Electronic Journal of Probability

Vol. 10 (2005), Paper no. 30, pages 988-1004.

Journal URL
http://www.math.washington.edu/~ejpecp/

A GENERAL STOCHASTIC MAXIMUM PRINCIPLE FOR SINGULAR CONTROL PROBLEMS[1]

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Abstract

We consider the stochastic control problem in which the control domain need not be convex, the control variable has two components, the first being absolutely continuous and the second singular.The coefficients of the state equation are non linear and depend explicitly on the absolutely continuous component of the control. We establish a maximum principle, by using a spike variation on the absolutely continuous part of the control and a convex perturbation on the singular one. This result is a generalization of Peng's maximum principle to singular control problems.

Keywords. Maximum principle, singular control, adjoint equation, variational inequality.

AMS Subject Classification (2000) 60H10, 60H20, 93E20.

Submitted to EJP on February 9, 2004. Final version accepted on May 25, 2005.

[1]This work is supported by MENA Swedish Algerian Research Partnership Program 348-2002-6874

1 Introduction

We consider in this paper stochastic control problems of nonlinear systems, where the control domain need not be convex and the control variable has two components, the first being absolutely continuous and the second singular. The system under consideration is governed by a stochastic differential equation of the following type

$$\begin{cases} dx_t = b\left(t, x_t, u_t\right) dt + \sigma\left(t, x_t, u_t\right) dB_t + G_t d\xi_t \\ x\left(0\right) = x_0, \end{cases}$$

where b, σ and G are given deterministic functions, x_0 is the initial state and $B = \left(B_t\right)_{t\geq 0}$ is a standard Brownian motion, defined on a filtered probability space $\left(\Omega, \mathcal{F}, \left(\mathcal{F}_t\right)_{t\geq 0}, P\right)$, satisfying the usual conditions. The control variable is a suitable process (u, ξ) where $u : [0,T] \times \Omega \longrightarrow A_1 \subset \mathbb{R}^k$, $\xi : [0,T] \times \Omega \longrightarrow A_2 = \left([0,\infty)\right)^m$ are $B\left[0,T\right] \otimes \mathcal{F}$-measurable, $(\mathcal{F}_t)$- adapted, and ξ is an increasing process, continuous on the left with limits on the right with $\xi_0 = 0$.

The criteria to be minimized over the class of admissible controls has the form

$$J\left(u, \xi\right) = \mathbb{E}\left[g\left(x_T\right) + \int_0^T h\left(t, x_t, u_t\right) dt + \int_0^T k_t d\xi_t\right].$$

A control process that solves this problem is called optimal. We suppose that an optimal control exists. Our main goal in this paper is to establish necessary conditions for optimality of the Pontriagin type for this kind of problems.

Singular control problems have been studied by many authors including Beněs, Shepp, and Witsenhausen [4] , Chow, Menaldi, and Robin [6] , Karatzas, Shreve [11] , Davis, Norman [7] , Haussmann, Suo [8, 9, 10] . See [8] for a complete list of references on the subject. The approaches used in these papers, to solve the problem are mainly based on dynamic programming. It was shown in particular that the value function is solution of a variational inequality, and the optimal state is a reflected diffusion at the free boundary. Note that in [8] , the authors apply the compactification method to show existence of an optimal singular control.

The other major approach to solve control problems is to derive necessary conditions satisfied by some optimal control, known as the stochastic maximum principle. The first version of the stochastic maximum principle that covers singular control problems was obtained by Cadenillas and Haussmann [5], in which they consider linear dynamics, convex cost criterion and convex state constraints. The method used in [5] is based on the known principle of convex analysis, related to the minimization of convex, Gâteaux-differentiable functionals defined on a convex closed set. Necessary conditions for optimality for non linear stochastic differential equations with convex state constraints or uncontrolled diffusion matrix and measure valued controls were obtained by Bahlali & Chala [1] and Bahlali & al [2].

In our situation, since the system is nonlinear and the control domain is not necessarily convex, the approach of convex analysis used in [5] is no longer valid. Moreover, since the diffusion coefficient depends explicitly on the control variable, the method of first order variation used in [2] cannot be applied. The approach that we use to establish our main result is based on a double perturbation of the optimal control $\left(\widehat{u},\widehat{\xi}\right)$. The first perturbation is a spike variation, on the absolutely continuous part of the control and the second one is convex, on the singular component. This perturbation is defined as follows

$$\left(u_t^\theta,\xi_t^\theta\right)=\begin{cases}\left(v,\widehat{\xi}_t+\theta\left(\eta_t-\widehat{\xi}_t\right)\right) & \text{if } t\in[\tau,\tau+\theta]\\ \left(\widehat{u}_t,\widehat{\xi}_t+\theta\left(\eta_t-\widehat{\xi}_t\right)\right), & \text{otherwise,}\end{cases}$$

where v is a A_1-valued, $\mathcal{F}_t-$measurable random variable and η is an increasing process with $\eta_0=0$.

The variational inequality is derived from the following inequality

$$0\leq J\left(u^\theta,\xi^\theta\right)-J\left(\widehat{u},\widehat{\xi}\right).$$

From the definition of our perturbation, it is difficult to derive directly the variational inequality. To handle this problem, it is necessary to separate the above inequality into two parts. Let

$$J_1=J\left(u^\theta,\xi^\theta\right)-J\left(u^\theta,\widehat{\xi}\right)$$
$$J_2=J\left(u^\theta,\widehat{\xi}\right)-J\left(\widehat{u},\widehat{\xi}\right)$$

The variational inequality will be obtained from the fact that

$$0\leq\lim_{\theta\to 0}\frac{1}{\theta}J_1+\lim_{\theta\to 0}\frac{1}{\theta}J_2$$

For the singular part of the control, we apply the method of Bensoussan [3], to derive a first order adjoint process and a variational inequality which reduces to the computation of a Gâteaux derivative. For the absolutely continuous part, we use the approach developed by Peng [12] to derive the first and second order adjoint processes and the second variational inequality. Putting together the adjoint processes and the variational inequalities, we obtain the stochastic maximum principle. Our result may be regarded as a generalization of Peng's maximum principle, to singular control problems.

Let us briefly describe the contents of this paper. In the second section, we formulate the problem and give the various assumptions used throughout the paper. The third section is devoted to some preliminary results, which will be used in the sequel. In the fourth section, we derive explicitly the first and second order adjoint processes and the variational inequalities. In the last section, we state the stochastic maximum principle which is our main result.

2 Formulation of the problem

Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\right)$ be a probability space equipped with a filtration satisfying the usual conditions, on which a d-dimensional Brownian motion $B = (B_t)_{t\geq 0}$ is defined. We assume that $(\mathcal{F}_t)$ is the P- augmentation of the natural filtration of $(B_t)_{t\geq 0}$.

Let T be a strictly positive real number and consider the following sets

A_1 is a non empty subset of $\mathbb{R}^k$ and $A_2 = ([0,\infty))^m$.

U_1 is the class of measurable, adapted processes $u : [0,T] \times \Omega \longrightarrow A_1$.

U_2 is the class of measurable, adapted processes $\xi : [0,T] \times \Omega \longrightarrow A_2$ such that ξ is nondecreasing, left-continuous with right limits and $\xi_0 = 0$.

Definition 1 *An admissible control is a $\mathcal{F}_t-$ adapted process $(u,\xi) \in U_1 \times U_2$ such that*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |u(t)|^2 + |\xi_T|^2\right] < \infty.$$

We denote by $\mathcal{U}$ the set of all admissible controls.

For any $(u,\xi) \in \mathcal{U}$, we consider the following stochastic equation

$$\begin{cases} dx(t) = b(t, x(t), u(t))\, dt + \sigma(t, x(t), u(t))\, dB_t + G(t)\, d\xi_t \\ x(0) = x_0, \end{cases} \tag{1}$$

where

$$\begin{aligned} &b : [0,T] \times \mathbb{R}^n \times A_1 \longrightarrow \mathbb{R}^n \\ &\sigma : [0,T] \times \mathbb{R}^n \times A_1 \longrightarrow \mathcal{M}_{n\times d}(\mathbb{R}) \\ &G : [0,T] \longrightarrow \mathcal{M}_{n\times m}(\mathbb{R}) \end{aligned}$$

The expected cost has the form

$$J(u,\xi)=\mathbb{E}\left[g(x(T))+\int_0^T h(t,x(t),u(t))\,dt+\int_0^T k(t)\,d\xi_t\right], \tag{2}$$

where

$$\begin{aligned}&g:\mathbb{R}^n\longrightarrow\mathbb{R}\\&h:[0,T]\times\mathbb{R}^n\times A_1\longrightarrow\mathbb{R}\\&k:[0,T]\longrightarrow([0,\infty))^m.\end{aligned}$$

The control problem is to minimize the functional $J(.)$ over $\mathcal{U}$. If $\left(\widehat{u},\widehat{\xi}\right)\in\mathcal{U}$ is an optimal solution, that is

$$J\left(\widehat{u},\widehat{\xi}\right)=\inf_{(u,\xi)\in\mathcal{U}}J(u,\xi),$$

we may ask, how we can characterize it, in other words what conditions must $\left(\widehat{u},\widehat{\xi}\right)$ necessarily satisfy?

The following assumptions will be in force throughout this paper

$$\begin{aligned}&b,\sigma,g,h \text{ are twice continuously differentiable with respect to } x.\\&\text{The derivatives } b_x,b_{xx},\sigma_x,\sigma_{xx},g_x,g_{xx},h_x,h_{xx} \text{ are continuous}\\&\text{in } (x,u) \text{ and uniformly bounded.}\\&b,\sigma \text{ are bounded by } C(1+|x|+|u|).\\&G \text{ and } k \text{ are continuous and } G \text{ is bounded.}\end{aligned} \tag{3}$$

Under the above hypothesis, for every $(u,\xi)\in\mathcal{U}$, equation (1) has a unique strong solution given by

$$x_t^{(u,\xi)}=x_0+\int_0^t b\left(s,x_s^{(u,\xi)},u(s)\right)ds+\int_0^t\sigma\left(s,x_s^{(u,\xi)},u(s)\right)dB_s+\int_0^t G_s d\xi_s,$$

and the cost functional J is well defined from $\mathcal{U}$ into $\mathbb{R}$.

We list some matrix notation that will be used throughout this paper. We denote by $\mathcal{M}_{n\times n}(\mathbb{R})$ the space of $n\times n$ real matrix and $\mathcal{M}_{n\times n}^d(\mathbb{R})$ the linear space of vectors $M=(M_1,...,M_d)$ where $M_i\in\mathcal{M}_{n\times n}(\mathbb{R})$.

For any $M,N\in\mathcal{M}_{n\times n}^d(\mathbb{R})$, $L\in\mathcal{M}_{n\times n}(\mathbb{R})$, $x,y\in\mathbb{R}^n$ and $z\in\mathbb{R}^d$, we use the following notation

$xy = \sum_{i=1}^{n} x_i y_i \in \mathbb{R}$ is the product scalar in $\mathbb{R}^n$.

$ML = \sum_{i=1}^{d} M_i L_i \in \mathbb{R}^n$.

$Mxz = \sum_{i=1}^{d} (M_i x) z_i \in \mathbb{R}^n$.

$MN = \sum_{i=1}^{d} M_i N_i \in \mathcal{M}_{n\times n}(\mathbb{R})$.

$MLN = \sum_{i=1}^{d} M_i L N_i \in \mathcal{M}_{n\times n}(\mathbb{R})$.

$MLz = \sum_{i=1}^{d} M_i L z_i \in \mathcal{M}_{n\times n}(\mathbb{R})$.

We denote by L^* the transpose of the matrix L and $M^* = (M_1^*, ..., M_d^*)$.

3 Preliminary Results

The purpose of the stochastic maximum principle is to find necessary conditions for optimality satisfied by an optimal control. Suppose that $\left(\widehat{u}, \widehat{\xi}\right) \in \mathcal{U}$ is an optimal control and $\widehat{x}(t)$ denotes the optimal trajectory, that is, the solution of (1) corresponding to $\left(\widehat{u}, \widehat{\xi}\right)$. Let us introduce the following perturbation of the optimal control $\left(\widehat{u}, \widehat{\xi}\right)$:

$$\left(u^\theta(t), \xi^\theta(t)\right) = \begin{cases} \left(v, \widehat{\xi}(t) + \theta\left[\eta(t) - \widehat{\xi}(t)\right]\right) & \text{if } t \in [\tau, \tau+\theta] \\ \left(\widehat{u}(t), \widehat{\xi}(t) + \theta\left[\eta(t) - \widehat{\xi}(t)\right]\right) & \text{otherwise,} \end{cases} \tag{4}$$

where $0 \leq \tau < T$ is fixed, $\theta > 0$ is sufficiently small, v is a $\mathcal{F}_t-$measurable random variable and η is an increasing process with $\eta_0 = 0$.

Since $\left(\widehat{u}, \widehat{\xi}\right)$ is optimal, we have

$$0 \leq J\left(u^\theta, \xi^\theta\right) - J\left(\widehat{u}, \widehat{\xi}\right).$$

Let

$$J_1 = J\left(u^\theta, \xi^\theta\right) - J\left(u^\theta, \widehat{\xi}\right) \tag{5}$$

$$J_2 = J\left(u^\theta, \widehat{\xi}\right) - J\left(\widehat{u}, \widehat{\xi}\right). \tag{6}$$

The variational inequality will be derived from the fact that

$$0 \leq \lim_{\theta \to 0} \frac{1}{\theta} J_1 + \lim_{\theta \to 0} \frac{1}{\theta} J_2 \tag{7}$$

Let x_t^θ, $x_t^{\left(u^\theta, \widehat{\xi}\right)}$ be the trajectories associated respectively with $\left(u^\theta, \xi^\theta\right)$, $\left(u^\theta, \widehat{\xi}\right)$. For simplicity of notation, we denote

$$f(t) = f(t, \widehat{x}, \widehat{u})$$
$$f^\theta(t) = f(t, \widehat{x}, u^\theta)$$

where f stands for one of the functions $b, b_x, b_{xx}, \sigma, \sigma_x, \sigma_{xx}, h, h_x, h_{xx}$.

We will proceed by separating the computation of the two limits in (7), and obtain a variational equality from (5) and a variational inequality from (6). To achieve this goal, we need the following technical lemmas.

Lemma 2 *Under assumptions* (3), *we have*

$$\lim_{\theta \to 0} \mathbb{E} \left| \frac{x^\theta(t) - x_t^{\left(u^\theta, \widehat{\xi}\right)}}{\theta} - z(t) \right|^2 = 0, \tag{8}$$

where z is the solution of the linear stochastic differential equation

$$z(t) = \int_0^t b_x(s) z(s) ds + \int_0^t \sigma_x(s) z(s) dB_s + \int_0^t G(s) d\left(\eta - \widehat{\xi}\right)_s \tag{9}$$

Proof. From $(3), (4)$ and by using Gronwall's lemma and Burkholder Davis Gundy inequality, we get

$$\lim_{\theta \to 0} \mathbb{E} \left[\sup_{t \in [0,T]} \left| x^\theta(t) - x_t^{\left(u^\theta, \widehat{\xi}\right)} \right|^2 \right] = 0 \tag{10}$$

$$\lim_{\theta \to 0} \mathbb{E} \left[\sup_{t \in [0,T]} \left| x_t^{\left(u^\theta, \widehat{\xi}\right)} - \widehat{x}(t) \right|^2 \right] = 0 \tag{11}$$

$$\mathbb{E}\left[|z(t)|^2 \right] < \infty. \tag{12}$$

Let

$$y^\theta(t) = \frac{x^\theta(t) - x_t^{\left(u^\theta, \widehat{\xi}\right)}}{\theta} - z(t),$$

it holds that

$$\begin{aligned}
&\mathbb{E}\left|y^{\theta}(t)\right|^{2} \\
&\leq 3\int_{0}^{t}\mathbb{E}\left|\int_{0}^{1}b_{x}\left(s,x_{s}^{\left(u^{\theta},\widehat{\xi}\right)}+\lambda\left[x^{\theta}(s)-x_{s}^{\left(u^{\theta},\widehat{\xi}\right)}\right],u^{\theta}(s)\right)y^{\theta}(s)\,d\lambda\right|^{2}ds \\
&+3\int_{0}^{t}\mathbb{E}\left|\int_{0}^{1}\sigma_{x}\left(s,x_{s}^{\left(u^{\theta},\widehat{\xi}\right)}+\lambda\left[x^{\theta}(s)-x_{s}^{\left(u^{\theta},\widehat{\xi}\right)}\right],u^{\theta}(s)\right)y^{\theta}(s)\,d\lambda\right|^{2}ds \\
&+3\mathbb{E}\left|\rho^{\theta}(t)\right|^{2},
\end{aligned}$$

where $\rho^{\theta}(t)$ is given by

$$\begin{aligned}
\rho^{\theta}(t)=&\int_{0}^{t}\int_{0}^{1}\left[b_{x}\left(s,x_{s}^{\left(u^{\theta},\widehat{\xi}\right)}+\lambda\left[x^{\theta}(s)-x_{s}^{\left(u^{\theta},\widehat{\xi}\right)}\right],u^{\theta}(s)\right)\right. \\
&\left.-b_{x}\left(s,\widehat{x}(s),\widehat{u}(s)\right)\right]z(s)\,d\lambda ds \\
&+\int_{0}^{t}\int_{0}^{1}\left[\sigma_{x}\left(s,x_{s}^{\left(u^{\theta},\widehat{\xi}\right)}+\lambda\left[x^{\theta}(s)-x_{s}^{\left(u^{\theta},\widehat{\xi}\right)}\right],u^{\theta}(s)\right)\right. \\
&\left.-\sigma_{x}\left(s,\widehat{x}(s),\widehat{u}(s)\right)\right]z(s)\,d\lambda dB_{s}.
\end{aligned}$$

Since b_x, σ_x are bounded, we have

$$\mathbb{E}\left|y^{\theta}(t)\right|^{2}\leq 6C\int_{0}^{t}\mathbb{E}\left|y^{\theta}(s)\right|^{2}ds+3\mathbb{E}\left|\rho^{\theta}(t)\right|^{2}.$$

b_x, σ_x being continuous and bounded, then using $(10),(11),(12)$ and the dominated convergence theorem, we get

$$\lim_{\theta\to 0}\mathbb{E}\left|\rho^{\theta}(t)\right|^{2}=0$$

We conclude by using Gronwall's lemma. ∎

Lemma 3 *Under assumption* (3)*, the following estimate holds*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|x_{t}^{\left(u^{\theta},\widehat{\xi}\right)}-\widehat{x}(t)-x_{1}(t)-x_{2}(t)\right|^{2}\right]\leq C\theta^{2}, \tag{13}$$

where x_1, x_2 *are solutions of*

$$\begin{aligned}
x_{1}(t)=&\int_{0}^{t}\left[b_{x}(s)\,x_{1}(s)+b^{\theta}(s)-b(s)\right]ds \\
&+\int_{0}^{t}\left[\sigma_{x}(s)\,x_{1}(s)+\sigma^{\theta}(s)-\sigma(s)\right]dB_{s}
\end{aligned} \tag{14}$$

$$x_2(t) = \int_0^t \left[b_x^\theta(s) - b_x(s)\right] x_1(s)\,ds \tag{15}$$
$$+ \int_0^t \left[b_x(s)\,x_2(s) + \frac{1}{2} b_{xx}(s)\,x_1(s)\,x_1(s)\right] ds$$
$$+ \int_0^t \left[\sigma_x^\theta(s) - \sigma_x(s)\right] x_1(s)\,dB_s$$
$$+ \int_0^t \left[\sigma_x(s)\,x_2(s) + \frac{1}{2}\sigma_{xx}(s)\,x_1(s)\,x_1(s)\right] dB_s$$

Proof. We put

$$\widetilde{x}(t) = \widehat{x}(t) - \int_0^t G(s)\,d\widehat{\xi}_s$$
$$\widetilde{x}_t^{\left(u^\theta,\widehat{\xi}\right)} = x_t^{\left(u^\theta,\widehat{\xi}\right)} - \int_0^t G(s)\,d\widehat{\xi}_s$$

It is clear that

$$x_t^{\left(u^\theta,\widehat{\xi}\right)} - \widehat{x}(t) - x_1(t) - x_2(t) = \widetilde{x}_t^{\left(u^\theta,\widehat{\xi}\right)} - \widetilde{x}(t) - x_1(t) - x_2(t)$$

By using the same proof as in [12], lemma 1 page 968, we show that

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\widetilde{x}_t^{\left(u^\theta,\widehat{\xi}\right)} - \widetilde{x}(t) - x_1(t) - x_2(t)\right|^2\right] \leq C\theta^2,$$

which prove the lemma. ∎

Remark 4 *Equations* (14) *and* (15) *are called the first and the second-order variational equations. Equation* (14) *is the variational equation in the usual sense. Since the diffusion coefficient* σ *depends explicitly on the control variable* u *and the control domain* $\mathcal{U}$ *is not convex, then by using only equation* (14) *we can only obtain the following estimation*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|x_t^{\left(u^\theta,\widehat{\xi}\right)} - \widehat{x}(t) - x_1(t)\right|^2\right] \leq C\theta.$$

This does not allow us to derive the variational inequality. The idea introduced by S. Peng [12] *is to use a second-order expansion to obtain an estimation of order* $o(\theta^2)$.

Lemma 5 *Under assumptions of lemma 2, we have*

$$\lim_{\theta\to 0}\frac{J_1}{\theta}=\mathbb{E}\left[z\left(T\right)g_x\left(\widehat{x}\left(T\right)\right)\right]+\mathbb{E}\int_0^T z\left(t\right)h_x\left(t\right)dt \tag{16}$$
$$+\mathbb{E}\int_0^T k\left(t\right)d\left(\eta-\widehat{\xi}\right)_t$$

Proof. From (5) we have

$$\frac{J_1}{\theta}=\mathbb{E}\int_0^T\int_0^1\left(\frac{x_t^\theta-x_t^{\left(u^\theta,\widehat{\xi}\right)}}{\theta}\right)h_x\left(t,x_t^{\left(u^\theta,\widehat{\xi}\right)}+\lambda\left[x_t^\theta-x_t^{\left(u^\theta,\widehat{\xi}\right)}\right],u_t^\theta\right)d\lambda dt$$
$$+\mathbb{E}\int_0^1\left(\frac{x_T^\theta-x_T^{\left(u^\theta,\widehat{\xi}\right)}}{\theta}\right)g_x\left(x_T^{\left(u^\theta,\widehat{\xi}\right)}+\lambda\left[x_T^\theta-x_T^{\left(u^\theta,\widehat{\xi}\right)}\right]\right)d\lambda$$
$$+\mathbb{E}\int_0^T k\left(t\right)d\left(\eta-\widehat{\xi}\right)_t$$

Since g_x and h_x are continuous and bounded, then from $(4),(8),(11)$ and by letting θ going to zero we conclude. ■

Lemma 6 *Under assumptions of lemma 3 we have*

$$J_2\leq\mathbb{E}\left[g_x(\widehat{x}(T))\left(x_1(T)+x_2(T)\right)+\int_0^T h_x(t)\left(x_1(t)+x_2(t)\right)dt\right] \tag{17}$$
$$+\frac{1}{2}\mathbb{E}\left[g_{xx}(\widehat{x}(T))x_1(T)x_1(T)+\int_0^T h_{xx}(t)x_1(t)x_1(t)dt\right]$$
$$+\mathbb{E}\int_0^T\left[h^\theta\left(t\right)-h\left(t\right)\right]dt+o\left(\theta\right)$$

Proof. From (6) we have

$$J_2=\mathbb{E}\left[g\left(x_T^{\left(u^\theta,\widehat{\xi}\right)}\right)-g\left(\widehat{x}\left(T\right)\right)\right]$$
$$+\mathbb{E}\int_0^T\left[h\left(t,x_t^{\left(u^\theta,\widehat{\xi}\right)},u^\theta\left(t\right)\right)-h\left(t,\widehat{x}\left(t\right),\widehat{u}\left(t\right)\right)\right]dt$$

By using the estimate (13), the result follows by mimicking the same proof as in [24] (lemma 2, page 970). ■

4 Variational inequalities and adjoint processes

In this section, we introduce the adjoint processes and we derive the variational inequalities from (16) and (17). The backward stochastic differential equations satisfied by the adjoint processes will be given in the next section.

4.1 The first-order expansion

The linear terms in (16) and (17) may be treated in the following way (see Bensoussan [6]). Let Φ_1 be the fundamental solution of the linear equation

$$\begin{cases} d\Phi_1(t) = b_x(t)\Phi_1(t)dt + \sigma_x(t)\Phi_1(t)dB_t \\ \Phi_1(0) = I_d \end{cases} \tag{18}$$

This equation is linear with bounded coefficients, then it admits a unique strong solution. This solution is invertible and its inverse $\Psi_1(t)$ is the unique solution of the following equation

$$\begin{cases} d\Psi_1(t) = \left[\sigma_x(t)\Psi_1(t)\sigma_x^*(t) - b_x(t)\Psi_1(t)\right] dt - \sigma_x(t)\Psi_1(t)dB_t \\ \Psi_1(0) = I_d. \end{cases} \tag{19}$$

Moreover, Φ_1 and Ψ_1 satisfy

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|\Phi_1(t)\right|^2\right] + \mathbb{E}\left[\sup_{t\in[0,T]} \left|\Psi_1(t)\right|^2\right] < \infty. \tag{20}$$

We introduce the following processes

$$\alpha_1(t) = \Psi_1(t)\left[x_1(t) + x_2(t)\right] \tag{21}$$

$$\beta_1(t) = \Psi_1(t)\, z(t), \tag{22}$$

$$X_1 = \Phi_1^*(T) g_x(\widehat{x}(T)) + \int_0^T \Phi_1^*(t) h_x(t)dt \tag{23}$$

$$Y_1(t) = \mathbb{E}\left[X_1 \,/\, \mathcal{F}_t\right] - \int_0^t \Phi_1^*(s) h_x(s)ds. \tag{24}$$

We remark from $(21), (22), (23), (24)$ that

$$\mathbb{E}\left[\alpha_1(T)\, Y_1(T)\right] = \mathbb{E}\left[g_x(\widehat{x}(T))\left(x_1(T) + x_2(T)\right)\right] \tag{25}$$

$$\mathbb{E}\left[\beta_1(T)\, Y_1(T)\right] = \mathbb{E}\left[g_x(\widehat{x}(T))\, z(T)\right] \tag{26}$$

Since g_x and h_x are bounded, then from (20), X_1 is square integrable. Hence $\left(\mathbb{E}\left[X_1 \ / \ \mathcal{F}_t\right]\right)_{t\geq 0}$ is a square integrable martingale with respect to the natural filtration of the Brownian motion (B_t). Then from Ito's representation theorem we have

$$Y_1\left(t\right) = \mathbb{E}\left[X_1\right] + \int_0^t Q_1\left(s\right) dB_s - \int_0^t \Phi_1^*\left(s\right) h_x\left(s\right) ds,$$

where $Q_1\left(s\right)$ is an adapted process such that $\mathbb{E}\int_0^T \left|Q_1\left(s\right)\right|^2 ds < \infty$.

By applying the Ito's formula to $\alpha_1(t)$ and $\beta_1\left(t\right)$ and using (25) and (26), we can rewrite (16) and (17) as

$$\lim_{\theta\to 0}\frac{J_1}{\theta} = \mathbb{E}\int_0^T \left[k\left(t\right) + G^*\left(t\right) p_1\left(t\right)\right] d\left(\eta - \widehat{\xi}\right)_t \tag{27}$$

$$\begin{aligned} J_2 \leq \mathbb{E}\int_0^T &\left\{H\left[t, \widehat{x}(t), u_\theta(t), p_1(t), q_1(t)\right] - H\left[t, \widehat{x}(t), \widehat{u}(t), p_1(t), q_1(t)\right]\right\} dt \\ &+ \frac{1}{2}\mathbb{E}\int_0^T x_1^*\left(t\right) H_{xx}\left[\widehat{x}(t), \widehat{u}(t), p_1(t), q_1(t)\right] x_1(t) dt \\ &+ \frac{1}{2}\mathbb{E}\left[x_1^*\left(T\right) g_{xx}(\widehat{x}(T)) x_1(T)\right] + o(\theta), \end{aligned} \tag{28}$$

where p_1 and q_1 are adapted processes given by

$$p_1(t) = \Psi_1^*(t) Y_1(t) \quad ; \quad p_1 \in \mathcal{L}^2\left(\left[0, T\right]; \mathbb{R}^n\right) \tag{29}$$

$$q_1(t) = \Psi_1^*(t) Q_1(t) - \sigma_x^*(t) p_1(t) \quad ; \quad q_1 \in \mathcal{L}^2\left(\left[0, T\right]; \mathbb{R}^{n\times d}\right). \tag{30}$$

and the Hamiltonian H is defined from $[0, T] \times \mathbb{R}^n \times A_1 \times \mathbb{R}^n \times \mathcal{M}_{n\times d}\left(\mathbb{R}\right)$ into $\mathbb{R}$ by

$$H\left[t, x\left(t\right), u\left(t\right), p\left(t\right), q\left(t\right)\right] = h\left(t\right) + p\left(t\right) b\left(t\right) + \sum_{i=1}^d \sigma_i\left(t\right) q_i\left(t\right),$$

where σ_i and q_i denote respectively the i^{th} columns of matrices σ and q.

The process p_1 is called the first order adjoint process and from (23), (24), (29), it is given explicitly by

$$p_1\left(t\right) = \mathbb{E}\left[\Psi_1^*\left(t\right)\Phi_1^*\left(T\right) g_x(\widehat{x}\left(T\right)) + \Psi_1^*\left(t\right)\int_t^T \Phi_1^*\left(s\right) h_x(s) ds / \ F_t\right]$$

where $\Phi_1(t)$ and $\Psi_1(t)$ are respectively the solutions of (18) and (19).

4.2 The second-order expansion

We now treat the quadratic terms of (28) by the same method. Let $Z = x_1 x_1^*$, by Ito's formula we obtain

$$\begin{aligned} dZ(t) &= \left[Z(t)b_x^*(t) + b_x(t)Z(t) + \sigma_x(t)Z(t)\sigma_x^*(t) + A_\theta(t)\right] dt \\ &\quad + \left[Z(t)\sigma_x^*(t) + \sigma_x(t)Z(t) + B_\theta(t)\right] dB_t, \end{aligned} \tag{31}$$

where $A_\theta(t)$ and $B_\theta(t)$ are given by

$$\begin{aligned} A_\theta(t) &= x_1(t)\left[b^\theta(t) - b(t)\right]^* + \left[b^\theta(t) - b(t)\right] x_1^* \\ &\quad + \sigma_x(t)x_1(t)\left[\sigma^\theta(t) - \sigma(t)\right]^* + \left[\sigma^\theta(t) - \sigma(t)\right] x_1^*(t)\sigma_x^\theta(t) \\ &\quad + \left[\sigma^\theta(t) - \sigma(t)\right]\left[\sigma^\theta(t) - \sigma(t)\right]^* \\ B_\theta(t) &= x_1(t)\left[\sigma^\theta(t) - \sigma(t)\right]^* + \left[\sigma^\theta(t) - \sigma(t)\right] x_1^*(t) \end{aligned}$$

We consider now the following symmetric matrix-valued linear equation

$$\begin{cases} d\Phi_2(t) = & \left[\Phi_2(t)b_x^*(t) + b_x(t)\Phi_2(t) + \sigma_x(t)\Phi_2(t)\sigma_x^*(t)\right] dt \\ & + \left[\Phi_2(t)\sigma_x^*(t) + \sigma_x(t)\Phi_2(t)\right] dB_t \\ \Phi_2(0) = & I_d. \end{cases} \tag{32}$$

This equation is linear with bounded coefficients, hence it admits a unique strong solution. $\Phi_2(t)$ is invertible and its inverse $\Psi_2(t)$ is the solution of the following equation

$$\begin{cases} d\Psi_2(t) = & \left[\sigma_x(t) + \sigma_x^*(t)\right]\Psi_2(t)\left[\sigma_x(t) + \sigma_x^*(t)\right]^* dt \\ & - \left[\Psi_2(t)b_x^*(t) + b_x(t)\Psi_2(t) + \sigma_x(t)\Psi_2(t)\sigma_x^*(t)\right] dt \\ & + \left[\Psi_2(t)\sigma_x^*(t) + \sigma_x(t)\Psi_2(t)\right] dB_t \\ \Psi_2(0) = & I_d. \end{cases} \tag{33}$$

Moreover, Φ_2 and Ψ_2 satisfy

$$\mathbb{E}\left[\sup_{t\in[0,T]} |\Phi_2(t)|^2\right] + \mathbb{E}\left[\sup_{t\in[0,T]} |\Psi_2(t)|^2\right] < \infty \tag{34}$$

We put

$$\alpha_2(t) = \Psi_2(t)Z(t) \tag{35}$$

$$X_2 = \Phi_2^*(T)g_{xx}(\widehat{x}(T)) + \int_0^T \Phi_2^*(t)H_{xx}\left(\widehat{x}(t), \widehat{u}(t), p_1(t), q_1(t)\right) dt \tag{36}$$

$$Y_2(t) = \mathbb{E}\left[X_2 \,/\, \mathcal{F}_t\right] - \int_0^t \Phi_2^*(s)H_{xx}\left(\widehat{x}(s), \widehat{u}(s), p_1(s), q_1(s)\right) ds \tag{37}$$

We remark from (35), (36) and (37) that

$$\mathbb{E}\left[x_1^*(T)g_{xx}(\widehat{x}\,(T))x_1(T)\right] = \mathbb{E}\left[\alpha_2(T)Y_2(T)\right] \tag{38}$$

Since g_{xx} and H_{xx} are bounded, then from (34), X_2 is square integrable, hence $\mathbb{E}\left[X_2 \;/\; \mathcal{F}_t\right]$ is a square integrable martingale with respect to the Brownian filtration. Then from Ito's representation theorem, we have

$$Y_2\,(t) = \mathbb{E}\left[X_2\right] + \int_0^t Q_2\,(s)\,dB_s - \int_0^t \Phi_2^*\,(s)\,H_{xx}\left[\widehat{x}(s), \widehat{u}(s), p_1(s), q_1(s)\right]ds$$

Where $Q_2\,(s)$ is an adapted process such that $\mathbb{E}\int_0^T \left|Q_2\,(s)\right|^2 ds < \infty$.

By applying Ito's formula to $\alpha_2(t)$ along with (38) and using the definition of $u^\theta\,(t)$, we can derive (28) as follows

$$\begin{aligned}\lim_{\theta\to 0}\frac{J_2}{\theta} &\leq \mathbb{E}\left\{H\left[\tau, \widehat{x}(\tau), v, p_1(\tau), q_1\,(\tau) - p_2\,(\tau)\,\sigma\left(\tau, \widehat{x}\,(\tau)\,, \widehat{u}\,(\tau)\right)\right]\right\} \\ &\quad + \frac{1}{2}\mathbb{E}\left\{Tr\left[\sigma\sigma^*\left(\tau, \widehat{x}\,(\tau)\,, v\right)\right]p_2\,(\tau)\right\} \\ &\quad - \mathbb{E}\left\{H\left[\tau, \widehat{x}(\tau), \widehat{u}\,(\tau)\,, p_1(\tau), q_1\,(\tau) - p_2\,(\tau)\,\sigma\left(\tau, \widehat{x}\,(\tau)\,, \widehat{u}\,(\tau)\right)\right]\right\} \\ &\quad - \frac{1}{2}\mathbb{E}\left\{Tr\left[\sigma\sigma^*\left(\tau, \widehat{x}\,(\tau)\,, \widehat{u}\,(\tau)\right)\right]p_2\,(\tau)\right\},\end{aligned} \tag{39}$$

where p_2 is an adapted process given by

$$p_2(t) = \Psi_2^*(t)Y_2(t) \quad ; \quad p_2 \in \mathcal{L}^2\left([0,T]\,; \mathbb{R}^{n\times n}\right) \tag{40}$$

The process p_2 is called the second order adjoint process and from (36), (37), (40), it is given explicitly by

$$\begin{aligned}p_2(t) &= \mathbb{E}\left[\Psi_2^*(t)\Phi_2^*(T)g_{xx}(x(T)) \;/\; \mathcal{F}_t\right] \\ &\quad + \mathbb{E}\left[\Psi_2^*(t)\int_t^T \Phi_2^*(s)H_{xx}(\widehat{x}(s), \widehat{u}(s), p_1(s), q_1(s))ds/\;\mathcal{F}_t\right]\end{aligned}$$

Where Φ_2 and Ψ_2 are respectively the solutions of (32) and (33).

5 Adjoint equations and the maximum principle

By applying Ito's formula to the adjoint processes p_1 in (29) and p_2 in (40), we obtain the first and second order adjoint equations which are linear backward stochastic differential equations, given by

$$\begin{cases} -dp_1(t) = H_x\left[\widehat{x}(t), \widehat{u}(t), p_1(t), q_1(t)\right]dt - q_1(t)dB_t \\ p_1(T) = g_x(\widehat{x}(T)), \end{cases} \tag{41}$$

$$\begin{cases} -dp_2(t) = & \left[b_x^*(t)p_2(t) + p_2(t)b_x(t) + \sigma_x^*(t)p_2(t)\sigma_x(t)\right]dt \\ & +\left[\sigma_x^*(t)q_2(t) + q_2(t)\sigma_x(t))\right]dt \\ & +H_{xx}\left[\widehat{x}(t), \widehat{u}(t), p_1(t), q_1(t)\right]dt - q_2(t)dB_t \\ p_2(T) = & g_{xx}(\widehat{x}(T)), \end{cases} \tag{42}$$

where $q_1(t)$ is given by (30) and $q_2(t)$ by

$$\begin{aligned} & q_2(t) = \left(q_2^1(t), ..., q_2^d(t)\right) \quad ; \quad q_2 \in \left(\mathcal{L}^2\left([0,T]; \mathbb{R}^{n\times n}\right)\right)^d \\ & q_2^i(t) = \Psi_2^*(t)Q_2^i(t) + p_2(t)\sigma_x^i(t) + \sigma_x^{i*}(t)p_2(t) \ ; \ i = 1, ..., d, \end{aligned} \tag{43}$$

and $Q_1(t), Q_2(t)$ satisfy respectively

$$\begin{aligned} \int_0^t Q_1(s)dBs = & \mathbb{E}\left[\Phi_1^*(T)g_x(\widehat{x}(T)) + \int_0^T \Phi_1^*(t)h_x(t)dt \ / \ \mathcal{F}_t\right] \\ & - \mathbb{E}\left[\Phi_1^*(T)g_x(\widehat{x}(T)) + \int_0^T \Phi_1^*(t)h_x(t)dt\right] \end{aligned}$$

$$\begin{aligned} & \int_0^t Q_2(s)dBs \\ & = \mathbb{E}\left[\Phi_2^*(T)g_{xx}(\widehat{x}(T)) + \int_0^T \Phi_2^*(t)H_{xx}\left[\widehat{x}(t), \widehat{u}(t), p_1(t), q_1(t)\right]dt \ / \ \mathcal{F}_t\right] \\ & - \mathbb{E}\left[\Phi_2^*(T)g_{xx}(\widehat{x}(T)) + \int_0^T \Phi_2^*(t)H_{xx}\left[\widehat{x}(t), \widehat{u}(t), p_1(t), q_1(t)\right]dt\right] \end{aligned}$$

We are ready now to state the main result of this paper.

Theorem 7 *(The Stochastic maximum principle) Let* $\left(\widehat{u}, \widehat{\xi}\right)$ *be an optimal control minimizing the cost* J *over* $\mathcal{U}$ *and* $\widehat{x}$ *denotes the corresponding optimal trajectory. Then there are two unique couples of adapted processes*

$$(p_1, q_1) \in \mathcal{L}^2\left([0,T]; \mathbb{R}^n\right) \times \mathcal{L}^2\left([0,T]; \mathbb{R}^{n\times d}\right)$$

$$(p_2, q_2) \in \mathcal{L}^2\left([0,T]; \mathbb{R}^{n\times n}\right) \times \left(\mathcal{L}^2\left([0,T]; \mathbb{R}^{n\times n}\right)\right)^d$$

which are respectively solutions of backward stochastic differential equations (41) *and* (42) *such that*

$$\begin{aligned} & H\left[\tau,\widehat{x}(\tau),\widehat{u}\left(\tau\right),p_1(\tau),q_1\left(\tau\right)-p_2\left(\tau\right)\sigma\left(\tau,\widehat{x}\left(\tau\right),\widehat{u}\left(\tau\right)\right)\right] \\ & \qquad +\frac{1}{2}Tr\left[\sigma\sigma^*\left(\tau,\widehat{x}\left(\tau\right),\widehat{u}\left(\tau\right)\right)\right]p_2\left(\tau\right) \\ & \leq H\left[\tau,\widehat{x}(\tau),v,p_1(\tau),q_1\left(\tau\right)-p_2\left(\tau\right)\sigma\left(\tau,\widehat{x}\left(\tau\right),\widehat{u}\left(\tau\right)\right)\right] \\ & \qquad +\frac{1}{2}Tr\left[\sigma\sigma^*\left(\tau,\widehat{x}\left(\tau\right),v\right)\right]p_2\left(\tau\right) \\ & \qquad \forall v\in A_1\ ;\ a.e,\ a.s \end{aligned} \tag{44}$$

$$P\left\{\forall t\in[0,T]\,,\ \forall i\ ;\ \left(k_t^i+G_i^*\left(t\right)p_1\left(t\right)\right)\geq 0\right\}=1 \tag{45}$$

$$P\left\{\sum_{i=1}^{d}\mathbf{1}_{\left\{k_t^i+G_i^*(t)p_1(t)\geq 0\right\}}d\widehat{\xi}_t^i=0\right\}=1 \tag{46}$$

Proof. From (7) , (27) , (39), we have for every $\mathcal{F}_t-$measurable random variable v, and every increasing process η with $\eta_0=0$

$$\begin{aligned} 0 & \leq \mathbb{E}\left\{H\left[\tau,\widehat{x}(\tau),v,p_1(\tau),q_1\left(\tau\right)-p_2\left(\tau\right)\sigma\left(\tau,\widehat{x}\left(\tau\right),\widehat{u}\left(\tau\right)\right)\right]\right\} \\ & +\frac{1}{2}\mathbb{E}\left\{Tr\left[\sigma\sigma^*\left(\tau,\widehat{x}\left(\tau\right),v\right)\right]p_2\left(\tau\right)\right\} \\ & -\mathbb{E}\left\{H\left[\tau,\widehat{x}(\tau),\widehat{u}\left(\tau\right),p_1(\tau),q_1\left(\tau\right)-p_2\left(\tau\right)\sigma\left(\tau,\widehat{x}\left(\tau\right),\widehat{u}\left(\tau\right)\right)\right]\right\} \\ & -\frac{1}{2}\mathbb{E}\left\{Tr\left[\sigma\sigma^*\left(\tau,\widehat{x}\left(\tau\right),\widehat{u}\left(\tau\right)\right)\right]p_2\left(\tau\right)\right\} \\ & +\mathbb{E}\int_0^T\left[k\left(t\right)+G^*\left(t\right)p_1\left(t\right)\right]d\left(\eta-\widehat{\xi}\right)_t \end{aligned}$$

If we put $\eta_t=\widehat{\xi}_t$ we obtain (44). On the other hand, if we choose $v=\widehat{u}(t)$ and using the same proof of in theorem 4.2 in [5] , we deduce (45) and (46). ■

Remark 8 *If we suppose that* $G=k=0$, *then we recover Peng's maximum principle* [12] .

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