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UNIQUENESS FOR THE SKOROKHOD EQUATION WITH NORMAL REFLECTION IN LIPSCHITZ DOMAINS

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Uniqueness for the Skorokhod equation with normal reflection in Lipschitz domains

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1. Introduction.

We consider the Skorokhod equation in a domain $D \subseteq \mathbb{R}^d$, $d \geq 2$:

$$dX_t = dW_t + \frac{1}{2}\nu(X_t)\, dL_t, \qquad X_0 = x_0, \tag{1.1}$$

where W_t is d-dimensional Brownian motion, L_t is the local time of X_t on the boundary of D, and ν is the inward pointing unit normal vector. It is well-known that in smooth domains X_t is reflecting Brownian motion with normal reflection.

There are various types of solutions to (1.1). Pathwise existence and uniqueness holds for (1.1) when the domain D is a C^2 domain. This was proved by Lions and Sznitman [LS]. In fact they considered domains slightly more general than C^2, but the class of domains they considered does not contain the class of $C^{1+\alpha}$ domains for any $\alpha \in (0,1)$. They also considered more general diffusion coefficients and considered oblique reflection as well as normal reflection. Their work was generalized by Dupuis and Ishii [DI], who considered C^1 domains, but required the angle of reflection to vary in almost a C^2 manner. For normal reflection, this implies the domains must be nearly C^2.

Another type of uniqueness is weak uniqueness. That means that there exist processes X_t and W_t satisfying (1.1) where W_t is a Brownian motion, but that X need not be measurable with respect to the σ-fields generated by W. In [BH1] reflecting Brownian motion in bounded Lipschitz domains with normal reflection was constructed using Dirichlet forms, and in [BH2] and [FOT], Ex. 5.2.2, it was shown that this process provides a weak solution to the Skorokhod equation. These results were extended in [FT].

Closely related to weak uniqueness is the submartingale problem of Stroock and Varadhan [SV1]. They proved existence and uniqueness of the submartingale problem corresponding to (1.1) with more general diffusion coefficients and with oblique reflection for C^2 domains.

Using Dirichlet forms techniques, Williams and Zheng [WZ] constructed reflecting Brownian motion that provides a weak solution to (1.1) for domains more irregular than Lipschitz domains. Further research along these lines was done by [CFW] and [C]. Uniqueness of reflecting Brownian motion corresponding to the Dirichlet form for Brownian motion can be proved for quite general domains by the techniques of [F].

In this paper we prove weak uniqueness of (1.1) for Lipschitz domains. We prove that there is only one probability measure $\mathbb{P}$ under which W_t is a d-dimensional Brownian motion, X_t spends 0 time on the boundary, L_t is the local time of X_t on the boundary (defined as a limit of occupation times), and (1.1) holds. See Theorem 2.2 for a precise statement.

The question of weak uniqueness is a natural one. In problems of weak convergence, (e.g., in proving convergence of penalty methods as in [LS]) one is led to solutions to the Skorokhod equation. If one knew *a priori* that the solution was associated to a Dirichlet form, the uniqueness would be easy, but in general one does not know in advance that the solution corresponds to a Dirichlet form or even that the solution is strong Markov. Submartingale problems are also a natural class to consider, but in Lipschitz domains there is considerable difficulty in formulating them; typically, the class of test functions one would want to consider is empty.

In Section 2 we give definitions and recall a few facts about the reflecting Brownian motion constructed in [BH1]. We also prove a few preliminary propositions.

The reason problems of weak uniqueness tend to be hard is the paucity of the right type of functions; this is also the reason problems involving Lipschitz domains are typically much harder than those involving smoother domains. Section 3 is devoted to constructing a sequence of functions satisfying certain conditions. An estimate of Dahlberg on harmonic measure and one of Jerison and Kenig for solutions to the Neumann problem play key roles.

Section 4 contains the proof of weak uniqueness for (1.1) for Lipschitz domains. The main idea is to show that any two solutions must have the same potentials.

In Section 5 we pose a question about the existence of strong solutions. An affirmative answer would imply that in fact pathwise uniqueness holds for (1.1) in Lipschitz domains. At the present time pathwise uniqueness is not known even for $C^{1+\alpha}$ domains in the plane.

2. Preliminaries.

Notation. We let $B(x,r)$ denote the open ball of radius r centered at x. The letter c with subscripts will denote constants; we begin renumbering anew at each proposition or theorem. Points $x=(x_1,\ldots,x_d)$ will sometimes be written $(\widetilde{x},y)$, where $\widetilde{x}=(x_1,\ldots,x_{d-1})\in\mathbb{R}^{d-1}$ and $y=x_d$. We will also use polar coordinates: $x=(r,\theta)$, where $r=|x|$ and $\theta=x/|x|\in\partial B(0,1)$. The inner product in $\mathbb{R}^d$ of x and y is written $x\cdot y$.

For a domain D with $x\in\partial D$, the boundary of D, we let $\nu(x)$ be the inward pointing normal vector and $\nu_o(x)=-\nu(x)$ the outward pointing normal vector. We write $\sigma(dx)$ for surface measure on ∂D.

Lipschitz domains. A function $f:\mathbb{R}^{d-1}\to\mathbb{R}$ or $f:\partial B(0,1)\to\mathbb{R}$ is Lipschitz if there exists M such that $|f(x)-f(y)|\le M|x-y|$ for all x,y in the domain of f. The smallest such M is the Lipschitz constant of f. A domain D is a Lipschitz domain if for all $z\in\partial D$ there exists a coordinate system CS_z, an $r_z>0$, and a Lipschitz function Γ_z such that

$$D\cap B(z,r_z)=\{x=(\widetilde{x},y)\text{ in }CS_z:y>\Gamma_z(\widetilde{x})\}\cap B(z,r_z),$$

i.e., locally D looks like the region above the graph of a Lipschitz function. A Lipschitz domain is star-like (relative to 0) if there exists a Lipschitz function $\varphi:\partial B(0,1)\to(0,\infty)$ such that $D=\{(r,\theta):0\le r<\varphi(\theta)\}$.

For each $z \in \partial D$, where D is a star-like Lipschitz domain, we let $V_\beta(z)$ denote the interior of the convex hull of $\{z\} \cup B(0, \beta)$. We fix β small enough so that $V_{2\beta}(z) \subseteq D$ for all $z \in \partial D$. When we need to emphasize the domain we write $V_\beta^D(z)$. Let

$$U_r = \{x \in D : \mathrm{dist}\,(x, \partial D) < r\}. \tag{2.1}$$

If u is a function on D, we let

$$N(u)(z) = N^D(u)(z) = \sup_{x \in V_\beta(z)} |u(x)|$$

and

$$N_r(u)(z) = N_r^D(u)(z) = \sup\{|u(x)| : x \in V_\beta(z) \cap U_r\}.$$

L^2 norms with respect to surface measure on ∂D will be denoted $\|f\|_{2,\partial D}$. Thus

$$\|f\|_{2,\partial D}^2 = \int_{\partial D} |f(z)|^2 \, \sigma(dz).$$

We will first prove our results for star-like Lipschitz domains, and then extend them to general Lipschitz domains. Let us describe the special set-up that we first consider.

(2.2) *Let D be a star-like Lipschitz domain, let $\rho < (\inf \varphi)/4$, and let $K = \overline{B(0, \rho)}$. We will consider open subsets G of $\partial B(0, 1)$ and we consider the corresponding open subsets $A = \varphi(G)$ of ∂D:*

$$A = \varphi(G) = \{(r, \theta) : r = \varphi(\theta), \theta \in G\}.$$

Reflecting Brownian motion. In this subsection let us suppose the dimension d is greater than or equal to 3. Let D be a Lipschitz domain with K a compact set contained in D such that K has smooth boundary. In [BH1] a strong Markov process $(\mathbb{Q}^x, X_t)$, $x \in \overline{D}$, was constructed that represents reflecting Brownian motion in $\overline{D}$ with absorption at K. We recall a few properties and derive some others. See [BH1] for details. Let

$$T_A = T(A) = \inf\{t > 0 : X_t \in A\}.$$

Reflecting Brownian motion in D has a Green function $g(x, y)$ that is symmetric in x and y for $x, y \in D - K$, harmonic in y in $D - K - \{x\}$, harmonic in x in $D - K - \{y\}$, vanishes as x or y tends to the boundary of K, and there exists c_1 depending only on D and K such that

$$g(x, y) \le c_1 |x - y|^{2-d}. \tag{2.3}$$

If D is star-like, the constant c_1 depends only on ρ, $\|\nabla\varphi\|_\infty$, $\inf\varphi$, and $\sup\varphi$. In particular, for each $\rho' > 0$, $g(x,\cdot)$ is bounded in $\overline{D} - K - B(x,\rho')$.

A consequence of (2.3) is that

$$\mathbb{E}^x T_K = \int_{\overline{D}-K} g(x,y)\,dy \le c_2,, \qquad x \in \overline{D}. \tag{2.4}$$

Another consequence of (2.3) is that

$$\mathbb{E}^x \int_0^{T_K} 1_{\overline{U}_r}(X_s)\,ds = \int_{\overline{U}_r} g(x,y)\,dy \to 0$$

as $r \to 0$, so X_t spends zero time in ∂D, and hence starting at $x \in \partial D$, the process leaves ∂D immediately.

In [BH1] it is proved that there exists a continuous additive functional L_t corresponding to the measure $\sigma(dy)$:

$$\mathbb{E}^x L_{T_K} = \int_{\partial D} g(x,y)\,\sigma(dy), \qquad x \in \overline{D},$$

and L_t increases only when X_t is in the support of σ, namely ∂D. It follows from (2.3) that $\mathbb{E}^x L_{T_K} \le c_3$, $x \in \overline{D}$, where c_3 depends on the domain D. When D is star-like, c_3 depends on ρ, $\|\nabla\varphi\|_\infty$, $\inf\varphi$, and $\sup\varphi$. Suppose f_m are nonnegative bounded functions supported in $D - K$ such that $f_m(y)\,dy$ converges weakly to $\sigma(dy)$ (this is the usual weak convergence of measures in probability theory, except that we do not assume the total mass is one) and also that there exist $c_4 > 0$ and $\gamma \in [0,1)$ such that

$$\int_{B(x,s)\cap\overline{D}} f_m(y)\,dy \le c_4(s \wedge 1)^{d-1-\gamma}, \qquad x \in D, s > 0. \tag{2.5}$$

An example of f_m satisfying (2.5) is $f_m(y) = a_m^{-1} 1_{U_{1/m}}(y)$, where a_m is the Lebesgue measure of $U_{1/m}$. If the f_m satisfy (2.5), we have by the proof of [BK], Section 2, that $\int g(x,y) f_m(y)\,dy \to \int g(x,y)\,\sigma(dy)$ uniformly in x. Let

$$A_m(t) = \int_0^t f_m(X_s)\,ds. \tag{2.6}$$

By [BK], Section 2,

$$\sup_{t \le T_K} |A_m(t) - L_t| \to 0$$

in probability as $m \to \infty$.

Suppose D is star-like, φ is smooth, $x_0 \in \overline{D}$, and $B = \overline{(D-K)} \cap B(x_0, r)$ for some $r > 0$. Then $h(x) = \mathbb{E}^x f(X_{T(B^c)})$ is harmonic in $D \cap B$ and has normal derivative 0 on

$(\partial D) \cap B$. There exist c_5 and α depending only on ρ, the Lipschitz constant of φ, the supremum and infimum of φ, and r such that

$$|h(x) - h(y)| \leq c_5 |x-y|^\alpha \|f\|_\infty, \qquad x, y \in \overline{(D-K)} \cap B(x_0, r/2). \tag{2.7}$$

Reflecting Brownian motion satisfies a tightness estimate similar to that of ordinary Brownian motion. By [BH1] there exist c_6 and c_7 such that if $x \in \overline{D}$ and $r > 0$,

$$\mathbb{P}^x(\sup_{s \leq t} |X_s - x| \geq \lambda) \leq c_6 e^{-c_7 \lambda^2/t}. \tag{2.8}$$

Lemma 2.1. *Suppose D is star-like and $\varepsilon, \eta > 0$. There exists δ depending only on $\varepsilon, \eta, \|\nabla\varphi\|_\infty, \sup\varphi$, and $\inf\varphi$ such that if $\operatorname{dist}(x, \partial D) < \delta$, then*

$$\mathbb{Q}^x(T_{\partial B(x,\eta)} < T_{\partial D}) < \varepsilon.$$

Proof. Let $r = \operatorname{dist}(x, \partial D)$. Since D is Lipschitz, there exists $c_1 > 0$ depending only on the Lipschitz constant of φ such that if $s \geq r$, then the surface measure of $\partial B(x, 2s) \cap D^c$ is greater than c_1 times the surface measure of $\partial B(x, 2s)$. Since the law of X_t up until time $T_{\partial D}$ is the same as that of standard d-dimensional Brownian motion and the distribution of Brownian motion on exiting a ball is uniform on the surface of the ball,

$$\mathbb{Q}^x(T_{\partial B(x,2r)} < T_{\partial D}) \leq 1 - c_1.$$

Any point y in $\partial B(x, 2r)$ is a distance $2r$ from x and hence no more than $3r$ from D^c. So if $y \in D \cap \partial B(x, 2r)$, the same reasoning tells us

$$\mathbb{Q}^y(T_{\partial B(y,6r)} < T_{\partial D}) \leq 1 - c_1.$$

By the strong Markov property and the fact that $B(y, 6r) \subseteq B(x, 8r)$ if $|y - x| = 2r$,

$$\mathbb{Q}^x(T_{\partial B(x,8r)} < T_{\partial D}) \leq (1 - c_1)^2.$$

We repeat the argument. A point in $\partial B(x, 8r)$ is a distance no more than $9r$ from D^c, and a ball of radius $18r$ about such a point is contained in $B(x, 26r)$, so using the strong Markov property,

$$\mathbb{Q}^x(T_{\partial B(x,26r)} < T_{\partial D}) \leq (1 - c_1)^3.$$

We continue by induction and obtain

$$\mathbb{Q}^x(T_{\partial B(x,(3^m-1)r)} < T_{\partial D}) \leq (1 - c_1)^m.$$

Now choose m so that $(1-c_1)^m < \varepsilon$ and then choose δ so that $(3^m - 1)\delta < \eta$. $\square$

Skorokhod equation. We now suppose that $d \geq 2$. In [BH2] and [FOT], Ex. 5.2.2, it was shown that the $(\mathbb{Q}^x, X_t)$ constructed in [BH1] satisfy the Skorokhod equation: there exists a d-dimensional Brownian motion W_t such that

$$dX_t = dW_t + \frac{1}{2}\nu(X_t)\,dL_t. \tag{2.9}$$

We want to show that the solution to (2.9) is unique in law. To be precise, let D be an arbitrary Lipschitz domain. We say that

(2.10) *a probability measure $\mathbb{P}$ is a solution to the Skorokhod equation (2.9) starting from $x_0 \in \overline{D}$ if*

(a)
$$\mathbb{P}(X_0 = x_0) = 1,$$

(b)
$$\int_0^\infty 1_{\partial D}(X_s)\,ds = 0,$$

(c) there exist nonnegative functions f_m with support in D such that $f_m(y)\,dy$ converges weakly to $\sigma(dy)$, the f_m satisfy (2.5), and for each t_0 we have

$$\sup_{t \leq t_0} |A_m(t) - L_t)| \to 0$$

in $\mathbb{P}$-probability as $m \to \infty$, where the A_m are defined by (2.6), and
(d) there exists a continuous process W_t which under $\mathbb{P}$ is a d-dimensional Brownian motion with respect to the filtration of X such that for all t, $X_t \in \overline{D}$ and

$$X_t - X_0 = W_t + \frac{1}{2}\int_0^t \nu(X_s)\,dL_s.$$

By our discussion above there exists at least one solution to (2.10), namely Q^{x_0}. Saying that W_t is a Brownian motion with respect to the filtration generated by X_t means that $W_t - W_s$ has the same distribution as that of a normal random variable with mean 0 and variance $t-s$ and $W_t - W_s$ is independent of $\sigma(X_r; r \leq s)$ whenever $s < t$.

Our main result is the following.

Theorem 2.2. *If D is a Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$, then there is exactly one solution to (2.10).*

The proof of Theorem 2.2 will take up Sections 3 and 4.

The condition (2.10)(c) is slightly stronger than the one sometimes seen in the literature, namely, that L_t be a nondecreasing continuous process that increases only when $X_t \in \partial D$. Here we are essentially requiring the local time L_t to be an additive functional corresponding to surface measure on the boundary.

We will need the following proposition. Let θ_t be shift operators so that $X_s \circ \theta_t = X_{s+t}$. By [B], Section I.2, we may always suppose such θ_t exist.

Proposition 2.3. *Let $\mathbb{P}$ be a solution to (2.10) started at $x_0 \in \overline{D}$, let S be a finite stopping time, and let $\mathbb{P}_S(\omega, d\omega')$ be a regular conditional probability for the law of $X_\cdot \circ \theta_S$ under $\mathbb{P}[\cdot \mid \mathcal{F}_S]$. Then $\mathbb{P}$-almost surely, $\mathbb{P}_S$ is a solution to (2.10) started at $X_S(\omega)$.*

Proof. The proof is standard. Let $A(\omega) = \{\omega' : X_0(\omega') = X_S(\omega)\}$. Then

$$A(\omega) \circ \theta_S = \{\omega' : X_0 \circ \theta_S(\omega') = X_S(\omega)\} = \{\omega' : X_S(\omega') = X_S(\omega)\}.$$

So

$$\mathbb{P}(A(\omega) \circ \theta_S \mid \mathcal{F}_S) = 1_{\{X_S(\omega)\}}(X_S) = 1, \qquad \text{a.s.}$$

If $B = \{L_t \text{ is the uniform limit of the } A_m(t)\}$, then

$$B \circ \theta_S = \{L_{t+S} - L_S \text{ is the uniform limit of } A_m(t+S) - A_m(S)\},$$

and so $\mathbb{P}(B \circ \theta_S \mid \mathcal{F}_S) = 1$, a.s. The proof that the process spends 0 time on the boundary under $\mathbb{P}_S$ is similar.

Finally, the law of $[X_t - X_0 - \frac{1}{2}\int_0^t \nu(X_s)\, dL_s] \circ \theta_S$ given $\mathcal{F}_S$ is the law of $[X_{t+S} - X_S - \frac{1}{2}\int_S^{S+t} \nu(X_s)\, dL_s]$ given $\mathcal{F}_S$. This is a Brownian motion by the strong Markov property of Brownian motion. $\square$

3. Some analytic estimates.

We suppose throughout this section that the dimension d is greater than or equal to 3. We start with an estimate on the normal derivative for a mixed boundary problem. We consider standard reflecting Brownian motion $(\mathbb{Q}^x, X_t)$ in D, and we kill this process on hitting K. Fix a point $x_0 \in D - K$ and choose ρ' small enough so that $\operatorname{dist}(x_0, \partial(D-K)) > 4\rho'$.

Proposition 3.1. *Suppose D satisfies (2.2) and in addition φ is C^∞. Suppose G consists of m components such that if $A = \varphi(G)$, then $\sigma(\overline{A} - A) = 0$. Let $g(\cdot)$ be the Green function for X_t killed on hitting $K \cup A$ with pole at x_0. Then $(\partial g/\partial \nu)(y)$ exists at almost every point of ∂D (with respect to σ) and there exists c_1 such that*

$$\int_A \Big(\frac{\partial g}{\partial \nu}(y)\Big)^2 \sigma(dy) \le c_1.$$

c_1 depends on $\rho, \rho', \sup\varphi, \inf\varphi$, and the Lipschitz constant of φ but does not otherwise depend on A. In particular, c_1 does not depend on m.

Proof. The function g is harmonic in $D - K - \{x_0\}$. By standard results from PDE on the solution to the Dirichlet problem (see [GT], Sections 6.3 and 6.4), g can be extended to be C^∞ at every point of A; this means that every point in A has a neighborhood in whose intersection with $\overline{D}$ the function g is C^∞. By standard results on the solution to the Neumann problem (see [GT], Section 6.7), g has a smooth extension up to the boundary in a neighborhood of each point in $\partial D - \overline{A}$. We make no claims at points in $\overline{A} - A$, but this set has surface measure 0.

Let us make the following assumptions about G and D. We will show they can be removed at the end of the proof. First we assume that each of the components of G has a piecewise smooth boundary (considered as a subset of the sphere $\partial B(0,1)$).

Let M be the Lipschitz constant of φ. Let $\theta_1 = (0, \dots, 0, -1)$. Choose r small (depending only on M, $\sup\varphi$, and $\inf\varphi$) such that there exists a Lipschitz function $\Gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ whose Lipschitz constant is less than $2M$ and with the property that the intersection of $B(\varphi(\theta_1), r)$ with the region above the graph of Γ is the same as $D \cap B(\varphi(\theta_1), r)$. Our second assumption is that $G \subseteq \partial B(0,1) \cap B(\theta_1, r)$.

If H is a $(d-1)$-dimensional hyperplane, let H^+ be the half space that contains $(\widetilde{0}, y)$ for all y sufficiently large. We want to be able to apply Green's identities in $D - K - B(x_0, \rho)$ with the function g, so to do so, we make the following assumption on D for now:

(3.1) *For each component E_i of G, there exists $\varepsilon_i > 0$ and a hyperplane H_i such that $\{\varphi(\theta) \in \partial D : \mathrm{dist}\,(\theta, E_i) \in (0, \varepsilon_i)\}$ is contained in H_i and $\varphi(\theta)$ lies in H_i^+ if $\mathrm{dist}\,(\theta, E_i^c) \in (0, \varepsilon_i)$.*

Consider the domain $C_i = \{(r, \theta) \in D - K : \mathrm{dist}\,(\theta, \partial E_i) < \varepsilon_i/2\}$. If we let C_i^R be the reflection of $C_i \cap H_i^+$ across H_i and let C_i^* be the interior of $\overline{C_i \cap H_i^+} \cup \overline{C_i^R}$, then C_i^* has Lipschitz boundaries. By the reflection principle, g may be extended across H_i. By Dahlberg's theorem ([B], Section III.5), $\partial g/\partial\nu$ is in L^2 with respect to surface measure on ∂C_i^*, from which is follows that $\partial g/\partial \nu$ is in L^2 with respect to surface measure on $\partial D \cap \{\varphi(\theta) : \mathrm{dist}\,(\theta, E_i^c) \in (0, \varepsilon_i/2)\}$.

We do this for each component, and conclude that we may apply Green's identities to the function g in $D - K - B(x_0, 2\rho')$. We can therefore conclude that $\partial g/\partial\nu$ is the density of harmonic measure on $D - K$ started at x_0; this may be proved exactly as in [B], pp. 217-218.

Let $S = \partial D \cap \overline{B(\varphi(\theta_1), r)}^c$. Let $F = D - B(0, 2\rho) - B(x_0, 2\rho')$ and let $y_0 \in B(x_0, 4\rho') - B(x_0, 2\rho')$. Since g is bounded and harmonic in F, then $|\nabla g|$ is bounded on $\partial B(0, 2\rho)$ and on $\partial B(x_0, 2\rho')$. By the PDE results mentioned in the first paragraph, $|\nabla g|$ is also bounded in a neighborhood of points of S. Let $(\mathbb{Q}^x, X_t)$ be the reflecting Brownian motion constructed in [BH1] and discussed in Section 2. Since $\partial g/\partial y$ is a harmonic function, we have by Doob's optional stopping theorem

$$\frac{\partial g}{\partial y}(y_0) = \mathbb{E}^{y_0} \frac{\partial g}{\partial y}(X_{T(\partial F)}). \tag{3.2}$$

By the fact that Γ is a Lipschitz curve (with Lipschitz constant $2M$) there exists c_2 depending only on M such that the ratio of $\partial g/\partial\nu$ to $\partial g/\partial y$ is bounded above by c_2 for $y \in A$. We have by the definition of harmonic measure that

$$\int_A \Big(\frac{\partial g}{\partial \nu}(z)\Big)^2 \sigma(dz) = \mathbb{E}^{x_0}\Big[\frac{\partial g}{\partial \nu}(X_{T(\partial F)}); X_{T(\partial F)} \in A\Big]. \tag{3.3}$$

Since $g \geq 0$ in D and $g = 0$ in A, then $\partial g/\partial y \geq 0$ in A. The function

$$z \mapsto \mathbb{E}^z\Big[\Big(\frac{\partial g}{\partial \nu} 1_A\Big)(X_{T(\partial F)})\Big]$$

is harmonic, so by Harnack's inequality, there exists a c_3 such that

$$\mathbb{E}^{x_0}\Big[\frac{\partial g}{\partial \nu}(X_{T(\partial F)}); X_{T(\partial F)} \in A\Big] \leq c_3 \mathbb{E}^{y_0}\Big[\frac{\partial g}{\partial \nu}(X_{T(\partial F)}); X_{T(\partial F)} \in A\Big]. \tag{3.4}$$

Combining (3.2)-(3.4),

$$\begin{aligned}
\int_A \Big(\frac{\partial g}{\partial \nu}(z)\Big)^2 \sigma(dz) &\leq c_3 \mathbb{E}^{y_0}\Big[\frac{\partial g}{\partial \nu}(X_{T(\partial F)}); X_{T(\partial F)} \in A\Big] \\
&\leq c_2 c_3 \mathbb{E}^{y_0}\Big[\frac{\partial g}{\partial y}(X_{T(\partial F)}); X_{T(\partial F)} \in A\Big] \\
&= c_4\Big(\frac{\partial g}{\partial y}(y_0) - \mathbb{E}^{y_0}\Big[\frac{\partial g}{\partial y}(X_{T(\partial F)}); X_{T(\partial F)} \in \partial B(x_0, 2\rho') \cup \partial B(0, 2\rho) \cup S\Big] \\
&\qquad - \mathbb{E}^{y_0}\Big[\frac{\partial g}{\partial y}(X_{T(\partial F)}; X_{T(\partial F)} \in \partial D - S - A\Big]\Big).
\end{aligned}$$

As we argued above, $\partial g/\partial y \geq 0$ a.e. in $\partial D - S$, while the first two terms on the right are bounded by constants depending only on ρ, ρ', and M. Therefore

$$\int_A \Big(\frac{\partial g}{\partial \nu}(z)\Big)^2 \sigma(dz) \leq c_5. \tag{3.5}$$

We now show how to eliminate the assumptions made near the beginning of the proof. Suppose that we no longer assume $G \subseteq B(\varphi(\theta_1), r)$. Let $A_0 = \varphi(G \cap B(\theta_1, r))$, let g be the Green function for reflecting Brownian motion killed on hitting $K \cup A$ with pole at x_0, and let g_0 be the Green function for reflecting Brownian motion killed on hitting $K \cup A_0$ with pole at x_0. Clearly

$$\mathbb{Q}^{x_0}(X_{T(K\cup A)} \in dy) \leq \mathbb{Q}^{x_0}(X_{T(K\cup A_0)} \in dy)$$

for $y \in A_0$, so on the set A_0 the density of harmonic measure for reflecting Brownian motion killed on hitting $K \cup A$, which is $\partial g / \partial \nu$, is less than or equal to the density of harmonic measure for reflecting Brownian motion killed on hitting $K \cup A_0$, which is $\partial g_0 / \partial \nu$. By this fact and (3.5) applied to g_0,

$$\int_{A \cap B(\varphi(\theta_1), r)} \Big(\frac{\partial g}{\partial \nu}(z)\Big)^2 \sigma(dz) \leq c_6. \tag{3.6}$$

By a rotation of the coordinate system, (3.6) holds when θ_1 is replaced by any other point of $\partial B(0,1)$. Since ∂D can be covered by finitely many balls of the form $B(\varphi(\theta), r)$ with $\theta \in \partial B(0,1)$, summing gives

$$\int_{A} \Big(\frac{\partial g}{\partial \nu}(z)\Big)^2 \sigma(dz) \leq c_6. \tag{3.7}$$

Recall the φ is C^∞. If the components E_i of G are each of the form $Q \cap \partial B(0,1)$, where Q is a cube of side length less than h, we can achieve (3.1) by modifying φ (and hence D) slightly. The smaller h is, the less we need to modify φ. Furthermore, we can approximate G as closely as we like by the union of such components. We can thus find a sequence of star-like domains D_m given by functions φ_m converging to D such that (3.7) holds when A is replaced by $A_m = \varphi_m(G)$ and g is replaced by the Green function for reflecting Brownian motion on D_m and c_6 is independent of m. By the limit argument of [B], pp. 217-218, we thus get (3.7) without any additional assumptions on G. $\square$

Corollary 3.2. *Let D, G and A be as in Proposition 3.1. Let H be C^∞ with support in $D - K$ and let*

$$u(x) = \mathbb{E}^x \int_0^{T_A \wedge T_K} H(X_s)\, ds.$$

Then $\partial u/\partial \nu$ exists a.e. on ∂D and there exists c_1 depending only on $\rho, \rho', \sup \varphi, \inf \varphi$, and the Lipschitz constant of φ such that

$$\int_{A} \Big(\frac{\partial u}{\partial \nu}(y)\Big)^2 \sigma(dy) \leq c_1.$$

Proof. The almost everywhere existence of $\partial u/\partial\nu$ follows by the same PDE results as were used in the first paragraph of the proof of Proposition 3.1. If $g(x,y)$ is the Green function for reflecting Brownian motion in D killed on hitting $A\cup K$ with pole at x, then $u(y)=\int g(x,y)H(x)\,dx$. Since the support of H is a positive distance from $\partial(D-K)$, the result now follows from Proposition 3.1, Fubini's theorem, and Fatou's lemma. $\square$

We next need an estimate that is essentially that of [JK], Section 4. Suppose D is a star-like Lipschitz domain: $D=\{(r,\theta): r<\varphi(\theta)\}$, where φ is a positive Lipschitz function. Let us suppose φ is also C^∞. Let $\psi(\theta)$ be another C^∞ positive function that is strictly less than φ for all θ and let $E=\{(r,\theta): r<\psi(\theta)\}$. Let $\delta<\operatorname{dist}(\partial D,\partial E)/4$ and let $E^\delta=\{x:\operatorname{dist}(x,\partial E)<\delta\}$. Recall that ν_o is the outward pointing unit normal vector and $\nu_o=-\nu$.

Proposition 3.3. *Let D and E be as above, suppose $f\in L^2(\partial D)$, and suppose u is harmonic in $(D-E)\cup E^\delta$ with $\partial u/\partial\nu_o=f$ on ∂D and $\int_{\partial D}u(z)\,\sigma(dz)=\int_{\partial E}u(z)\,\sigma(dz)$. There exists c_1 depending only on $\delta,\sup\varphi,\inf\varphi$, and the Lipschitz constants of φ and ψ such that*

$$\|N(\nabla u)\|_{2,\partial D}\le c_1\|f\|_{2,\partial D}+c_1\sup_{E^\delta}|u|.$$

Proof. The proof follows [JK], Section 4, closely. First let us suppose f is smooth. Let $h=\partial u/\partial\nu_o$ on ∂E, where by ν_o on ∂E we mean the outward normal vector with respect to the domain $D-E$. Then clearly u is the solution to the Neumann problem in $D-E$ with boundary functions f on ∂D and h on ∂E; by Green's identity, $\int_{\partial D}f=\int_{\partial E}h$. Hence by [GT], Chapter 6, u is smooth on $\overline{D-E}$. Let x be the vector from the point 0 to the point x. If

$$R(x)=|\nabla u(x)|^2x-2\big(x\cdot\nabla u(x)\big)\nabla u(x)-(d-2)u(x)\nabla u(x),$$

a calculation shows that $\operatorname{div}R(x)=0$ in $D-E$ since u is harmonic there. So by the divergence theorem,

$$\int_{\partial D}(R\cdot\nu_o)(z)\,\sigma(dz)=\int_{\partial E}(R\cdot\nu_o)(z)\,\sigma(dz). \tag{3.8}$$

Let us let $K=\sup_{x\in E^\delta}|u(x)|$. Since u is harmonic in E^δ, then ∇u is bounded by c_2K there, and so the right hand side of (3.8) is bounded by c_3K^2.

Let $a(x)=x-(x\cdot\nu_o(x))\nu_o(x)$, so that

$$(x\cdot\nabla u)=(a(x)\cdot\nabla u)+(x\cdot\nu_o)\frac{\partial u}{\partial\nu_o}.$$

Let $\nabla_t u$ denote the tangential component of ∇u, that is,

$$\nabla_t u(x) = (\nabla u(x) \cdot v_1(x), \ldots, \nabla u(x) \cdot v_{d-1}(x)),$$

where $(v_1(x), \ldots, v_{d-1}(x), \nu_o(x))$ forms an orthonormal set of vectors at $x \in \partial(D-E)$. Then

$$|\nabla u|^2 = |\nabla_t u|^2 + \Big(\frac{\partial u}{\partial \nu_o}\Big)^2.$$

Since $\partial u/\partial \nu_o = f$ on ∂D, we have

$$\int_{\partial D} |\nabla_t u|^2 (x \cdot \nu_o(x)) \sigma(dx) \le \int_{\partial D} |f^2(x \cdot \nu_o)| + 2 \int_{\partial D} |(a \cdot \nabla u) f| + (d-2) \int_{\partial D} |uf| + c_3 K^2.$$

The domain is bounded, so $x \cdot \nu_o$ and a are bounded, and because D is star-like, there exists c_4 such that $x \cdot \nu_o \ge c_4$ on ∂D. Hence

$$\int_{\partial D} |\nabla_t u|^2 \le c_5 \Big[\int_{\partial D} f^2 + \int_{\partial D} |\nabla_t u| \, |f| + \int_{\partial D} |u| \, |f| + K^2 \Big]. \tag{3.9}$$

We said that $\int_{\partial D} u = \int_{\partial E} u$ and $|u| \le K$ on ∂E. By the Poincaré inequality [M],

$$\begin{aligned} \frac{1}{\sigma(\partial D)} \int_{\partial D} u^2 &= \frac{1}{\sigma(\partial D)} \int_{\partial D} \Big(u - \frac{1}{\sigma(\partial D)} \int_{\partial D} u\Big)^2 + \Big(\frac{1}{\sigma(\partial D)} \int_{\partial D} u\Big)^2 \\ &\le c_6 \Big[\int_{\partial D} |\nabla_t u|^2 + K^2 \Big]. \end{aligned}$$

If we write F for $\int_{\partial D} f^2$ and I for $\int_{\partial D} |\nabla_t u|^2$, then by the Cauchy-Schwarz inequality we have

$$\int_{\partial D} |uf| \le c_7 (I + K^2)^{1/2} F^{1/2}.$$

Substituting in (3.9),

$$I \le c_8 \Big[F + I^{1/2} F^{1/2} + K F^{1/2} + K^2 \Big].$$

This implies there exists c_9 depending only on c_8 such that $I \le c_9 [K^2 + F]$. Since $\int_{\partial D} |\nabla u|^2 = I + F$, we get $\|\nabla u\|_{2,\partial D} \le c_{10} [K^2 + F]$. The result now follows for smooth f since the nontangential maximal function is bounded in L^2 norm by the L^2 norm of the function on the boundary (see [B], Section III.4, or [JK]). Finally we remove the restriction that f be smooth exactly as in [JK], pp. 39-42. $\square$

Suppose D is a domain satisfying the hypotheses of Proposition 3.1, except that now we only assume φ is Lipschitz, not necessarily C^∞. Let K be as above. Let H be a nonnegative C^∞ function with support in $D - K$; let E be a smooth domain whose closure is contained in D, which contains the support of H, and which is star-like with respect to 0. Let G be an open subset of $\partial B(0,1)$ consisting of finitely many components such that if $A = \varphi(G)$, then $\sigma(\overline{A} - A) = 0$.

Proposition 3.4. *Let D, K, E, G, A, and H be as above. There exists a function u that is nonnegative and bounded, $-(1/2)\Delta u = H$ in $D - K$, $\partial u/\partial\nu_o$ exists a.e., $u = 0$ a.e. on A, $\partial u/\partial\nu_o = 0$ a.e. on $\partial D - A$, and $\|N(\nabla u)\|_{2,\partial D} < \infty$.*

Proof. Let $D_n = \{(r,\theta) : r < \varphi_n(\theta)\}$ be domains that are star-like with respect to 0, where the φ_n are C^∞ and that decrease to D; we suppose also that $\sup_n \|\nabla\varphi_n\|_\infty$ is finite.

Let $A_n = \varphi_n(G)$. Let $(\mathbb{Q}_n^x, X_t)$ be standard reflecting Brownian motion in D_n, let expectation with respect to $\mathbb{Q}_n^x$ be written $\mathbb{E}_n^x$, and let

$$u_n(x) = \mathbb{E}_n^x \int_0^{T_{A_n}\wedge T_K} H(X_s)\,ds.$$

Since the support of H is a compact subset of D and D is open, u_n is harmonic in a neighborhood of ∂D_n. By the discussion in Section 2, $\partial u_n/\partial\nu = 0$ a.e. on $(\partial D_n) - A_n$. By Corollary 3.2, $\partial u_n/\partial\nu$ exists a.e. on ∂D_n with $L^2(\partial D_n)$ norm not depending on n. So by Proposition 3.3, $N(\nabla u_n)$ is in $L^2(\partial D_n)$ with a norm not depending on n.

We will show that a subsequence of the u_n converges to a function u that satisfies $-(1/2)\Delta u = H$ in D, u is nonnegative and bounded, $u = 0$ a.e. on A, $\partial u/\partial\nu_o = 0$ a.e. on $\partial D - A$, and $\|N(\nabla u)\|_{2,\partial D} < \infty$.

Note each $u_n(x)$ is nonnegative and

$$\|u_n\|_\infty \le \|H\|_\infty \sup_{n,y} \mathbb{E}_n^y T_K.$$

By (2.4), the right hand side is finite. By the strong Markov property,

$$u_n(x) = \mathbb{E}_n^x \int_0^{T(\partial E)\wedge T_K} H(X_s)\,ds + \mathbb{E}_n^x u_n(X_{T(\partial E)\wedge T_K}).$$

The second term is harmonic inside $E - K$ and the first is uniformly smooth inside $E - K$ by standard results from PDE (see [GT], Section 6.4), since E is smooth. So the u_n are equicontinuous inside $E - K$. On the other hand, each u_n is harmonic outside the support of H. Therefore the u_n are equicontinuous on compact subsets of D and are uniformly bounded; hence there exists a subsequence which converges uniformly on compact subsets of D, say to u. By relabeling the D_n, let us suppose that the original sequence u_n converges.

Observe that if $\theta \in \partial B(0,1)$, then $V_\beta^D(\varphi(\theta)) \subseteq V_\beta^{D_n}(\varphi_n(\theta))$, so $N^D(\nabla u_n)(\varphi(\theta)) \le N^{D_n}(\nabla u_n)(\varphi_n(\theta))$. Since ∇u_n converges uniformly to ∇u in compact subsets of $D - E$, it follows easily by Fatou's lemma (cf. [JK], Section 4) that for $\delta < \operatorname{dist}(\partial D, \partial E)$, we have $\|N_\delta^D(\nabla u)\|_{2,\partial D} \le c_1$, where c_1 depends only on δ and the Lipschitz constant of φ but not on G. It is easy to see that ∇u is bounded in $D - U_\delta$ since H is smooth and u is the uniform limit of the u_n there, so we have

$$\|N^D(\nabla u)\|_{2,\partial D} \le c_2. \tag{3.10}$$

Since u is bounded, it has nontangential limits a.e. We show the nontangential limit is 0 a.e. on A. Let $v_n(x) = \mathbb{Q}_n^x(X_t \text{ hits } E \text{ before } A_n)$. This is a harmonic function in $D_n - E$ which equals 1 on E and has nontangential limits 0 on A_n. Let $\varepsilon > 0$ and $\theta \in G$. Since v_n is bounded by 1 and is 0 on A_n, by Lemma 2.1 there exists δ such that if $x \in V_\beta^{D_n}(\varphi_n(\theta))$ and $|x - \varphi_n(\theta)| < \delta$, then $v_n(x) < \varepsilon$. So if $x \in V_\beta^D(\varphi(\theta)) \subseteq V_\beta^{D_n}(\varphi_n(\theta))$ and $|x - \varphi(\theta)| < \delta/2$, then for n large, $|x - \varphi_n(\theta)| < \delta$. Since u_n has nontangential limits 0 on A_n and is harmonic in $D_n - E$,

$$0 \leq u_n(x) \leq \sup_n \|u_n\|_\infty \mathbb{Q}_n^x(T_E < T_{A_n}) \leq v_n(x) \sup_n \|u_n\|_\infty.$$

Hence $u(x) \leq \varepsilon \sup_n \|u_n\|_\infty$. This shows $u(x) \to 0$ as $x \to \varphi(\theta)$ nontangentially.

Since $\|N_D(\nabla u)\|_{2,\partial D} \leq c_2$, then ∇u converges nontangentially a.e. and so $\partial u/\partial \nu_o$ exists a.e. It remains to show that $\partial u/\partial \nu_o = 0$ a.e. on $\partial D - A$. Let $h(\theta)$ be a smooth function with support in G^c. Let $f_n(x) = \mathbb{E}_n^x h(X_{T(\partial D_n)})$, $f(x) = \mathbb{E}^x h(X_{T(\partial D)})$, where $(\mathbb{Q}^x, X_t)$ is reflecting Brownian motion on D. Note the restriction of f_n to ∂D_n is supported on A_n^c and the restriction of f to ∂D is supported on A^c.

Let $(\mathbb{P}^x, W_t)$ be standard Brownian motion on $\mathbb{R}^d$. Up until times $T(\partial D_n)$ and $T(\partial D)$, $(\mathbb{Q}_n^x, X_t)$ and $(\mathbb{Q}^x, X_t)$, respectively, have the same law as $(\mathbb{P}^x, W_t)$. So $f_n(x) = \mathbb{E}_{\mathbb{P}^x} h(W_{T(\partial D_n)})$ and $f(x) = \mathbb{E}_{\mathbb{P}^x} h(W_{T(\partial D)})$. Since $T_{\partial D_n} \downarrow T_{\partial D}$ and h is smooth, it follows that f_n converges to f on D. Since f_n and f are harmonic, the convergence is uniform on compact subsets of D.

By Green's first identity, since f_n and f are harmonic in $D_n - E$ and $D - E$, respectively, and $\partial u_n/\partial \nu_o = 0$ on $(\partial D_n) - A_n$,

$$\int_{D_n - E} \nabla f_n \cdot \nabla u_n = \int_{\partial D_n} f_n \frac{\partial u_n}{\partial \nu_o} + \int_{\partial E} f_n \frac{\partial u_n}{\partial \nu_o} = \int_{\partial E} f_n \frac{\partial u_n}{\partial \nu_o}$$

and

$$\int_{D-E} \nabla f \cdot \nabla u = \int_{\partial D} f \frac{\partial u}{\partial \nu_o} + \int_{\partial E} f \frac{\partial f}{\partial \nu_o}.$$

(Recall ν_o is the outward normal vector for the domains $D_n - E$ or $D - E$.)

We will show

$$\int_{D_n - E} \nabla f \cdot \nabla u_n \to \int_{D-E} \nabla f \cdot \nabla u. \tag{3.11}$$

Since $f_n \to f$ and u_n is harmonic on ∂E and uniformly bounded in n in a neighborhood of E, $\partial u_n/\partial \nu_o \to \partial u/\partial \nu_o$ on E. So if (3.11) holds, $\int_{\partial D} f(\partial u/\partial \nu_o) = 0$. If this holds for all such h, then $\partial u/\partial \nu_o = 0$ a.e. on A^c. So it remains to show (3.11).

Recall the definition of U_r from (2.1). We have $\nabla f_n \to \nabla f$ uniformly on $D - U_r$ and $\nabla u_n \to \nabla u$ uniformly on $D - U_r$, so

$$\int_{D-U_r} \nabla f_n \cdot \nabla u_n \to \int_{D-U_r} \nabla f \cdot \nabla u. \tag{3.12}$$

Since h is smooth, by [JK], Theorem 4.13, there exists c_3 independent of n such that

$$\int_{\partial D_n} (N(\nabla f_n)(y))^2\, \sigma(dy) \le c_3$$

and

$$\int_{\partial D} (N(\nabla f)(y))^2\, \sigma(dy) \le c_3.$$

Let $\varepsilon > 0$. If $r < \varphi_n(\theta)$, then $|\nabla f_n(r,\theta)| \le N(\nabla f_n)(\varphi_n(\theta))$. So $\int_{(D_n-D)\cup U_r} |\nabla f_n|^2$ can be made less than ε if r is small enough and n is large enough, and similarly $\int_{U_r} |\nabla f|^2 < \varepsilon$ if r is small enough. We have

$$\begin{aligned}\int_{(D_n-D)\cup U_r} |\nabla f_n \cdot \nabla u_n| &\le \Big(\int_{(D_n-D)\cup U_r} |\nabla f_n|^2\Big)^{1/2} \Big(\int_{(D_n-D)\cup U_r} |\nabla u_n|^2\Big)^{1/2}\\ &\le c_4 \sup_n \varepsilon \|N(\nabla u_n)\|_{2,\partial D}\end{aligned}$$

and

$$\int_{U_r} |\nabla f \cdot \nabla u| \le \Big(\int_{U_r} |\nabla f|^2\Big)^{1/2} \Big(\int_{U_r} |\nabla u|\Big)^{1/2} \le c_4 \varepsilon \|N(\nabla u)\|_{2,\partial D}.$$

Combining with (3.12) and using the fact that ε is arbitrary gives (3.11). $\square$

Proposition 3.5. *Let D, K, E, and H be as in Proposition 3.4. Let G be an arbitrary open set in $\partial B(0,1)$ and $A = \varphi(G)$. Then there exists a function $u(x)$ satisfying the conclusions of Proposition 3.4.*

Proof. Let G_m be open sets satisfying the hypotheses of Proposition 3.1 and increasing to an open set G and let $A_m = \varphi(G_m)$ and $A = \varphi(G)$. Let $u_m(x)$ be the corresponding functions given by Proposition 3.4. The u_m are uniformly bounded, harmonic in $D - E$, and satisfy $-(1/2)\Delta u_m = H$ in $E - K$. As m increases, $A_m \uparrow A$. Using the notation of the proof of Proposition 3.4, observe that as m increases, T_{A_m} decreases, and so $\mathbb{E}_n^x \int_0^{T_{A_m}\wedge T_K} H(X_s)\, ds$ decreases. It follows that $u_m(x)$ decreases as m increases for each x. Let $u(x) = \lim_{m\to\infty} u_m(x)$. By the harmonicity and boundedness of the u_m, the convergence is uniform on compact subsets of $D - E$. Therefore u is harmonic in $D - E$, bounded in D, 0 on K, and $-(1/2)\Delta u = H$ in $E - K$.

Suppose $x \in V_\beta(z)$ for some $z \in A$. Then $z \in A_m$ for some m and given ε, there exists δ such that if $|x - z| < \delta$, then $0 \le u_m(x) < \varepsilon$. Therefore $u(x) \le u_m(x) < \varepsilon$. This shows that u has nontangential limits 0 a.e. on A.

By Fatou's lemma and the corresponding result for the u_m, $\|N(\nabla u)\|_{2,\partial D} \le c_1$. So $\partial u/\partial \nu_o$ exists a.e., and we must show that it is 0 a.e. on A^c. Suppose there exists a set B of positive surface measure contained in A^c on which $\partial u/\partial \nu_o > r$ for some $r > 0$. (The case where $\partial u/\partial \nu_o$ is negative is treated similarly.) Pick f smooth on ∂D such that $f=1$ on B and $\int_{\partial D} f(\partial u/\partial \nu_o) > r\sigma(B)/2$. We also require

$$\Big(\int_{\partial D - B} f^2\Big)^{1/2} < \frac{r\sigma(B)}{4\sup_m \|N(\nabla u_m)\|_{2,\partial D}}.$$

We can find such a f by taking smooth f decreasing to 1_B. Extend f to D be defining $f(x) = \mathbb{E}^x f(X_{T(\partial D)})$, where $(\mathbb{Q}^x, X_t)$ is reflecting Brownian motion on D. Since $B \subseteq A^c \subseteq A_m^c$, $\partial u_m/\partial \nu_o = 0$ on B and

$$\Big|\int_{\partial D} f \frac{\partial u_m}{\partial \nu_o}\Big| = \Big|\int_{\partial D - B} f \frac{\partial u_m}{\partial \nu_o}\Big| \le \Big(\int_{\partial D - B} f^2\Big)^{1/2} \Big(\int_{\partial D} \Big|\frac{\partial u_m}{\partial \nu_o}\Big|^2\Big)^{1/2} < r\sigma(B)/4.$$

Now by Green's identity on D and the fact that f is harmonic in D,

$$\int_{\partial D} f \frac{\partial u_m}{\partial \nu_o} = \int_D \nabla f \cdot \nabla u_m,$$

and similarly to Proposition 3.4 but easier, this converges to $\int_D \nabla f\cdot \nabla u = \int_{\partial D} f(\partial u/\partial \nu_o)$. This implies that

$$\frac{r\sigma(B)}{4} \ge \lim_m \int_{\partial D} f \frac{\partial u_m}{\partial \nu_o} = \int_{\partial D} f \frac{\partial u}{\partial \nu_o} > \frac{r\sigma(B)}{2},$$

a contradiction. Therefore there exists no such subset B. □

Corollary 3.6. *Let $D, K, E, H, G,$ and A be as in Proposition 3.5. There exist reals $r_n \uparrow 1$ and functions $F_n : D \to \mathbb{R}$ that are C^∞, the F_n are nonnegative and uniformly bounded, $-(1/2)\Delta F_n \to H$ uniformly in $E - K$, $F_n = 0$ on $B(0, \rho/r_n)$, $F_n \to 0$ a.e. on A, $\partial F_n/\partial \nu_o \to 0$ a.e. on A^c, and $\sup_n \|N(\nabla F_n)\|_{2,\partial D} < \infty$.*

Proof. Let u be the function constructed in Proposition 3.5, let $r_n \uparrow 1$ and let

$$F_n(x) = u(r_n x), \qquad x \in D.$$ □

4. Uniqueness.

For now we suppose D satisfies (2.2) and the dimension $d \ge 3$. Let $(\mathbb{Q}^x, X_t)$ denote a standard reflecting Brownian motion, let $x_0 \in D$, and let $\mathbb{P}$ be a probability measure that is a solution to (2.10). Without loss of generality we may suppose $x_0 \ne 0$. Our main goal is to show $\mathbb{P} = \mathbb{Q}^{x_0}$. Let $\rho < \min(|x_0|, \operatorname{dist}(0, \partial D))/4$ and define $K = B(0,\rho)$. Let θ_t be the usual shift operators.

The process L_t is a continuous process, so if $M > 0$ and $\xi_1(M) = \inf\{t : L_t \ge M\}$, then $\xi_1(M) > 0$. Since $L_{t\wedge\xi_1(M)}$ is the uniform limit of $A_m(t \wedge \xi_1(M))$, where A_m is defined by (2.6), it follows that $\xi_2(M) = \inf\{t : \sup_m A_m(t \wedge \xi_1(M)) \ge 2M\}$ is also strictly positive, a.s. We let $\xi_3 = \xi_3(M) = \xi_1(M) \wedge \xi_2(M) \wedge M$, and observe that $0 < \xi_3(M)$ and $\xi_3(M) \to \infty$ as $M \to \infty$.

We define a new probability measure $\mathbb{P}'$ that agrees with $\mathbb{P}$ up to time ξ_3 and agrees with $\mathbb{Q}^{x_0}$ after time ξ_3 as follows. If $A \in \mathcal{F}_{\xi_3}$ and $B \in \mathcal{F}_\infty$, let

$$\mathbb{P}'(B \circ \theta_{\xi_3} \cap A) = \mathbb{E}_{\mathbb{P}}(\mathbb{Q}^{X(\xi_3)}(B); A).$$

This determines the probability measure $\mathbb{P}'$ on $\mathcal{F}_\infty$ ([SV2], Chapter 6). $\mathbb{P}'$ is a solution to (2.10) up to time ξ_3 since it agrees with $\mathbb{P}$ on $\mathcal{F}_{\xi_3}$. $\mathbb{P}'$ solves (2.10) shifted by an amount ξ_3 by the fact that for each x, $\mathbb{Q}^x$ is a solution to (2.10) starting at x. If we show that $\mathbb{P}' = \mathbb{Q}^{x_0}$, then $\mathbb{P} = \mathbb{Q}^{x_0}$ on $\mathcal{F}_{\xi_3(M)}$, and letting $M \to \infty$, we obtain $\mathbb{P} = \mathbb{Q}^{x_0}$.

The reason we work with $\mathbb{P}'$ is the following.

Proposition 4.1. *(a)* $\mathbb{E}_{\mathbb{P}'} L_{T_K} < \infty$.
(b) $\mathbb{E}_{\mathbb{P}'} \sup_m A_m(T_k) < \infty$.
(c) $\mathbb{E}_{\mathbb{P}'} T_K < \infty$.

Proof. We have $A_m(T_K) = A_m(\xi_3) + A_m(T_K) \circ \theta_{\xi_3}$ and letting $m \to \infty$ we obtain $L_{T_K} = L_{\xi_3} + L_{T_K} \circ \theta_{\xi_3}$. So by the definition of $\mathbb{P}'$,

$$\begin{aligned}\mathbb{E}_{\mathbb{P}'} L_{T_K} &= \mathbb{E}_{\mathbb{P}} L_{\xi_3} + \mathbb{E}_{\mathbb{P}'}(L_{T_K} \circ \theta_{\xi_3})\\ &= \mathbb{E}_{\mathbb{P}} L_{\xi_3} + \mathbb{E}_{\mathbb{P}}(\mathbb{E}^{X(\xi_3)} L_{T_K}).\end{aligned}$$

By the definition of ξ_3, the first term is bounded by M, while the second term is finite by the discussion following (2.4). The proof of assertion (c) is essentially the same.

Since

$$\sup_m A_m(T_K) \le \sup_m A_m(\xi_3) + (\sup_m A_m(T_K)) \circ \theta_{\xi_3},$$

the proof of (b) is similar. □

We now drop the primes from $\mathbb{P}'$, and without loss of generality we may suppose that $\mathbb{E}_{\mathbb{P}} L_{T_K} < \infty$, $\mathbb{E}_{\mathbb{P}} \sup_m A_m(T_K) < \infty$, and $\mathbb{E}_{\mathbb{P}} T_K < \infty$.

Define a measure μ on $D - K$ by

$$\mu(B) = \mathbb{E}_{\mathbb{P}} \int_0^{T_K} 1_B(X_s)\, ds,$$

the amount of time spent in B before hitting K.

Proposition 4.2. *μ is absolutely continuous with respect to Lebesgue measure. If we let h denote the Radon-Nikodym derivative, we may choose h to be finite, nonnegative, and harmonic in $D - K - \{x_0\}$.*

Proof. The nonnegativity is clear. Let $B_0 \subseteq B_1 \subseteq B_2$ be any three concentric balls contained in $D - K - \{x_0\}$ such that $\overline{B}_0 \subseteq B_1$, $\overline{B}_1 \subseteq B_2$, and $\overline{B}_2 \subseteq D - K - \{x_0\}$. If we prove that μ restricted to B_0 has a Radon-Nikodym derivative with respect to Lebesgue measure that may be chosen to be harmonic in B_0, then since B_0 is arbitrary we will obtain our result.

Let $S_1 = \inf\{t : X_t \in B_1\}$, $T_1 = \inf\{t > S_1 : X_t \notin B_2\}$, and for $i \geq 1$ let $S_{i+1} = \inf\{t > T_i : X_t \in B_1\}$ and $T_{i+1} = \inf\{t > S_{i+1} : X_t \notin B_2\}$. If $B \subseteq B_0$, then

$$\mathbb{E}_{\mathbb{P}} \int_0^{T_K} 1_B(X_s)\, ds = \sum_{i=1}^{\infty} \mathbb{E}_{\mathbb{P}} \Big[\int_{S_i}^{T_i} 1_B(X_s)\, ds; S_i < T_K \Big]. \tag{4.1}$$

Note that under $\mathbb{P}$, the process $\int_0^t 1_B(X_s)\, ds$ cannot increase before time S_1 because $x_0 \notin B_2$; since $K \cap B_2 = \emptyset$, if $S_i < T_K$, then $T_i < T_K$.

The law of $X_s \circ \theta_{S_i}$ under a regular conditional probability for $\mathbb{E}[\cdot \mid \mathcal{F}_{S_i}]$ is by Proposition 2.3 a solution to (2.10) started at X_{S_i}. Started at $x \in D - K$, the law of X_t is the same as that of a standard d-dimensional Brownian motion up to time $T_{\partial D}$. Therefore $\mathbb{E}_{\mathbb{P}}[\int_{S_i}^{T_i} 1_B(X_s)\, ds \mid \mathcal{F}_{S_i}]$ is the same as the amount of time Brownian motion started at X_{S_i} spends in B up until leaving B_2. If $g_{B_2}(x,y)$ is the Green function for standard d-dimensional Brownian motion killed on exiting B_2, we have then

$$\mathbb{E}_{\mathbb{P}} \Big[\int_{S_i}^{T_i} 1_B(X_s)\, ds \mid \mathcal{F}_{S_i} \Big] = \int_B g_{B_2}(X_{S_i}, y)\, dy. \tag{4.2}$$

The law of $X \circ \theta_{T_i}$ under a regular conditional probability for $\mathbb{E}_{\mathbb{P}}[\cdot \mid \mathcal{F}_{T_i}]$ is a solution to (2.10) started at X_{T_i}. A solution started at X_{T_i} is a standard Brownian motion up until hitting ∂D. By the support theorem for Brownian motion ([B], p. 59), there exists $\rho < 1$ such that

$$\mathbb{Q}^y(T_{B_1} < T_K) \leq \rho, \qquad y \in \partial B_2.$$

Hence

$$\mathbb{Q}^{X(T_i)}(T_{B_1} < T_K) \leq \rho,$$

or

$$\mathbb{P}(S_{i+1} < T_K \mid \mathcal{F}_{T_i}) \leq \rho, \quad \text{a.s.}$$

From this we deduce

$$\begin{aligned}\mathbb{P}(S_{i+1} < T_K) &= \mathbb{P}(S_{i+1} < T_K, S_i < T_K) = \mathbb{E}_{\mathbb{P}}[\mathbb{P}(S_{i+1} < T_K \mid \mathcal{F}_{T_i}); S_i < T_K] \\ &\leq \rho \mathbb{P}(S_i < T_K).\end{aligned}$$

By induction, $\mathbb{P}(S_i < T_K) \le \rho^i$.

Combining with (4.1) and (4.2) and using Fubini's theorem,

$$\mathbb{E}_{\mathbb{P}} \int_0^{T_K} 1_B(X_s)\, ds = \int_B \sum_{i=1}^{\infty} \mathbb{E}_{\mathbb{P}}[g_{B_2}(X_{S_i}, y); S_i < T_K]\, dy.$$

$g_{B_2}(x, y)$ is harmonic in $y \in B_0$ when $x \in \partial B_1$; therefore $\mathbb{E}_{\mathbb{P}}[g_{B_2}(X_{S_i}, y); S_i < T_K]$ is harmonic. Since $g_{B_2}(x, y)$ is bounded over $x \in \partial B_1, y \in B_0$, then

$$\sum_{i=i_0}^{\infty} \mathbb{E}_{\mathbb{P}}[g_{B_2}(X_{S_i}, y); S_i < T_K] \le \sum_{i=i_0}^{\infty} c_1 \mathbb{P}(S_i < T_K) \le \sum_{i=i_0}^{\infty} c_1 \rho^i < \infty. \tag{4.3}$$

Let

$$h(y) = \sum_{i=1}^{\infty} \mathbb{E}_{\mathbb{P}}[g_{B_2}(X_{S_i}, y); S_i < T_K].$$

In view of (4.3) the sum converges uniformly over $y \in B_0$ and hence h is finite and harmonic in B_0. $\square$

Since h is nonnegative and harmonic in $D - K - \{x_0\}$, the nontangential maximal function of h is finite a.e. in a neighborhood of ∂D, i.e., for ε less than ρ',

$$N_\varepsilon(h)(z) < \infty, \qquad \text{for almost every } z \in \partial D.$$

By (2.3), the Green function for $D - K$ with pole at x_0 is bounded in $D - B(x_0, \rho')$, say by R. We construct a sawtooth domain

$$D_0 = \bigcup \{V_\beta(z) : z \in \partial D, N_\varepsilon(h)(z) \le 3R\}.$$

D_0 is a Lipschitz domain, and since it is contained in D, still star-like with respect to 0. Let $A = \partial D_0 - \partial D$.

Lemma 4.3. *There exists c_1 depending only on ε and R such that*

$$\mathbb{E}_{\mathbb{P}} \int_0^{T_K \wedge T_A} |\varphi(X_t)|\, dL_t \le c_1 \int_{\partial D - A} |\varphi(y)|\, \sigma(dy).$$

Proof. First suppose φ is nonnegative and continuous on ∂D and extend φ to be nonnegative and continuous in D as well. Since L_t is the uniform limit of the $A_m(t)$,

$$\int_0^{T_K \wedge T_A \wedge t} \varphi(X_s)\, dA_m(s) \to \int_0^{T_K \wedge T_A \wedge t} \varphi(X_s)\, dL_s.$$

By Proposition 4.1(b) and dominated convergence,

$$\mathbb{E}_{\mathbb{P}} \int_0^{T_K \wedge T_A \wedge t} \varphi(X_s)\, dL_s = \lim_{m\to\infty} \mathbb{E}_{\mathbb{P}} \int_0^{T_K \wedge T_A \wedge t} \varphi(X_s)\, dA_m(s). \tag{4.4}$$

h is bounded by $3R$ in a neighborhood of ∂D_0. Recall the definition of U_r from (2.1). Because h is harmonic in $D - K - B(x_0, \rho') - U_r$, it is bounded by a constant c_2 there. Since $dA_m(s) = f_m(X_s)\, ds$, the right hand side in (4.4) is bounded by

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} \int_0^{T_A \wedge T_K \wedge t} \varphi(X_s) f_m(X_s)\, ds &\le \int 1_{D_0}(y) \varphi(y) f_m(y) h(y)\, dy \\ &\le c_3 \int_D \varphi(y) f_m(y)\, dy, \end{aligned}$$

where $c_3 = c_2 \vee R$. Since $f_m(y)\, dy$ converges weakly to $\sigma(dy)$ and φ is continuous,

$$\mathbb{E}_{\mathbb{P}} \int_0^{T_A \wedge T_K \wedge t} \varphi(X_s)\, dL_s \le c_3 \int_{\partial D} \varphi(y)\, \sigma(dy).$$

Now let $t \to \infty$. By linearity and a limit argument,

$$\mathbb{E}_{\mathbb{P}} \int_0^{T_A \wedge T_K} |\varphi(X_s)|\, dL_s \le c_3 \int_{\partial D} |\varphi(y)|\, \sigma(dy)$$

for all φ bounded and measurable on ∂D. Finally, since L_t grows only when $X_t \in \partial D$,

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} \int_0^{T_K \wedge T_A} |\varphi(X_s)|\, dL_s &= \mathbb{E}_{\mathbb{P}} \int_0^{T_A \wedge T_K} |(\varphi 1_{(\partial D - A)})(X_s)|\, dL_s \\ &\le c_3 \int |(\varphi 1_{(\partial D - A)})(y)|\, \sigma(dy). \end{aligned}$$

This completes the proof. □

Let H be a C^∞ function with support in $D - K - \{x_0\}$. The key proposition is the following.

Proposition 4.4. *Let $u(x) = \mathbb{E}^x \int_0^{T_A \wedge T_K} H(X_s)\, ds$. Then if $x_0 \in D - K$,*

$$\mathbb{E}_{\mathbb{P}} \int_0^{T_A \wedge T_K} H(X_s)\, ds = u(x_0).$$

Proof. Let E be star-like, contained in D, and containing the support of H. Construct the F_n as in Corollary 3.6. For $r > 1$ let $K_r = B(0, r\rho)$. F_n is C^2 on $D - K_r$, so by Ito's formula,

$$\begin{aligned} F_n(X(T_{K_r} \wedge T_A \wedge t)) - F_n(X_0) = \text{ martingale } &+ \frac{1}{2} \int_0^{T_{K_r} \wedge T_A \wedge t} \Delta F_n(X_s)\, ds \\ &+ \int_0^{T_{K_r} \wedge T_A \wedge t} \frac{\partial F_n}{\partial \nu}(X_s)\, dL_s. \end{aligned} \tag{4.5}$$

Take expectations with respect to $\mathbb{P}$. By Lemma 4.3, the expectation of the local time term in (4.5) (i.e., the last term on the right) is bounded by

$$c_1 \int_{\partial D - A} \Big| \frac{\partial F_n}{\partial \nu}(y) \Big| \, \sigma(dy).$$

F_n is bounded and ΔF_n is bounded in D, so letting $t \to \infty$, we obtain

$$\mathbb{E}_{\mathbb{P}} F_n(X(T_{K_r} \wedge T_A)) - F_n(x_0) = \frac{1}{2} \mathbb{E}_{\mathbb{P}} \int_0^{T_{K_r} \wedge T_A} \Delta F_n(X_s)\, ds + R,$$

where

$$|R| \le c_1 \int_{\partial D - A} \Big| \frac{\partial F_n}{\partial \nu}(y) \Big| \, \sigma(dy).$$

Next let $n \to \infty$. By Corollary 3.6 $\partial F_n / \partial \nu$ is in $L^2(\sigma)$ with a bound independent of n and $\partial F_n / \partial \nu \to 0$ a.e. on $\partial D - A$. We also have that $(1/2)\Delta F_n \to -H$ uniformly on $D - K_r$ and $F_n \to u$ uniformly. So we obtain

$$\mathbb{E}_{\mathbb{P}} u(X(T_{K_r} \wedge T_A)) - u(x_0) = -\mathbb{E}_{\mathbb{P}} \int_0^{T_{K_r} \wedge T_A} H(X_s)\, ds.$$

The function u is 0 on A and on K and H is bounded. So by dominated convergence on the left and monotone convergence on the right, letting $r \downarrow 1$,

$$-u(x_0) = -\mathbb{E}_{\mathbb{P}} \int_0^{T_K \wedge T_A} H(X_s)\, ds. \qquad \square$$

Corollary 4.5. *If $x_0 \in D - K$,*

$$\mathbb{E}_{\mathbb{P}} \int_0^{T_A \wedge T_K} H(X_s)\, ds = \mathbb{E}^{x_0} \int_0^{T_A \wedge T_K} H(X_s)\, ds$$

for all bounded functions H.

Proof. This follows from Proposition 4.4 by a limit argument and the fact that the quantity $\mathbb{E}_{\mathbb{P}} \int_0^{T_K \wedge T_A} 1_{\{x_0\}}(X_s)\, ds$ is 0 since X_t behaves like a Brownian motion in a neighborhood of x_0. $\square$

Corollary 4.6. *If $x_0 \in D - K$,*

$$\mathbb{E}_{\mathbb{P}} \int_0^{T_K} H(X_s)\, ds = \mathbb{E}^{x_0} \int_0^{T_K} H(X_s)\, ds$$

for all bounded functions H.

Proof. Recall the definition of R following the proof of Proposition 4.2. If H has support in $D-K-B(x_0,\rho')$, then by the definition of R,

$$\mathbb{E}^{x_0}\int_0^{T_K} H(X_s)1_B(X_s)\,ds \le R\|H\|_\infty|B|$$

for $B\subseteq D$. This and Corollary 4.5 imply that $h\le R$, a.e., on $D-B(x_0,\rho')$. Since h is harmonic, it is continuous, or $h\le R$ on $D-B(x_0,\rho')$. This implies $N_\varepsilon(h)(z)\le R$, so $D_0=D$ and hence $A=\emptyset$. $\square$

We would like the conclusion of Corollary 4.6 to hold for all x_0, even for $x_0\in\partial D$.

Proposition 4.7. *If $x_0\in\partial D$,*

$$\mathbb{E}_{\mathbb{P}}\int_0^{T_K} H(X_s)\,ds = \mathbb{E}^{x_0}\int_0^{T_K} H(X_s)\,ds.$$

Proof. Let $x_0\in\partial D$. Let $\xi_4(n)=\inf\{t:|X_t-x_0|\ge 1/n\}$ and $\xi_5(m)=\inf\{t:\operatorname{dist}(X_t,\partial D)\ge 1/m\}$. Choose m_n such that $\mathbb{P}(\xi_5(m_n)>\xi_4(n))<1/n$; this is possible since starting at x_0 the process under $\mathbb{P}$ leaves ∂D immediately. Let $\xi_6(n)=\xi_4(n)\wedge\xi_5(m_n)\wedge 1/n$. As in Corollary 4.5 it suffices to prove the proposition for H in C^∞ with support in $D-K$. So for n sufficiently large, X_t will not be in the support of H when $t\le\xi_6(n)$ and

$$\mathbb{E}_{\mathbb{P}}\int_0^{T_K} H(X_s)\,ds = \mathbb{E}_{\mathbb{P}}\int_{\xi_6(n)}^{T_K} H(X_s)\,ds.$$

The law of the process $X_s\circ\theta_{\xi_6(n)}$ under a regular conditional probability for $\mathbb{E}_{\mathbb{P}}[\cdot\mid\mathcal{F}_{\xi_6(n)}]$ is a solution to (2.10) started at $X_{\xi_6(n)}$. On the set where $X_{\xi_6(n)}\notin\partial D$, by Corollary 4.6 we have $\mathbb{E}_{\mathbb{P}}[\int_0^{T_K} H(X_s)\,ds\mid\mathcal{F}_{\xi_6(n)}]=u(X_{\xi_6(n)})$, where

$$u(x)=\mathbb{E}^x\int_0^{T_K} H(X_s)\,ds. \tag{4.6}$$

So

$$\begin{aligned}\Big|\mathbb{E}_{\mathbb{P}}\int_{\xi_6(n)}^{T_K} H(X_s)\,ds-\mathbb{E}_{\mathbb{P}}u(X_{\xi_6(n)})\Big| &\le (\|H\|_\infty\mathbb{E}_{\mathbb{P}}T_K+\|u\|_\infty)\mathbb{P}(X_{\xi_6(n)}\in\partial D)\\ &\le c_1/n.\end{aligned}$$

By (2.7), u is continuous in $B(x_0, 1/n) \cap \overline{D}$, so letting $n \to \infty$, we have $u(X_{\xi_6(n)}) \to u(x_0)$, and hence

$$\mathbb{E}_{\mathbb{P}} \int_0^{T_K} H(X_s)\,ds = \mathbb{E}_{\mathbb{P}} u(x_0) = u(x_0). \qquad \square$$

Proposition 4.8. *Let $S_\lambda H = \mathbb{E}_{\mathbb{P}} \int_0^{T_K} e^{-\lambda t} H(X_t)\,dt$. Then for all $x_0 \in \overline{D} - K$ and for all $\lambda < 1/(2 \sup_y \mathbb{E}^y T_K)$,*

$$S_\lambda H = \mathbb{E}^{x_0} \int_0^{T_K} e^{-\lambda t} H(X_t)\,dt.$$

Proof. It is enough to consider H that are C^∞ with support in $D - K$. Let us kill the process on hitting K. Since H is 0 there, we can let the integrals run from 0 to ∞. Let u be defined by (4.6). Under a regular conditional probability for $\mathcal{F}_t$, the law of the process $X_s \circ \theta_t$ is a solution to (2.10) started at X_t. Therefore by Proposition 4.7

$$\mathbb{E}_{\mathbb{P}} \Big[\int_0^\infty H(X_s \circ \theta_t)\,ds \mid \mathcal{F}_t \Big] = \mathbb{E}^{X_t} \int_0^\infty H(X_s)\,ds = u(X_t).$$

We then have

$$\begin{aligned}
S_\lambda u &= \mathbb{E}_{\mathbb{P}} \int_0^\infty e^{-\lambda t} u(X_t)\,dt \\
&= \mathbb{E}_{\mathbb{P}} \int_0^\infty e^{-\lambda t} \mathbb{E}_{\mathbb{P}} \Big[\int_0^\infty H(X_{s+t})\,ds \mid \mathcal{F}_t \Big]\,dt \\
&= \mathbb{E}_{\mathbb{P}} \int_0^\infty e^{-\lambda t} \int_t^\infty H(X_s)\,ds\,dt \\
&= \mathbb{E}_{\mathbb{P}} \int_0^\infty H(X_s) \int_0^s e^{-\lambda t}\,dt\,ds \\
&= \mathbb{E}_{\mathbb{P}} \int_0^\infty H(X_s) \frac{1 - e^{-\lambda s}}{\lambda}\,ds \\
&= \frac{1}{\lambda} u(x_0) - \frac{1}{\lambda} S_\lambda H,
\end{aligned} \tag{4.7}$$

or $S_\lambda H = u(x_0) - \lambda S_\lambda u$. Define the operator R_λ by

$$R_\lambda f(x) = \mathbb{E}^x \int_0^\infty e^{-\lambda t} f(X_t)\,dt. \tag{4.8}$$

We thus have $u = R_0 H$ and so

$$S_\lambda H = R_0 H(x_0) - \lambda S_\lambda R_0 H. \tag{4.9}$$

Let

$$\Theta = \sup_{\|H\|_\infty \le 1} |S_\lambda H - R_\lambda H(x_0)|.$$

Note $\|R_\lambda H\|_\infty \le \lambda^{-1}\|H\|_\infty$ and $\|R_0 H\|_\infty \le c_1\|H\|_\infty$, where $c_1 = \sup_y \mathbb{E}^y T_K$. From the semigroup property of $\mathbb{Q}^x$ (cf. [B], p. 19),

$$R_\lambda H(x_0) = R_0 H(x_0) - \lambda R_\lambda R_0 H(x_0). \tag{4.10}$$

Subtracting (4.10) from (4.9),

$$|S_\lambda H - R_\lambda H(x_0)| = |\lambda(S_\lambda R_0 H - R_\lambda R_0 H(x_0))| \le \lambda\Theta\|R_0 H\|_\infty \le \lambda\Theta c_1\|H\|_\infty.$$

Taking the supremum over H with $\|H\|_\infty \le 1$, if $\lambda \le 1/2c_1$,

$$\Theta \le \lambda\Theta c_1 \le \Theta/2.$$

Since

$$\Theta \le \sup_{\|H\|_\infty \le 1} (|S_\lambda H| + \|R_\lambda H\|_\infty) \le 2/\lambda < \infty,$$

we have $\Theta = 0$ or $S_\lambda H = R_\lambda H(x_0)$. $\square$

Proof of Theorem 2.2. First suppose that $d \ge 3$ and D satisfies (2.2). Recall the notation of Proposition 4.8 and that $S_\lambda H = R_\lambda H(x_0)$. By the uniqueness of the Laplace transform and the continuity of $H(X_t)$ when H is continuous, $\mathbb{E}_{\mathbb{P}} H(X_{t\wedge T_K}) = \mathbb{E}^{x_0} H(X_{t\wedge T_K})$. As $\rho \to 0$, then $T_K \to T_{\{x_0\}}$. Since X_t behaves like a Brownian motion up until time $T(\partial D)$, then $T_{\{x_0\}}$ is identically infinite. So $\mathbb{E}_{\mathbb{P}} H(X_t) = \mathbb{E}^{x_0} H(X_t)$. By standard arguments (see [SV2], Chapter 6), this implies that the finite dimensional distributions of X_t under $\mathbb{P}$ and under $\mathbb{Q}^{x_0}$ agree. Therefore $\mathbb{P} = \mathbb{Q}^{x_0}$.

Now let D be an arbitrary Lipschitz domain. By standard piecing-together arguments (see [SV2]) and (2.8), it suffices to show that for each $x_0 \in \overline{D}$, any two solutions $\mathbb{P}_1$ and $\mathbb{P}_2$ agree in a neighborhood of x_0. That is, if $x_0 \in \overline{D}$, there exists $r > 0$ such that $\mathbb{P}_1$ and $\mathbb{P}_2$ agree on $\mathcal{F}_{T(\partial B(x_0,r))}$. Inside D, X_t under both $\mathbb{P}_1$ and $\mathbb{P}_2$ behaves like ordinary Brownian motion, so we need only consider $x_0 \in \partial D$. Let a coordinate system and a domain D' satisfying (2.2) be chosen so that D' agrees with D in a neighborhood $B(x_0, r) \cap D$ of x_0. Define $\mathbb{P}'_i$ for $i = 1, 2$ by

$$\mathbb{P}'_i(B \circ \theta_{T(\partial B(x_0,r))} \cap A) = \mathbb{E}_{\mathbb{P}_i}(\mathbb{Q}_{D'}^{X(T(\partial B(x_0,r)))}(B); A), \qquad A \in \mathcal{F}_{T(\partial B(x_0,r))}, \quad B \in \mathcal{F}_\infty,$$

where here $(\mathbb{Q}^x_{D'}, X_t)$ is the law of reflecting Brownian motion in D' started at x. As in the discussion preceding Proposition 4.1, $\mathbb{P}'_i$ is a solution to (2.10) in D' for $i = 1, 2$. By the uniqueness result for domains satisfying (2.2), $\mathbb{P}'_1 = \mathbb{P}'_2 = \mathbb{Q}^{x_0}_{D'}$. So if $A \in \mathcal{F}_{T(\partial B(x_0,r))}$, then $\mathbb{P}_1(A) = \mathbb{P}'_1(A) = \mathbb{P}'_2(A) = \mathbb{P}_2(A)$.

Finally we consider the case of $d = 2$. Let W_t be a one-dimensional Brownian motion reflecting at -1 and 1 and independent of X_t. Then the law of (X_t, W_t) is a solution to (2.10) for the Lipschitz domain $D \times (-1, 1)$, and so is unique. The uniqueness of the law of X_t follows easily. $\square$

5. Strong solutions.

A strong solution to (2.9) exists if there exists a process X_t satisfying (2.9) such that X is measurable with respect to the σ-fields of W. An interesting open problem is the following.

Problem 5.1. *Does there exist a strong solution to (2.9)?*

The reason for our interest is that if a strong solution exists, then in fact pathwise uniqueness holds for (2.9). That is, any two solutions to (2.9) must be identical. The proof of this is simple; cf. [K], Lemma 2.1.

Proposition 5.2. *Suppose a strong solution to (2.9) exists satisfying (2.10)(a)-(c). Then any two solutions to (2.9) that satisfy (2.10)(a)-(c) agree pathwise, a.s.*

Proof. Suppose $dY_t = dW_t + (1/2)\nu(Y_t)\, dL_t$ and Y_t is a strong solution. Then there exists a measurable map F from $C[0, \infty)$ to $C[0, \infty)$ such that $Y = F(W)$. Let X_t be another solution. We have

$$W_t = Y_t - \frac{1}{2}\int_0^t \nu(Y_s)\, dL_s, \qquad W_t = X_t - \frac{1}{2}\int_0^t \nu(X_s)\, dL_s. \tag{5.1}$$

The uniqueness in law (Theorem 2.2) says that the law of Y is equal to the law of X, so using (5.1) the law of the pair (Y, W) is equal to the law of the pair (X, W). Since $Y = F(W)$, then $X = F(W)$, a.s., and we then conclude that $X = F(W) = Y$, a.s. $\square$

Remark. We do not know the answer to Problem 5.1 even when D is a $C^{1+\alpha}$ domain and even when the dimension d is 2. (The obvious conformal mapping argument does not appear to help). A $C^{1+\alpha}$ domain is defined analogously to a Lipschitz domain, where we replace Lipschitz functions in the definition by functions whose gradient is Hölder continuous of order α.

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