uparrow\kappa} L_1(\beta) < \infty$, then dominated convergence yields $L_h(\beta_{cr}^h) \to L_1(\kappa) = \langle \log \frac{\kappa}{-\xi(1)} \rangle$ as $h \uparrow 1$, and the claim follows. □

Proof of Proposition 2.6. (a) We first show convexity. Writing $u_\kappa(t,x)$ to emphasise the dependence of the solution to (1.1) on κ, we have for a random walk $(X_t)_{t\in\mathbb{R}_+}$ with generator Δ_h:

$$\begin{aligned}
\lambda_0(\kappa) &= \lim_{t\to\infty} \frac{1}{t} \log u_\kappa(t,x) = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_0 \exp\Big\{ \int_0^t \xi(X_{\kappa s})\, ds \Big\} \\
&= \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_0 \exp\Big\{ \frac{1}{\kappa} \int_0^{\kappa t} \xi(X_s)\, ds \Big\} \\
&= \kappa \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_0 \exp\Big\{ \frac{1}{\kappa} \int_0^{t} \xi(X_s)\, ds \Big\} = \kappa \Psi(1/\kappa),
\end{aligned} \tag{7.4}$$

where

$$\Psi(x) := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_0 \exp\Big\{ x \int_0^t \xi(X_s)\, ds \Big\}$$

for $x \geq 0$. Note that the limit defining $\Psi(x)$ exists in $[-1, 0]$ for all $x \in \mathbb{R}_+$ due to Theorem 2.1 and (1.8). Hölder's inequality now tells us that Ψ is convex and choosing $\alpha := \frac{\beta x}{\beta x + (1-\beta)y}$ and $\gamma := \frac{(1-\beta)y}{\beta x + (1-\beta)y}$, we obtain the convexity of $x\Psi(1/x)$ in a similar manner:

$$\begin{aligned}
(\beta x + (1-\beta)y)\Psi\Big(\frac{1}{\beta x + (1-\beta)y}\Big) &= (\beta x + (1-\beta)y)\Psi\Big(\frac{\alpha}{x} + \frac{\gamma}{y}\Big) \\
&\leq \alpha(\beta x + (1-\beta)y)\Psi(1/x) + \gamma(\beta x + (1-\beta)y)\Psi(1/y) \\
&= \beta x \Psi(1/x) + (1-\beta)y\Psi(1/y).
\end{aligned}$$

In combination with (7.4) the convexity of $\kappa \mapsto \lambda_0(\kappa)$ follows.

To show that $\lambda_0(\kappa)$ is nonincreasing in $\kappa \in (0,\infty)$ assume to the contrary that there is $0 < \kappa_1 < \kappa_2$ such that $\lambda_0(\kappa_1) < \lambda_0(\kappa_2)$. The convexity of λ_0 would then imply $\lim_{\kappa\to\infty} \lambda_0(\kappa) = \infty$ which is impossible since we clearly have $\lambda_0(\kappa) \le 0$ for all $\kappa \in (0,\infty)$.

(b) From Theorem 2.1 we deduce

$$\lambda_0(\kappa) \in [-\beta_{cr}, 0], \tag{7.5}$$

and using (1.8) the claim follows.

(c) (2.7) follows from (7.4) and the fact that $\lim_{x\downarrow 0} \Psi(x) = 0$, which is due to the boundedness of ξ from below. (2.8) follows using $\Psi(1) = \lambda_0(1) < 0$, the fact that $\Psi(x) \in [-1,0]$ (due to Theorem 2.1 and (1.8)), as well as (7.4) and the monotonicity of Ψ. $\square$

7.2 Annealed regime

In this subsection we primarily deal with proofs concerning the annealed Lyapunov exponents, i.e. in particular we assume (2.9).

Proof of Proposition 2.11. (a) For $p > 0$ we directly obtain $\lambda_0 \le \lambda_p$ from the corresponding formulae given in Theorems 2.1 and 2.7. If $0 < p < q$, then Jensen's inequality supplies us with $\langle u(t,0)^p\rangle^{\frac{1}{p}} \le \langle u(t,0)^q\rangle^{\frac{1}{q}}$ and the statement follows from the definition of λ_p.

(b) For $\beta \in (0,1)$ and $0 < p < q$ we get

$$\langle u(t,0)^{\beta p+(1-\beta)q}\rangle \le \langle u(t,0)^p\rangle^{\beta} \langle u(t,0)^q\rangle^{1-\beta}$$

by Hölder's inequality, which implies the desired convexity on $(0,\infty)$.

(c) For $p = 0$ this follows from Proposition 2.6 (a); for $p \in (0,\infty)$ the proof proceeds in complete analogy to the corresponding part of the proof of Proposition 2.6 (a).

(d) Assume to the contrary that u is p-intermittent but not q-intermittent for some $q > p$. Then, by the definition of p-intermittency and part (a) of this same proposition we have $\lambda_p < \lambda_{p+\varepsilon}$ for all $\varepsilon > 0$ and there exists $\varepsilon^* > 0$ such that $\lambda_q = \lambda_{q+\varepsilon^*}$. Fixing $\varepsilon := (q-p)/2 \wedge \varepsilon^*$, we get using the convexity statement of part (b) and $\lambda_p < \lambda_q$:

$$\begin{aligned} q\lambda_q &\le \frac{\varepsilon}{q+\varepsilon-p} p\lambda_p + \frac{q-p}{q+\varepsilon-p}(q+\varepsilon)\lambda_{q+\varepsilon} < \frac{\varepsilon}{q+\varepsilon-p} p\lambda_q + \frac{q-p}{q+\varepsilon-p}(q+\varepsilon)\lambda_{q+\varepsilon} \\ &= \frac{\varepsilon p/q}{q+\varepsilon-p} q\lambda_q + \frac{(q-p)(q+\varepsilon)/q}{q+\varepsilon-p} q\lambda_q = q\lambda_q, \end{aligned}$$

a contradiction. Hence, u must be q-intermittent as well. $\square$

Proof of Proposition 2.9. We first show that L_p^{sup} has a zero in $(0,\beta_{cr})$ for $p > 0$ large enough and then invoke Lemma 5.10 to conclude the proof.

To show the existence of such a zero, let $\mu \in \mathscr{M}_1([b,0])$ such that $H(\mu|\eta) < \infty$ and $\mu([-\beta_{cr}/3,0]) = 1$. Then due to (5.1) and Proposition 5.2 we have

$$\mathscr{I}(\mu^{\mathbb{N}_0}) = \lim_{n\to\infty} H(\mu^n|\mu^{n-1} \otimes \eta) = H(\mu|\eta) < \infty$$

as well as

$$L(\beta_{cr}/2, \mu^{\mathbb{N}_0}) = \int_{\Sigma_b^+} \log \mathbb{E}_1 \exp\Big\{ \int_0^{T_0} (\zeta(Y_s) + \beta_{cr}/2)\, ds \Big\} \mu^{\mathbb{N}_0}(d\zeta)$$
$$\geq \log \mathbb{E}_1 \exp\{(-\beta_{cr}/3 + \beta_{cr}/2) T_0\} > 0.$$

We deduce

$$L_p^{sup}(\beta_{cr}/2) \geq L(\beta_{cr}/2, \mu^{\mathbb{N}_0}) - \mathscr{I}(\mu^{\mathbb{N}_0})/p > 0$$

for $p > 0$ large enough, in which case L_p^{sup} has zero $-\lambda_p \in (0, \beta_{cr})$, cf. Theorem 2.7.

Lemma 5.10 now tells us that we find $\nu_p \in \mathscr{M}_1^s(\Sigma_b^+)$ with $L_p^{sup}(-\lambda_p) = L(-\lambda_p, \nu_p) - \mathscr{I}(\nu_p)/p$. Since Prob can be assumed to be non-degenerate, one can show that for p large enough we have $\nu_p \neq \text{Prob}$. We then have $\mathscr{I}(\nu_p) \in (0, \infty)$ and for $\varepsilon > 0$ we obtain

$$L_{p+\varepsilon}^{sup}(-\lambda_p) \geq L(-\lambda_p, \nu_p) - \mathscr{I}(\nu_p)/(p + \varepsilon) > L(-\lambda_p, \nu_p) - \mathscr{I}(\nu_p)/p = L_p^{sup}(-\lambda_p) = 0.$$

Therefore, $L_{p+\varepsilon}^{sup}$ has a zero in $(0, -\lambda_p)$, whence due to Theorem 2.7 we have $\lambda_{p+\varepsilon} > \lambda_p$ and u is p-intermittent. □

The following claim is employed in the proof of Theorem 2.10.

Claim. *For each neighbourhood U of* Prob $= \eta^{\mathbb{N}_0}$ *in $\mathscr{M}_1(\Sigma_b^+)$, there exists $\varepsilon > 0$ such that $\{\mathscr{I} \leq \varepsilon\} \subseteq U$.*

Proof. Indeed, if this was not the case, we would find an open neighbourhood U of Prob such that $\{\mathscr{I} \leq \varepsilon\} \nsubseteq U$ for all $\varepsilon > 0$. Now since $\mathscr{I}$ is a good rate function (cf. Corollary 6.5.15 in [DZ98]) $\{\mathscr{I} \leq \varepsilon\} \cap U^c$ is compact and non-empty whence there exists $\nu \in \mathscr{M}_1(\Sigma_b^+)$ with $\mathscr{I}(\nu) = 0$ and $\nu \notin U$. But due to Corollary 5.3, $\eta^{\mathbb{N}_0}$ is the only zero of $\mathscr{I}$, contradicting $\nu \notin U$. □

Proof of Theorem 2.10. The continuity on $(0, \infty)$ follows from Proposition 2.11 (b). It therefore remains to show the continuity in 0.

For this purpose, we first show that $L_p^{sup} \downarrow L$ pointwise as $p \downarrow 0$ on $(0, \beta_{cr})$.

Fix $\beta \in (0, \beta_{cr})$. Then $M := \sup_{\nu \in \mathscr{M}_1^s(\Sigma_b^+)} L(\beta, \nu) < \infty$ due to Corollary 5.7 and for $\varepsilon > 0$ we may therefore find a neighbourhood $U(\text{Prob})$ of Prob such that $|L(\beta, \nu) - L(\beta)| < \varepsilon$ for all $\nu \in U(\text{Prob})$. Choosing $\delta > 0$ small enough such that $\{\mathscr{I} \leq \delta\} \subset U(\text{Prob})$ (which is possible due to the above claim), we set $p_\varepsilon := \delta/(M - L(\beta))$. Then for $p \in (0, p_\varepsilon)$, we have $|L_p^{sup}(\beta) - L(\beta)| \leq \varepsilon$. This proves the above convergence.

The continuity of $p \mapsto \lambda_p$ in zero now follows from Theorems 2.1 and 2.7 where we may distinguish the cases that L does or does not have a zero in $(0, \beta_{cr})$. □

8 The case of maximal drift

In subsection 8.1 we will give the modifications necessary to adapt the proofs leading to the results of section 2 to the case $h = 1$.

Subsequently, in subsection 8.2 we will provide an alternative approach to establish the existence of the first annealed Lyapunov exponent using a modified subadditivity argument. By means of the

Laplace transform we will then retrieve an easy formula for the p-th annealed Lyapunov exponent for $p \in \mathbb{N}$.

Note that there have been some initial investigations of the first annealed Lyapunov exponent in the case $h = 1$ using a large deviations approach to establish its existence (cf. [Sch05]).

8.1 Modifications in proofs for maximal drift

As one may have noticed, some of the results and proofs given so far depended on h being strictly smaller than 1. Already Proposition 3.1 does not hold true anymore in the case of maximal drift. Indeed, with the previous definitions one computes

$$\beta_{cr} = \kappa \leq \kappa - \xi(0) = -c^*; \tag{8.1}$$

in particular, c^* is in general a non-degenerate random variable. On the other hand, in the case $h = 1$ we have the simple representations

$$L(\beta) = \left\langle \log \frac{\kappa}{\kappa - \xi(1) - \beta} \right\rangle \quad \text{and} \quad \Lambda(\beta) = \left\langle \log \frac{\kappa - \xi(1)}{\kappa - \xi(1) - \beta} \right\rangle, \quad \beta \in (-\infty, \kappa).$$

Notwithstanding these differences between the cases of $h = 1$ and $h \in (0,1)$, our main results are still valid in the case $h = 1$. To verify this, we make use of the identity

$$\mathbb{E}_0 \exp\left\{ \int_0^t \xi(Y_s)\, ds \right\} \mathbb{1}_{Y_t = 0} = \exp\{(-\kappa + \xi(0))t\}. \tag{8.2}$$

We will now exhibit the modifications necessary to derive the results of section 2.

The proof of Lemma 4.1 is as follows:

Proof. For $\alpha > 0$ and bearing in mind (8.1) and (8.2), the supremum on the right-hand side of (4.1) is obtained as in the case $h \in (0,1)$. For $\alpha = 0$ it evaluates to $-\beta_{cr} = -\kappa$ and in this case, choosing for arbitrary $\varepsilon > 0$ an $n \in \mathbb{N}$ such that $\xi(n) > -\varepsilon$ yields in combination with the Markov property applied at time $t - 1$:

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}_n \exp\left\{ \int_0^t \xi(Y_s)\, ds \right\} \mathbb{1}_{Y_t=0} \\
&\geq \liminf_{t\to\infty} \frac{1}{t} \log \Big(\mathbb{E}_n \exp\{\xi(n)t\} \mathbb{1}_{Y_{t-1}=n} \big(\min_{k\in\{0,\dots,n\}} \exp\{\xi(k)\} \mathbb{P}_n(Y_0 = n, Y_1 = 0) \big) \Big) \\
&= -\kappa - \varepsilon.
\end{aligned} \tag{8.3}$$

Since $\varepsilon > 0$ was chosen arbitrarily, this finishes the proof. ☐

Bearing in mind (8.1) and (8.2) again, the proof of Lemma 4.2 proceeds very similarly to the case $h \in (0,1)$; note that, as it will frequently be the case, the proof facilitates lightly since for $h = 1$ we do not have to consider the negative summands appearing in (2.3). This is also the reason why Lemma 4.3 is not required for $h = 1$. The proof of Lemma 4.4 does not depend on h at all, whence no modifications are required. With these results at hand, Corollary 4.5 is proven as before and the same applies to Theorem 2.1 and Corollary 2.3.

When turning to section 5, we note that Lemma 5.5 is not needed in the case $h=1$. Furthermore, for $h=1$ we note that $\beta_{cr}=\kappa$, whence Lemma 5.6 can be easily verified to hold true using

$$\mathbb{E}_1 \exp\Big\{ \int_0^{T_0} (\zeta(Y_s)+\beta)\,ds \Big\} = \frac{\kappa}{\kappa-\zeta(1)-\beta}, \quad \beta<\kappa,\ \zeta\in\Sigma_b^+.$$

With respect to section 6, we note that to derive Lemma 6.1 we just have to employ the relations (8.1) and (8.2) in the proof to obtain the same result. When it comes to Lemmas 6.2 and 6.3, we observe that the proof goes along similar lines but facilitates at different steps. But note that e.g. in (6.18) the infimum in β should be taken over $[c,\beta_{cr}-\delta]$ for some $\delta>0$ small enough since $L(\beta_{cr})$ might be infinite (whereas the quoted minimax theorem is applicable to real-valued functions only).

Corollary 6.6 and Lemma 5.11 are proven analogously, whence the same applies to Theorem 2.7.

8.2 Analysis of the maximal drift case

When considering annealed Lyapunov exponents for an i.i.d. medium, the situation that $h=1$ is much easier to analyse than the case of $h\in(0,1)$. This is the case since in this setting the independence of the medium yields a product structure for expressions such as

$$\Big\langle \mathbb{E}_0 \exp\Big\{ \int_0^{T_n} \xi(X_s)\,ds \Big\} \Big\rangle,$$

which evaluates to $\langle \kappa/(\kappa-\xi(0))\rangle^n$.

8.2.1 Additional derivations for the annealed regime

While in general even showing the mere existence of the Lyapunov exponents requires quite some effort, in the case of maximal drift and an i.i.d. potential, the existence of λ_1 can be retrieved by a modified subadditivity argument.

Lemma 8.1. *Let $f:\mathbb{R}_+\to\mathbb{R}$ be a continuous function fulfilling the following property: For all $\delta>0$ there exists $K_\delta>0$ such that for all $s,t\in\mathbb{R}_+$ we have*

$$f(s+t)\le K_\delta+\delta s+f(s)+f(t). \tag{8.4}$$

Then $\lim_{t\to\infty} f(t)/t$ exists in $[-\infty,\infty)$.

Proof. For t and T such that $0<t<T$ choose $n\in\mathbb{N}$ and $r\in[0,t)$ such that $T=nt+r$. We infer using (8.4) that

$$\begin{aligned}\frac{f(T)}{T} &\le \frac{1}{T}\big((K_\delta+\delta t+f(t))n+f(r)\big)\\ &\le \frac{1}{t}(K_\delta+\delta t+f(t))+\frac{f(r)}{T}.\end{aligned}$$

It follows that

$$\limsup_{T\to\infty}\frac{f(T)}{T}\le \liminf_{t\to\infty}\frac{f(t)}{t}+\delta<\infty$$

for all $\delta>0$ and thus $\lim_{t\to\infty} f(t)/t$ exists $[-\infty,\infty)$. □

When trying to apply this lemma to the function $t \to \log\langle u(t,0)\rangle$ we compute writing $H(t) := \log\langle e^{t\xi(0)}\rangle$, denoting by T_n the first hitting time of n by X, and employing the strong Markov property:

$$\begin{aligned}
\langle u(s+t,0)\rangle &= \sum_{n\in\mathbb{N}_0} \Big\langle \mathbb{E}_0 \exp\Big\{\int_0^{s+t} \xi(X_r)\,dr\Big\} \mathbb{1}_{T_n\le s<T_{n+1}}\Big\rangle \\
&\le \sum_{n\in\mathbb{N}_0} \mathbb{E}_0\Big(\Big\langle \exp\Big\{\int_0^{T_n} \xi(X_r)\,dr\Big\}\Big\rangle \Big\langle \mathbb{1}_{T_n\le s<T_{n+1}} \mathbb{E}_0\Big(\exp\Big\{\int_s^{s+t} \xi(X_u)\,du\Big\}\Big|\mathscr{F}_s\Big)\Big\rangle\Big) \\
&= \sum_{n\in\mathbb{N}_0} \mathbb{E}_0\Big(\Big\langle \exp\Big\{\int_0^{s} \xi(X_r)\,dr\Big\}\Big\rangle e^{-H(s-T_n)} \mathbb{1}_{T_n\le s<T_{n+1}} \Big\langle \mathbb{E}_n \exp\Big\{\int_0^{t} \xi(X_u)\,du\Big\}\Big\rangle\Big) \qquad (8.5)\\
&\le \sum_{n\in\mathbb{N}_0} \mathbb{E}_0\Big(\Big\langle \exp\Big\{\int_0^{s} \xi(X_r)\,dr\Big\}\Big\rangle \mathbb{1}_{T_n\le s<T_{n+1}}\Big) e^{-H(s)} \langle u(t,0)\rangle \\
&\le e^{K_\delta+\delta s} \langle u(s,0)\rangle \langle u(t,0)\rangle,
\end{aligned}$$

where to obtain the last line we used $0 \ge \frac{H(t)}{t} \to 0$ as $t \to \infty$, which implies that for all $\delta > 0$ there exists $K_\delta > 0$ such that $-H(t) \le K_\delta + \delta t$ for all $t > 0$. Taking logarithms on both sides of (8.5), Lemma 8.1 is applicable and yields the existence of λ_1. It is now promising to consider the Laplace transform

$$\mathbb{R} \ni \beta \mapsto \int_0^\infty e^{-\beta t} \langle u(t,0)\rangle\,dt; \qquad (8.6)$$

observe that λ_1 is given as the critical value of β for the divergence of this integral. By direct computation, the integral in (8.6) can be shown to equal

$$\frac{1}{\kappa} \sum_{n\in\mathbb{N}} \Big\langle \frac{\kappa}{\kappa+\beta-\xi(0)} \Big\rangle^n$$

for $\beta \ge -\kappa$, see also Lemma 3.2 in [Sch05]. Thus, given the existence of λ_1 and using (8.6), we observe that λ_1 is given as the zero of

$$\beta \mapsto \log\Big\langle \frac{\kappa}{\kappa+\beta-\xi(0)} \Big\rangle \qquad (8.7)$$

in $(-\kappa, 0)$ if this zero exists; otherwise, we conclude $\lambda_1 \le -\kappa$ and by considering realisations of X in (2.3) which stay at sites n with $\xi(n) \approx 0$ for nearly all the time, we may conclude $\lambda_1 \ge -\kappa$, cf. (8.3). Thus, we get $\lambda_1 = -\kappa$ in this situation. We have therefore proven the following proposition for $p = 1$:

Proposition 8.2. *Assume (1.2) as well as (2.9) to hold. Then for $h = 1$ and $p \in \mathbb{N}$, the p-th annealed Lyapunov exponent λ_p is given as the zero of*

$$\beta \mapsto \log\Big\langle \Big(\frac{\kappa}{\kappa+\beta-\xi(0)}\Big)^p \Big\rangle \qquad (8.8)$$

in $(-\kappa, 0)$ if this zero exists and $-\kappa$ otherwise.

Remark 8.3. While Theorem 2.7 yields the existence and implicit formulae for all λ_p, $p \in (0,\infty)$, simultaneously, Proposition 8.2 provides a nicer representation in the cases $p \in \mathbb{N}$ with $h = 1$.

Proof. While for $p = 1$ we showed how to employ a subadditivity argument to infer existence of λ_1, for general $p \in \mathbb{N}$ we now refer to Theorem 2.7 for this purpose. We can then use the Laplace transform again to deduce a more convenient representation of λ_p. For the sake of simplicity, we prove the proposition for $p = 2$ and give corresponding remarks where generalisations to arbitrary $p \in \mathbb{N}$ are not straightforward.

Denote by $X^{(1)}$ and $X^{(2)}$ two independent copies of X and by $\mathbb{P}_{0,0}$ and $\mathbb{E}_{0,0}$ we denote the probability and expectation, respectively, of these processes both starting in 0. Note that since $h = 1$, these are Poisson processes with intensity κ. We set $\tau_0^{(j)} := 0$, $\tau_k^{(j)} := T_k^{(j)} - T_{k-1}^{(j)}$ for $k \in \mathbb{N}$ and $j \in \{1,2\}$, where by T_k^j we denote the first hitting time of k by $X^{(j)}$. Note that the $\tau_k^{(j)}$ are i.i.d. exponentially distributed with parameter κ. We distinguish cases:

(i) Assume that for $p = 2$ the function of (8.8) has a zero in $(-\kappa, 0)$.

Using Hölder's inequality[4] we estimate for

$$\beta > -2\kappa : \tag{8.9}$$

$$\begin{aligned}
&\int_0^\infty e^{-\beta t} \langle u(t,0)^2 \rangle \, dt \\
&= \int_0^\infty e^{-\beta t} \sum_{m,n \in \mathbb{N}_0} \Big\langle \mathbb{E}_{0,0} \Big(\exp\Big\{ \sum_{k=1}^m \tau_k^{(1)} \xi(k-1) + (t - T_m^{(1)}) \xi(m) \Big\} \\
&\quad \times \exp\Big\{ \sum_{k=1}^n \tau_k^{(2)} \xi(k-1) + (t - T_n^{(2)}) \xi(n) \Big\} \mathbb{1}_{X_t^{(1)} = m} \mathbb{1}_{X_t^{(2)} = n} \Big) \Big\rangle \, dt \qquad (8.10) \\
&\leq \sum_{m,n \in \mathbb{N}_0} \Big\langle \int_0^\infty e^{-\beta t} \Big(\mathbb{E}_{0,0} \exp\Big\{ \sum_{k=1}^m \tau_k^{(1)} \xi(k-1) + (t - T_m^{(1)}) \xi(m) \Big\} \mathbb{1}_{X_t^{(1)} = m} \Big)^2 dt \Big\rangle^{\frac{1}{2}} \\
&\quad \times \Big\langle \int_0^\infty e^{-\beta t} \Big(\mathbb{E}_{0,0} \exp\Big\{ \sum_{k=1}^n \tau_k^{(2)} \xi(k-1) + (t - T_n^{(2)}) \xi(n) \Big\} \mathbb{1}_{X_t^{(2)} = n} \Big)^2 dt \Big\rangle^{\frac{1}{2}}.
\end{aligned}$$

[4]For arbitrary p we retreat to the generalised Hölder inequality with the p exponents $1/p, \ldots, 1/p$.

We now can estimate the diagonal summands as follows:

$$
\begin{aligned}
&\Big\langle \int_0^\infty e^{-\beta t}\mathbb{E}_{0,0}\Big(\exp\Big\{\sum_{k=1}^m (\tau_k^{(1)}+\tau_k^{(2)})\xi(k-1)\Big\} \exp\{(2t-T_m^{(1)}-T_m^{(2)})\xi(m)\}\mathbb{1}_{X_t^{(1)}=X_t^{(2)}=m}\Big)\,dt\Big\rangle \\
&= \Big\langle \int_0^\infty e^{-\beta t}\mathbb{E}_{0,0}\Big(\exp\Big\{\sum_{k=1}^m (\tau_k^{(1)}+\tau_k^{(2)})\xi(k-1) + (2t-T_m^{(1)}-T_m^{(2)})\xi(m)\Big\} \\
&\quad \times \exp\Big\{-\kappa(2t-T_m^{(1)}-T_m^{(2)})\Big\}\mathbb{1}_{T_m^{(1)}\le t}\mathbb{1}_{T_m^{(2)}\le t}\Big)\,dt\Big\rangle \\
&\le \mathbb{E}_{0,0}\Big\langle \int_0^\infty \exp\Big\{\sum_{k=1}^m (\tau_k^{(1)}+\tau_k^{(2)})(\xi(k-1)-\beta/2)\Big\} \\
&\quad \times \exp\big\{(2t-T_m^{(1)}-T_m^{(2)})(\xi(m)-\kappa-\beta/2)\big\}\mathbb{1}_{T_m^{(1)}+T_m^{(2)}\le 2t}\,dt\Big\rangle \\
&\overset{t\mapsto t+\frac{T_m^{(1)}+T_m^{(2)}}{2}}{=} \mathbb{E}_{0,0}\Big\langle \exp\Big\{\sum_{k=1}^m (\tau_k^{(1)}+\tau_k^{(2)})(\xi(k-1)-\beta/2)\Big\}\Big\rangle \underbrace{\Big\langle \int_0^\infty \exp\{2t(\xi(0)-\kappa-\beta/2)\}\,dt\Big\rangle}_{=:C<\infty,\text{ since }\beta>-2\kappa} \\
&= C\Big\langle\Big(\frac{\kappa}{\kappa+\beta/2-\xi(0)}\Big)^2\Big\rangle^m .
\end{aligned}
\tag{8.11}
$$

Hence, combining (8.10) and (8.11) we have

$$
\begin{aligned}
\int_0^\infty e^{-\beta t}\langle u(t,0)^2\rangle\,dt &\le C^2\sum_{m,n\in\mathbb{N}_0}\Big\langle\Big(\frac{\kappa}{\kappa+\beta/2-\xi(0)}\Big)^2\Big\rangle^{\frac{m+n}{2}} \\
&= C^2\Big(\sum_{m\in\mathbb{N}_0}\Big\langle\Big(\frac{\kappa}{\kappa+\beta/2-\xi(0)}\Big)^2\Big\rangle^{\frac{m}{2}}\Big)^2 .
\end{aligned}
\tag{8.12}
$$

For the lower bound we compute

$$\int_0^\infty e^{-\beta t}\langle u(t,0)^2\rangle\,dt \geq \sum_{m\in\mathbb{N}_0}\int_0^\infty \Big\langle \mathbb{E}_{0,0}\exp\Big\{\sum_{k=1}^m(\tau_k^{(1)}+\tau_k^{(2)})(\xi(k-1)-\beta/2)\Big\}$$

$$\times\exp\big\{(2t-T_m^{(1)}-T_m^{(2)})(\xi(m)-\beta/2)\big\}\mathbb{1}_{X_t^{(1)}=X_t^{(2)}=m}\Big\rangle\,dt$$

$$=\sum_{m\in\mathbb{N}_0}\Big\langle \mathbb{E}_{0,0}\Big(\int_0^\infty \exp\Big\{\sum_{k=1}^m(\tau_k^{(1)}+\tau_k^{(2)})(\xi(k-1)-\beta/2)\Big\}$$

$$\times\exp\big\{(2t-T_m^{(1)}-T_m^{(2)})(\xi(m)-\kappa-\beta/2)\big\}\mathbb{1}_{T_m^{(1)}\leq t}\mathbb{1}_{T_m^{(2)}\leq t}\,dt\Big)\Big\rangle$$

$$\overset{t\mapsto t+\frac{T_m^{(1)}+T_m^{(2)}}{2}}{=}\sum_{m\in\mathbb{N}_0}\Big\langle \mathbb{E}_{0,0}\Big(\int_0^\infty \exp\Big\{\sum_{k=1}^m(\tau_k^{(1)}+\tau_k^{(2)})(\xi(k-1)-\beta/2)\Big\}$$

$$\times\exp\{2t(\xi(m)-\kappa-\beta/2)\}\mathbb{1}_{\frac{T_m^{(1)}-T_m^{(2)}}{2}\leq t}\mathbb{1}_{\frac{T_m^{(2)}-T_m^{(1)}}{2}\leq t}\,dt\Big)\Big\rangle \tag{8.13}$$

$$\overset{t\mapsto t+\frac{|T_m^{(1)}-T_m^{(2)}|}{2}}{=}\sum_{m\in\mathbb{N}_0}\Big\langle \mathbb{E}_{0,0}\Big(\int_0^\infty \exp\Big\{\sum_{k=1}^m(\tau_k^{(1)}+\tau_k^{(2)})(\xi(k-1)-\beta/2)\Big\}$$

$$\times\exp\big\{(2t+|T_m^{(1)}-T_m^{(2)}|)(\xi(m)-\kappa-\beta/2)\big\}\,dt\Big)\Big\rangle$$

$$\geq\sum_{m\in\mathbb{N}_0}\Big\langle \mathbb{E}_{0,0}\Big(\mathbb{1}_{|T_m^{(1)}-T_m^{(2)}|\leq m\delta}\exp\Big\{\sum_{k=1}^m(\tau_k^{(1)}+\tau_k^{(2)})(\xi(k-1)-\beta/2)\Big\}\Big)\Big\rangle$$

$$\times\Big\langle\exp\{m\delta(\xi(m)-\kappa-\beta/2)\}\int_0^\infty\exp\{2t(\xi(m)-\kappa-\beta/2)\}\,dt\Big\rangle. \tag{8.14}$$

Note here that for arbitrary $p\in\mathbb{N}$ the indicators appearing in (8.13) are replaced by

$$\prod_{j=1}^p\mathbb{1}_{T_m^{(j)}-\frac{\sum_{1\leq k\leq p}T_m^{(k)}}{p}\leq t}$$

which can be estimated from below by

$$\mathbb{1}_{\max_{1\leq j,k\leq p}|T_m^{(j)}-T_m^{(k)}|\leq t}.$$

The subsequent substitution can duely be replaced by

$$t\mapsto t+\frac{\max_{1\leq j,k\leq p}|T_m^{(j)}-T_m^{(k)}|}{p},$$

and the remaining steps are analogous to $p=2$.

Now we continue (8.14) with $p=2$ and bearing in mind that $\beta>-2\kappa$, we estimate the right-hand side factor using Jensen's inequality to get

$$\Big\langle \exp\{m\delta(\xi(m)-\kappa-\beta/2)\}\int_0^\infty \exp\{2t(\xi(m)-\kappa-\beta/2)\}\,dt\Big\rangle$$
$$\geq \frac{1}{2}\Big\langle \underbrace{\exp\{\xi(m)-\kappa-\beta/2\}^{\delta}}_{\to 1 \text{ Prob-a.s. as } \delta\downarrow 0} \underbrace{\Big(\frac{1}{\kappa+\beta/2-\xi(m)}\Big)^{\frac{1}{m}}}_{\to 1 \text{ Prob-a.s. as } m\to\infty}\Big\rangle^{m} \tag{8.15}$$
$$\geq \frac{(1-\varepsilon)^m}{2},$$

where the last inequality holds for arbitrary $\varepsilon > 0$ and all $m \geq m_{\delta,\varepsilon}$ large enough. Writing $H(t) := \log\langle e^{t\xi(0)}\rangle$ again, we obtain combining (8.14) and (8.15):

$$\int_0^\infty e^{-\beta t}\langle u(t,0)^2\rangle\,dt \geq \frac{1}{2}\sum_{m\geq m_{\delta,\varepsilon}}(1-\varepsilon)^m \mathbb{E}_{0,0}\Big(\mathbb{1}_{|T_m^{(1)}-T_m^{(2)}|\leq m\delta}$$
$$\times \exp\Big\{\sum_{k=1}^{m}\big(H(\tau_k^{(1)}+\tau_k^{(2)})-\beta/2(\tau_k^{(1)}+\tau_k^{(2)})\big)\Big\}\Big).$$

Next we define

$$\hat{\mathbb{P}}_m(A) := \frac{\mathbb{E}_{0,0}(\exp\{\sum_{k=1}^m H(\tau_k^{(1)}+\tau_k^{(2)})-\beta/2(\tau_k^{(1)}+\tau_k^{(2)})\}\mathbb{1}_A)}{\underbrace{\mathbb{E}_{0,0}\big(\exp\big\{\sum_{k=1}^{m} H(\tau_k^{(1)}+\tau_k^{(2)})-\beta/2(\tau_k^{(1)}+\tau_k^{(2)})\}\big)}_{=\langle(\frac{\kappa}{\kappa+\beta/2-\xi(0)})^2\rangle^m<\infty \text{ as } \beta>-2\kappa, \text{ cf. } (8.9)}}$$

for $m\in\mathbb{N}$ and measurable A. Now $(\tau_k^{(1)}-\tau_k^{(2)})_{k\in\{1,\dots,m\}}$ have mean 0 and are square integrable and i.i.d. with respect to $\hat{\mathbb{P}}_m$. Thus, a weak law of large numbers supplies us with

$$\int_0^\infty e^{-\beta t}\langle u(t,0)^2\rangle\,dt \geq \sum_{m\geq m_{\delta,\varepsilon}}\hat{\mathbb{P}}_m(|T_m^{(1)}-T_m^{(2)}|\leq\delta m)\Big\langle\Big(\frac{\kappa}{\kappa+\beta/2-\xi(0)}\Big)^2\Big\rangle^m \tag{8.16}$$
$$\times\frac{(1-\varepsilon)^m}{2}$$
$$\geq\frac{1}{4}\sum_{m\geq m_{\delta,\varepsilon}}\Big(\Big\langle\Big(\frac{\kappa}{\kappa+\beta/2-\xi(0)}\Big)^2\Big\rangle(1-\varepsilon)\Big)^m, \tag{8.17}$$

where we choose $m_{\delta,\varepsilon}$ large enough such that $\hat{\mathbb{P}}_m(|T_m^{(1)}-T_m^{(2)}|\leq\delta m)\geq 1/2$ for all $m\geq m_{\delta,\varepsilon}$ due to the law of large numbers.

Since $\varepsilon>0$ was chosen arbitrarily, we infer combining (8.12) and (8.17) that λ_2 equals the zero of

$$\beta\mapsto\log\Big\langle\Big(\frac{\kappa}{\kappa+\beta-\xi(0)}\Big)^2\Big\rangle.$$

(ii) Now assume that for $p=2$ the function of (8.8) does not have a zero in $(-\kappa,0)$.

Again, considering realisations of X in (2.3) which stay at sites n with $|\xi(n)|$ small for nearly all the time, we arrive at $\lambda_2\geq-\kappa$, cf. (8.3). But from (8.12) we infer $\lambda_2\leq-\kappa$ if (8.8) does not have a zero in $(-\kappa,0)$ for $p=2$, which finishes the proof. □

It is inherent to the approach along which we proved Proposition 8.2 that it applies to natural p only. Nevertheless, we expect the formula to hold true for general $p \in (0, \infty)$.

Conjecture 8.4. *Assume (1.2) as well as (2.9) to hold. Then for $h = 1$ and $p \in (0, \infty)$, the p-th annealed Lyapunov exponent λ_p is given as the zero of*

$$\beta \mapsto \log \left\langle \left(\frac{\kappa}{\kappa + \beta - \xi(0)} \right)^p \right\rangle$$

in $(-\kappa, 0)$ if this zero exists and $-\kappa$ otherwise.

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