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Local extinction for superprocesses in random environments

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Abstract

We consider a superprocess in a random environment represented by a random measure which is white in time and colored in space with correlation kernel $g(x, y)$. Suppose that $g(x, y)$ decays at a rate of $|x - y|^{-\alpha}$, $0 \leq \alpha \leq 2$, as $|x - y| \to \infty$. We show that the process, starting from Lebesgue measure, suffers longterm local extinction. If $0 \leq \alpha < 2$, then it even suffers *finite* time local extinction. This property is in contrast with the classical super-Brownian motion which has a non-trivial limit when the spatial dimension is higher than 2. We also show in this paper that in dimensions $d = 1, 2$ superprocess in random environment suffers

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local extinction for *any* bounded function g.

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1 Introduction

A system of branching particles whose branching probabilities depend on the random environment the particles are in is studied by Mytnik [20]. More specifically, the particles in this system move according to independent Brownian motions (with diffusion coefficient 2κ) in $\mathbb{R}^d$. At conditionally (given environment ξ) independent exponential times, each particle will split into two with probability $\frac{1}{2} + \frac{1}{2\sqrt{n}}((-\sqrt{n}) \vee \xi(t,x) \wedge \sqrt{n})$, and will die with probability $\frac{1}{2} - \frac{1}{2\sqrt{n}}((-\sqrt{n}) \vee \xi(t,x) \wedge \sqrt{n})$, where $\xi(t,x)$ is a random field satisfying

$$\mathbb{E}\xi(t,x) = 0, \qquad \mathbb{E}\xi(t,x)\xi(s,y) = \delta(t-s)g(x,y), \quad t \geq 0, x \in \mathbb{R}^d$$

g is a covariance function. Let $\mathcal{C}_b^k(\mathbb{R}^d)$ (respectively $\mathcal{C}_b^\infty(\mathbb{R}^d)$) denote the collection of all bounded continuous functions on $\mathbb{R}^d$ with bounded continuous derivatives up to order k (respectively with bounded derivatives of all orders). For all $\phi \in \mathcal{C}_b(\mathbb{R}^d)$, let $\langle \mu, \phi \rangle = \langle \phi, \mu \rangle$ denote the integral of ϕ with respect to the measure μ. $\langle f, g \rangle$ also means the integral of fg with respect to Lebesgue measure whenever it exists. Also let Δ be the d-dimensional Laplacian operator. It was proved in [20] that the high-density limit of the above system converges to a measure-valued process X which is a solution to the following martingale problem (MP): $\forall\ \phi \in \mathcal{C}_b^2(\mathbb{R}^d)$,

$$M_t^\phi \equiv \langle X_t, \phi \rangle - \langle \mu, \phi \rangle - \int_0^t \langle X_s, \kappa\Delta\phi \rangle \, ds, \qquad t \geq 0 \tag{1.1}$$

is a continuous martingale with quadratic variation process

$$\begin{aligned} \left\langle M^\phi \right\rangle_t &= 2\sigma^2 \int_0^t \left\langle X_s, \phi^2 \right\rangle ds \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(x,y)\phi(x)\phi(y) X_s(dx) X_s(dy) ds, \qquad t \geq 0. \end{aligned} \tag{1.2}$$

The uniqueness of the solution to the MP (1.1-1.2) is established in [20] by a limiting duality argument. Later, for the case of

$$g(x,y) = \sum_{i=1}^n h_i(x) h_i(y),$$

the uniqueness is re-established in Crisan [2] by conditional log-Laplace transform.

It is well known that the SBM starting from Lebesgue measure suffers longtime local extinction when $d = 2$, and finite time local extinction when $d = 1$. It is persistent for $d \geq 3$. The *aim* of this paper is to study the local extinction of this process X. To this end, we need to study the conditional log-Laplace transform for the process X as well as that of its occupation time process $\int_0^t X_s ds$.

For the rest of the paper let us fix the following assumption on the function g.

Assumption 1 There exist some constants $\tilde{c}_1, \tilde{c}_2 \geq 0$, and $\alpha \geq 0$ such that for all $x,\ y \in \mathbb{R}^d$,

$$\tilde{c}_1(|x-y|^{-\alpha} \wedge 1) \leq g(x,y) \leq \tilde{c}_2.$$

We now come to the *main* results of this paper which describe the conditions for the longtime local extinction for the superprocess in random environments denoted by X.

Theorem 1.1. *Fix $\kappa, \sigma^2 > 0$ and g satisfying Assumption 1 for some*

$$\alpha \in [0, 2),$$

and $\tilde{c}_1, \tilde{c}_2 > 0$. Let X be a solution to the martingale problem (MP) with $X_0 = \mu$ being the Lebesgue measure. Then X suffers local extinction in finite time, that is for any compact subset K of $\mathbb{R}^d$, there exists a random time N, such that

$$X_t(K) = 0, \quad \forall t \geq N.$$

Unfortunately for $\alpha = 2$ we have a weaker result.

Theorem 1.2. *Fix $\kappa, \sigma^2 > 0$, and g satisfying Assumption 1 for*

$$\alpha = 2$$

and some constants $\tilde{c}_1, \tilde{c}_2 > 0$. Let X be a solution to the martingale problem (MP) with $X_0 = \mu$ being the Lebesgue measure. Then, for any compact subset K of $\mathbb{R}^d$,

$$\lim_{t \to \infty} X_t(K) = 0, \quad \text{in probability.}$$

The above theorems hold in any dimension. This contrasts already mentioned behavior of the classical super-Brownian motion starting at Lebesgue measure — finite time local extinction in dimension $d = 1$, long-term local extinction in dimension $d = 2$ and persistence in dimensions $d \geq 3$. In fact in low dimensions we can recover the extinction results for any bounded function g.

Theorem 1.3. *Fix $\kappa, \sigma^2 > 0$ and g satisfying Assumption 1 for $\tilde{c}_1 = 0$ and some $\tilde{c}_2 \geq 0$. Let X be a solution to the martingale problem (MP) with $X_0 = \mu$ being the Lebesgue measure.*

(a) *Let $d = 1$. Then X suffers local extinction in finite time, that is for any compact subset K of $\mathbb{R}^d$, there exists a random time N, such that*

$$X_t(K) = 0, \quad \forall t \geq N.$$

(b) *Let $d = 2$. Then, for any compact subset K of $\mathbb{R}^d$,*

$$\lim_{t \to \infty} X_t(K) = 0, \quad \text{in probability.}$$

The rest of the paper is devoted to the proof of the above results.

Here we would like to give several comments on the results and the strategy of the proof. The proof of Theorem 1.3 is relatively easy and is based on comparison with the classical super-Brownian motion. The important part of the proofs of Theorems 1.1, 1.2 relies on the comparison of the process X with another process which can be formally described as a solution to the following SPDE

$$\frac{\partial \phi(t,x)}{\partial t} = \kappa \Delta \phi(t,x) + \phi(t,x)\dot{W}(t,x), \tag{1.3}$$

where $\dot{W}$ is a Gaussian noise white in time whose covariance function in space is given by g. For the precise definition of ϕ go to (2.7) in Section 2. From Corollary 2.16 and Lemma 2.19 one may easily conclude that X converges to zero whenever ϕ converges to zero as time goes to infinity. To get the local extinction of ϕ we adopt the ideas from Mueller and Tribe [19] (see Proposition 2 there). The crucial estimate is proved in Lemma 2.20. In Section 3 we use that lemma to show the local extinction of ϕ which allows to complete the proof of Theorem 1.2.

The proof of *finite time* local extinction of X (Theorem 1.1) essentially requires a finer analysis of the rate of convergence of ϕ to zero as time goes to infinity. This is done in Section 4 again with the help of the estimate from Lemma 2.20. In fact here we also have to use the branching structure of the process X and to play a bit with its Laplace transform in order to push through the Borel-Cantelli argument.

A few words about the possible extensions of the above result are in order. As we have mentioned already, an essential part of the argument is based on the analysis of the longterm behavior of the process ϕ satisfying (1.3). This equation has been extensively studied in the literature under name of Anderson model. In the recent years there were a number of papers studying the Lyapunov exponent for this model (see e.g. Carmona and Viens [1], Tindel and Viens [22], Florescu and Viens [8]). It is easy to conclude from the above results that for a large class of homogeneous noises and for all κ sufficiently small

$$\phi_t(x) \to 0, \ \text{ a.s., as } t \to \infty,$$

for any fixed x. If one can extend this for the integral setting, that is, to show that

$$\phi_t(K) \to 0, \ \text{ a.s., as } t \to \infty$$

then it seems possible also to extend Theorem 1.2 for a larger class of noises and small κ (we also discuss this issue in Remark 3.2). In order to extend Theorem 1.1 one may need even more delicate estimates (see Remark 4.3).

Here we would like to say a few words about what we expect to happen in the case of more rapid decay of correlation function g. Suppose the decay is faster than $|x-y|^{-\alpha}$ for some $\alpha > 2$. As we have established in Theorem 1.3, in dimensions $d = 1, 2$ nothing different from the super-Brownian motion case happens. As for the case of $d \geq 3$, the situation is a bit more complicated. Note that the equation (1.3) was studied by Dawson and Salehi [6] in the case of homogeneous noise $\dot{W}$, that is, $g(x,y) = q(x-y)$ for some function q. They proved (see Theorem 3.4 and Remark 4.2 in [6]) that if $q(0)$ is sufficiently small, then there exists a *non-trivial* limiting longterm distribution of a solution to (1.3). This and the fact that super-Brownian motion starting at the Lebesgue measure persists in dimensions $d \geq 3$ allows us to make the following conjecture

Conjecture 1.4. *Let $d \geq 3$. Fix $\alpha > 2$, $\kappa, \sigma^2 > 0$ and*

$$g(x,y) = q(x-y) \leq \tilde{c}_2(|x-y|^{-\alpha} \wedge 1).$$

Let X be a solution to the martingale problem (MP) with $X_0 = \mu$ being the Lebesgue measure. Then for $\tilde{c}_2$ sufficiently small X_t survives as $t \to \infty$, namely, X_t converges weakly to some random measure X_∞ such that $\mathbb{E}(X_\infty) = \mu$.

We do not settle this conjecture in the current paper. Another interesting question that is left unresolved in the current paper: whether it is possible to get a stronger result in Theorem 1.2, namely

Does the finite time local extinction hold in the case of $\alpha = 2$ and $d \geq 3$?

Unfortunately our method of proof does not allow us to answer this question. In fact, we even do not have a conjecture here. So this is left for a future investigation.

In this paper, we shall use $c, c_1, c_2, \ldots$ to denote non-negative constants whose values are not of a concern and can be changed from place to place. We also will need the following notation. For a set $\Gamma \subset \mathbb{R}^d$, let Γ^c be the complement of Γ. Let $\mathcal{B}_b = \mathcal{B}_b(\mathbb{R}^d)$ be the family of all bounded Borel measurable functions on $\mathbb{R}^d$ and $\mathcal{M}(\mathbb{R}^d)$ be the set of Radon measures on R^d. Let $\mathcal{C}_0^\infty = \mathcal{C}_0^\infty(\mathbb{R}^d)$ be the set of continuous infinitely differentiable functions with compact support. Let E_1, E_2 be two metric spaces. Then $\mathcal{C}(E_1, E_2)$ denotes the collection of all continuous functions from E_1 to E_2. In general if F is a set of functions, write F_+ or F^+ for non-negative functions in F.

The rest of the paper is organized as follows. In Section 2 we deduce the log-Laplace equation for the process X and study its properties. The proof of Theorems 1.2, 1.3 is given in Section 3. Theorem 1.1 is proved in Section 4.

2 Stochastic log-Laplace equation

Conditional log-Laplace transforms for a special case of g have been studied by Crisan [2]. For a related model, they have been investigated by Xiong [23]. It is demonstrated in Xiong [24] that the conditional log-Laplace transform is a powerful tool in the study of the longterm behavior for the superprocess under a stochastic flow.

In this paper, we shall study the current model by making use of the log-Laplace transform. In this section, we derive the stochastic log-Laplace equation for X as well as for its occupation measure process. We shall see in this paper that the conditional log-Laplace transform plays an important role in the study of the local extinction for the current model.

Let S_0 be the linear span of the set of functions $\{g(x, \cdot) : x \in \mathbb{R}^d\}$. We define an inner product on S_0 by

$$\langle g(x, \cdot), g(y, \cdot) \rangle_{\mathbb{H}} = g(x, y). \tag{2.1}$$

Let $\mathbb{H}$ be the completion of S_0 with respect to the norm $\|\cdot\|_{\mathbb{H}}$ corresponding to the inner product $\langle \cdot, \cdot \rangle_{\mathbb{H}}$. Then $\mathbb{H}$ is a Hilbert space which is called the reproducing kernel Hilbert space (RKHS) corresponding to the covariance function g. We refer the reader to Kallianpur [13], p. 139 for more details on RKHS.

By (2.1), it is easy to show that $\forall\, \phi \in \mathcal{C}_b(\mathbb{R}^d)$,

$$\left\| \int_{\mathbb{R}^d} g(x, \cdot)\phi(x) X_s(dx) \right\|_{\mathbb{H}}^2 = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(x, y)\phi(x)\phi(y) X_s(dx) X_s(dy).$$

Then the MP (1.1-1.2) becomes

$$M_t^\phi \equiv \langle X_t, \phi\rangle - \langle \mu, \phi\rangle - \int_0^t \langle X_s, \kappa\Delta\phi\rangle\, ds$$

is a continuous martingale with quadratic variation process

$$\left\langle M^\phi \right\rangle_t = 2\sigma^2 \int_0^t \left\langle X_s, \phi^2\right\rangle ds + \int_0^t \left\| \int_{\mathbb{R}^d} g(x,\cdot)\phi(x) X_s(dx)\right\|_{\mathbb{H}}^2 ds.$$

For the convenience of the reader, we roughly recall the definition of an $\mathbb{H}$-cylindrical Brownian motion ($\mathbb{H}$-CBM) and its stochastic integrals. We refer the reader to the book of Kallianpur and Xiong [14] for more details.

An $\mathbb{H}$-CBM W_t is a family of real-valued Brownian motions $\{B_t^h : \ h \in \mathbb{H}\}$ such that $\forall\ \beta_i \in \mathbb{R},\ h_i \in \mathbb{H},\ i = 1, 2,\ t \geq 0$,

$$B_t^{\beta_1 h_1 + \beta_2 h_2} = \beta_1 B_t^{h_1} + \beta_2 B_t^{h_2} \qquad a.s.$$

and

$$\left\langle B^{h_1}, B^{h_2}\right\rangle_t = \langle h_1, h_2\rangle_{\mathbb{H}}\, t.$$

For an $\mathbb{H}$-valued square-integrable predictable process f, the stochastic integral with respect to W is

$$\int_0^t \langle f(s,\cdot), dW_s\rangle_{\mathbb{H}} = \sum_{i=1}^\infty \int_0^t \langle f(s,\cdot), h_i\rangle_{\mathbb{H}}\, dB_s^{h_i}$$

where $\{h_i : \ i = 1, 2, \cdots\}$ is a complete orthonormal basis of $\mathbb{H}$.

Definition 2.1. *Let X be a measure-valued process and let W_t be an $\mathbb{H}$-CBM defined on the same stochastic basis $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$. Denote by $\mathbb{P}^W$ the conditional probability given W and define the σ-field $\mathcal{G}_t \equiv \mathcal{F}_t \vee \mathcal{F}_\infty^W$, where $\mathcal{F}_\infty^W$ is the σ-field generated by W. X is a solution to CMP with W_t if*

$$\begin{aligned} N_t^\phi \ &\equiv\ \langle X_t, \phi\rangle - \langle \mu, \phi\rangle - \int_0^t \langle X_s, \kappa\Delta\phi\rangle\, ds \\ &\quad - \int_0^t \left\langle \int_{\mathbb{R}^d} g(x,\cdot)\phi(x) X_s(dx), dW_s \right\rangle_{\mathbb{H}} \end{aligned} \tag{2.2}$$

is a continuous $(\mathbb{P}, \mathcal{G}_t)$-martingale with quadratic variation process

$$\left\langle N^\phi \right\rangle_t = 2\sigma^2 \int_0^t \left\langle X_s, \phi^2\right\rangle ds. \tag{2.3}$$

The proof of the following proposition will be postponed later in Proposition 2.17.

Proposition 2.2. *The CMP has a solution.*

Now we discuss the relation between CMP and the MP.

Lemma 2.3. *If X is a solution to the CMP with W_t, then X is a solution to the MP.*

Proof: It is clear that N_t^ϕ is also a $(\mathbb{P},\mathcal{F}_t)$-martingale. Denote

$$\hat{N}_t^\phi = \int_0^t \left\langle \int_{\mathbb{R}^d} g(x,\cdot)\phi(x)X_s(dx), dW_s \right\rangle_{\mathbb{H}}.$$

Then $M_t^\phi = N_t^\phi + \hat{N}_t^\phi$ is a $(\mathbb{P},\mathcal{F}_t)$-martingale. Note that for $t > s$ and $h \in \mathbb{H}$,

$$\begin{aligned}
\mathbb{E}(N_t^\phi B_t^h|\mathcal{F}_s) &= \mathbb{E}\left(\mathbb{E}(N_t^\phi B_t^h|\mathcal{G}_s)|\mathcal{F}_s\right) \\
&= \mathbb{E}\left(B_t^h \mathbb{E}(N_t^\phi|\mathcal{G}_s)|\mathcal{F}_s\right) \\
&= \mathbb{E}\left(B_t^h N_s^\phi|\mathcal{F}_s\right) \\
&= N_s^\phi B_s^h.
\end{aligned}$$

Thus $\langle N^\phi, W\rangle_t = 0$ which implies that $\left\langle N^\phi, \hat{N}^\phi\right\rangle_t = 0$. Therefore

$$\begin{aligned}
\left\langle M^\phi\right\rangle_t &= \left\langle N^\phi\right\rangle_t + \left\langle \hat{N}^\phi\right\rangle_t \\
&= 2\sigma^2 \int_0^t \left\langle X_s, \phi^2\right\rangle ds + \int_0^t \left\| \int_{\mathbb{R}^d} g(x,\cdot)\phi(x)X_s(dx)\right\|_{\mathbb{H}}^2 ds.
\end{aligned}$$

□

Corollary 2.4. *If X is a solution to the MP, then X is equal in distribution to a process Y that solves CMP (Y may be defined on a different probability space).*

Proof: Let Y be a solution to the CMP with the same initial distribution as X. Note that Y exists by Proposition 2.2. Then, by the previous lemma, Y is also a solution to the MP. By the uniqueness of Theorem 3.1 in Mytnik [20], we see that $X = Y$ in distribution. □

Next, we consider the following backward SPDE:

$$\begin{aligned}
\psi_{s,t}(x) &= \phi(x) + \int_s^t \kappa\Delta\psi_{r,t}(x)dr \\
&\quad + \int_s^t \left\langle g(x,\cdot)\psi_{r,t}(x), \hat{d}W_r\right\rangle_{\mathbb{H}} - \int_s^t \sigma^2\psi_{r,t}^2(x)dr, \qquad 0 \le s \le t
\end{aligned} \tag{2.4}$$

where $\int \ldots \hat{d}W_r$ is the backward Itô integral as defined by (e.g. [16], Section 3.4), i.e., for $f:\ [0,t]\times\Omega \to \mathbb{H}$ being square-integrable and $\tilde{f}(s) \equiv f(t-s)$ being predictable, we define

$$\int_0^t f(s)\hat{d}W_s = \int_0^t \tilde{f}(s)d\mathbb{W}_s$$

where $\mathbb{W}_s \equiv W_{t-s} - W_t$ is a backward $\mathbb{H}$-CBM. Roughly speaking, backward Itô integral is obtained by taking the right endpoints instead of the left ones in the approximating Riemann sum.

Remark 2.5. *Here we would like to make the following convention. Throughout this paper, by a solution to the SPDE we mean solution to the "weak" form of the equation. That is, when we claim that $\psi_{s,t}$ solves (2.4), rigorously we mean that for any test function $f \in \mathcal{C}_b^\infty(\mathbb{R}^d)$,*

$$\begin{aligned}\langle \psi_{s,t}, f\rangle &= \langle \phi, f\rangle + \int_s^t \langle \psi_{r,t}, \kappa\Delta f\rangle \, dr \\ &\quad + \int_s^t \left\langle \int_{\mathbb{R}^d} g(x,\cdot)\psi_{r,t}(x) f(x)\, dx, \hat{d}W_r \right\rangle_{\mathbb{H}} - \int_s^t \sigma^2 \left\langle \psi_{r,t}^2, f\right\rangle dr.\end{aligned} \tag{2.5}$$

Let us fix $t > 0$ and define

$$\psi_s \equiv \psi_{t-s,t}\,, \quad 0 \le s \le t.$$

Then it is easy to check that ψ_s solves the following forward version of (2.4):

$$\begin{aligned}\psi_s(x) &= \phi(x) + \int_0^s \left(\kappa\Delta\psi_r(x) - \sigma^2(\psi_r(x))^2\right) dr \\ &\quad + \int_0^s \langle g(x,\cdot)\psi_r(x), d\mathbb{W}_r\rangle_{\mathbb{H}}\,, \quad s \in [0,t],\end{aligned} \tag{2.6}$$

for some $\mathbb{H}$-cylindrical Brownian motion $\mathbb{W}$. For our purposes we will often use version (2.6) to simplify the exposition. Also note, (2.6) can be considered on all $s \ge 0$.

To make use of Kotelenez's results (Theorems 3.2, 3.3 and 3.4 in [15]), we introduce some notations. For $\rho > d$, let

$$\mathbb{H}_\rho = \left\{ \begin{array}{ll} f : \mathbb{R}^d \to \mathbb{R}; & f \text{ is measurable and} \\ \|f\|_\rho^2 = \int_{\mathbb{R}^d} f^2(x)\left(1+|x|^2\right)^{-\frac{\rho}{2}} dx < \infty & \end{array} \right\}.$$

(Note that there is a typo in [15] for this definition.) Since $\rho > d$, we have $1 \in \mathbb{H}_\rho$.

Let

$$W_t'(x) = \langle g(x,\cdot), \mathbb{W}_t\rangle_{\mathbb{H}}\,, \qquad t \ge 0.$$

Note that

$$\mathbb{E}\int_{\mathbb{R}^d} |W_t'(x)|^2 \left(1+|x|^2\right)^{-\frac{\rho}{2}} dx = t\int_{\mathbb{R}^d} g(x,x)\left(1+|x|^2\right)^{-\frac{\rho}{2}} dx < \infty.$$

Thus W_t' is a regular $\mathbb{H}_\rho$-valued cylindrical Brownian motion and (2.6) can be written in the form of (1.3) in [15]:

$$\begin{aligned}\psi_t(x) &= \phi(x) + \int_0^t \left(\kappa\Delta\psi_s(x) - \sigma^2(\psi_s(x))^2\right) ds \\ &\quad + \int_0^t \psi_s(x) dW_s'(x)\,, \quad t \in [0,t].\end{aligned}$$

In (1.3) of [15], two equations are considered. In our case, the second equation will be the following linear equation

$$\phi_t(x) = \phi(x) + \int_0^t \kappa\Delta\phi_s(x) ds + \int_0^t \langle g(x,\cdot)\phi_s(x), d\mathbb{W}_s\rangle_{\mathbb{H}}\,, \qquad t \ge 0. \tag{2.7}$$

This equation can also be written in the form of (1.3) in [15]:

$$\phi_t(x) = \phi(x) + \int_0^t \kappa\Delta\phi_s(x)ds + \int_0^t \phi_s(x)dW'_s(x), \qquad t \geq 0.$$

It is easy to verify the conditions in Theorems 3.2, 3.3 and 3.4 in [15]. Thus, we have the following

Lemma 2.6. *The SPDEs (2.6) and (2.7) have unique $\mathbb{H}^+_\rho$-valued solutions ψ_t and ϕ_t such that*

$$\psi_t(x) \leq \phi_t(x), \qquad \forall\ t \geq 0 \text{ and } x \in \mathbb{R}^d, \text{ a.s..}$$

Now we proceed to proving the continuity of ψ_t. Let $\mathcal{X}_0$ be the collection of measurable functions ϕ such that

$$0 \leq \phi(x) \leq c_\varphi\varphi_1(x), \qquad \forall\ x \in \mathbb{R}^d,$$

where c_φ is a constant (may depend on ϕ) and $\varphi_t(x)$ is the density of a normal random vector with mean 0 and covariance matrix tI.

Lemma 2.7. *Suppose that $\phi \in \mathcal{X}_0$, then for some $c_1 = c_1(t)$,*

$$\mathbb{E}\left(\psi_t(x_1)\cdots\psi_t(x_n)\right) \leq c_1\Pi_{i=1}^n\varphi_{2\kappa t+1}(x_i).$$

Proof: For simplicity of notation, we take $n = 2$, $x_1 = x$ and $x_2 = y$. By Itô's formula (use test function if necessary), we have

$$\begin{aligned} d\left(\phi_t(x)\phi_t(y)\right) &= \kappa\left(\phi_t(x)\Delta\phi_t(y) + \phi_t(y)\Delta\phi_t(x)\right)dt \\ &\quad + g(x,y)\phi_t(x)\phi_t(y)dt + d(mart.) \end{aligned}$$

Let

$$u_t(x,y) = \mathbb{E}\left(\phi_t(x)\phi_t(y)\right).$$

Then

$$\partial_t u_t(x,y) = \kappa\Delta_{2d}u_t(x,y) + g(x,y)u_t(x,y),$$

where Δ_{2d} is the $2d$-dimensional Laplacian operator. By Feymann-Kac formula, we get

$$\begin{aligned} u_t(x,y) &= \mathbb{E}_{(x,y)}\left(\phi(X_t)\phi(Y_t)\exp\left(\int_0^t g(X_s,Y_s)ds\right)\right) \\ &\leq e^{\tilde{c}_2 t}\mathbb{E}_{(x,y)}\left(\phi(X_t)\phi(Y_t)\right) \\ &\leq c_1\varphi_{2\kappa t+1}(x)\varphi_{2\kappa t+1}(y), \end{aligned}$$

where X and Y are independent d-dimensional Brownian motions with diffusion coefficient 2κ, and $c_1 = c_\varphi^2 e^{\tilde{c}_2 t}$. Now we are done by Lemma 2.6. □

Theorem 2.8. *Suppose that $\phi \in \mathcal{X}_0$, then (2.6) has a unique non-negative solution $\psi \in \mathcal{C}((0,\infty)\times\mathbb{R}^d, \mathbb{R}_+)$ a.s. Further, for any $T > 0$ and $\lambda \in \mathbb{R}$, we have*

$$\mathbb{E}\left(\sup_{t\leq T,\ x\in\mathbb{R}^d} e^{\lambda|x|}\psi_t(x)^p\right) < \infty. \tag{2.8}$$

Proof: The existence and uniqueness of the solution to (2.6) is proved in Lemma 2.6. Now we prove its smoothness and (2.8). By the convolution form, we have

$$\begin{aligned}\psi_t(x) &= T_t^\kappa \phi(x) - \int_0^t T_{t-s}^\kappa \psi_s^2(x) ds \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \varphi_{2\kappa(t-s)}(x-y)\psi_s(y) \left\langle g(y,\cdot), d\mathbb{W}_s \right\rangle_{\mathbb{H}} dy, \end{aligned} \tag{2.9}$$

where $\{T_t^\kappa,\ t \geq 0\}$ is the semigroup with generator $\kappa\Delta$. The continuity of the first two terms is easy. We denote the last term by $Y(t,x)$. Note that for any $\alpha \in \left(0, \frac{1}{2}\right)$ and $|x| < |y|$, we have

$$|\varphi_t(x) - \varphi_t(y)| \leq c\varphi_t(x) t^{-\alpha} |y-x|^\alpha.$$

Since the quadratic variation process of

$$\begin{aligned} & Y(t,x+h) - Y(t,x) \\ = & \int_0^t \int_{\mathbb{R}^d} \left(\varphi_{2\kappa(t-s)}(x+h-y) - \varphi_{2\kappa(t-s)}(x-y)\right) \psi_s(y) \left\langle g(y,\cdot), d\mathbb{W}_s \right\rangle_{\mathbb{H}} dy \end{aligned}$$

equals to

$$\int_0^t \left| \int_{\mathbb{R}^d} \left(\varphi_{2\kappa(t-s)}(x-y+h) - \varphi_{2\kappa(t-s)}(x-y)\right) \psi_s(y) g(y,\cdot) dy \right|_{\mathbb{H}}^2 ds,$$

we have

$$\begin{aligned} & \mathbb{E}\left(|Y(t,x+h) - Y(t,x)|^{2n}\right) \\ \leq & c\mathbb{E}\left\{ \left(\int_0^t \left| \int_{\mathbb{R}^d} \left(\varphi_{2\kappa(t-s)}(x-y+h) - \varphi_{2\kappa(t-s)}(x-y)\right) \psi_s(y) g(y,\cdot) dy \right|_{\mathbb{H}}^2 ds \right)^n \right\} \\ = & c\mathbb{E}\{ (\int_0^t \int_{\mathbb{R}^{2d}} \left(\varphi_{2\kappa(t-s)}(x-y+h) - \varphi_{2\kappa(t-s)}(x-y)\right) \\ & \quad \times \left(\varphi_{2\kappa(t-s)}(x-z+h) - \varphi_{2\kappa(t-s)}(x-z)\right) \psi_s(y)\psi_s(z) g(y,z) d(y,z) ds)^n \} \\ \leq & c \int_0^t ds_1 \cdots \int_0^t ds_n \int_{(\mathbb{R}^d)^{2n}} \Pi_{i=1}^n \varphi_{2\kappa(t-s_i)}(x-y_i)\varphi_{2\kappa(t-s_i)}(x-z_i)(t-s_i)^{-2\alpha}|h|^{2\alpha} \\ & \quad \times \mathbb{E}\Pi_{i=1}^n \psi_{s_i}(y_i)\psi_{s_i}(z_i) d(y,z). \end{aligned}$$

By Lemma 2.7, we can continue the above estimate with

$$\begin{aligned} & \mathbb{E}\left(|Y(t,x+h) - Y(t,x)|^{2n}\right) \\ \leq & c \int_0^t ds_1 \cdots \int_0^t ds_n \int_{(\mathbb{R}^d)^{2n}} \Pi_{i=1}^n \varphi_{2\kappa(t-s_i)}(x-y_i)\varphi_{2\kappa(t-s_i)}(x-z_i)(t-s_i)^{-2\alpha}|h|^{2\alpha} \\ & \quad \times \Pi_{i=1}^n \varphi_{1+2\kappa s_i}(y_i)\varphi_{1+2\kappa s_i}(z_i) d(y,z) \\ = & c \int_0^t ds_1 \cdots \int_0^t ds_n \left(\Pi_{i=1}^n (t-s_i)^{-2\alpha}\right) |h|^{2n\alpha} \varphi_{1+2\kappa t}(x)^{2n} \\ \leq & c|h|^{2n\alpha} e^{-n|x|^2}. \end{aligned}$$

Similarly, we can prove that

$$\mathbb{E}\left(|Y(t+u,x) - Y(t,x)|^n\right) \leq c|u|^{n\alpha} e^{-n|x|^2}.$$

Take n large such that $n\alpha > 2+d$, by a generalized Kolmogorov's theorem, we get the continuity of Y in (t,x). Further, for any $\lambda \in \mathbb{R}$, we have

$$\mathbb{E}\left(\sup_{0\le t\le T,\ x\in\mathbb{R}^d}\left(|Y(t,x)|^p e^{\lambda|x|}\right)\right) < \infty.$$

The same inequality holds for the first term of (2.9) and the second term is negative. (2.8) follows by taking $\lambda = 0$. $\square$

Now we proceed by constructing a process X that solves the CMP from Definition 2.1. We will also show that $\psi_{0,t}$ is the conditional log-Laplace transform of the process X. To this end, we consider the approximations for X and $\psi_{s,t}$.

Let $\epsilon > 0$. In the intervals $[2i\epsilon, (2i+1)\epsilon]$, $i = 0, 1, 2, \cdots$, X^ϵ is a SBM with initial $X^\epsilon_{2i\epsilon}$, i.e.,

$$M^\epsilon_t(\phi) \equiv \langle X^\epsilon_t, \phi\rangle - \langle X^\epsilon_{2i\epsilon}, \phi\rangle - \int_{2i\epsilon}^t \langle X^\epsilon_s, \kappa\Delta\phi\rangle\, ds$$

is a continuous martingale with quadratic variation process

$$\langle M^\epsilon(\phi)\rangle_t = 4\sigma^2\int_{2i\epsilon}^t \left\langle X^\epsilon_s, \phi^2\right\rangle ds;$$

and in the intervals $[(2i+1)\epsilon, 2(i+1)\epsilon]$, it is the solution to the following linear SPDE:

$$\langle X^\epsilon_t, \phi\rangle = \left\langle X^\epsilon_{(2i+1)\epsilon}, \phi\right\rangle + \int_{(2i+1)\epsilon}^t \langle X^\epsilon_s, \kappa\Delta\phi\rangle\, ds + \int_{(2i+1)\epsilon}^t \langle X^\epsilon_s(g\phi), dW^\epsilon_s\rangle_{\mathbb{H}},$$

here $\mu(f)$ also denote the integral of f with respect to the measure μ, and

$$W^\epsilon_t = \sqrt{2}\int_0^t 1_{A^c}(s)dW_s, \tag{2.10}$$

while $A = \{s :\ 2i\epsilon \le s \le (2i+1)\epsilon,\ i = 0, 1, 2, \cdots\}$. It is easy to see that $\{X^\epsilon, W^\epsilon\}$ is a solution to the following approximate martingale problem for X (AMPX): W^ϵ is as in (2.10);

$$B^{\epsilon,h}_t = \sqrt{2}\int_0^t 1_{A^c}(s)dB^h_s$$

and

$$\langle X^\epsilon_t, \phi\rangle = \langle \mu, \phi\rangle + \int_0^t \langle X^\epsilon_s, \kappa\Delta\phi\rangle\, ds + M^{1,\epsilon}_t(\phi) + M^{2,\epsilon}_t(\phi)$$

where $M^{1,\epsilon}_t(\phi)$, $M^{2,\epsilon}_t(\psi)$ are uncorrelated martingales satisfying

$$\left\langle M^{1,\epsilon}(\phi)\right\rangle_t = 4\sigma^2\int_0^t \left\langle X^\epsilon_s, \phi^2\right\rangle 1_A(s)ds,$$

$$\left\langle M^{1,\epsilon}(\phi), B^{\epsilon,h}\right\rangle_t = 0,$$

$$\left\langle M^{2,\epsilon}(\phi)\right\rangle_t = 2\int_0^t \|X^\epsilon_s(g\phi)\|^2_{\mathbb{H}} 1_{A^c}(s)ds$$

and

$$\left\langle M^{2,\epsilon}(\phi), B^{\epsilon,h}\right\rangle_t = 2\int_0^t X^\epsilon_s\left(\langle g, h\rangle_{\mathbb{H}}\,\phi\right) 1_{A^c}(s)ds$$

where $g(x) \equiv g(x, \cdot) \in \mathbb{H}$.

Lemma 2.9. *Suppose that $\mu(\mathbb{R}^d) < \infty$. Then, $\{X^\epsilon\}$ is tight in $\mathcal{C}(\mathbb{R}_+, \mathcal{M}_F(\mathbb{R}^d))$.*

Proof: Note that

$$\langle X_t^\epsilon, 1\rangle = \langle \mu, 1\rangle + M_t^{1,\epsilon}(1) + M_t^{2,\epsilon}(1).$$

By Burkholder-Davis-Gundy inequality, we get

$$\begin{aligned}
\mathbb{E}\sup_{s\le t}\langle X_t^\epsilon, 1\rangle^4 &\le 8\langle \mu, 1\rangle^4 + 8\left(\frac{4}{4-1}\right)^4 \mathbb{E}\left(4\sigma^2\int_0^t \langle X_s^\epsilon, 1\rangle 1_A(s)ds\right)^2 \\
&\quad +8\left(\frac{4}{4-1}\right)^4 \mathbb{E}\left(\int_0^t \tilde{c}_2^2 \langle X_s^\epsilon, 1\rangle^2 1_{A^c}(s)ds\right)^2 \\
&\le c_1 + c_2\int_0^t \mathbb{E}\langle X_s^\epsilon, 1\rangle^4 ds.
\end{aligned}$$

It follows from Gronwall's inequality that

$$\mathbb{E}\sup_{s\le t}\langle X_t^\epsilon, 1\rangle^4 \le c_3. \tag{2.11}$$

Next, for $0 < s < t$ and $\phi \in \mathcal{C}_\ell^2(\mathbb{R}^d)$, we have

$$\begin{aligned}
&\mathbb{E}\left|\langle X_t^\epsilon, \phi\rangle - \langle X_s^\epsilon, \phi\rangle\right|^4 \\
= \ &\mathbb{E}\left|\int_s^t \langle X_r^\epsilon, \kappa\Delta\phi\rangle dr + \int_s^t 1_A(r)\langle X_r^\epsilon(g\phi), dW_r\rangle_{\mathbb{H}} + \int_s^t 1_{A^c}(r)dM_r^\epsilon(\phi)\right|^4 \\
\le \ &8\mathbb{E}\left|\int_s^t \langle X_r^\epsilon, \kappa\Delta\phi\rangle dr\right|^4 + 8\left(\frac{4}{4-1}\right)^4 \mathbb{E}\left(\int_s^t \|X_s^\epsilon(g\phi)\|_{\mathbb{H}}^2 1_{A^c}(s)ds\right)^2 \\
&+8\left(\frac{4}{4-1}\right)^4 \mathbb{E}\left(4\sigma^2\int_s^t \left\langle X_s^\epsilon, \phi^2\right\rangle 1_A(s)ds\right)^2 \\
\le \ &c_4|t-s|^2,
\end{aligned} \tag{2.12}$$

where $\mathcal{C}_\ell^2(\mathbb{R}^d)$ is the collection of $C_b^2(\mathbb{R}^d)$ functions with limit at ∞. Note that the constant c_4 in (2.12) depends on $\|\phi\|_{2,\infty}$ only, where

$$\|\phi\|_{2,\infty} = \sum_{|\beta|\le 2}\sup_x |D^\beta\phi(x)|,$$

$\beta = (\beta_1, \cdots, \beta_d)$ is a multiindex and $|\beta| = \beta_1 + \cdots + \beta_d$. We can take a sequence $\{f_n\}$ in $\mathcal{C}_\ell^2(\mathbb{R}^d)$ such that $\|f_n\|_{2,\infty} \le 1$ for all $n \ge 1$. The weak topology of $\mathcal{M}_F(\bar{\mathbb{R}}^d)$ is given by the metric ρ defined by

$$\rho(\mu, \nu) \equiv \sum_{n=1}^\infty e^{-n}\left(|\langle \mu - \nu, f_n\rangle| \wedge 1\right),$$

where $\bar{\mathbb{R}}^d$ is the compactification of $\mathbb{R}^d$. By (2.12), it is easy to show that

$$\mathbb{E}\rho(X_t^\epsilon, X_s^\epsilon)^4 \le c_4|t-s|^2. \tag{2.13}$$

The tightness of X^ϵ in $\mathcal{C}(\mathbb{R}_+, \mathcal{M}_F(\bar{\mathbb{R}}^d))$ follows from (2.11) and (2.13) (cf. Corollary 16.9 in Kallenberg [12], p313).

If $R \to \infty$, then

$$\mathbb{E}\left(X_t^\epsilon(\phi_R)\right) = X_0(T_t^\kappa \phi_R) \to 0,$$

where $\phi_R(x) = 0$ for $|x| \le R$, $\phi_R(x) = 1$ for $|x| \ge R+1$ and connected by lines for x in between. Then for any $t \ge 0$, we have $X_t(\{\infty\}) = 0$. By the continuity of X, we get that almost surely, $X_t(\{\infty\}) = 0$ for all $t \ge 0$. Thus, X^ϵ is tight in $\mathcal{C}(\mathbb{R}_+, \mathcal{M}_F(\mathbb{R}^d))$. □

To study the tightness of W^ϵ, we need to define a space for which W^ϵ take their values. Let $\{h_j,\ j = 1, 2, \cdots\}$ be a complete orthonormal basis of $\mathbb{H}$. For any $h \in \mathbb{H}$, we define norms $\|h\|_i$, $i = 1,\ 2$, by

$$\|h\|_i^2 \equiv \sum_{j=1}^\infty j^{-2i} \left\langle h, h_j\right\rangle_{\mathbb{H}}^2 .$$

Let $\mathbb{B}_i$ be the completion of $\mathbb{H}$ with respect to $\|\cdot\|_i$. Then $\mathbb{H} \subset \mathbb{B}_1 \subset \mathbb{B}_2$ and the injections are compact.

Lemma 2.10. *$\{W^\epsilon\}$ is tight in $\mathcal{C}(\mathbb{R}_+, \mathbb{B}_2)$.*

Proof: Note that

$$\begin{aligned}
\mathbb{E}\sup_{r\le t}\|W_r^\epsilon\|_1^2 &= \mathbb{E}\sup_{r\le t}\sum_{j=1}^\infty j^{-2}\left(\sqrt{2}\int_0^r 1_{A^c}(s)dB_s^{h_j}\right)^2 \\
&\le 2\sum_{j=1}^\infty j^{-2}\mathbb{E}\sup_{r\le t}\left(\sqrt{2}\int_0^r 1_{A^c}(s)dB_s^{h_j}\right)^2 \\
&\le 2\sum_{j=1}^\infty j^{-2}4\mathbb{E}\int_0^t 1_{A^c}(s)ds \\
&\le 16t.
\end{aligned}$$

Since the injection from $\mathbb{B}_1$ to $\mathbb{B}_2$ is compact, $\{x \in \mathbb{B}_2 :\ \ \|x\|_1 \le K\}$ is compact in $\mathbb{B}_2$. Thus, $\{W^\epsilon\}$ satisfies the compact containment condition in $\mathbb{B}_2$.

Similarly, we can prove that, for any $s < t$ and $\epsilon > 0$,

$$\mathbb{E}\left(\|W_t^\epsilon - W_s^\epsilon\|_2^4\right) \le \mathbb{E}\left(\|W_t^\epsilon - W_s^\epsilon\|_1^4\right) \le c(t)|t-s|^2.$$

The tightness of $\{W^\epsilon\}$ in $\mathcal{C}(\mathbb{R}_+, \mathbb{B}_2)$ follows easily. □

Lemma 2.11. *Suppose that μ is finite. Let (X^0, W^0) be any limit point of $\{(X^\epsilon, W^\epsilon)\}$. Then (X^0, W^0) satisfy the following joint martingale problem (JMP): $W^0 = \{B^{0,h} :\ \ h \in \mathbb{H}\}$ is an $\mathbb{H}$-CBM and*

$$M_t(\phi) \equiv \left\langle X_t^0, \phi\right\rangle - \left\langle \mu, \phi\right\rangle - \int_0^t \left\langle X_s^0, \kappa\Delta\phi\right\rangle ds$$

is a continuous martingale with quadratic covariation processes

$$\left\langle M(\phi)\right\rangle_t = \int_0^t \left(2\sigma^2\left\langle X_s^0, \phi^2\right\rangle + \|X_s^0(g\phi)\|_{\mathbb{H}}^2\right) ds$$

and

$$\left\langle M(\phi), B^{0,h}\right\rangle_t = \int_0^t X_s^0\left(\left\langle g, h\right\rangle_{\mathbb{H}}\phi\right) ds.$$

Moreover, X^0 satisfies MP (1.1-1.2).

Proof: Immediate from the previous lemmas and the convergence of all the terms in AMPX to the corresponding terms in JMP. $\square$

Now we define an approximation $\psi^\epsilon_{s,t}$ of $\psi_{s,t}$ in two cases. Note that in our construction the process $\psi^\epsilon_{s,t}$ will be independent of X^ϵ *conditionally on* W^ϵ. First, we suppose that $2k\epsilon \le t < (2k+1)\epsilon$. Then for $2k\epsilon \le s \le t$, we define

$$\psi^\epsilon_{s,t} = \phi + \int_s^t \left(\kappa\Delta\psi^\epsilon_{r,t} - 2\sigma^2(\psi^\epsilon_{r,t})^2\right) dr.$$

For $(2k-1)\epsilon \le s \le 2k\epsilon$,

$$\psi^\epsilon_{s,t} = \psi^\epsilon_{2k\epsilon,t} + \int_s^{2k\epsilon} \kappa\Delta\psi^\epsilon_{r,t}dr + \int_s^{2k\epsilon} \psi^\epsilon_{r,t}\left\langle g, \hat{d}W^\epsilon_r\right\rangle_{\mathbb{H}}.$$

The definition continues in this pattern. For the case of $(2k+1)\epsilon \le t < 2(k+1)\epsilon$, the definition is modified in an obvious manner.

Since the behavior of the processes X^ϵ_s and $\psi^\epsilon_{s,t}$ does not depend on W^ϵ we get

$$\mathbb{E}^{W^\epsilon}_\mu\left(e^{-\langle X^\epsilon_t,\phi\rangle}|X^\epsilon_{2k\epsilon}\right) = e^{-\left\langle X^\epsilon_{2k\epsilon},\psi^\epsilon_{2k\epsilon,t}\right\rangle}.$$

Hence,

$$\mathbb{E}^{W^\epsilon}_\mu e^{-\langle X^\epsilon_t,\phi\rangle} = \mathbb{E}^{W^\epsilon}_\mu e^{-\left\langle X^\epsilon_{2k\epsilon},\psi^\epsilon_{2k\epsilon,t}\right\rangle}.$$

By Corollary 3.3 in Crisan and Xiong [3] (cf. Corollary 6.6.21 in Xiong [25] for the detailed proof for the case when W_t is a finite dimensional Brownian motion), we have

$$\left\langle X^\epsilon_{2k\epsilon}, \psi^\epsilon_{2k\epsilon,t}\right\rangle = \left\langle X^\epsilon_{(2k-1)\epsilon}, \psi^\epsilon_{(2k-1)\epsilon,t}\right\rangle,\ P-\text{a.s.}.$$

Therefore,

$$\mathbb{E}^{W^\epsilon}_\mu e^{-\langle X^\epsilon_t,\phi\rangle} = \mathbb{E}^{W^\epsilon}_\mu e^{-\left\langle X^\epsilon_{(2k-1)\epsilon},\psi^\epsilon_{(2k-1)\epsilon,t}\right\rangle}.$$

Continuing this pattern, we get

$$\mathbb{E}^{W^\epsilon}_\mu e^{-\langle X^\epsilon_t,\phi\rangle} = e^{-\left\langle \mu,\psi^\epsilon_{0,t}\right\rangle}. \tag{2.14}$$

Note again that in our construction the process $\psi^\epsilon_{s,t}$ is independent of X^ϵ conditionally on W^ϵ.

Lemma 2.12. *We endow $\mathbb{H}_0$ with weak topology. Then for any $t > 0$, $\{\psi^\epsilon_{\cdot,t}\}$ is tight in $\mathcal{C}([0,t],\mathbb{H}_0)$.*

Proof: For simplicity of presentation, we will consider the forward version of the equations. Also, we assume that $t = k'\epsilon$. We will consider the case of $k' = 2k$ only, since the other case can be treated similarly. Let $\mathbb{W}^\epsilon_s = W^\epsilon_{t-s} - W^\epsilon_t$. Then, for $2i\epsilon \le s \le (2i+1)\epsilon$, $0 \le i < k$,

$$\psi^\epsilon_s = \psi^\epsilon_{2i\epsilon} + \int_{2i\epsilon}^s \kappa\Delta\psi^\epsilon_r dr + \int_{2i\epsilon}^s \psi^\epsilon_r \langle g, d\mathbb{W}^\epsilon_r\rangle_{\mathbb{H}}, \tag{2.15}$$

and for $(2i+1)\epsilon \le s \le 2(i+1)\epsilon$, $0 \le i < k$,

$$\psi^\epsilon_s = \psi^\epsilon_{(2i+1)\epsilon} + \int_{(2i+1)\epsilon}^s \left(\kappa\Delta\psi^\epsilon_r - 2\sigma^2(\psi^\epsilon_r)^2\right) dr. \tag{2.16}$$

It is easy to show that the solution of (2.15) is an increasing functional of the initial condition $\psi^\epsilon_{2i\epsilon}$; and the solution of (2.16) is less than ϕ^ϵ_s given by

$$\phi^\epsilon_s = \phi^\epsilon_{(2i+1)\epsilon} + \int_{(2i+1)\epsilon}^s \kappa\Delta\phi^\epsilon_r dr, \qquad (2i+1)\epsilon \le s \le 2(i+1)\epsilon.$$

provided that $\phi^\epsilon_{(2i+1)\epsilon} \ge \psi^\epsilon_{(2i+1)\epsilon}$. For $2i\epsilon \le s \le (2i+1)\epsilon$, we define ψ^ϵ_s by (2.15) with $\psi^\epsilon_{2i\epsilon}$ replaced by $\phi^\epsilon_{2i\epsilon}$. Then, $\psi^\epsilon_s \le \phi^\epsilon_s$ for all $s \in [0,t]$.

Note that

$$\phi^\epsilon_s = \phi + \int_0^s \kappa\Delta\phi^\epsilon_r dr + \int_0^s 1_{A^c}(r)\phi^\epsilon_r \langle g, d\mathbb{W}^\epsilon_r\rangle_{\mathbb{H}}.$$

Applying Theorem 2.8 (taking $\sigma^2 = 0$) in each small interval of length ϵ used in the definition of ψ^ϵ, we get

$$\sup_{s\le t} \mathbb{E}\|\phi^\epsilon_s\|_0^{2p} < \infty, \quad \forall p > 0. \tag{2.17}$$

Let $Z^{\delta,\epsilon}_s = T^\kappa_\delta \phi^\epsilon_s$. By applying Itô's formula in order to get an expression for $(Z^{\delta,\epsilon}_s)^2$ and integrating we get

$$\begin{aligned}
\|Z^{\delta,\epsilon}_s\|_0^2 &= \|T^\kappa_\delta\phi\|_0^2 + 2\int_0^s \left\langle Z^{\delta,\epsilon}_r, \kappa\Delta Z^{\delta,\epsilon}_r\right\rangle dr \\
&\quad +2\int_0^s 1_{A^c}(r)\left\langle\left\langle T^\kappa_\delta(\phi^\epsilon_r g), Z^{\delta,\epsilon}_r\right\rangle, d\mathbb{W}^\epsilon_r\right\rangle_{\mathbb{H}} \\
&\quad +\int_0^s 1_{A^c}(r)\|T^\kappa_\delta(\phi^\epsilon_r g)\|^2_{\mathbb{H}_0\otimes\mathbb{H}} dr \\
&\le \|T^\kappa_\delta\phi\|_0^2 + 2\int_0^s \left\langle\left\langle T^\kappa_\delta(\phi^\epsilon_r g), Z^{\delta,\epsilon}_r\right\rangle, d\mathbb{W}^\epsilon_r\right\rangle_{\mathbb{H}} \\
&\quad +\int_0^s \|T^\kappa_\delta(\phi^\epsilon_r g)\|^2_{\mathbb{H}_0\otimes\mathbb{H}} dr.
\end{aligned}$$

Taking $\delta \to 0$, it can be easily derived with the help of (2.17) that all the terms in the above inequality converge and we get

$$\|\phi^\epsilon_s\|_0^2 \le \|\phi\|_0^2 + 2\int_0^s \langle\langle\phi^\epsilon_r g, \phi^\epsilon_r\rangle, d\mathbb{W}^\epsilon_r\rangle_{\mathbb{H}} + \int_0^s \|\phi^\epsilon_r\|_0^2 \tilde{c}_2^2 dr.$$

By Burkholder-Davis-Gundy inequality, we get

$$\mathbb{E}\sup_{r\le s}\|\phi^\epsilon_r\|_0^{2p} \le c_1 + c_2\int_0^s \mathbb{E}\|\phi^\epsilon_r\|_0^{2p} dr. \tag{2.18}$$

By (2.17) we immediately see that the last term in (2.18) is bounded, hence, by Gronwall's inequality, we have

$$\mathbb{E}\sup_{r\le s}\|\psi^\epsilon_r\|_0^{2p} \le \mathbb{E}\sup_{r\le s}\|\phi^\epsilon_r\|_0^{2p} \le c_1 e^{c_2 t}. \tag{2.19}$$

Note that

$$\psi^\epsilon_t = \phi + \int_0^t \kappa\Delta\psi^\epsilon_r dr + \int_0^t 1_{A^c}(r)\psi^\epsilon_r \langle g, d\mathbb{W}^\epsilon_r\rangle_{\mathbb{H}} - 2\sigma^2\int_0^t 1_A(r)(\psi^\epsilon_r)^2 dr,$$

and (2.8) easily holds with ψ_t replaced by ψ_t^ϵ. Then for any $f \in \mathbb{H}_0 \cap \mathcal{C}_b^2(\mathbb{R}^d)$, we have

$$\begin{aligned} \mathbb{E}\,|\langle \psi_t^\epsilon - \psi_s^\epsilon, f\rangle|_0^{2p} &\leq 3^{2p-1}\mathbb{E}\left|\int_s^t \langle \psi_r^\epsilon, \kappa\Delta f\rangle\, dr\right|^{2p} \\ &\quad +3^{2p-1}\mathbb{E}\left|\int_s^t 1_{A^c}(r)\,\langle\langle \psi_r^\epsilon g, f\rangle, d\mathbb{W}_r^\epsilon\rangle_{\mathbb{H}}\right|^{2p} \\ &\quad +3^{2p-1}\mathbb{E}\left|2\sigma^2\int_s^t 1_A(r)\int \psi_r^\epsilon(x)^2|f(x)|dxdr\right|^{2p} \\ &\leq c_3|t-s|^p. \end{aligned} \tag{2.20}$$

The tightness of $\{\psi^\epsilon\}$ then follows from (2.19) and (2.20) with $p > 1$. □

Corollary 2.13. *Let ψ^ϵ be a solution to (2.15)-(2.16). Then $\{\psi^\epsilon\}$ is tight in $\mathcal{C}(\mathbb{R}_+, \mathbb{H}_0)$.*

Proof: Immediate from the previous lemma. □

Lemma 2.14. *Suppose that $(\psi^0, \mathbb{W}^0)$ be a limit point of $(\psi^\epsilon, \mathbb{W}^\epsilon)$. Then*

$$\psi_s^0 = \phi + \int_0^s \left(\kappa\Delta\psi_r^0 - \sigma^2(\psi_r^0)^2\right) dr + \int_0^s \psi_r^0\,\langle g, d\mathbb{W}_r^0\rangle_{\mathbb{H}}\,. \tag{2.21}$$

Similarly, let $\{\psi_{\cdot,t}^0, W^0)$ be a limit point of $\{\psi_{\cdot,t}^\epsilon, W^\epsilon)$. Then

$$\psi_{s,t}^0 = \phi + \int_s^t \left(\kappa\Delta\psi_{r,t}^0 - \sigma^2(\psi_{r,t}^0)^2\right) dr + \int_s^t \psi_{r,t}^0 \left\langle g, \hat{d}W_{r,t}^0\right\rangle_{\mathbb{H}}.$$

Proof: Note that for any $f \in \mathcal{C}_0^2(\mathbb{R}^d)$,

$$N_t^\epsilon(f) \equiv \langle \psi_t^\epsilon, f\rangle - \langle \phi, f\rangle - \int^t \left(\langle \psi_r^\epsilon, \kappa\Delta f\rangle - 1_A(r)\left\langle 2\sigma^2(\psi_r^\epsilon)^2, f\right\rangle\right) dr$$

is a martingale with

$$\langle N^\epsilon(f)\rangle_t = 2\int_0^t 1_{A^c}(r)\,\langle \psi_r^\epsilon, fg\rangle_{\mathbb{H}_0\otimes\mathbb{H}}^2\, dr$$

and for any $h \in \mathbb{H}$,

$$\left\langle N^\epsilon(f), \mathbb{B}^{\epsilon,h}\right\rangle_t = 2\int_0^t 1_{A^c}(r)\,\langle \psi_r^\epsilon, \langle h, g\rangle_{\mathbb{H}}\, f\rangle\, dr.$$

Passing to the limit, we see that

$$N_t(f) \equiv \left\langle \psi_t^0, f\right\rangle - \langle \phi, f\rangle - \int_0^t \left(\left\langle \psi_r^0, \kappa\Delta f\right\rangle - \left\langle \sigma^2(\psi_r^0)^2, f\right\rangle\right) dr$$

is a martingale with

$$\langle N(f)\rangle_t = \sum_{j=1}^\infty \int_0^t \left\langle \psi_r^0, fg_j\right\rangle_0^2 dr,$$

$$\left\langle \mathbb{B}^{0,h_j}, \mathbb{B}^{0,h_k}\right\rangle_t = \delta_{jk}t$$

and

$$\left\langle N(f), \mathbb{B}^{0,h_j} \right\rangle_t = \int_0^t \left\langle \psi_r^0, g_j f \right\rangle dr,$$

where $g_j = \langle g, h_j \rangle_{\mathbb{H}}$. Similar to Theorem 3.3.6 in Kallianpur and Xiong [14], there exists an $\mathbb{H}$-CBM $\mathbb{W}$ such that

$$N_t(f) = \sum_{j=1}^{\infty} \int_0^t \left\langle \psi_r^0, g_j f \right\rangle d\mathbb{B}_r^{h_j}$$

and

$$\mathbb{B}_t^{0,h_j} = \mathbb{B}_t^{h_j}.$$

Thus, $\mathbb{W} = W^0$ and hence, (2.21) holds. □

With the above preparation, we can prove the following theorem for the Laplace transform of X.

Theorem 2.15. *The backward SPDE (2.4) has a pathwise unique non-negative solution $\psi_{s,t}(x)$ for any $\phi \in \mathcal{X}_0$. Moreover, there exists a triple (X^0, W^0, ψ^0) such that X and X^0 have the same law, and for any $\mu \in \mathcal{M}(\mathbb{R}^d)$, we have*

$$\mathbb{E}_\mu^{W^0} \exp\left(-\left\langle X_t^0, \phi \right\rangle\right) = \exp\left(-\left\langle \mu, \psi_{0,t}^0 \right\rangle\right). \tag{2.22}$$

Proof: First, we suppose μ is finite. Making use of Lemmas 2.9, 2.10, 2.11, 2.12 and 2.14, it follows from (2.14) that for F being a real valued continuous function on $\mathcal{C}([0,t], \mathbb{B}_2)$, we have

$$\begin{aligned}
\mathbb{E}\left(\exp\left(-\left\langle \mu, \psi_{0,t}^0 \right\rangle\right) F(W^0)\right) &= \lim_{\epsilon \to 0} \mathbb{E}\left(\exp\left(-\left\langle \mu, \psi_{0,t}^\epsilon \right\rangle\right) F(W^\epsilon)\right) \\
&= \lim_{\epsilon \to 0} \mathbb{E}\left(\exp\left(-\left\langle X_t^\epsilon, \phi \right\rangle\right) F(W^\epsilon)\right) \\
&= \mathbb{E}\left(\exp\left(-\left\langle X_t^0, \phi \right\rangle\right) F(W^0)\right).
\end{aligned}$$

The conclusion for general μ follows from a limiting argument. □

We will abuse the notation a bit by dropping the superscript 0 in (2.22).

Corollary 2.16. *For any $f \in \mathcal{X}_0$,*

$$\mathbb{E}_\mu e^{-\langle X_t, f \rangle} \geq \mathbb{E} e^{-\langle \mu, \phi_t \rangle}. \tag{2.23}$$

where ϕ is a solution to (2.7) with initial condition $\phi_0 = f$.

Proof By Lemma 2.6 and Theorem 2.15, the result is immediate. □

As another consequence of the theorem, we now prove that the CMP has a solution.

Proposition 2.17. *If (X, W) is a solution to the JMP, then it is a solution to the CMP.*

Proof: Let $M_t(\phi)$ be defined by JMP with superscript 0 dropped. Define

$$M_t^1(\phi) = M_t(\phi) - \int_0^t \left\langle X_s(g\phi), dW_s \right\rangle_{\mathbb{H}}.$$

Then for any $\phi \in \mathcal{C}_b(\mathbb{R}^d)$ and $h \in \mathbb{H}$, we have

$$\left\langle M^1(\phi), B^h \right\rangle_t = 0.$$

Note that M_t^1 can be regarded as a martingale in the dual of a nuclear space Φ' (e.g., the space of Schwarz distributions). It follows from the martingale representation theorem (cf. Theorem 3.3.6 in Kallianpur and Xiong [14]) that there exists a Φ'-valued Brownian motion $\mathcal{W}$ such that

$$M_t^1 = \int_0^t f(s) d\mathcal{W}_s$$

where f is an appropriate integrand. Further, $\mathcal{W}$ and W are uncorrelated and hence, independent. Thus, M_t^1 is a conditional martingale given W. Thus, (X, W) is a solution to the CMP. $\square$

Proof of Proposition 2.2: It follows immediately from the above Proposition 2.17 and Lemma 2.11.

Now we consider the log-Laplace transform for the occupation measure process.

Theorem 2.18. *$\forall f, \varphi \in \mathcal{X}_0$ and $\mu \in \mathcal{M}(\mathbb{R}^d)$*

$$\mathbb{E}_\mu \exp\left(- \langle X_t, f\rangle - \int_0^t \langle X_s, \varphi\rangle \, ds\right) \geq \exp\left(- \langle \mu, V_t\rangle\right). \tag{2.24}$$

where $V_t(\cdot) \equiv V_t(\varphi, f, \cdot)$ is the unique solution to the following nonlinear PDE:

$$\begin{cases} \frac{\partial}{\partial t} V_t = \kappa \Delta V_t - \sigma^2 V_t^2 + \varphi \\ V_0 = f. \end{cases} \tag{2.25}$$

Proof To simplify the notation and the exposition we present the proof only for the case of $f = 0$. However for a general non-negative f the proof goes along the same lines.

Fix arbitrary $\varphi \in \mathcal{B}_b^+$ and $t > 0, n > 1$. Denote $t_i = \frac{i}{n} t,\ i = 1, \ldots, n$. First we show that there exists a non-negative function-valued functional $V_s^{(n)}(\varphi)$ such that

$$\mathbb{E}_\mu \exp\left(-\frac{1}{n} \sum_{i=1}^n \langle X_{t_i}, \varphi\rangle\right) \geq \exp\left(- \left\langle \mu, V_t^{(n)}(\varphi)\right\rangle\right), \tag{2.26}$$

where $V^{(n)}$ satisfies:

$$V_{t-s}^{(n)} = \frac{n-i}{n}\varphi + \int_s^t \kappa \Delta V_{t-r}^{(n)} dr - \int_s^t \sigma^2 (V_{t-r}^{(n)})^2 dr, \tag{2.27}$$

for $s \in [t_i, t_{i+1})$, $i = 0, 1, \cdots, n-1$.

For simplicity of notation, we consider the case $n = 2$, and replace φ by 2φ. By Theorem 2.15,

$$\mathbb{E}_\mu^W \exp\left(- \langle X_{t_1}, \varphi\rangle - \langle X_{t_2}, \varphi\rangle\right) = \mathbb{E}_\mu^W \exp\left(- \langle X_{t_1}, \varphi + \psi_{t_1,t_2}(\varphi)\rangle\right) \tag{2.28}$$

where ψ_{s,t_2}, $s \in [t_1, t_2]$, is the unique solution to the following backward SPDE:

$$\begin{aligned}\psi_{s,t_2}(x) &= \varphi(x) + \int_s^{t_2} \kappa \Delta \psi_{r,t_2}(x) dr \\ &\quad + \int_s^{t_2} \left\langle g(x,\cdot)\psi_{r,t_2}(x), \hat{d}W_r \right\rangle_{\mathbb{H}} - \int_s^{t_2} \sigma^2 \psi_{r,t_2}^2(x) dr.\end{aligned} \tag{2.29}$$

By (2.8) and Assumption 1, we have

$$\begin{aligned}\mathbb{E}\int_0^{t_2} \|g(x,\cdot)\psi_{r,t_2}(x)\|_{\mathbb{H}}^2 dr &= \mathbb{E}\int_0^{t_2} g(x,x)\psi_{r,t_2}(x)^2 dr \\ &\leq T\tilde{c}_2 \sup_{0\leq t\leq T} \sup_{x\in\mathbb{R}^d} \mathbb{E}\psi_t(x)^2 < \infty.\end{aligned}$$

Thus,

$$s \mapsto \int_s^{t_2} \left\langle g(x,\cdot)\psi_{r,t_2}(x), \hat{d}W_r \right\rangle_{\mathbb{H}}$$

is a backward martingale (not just a local martingale) on $[0, t_2]$ and hence we can take expectation on both sides of (2.29) to get

$$\mathbb{E}\psi_{s,t_2}(x) \leq \varphi(x) + \int_s^{t_2} \kappa\Delta\mathbb{E}\psi_{r,t_2}(x)dr - \int_s^{t_2} \sigma^2(\mathbb{E}\psi_{r,t_2}(x))^2 dr.$$

Hence

$$\mathbb{E}\psi_{s,t_2}(x) \leq V_{t_2-s}(0,\varphi,x). \tag{2.30}$$

By Jensen's inequality, we can continue (2.28) with

$$\begin{aligned}\mathbb{E}_\mu^W \exp\left(-\langle X_{t_1},\varphi\rangle - \langle X_{t_2},\varphi\rangle\right) &= \mathbb{E}_\mu^W \mathbb{E}_\mu^W \left(\exp\left(-\langle X_{t_1},\varphi+\psi_{t_1,t_2}(\varphi)\rangle\right) | \mathcal{F}_{t_1}\right) \\ &\geq \mathbb{E}_\mu^W \exp\left(-\mathbb{E}_\mu^W \left(\langle X_{t_1},\varphi+\psi_{t_1,t_2}(\varphi)\rangle | \mathcal{F}_{t_1}\right)\right) \\ &\geq \mathbb{E}_\mu^W \exp\left(-\langle X_{t_1},\varphi+V_{t_2-t_1}(0,\varphi)\rangle\right) \\ &= \exp\left(-\langle \mu, \psi_{0,t_1}(\varphi+V_{t_2-t_1}(0,\varphi))\rangle\right).\end{aligned}$$

Similar to (2.30), we then have

$$\mathbb{E}_\mu \exp\left(-\langle X_{t_1},\varphi\rangle - \langle X_{t_2},\varphi\rangle\right) \geq \exp\left(-\langle \mu, V_{t_1}(0,\varphi+V_{t_2-t_1}(0,\varphi))\rangle\right).$$

Let

$$V_s^{(2)} = \begin{cases} V_{t_2-s}(0,\varphi) & \text{if } t_1 \leq s \leq t_2 \\ V_{t_1-s}(0,\varphi+V_{t_2-t_1}(0,\varphi)) & \text{if } 0 \leq s \leq t_1. \end{cases}$$

This proves (2.27) for $n = 2$. The general case follows by induction. As we have mentioned the proof for arbitrary $f, \varphi \in \mathcal{B}_b^+$ follows along the same lines.

□

Next we prove the following self-duality for ϕ_t.

Lemma 2.19. *Suppose that $\tilde{\phi}_t, \phi_t$ satisfy (2.7) with initial conditions ϕ_0, $\tilde{\phi}_0 \in \mathcal{B}_b^+$. Then* $\forall\ \lambda > 0$,

$$\mathbb{E}e^{-\lambda\langle \phi_t, \tilde{\phi}_0\rangle} = \mathbb{E}e^{-\lambda\langle \tilde{\phi}_t, \phi_0\rangle}. \tag{2.31}$$

Proof The result is immediate from Theorems 2.15 with $\sigma^2 = 0$. $\square$

The next lemma is crucial for the proof of Theorems 1.1 and 1.2. Let $\{R_t\}_{t\geq 0}$ be a strictly positive non-decreasing function. Also define

$$\Gamma_t = \{x \in \mathbb{R}^d : \ |x| < R_t\}.$$

Lemma 2.20. *Let ϕ_t be a solution to (2.7), and $\phi_0 \in \mathcal{B}_b^+$ be with compact support. Then there exist constants $c_1, c_2 > 0$ such that*

$$\mathbb{E}\,\langle \phi_t, 1\rangle^{\frac{1}{2}} \leq \langle \phi_0, 1\rangle^{\frac{1}{2}}\, e^{-c_2 \int_0^t \frac{1}{R_s^\alpha} ds} + c_1 \int_0^t \frac{1}{R_s^\alpha} \left(\int_{\Gamma_s^c} T_s^\kappa \phi_0(x) dx \right)^{\frac{1}{2}} e^{-c_2 \int_s^t \frac{1}{R_r^\alpha} dr} ds. \tag{2.32}$$

Proof Apply Itô's formula to obtain

$$\begin{aligned}
\langle \phi_t, 1\rangle^{\frac{1}{2}} &= \langle \phi_0, 1\rangle^{\frac{1}{2}} + \frac{1}{2}\int_0^t \langle \phi_t, 1\rangle^{-\frac{1}{2}} \left\langle \int_{\mathbb{R}^d} g(x,\cdot)\phi_s(x)dx, d\mathbb{W}_s \right\rangle_{\mathbb{H}} \\
&\quad -\frac{1}{8}\int_0^t \langle \phi_s, 1\rangle^{-\frac{3}{2}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(x,y)\phi_s(x)\phi_s(y)dxdyds.
\end{aligned}$$

Let $\pi_t(x) = \frac{\phi_t(x)}{\langle \phi_t, 1\rangle}$. Then, by Assumption 1,

$$\begin{aligned}
\mathbb{E}\,\langle \phi_t, 1\rangle^{\frac{1}{2}} &\leq \langle \phi_0, 1\rangle^{\frac{1}{2}} - \frac{1}{8}\mathbb{E}\int_0^t \langle \phi_s, 1\rangle^{\frac{1}{2}} \left(\int_{\Gamma_s}\int_{\Gamma_s} c(2R_s)^{-\alpha} \pi_s(x)\pi_s(y)dxdy \right) ds \\
&\leq \langle \phi_0, 1\rangle^{\frac{1}{2}} - c_2 \int_0^t \frac{1}{R_s^\alpha}\mathbb{E}\,\langle \phi_s, 1\rangle^{\frac{1}{2}} ds \\
&\quad + c_1 \int_0^t \frac{1}{R_s^\alpha}\mathbb{E}\left(\langle \phi_s, 1\rangle^{\frac{1}{2}} \int_{\Gamma_s^c} \pi_s(x)dx \right) ds \\
&\leq \langle \phi_0, 1\rangle^{\frac{1}{2}} - c_2 \int_0^t \frac{1}{R_s^\alpha}\mathbb{E}\,\langle \phi_s, 1\rangle^{\frac{1}{2}} ds \\
&\quad + c_1 \int_0^t \frac{1}{R_s^\alpha}\mathbb{E}\left\{ \left(\int_{\Gamma_s^c} \phi_s(x)dx \right)^{\frac{1}{2}} \right\} ds \\
&\leq \langle \phi_0, 1\rangle^{\frac{1}{2}} - c_2 \int_0^t \frac{1}{R_s^\alpha}\mathbb{E}\,\langle \phi_s, 1\rangle^{\frac{1}{2}} ds \\
&\quad + c_1 \int_0^t \frac{1}{R_s^\alpha} \left(\int_{\Gamma_s^c} T_s^\kappa \phi_0(x)dx \right)^{\frac{1}{2}} ds
\end{aligned}$$

Then, by Gronwall's inequality, we are done. $\square$

3 Proof of Theorems 1.2, 1.3

Without loss of generality, we may assume that $K = B(0,1)$, the unit ball in $\mathbb{R}^d$. Fix $\psi_0 = 1_{B(0,1)}$. Recall that $X_0 = \mu$ is Lebesgue measure.

Let Y be the classical super-Brownian motion starting at Lebesgue measure μ, that is Y is a solution to the martingale problem (MP) with $g \equiv 0$. Let $V_t(\cdot)$ be as in Theorem 2.18. It is well known (see e.g. Theorem 3.1 of Iscoe [10]) that V_t is a log-Laplace transform of Y_t. Hence from Theorem 2.18 we immediately get

$$\begin{aligned}
&\mathbb{E}_\mu \exp\left(- \langle X_t, f\rangle - \int_0^t \langle X_s, \phi\rangle \, ds \right) \qquad (3.1)\\
&\geq \quad \mathbb{E}_\mu \exp\left(- \langle Y_t, f\rangle - \int_0^t \langle Y_s, \phi\rangle \, ds \right), \quad \forall f, \phi \in \mathcal{X}_0,\ t \geq 0.
\end{aligned}$$

Now we are ready to give a

Proof of Theorem 1.3 (b) Fix $f = \psi_0 = 1_{B(0,1)}, \phi = 0$ in (3.1). By Theorem 3.1 of Dawson [4], the classical super-Brownian motion exhibits longterm local extinction, and hence the right hand side converges to 1 as $t \to \infty$. Hence, $\langle X_t, f\rangle$ convergence 0, and the part **(b)** of the theorem follows. □

Proof of Theorem 1.3 (a) It follows from the proof of Theorem 3 of Iscoe [11] that it is enough to show

$$\lim_{t\to\infty} \mathbb{P}\left(\int_t^\infty X_s(\psi_0)\, ds = 0 \right) = 1. \qquad (3.2)$$

However the left hand side of (3.2) equals to

$$\lim_{t\to\infty} \lim_{m\to\infty} \mathbb{E}_\mu \exp\left(-m \int_t^\infty \langle X_s, \psi_0\rangle \right) ds. \qquad (3.3)$$

Hence by (3.1) it is enough to show that

$$\lim_{t\to\infty} \lim_{m\to\infty} \mathbb{E}_\mu \exp\left(-m \int_t^\infty \langle Y_s, \psi_0\rangle \right) ds = 1. \qquad (3.4)$$

(3.4) follows from the proof of Theorem 3 of Iscoe [11]. □

Proof of Theorem 1.2 By Theorem 2.15, we have

$$\mathbb{E}_\mu \exp\left(- \langle X_t, \psi_0\rangle\right) = \mathbb{E} \exp\left(- \langle \psi_t, 1\rangle\right) \qquad (3.5)$$

and $\langle \psi_t, 1\rangle$ satisfies the following equation:

$$\langle \psi_t, 1\rangle = \langle \psi_0, 1\rangle + \int_0^t \left\langle \int_{\mathbb{R}^d} g(x,\cdot)\psi_s(x)dx, d\mathbb{W}_s \right\rangle_{\mathbb{H}} - \int_0^t \int_{\mathbb{R}^d} \sigma^2 \psi_s^2(x) dx ds.$$

By Lemma 2.6 we know that

$$\psi_t \leq \phi_t\,, \quad t \geq 0,$$

where ϕ_t solves (2.7) with $\phi_0 = \psi_0$. This and Lemma 2.20 imply

$$\mathbb{E}\left\langle \psi_t, 1\right\rangle^{\frac{1}{2}} \le \left\langle \psi_0, 1\right\rangle^{\frac{1}{2}} e^{-c_2\int_0^t \frac{1}{R_s^\alpha}ds} + c_1 \int_0^t \frac{1}{R_s^\alpha}\left(\int_{\Gamma_s^c} T_s^\kappa \psi_0(x)dx\right)^{\frac{1}{2}} e^{-c_2\int_s^t \frac{1}{R_r^\alpha}dr}ds. \tag{3.6}$$

Take $R_t = \max\{\sqrt{|t\log\log t|}, 1\}$. Fix arbitrary $\epsilon \in (0,1)$. Then it is easy to check that there exists a constant c such that

$$\int_s^t \frac{1}{R_r^2}dr \ge c\left((\log t)^\epsilon - (\log s)^\epsilon\right).$$

for all t sufficiently large. Hence, by (3.6),

$$\begin{aligned}
\mathbb{E}\left\langle \psi_t, 1\right\rangle^{\frac{1}{2}} &\le \left\langle \psi_0, 1\right\rangle^{\frac{1}{2}} e^{-c(\log t)^\epsilon} \\
&\quad + c_1 e^{-c_2(\log t)^\epsilon}\int_0^t \frac{1}{\max\{|s\log\log s|, 1\}} e^{c|\log s|^\epsilon}ds \\
&\to 0, \ \text{ as } t\to\infty.
\end{aligned} \tag{3.7}$$

(3.7) implies that $Z_t \equiv \langle \psi_t, 1\rangle$ converges to 0 in probability, and hence also weakly. Z_t is non-negative, therefore from the definition of the weak convergence we get

$$\begin{aligned}
\lim_{t\to\infty}\mathbb{E}\left[e^{-Z_t}\right] &= \mathbb{E}\left[e^{-Z_\infty}\right] \\
&= 1.
\end{aligned}$$

This and (3.5) in turn imply that

$$\mathbb{E}e^{-X_t(\psi_0)} \to 1, \qquad \text{as } t\to\infty.$$

Hence $X_t(B(0,1)) \to 0$ in probability as $t\to\infty$ and we are done. □

Remark 3.1. *When ψ_t is replaced by the Anderson model (cf. (2.7)) ϕ_t, the estimate (3.7) was given by Mueller and Tribe [19]. In fact, the proof of the theorem is inspired by this estimate.*

Remark 3.2. *Here we would like to make a remark about the possible extensions of Theorem 1.2. As we have already mentioned in the Introduction, the Anderson model (2.7) was considered in a number of papers in the recent years (see e.g. Carmona and Viens [1], Tindel and Viens [22], Florescu and Viens [8]). Although it was investigated in the Stratonovich setting the reformulation of their results for the Itô equation considered here is possible in some cases. For example, let the Gaussian noise be homogeneous, that is $g(x,y) = g(x-y)$, and the Fourier transform $\hat{g}$ of g satisfies*

$$\int_{\mathbb{R}^d} |\lambda|^\beta \hat{g}(d\lambda) < \infty \tag{3.8}$$

for some $\beta > 0$. Then it can be deduced from Theorem 2 of Carmona and Viens [1] that there exists a constant c such that for all κ sufficiently small

$$\limsup_{t\to\infty} t^{-1}\log\phi(t,x) \le \frac{c}{\log\kappa^{-1}} - \frac{g(0)}{2}, \ \text{ a.s., } \forall x. \tag{3.9}$$

Note that the correction term $g(0)/2$ comes from the Itô formulation of the equation. From (3.9) we get that for κ sufficiently small

$$\lim_{t\to\infty} \phi(t,x) = 0, \quad \text{a.s.}, \ \forall x. \tag{3.10}$$

Therefore to prove Theorem 1.2 in the case when κ is small and the noise satisfy condition (3.8) it is sufficient to extend (3.9) to the integral setting:

$$\limsup_{t\to\infty} t^{-1} \log\left(\int_K \phi(t,x)\,dx\right) \le \frac{c}{\log \kappa^{-1}} - \frac{g(0)}{2}, \quad \text{a.s.}, \tag{3.11}$$

for any compact set K. (3.11) is an open problem, and if it is resolved one would be able to establish longterm local extinction for a larger class of random environments.

4 Finite time local extinction: Proof of Theorem 1.1

In the following lemma we state that the superprocess in random environment possesses the "branching" property.

Lemma 4.1. *Let X^1, X^2 be P^W-conditionally independent solutions to the conditional martingale problem (2.2), (2.3) with initial conditions $\mu_1, \mu_2 \in \mathcal{M}(\mathbb{R}^d)$ respectively. Then*

$$X \equiv X^1 + X^2$$

solves the conditional martingale problem (2.2), (2.3) with initial condition $\mu = \mu_1 + \mu_2$.

Proof: By (2.2), for $i = 1,\ 2$,

$$\begin{aligned} N_t^{i,\phi} &\equiv \left\langle X_t^i, \phi\right\rangle - \langle \mu_i, \phi\rangle - \int_0^t \left\langle X_s^i, \kappa\Delta\phi\right\rangle ds \\ &\quad - \int_0^t \left\langle \int_{\mathbb{R}^d} g(x,\cdot)\phi(x) X_s^i(dx), dW_s \right\rangle_{\mathbb{H}} \end{aligned}$$

are continuous conditionally independent $\mathbb{P}^W$-martingales with quadratic variation processes

$$\left\langle N^{i,\phi}\right\rangle_t = 2\sigma^2 \int_0^t \left\langle X_s^i, \phi^2\right\rangle ds.$$

Hence, $N_t^{\phi} = N_t^{1,\phi} + N_t^{2,\phi}$ is a continuous $\mathbb{P}^W$-martingales with quadratic variation process

$$\begin{aligned} \left\langle N^{\phi}\right\rangle_t &= \left\langle N^{1,\phi}\right\rangle_t + \left\langle N^{2,\phi}\right\rangle_t \\ &= 2\sigma^2 \int_0^t \left\langle X_s, \phi^2\right\rangle ds. \end{aligned}$$

$\square$.

By the previous lemma we can represent our process X starting with Lebesgue initial conditions as a sum of two processes:

$$X \stackrel{d}{=} X^n + \tilde{X}^n\,, \tag{4.1}$$

where

$$X_0^n(dx) = 1_{B(0,n)}(x)dx, \quad \tilde{X}_0^n(dx) = 1_{B(0,n)^c}(x)dx, \tag{4.2}$$

and $B(0,n)$ is the ball in $\mathbb{R}^d$ with center 0 and radius n.

Lemma 4.2. *Suppose $0 \le \alpha < 2$ and η is a constant satisfying*

$$\eta > \frac{2-\alpha}{2+\alpha} \text{ and } \frac{\alpha(1+\eta)}{2} < 1. \tag{4.3}$$

Also fix $\theta > 1$ such that

$$\theta\frac{1+\eta}{2} > 1. \tag{4.4}$$

Then

$$\mathbb{P}(X_{n^\theta}^n \neq 0) \le cn^{-\theta},$$

for all n sufficiently large.

Proof Fix arbitrary $m > 0$. By Theorem 2.18 and the Markov property we have

$$\mathbb{E}e^{-\langle X_t^n, 1\rangle m} \quad \ge \quad \mathbb{E}e^{-\left\langle X_{t/2}^n, v_{t/2,t}^m\right\rangle} \tag{4.5}$$

where $v_{s,t}^m$ solves the following equation:

$$v_{s,t}^m(x) = m + \int_s^t \kappa\Delta v_{r,t}^m(x)dr - \int_s^t \sigma^2 v_{r,t}^m(x)^2 dr, \;\; 0 \le s \le t, \; x \in \mathbb{R}^d.$$

Note that (see e.g. (II.5.12) of [21])

$$\lim_{m\to\infty} v_{s,t}^m = \frac{1}{\sigma^2(t-s)}. \tag{4.6}$$

By (4.5) and (4.6), we have

$$\begin{aligned} \mathbb{P}(X_t^n = 0) &= \lim_{m\to\infty} \mathbb{E}e^{-\langle X_t^n, 1\rangle m} \\ &\ge \mathbb{E}e^{-\frac{2}{\sigma^2 t}\left\langle X_{t/2}^n, 1\right\rangle}. \end{aligned} \tag{4.7}$$

Set $l_t \equiv 2/t\sigma^2$. Then, by Corollary 2.16 we can continue (4.7) with

$$\mathbb{P}(X_t^n = 0) \ge \mathbb{E}e^{-\left\langle \phi_{t/2}^{l_t}, 1_{B(0,n)}\right\rangle}$$

where ϕ^{l_t} is a solution to (2.7) with $\phi_0 = l_t$. Let $\tilde{\phi}_0(\cdot) = 1_{B(0,n)}(\cdot)$. Then by the self-duality Lemma 2.19 we get

$$\mathbb{P}(X_t^n = 0) \ge \mathbb{E}e^{-l_t\left\langle \tilde{\phi}_{t/2}, 1\right\rangle},$$

where $\tilde{\phi}$ is a solution to (2.7) with initial condition $\tilde{\phi}_0$.

Therefore,

$$\begin{aligned}
\mathbb{P}(X^n_{n^\theta} \neq 0) &\leq \mathbb{E}\left(1 - e^{-l_{n^\theta}\left\langle \tilde{\phi}_{n^\theta/2}, 1\right\rangle}\right) \\
&\leq 1 - e^{-l_{n^\theta}} + \mathbb{P}\left(\left\langle \tilde{\phi}_{n^\theta/2}, 1\right\rangle^{\frac{1}{2}} > 1\right) \\
&\leq l_{n^\theta} + \mathbb{E}\left[\left\langle \tilde{\phi}_{n^\theta/2}, 1\right\rangle^{\frac{1}{2}}\right]. \qquad (4.8)
\end{aligned}$$

Let $\delta = 1 - \frac{\alpha(1+\eta)}{2}$. Fix

$$R_t = \max\{t^{(1+\eta)/2}, 1\} \qquad (4.9)$$

where η is a constant satisfying (4.3). Hence from Lemma 2.20 we get

$$\begin{aligned}
&\mathbb{E}\left[\left\langle \tilde{\phi}_{n^\theta/2}, 1\right\rangle^{\frac{1}{2}}\right] \leq cn^{d/2} e^{-c_2 n^{\theta\delta}} \\
&\qquad + c_1 \int_0^{n^\theta/2} s^{-\frac{\alpha(1+\eta)}{2}} \left(\int_{\Gamma_s^c} T_s^\kappa \tilde{\phi}_0(x)dx\right)^{\frac{1}{2}} e^{-c_2\left((n^\theta/2)^\delta - s^\delta\right)}\, ds \\
&= cn^{d/2} e^{-c_2 n^{\theta\delta}} + c_1 \int_0^{n^{\theta'}} s^{-\frac{\alpha(1+\eta)}{2}} \left(\int_{\Gamma_s^c} T_s^\kappa \tilde{\phi}_0(x)dx\right)^{\frac{1}{2}} e^{-c_2\left((n^\theta/2)^\delta - s^\delta\right)}\, ds \\
&\qquad + c_1 \int_{n^{\theta'}}^{n^\theta/2} s^{-\frac{\alpha(1+\eta)}{2}} \left(\int_{\Gamma_s^c} T_s^\kappa \tilde{\phi}_0(x)dx\right)^{\frac{1}{2}} e^{-c_2\left((n^\theta/2)^\delta - s^\delta\right)}\, ds \\
&\equiv I^{1,n} + I^{2,n} + I^{3,n} \qquad (4.10)
\end{aligned}$$

where $\theta' \in (0, \theta)$ is chosen such that

$$\theta' \frac{1+\eta}{2} \quad > \quad 1. \qquad (4.11)$$

Since $\int_{\mathbb{R}^d} T_s^\kappa \tilde{\phi}_0(x)dx \leq cn^d$, we can easily get

$$I^{2,n} \quad \leq \quad n^{d/2+\theta'\delta} e^{-c_2 n^{\theta\delta}}\,. \qquad (4.12)$$

Now we will bound $I^{3,n}$. For this purpose we have to bound $\int_{\Gamma_s^c} T_s^\kappa \tilde{\phi}_0(x)dx$. First, note that for $s \geq n^{\theta'}$ and large n, we have

$$R_s = s^{\frac{1+\eta}{2}} \geq n^{\theta' \frac{1+\eta}{2}} >> n \qquad (4.13)$$

where the last inequality follows by (4.11). Let ξ_t be the Brownian motion with diffusion coefficient 2κ. Then we can easily get that

$$\int_{\Gamma_s^c} T_s^\kappa \tilde{\phi}_0(x)dx = \int_{B(0,n)} \mathbb{P}\left(|\xi_s| > R_s |\, \xi_0 = y\right)\, dy. \qquad (4.14)$$

It follows from (4.11) and (4.13) that there exists a positive constant c such that

$$|R_s - y| > cR_s\,, \quad \forall y \in B(0,n),\ s \geq n^{\theta'}\,. \tag{4.15}$$

Also from (4.13) we have

$$R_s >> \sqrt{s}. \tag{4.16}$$

(4.15), (4.16) imply that

$$\mathbb{P}\left(|\xi_s| > R_s |\, \xi_0 = y\right) \quad \leq \quad c_1 e^{-c_2 (R_s/\sqrt{s})^2}, \quad \forall y \in B(0,n),$$

and hence from (4.14) we immediately get

$$\int_{\Gamma_s^c} T_s^\kappa \tilde{\phi}_0(x)dx \leq c_1 n^d e^{-c_2 (R_s/\sqrt{s})^2}.$$

This implies that

$$\begin{aligned} I^{3,n} \quad &\leq \quad c_1 \int_{n^{\theta'}}^{n^\theta/2} s^{-\frac{\alpha(1+\eta)}{2}} n^{d/2} e^{c_3 n^{\theta'\eta}} ds \\ &\leq \quad n^{d/2+\theta\delta} e^{-c_3 n^{\theta'\eta}}\,. \end{aligned} \tag{4.17}$$

Now combine (4.10), (4.12), (4.17) to get

$$\mathbb{E}\left[\left\langle \tilde{\phi}_{n^\theta/2}, 1\right\rangle^{\frac{1}{2}}\right] \leq cn^{-\theta} \tag{4.18}$$

for all n sufficiently large. Now substitute this bound into (4.8), recall that $l_{n^\theta} = cn^{-\theta}$ and this finishes the proof.

□

Proof of Theorem 1.1 As in the proof of Theorem 1.2, we take the compact set to be the unit ball. Recall $\tilde{X}^n$ is the process starting at $\tilde{X}_0^n(dx) = 1_{B(0,n)^c}(x)dx$. Fix $\phi \in \mathcal{C}_0^\infty$ such that

$$\phi(x) \quad = \quad \begin{cases} 1, & x \in B(0,1), \\ 0, & x \in B^c(0,2). \end{cases}$$

Then

$$\begin{aligned} &\mathbb{P}\left(\int_0^{(n+1)^\theta} \tilde{X}_s^n(B(0,1))ds = 0\right) \\ = \quad &\lim_{m\to\infty} \mathbb{E}\left(\exp\left(-m\int_0^{(n+1)^\theta} \left\langle \tilde{X}_s^n, 1_{B(0,1)}\right\rangle ds\right)\right) \\ \geq \quad &\lim_{m\to\infty} \mathbb{E}\left(\exp\left(-m\int_0^{(n+1)^\theta} \left\langle \tilde{X}_s^n, \phi\right\rangle ds\right)\right) \\ \geq \quad &\lim_{m\to\infty} \mathbb{E}\left(\exp\left(-\int_{B(0,n)^c} V_{(n+1)^\theta}^m(x)dx\right)\right) \end{aligned} \tag{4.19}$$

where (4.19) follows by Theorem 2.18 and $V_t^m(\cdot) = V_t(m\phi, 0, \cdot)$ (recall that V_t satisfies (2.25)). Note that V_t^m increases to V_t^∞. In the end we will be interested in the limiting behavior of the expression (4.19) as $n \to \infty$. Hence we may assume without loss of generality that $n \geq 10$. Now we will apply Lemma 3.5 of Dawson *et al* [5]. Although that lemma was proved for the particular case of $\kappa = \sigma^2 = 1/2$, it can be easily generalized to our case, in a way that it implies the following (we fix $R = 2, r = 4$ in that lemma):

$$V_t^\infty(x) \leq c\mathbb{P}(T_4 \leq t | \xi_0 = x), \quad \forall |x| > 4$$

where

$$T_4 = \inf\{t : \ |\xi_t| \leq 4\},$$

and ξ_t is a Brownian motion with diffusion coefficient 2κ. As

$$\begin{aligned} \mathbb{P}(T_4 \leq t | \xi_0 = x) &\leq \mathbb{P}\left(\sup_{s \leq t} |\xi_s| > |x| - 4 \,\middle|\, \xi_0 = 0 \right) \\ &\leq c_1 e^{-c_2 |x|^2/t}, \quad \forall |x| \geq 10, \end{aligned}$$

we have

$$V_t^\infty(x) \leq c_1 e^{-c_2|x|^2/t}, \quad \forall |x| \geq 10. \tag{4.20}$$

>From (4.19) and (4.20), we get

$$\begin{aligned} &\mathbb{P}\left(\int_0^{(n+1)^\theta} \tilde{X}_s^n(B(0,1))ds = 0 \right) \\ &\geq \exp\left(- \int_{B(0,n)^c} V_{(n+1)^\theta}^\infty(x)dx \right) \\ &\geq \exp\left(- \int_{B(0,n)^c} c_1 e^{-c_2|x|^2/(n+1)^\theta} dx \right) \\ &\geq e^{-e^{-n^{\delta'}}} \end{aligned} \tag{4.21}$$

for some $0 < \delta' < 2 - \theta$.

Now put together Lemma 4.2 and (4.1), (4.2), (4.21) to get

$$\begin{aligned} &\mathbb{P}\left(X_t(B(0,1)) \neq 0, \ \exists t \in [n^\theta, (n+1)^\theta] \right) \\ &\leq \mathbb{P}\left(X_{n^\theta}^n \neq 0 \text{ or } \int_0^{(n+1)^\theta} \tilde{X}_s^n(B(0,1))ds \neq 0 \right) \\ &\leq cn^{-\theta} + c(1 - e^{-e^{-n^{\delta'}}}). \end{aligned} \tag{4.22}$$

The last expression is summable in n. Hence, by Borel-Cantelli's lemma, a.s., there exists $N(\omega)$ such that

$$X_t(B(0,1)) = 0, \qquad \forall t \geq N(\omega).$$

□

Remark 4.3. *As in Remark 3.2 we would like to mention a possibility to extend Theorem 1.1 to the case where* κ *is small and the homogeneous noise satisfies condition (3.8). Here it is not enough to show (3.11). One should also prove that* $\mathbb{P}\left(\left\langle \tilde{\phi}_{n^{\theta}/2}, 1\right\rangle > 1\right)$ *converges to zero sufficiently fast in order to apply Borel-Cantelli argument.*

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