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PERCOLATION OF ARBITRARY WORDS ON THE CLOSE-PACKED GRAPH OF $\mathbb{Z}^2$

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Abstract Let $\mathbb{Z}^2_{cp}$ be the close-packed graph of $\mathbb{Z}^2$, that is, the graph obtained by adding to each face of $\mathbb{Z}^2$ its diagonal edges. We consider site percolation on $\mathbb{Z}^2_{cp}$, namely, for each v we choose $X(v) = 1$ or 0 with probability p or $1-p$, respectively, independently for all vertices v of $\mathbb{Z}^2_{cp}$. We say that a word $(\xi_1, \xi_2, \dots) \in \{0,1\}^{\mathbb{N}}$ is seen in the percolation configuration if there exists a selfavoiding path $(v_1, v_2, \dots)$ on $\mathbb{Z}^2_{cp}$ with $X(v_i) = \xi_i, i \geq 1$. $p_c(\mathbb{Z}^2, \text{site})$ denotes the critical probability for site-percolation on $\mathbb{Z}^2$. We prove that for each fixed $p \in \big(1 - p_c(\mathbb{Z}^2, \text{site}), p_c(\mathbb{Z}^2, \text{site})\big)$, with probability 1 all words are seen. We also show that for some constants $C_i > 0$ there is a probability of at least C_1 that all words of length $C_0 n^2$ are seen along a path which starts at a neighbor of the origin and is contained in the square $[-n, n]^2$.

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1. Introduction.

Benjamini and Kesten (1995) introduced the problem whether 'all words are seen in percolation on a graph $\mathcal{G}$'. The set-up is as follows. $\mathcal{G}$ is an infinite connected graph and the vertices of $\mathcal{G}$ are independently chosen to be *occupied* with probability p and *vacant* with probability $1-p$. The resulting probability measure on configurations of occupied and vacant vertices of $\mathcal{G}$ is denoted by P_p. We set

$$X(v) = \begin{cases} 1 & \text{if } v \text{ is occupied} \\ 0 & \text{if } v \text{ is vacant.} \end{cases}$$

Under P_p the $X(v)$, v a vertex of $\mathcal{G}$, are i. i. d. binomial variables with $P_p\{X(v) = 1\} = p$.

A *path* on $\mathcal{G}$ will be a sequence $\pi = (v_0, v_1, \dots,)$, with $v_0, v_1, \dots$ vertices of $\mathcal{G}$, such that v_i and v_{i+1} are adjacent for $i \geq 0$. The path π is called *self-avoiding* if all its vertices are distinct. A path may be finite or infinite. A *word* is an (finite or infinite) sequence $\xi = (\xi_1, \xi_2, \dots)$ of zeroes and ones. A finite word $(\xi_1, \dots, \xi_n)$ of n zeroes and ones is said to have *length* n. The space of all infinite words is denoted by

$$\Xi = \{0,1\}^{\mathbb{N}}.$$

We will say that the word $\xi = (\xi_1, \xi_2, \dots)$, is *seen along a path* $\pi = (v_0, v_1, \dots)$ on $\mathcal{G}$ if π is self-avoiding and $X(v_i) = \xi_i$, $i \geq 1$ (note that the state of the initial vertex v_0 doesn't figure in this definition).

Benjamini and Kesten (1995) investigated when all words in Ξ are seen along some self-avoiding path on $\mathcal{G}$. They also considered the even stronger requirement that all words of Ξ are seen along some self-avoiding path which starts at a fixed vertex v_0. They showed that even this stronger phenomenon occurs with positive probability when $p = 1/2$ and $\mathcal{G} = \mathbb{Z}^d$ for sufficiently high d. A weaker phenomenon is that *almost all* words from Ξ are seen along some self-avoiding path. Here 'almost all' is with respect to some measure μ on Ξ. In all investigations so far one has taken μ to be a product measure

$$\mu = \prod_{i=1}^{\infty} \mu_i \,, \tag{1.1}$$

with $\mu(\{0\}) = 1 - \mu(\{1\}) = \beta$ for some $0 < \beta < 1$. Kesten, Sidoravicius and Zhang (1998) proved that if $p = 1/2$ and $\mathcal{G}$ is the triangular lattice, then almost all words are seen in this sense (for any $0 < \beta < 1$).

This paper deals only with the graph $\mathbb{Z}^2_{cp}$, which is obtained by 'close packing' the faces of $\mathbb{Z}^2$. Throughout this paper we think of the square latice $\mathbb{Z}^2$ as being imbedded in $\mathbb{R}^2$ in the standard way. Let F be a face of $\mathbb{Z}^2$. *Close-packing* F means adding an edge to $\mathbb{Z}^2$ between any pair of vertices on the perimeter of F which are not yet adjacent. The vertex set of $\mathbb{Z}^2_{cp}$ is therefore the same as the vertex set of $\mathbb{Z}^2$, and the edge set of $\mathbb{Z}^2_{cp}$ consists of the edge set of $\mathbb{Z}^2$ plus, for each face F of $\mathbb{Z}^2$, two 'diagonal' edges between pairs of vertices on the perimeter of F. $\mathbb{Z}^2_{cp}$ and $\mathbb{Z}^2$ are a matching pair of graphs in the terminology of Kesten (1982), Section 2.2. (In the notation of that section $\mathbb{Z}^2_{cp}$ and $\mathbb{Z}^2$ are based on the 'mosaic' $\mathbb{Z}^2$ and the collection of all its faces.) Note that $\mathbb{Z}^2_{cp}$ is not a planar graph anymore. $p_c(\mathbb{Z}^2, \text{site})$ will denote the critical probability for site percolation on $\mathbb{Z}^2$. It is known that

$$p_c(\mathbb{Z}^2, \text{site}) \geq .556$$

(see van den Berg and Ermakov (1996)). Moreover, it follows from Theorem 3.1 and Corollary 3.1 in Kesten (1982) (see also pp. 54-56 there) or from Russo (1981) that the critical probability for site percolation on $\mathbb{Z}^2_{cp}$ is $1 - p_c(\mathbb{Z}^2, \text{site})$. Throughout we will fix p so that

$$1 - p_c(\mathbb{Z}^2, \text{site}) < p < p_c(\mathbb{Z}^2, \text{site}). \tag{1.2}$$

In this note we will prove the following result.

Theorem. *Let p satisfy (1.2). Then*

$$P_p\{\textit{all } \xi \in \Xi \textit{ are seen along some path on } \mathbb{Z}^2_{cp} \textit{ from the origin}\} > 0 \tag{1.3}$$

and

$$P_p\{\textit{all } \xi \in \Xi \textit{ are seen along some path on } \mathbb{Z}^2_{cp}\} = 1. \tag{1.4}$$

Moreover, there exist constants $C_i = C_i(p) > 0$ such that

$$\begin{aligned} P_p\{&\textit{for all large } n \textit{ all words of length } C_2 n^2 \textit{ are seen along some path on} \\ &\mathbb{Z}^2_{cp} \textit{ from the origin and inside } [-n, n]^2\} \ge C_1. \end{aligned} \tag{1.5}$$

We note that the triangular lattice is a sublattice of $\mathbb{Z}^2_{cp}$. Thus, at $p = 1/2$, the result of Kesten, Sidoravicius and Zhang (1998) already shows that almost all words (with respect to a measure μ of the form (1.1)) are seen somewhere on $\mathbb{Z}^2_{cp}$. (1.3) is of course a much stronger statement (which does *not* hold on the triangular lattice, even at $p = 1/2$).

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2. "Double paths" and their properties.

It is convenient for our proofs to introduce a planar graph $\mathcal{M}$ which is closely related to $\mathbb{Z}^2_{cp}$. $\mathcal{M}$ is obtained from $\mathbb{Z}^2$ by adding in each face F of $\mathbb{Z}^2$ a vertex which is connected by an edge to each of the four vertices on the perimeter of F. We call the added vertices *central vertices* and to help us in picturing these we think of them as being located at the points $(i + \frac{1}{2}, j + \frac{1}{2}),\ i, j \in \mathbb{Z}^2$. Another way of picturing $\mathcal{M}$ is to make the crossing of two 'diagonal' edges, which are added to a face F when forming $\mathbb{Z}^2_{cp}$ from $\mathbb{Z}^2$, into a vertex of $\mathcal{M}$. A path and a self-avoiding path on $\mathcal{M}$ are defined in the obvious way.

We shall frequently associate a self-avoiding path $(w_1, w_2, \dots)$ on $\mathbb{Z}^2_{cp}$ to a self-avoiding path $(v_0, v_1, \dots)$ on $\mathcal{M}$. This is done in the following unique way. We take

$$w_1, w_2 \dots \text{ as the successive noncentral vertices among } v_1, v_2, \dots. \tag{2.1}$$

We regard the w_i also as vertices on $\mathbb{Z}^2_{cp}$. With this interpretation it is clear that $(w_1, w_2, \dots)$ is a self-avoiding path on $\mathbb{Z}^2_{cp}$. We call this path the path *associated to* $(v_0, v_1, \dots)$. Note that v_0 never is a vertex of the associated path, even if v_0 is a noncentral vertex.

We will call a self-avoiding path $(v_0, v_1, \dots, v_n)$ on $\mathcal{M}$ *occupied (vacant)* if and only if all noncentral vertices among the v_i are occupied (respectively, vacant). The occupancy or vacancy of the central vertices will not be significant for our purposes, since we only want to discuss words seen on $\mathbb{Z}^2_{cp}$. Accordingly, if $\pi = (v_0, v_1, \dots)$ is a path on $\mathcal{M}$ and $\xi = (\xi_1, \xi_2, \dots)$ is a word, then we say that ξ is *seen along* π (*on* $\mathcal{M}$) if and only if

$$X(w_i) = \xi_i, \ i \geq 1. \tag{2.2}$$

Intuitively speaking this says that ξ is seen along the path associated to π. This is not quite accurate though, because when we consider whether ξ is seen along a path on $\mathbb{Z}^2_{cp}$ we ignore the initial point of the path. Because of this technicality, (2.2) really says that $X(w_1) = \xi_1$ and $(\xi_2, \xi_3, \dots)$ is seen along the path associated to π.

Next we define a *double path* on $\mathcal{M}$. This is defined to be a pair of self-avoiding paths π', π'' on $\mathcal{M}$ which satisfy the following properties (2.3)-(2.7):

$$\pi' \text{ and } \pi'' \text{ have no vertices in common;} \tag{2.3}$$

$$\pi' \text{ is occupied and } \pi'' \text{ is vacant;} \tag{2.4}$$

$$\text{the initial points of } \pi' \text{ and } \pi'',\ u' \text{ and } u'',\ \text{are neighbors on } \mathcal{M}; \tag{2.5}$$

$$\text{the final points of } \pi' \text{ and } \pi'',\ v' \text{ and } v'',\ \text{are neighbors on } \mathcal{M}. \tag{2.6}$$

In addition we will require the minimality property (2.7) below. Let $\widetilde{\pi}'$ and $\widetilde{\pi}''$ be a pair of paths on $\mathcal{M}$ which satisfy (2.3)-(2.6) with π' replaced by $\widetilde{\pi}'$ and π'' replaced by $\widetilde{\pi}''$. Denote by $R(\widetilde{\pi}', \widetilde{\pi}'')$ the interior of the Jordan curve formed by concatenating $\widetilde{\pi}'$, $\{v', v''\}$, (the reverse of) $\widetilde{\pi}''$ and $\{u'', u'\}$. Let $\overline{R}(\widetilde{\pi}', \widetilde{\pi}'')$ be the union of $R(\widetilde{\pi}', \widetilde{\pi}'')$ and its boundary (that is, the above Jordan curve). Then we further require that

$$\overline{R}(\pi', \pi'') \text{ is minimal among all such } \overline{R}(\widetilde{\pi}', \widetilde{\pi}''), \tag{2.7}$$

that is, there does not exist a pair $\widetilde{\pi}', \widetilde{\pi}''$ satisfying (2.3)-(2.6) and such that $\overline{R}(\widetilde{\pi}', \widetilde{\pi}'')$ is strictly contained in $R(\widetilde{\pi}', \widetilde{\pi}'')$.

In order to find double paths, the following observation will be useful. For any given pair $\widetilde{\pi}', \widetilde{\pi}''$ which satisfies (2.3) - (2.6), there exist at most finitely many pairs $\widehat{\pi}', \widehat{\pi}''$ of paths on $\mathcal{M}$ which also satisfy (2.3)-(2.6) as well as

$$\overline{R}(\widehat{\pi}', \widehat{\pi}'') \subset \overline{R}(\widetilde{\pi}', \widetilde{\pi}'').$$

(Throughout this paper $A \subset B$ will mean that A is contained in B, but not necessarily strictly; thus $A = B$ is possible if $A \subset B$). Now, for any pair $\widetilde{\pi}', \widetilde{\pi}''$ which satisfy (2.3)-(2.6), there exists a pair π', π'' which satisfies (2.3)-(2.7) (that is, a double path) which in addition satisfies

$$\overline{R}(\pi', \pi'') \subset \overline{R}(\widetilde{\pi}', \widetilde{\pi}''). \tag{2.8}$$

The following lemma gives an important necessary condition for π', π'' to be a double path.

Lemma 1. *If π', π'' is a double path, and if v is a vertex on π', then there exists a vertex w adjacent (on $\mathcal{M}$) to v, such that w is connected by a vacant path on $\mathcal{M}$ in $\overline{R}(\pi', \pi'') \setminus \pi'$ to π''.*

Similarly, if v is a vertex on π'', then there exists a vertex w adjacent (on $\mathcal{M}$) to v, such that w is connected by an occupied path on $\mathcal{M}$ in $\overline{R}(\pi', \pi'') \setminus \pi''$ to π'.

Note that it is possible in the first part of the lemma that $w \in \pi''$. In this case, the vacant path from w to π'' consists of w only. A similar comment applies to the second part.

Proof of Lemma 1. Because of the symmetric roles of π', π'' we only need to prove the first part of the Lemma. We then apply Proposition 2.2 of Kesten (1982) with 'occupied' and 'vacant' interchanged (note that J should be $\overline{J}$ on lines 1 and 2 from bottom on p. 30 of Kesten (1982)). We make the following choices: For the mosaic $\mathcal{M}$ in Proposition 2.2 we take the present $\mathcal{M}$. The graphs $\mathcal{G}$ and $\mathcal{G}^*$ are both taken equal to $\mathcal{M}$ (this is indeed a matching pair based on the mosaic $\mathcal{M}$, because all faces of $\mathcal{M}$ are already close packed; see Def. 4 on p. 18 of Kesten (1982)). For the occupancy configuration on $\mathcal{M}$ we extend the existing occupancy configuration on the noncentral vertices by declaring each central vertex on π'' to be vacant and all central vertices not on π'' to be occupied. J is the Jordan curve formed by concatenating π', $\{v', v''\}$, (the reverse of) π'' and $\{u'', u'\}$. Finally, $A_1 = \{v\}$, $A_2 =$ the piece of π' from v (including v) to v', followed by the edge $\{v', v''\}$, $A_3 = \pi''$, and finally $A_4 =$the edge $\{u'', u'\}$ followed by the piece of π' from u' to v (including v). Then $\overline{J} = \overline{R}(\pi', \pi'')$ and by the minimality property (2.7) there does not exist an occupied path r^* on $\mathcal{G}^* = \mathcal{M}$ inside $\overline{R}(\pi', \pi'') \setminus A_1 \cup A_3$ from a vertex of $\mathring{A}_2$ to a vertex of $\mathring{A}_4$. (As in Kesten (1982), $\mathring{A}_i$ stands for A_i minus its endpoints.) Indeed, if such a path r^* would exist, then it would contain a crosscut $\widetilde{r}$ of $R(\pi', \pi'')$ and we could change π' to a new path $\widetilde{\pi}'$ by replacing a piece of π' by $\widetilde{r}$, such that

$$\overline{R}(\widetilde{\pi}', \pi'') \subsetneq \overline{R}(\pi', \pi'').$$

(As in Newman (1951) we define a crosscut of R to be a simple curve in $\overline{R}$ with only its endpoints on the boundary of R.) This would contradict (2.7). Proposition 2.2 of Kesten (1982) now gives the existence of a path r on $\mathcal{G} = \mathcal{M}$ from v to π'' in $\overline{R}(\pi', \pi'') \setminus A_1 \cup A_3$ such that all vertices of r in $\overline{J} \setminus A_1 \cup A_3$ are vacant. In fact, r minus its initial point v must be disjoint from π', because all vertices on π' are occupied. We now take for w the first vertex on r after v. The path required in the lemma is then the piece of r from w to π''. □

Now let (π', π'') be a double path starting at (u', u'') and ending at (v', v''), and define

$$\Theta = \Theta(\pi', \pi'') = \lfloor \frac{1}{4} \min(\|v' - u'\|, \|v'' - u''\|) \rfloor, \tag{2.9}$$

where $\|\cdot\|$ denotes the Euclidean norm. (This definition differs slightly from the corresponding one in Kesten, Sidoravicius and Zhang (1998).) Finally, let $\xi = (\xi_1, \dots) \in \Xi$ be any infinite word. The next proposition (which is purely deterministic) shows how one can 'see' an initial segment of the word ξ inside $\overline{R}(\pi', \pi'')$. This proposition is our principal technical step. It is entirely analogous to Lemma 8 in Kesten, Sidoravicius and Zhang (1998).

Proposition 2. *Let (π', π'') be a double path starting at (u', u'') and ending at (v', v'') and let $\xi \in \Xi$ be arbitrary. Then there exist paths $\sigma' = (\sigma'_0 = u', \sigma'_1, \dots), \sigma'' = (\sigma''_0 = u'', \sigma''_1, \dots)$ on $\mathcal{M}$ with the following properties:*

$$\sigma',\ \sigma'' \subset \overline{R}(\pi', \pi''); \tag{2.10}$$

$$\sigma' \text{ and } \sigma'' \text{ start at } u' \text{ and } u'', \text{ respectively}; \tag{2.11}$$

$$\text{the endpoints of } \sigma' \text{ and } \sigma'' \text{ belong to } \{v', v''\}; \tag{2.12}$$

one sees an initial segment of ξ on $\mathcal{M}$ from $u'(u'')$ along $\sigma'(\sigma'')$.
This initial segment of ξ contains at least $(\xi_1, \dots, \xi_{\Theta-1})$. (2.13)

Proof. We only prove that we can find the path σ' from u'. The argument for σ'' is the same except for an interchange of the roles of 'occupied' and 'vacant'. For brevity we shall suppress the primes on σ. We have to show that one can choose a path $\sigma = (\sigma_0 = u', \sigma_1, \dots, \sigma_\nu)$ on $\mathcal{M}$ contained in $\overline{R}(\pi', \pi'')$, such that σ_1 is adjacent to $u', \sigma_\nu \in \{v', v''\}$, and such that along σ one sees on $\mathcal{M}$ an initial segment containing at least the first $\Theta - 1$ components of ξ. We prove this in the following recursive way. The steps differ slightly, depending on whether u' is a central vertex or not. If u' is a central vertex, then we find a vertex σ_1 of $\mathbb{Z}^2_{cp}$ which is adjacent on $\mathcal{M}$ to u' and which lies in $\overline{R}(\pi', \pi'')$ and is such that σ_1 is occupied (vacant) if $\xi_1 = 1$ ($\xi_1 = 0$). We further find a new double path (π'_1, π''_1) with σ_1 the initial point of one of them and with endpoints (v', v''), and such that

$$\overline{R}(\pi'_1, \pi''_1) \subset \overline{R}(\pi', \pi'') \setminus \{u'\}. \tag{2.14}$$

If u' is noncentral, then we either use the same construction as just outlined or take σ_1, σ_2 such that σ_1 is a central vertex adjacent on $\mathcal{M}$ to u' and such that σ_2 is adjacent on $\mathcal{M}$ to σ_1. Moreover σ_2 has to be a vertex of $\mathbb{Z}^2_{cp}$ which lies in $\overline{R}(\pi', \pi'')$ and such that σ_2 is occupied (vacant) if $\xi_1 = 1$ ($\xi_1 = 0$). This time we find a new double path (π'_1, π''_1) with σ_2 the initial point of one of them and with endpoints (v', v''), and such that

$$\overline{R}(\pi'_1, \pi''_1) \subset \overline{R}(\pi', \pi'') \setminus \{u', \sigma_1\} \tag{2.15}$$

holds (instead of (2.14)).

We then repeat this step with (π'_1, π''_1) replacing (π', π''). This construction will continue until for the first time we come to a double path (π'_ν, π''_ν) with

$$\Theta(\pi'_\nu, \pi''_\nu) \le 1.$$

Our construction is such that the initial point of one of π'_{k+1}, π''_{k+1} is within distance 2 of one of the initial points of π'_k, π''_k, and the endpoints of all the double paths are v' and v''. Therefore,

$$\Theta(\pi'_{k+1}, \pi''_{k+1}) \ge \Theta(\pi'_k, \pi''_k) - 1$$

and it takes at least $\Theta(\pi', \pi'') - 1$ steps before we stop. Since at each recursive step one ξ_i is used, it is clear that we will see at least $(\xi_1, \cdots, \xi_{\Theta-1})$ when our process stops.

Several cases have to be distinguished in our construction, depending on the value of ξ_1 and on whether u' is a central vertex or not.

Case (Ia) $\xi_1 = 1$ and u' is a central vertex. This case is treated in exactly the same way as Case (i) in Lemma 8 of Kesten, Sidoravicius and Zhang (1998). We briefly state the essentials. We take σ_1 to be the first vertex of $\pi' \setminus \{u'\}$. Since u' is a central vertex, σ_1 is necessarily a vertex of $\mathbb{Z}^2_{cp}$. Since $\sigma_1 \in \pi'$, it is occupied, in agreement with the requirement $X(\sigma_1) = \xi_1 = 1$.

We next take $\widehat{\pi}'$ to be the piece of π' from σ_1 to v' (this is just π' minus its first edge). We also want a vacant path $\widehat{\pi}''$. To choose this, we observe that by Lemma 1 there must exist a vacant path π_2 on $\mathcal{M}$ from a neighbor u_1 on $\mathcal{M}$ of σ_1 to π'', and such that

$$\pi_2 \in \overline{R}(\pi', \pi'') \setminus \pi'. \tag{2.16}$$

Now form the vacant path $\widehat{\pi}''$ on $\mathcal{M}$ from u_1 to v'' which consists of π_2 followed by the piece of π'' from the endpoint of π_2 to v''. Then $(\widehat{\pi}', \widehat{\pi}'')$ is a pair of paths, occupied and vacant, respectively, from (σ_1, u_1) to (v', v''). $\widehat{\pi}'$ and $\widehat{\pi}''$ have no vertex in common, by virtue of (2.16) and (2.3). Finally, by construction, $\widehat{\pi}', \widehat{\pi}''$ and $\{\sigma_1, u_1\}$ are contained in $\overline{R}(\pi', \pi'') \setminus \{u'\}$, so that

$$\overline{R}(\widehat{\pi}', \widehat{\pi}'') \subset \overline{R}(\pi', \pi'') \setminus \{u'\}.$$

It is not clear that $(\widehat{\pi}', \widehat{\pi}'')$ itself has the minimality property corresponding to (2.7). However, using the observation before Lemma 1, we can take for (π_1', π_1'') an occupied and vacant pair of paths from (σ_1, u_1) to (v', v'') which makes $\overline{R}(\pi_1', \pi_1'')$ minimal. This will automatically satisfy

$$\overline{R}(\pi_1', \pi_1'') \subset \overline{R}(\widehat{\pi}', \widehat{\pi}'') \subset \overline{R}(\pi', \pi'') \setminus \{u'\}. \tag{2.17}$$

Thus (2.14) will be satisfied and we are done with our recursive step in Case (Ia).

Case (Ib) $\xi_1 = 1$ and u' is not a central vertex. Now u' is a vertex of $\mathbb{Z}^2_{cp}$. It may be that the first vertex of $\pi' \setminus \{u'\}$ is also a vertex of $\mathbb{Z}^2_{cp}$. Then we take σ_1 equal to this first vertex of $\pi' \setminus \{u'\}$ and proceed as in case (Ia).

The only other possibility is that π' begins with u', w_1, w_2 with w_1 some central vertex and w_2 some vertex of $\mathbb{Z}^2_{cp}$ such that u' and w_2 are adjacent on $\mathbb{Z}^2_{cp}$, but not necessarily on $\mathcal{M}$. In this case we take $\sigma_1 = w_1, \sigma_2 = w_2$. We then again proceed as in case (Ia), but now with σ_1 replaced by σ_2. That is, we take for $\widehat{\pi}'$ the piece of π' from σ_2 to v'. We further form $\widehat{\pi}''$ by concatenating a vacant path in $\overline{R}(\pi', \pi'') \setminus \pi'$ from a neighbor of σ_2 to π'' with a piece of π'' ending at v''. We then find the new double path (π_1', π_1'') which satisfies (2.15) by applying the observation before Lemma 1, as in Case (Ia).

Case (IIa) $\xi_1 = 0$ and u' is a central vertex. This case closely follows case (ii) of Lemma 8 in Kesten, Sidoravicius and Zhang (1998). By a translation we may assume that $u' = (\frac{1}{2}, \frac{1}{2})$. We define H to be the unit square centered at u', that is,

$$H = \{(x, y) : |x - \frac{1}{2}| < \frac{1}{2}, |y - \frac{1}{2}| < \frac{1}{2}\}.$$

We denote the first vertex of $\pi' \setminus \{u'\}$ by u_1 and consider the four neighbors of u' on $\mathcal{M}$. These are all vertices of $\mathbb{Z}^2_{cp}$ at the four 'corners' of H (see Figure 1).

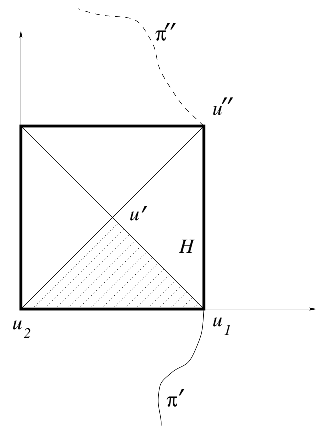

FIGURE 1. The solidly drawn square is the boundary of H. The interior of the hatched triangle is contained in $R(\pi', \pi'')$.

One of them is u_1 (which is occupied) and another one is u'' (which is vacant). Without loss of generality we take $u_1 = (1, 0)$. Now the interior of one of the triangles with vertices u', u_1 and (0,0) or with vertices u', u_1 and (1,1) must be contained in $R(\pi', \pi'')$. For the sake of argument assume that

$$\text{the interior of the triangle with vertices at } u', u_1 \text{ and } (0,0)$$
$$\text{is contained in } R(\pi', \pi'').$$

Write u_2 for the third vertex of this triangle, that is, $u_2 = (0, 0)$. We claim that u_2 must be vacant. Assume, to arrive at a contradiction, that u_2 is occupied (and hence $u_2 \notin \pi''$). If u_2 is not a vertex on π', then we can replace the first edge of π' (i.e., the edge $\{u', u_1\}$) by the two edges $\{u', u_2\}, \{u_2, u_1\}$. This will remove the triangle with vertices u', u_1, u_2 from $R(\pi', \pi'')$, in contradiction to the minimality property (2.7). If, on the other hand, u_2 is a vertex of π', then we can replace the piece of π' from u' to u_2 by the single edge $\{u', u_2\}$. Since this last edge is a crosscut of $R(\pi', \pi'')$, this replacement will again strictly decrease $R(\pi', \pi'')$. This is impossible, again by (2.7). This proves our claim that u_2 is vacant.

We next claim that there exists a vacant connection r on $\mathcal{M}$ from u_2 to π''. It may be that u_2 already lies on π'', in which case the sought vacant connection r consists of $\{u_2\}$ only. To see that r exists in general, we move along the arc of the perimeter of H from u_2 to u'' which does not contain u_1. Let u_3 be the first vertex of $\pi' \cup \pi''$ we meet while moving along this arc and denote by A_2 the piece from u_2 to u_3 of this arc (including its endpoints u_2 and u_3). As in case (ii) of Lemma 8 in Kesten, Sidoravicius and Zhang (1998) it cannot be the case that $u_3 \in \pi'$. Indeed, if u_3 were a vertex of π', then the edge $\{u', u_3\}$ would be a crosscut of $R(\pi', \pi'')$, because its endpoints would lie on π', and it could be connected in H to $\mathring{A}_2 \subset R(\pi', \pi'')$ without intersecting $\pi' \cup \pi''$. But if $\{u', u_3\}$ is a crosscut of $R(\pi', \pi'')$, then we can replace the piece of π' from u' to u_3 by the single edge $\{u', u_3\}$ and strictly decrease $R(\pi', \pi'')$. Since this contradicts (2.7), we must have

$u_3 \in \pi''$. If A_2 is vacant, then we can take $r = A_2$. In the other cases we prove the existence of r by an application of Proposition 2.2 of Kesten (1982). We make the following choices. For the occupancy configuration of $\mathcal{M}$ we extend the occupancy configuration on $\mathbb{Z}^2_{cp}$ by taking all central vertices on π'' vacant and all vertices off π'' occupied. Further we take $A_1 = \{u_2\}$, A_2 as above, $A_3 =$ the piece of π'' from u_3 to v'', and $A_4 =$ the edge $\{v'', v'\}$, followed by the piece of (the reverse of) π' from v' to u_1, followed by the edge $\{u_1, u_2\}$. These arcs make up a Jordan curve J which is contained in $\overline{R}(\pi', \pi'')$. Therefore the interior of J is contained in $R(\pi', \pi'')$. Moreover, there cannot exist in $\overline{J}$ an occupied path r^* on $\mathcal{M}$ from some vertex w in $\mathring{A}_2$ to some vertex in $\mathring{A}_4$ (a vertex in $\mathring{A}_4$ is a vertex in π'). Indeed, since $w \in \mathring{A}_2$, w would be a neighbor on $\mathcal{M}$ of u'. Therefore, the edge $\{u', w\}$ followed by such a path r^* would form an occupied crosscut of $R(\pi', \pi'')$, and no such crosscut can exist by the minimality property (2.7). We can therefore apply Proposition 2.2 of Kesten (1982) (with 'occupied' and 'vacant' interchanged and corrected by replacing J by $\overline{J}$ in lines 1 and 2 from bottom on p. 30). This guarantees the existence of a vacant path r on $\mathcal{M}$ in $\overline{J} \subset \overline{R}(\pi', \pi'')$ from u_2 to $A_3 \subset \pi''$. r is necessarily disjoint from π', since all vertices on π' are occupied and all vertices on r are vacant. This proves that r has the desired properties.

We can now complete our choices for this recursive step. We take $\sigma_1 = u_2$ (which satisfies the requirement $X(\sigma_1) = 0 = \xi_1$). We further take $\widehat{\pi}'_1$ =piece of π' from u_1 to v', and $\widehat{\pi}'' =$ the path r followed by the piece of π'' from the endpoint of r to v''. Finally, we again choose π'_1, π''_1 such that (2.17) holds and such that $\overline{R}(\pi'_1, \pi''_1)$ is minimal in the family of possible $\overline{R}(\pi'_1, \pi''_1)$ satisfying (2.17).

Case (IIb) $\xi_1 = 0$ and u' is not a central vertex, but $u_1 :=$ first vertex of $\pi' \setminus \{u'\}$ is central. The argument is similar to that of the last case. By translation and rotation we may assume that $u' = (0,0)$ and $u_1 = (\frac{1}{2}, \frac{1}{2})$. One 'side' of the edge $\{u', u_1\}$ must lie in $R(\pi', \pi'')$. By symmetry we may therefore assume that the interior of the triangle with vertices u', u_1 and $(1,0)$ is contained in $R(\pi', \pi'')$.

We now follow the argument of the preceding case. This time we take H to be the 'diamond' $\{(x,y) : |x| + |y| = 1\}$ (see Figure 2).

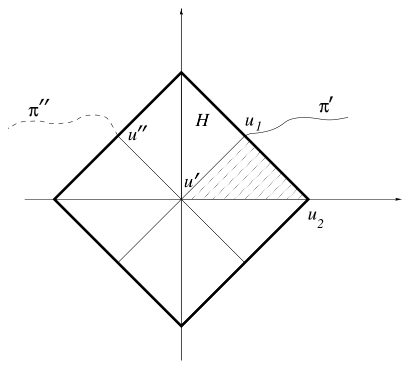

FIGURE 2. The solidly drawn diamond is the boundary of H. The interior of the hatched triangle is contained in $R(\pi', \pi'')$.

u_2 will denote the vertex $(1,0)$. The boundary of H is a Jordan curve which necessarily contains the initial point u'' of π''. When one moves from u_2 to u''

along the arc of the boundary of H which does not contain u_1, then one will meet a first vertex, u_3 say, of $\pi' \cup \pi''$. We denote by A_2 the arc of the boundary of H from u_2 to u_3. As in the preceding case we now show that u_2 must be vacant and $u_3 \in \pi''$. Moreover, u_2 must have a vacant connection r on $\mathcal{M}$ in $\overline{R}(\pi', \pi'')$ to π''.

Finally we complete this recursive step when $u_1 = (\frac{1}{2}, \frac{1}{2})$, by taking $\sigma_1 = u_2$, $\widehat{\pi}'$ = the piece of π' from u_1 to v' and $\widehat{\pi}''$ = the path r followed by the piece of π'' from the endpoint of r to v''.

Case (IIc) $\xi_1 = 0$ and neither u' nor u_1 are central. Now we may assume that $u' = (0,0)$ and $u_1 = (1,0)$. In this case there is again a triangle adjacent to the edge $\{u', u_1\}$ whose interior is contained in $R(\pi', \pi'')$. Without loss of generality, let this be the triangle with vertices u', u_1 and $(\frac{1}{2}, -\frac{1}{2})$. We then take $u_2 = (\frac{1}{2}, -\frac{1}{2})$. This vertex does not lie on π' for the same reasons as in case (IIa). It is a central vertex. Since it does not lie on π' we may declare this site to be vacant. We further declare all central vertices on π'' vacant and all central vertices off π'' other than u_2 occupied. We can now use the same proof as in case (IIb) or case (IIa) to show that there exists a vacant connection r on $\mathcal{M}$ in $\overline{R}(\pi', \pi'') \setminus \pi'$ from u_2 to π''. However, σ_1 and the paths $\widehat{\pi}', \widehat{\pi}''$ have to be chosen slightly differently than in the previous cases. Let $\widetilde{\pi}$ be the (vacant) path consisting of r and the piece of π'' from the endpoint of r to v''. Denote the first vertex of $\widetilde{\pi}$ after its initial point by ρ and take $\widehat{\pi}''$ to be the piece of $\widetilde{\pi}$ from ρ to v''. Thus, basically, $\widehat{\pi}''$ equals $\widetilde{\pi}$ minus its first edge.

Since r starts at $(\frac{1}{2}, -\frac{1}{2})$ and $u' = (0,0)$ and $u_1 = (1,0)$ lie on π', ρ can only take the values $(0,-1)$ and $(1,-1)$. If $\rho = (1,-1)$, then ρ is adjacent to u_1 on $\mathcal{M}$. We now take $\sigma_1 = u_2 = (\frac{1}{2}, -\frac{1}{2}), \sigma_2 = \rho = (1,-1)$, $\widehat{\pi}'$ = the piece of π' from u_1 to v', and $\widehat{\pi}'' = \widetilde{\pi}$. Then (2.15) holds. This finishes the recursive step when $\rho = (1,-1)$.

If $\rho = (0,-1)$, then ρ is not adjacent to u_1 on $\mathcal{M}$. In this situation we therefore take for $\widehat{\pi}'$ the path consisting of the edge from u_2 to u_1 followed by the piece of π' from u_1 to v'. We further take $\sigma_1 = \rho = (0,-1)$ and $\widehat{\pi}'' = \widetilde{\pi}$, as before. Now $\widehat{\pi}'$ and $\widehat{\pi}''$ again start at adjacent points on $\mathcal{M}$ (namely at u_2 and $\rho = \sigma_1$) and end at v' and v''. Also (2.14) holds. Moreover σ_1 is adjacent to u' on $\mathcal{M}$ and $X(\sigma_1) = 0$, as required. This finishes the recursive step for this last case. $\square$

3. Proof of the first part of Theorem 1.

With proposition 2 we can prove (1.3) and (1.4) quickly from known facts about supercritical percolation. Even though (1.3), and consequently also (1.4), is contained in (1.5), we give first a direct proof of (1.3) and (1.4), because this is much easier than (1.5), and most readers will be satisfied with (1.3) and (1.4). The proof of (1.5) will be given in a separate section.

Throughout this section C_i will denote a strictly positive, finite constant (independent of n). Define the following events:

$$E_1 = \{\text{there exists an infinite occupied path } \widetilde{\pi}' \text{ on } \mathcal{M} \text{ from the origin in (the fourth quadrant) } \cup \mathbf{0} = (0,\infty) \times (-\infty, 0) \cup \{\mathbf{0}\}\},$$

$$E_2 = \{\text{there exists an infinite vacant path } \widehat{\pi}'' \text{ on } \mathcal{M} \text{ from a neighbor of the origin and inside the first quadrant } (0,\infty)^2\},$$

and

$$E_3 = \{\text{for infinitely many } k \text{ there exists a vacant circuit on } \mathcal{M} \text{ surrounding the origin in the annulus } [-2^{k+1}, 2^{k+1}]^2 \setminus [-2^k, 2^k]^2]\}.$$

Then (1.2) implies that $P_p\{E_1\} > 0$, $P_p\{E_2\} > 0$ and $P_p\{E_3\} = 1$ (see Grimmett (1999), Theorem 11.70, Smythe and Wierman (1978), Section 3.4). Since E_1 and E_2 are defined in terms of disjoint sets of vertices, they are independent and

$$P_p\{E_1 \cap E_2 \cap E_3\} = P_p\{E_1 \cap E_2\} = P_p\{E_1\}P_p\{E_2\} > 0. \tag{3.1}$$

Now assume that $E_1 \cap E_2 \cap E_3$ occurs, and that $\widetilde{\pi}', \widehat{\pi}''$ are an occupied and a vacant path as described in E_1 and E_2, respectively. Let u'' be the initial point of $\widehat{\pi}''$. Also let $\mathcal{C}_k$ be a vacant circuit surrounding the origin in $[-2^{k+1}, 2^{k+1}]^2 \setminus [-2^k, 2^k]^2$. Then both paths $\widetilde{\pi}'$ and $\widehat{\pi}''$ must intersect $\mathcal{C}_k$. One can then construct from a piece of $\widehat{\pi}''$ and a piece of $\mathcal{C}_k$ a vacant path $\widetilde{\pi}''$ on $\mathcal{M}$ from u'' to a vertex v'', adjacent on $\mathcal{M}$ to a vertex $v' \in \widetilde{\pi}'$. We can and will even choose $\widetilde{\pi}'$ disjoint from $\widetilde{\pi}''$. As in the argument following (2.6) there then exists a double path (π', π'') from $(u' = \mathbf{0}, u'')$ to (v', v'') so that (2.8) holds. By Proposition 2 there then exists for each infinite word ξ a path σ from $\mathbf{0} = u'$ to v' or v'' such that an initial piece of ξ is seen along σ. Since $v' \in \mathcal{C}_k$, any path from $\mathbf{0}$ to v' or v'' must contain at least $2^k + 1$ noncentral vertices. Thus the length of the piece of ξ which is seen along σ is at least 2^k.

This argument works for all ξ and all k. Therefore, the left hand side of (1.3) is at least $P_p\{E_1 \cap E_2 \cap E_3\} > 0$. This proves (1.3). In turn, (1.4) then follows from (1.3) and the ergodic theorem (compare Harris (1960), Lemmas 3.1 and 5.1).

4. Proof of (1.5).

Before we turn to the details of the proof of (1.5) we give a brief outline. The proof is based on the construction of a "snake" inside of which the finite words will be seen. A snake will be the region $\overline{R}(\widetilde{\pi}', \widetilde{\pi}'')$ between an occupied path $\widetilde{\pi}'$ and a vacant path $\widetilde{\pi}''$ which "wiggle a lot" (see Figure 3). More specifically, if u', v' (u'', v'') are the initial and endpoint of $\widetilde{\pi}'$ (of $\widetilde{\pi}''$, respectively), then any path on $\mathcal{M}$ from u' to $\{v', v''\}$ and contained in $\overline{R}(\widetilde{\pi}', \widetilde{\pi}'')$ will be forced to go back and forth between the strips $[0, n/4] \times \mathbb{R}$ and $[3n/4, n] \times \mathbb{R}$ at least $C_3 n$ times. Thus, any such path will have length at least $C_3 n \cdot n/2$. If (π', π'') is a double path from (u', u'') to (v', v'') with $\overline{R}(\pi', \pi'') \subset \overline{R}(\widetilde{\pi}', \widetilde{\pi}'')$, then an initial piece of any word can be seen along a path from u' to (v', v'') in $\overline{R}(\pi', \pi'')$ (by Proposition 2). Because such a path must have length at least $C_3 n^2/2$, we will actually see any word of length $C_4 n^2$ along some path from u' to (v', v'') inside $\overline{R}(\pi', \pi'')$.

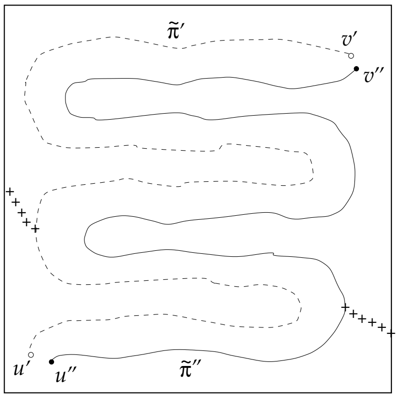

FIGURE 3. A schematic diagram of a snake. $\widetilde{\pi}'$ and $\widetilde{\pi}''$ are the solidly drawn and dashed curves in the boundary of the snake and the curves ++++ represent two typical crosscuts λ_{2i-1}.

A somewhat more topological description of the situation is as follows. Let J_1 be the Jordan curve formed by concatenating $\widetilde{\pi}'$, $\{v', v''\}$, the reverse of $\widetilde{\pi}''$ and $\{u'', u'\}$. $R(\widetilde{\pi}', \widetilde{\pi}'')$ is the interior of J_1. If J is a Jordan curve, we will denote its interior by $\overset{\circ}{J}$. (This should not be confused with $\overset{\circ}{A}$ when A is an arc as in Section 2.) The way we will force a path from u' to $\{v', v''\}$ to go back and forth between the strips $[0, n/4] \times \mathbb{R}$ and $[3n/4, n] \times \mathbb{R}$ is by constructing disjoint crosscuts $\lambda_{2i-1}, 1 \le i \le C_3 n$, from $\widetilde{\pi}'$ to $\widetilde{\pi}''$ in $R(\widetilde{\pi}', \widetilde{\pi}'')$ with $\lambda_{2i-1} \subset [0, n/4] \times \mathbb{R}$ when i is even, and $\lambda_{2i-1} \subset [3n/4, n] \times \mathbb{R}$ when i is odd. Each such crosscut λ_{2i-1} divides $\overset{\circ}{J}_1$ into two components, one of which has $\{u', u''\}$ in its boundary, and the other of which has $\{v', v''\}$ in its boundary. This will force a path from u' to $\{v', v''\}$ to cross each λ_{2i-1}. In fact, these λ's will be constructed in such a way that a path from u' to $\{v', v''\}$ has to cross $\lambda_1, \lambda_3, \dots$ in succession. Because the crosscuts λ_{2i-1} are alternately located in the strips $[0, n/4] \times \mathbb{R}$ and $[3n/4, n] \times \mathbb{R}$ this will also force any path from u' to $\{v', v''\}$ to go back and forth between these strips, as desired.

In our construction, we will take u' = the origin and the crosscuts λ_{2i-1} will be constructed from appropriate left-right and top-bottom crossings of $[-n, n]^2$. The next lemma gives the deterministic (topological) part of the proof. As usual, the details of these arguments are messier than the simple intuitive picture suggests. The last lemma will then estimate the probability that the various required crossings of $[-n, n]^2$ exist. Let $R = [a, b] \times [c, d]$ be a rectangle and $\overset{\circ}{R} = (a, b) \times (c, d)$ be its interior. A *left-right crossing of* R is a self-avoiding path on $\mathcal{M}$ inside $[a, b] \times (c, d)$ with one endpoint on $\{a\} \times (c, d)$ and the other endpoint on $\{b\} \times (c, d)$. If ρ is a left-right crossing of R, then $\overset{\circ}{R} \setminus \rho$ consists of two components which contain $[a, b] \times \{d\}$ and $[a, b] \times \{c\}$ in their boundary, respectively. We shall denote these components by $\rho^+ = \rho^+(R)$ and $\rho^- = \rho^-(R)$, respectively. In a similar way one defines a *top-bottom crossing of* R and its *left and right component*. If σ is the top-bottom crossing, its left and right component will be denoted by $\sigma^\ell = \sigma^\ell(R)$ and $\sigma^r = \sigma^r(R)$, respectively. All paths in the next lemma are paths on $\mathcal{M}$.

Lemma 2. *Let* $S(n) = [-n, n]^2$ *and let* n *be divisible by* 4*. Assume that the following conditions (4.1)-(4.5) are satisfied:*

$$\exists \textit{ left-right crossings } \rho_1, \dots, \rho_{4k-1} \textit{ of the rectangle } [-n, n] \times [0, n] \tag{4.1}$$

such that

$$\text{all } \rho_i \text{ are disjoint,} \tag{4.2}$$

$$\begin{aligned} &\rho_i \text{ is vacant if } i \equiv 0, 3 (mod\ 4),\\ &\rho_i \text{ is occupied if } i \equiv 1, 2 (mod\ 4), \end{aligned} \tag{4.3}$$

$$\rho^+_{i+1}(S(n)) \subset \rho^+_i(S(n)), \quad 1 \le i \le 4k-2, \tag{4.4}$$

and

$$\begin{aligned} &\rho_i \text{ has only one point on the vertical line } x = n/4 \text{ if } i \equiv 1, 2 (mod\ 4)\\ &\quad \text{and has only one point on the vertical line } x = 3n/4 \text{ if } i \equiv 0, 3 (mod\ 4). \end{aligned} \tag{4.5}$$

Assume further that

$$\begin{aligned} &\exists \text{ a top-bottom crossings } \tau_1 \text{ of } [0, n/4] \times [-n, n]\\ &\quad \text{and top-bottom crossings } \tau_2, \tau_3 \text{ of } [3n/4, n] \times [-n, n] \end{aligned} \tag{4.6}$$

such that

$$\tau_1, \tau_2, \tau_3 \text{ are disjoint,} \tag{4.7}$$

$$\tau_1 \text{ and } \tau_3 \text{ are occupied and } \tau_2 \text{ is vacant,} \tag{4.8}$$

and

$$\tau_3^r([3n/4, n] \times [-n, n]) \subset \tau_2^r([3n/4, n] \times [-n, n]). \tag{4.9}$$

Finally, assume that there exists an occupied path γ on $\mathcal{M}$ from $u' = \mathbf{0}$ to the vertical line $x = n$ inside $[0, n] \times [-n, 0]$ and a vacant path δ on $\mathcal{M}$ from a neighbor u'' of u' to the horizontal line $y = n$ inside $(-n, 0) \times [0, n]$. Then there exist a constant $C_5 > 0$, an occupied path $\widetilde{\pi}'$ on $\mathcal{M}$ from u' to some v' and a vacant path $\widetilde{\pi}''$ on $\mathcal{M}$ from u'' to a neighbor v'' of v' such that

$$\widetilde{\pi}' \text{ and } \widetilde{\pi}'' \text{ are disjoint,} \tag{4.10}$$

$$\overline{R}(\widetilde{\pi}', \widetilde{\pi}'') \subset S(n), \tag{4.11}$$

and

$$\text{every path from } u' \text{ to } \{v', v''\} \text{ inside } \overline{R}(\widetilde{\pi}', \widetilde{\pi}'') \text{ has length } \ge C_5 n(k-1). \tag{4.12}$$

Proof. This proof will be broken down into a number of steps.

Step 1. This step is a simple observation which we shall use repeatedly. Let J be a Jordan curve with interior $\overset{\circ}{J}$, and let τ be a crosscut of $\overset{\circ}{J}$ with endpoints a and b on J. Then J is made up of two closed arcs from a to b, which have only the points a and b in common. If these two arcs are denoted by $\mathcal{A}_1, \mathcal{A}_2$, then $\overset{\circ}{J} \setminus \tau$ consists of two components, $\mathcal{K}_1$ and $\mathcal{K}_2$ say, and the boundary of $\mathcal{K}_i$ consists of $\mathcal{A}_i \cup \tau$ (see

Newman (1951), Theorem V.11.8). Now let σ be a further simple curve with one endpoint, c, on τ and the other endpoint, d, on J, such that

$$\sigma \setminus \{c,d\} \subset \mathring{J} \setminus \tau. \tag{4.13}$$

Then

$$d \in \mathcal{A}_i \setminus \{a,b\} \text{ implies that } \sigma \text{ is a crosscut of } \mathcal{K}_i. \tag{4.14}$$

This is rather obvious, because $\sigma \setminus \{c,d\} \subset \mathcal{K}_1 \cup \mathcal{K}_2$ by virtue of (4.13). Also by (4.13), $\sigma \setminus \{c,d\}$ does not intersect the boundary of $\mathcal{K}_1$, nor the boundary of $\mathcal{K}_2$. Thus $\sigma \setminus \{c,d\}$ is entirely contained in $\mathcal{K}_1$ or entirely contained in $\mathcal{K}_2$. But if $d \in \mathcal{A}_i \setminus \{a,b\}$, then d lies only in the boundary of $\mathcal{K}_i$ and therefore near d, σ cannot contain points of $\mathcal{K}_{3-i}$. Thus $\sigma \setminus \{c,d\} \subset \mathcal{K}_i$, which proves (4.14).

Step 2. In this step we construct a first Jordan curve which will surround our snake. This Jordan curve will consist of pieces of γ, δ, τ_3 and ρ_{4k-1}. We begin with a Jordan curve J_1 which is constructed by concatenating the following curves: γ from $u' = \mathbf{0}$ to its endpont a_1 on $\{n\} \times [-n,n]$, the segment of $\{n\} \times [-n,n]$ from a_1 to the upper right hand corner of $S(n)$, (n,n), the segment, s say, of $[-n,n] \times \{n\}$ from (n,n) to the endpoint b_1 of δ, the reverse of δ from b_1 to u'' and finally $\{u'',u'\}$. Then one sees from the location of γ and δ that $\mathring{J}_1 \supset (0,n)^2$. It also follows from the fact that ρ_{4k-1} is a left-right crossing of $[-n,n] \times [0,n]$, that ρ_{4k-1} contains a piece ρ'_{4k-1} which connects a point c_{4k-1} on δ to the segment from a_1 to (n,n) on the right edge of $S(n)$. In fact we shall take for ρ'_{4k-1} the piece of ρ_{4k-1} from its last intersection with δ (when starting on $\{-n\} \times [0,n]$) to its endpoint d_{4k-1} on $\{n\} \times [-n,n]$. Clearly ρ'_{4k-1} minus its endpoints is contained in $\mathring{S}(n)$. Thus by (4.14) (with J taken as the boundary of $S(n)$, viewed as a Jordan curve, and $\tau =$ the reverse of δ, followed by $\{u'',u'\}$ and γ), we see that

$$\rho'_{4k-1} \text{ is a crosscut of } \mathring{J}_1 \tag{4.15}$$

(see Figure 4). For the same reason, for $i = 1,2,3$, τ_i contains a piece τ'_i from a point e_i on γ to a point f_i of s on the top edge of $S(n)$, such that τ'_i is a crosscut of $\mathring{J}_1$. In particular the piece of τ_i between e_i and f_i does not intersect γ except at e_i (see Figure 4 again).

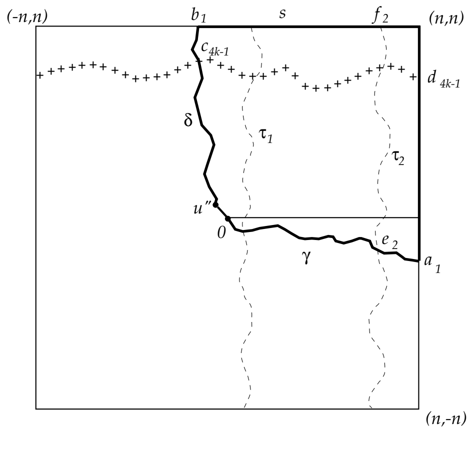

FIGURE 4. The Jordan curve J_1 and various crosscuts of $\mathring{J}_1$.

Now, by (4.15), ρ'_{4k-1} divides $\mathring{J}_1$ into two components, which we denote by $\mathcal{K}^\gamma$ and $\mathcal{K}^s$. The boundary of $\mathcal{K}^\gamma$ consists of γ, followed by the segment of $\{n\} \times [-n, n]$ from a_1 to d_{4k-1} along the right edge of $S(n)$, then by ρ'_{4k-1} to c_{4k-1} and by the piece of δ from c_{4k-1} to u'' and finally by $\{u'', u'\}$. $\mathcal{K}^s$ has s in its boundary. τ'_i starts on γ which is in the boundary of $\mathcal{K}^\gamma$ and ends on s, which is contained in the boundary of $\mathcal{K}^s$. Thus the crosscut τ'_i of $\mathring{J}_1$ runs from one component of $\mathring{J}_1 \setminus \rho'_{4k-1}$ to the other. Consequently, τ'_i must have a first intersection g_i with ρ'_{4k-1} (when starting at e_i). Call the piece of τ'_i from e_i to g_i, τ''_i. Again by (4.14) (with J_1 taken for J),

$$\tau''_i \text{ is a crosscut of } \mathcal{K}^\gamma,\ i = 1, 2, 3. \tag{4.16}$$

Hereafter the construction of the snake will take place in the component $\mathcal{K}$ of $\mathcal{K}^\gamma$ whose boundary consists of the following pieces: γ from $u' = \mathbf{0}$ to e_3, τ''_3 (which runs from e_3 to g_3), the piece of ρ'_{4k-1} from g_3 to the endpoint c_{4k-1} of ρ'_{4k-1} on δ, the piece of δ from this endpoint to u'', and $\{u'', u'\}$. We denote this boundary (when viewed as a Jordan curve) by J_2. For later use we point out that the first two pieces of J_2 (i.e., the piece of γ and τ''_3) are occupied while the pieces of ρ'_{4k-1} and δ are vacant.

Step 3. Here we exhibit some crosscuts of $\mathcal{K}$ which will be used in the construction of our snake. We claim that

$$\text{the piece } \tau''_i \text{ of } \tau'_i \text{ between } e_i \text{ and } g_i \text{ is a crosscut of } \mathcal{K},\ i = 1, 2. \tag{4.17}$$

This is a consequence of the locations of the τ_i and (4.9), as we shall now demonstrate. The fact that $\tau^r_3([3n/4, n] \times [-n, n]) \subset \tau^r_2([3n/4, n] \times [-n, n])$ implies that if we move along the top edge of $S(n)$ from (n, n) to b_1, then we must meet f_3 before f_2 (because in the neighborhood of f_3 there are points which lie in $\tau^r_3([3n/4, n] \times [-n, n])$ and hence in $\tau^r_2([3n/4, n] \times [-n, n])$; thus when we reach f_3 we cannot have left $\tau^r_2([3n/4, n] \times [-n, n])$ yet). This says that f_2 lies on the segment of s between f_3 and b_1. Near f_2 there are therefore points of τ'_2 which lie in the component of $\mathring{J}_1 \setminus \tau'_3$ whose boundary consists of the piece of s from f_3 to b_1, (the reverse of) $\delta, \{u'', u'\}$, the piece of γ from u' to e_3 and τ'_3. Since τ'_2 does not intersect the boundary of this component as one moves from e_2 to f_2, also e_2 lies in the part of γ which belongs to the boundary of this component, i.e., the part of γ between u' and e_3. But this part of γ also belongs to the boundary of $\mathcal{K}$, so that e_2 lies in the boundary of $\mathcal{K}$. Then τ''_2 must be a crosscut of $\mathcal{K}$ (since we already know from (4.16) that τ''_2 either lies entirely in $\mathcal{K}$ or entirely in the other component of $\mathcal{K}^\gamma \setminus \tau''_3$). The same argument can be made with the subscript 1 replacing the subscript 2, if we take into account that one meets f_3 before f_1 as one moves along s from $(-n, n)$ to b_1. Indeed τ_3 and f_3 lie to the right of the vertical line $x = 3n/4$ and τ_1 and f_1 lie to the left of the vertical line $x = n/4$. This establishes our claim (4.17).

Since the τ_i are top-bottom crossings of $S(n)$ we shall think of τ''_1 and τ''_2 as "vertical crossings" of $\mathcal{K}$. This terminology is merely a crutch for us to form some mental picture of the τ''_i and also to distinguish them from the "horizontal crossings" of $\mathcal{K}$ which we now construct from the ρ_j, $1 \le j \le 4k - 2$. In fact, it follows from the fact that ρ_j is a left-right crossing of $[-n, n] \times [0, n]$, that ρ_j must intersect δ as well as τ''_3. Let c_j be the last intersection of ρ_j with δ and let h_j be the first intersection of ρ_j and τ_3'' after c_j, as one moves from $\{-n\} \times [0, n]$ to $\{n\} \times [0, n]$ along ρ_j. Finally, let d_j be the endpoint of ρ_j on $\{n\} \times [0, n]$ (which is part of the right edge of $S(n)$). As in (4.15), the piece of ρ_j between c_j and d_j is a crosscut of $\mathring{J}_1$. The same argument as used for (4.17) now shows that for each $j \le 4k-2$, the piece ρ_j'' of ρ_j from c_j to h_j is a crosscut of $\mathcal{K}$. The role of the relation (4.9) in the argument will now be taken over by the relation

$$\rho_{4k-1}^+(S(n)) \subset \rho_j^+(S(n)),$$

which follows from (4.4). In fact, $\mathcal{K} \setminus \rho_j''$ consists of two components, which we shall denote by $\mathcal{K}_j^{\pm}$, where $\mathcal{K}_j^+$ ($\mathcal{K}_j^-$) contains a piece of ρ'_{4k-1} (a piece of γ, respectively) in its boundary (see Figure 5). Then the argument which shows that ρ_j'' is a crosscut of $\mathcal{K}$ shows at the same time that

$$\mathcal{K}_{j+1}^+ \subset \mathcal{K}_j^+, \quad 1 \le j \le 4k-2. \tag{4.18}$$

The intuitive picture is now as indicated in Figure 5. We have the component $\mathcal{K}$, which is "close to" the square $[0, n]^2$, and whose boundary consists of an occupied path on $\mathcal{M}$ from u' to a neighbor of g_3 (which we shall call v'), and a vacant path from u'' to g_3; g_3 will be v''. (Note that g_3 is necessarily a central vertex, for otherwise it would have to be vacant, as a vertex on ρ_{4k-1}, as well as occupied, as a vertex of τ_3. We therefore can take $v'' = g_3$ without having to check that g_3 is vacant.) To the left of $x = n/4$ (right of 3n/4) we have an occupied (vacant) vertical crossing τ_1'' (τ_2'', respectively). We also have a sequence of horizontal crossings ρ_j'' with ρ_{j+1}'' "above" ρ_j''.

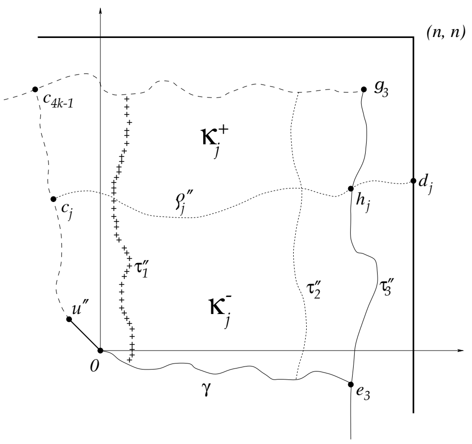

FIGURE 5. The component $\mathcal{K}$ with the crossings τ_1'', τ_2'' and some ρ_j''.

Step 4. We now define further paths σ_ℓ, pieces of which will go into the boundary of the snake. We shall use the notation

$$A^{cl} = \text{ (topological) closure of } A$$

for $A \subset \mathbb{R}^2$.

The definition of σ_ℓ depends on the parity of ℓ. Let us first consider the case when ℓ is odd. Then note that τ_1'' runs between e_1 and g_1. These points lie in different components of $\mathcal{K} \setminus \rho_j''$; $e_1 \in \mathcal{K}_j^-$ and $g_1 \in \mathcal{K}_j^+$. Therefore, as one traverses τ_1'' from e_1 to g_1 there is a last point in the $\left(\mathcal{K}_{2\ell-1}^-\right)^{cl}$. Call this point, which is an intersection of τ_1'' and $\rho_{2\ell-1}$, $y_{2\ell-1}$. After this point τ_1'' lies in $\mathcal{K}_{2\ell-1}^+$, but in some neighborhood of $y_{2\ell-1}$, τ_1'' is still in $\mathcal{K}_{2\ell}^-$. The first point after $y_{2\ell-1}$ where τ_1'' hits $\rho_{2\ell}''$ we shall denote by $z_{2\ell}$. Thus,

$$\text{the piece of } \tau_1'' \text{ from } y_{2\ell-1} \text{ to } z_{2\ell} \text{ (minus its endpoints } y_{2\ell-1} \text{ and } z_{2\ell}) \subset \mathcal{K}_{2\ell}^- \cap \mathcal{K}_{2\ell-1}^+. \tag{4.19}$$

We now define σ_ℓ as the selfavoiding path which consists of the concatenation of the following three paths: the piece of $\rho_{2\ell-1}''$ from $h_{2\ell-1}$ to $y_{2\ell-1}$, the piece of τ_1'' from $y_{2\ell-1}$ to $z_{2\ell}$ and the piece of $\rho_{2\ell}''$ from $z_{2\ell}$ to $h_{2\ell}$. For odd ℓ all these pieces are occupied (by (4.3) and (4.8)) so that

$$\sigma_\ell \text{ is occupied for odd } \ell. \tag{4.20}$$

The endpoints $h_{2\ell-1}$ and $h_{2\ell}$ of σ_ℓ lie on the boundary of $\mathcal{K}$ and σ_ℓ contains points of τ_1'' which lie in $\mathcal{K}$. Thus,

$$\sigma_\ell \text{ is a crosscut of } \mathcal{K}. \tag{4.21}$$

For ℓ even we interchange the roles of left and right. We now define σ_ℓ as a path consisting of pieces of $\rho_{2\ell-1}$ and $\rho_{2\ell}$ which run from $c_{2\ell-1}$ to τ_2'' and from τ_2'' to $c_{2\ell}$ with a connecting piece of τ_2''. We will again have (4.21), but

$$\sigma_\ell \text{ is vacant for even } \ell. \tag{4.22}$$

For odd ℓ we shall denote by $\mathcal{H}_\ell$ that component of $\mathcal{K} \setminus \sigma_\ell$ whose boundary consists of σ_ℓ and the piece of τ_3'' between $h_{2\ell-1}$ and $h_{2\ell}$ (but no pieces of ρ_{4k-1}, δ or γ). Note that this boundary lies in

$$\left(\mathcal{K}_{2\ell-1}^+\right)^{cl} \cap \left(\mathcal{K}_{2\ell}^-\right)^{cl}. \tag{4.23}$$

In fact the parts of the boundary of $\mathcal{H}_\ell$ on τ_1'' and on $\rho_{2\ell}$ together also form a crosscut of $\mathcal{K}_{2\ell-1}^+$ and an alternative description of $\mathcal{H}_\ell$ is therefore that

$$\begin{gathered}\mathcal{H}_\ell \text{ is the component of } \mathcal{K}_{2\ell-1}^+ \setminus \sigma_\ell \text{ whose boundary consists of} \\ \sigma_\ell \text{ and the piece of } \tau_3'' \text{ between } h_{2\ell-1} \text{ and } h_{2\ell}.\end{gathered} \tag{4.24}$$

Since the boundary of $\mathcal{H}_\ell$ lies in the set (4.23) we also have (see Newman (1951), Theorem V.11. 1 and its proof)

$$\mathcal{H}_\ell \subset \mathcal{K}_{2\ell-1}^+ \cap \mathcal{K}_{2\ell}^-. \tag{4.25}$$

For even ℓ we take for $\mathcal{H}_\ell$ that component of $\mathcal{K} \setminus \sigma_\ell$ whose boundary consists of σ_ℓ and the piece of δ between $c_{2\ell-1}$ and $c_{2\ell}$ (but no pieces of ρ_{4k-1}, τ_3'' or γ). Then (4.25) also holds for even ℓ.

Step 5. We shall now define a path ζ, which will in fact be the boundary of our snake. Roughly speaking, ζ is the Jordan curve J_2, which is the boundary of $\mathcal{K}$, except that the pieces between $h_{2\ell-1}$ and $h_{2\ell}$ are replaced by σ_ℓ for ℓ odd, and the pieces between $c_{2\ell-1}$ and $c_{2\ell}$ are replaced by σ_ℓ for ℓ even. Somewhat more formally, to traverse ζ we start at $u' = \mathbf{0}$ and move along γ to e_3. We then move along τ_3'' from e_3 to h_1. From h_1 to h_2 we do not move along τ_3'', but instead follow σ_1. After arriving at h_2 we continue along the piece of τ_3'' between h_2 and h_5. We then move from h_5 to h_6 along σ_2. We continue in this way till we reach h_{4k-2}. From there we go along τ_3'' till its intersection with ρ_{4k-1}'' (which we called g_3). The part of ζ described so far is occupied. We now go back to u'' by a vacant piece of ζ. This part of ζ consists first of ρ_{4k-1}'' from g_3 to c_{4k-1}, the intersection of ρ_{4k-1}'' with δ. We then move along the reverse of δ to c_{4k-4}. Instead of following the piece of δ from c_{4k-4} to c_{4k-5} we follow the reverse of σ_{2k-2} from c_{4k-4} to c_{4k-5}. We then go along the reverse of δ to c_{4k-8} and follow the reverse of σ_{2k-4} etc., until we arrive at c_3. From there we move along the reverse of δ to u''. Finally, to make ζ into a closed curve we add the segment between the adjacent points (on $\mathcal{M}$) u'' and u'.

We claim that ζ is a Jordan curve, located in the closure of $\mathcal{K}$. This follows quickly from the construction of the σ_ℓ. Each σ_ℓ is a self-avoiding path which lies in $\mathcal{K}$ except for its endpoints $h_{2\ell-1}, h_{2\ell}$ or $c_{2\ell-1}, c_{2\ell}$, which lie on the boundary of $\mathcal{K}$, so that indeed $\zeta \subset \big(\mathcal{K}\big)^{cl}$. We therefore only have to prove that the different σ_ℓ are disjoint. But we already saw that σ_ℓ is contained in the set (4.23) and we claim that these regions are disjoint for different ℓ. Indeed

$$\big(\mathcal{K}_p^+\big)^{cl} \cap \big(\mathcal{K}_p^-\big)^{cl} = \rho_p'' \tag{4.26}$$

essentially by definition (see Newman (1951), Theorem V.11.8). Moreover, for $j > p$,

$$\rho_p'' \cap \big(\mathcal{K}_j^+\big)^{cl} = \emptyset. \tag{4.27}$$

To see this note that ρ_p'' cannot contain any point of $\mathcal{K}_j^+$, because in the neighborhood of such a point there would be points outside $\mathcal{K}_p^+$, and hence outside $\mathcal{K}_j^+$ (by (4.18)). Neither can ρ_p'' intersect ρ_j'', by assumption (4.2). Therefore $\rho_p'' \cap \big(\mathcal{K}_j^+\big)^{cl}$ can only consist of points in (boundary of $\mathcal{K}_j^+) \setminus \rho_j''$. This is an arc of J_2, the boundary of $\mathcal{K}$. But if there were a point in $\rho_p'' \cap ($ boundary of $\mathcal{K}_j^+) \setminus \rho_j''$, then again any neighborhood of that point would contain points outside $\mathcal{K}_p^+$ but inside $\mathcal{K}_j^+$. As we just saw this is impossible, so that (4.26) holds. Finally, (4.18), (4.26) and (4.27) together show that

$$\big(\mathcal{K}_j^+\big)^{cl} \cap \big(\mathcal{K}_p^-\big)^{cl} \subset \big(\mathcal{K}_j^+\big)^{cl} \cap \big(\mathcal{K}_p^+\big)^{cl} \cap \big(\mathcal{K}_p^-\big)^{cl} = \emptyset \text{ for } j > p. \tag{4.28}$$

Thus the regions in (4.23) and the σ_ℓ for different ℓ are indeed disjoint .

Step 6. In this step we complete the proof of the lemma. Let $\mathring{\zeta}$ denote the interior of ζ. Since $\zeta \subset \big(\mathcal{K}\big)^{cl}$, it must be the case that $\mathring{\zeta} \subset \mathcal{K}$ (see Newman (1951), proof of Theorem V.11.1), and

$$\big(\mathring{\zeta}\big)^{cl} = \mathring{\zeta} \cup \zeta \subset \big(\mathcal{K}\big)^{cl}.$$

Now let $\phi : [0,1] \to \mathring{\zeta} \cup \zeta$ be a path in $\mathring{\zeta} \cup \zeta$ which starts at $u' = \mathbf{0}$ and ends at or adjacent to g_3. Then ϕ is also a path in the closure of $\mathcal{K}$. It begins at $u' \in (\mathcal{K}_1^-)^{cl} \subset (\mathcal{K}_j^-)^{cl}$ for all j, and ends near g_3 and hence in $(\mathcal{K}_{4k-2}^+)^{cl} \subset (\mathcal{K}_j^+)^{cl} \setminus \rho_j''$ for all $j \le 4k-2$ (note that $g_3 \in \rho_{4k-1}$ and hence $g_3 \notin \rho_j$ for $j \le 4k-2$) . ϕ must therefore intersect each of the ρ_j'' which separate $\mathcal{K}_j^+$ from $\mathcal{K}_j^-$. In fact more is true. If w_p is the last point of ϕ on ρ_p'', then ϕ must still intersect ρ_j'' for all $j > p$ after w_p (because also $w_p \in \rho_p'' \subset (\mathcal{K}_p^-)^{cl} \subset \mathcal{K}_j^-$). For $j = 2q-1$ with q odd, denote the part of ρ_j'' between y_{2q-1} (on τ_1'') and c_{2q-1} (on δ) by λ_{2q-1} (see Figure 6). When q is even

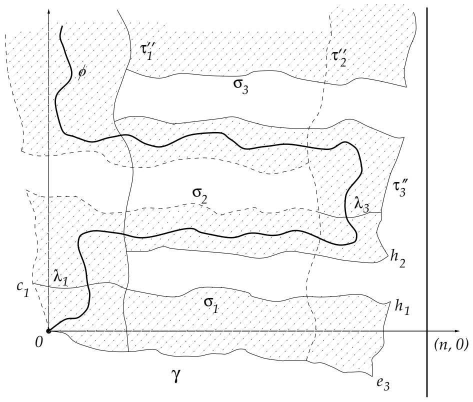

FIGURE 6. A path ϕ (solidly drawn) inside the "snake".

let λ_{2q-1} be the piece of ρ_{2q-1} between its first intersection with τ_2'' (when starting at c_{2q-1}) and h_{2q-1}.

The most important part of our argument is now that

$$\phi \text{ must intersect } \lambda_{2q-1} \text{ after } w_p \text{ for } 2q-1 > p. \tag{4.29}$$

For the sake of argument we prove this for odd q. We first observe that the boundary of each $\mathcal{H}_q$ is disjoint from $\mathring{\zeta}$. Indeed, this boundary consists of $\sigma_q \subset \zeta$, which certainly lies outside $\mathring{\zeta}$, and the piece of τ_3'' strictly between h_{2q-1} and h_{2q}. The latter piece is disjoint from ζ, because this piece was replaced by σ_q in the construction of ζ. But then this piece of τ_3'' either lies entirely in $\mathring{\zeta}$ or entirely in the exterior of ζ. Because points of τ_3'' can be connected to infinity by paths which lie outside the closure of $\mathcal{K}$ (except for their initial point), and hence in the exterior of ζ, the piece of τ_3'' between h_{2q-1} and h_{2q} lies in the exterior of ζ. This proves our claim that the boundary of $\mathcal{H}_q$ is disjoint from $\mathring{\zeta}$.

The fact that the boundary of $\mathcal{H}_q$ is disjoint from $\mathring{\zeta}$ implies that any two points of $\mathring{\zeta}$ can be connected by a path in $\mathring{\zeta}$ which does not intersect the boundary of $\mathcal{H}_q$. Consequently, all of $\mathring{\zeta}$ either lies in $\mathcal{H}_q$ or in the complement of the $(\mathcal{H}_q)^{cl}$. The latter case must occur, because $\mathring{\zeta}$ contains points arbitrarily close to $u' = \mathbf{0}$ which can be connected to infinity outside $\mathcal{H}_q$ (in fact, $\mathbf{0}$ can be connected to infinity outside $\mathcal{K}$). We conclude that $\overset{\circ}{\zeta} \cap \mathcal{H}_q = \emptyset$. This of course implies that not even ζ can intersect the open set $\mathcal{H}_q$, so that

$$(\overset{\circ}{\zeta} \cup \zeta) \cap \mathcal{H}_q = \emptyset. \tag{4.30}$$

Now assume, to derive a contradiction, that (4.29) fails. Since we already know that ϕ must leave $(\mathcal{K}^-_{2q-1})^{cl}$ and enter $(\mathcal{K}^+_{2q-1})^{cl}$, ϕ must cross ρ''_{2q-1} after w_p, that is, there must exist a t_q such that

$$\begin{aligned} \phi(t_q) \in & \rho''_{2q-1} \text{ and } \phi(t) \in \left(\mathcal{K}^+_{2q-1}\right)^{cl} \setminus \rho''_{2q-1} \\ & \text{for some } t > t_q, \text{ arbitrarily close to } t_q. \end{aligned} \tag{4.31}$$

If (4.29) fails, then

$$\phi(t_q) \in \rho''_{2q-1} \setminus \lambda_{2q-1}. \tag{4.32}$$

This says that $\phi(t_q)$ lies in the piece of ρ''_{2q-1} between y_{2q-1} and h_{2q-1}, that is the piece of ρ''_{2q-1} in the boundary of $\mathcal{H}_q$. Also, $\phi(t_q) \neq y_{2q-1} \in \lambda_{2q-1}$. But, as we saw in the lines before (4.25), the parts of σ_q on ρ''_{2q-1} and on τ''_1 form a crosscut of $\mathcal{K}^+_{2q-1}$. $\mathcal{H}_q$ is one of the components of $\mathcal{K}^+_{2q-1}$ after this crosscut is removed from $\mathcal{K}^+_{2q-1}$, and (4.29) puts $\phi(t_q)$ in the boundary of $\mathcal{H}_q$, but not in boundary of the other component of ($\mathcal{K}^+_{2q-1}$ minus the crosscut). Therefore, $\phi(t_q)$ has some neighborhood U so that

$$U \cap \mathcal{K}^+_{2q-1} \subset \mathcal{H}_q. \tag{4.33}$$

But, by virtue of $\phi(t) \in \overset{\circ}{\zeta} \cup \zeta$ and of (4.30), we cannot have $\phi(t) \in \mathcal{H}_q$. Thus for some $t > t_q$, but arbitrarily close to t_q it must be the case that

$$\phi(t) \in \left(\mathcal{K}^+_{2q-1}\right)^{cl} \setminus \left(\rho''_{2q-1} \cup \mathcal{K}^+_{2q-1}\right) \tag{4.34}$$

(see (4.31) and (4.33)). Since $\mathcal{K}^+_{2q-1}$ is a component of $\mathcal{K} \setminus \rho''_{2q-1}$, the right hand side here is contained in the arc of the boundary of $\mathcal{K}$ from h_{2q-1} to c_{2q-1} which contains ρ''_{4k-1}. But as $t \downarrow t_q$, the points $\phi(t)$ here have to approach $\phi(t_q) \in \rho''_{2q-1}$. By our assumption, $\phi(t_q) \notin \lambda_q$, so that in particular, $\phi(t_q) \neq c_{2q-1}$. This forces $\phi(t)$ to take values on the segment of τ''_3 strictly between h_{2q-1} and h_{2q}. This, however, is also impossible, because we already proved in the lines following (4.29) that this arc lies in the exterior of ζ. Thus (4.29) must hold.

It is now easy to complete the proof. (4.29) also holds when $j = 2q-1$ for an even q, by the same proof as for odd q, with only the roles of left and right interchanged. Now (4.29) implies that a path ϕ from u' to $\{v', v''\}$ must successively intersect $\lambda_1, \lambda_3, \dots, \lambda_{2k-1}$. But τ''_1 lies to the left of the vertical line $x = n/4$. Thus as one traverses ρ''_{2q-1} from h_{2q-1} to c_{2q-1} one first hits the vertical line $x = n/4$ before one hits τ''_1, and also the point y_{2q-1} of τ''_1 must lie on or to the left of the line $x = n/4$. Since ρ''_1 has only one point in common with the vertical line $x = n/4$ (by assumption (4.4)), the piece λ_{2q-1} from y_{2q-1} to c_{2q-1} must lie entirely in the half plane $(-\infty, n/4] \times \mathbb{R}$. For similar reasons, for even q, $\lambda_{2q-1} \subset [3n/4, \infty) \times \mathbb{R}$. Thus, the distance between λ_{2r-1} and λ_{2r+1} is at least $n/2$, for each $1 \leq r \leq 2k-2$.

Consequently, if ϕ successively intersects $\lambda_1, \lambda_3, \dots, \lambda_{4k-3}$, then its length is at least $(2k-2)n/2$. $\square$

The remainder of the proof uses fairly standard block techniques from percolation to show that the conditions of Lemma 2 with $k \sim \lfloor C_6 n \rfloor$ are fulfilled with a probability which rapidly approaches 1 as $n \to \infty$. First we introduce our blocks or renormalized sites. The renormalized site (i,j) will depend on the configuration of occupied and vacant sites in the square $[(14i-13)N, (14i+13)N-1] \times [(14j-13)N, (14j+13)N-1]$ for an N to be determined below. The renormalized site $(0,0)$ will be colored *white* if all the paths listed in (4.35)-(4.40) exist on $\mathcal{M}$. The reader is advised to look at Figure 7 for these paths, before reading the formal description.

$$\text{there exists an occupied circuit surrounding the origin in each of the 2 annuli } S(4N) \setminus S(3N), S(5N) \setminus S(4N). \tag{4.35}$$

$$\text{there exists a vacant circuit surrounding the origin in each of the 4 annuli } S(2N) \setminus S(N), S(3N) \setminus S(2N), S(6N) \setminus S(5N) \text{ and } \overset{\circ}{S}(7N) \setminus S(6N); \tag{4.36}$$

$$\text{there exist occupied left-right crossings of each of the 4 rectangles } [-13N, -N+1] \times [-N/3, -1], [-13N, -N+1] \times [0, N/3-1], [N, 13N-1] \times [-N/3, -1] \text{ and } [N, 13N-1] \times [0, N/3-1]; \tag{4.37}$$

$$\text{there exist occupied top-bottom crossings of each of the 4 rectangles } [-N/3, -1] \times [N, 13N-1], [0, N/3-1] \times [N, 13N-1], [-N/3, -1] \times [-13N, -N+1] \text{ and } [0, N/3-1] \times [-13N, -N+1]; \tag{4.38}$$

$$\text{there exist vacant left-right crossings of each of the 8 rectangles } [-13N, -N+1] \times [-N, -2N/3-1], [-13N, -N+1] \times [-2N/3, -N/3-1], [-13N, -N+1] \times [N/3, 2N/3-1], [-13N, -N+1] \times [2N/3, N], [N, 13N-1] \times [-N, -2N/3-1], [N, 13N-1] \times [-2N/3, -N/3-1], [N, 13N-1] \times [N/3, 2N/3-1] \text{ and } [N, 13N-1] \times [2N/3, N]; \tag{4.39}$$

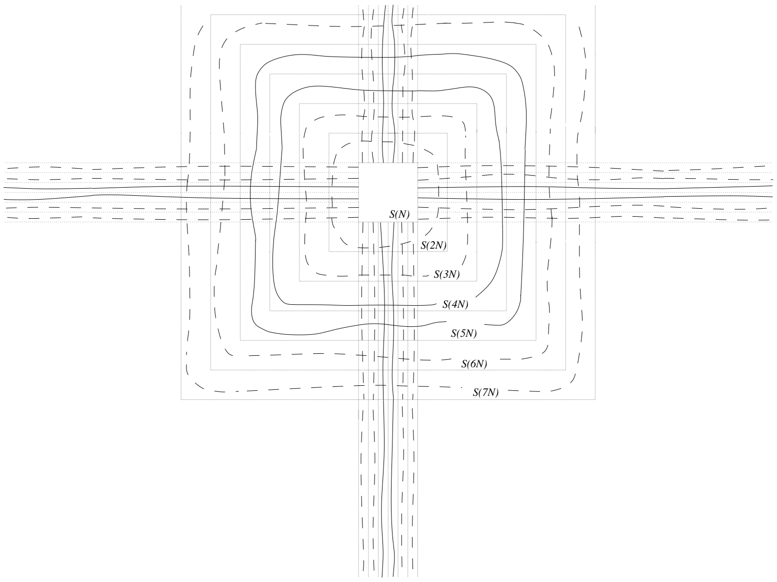

FIGURE 7. A schematic diagram of the paths which are needed for $(0,0)$ to be white. Occupied paths are solidly drawn and vacant paths are dashed. The squares in the figure are (from the inside out) $S(N)$ – $S(7N)$.

there exist vacant top-bottom crossings of each of the 8 rectangles

$$\begin{aligned}&[-N,-2N/3-1]\times[N,13N-1],[-2N/3,-N/3-1]\times[N,13N-1],\\&[N/3,2N/3-1]\times[N,13N-1],[2N/3,N]\times[N,13N-1],\\&[-N,-2N/3-1]\times[-13N,-N+1],[-2N/3,-N/3-1]\times[-13N,-N+1],\\&[N/3,2N/3-1]\times[-13N,-N+1]\text{ and }[2N/3,N]\times[-13N,-N+1];\end{aligned}\tag{4.40}$$

The renormalized site (i,j) is colored white if translates by $(14iN,14jN)$ of the paths in (4.35)-(4.40) exist in $(14iN,14jN)+[-13N,13N-1]^2$. All renormalized sites which are not colored white are colored *black*. Two renormalized sites (i,j) and (i',j') will be adjacent if $|i-i'|+|j-j'|=1$, so that the renormalized sites can be viewed as the sites of $\mathbb{Z}^2$. Thus the colorings can be viewed as a site percolation process on $\mathbb{Z}^2$. Since the color of (i,j) depends on the configuration in $[(14i-13)N,(14i+13)N-1]\times[(14j-13)N,(14j+13)N-1]\subset(14iN,14jN)+S(14N)$ the coloring of the sites is not an not an independent percolation process on $\mathbb{Z}^2$, but a 1-dependent one. That is, the color configurations of two collections of sites A_1 and A_2 are independent if $|(i,j)-(i',j')|_\infty>1$ for all $(i,j)\in A_1,(i',j')\in A_2$.

It is clear that

$$\alpha_N:=P_p\{(i,j)\text{ is white}\}\text{ is independent of }(i,j).\tag{4.41}$$

It is also standard that

$$\lim_{N\to\infty}\alpha_N=1,\tag{4.42}$$

because for our percolation process on the original lattice $\mathbb{Z}^2_{cp}$, under (1.2),

$$\begin{aligned}
&P_p\{\nexists \text{ occupied left-right crossing of } [0,\lfloor xN\rfloor]\times[0,N] \text{ on } \mathbb{Z}^2_{cp}\}\\
&= P_p\{\exists \text{ vacant path on } \mathbb{Z}^2 \text{ from } [0,\lfloor xN\rfloor]\times\{1\} \text{ to } [0,\lfloor xN\rfloor]\times\{N-1\}\\
&\qquad \text{in } [0,\lfloor Nx\rfloor]\times[1,N-1]\}\\
&\le C_7\exp(-C_8N) \qquad\qquad (4.43)
\end{aligned}$$

for some constants $C_i \in (0,\infty)$ which depend on p and x only (see Kesten (1982), Proposition 2.2 (corrected to replace $J\setminus A_1\cup A_3$ by $\overline{J}\setminus A_1\cup A_3$ on p. 30, L. 1, 2 f.b.) and Theorem 5.1; see also Grimmett (1999), p.293 and Theorem 5.4). The same statement is true for vacant crossings and for occupied and vacant top-bottom crossings of $[0,N]\times[0,\lfloor xN\rfloor]$. These exponential bounds for the non-existence of paths on $\mathbb{Z}^2_{cp}$ immediately imply similar bounds on $\mathcal{M}$. An exponential upper bound for the nonexistence of circuits in the annuli $S((i+1)N)\setminus S(iN),\ 1\le i\le 6$, also follows from (4.43) (see Grimmett (1999), Theorem 11.70).

Our definition of the coloring also has the following geometric (deterministic) consequence: Assume that

$$\begin{aligned}
&\text{there is a white path } v_1,v_2,\dots,v_r, \text{ with } v_k=(i_k,j_k)\\
&\text{which forms a left-right crossing of } [a,b]\times[c,d] \text{ on } \mathbb{Z}^2. \qquad (4.44)
\end{aligned}$$

Then

$$\begin{aligned}
&\text{there exist on } \mathcal{M} \text{ two occupied left-right crossings, } \rho_1,\rho_2,\\
&\text{and two vacant left-right crossings, } \rho_3,\rho_4, \text{ of}\\
&[(14a-13)N,(14b+13)N-1]\times[(14c-13)N,(14d+13)N-1]\\
&\text{in the set } \Lambda(v_1,\dots,v_r) := \bigcup_{k\le r}\big(14v_k+[-7N,7N-1]\times[7N,7N-1]\big)\\
&\cup[(14i_1-13)N,(14i_1-7)N-1]\times[(14j_1-7)N,(14j_1+7)N-1]\\
&\cup[(14i_r+7)N,(14i_r+13)N-1]\times[(14j_r-7)N,(14j_r+7)N-1]\\
&\qquad\qquad (4.45)
\end{aligned}$$

(note that $i_1=a, i_r=b$); also

$$\rho_1-\rho_4 \text{ will be disjoint;} \qquad (4.46)$$

in addition these crossings will be such that

$$\rho_1 \text{ and } \rho_2 \text{ lie “below” } \rho_3,\rho_4. \qquad (4.47)$$

If we write $\rho_i^+=\rho_i^+(a,b,c,d)$ for the component of

$$[(14a-13)N,(14b+13)N-1]\times[(14c-13)N,(14d+13)N-1]\setminus\rho_i,$$

then the precise meaning of (4.47) is that

$$\rho_4^+\subset\rho_3^+\subset\rho_2^+\subset\rho_1^+. \qquad (4.48)$$

If $((14a-13)N,\ell_i)$ and $((14b+13)N-1,r_i)$ are the endpoints of ρ_i on the left and right edge of $[(14a-13)N,(14b+13)N-1]\times[(14c-7)N,(14d+7)N-1]$, respectively, then an equivalent way of expressing (4.46) (under (4.34)) is that

$$\ell_1<\ell_2<\ell_3<\ell_4, \tag{4.49}$$

or that

$$r_1<r_2<r_3<r_4. \tag{4.50}$$

We leave a formal proof of this to the reader. We merely illustrate in Figure 8 how two disjoint vacant and two disjoint occupied left-right crossings of $[(14i-7)N,(14i-1)N-1]\times[(14j-1)N,(14j+1)N]$ can be continued through $[(14i-7)N,(14i+7)N-1]\times[(14j-7)N,(14j+7)N-1]$ and end in top-bottom crossings of $[(14i-1)N,(14i+1)N]\times[(14j-13)N,(14j-1)N-1]$, or of $[(14i-1)N,(14i+1)N]\times[(14j+1)N,(14j+13)N-1]$, provided the renormalized site (i,j) is white.

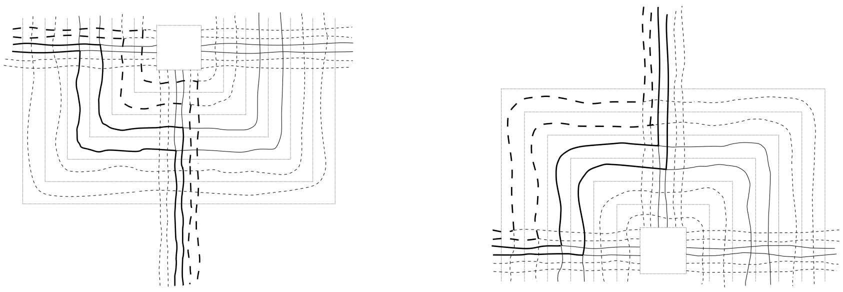

FIGURE 8. Two illustrations of the continuation of two disjoint occupied and two disjoint vacant paths through a white site.

We now show that the properties listed above are enough to give us (1.5). The following lemma is the main step.

Lemma 3. *There exists an N and constants $C_i=C_i(p)\in(0,\infty)$ such that for all large n,*

$$P_p\{\text{(4.1)-(4.5) of Lemma 2 with } n \text{ replaced by } 56Nn \text{ and with } k=C_9n \text{ hold}\}$$
$$\geq 1-C_{10}\exp(-C_{11}n).$$

Proof. Define the following events for the renormalized sites:

$$F_1=\{\text{there exist at least } 3n \text{ disjoint white left-right crossings of } [-4n,n-2]\times[1,4n-1]\};$$

$$F_2=\{\text{there exist at least } 3n \text{ disjoint white left-right crossings of } [n+2,3n-2]\times[1,4n-1]\};$$

and

$$F_3 = \{\text{there exist at least } 3n \text{ disjoint white left-right crossings of } [3n+2, 4n] \times [1, 4n-1]\}.$$

Assume that F_1 occurs. Then there exist integers $a_1 < a_2 < \cdots < a_{3n} \subset [1, 4n-1]$ and $3n$ disjoint white left-right crossings of $[-4n, n-2] \times [1, 4n-1]$ whose endpoints on $\{n-2\} \times [1, 4n-1]$ are $(n-2, a_1), (n-2, a_2), \ldots, (n-2, a_{3n})$. Similarly, if also F_2 occurs, then there are $3n$ disjoint white left-right crossings of $[n+2, 3n-2] \times [1, 4n-1]$ with left and right endpoints $(n+2, b_i)$ and $(3n-2, c_i)$, respectively, with $1 \le b_1 < b_2 < \cdots < b_{3n} \le 4n-1$ and $1 \le c_1 < c_2 < \cdots < c_{3n} \le 4n-1$. Finally, if F_3 occurs, then there exist $3n$ disjoint white left-right crossings of $[3n+2, 4n]$ with left endpoints $(3n+2, d_i)$ with $1 \le d_1 < \cdots < d_{3n} \le 4n-1$.

Now let $F_1 \cap F_2 \cap F_3$ occur and let a_i, b_i, c_i and d_i be as in the prceding paragraph. The number of integers in $[1, 4n-1]$ which are not equal to one of the a_j is at most $n-1$. Thus, at least $3n-(n-1) \ge 2n$ of the b_i are also equal to some a_j. For similar reasons, among $2n$ of the i for which b_i equals some a_j, there are at least n values of i for which c_i equals some d_j. Thus, there exist at least n pairs (b_i, c_i) such that b_i equals some a_j and c_i equals some d_j. By discarding some of the $a_i - d_i$ and renumbering the remaining ones, we therefore can find $b_1 < \cdots < b_n$ and $c_1 < \cdots < c_n$ for which there exist disjoint white left-right crossings of $[-4n, n-2] \times [1, 4n-1]$ with right endpoints $(n-2, b_i)$, disjoint white left-right crossings of $[n+2, 3n-2] \times [1, 4n-1]$ from $(n+2, b_i)$ to $(3n-2, c_i)$ and disjoint white left-right crossings of $[3n+2, 4n] \times [1, 4n-1]$ with left endpoints $(3n+2, c_i)$. By properties (4.45)-(4.47) this means that there exist for each $1 \le i \le n$ on $\mathbb{Z}^2_{cp}$ two occupied left-right crossings $\rho_{i,1}, \rho_{i,2}$ and two vacant left-right crossings $\rho_{i,3}, \rho_{i,4}$ of

$$[(-4n(14)-13)N, ((n-2)(14)+13)N-1] \times [0, 4n(14)N]$$

whose right endpoint lies in $\{((n-2)(14)+13)N-1\} \times [(b_i(14)-7)N, (b_i(14)+7)N-1]$, and all four of them disjoint. Similarly, the white path from $(n+2, b_i)$ to $(3n-2, c_i)$ gives us two occupied left-right crossings $\rho'_{i,1}, \rho'_{i,2}$ and two disjoint left-right crossings $\rho'_{i,3}, \rho'_{i,4}$ of $[((n+2)(14)-13)N, ((3n-2)(14)+13)N-1] \times [0, 4n(14)N]$ with starting point in $\{((n+2)(14)-13)N\} \times [(b_i(14)-7)N-N, (b_i(14)+7)N-1]$ and endpoint in $\{((3n-2)(14)+13)N-1\} \times [(c_i(14)-7)N, (c_i(14)+7)N-1]$. All four of these paths are disjoint. Finally, there exist similar left-right crossings $\rho''_{i,j}$ of $[((3n+2)(14)-13)N, (4n(14)+13)N-1] \times [0, 4n(14)N]$ with starting points in $\{((3n+2)(14)-13)N\} \times [(c_i(14)-7)N, (c_i(14)+7)N-1]$. Moreover, $\rho_{i,1}$ and $\rho_{i,2}$ lie below $\rho_{i,3}$ and $\rho_{i,4}$ and the same statement holds for the ρ' and for the ρ''.

We now want to connect $\rho_{i,j}, \rho'_{i.j}, \rho''_{i,j}$ to form a left-right crossing of the whole rectangle $[(-4n(14)-13)N, (4n(14)+13)N-1] \times [0, 4n(14)N]$, in such a way that the resulting family of crossings (when suitably renumbered) satisfies (4.1)-(4.5). It is clear that there exists a path $\psi_{i,j}$ on $\mathcal{M}$ in

$$[((n-2)(14)+13)N, ((n+2)(14)-13)N] \times [(b_i(14)-7)N, (b_i(14)+7)N-1] \quad (4.51)$$

which connects $\rho_{i,j}$ to $\rho'_{i,j}$. It is even possible to choose $\psi_{i,j}$ such that it intersects the vertical line $\{x = n(14)N\}$ in one point only. By our construction the order of the right endpoints of the $\rho_{i,j}$, $1 \le j \le 4$, is the same as that of the left endpoints of the $\rho'_{i,j}$ (see (4.49) and (4.50)). It is not hard to see that, for sufficiently large N, this allows us to even choose the $\psi_{i,j}$, $1 \le j \le 4$, disjoint. In a similar way we can choose disjoint paths $\psi'_{i,j}$, $1 \le j \le 4$, on $\mathcal{M}$, inside

$$[((3n-2)(14)+13)N, ((3n+2)(14)-13)N-1] \times [(c_i(14)-7)N, (c_i(14)+7)N-1] \tag{4.52}$$

which connect $\rho'_{i,j}$ and $\rho''_{i,j}$ in such a way that $\psi'_{i,j}$ intersects the vertical line $\{x = 3n(14)N\}$ in one point only. We shall denote by by $G(i)$ the event that $\psi_{i,j}$ and $\psi'_{i,j}$ are occupied for $j = 1, 2$ and vacant for $j = 3, 4$. Now note that the occurrence of $F_1 \cap F_2 \cap F_3$ tell us nothing about the occupancy of vertices in

$$\Big([((n-2)(14)+13)N, ((n+2)(14)-13)N-1] \times \mathbb{Z}\Big) \\ \cup \Big([((3n-2)(14)+13)N, ((3n+2)(14)-13)N-1] \times \mathbb{Z}\Big).$$

There exists therefore a $C_{12} > 0$ such that

$$P_p\{G(i) | F_1 \cap F_2 \cap F_3\} \ge C_{12}.$$

If $F_1 \cap F_2 \cap F_3$ and $G(i)$ occur, then the $\rho_{i,j}, \rho'_{i,j}, \rho''_{i,j}$ with their connections $\psi_{i,j}, \psi'_{i,j}$ form four disjoint left-right crossings of $[(-4n(14)-13)N, (4n(14)+13)N-1] \times [0, 4n(14)N]$, the lower two of which are occupied and the top two of which are vacant. Moreover, these crossings have only one point in common with each of the vertical lines $\{x = n(14)N\}$ and $\{x = 3n(14)N\}$.

Next we note that for given distinct b_i the regions (4.51) are disjoint for different i. Similarly, the regions in (4.52) are disjoint. Thus, conditionally on $F_1 \cap F_2 \cap F_3$ and a choice of the b_i, c_i, the events $G(i), 1 \le i \le n$, are independent. Since there are n possible choices for i, standard large deviation estimates for the binomial distribution tell us that

$$P_p\{G(i) \text{ occurs for at least } C_6 n/2 \text{ values of } i | F_1 \cap F_2 \cap F_3\} \\ \ge 1 - C_{13} \exp(-C_{14} n). \tag{4.53}$$

We claim further that the paths which are constructed from the $\rho_{i,j}, \rho'_{i,j}, \rho''_{i,j}, \psi_{i,j}, \psi'_{i,j}$ for different values of i are disjoint. This is so, because for two disjoint paths $v_1, \dots, v_r$ and $w_1, \dots, w_s$ on $\mathbb{Z}^2$, the sets $\Lambda(v_1, \dots, v_r)$ and $\Lambda(w_1, \dots, w_s)$, in which the paths of (4.45) are located, are disjoint. It follows that if $F_1 \cap F_2 \cap F_3$ occurs, and $G(i)$ occurs for at least $C_{12} n/2$ values of i, then we can renumber the resulting left-right crossings of $[(-4n(14)-13)N, (4n(14)+13)N-1] \times [0, 4n(14)N]$ and obtain that (4.1)-(4.5) are satisfied with n replaced by $56Nn$ and k by $C_{12} n/2$. (Note that (4.4) is merely a question of numbering of the crossings, because the components $\rho_i^+(S(n))$ are automatically nested if the ρ_i are disjoint). In view of (4.53) it therefore suffices for this lemma to show that for large enough N

$$P_p\{F_\ell\} \ge 1 - C_{15} \exp(-C_{16} n), \quad \ell = 1, 2, 3. \tag{4.54}$$

Fortunately it is well known how to prove (4.54) (see Grimmett (1999), Lemma 11.22 and the argument following Theorem 2.45, or Kesten, Theorem 11.1). Since we do not have independent percolation for the white sites we add a few words. For the sake of argument take $\ell = 1$. Note that $\mathbb{Z}^2$ and $\mathbb{Z}^2_{cp}$ are a matching pair in the terminology of Kesten (1982). Then, as in the references just mentioned, by Menger's theorem (and Proposition 2.2 in Kesten (1982)), the maximal number of disjoint white left-right crossings of $[-4n, n-2] \times [1, 4n-1]$ on $\mathbb{Z}^2$ equals the minimal number of white sites on any self-avoiding path on $\mathbb{Z}^2_{cp}$ from $[-4n, n-2] \times \{2\}$ to $[-4n, n-2] \times \{4n-2\}$ inside $[-4n, n-2] \times [2, 4n-2]$. Thus, by a simple Peierls argument

$$
\begin{aligned}
&P_p\{F_1 \text{ fails}\} \\
&\le P_p\{\exists \text{ a self-avoiding path on } \mathbb{Z}^2_{cp} \text{ starting on } [-4n, n-2] \times \{2\} \text{ and} \\
&\qquad \text{containing } 4n-3 \text{ sites, but with fewer than } 3n \text{ white sites}\} \\
&\le (5n-1) 8^{4n-4} 2^{4n-3} (1-\alpha_N)^{(n-3)/64}. \qquad (4.55)
\end{aligned}
$$

Here $(5n-1)$ is a bound for the number of starting points of the paths, 8^{4n-4} is a bound on the number of paths of $4n-3$ sites from a given point, 2^{4n-3} a bound on the number of choices for the subset of the vertices which have to be black. Finally, $(1-\alpha_N)^{(n-3)/64}$ is a bound for the probability that a given subset of $n-3$ vertices is black, because any such set contains a further subset of at least $(n-3)/5$ vertices, any two of which have distance of at least 2 between them, and have therefore independent colors. (4.54) for $\ell = 1$ and large N now follows from (4.42). $\square$

The remainder of the proof of (1.5) is now easy. The estimate (4.43) shows that under (1.2)

$$
P_p\{(4.6) - (4.9) \text{ with } n \text{ replaced by } 56Nn \text{ hold}\} \ge 1 - C_{17}\exp(-C_{18}n).
$$

Also the γ of the hypotheses of Lemma 2 exist for all n if there exists an occupied path on $\mathcal{M}$ from $\mathbf{0}$ to infinity in the sector $\{(x, y) : x \ge 0, -x \le y \le 0\}$, and this event has a strictly positive probability (see Grimmett (1999), Theorem 11.55 and its proof). A similar argument applies to the existence of both γ and δ, as required in the hypotheses of Lemma 2. We conclude that there exists a constant $C_{19} = C_{19}(p) > 0$ so that the probability that all hypotheses of Lemma 2 are fulfilled for n replaced by $56Nn$ and k replaced by $C_9 n$ and for all $n \ge n_0$ simultaneously, is at least

$$
C_{19} - \sum_{n \ge n_0} \left[C_{10} e^{-C_{11}n} + C_{17} e^{-C_{18}n}\right].
$$

Fix n_0 so large that the right hand side here is at least $C_{19}/2$. Then by Lemmas 2 and 1, there is a probability of at least $C_{19}/2$ that for all $n \ge n_0$ all words of length $C_5 C_9 (56) N n^2$ are seen in $S(56Nn)$. This proves (1.5).

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