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Erdős-Rényi random graphs + forest fires = self-organized criticality*

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Abstract

We modify the usual Erdős-Rényi random graph evolution by letting connected clusters 'burn down' (i.e. fall apart to disconnected single sites) due to a Poisson flow of lightnings. In a range of the intensity of rate of lightnings the system sticks to a permanent critical state.

Key words: forest fire model, Erdős-Rényi random graph, Smoluchowski coagulation equations, self-organized criticality.

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Contents

1 Introduction

1.1 Context

In conventional models of equilibrium statistical physics, such as Bernoulli percolation, random cluster models, the Ising model or the Heisenberg model there is always a parameter which controls the character of the equilibrium Gibbs measure: in percolation and random cluster-type models this is the density of open sites/edges, in the Ising or Heisenberg models the inverse temperature. Typically the following happens: tuning the control parameter at a particular value (the critical density or the critical inverse temperature) the system exhibits critical behavior in the thermodynamical limit, manifesting e.g. in power law rather than exponential decay of the upper tail of the distribution of the size of connected clusters. Off this particular critical value of the control parameter these distributions decay exponentially. We emphasize here that the critical behavior is observed only at this particular critical value of the control parameter.

As opposed to this, in some dynamically defined models of interacting microscopic units one expects the following robust manifestation of criticality: In some systems dynamics defined naturally in terms of local interactions some effects can propagate instantaneously through macroscopic distances in the system. This behavior may have dramatic effects on the global behavior, driving the system to a permanent critical state. The point is that without tuning finely some parameter of the interaction the dynamics drives the system to criticality. This kind of behavior is called *self-organized criticality (SOC)* in the physics literature. The two best known examples are the sandpile models where so called avalanches spread over macroscopic distances instantaneously, and the forest fire models where beside the Poissonian flow of switching sites/edges from "empty" to "occupied" state (i.e. trees being grown), at some instants connected clusters of occupied sites/edges (forests of trees) are turned from "occupied" to "empty" state instantaneously (i.e. forests hit by lightnings are burnt down on a much faster time scale than the growth of trees). These models and these phenomena prove to be difficult to analyze mathematically rigorously due to the following two facts: (1) There are always two competing components of the dynamics (in the forest fire models: growing trees and burning down forests) causing lack of any kind of monotonicity of the models. (2) Long range effects due to instantaneous propagation of short range interactions are very difficult to be controlled.

Regarding forest fire models there are very few mathematically rigorous results describing SOC. The best known and most studied model of forest fires is the so-called Drossel-Schwabl model. For the original formulation see [11], or the more recent survey [16]. We formulate here a related variant.

Let $\Lambda_n := \mathbb{Z}^d \cap [-n, n]^d$. The state space of the model of size n is $\Omega_n := \{0, 1\}^{\Lambda_n}$: sites of Λ_n can be occupied by a tree (1) or empty (0). The dynamics consists of two competing mechanisms:
(A) Empty (0) sites turn occupied (1) with rate one, independently of whatever else happens in the system.
(B) Sites get hit by "lightnings" with rate $\lambda(n)$, independently of whatever else happens in the system. When site is hit by lightning its whole connected cluster of occupied sites turns instantaneously from "occupied" (1) to "empty" (0) state. (That is: when a tree is hit by lightning the whole forest to which it belongs burns down instantaneously.)
The dynamics goes on indefinitely.

As long as n is kept fixed the mechanism $A+B$ defines a decent finite state Markov process – though a rather complicated one. The main question is: what happens in the thermodynamic limit, when $n \to \infty$, $\Lambda_n \nearrow \mathbb{Z}^d$? Can one specify a dynamics on the state space $\Omega_\infty := \{0, 1\}^{\mathbb{Z}^d}$ which could be identified with the infinite volume limit of the systems defined above?

In order to make some guesses, one has first to specify the lightning rate $\lambda(n)$. Intuitively one expects four regimes of the rate $\lambda(n)$ with essentially different asymptotic behavior of the system in the limit of infinite volume:

I. If $\lambda(n) \ll |\Lambda_n|^{-1}$ then the effect of lightning is simply not felt in the thermodynamic limit: in macroscopic time intervals of any fixed length no lightning will hit the entire system. Thus, in this regime the system will simply be the dynamical formulation of Bernoulli percolation.

II. If $\lambda(n) = |\Lambda_n|^{-1}\lambda$ with some fixed $\lambda \in (0, \infty)$ then one expects in the thermodynamic limit the following dynamics (described in plain, non-technical terms). The system evolves as dynamical site percolation, with independent Poisson evolutions on sites, and with rate $\lambda\theta(t)$, where $\theta(t)$ is the density of the (unique) infinite cluster, the sites of this (unique) infinite cluster are turned from occupied to empty. After this forest fire the system keeps on evolving like dynamical percolation until a new infinite component is born, and the dynamics goes on indefinitely.

III. If $|\Lambda_n|^{-1} \ll \lambda(n) \ll 1$ then in the infinite volume limit - if it makes any sense - something really interesting must happen: The lightning rate is too small to hit finite clusters within any finite horizon. But it is too large to let the infinite percolating cluster to be born. One can expect (somewhat naively) that in this regime in the thermodynamic limit a dynamics will be defined on Ω_∞ in which *in plain words* the following happens:
- empty (0) sites turn occupied (1) with rate one, independently of whatever else happens in the system;
- when the *incipient infinite percolating cluster* is about to be born, it is switched from "occupied" (1) to "empty" (0) state;
- the dynamics goes on indefinitely.
In this way this presumed infinitely extended dynamics would stick to a permanent critical state when the infinite incipient critical cluster is always about to be born, but not let to grow beyond criticality.

IV. If $\lambda(n) = \lambda \in (0, \infty)$ then lightning will hit regularly even small clusters and thus, one may expect that - if the infinitely extended dynamics is well defined - the system will stay subcritical indefinitely.

There is no problem with the mathematically rigorous definition of the infinitely extended dynamics in regimes I. and II. But these plain descriptions don't necessarily make mathematical sense and it is not at all clear that such infinitely extended critical forest fire models can at all be defined in a mathematically satisfactory way.

In our understanding, the most interesting open questions are the existence and characterization of the infinitely extended dynamics in regime III. and/or the $\lambda \to \infty$ limit in regime II. and/or the $\lambda \to 0$ limit in regime IV., after the thermodynamic limit.

There are however some deep results regarding these (or some other related) models of forest fires, though clarification of the above questions seems to be far out of reach at present.

Here follows a (necessarily incomplete) list of some important results related to these questions:

– M. Dürre proves existence of infinitely extended forest fire dynamics in a related model in the subcritical regime IV. , [12]. In a companion paper he also proves that under some regularity conditions assumed the dynamics is uniquely defined, [13].

– J. van den Berg and R. Brouwer, respectively R. Brouwer consider the so called *self-destructive percolation* model, which is very closely related to what we called regime II. above. They prove various deep technical results and formulate some intriguing conjectures related to the $\lambda \to \infty$ limit in regime II. (of the already infinitely extended dynamics), see [2], [3], [8]

– J. van den Berg and A. Járai analyze the $\lambda \to 0$ asymptotics of the (infinitely extended) model in regime IV. in dimension 1, [4].

– J. van den Berg and B. Tóth consider an *inhomogeneous* one dimensional model which indeed exhibits SOC, see [5]. (In one dimensional space-homogeneous models of course there is no critical behavior)

1.2 The model

We investigate a modification of the dynamical formulation of the Erdős-Rényi random graph model, adding "forest fires" caused by "lightning" to the conventional Erdős-Rényi coagulation mechanism. Actually our model will be a particular coagulation-fragmentation dynamics exhibiting robust self-organized criticality.

Let $\mathscr{S}_n := \{1, 2, \ldots, n\}$ and $\mathscr{B}_n := \{(i,j) = (j,i) : i, j \in \mathscr{S}_n,\ i \neq j\}$ be the set of vertices, respectively, unoriented edges of the complete graph $\mathscr{K}_n$. We define a dynamical random graph model as follows. The state space of our Markov process is $\{0,1\}^{\mathscr{B}_n}$.

Edges (i,j) of $\mathscr{K}_n$ will be called occupied or empty according whether $\omega(i,j) = 1$ or $\omega(i,j) = 0$. As usual, we call clusters the maximal subsets connected by occupied edges.

Assume that initially, at time $t = 0$, all edges are empty. The dynamics consists of the following

(A) Empty edges turn occupied with rate $1/n$, independently of whatever else happens in the system.

(B) Sites of $\mathscr{K}_n$ get hit by lightnings with rate $\lambda(n)$, independently of whatever else happens in the system. When a site is hit by lightning, all edges which belong to its connected occupied cluster turn instantaneously empty.

In this way a random graph dynamics is defined. The coagulation mechanism (A) alone defines the well understood Erdős-Rényi random graph model. For basic facts and refined details of the Erdős-Rényi random graph problem see [14], [6], [15]. As we shall see soon, adding the fragmentation mechanism (B) may cause essential changes in the behavior of the system.

We are interested of course in the asymptotic behavior of the system when $n \to \infty$. In order to formulate our problem first have to introduce the proper spaces on which our processes are defined.

We denote

$$\mathscr{V} := \{\mathbf{v} = (v_k)_{k\in\mathbb{N}} : v_k \geq 0, \quad \sum_{k\in\mathbb{N}} v_k \leq 1\}, \quad \theta(\mathbf{v}) := 1 - \sum_{k\in\mathbb{N}} v_k, \tag{1}$$

$$\mathscr{V}_1 := \{\mathbf{v} \in \mathscr{V} : \theta(\mathbf{v}) = 0\}. \tag{2}$$

We endow $\mathscr{V}$ with the (weak) topology of component-wise convergence. We may interpret θ as the density of the giant component.

A map $[0,\infty) \ni t \mapsto \mathbf{v}(t) \in \mathscr{V}$ which is component-wise of bounded variation on compact intervals of time and continuous from the left in $[0,\infty)$, will be called *a forest fire evolution (FFE)*. If $\mathbf{v}(t) \in \mathscr{V}_1$ for all $t \in [0,\infty)$ we call the FFE *conservative*. Denote the space of FFE-s and conservative FFE-s by $\mathscr{E}$, respectively, $\mathscr{E}_1$. The space $\mathscr{E}$ is endowed with the topology of component-wise weak convergence of the signed measures corresponding to the functions $v_k(\cdot)$ on compact intervals of time. This topology is metrizable and the space $\mathscr{E}$ endowed with this topology is complete and separable.

Now, we define the *cluster size distribution* in our random graph process as follows

$$v_{n,k}(t) := n^{-1}\#\{j \in \mathscr{S}_n : j \text{ belongs to a cluster of size } k \text{ at time } t\} =: n^{-1} V_{n,k}(t), \tag{3}$$

$$\mathbf{v}_n(t) := (v_{n,k}(t))_{k\in\mathbb{N}}. \tag{4}$$

This means that $\mathbf{v}_n(t)$ is the cluster size distribution of a uniformly selected site from $\mathscr{S}_n$, at time t. Clearly, the random trajectory $t \mapsto \mathbf{v}_n(t)$ is a (conservative) FFE. We consider the left-continuous version of $t \mapsto \mathbf{v}_n(t)$ instead of the traditional c.à.d.l.à.g., for technical reasons discussed in Subsection 2.1.

We investigate the asymptotics of this process, as $n \to \infty$.

It is well known (see e.g. [9], [10], [1]) that in the Erdős-Rényi case – that is: if $\lambda(n) = 0$

$$\mathbf{v}_n(\cdot) \xrightarrow{\mathbf{P}} \mathbf{v}(\cdot) = \big(v_k(\cdot)\big)_{k\in\mathbb{N}} \quad \text{as} \quad n \to \infty, \tag{5}$$

where the deterministic functions $t \mapsto v_k(t)$ are solutions of the infinite system of ODE-s

$$\dot{v}_k(t) = \frac{k}{2}\sum_{l=1}^{k-1} v_l(t)v_{k-l}(t) - kv_k(t), \qquad k \geq 1, \tag{6}$$

with initial conditions

$$v_k(0) = \delta_{k,1}. \tag{7}$$

The infinite system of ODE-s (6) are the *Smoluchowski coagulation equations*, the initial conditions (7) are usually called *monodisperse*. The system (6) is actually not very scary: it can be solved one-by-one for $k = 1, 2, \dots$ in turn. For the initial conditions (7) the solution is known explicitly:

$$v_k(t) = \frac{k^{k-1}}{k!} e^{-kt} t^{k-1}.$$

$\big(v_k(t)\big)_{k=1}^{\infty} \in \mathscr{V}$ is a (possibly defected) probability distribution called the Borel distribution: in a Galton-Watson branching process with offspring distribution $POI(t)$ the resulting random tree has k vertices with probability $v_k(t)$. Thus the branching process is subcritical, critical and supercritical for $t < 1$, $t = 1$ and $t > 1$, respectively.

For general initial conditions $v_k(0)$ satisfying

$$\sum_{k=1}^{\infty} v_k(0) = 1, \qquad \sum_{k=1}^{\infty} k^2 v_k(0) < \infty,$$

the qualitative behavior of the solution of (6) is similar: Define the *gelation time*

$$T_{\text{gel}} := \Big(\sum_{k=1}^{\infty} k v_k(0)\Big)^{-1} \tag{8}$$

- For $0 \leq t < T_{\text{gel}}$ the system is subcritical: $\theta(\mathbf{v}(t)) = 0$ and, $k \mapsto v_k(t)$ decay exponentially with k.
- For $T_{\text{gel}} < t < \infty$ the system is supercritical: $\theta(\mathbf{v}(t)) > 0$ and $k \mapsto v_k(t)$ decay exponentially with k. Further on: $t \mapsto \theta(\mathbf{v}(t))$ is smooth and strictly increasing with $\lim_{t\to\infty} \theta(\mathbf{v}(t)) = 1$.
- Finally, at $t = T_{\text{gel}}$ the system is critical: $\theta(\mathbf{v}(T_{\text{gel}})) = 0$ and

$$\sum_{l=k}^{\infty} v_l(T_{\text{gel}}) \asymp k^{-1/2} \quad \text{as} \quad k \to \infty. \tag{9}$$

Our aim is to understand in similar terms the asymptotic behavior of the system when, beside the Erdős-Rényi coagulation mechanism, the fragmentation due to forest fires also take place.

Similarly to the Drossel-Schwabl case presented in subsection 1.1 we have four regimes of the lightning rate $\lambda(n)$, in which the asymptotic behavior is different:

$$\begin{array}{lll} \text{I.:} & \lambda(n) \ll n^{-1}, & \\ \text{II.:} & \lambda(n) = n^{-1}\lambda, & \lambda \in (0,\infty), \\ \text{III.:} & n^{-1} \ll \lambda(n) \ll 1, & \\ \text{IV.:} & \lambda(n) = \lambda \in (0,\infty). & \end{array}$$

The $n \to \infty$ asymptotics of the processes $t \mapsto \mathbf{v}_n(t)$ in the four regimes is summarized as follows:

I. The effect of lightnings is simply not felt in the $n \to \infty$ limit. In this regime the system will be the dynamical formulation of the Erdős-Rényi random graph model, the asymptotic description presented in the previous paragraph is valid.

II. In the $n \to \infty$ limit the sequence of processes $t \mapsto \mathbf{v}_n(t)$ converges weakly (in distribution) in the topology of the space $\mathscr{E}$ to a process $t \mapsto \mathbf{v}(t)$ described as follows: The process $t \mapsto \mathbf{v}(t)$ evolves deterministically, driven by the Smoluchovski equations (6) (exactly as in the limit of the dynamical Erdős-Rényi model) with the following Markovian random jumps added to the dynamics:

$$\mathbf{P}\big(\,\mathbf{v}(t+dt) = J\mathbf{v}\,\big|\,\mathbf{v}(t) = \mathbf{v}\,\big) = \lambda\theta(\mathbf{v})dt + o(dt) \tag{10}$$

$$\text{where} \quad J : \mathscr{V} \to \mathscr{V}, \quad (J\mathbf{v})_k = v_k + \delta_{k,1}\theta(\mathbf{v}). \tag{11}$$

In plain words: with rate $\lambda\theta(\mathbf{v}(t))$ the amount of mass $\theta(\mathbf{v}(t))$ contained in the gel (i.e. the unique giant component) is instantaneously pushed into the singletons.

III. This is the most interesting regime and *technically the content of the present paper*. In the $n \to \infty$ limit (5) holds, where now the deterministic functions $t \mapsto v_k(t)$ are solutions of the infinite system of *constrained ODE-s*

$$\dot{v}_k(t) = \frac{k}{2}\sum_{l=1}^{k-1} v_l(t)v_{k-l}(t) - kv_k(t), \qquad k \geq 2, \tag{12}$$

$$\sum_{k\in\mathbb{N}} v_k(t) = 1, \tag{13}$$

with the initial conditions (7). Mind the difference between the system (6) at one hand and the constrained system (12)+(13) at the other: the first equation from (6) is replaced by the global constraint (13). A first consequence is that it is no more true that the ODE-s in (12) can be solved for $k = 1, 2, \ldots,$ one-by-one, in turn. The system of ODE-s is *genuinely infinite*. Up to T_{gel} the solutions of (6), respectively, of (12)+(13) coincide, of course. But dramatic differences arise beyond this critical time. We prove that the system (12)+(13) admits a *unique solution* and for $t \geq T_{\text{gel}}$

$$\sum_{l=k}^{\infty} v_l(t) \sim \sqrt{\frac{2\varphi(t)}{\pi}} k^{-1/2}, \quad \text{as} \quad k \to \infty, \tag{14}$$

where $[T_{\text{gel}}, \infty) \ni t \mapsto \varphi(t)$ is strictly positive, bounded and Lipschitz continuous. This shows that in this regime the random graph dynamics exhibits indeed *self-organized critical behavior*: beyond the critical time T_{gel} it stays critical for ever. The unique stationary solution of the system (12)+(13) is easily found

$$v_k(\infty) = 2\binom{2n-2}{n-1}\frac{1}{n}4^{-n} \approx \frac{1}{\sqrt{4\pi}} k^{-3/2}. \tag{15}$$

IV. In the $n \to \infty$ limit (5) holds again, where now the deterministic functions $t \mapsto v_k(t)$ are solutions of the infinite system of ODE-s

$$\dot{v}_k(t) = \frac{k}{2}\sum_{l=1}^{k-1} v_l(t)v_{k-l}(t) - kv_k(t) - \lambda k v_k(t) + \lambda \delta_{k,1}\sum_{l=1}^{\infty} l v_l(t), \qquad k \geq 1, \tag{16}$$

with the initial conditions in $\mathcal{V}_1$. The system (16) is again a genuine infinite system (it can't be solved one-by-one for $k = 1, 2, \ldots$ in turn). The Cauchy problem (16) with initial condition in $\mathcal{V}_1$ has a unique solution, which stays *subcritical*, i.e. for any $t \in (0, \infty)$ $k \mapsto v_k(t)$ decays exponentially. The unique stationary solution is closely related to that of (15):

$$v_{\lambda,k}(\infty) = (\lambda + 1)\left(1 - \frac{\lambda^2}{(1+\lambda)^2}\right)^k v_k(\infty)$$

1.3 The main results

We present the results formulated and proved only for the regime III: $n^{-1} \ll \lambda(n) \ll 1$, which shows *self-organized critical* asymptotic behaviour. The methods developed along the proofs are sufficient to prove the asymptotic behaviour in the other regimes, described in items I, II and IV but we omit these (in our opinion less interesting) details.

Theorem 1. *If the initial condition* $\mathbf{v}(0) \in \mathcal{V}_1$ *is such that* $\sum_{k=1}^{\infty} k^3 v_k(0) < +\infty$*, and* T_{gel} *is defined by* (8) *then the critical forest fire equations* (12)+(13) *have a unique solution with the following properties:*

1. *For* $t \leq T_{gel}$ *the solution coincides with that of* (6).
2. *For* $t \geq T_{gel}$ *there exists a positive, locally Lipschitz-continuous function* φ *such that*

$$\dot{v}_1(t) = -v_1(t) + \varphi(t) \tag{17}$$

and (14) *holds.*

Theorem 2. *Let $\mathbb{P}_n$ denote the law of the random FFE of the forest fire Markov chain $\mathbf{v}_n(t)$ with initial condition $\mathbf{v}_n(0)$ and lightning rate parameter $n^{-1} \ll \lambda(n) \ll 1$. If $\mathbf{v}_n(0) \to \mathbf{v}(0) \in \mathscr{V}_1$ component-wise where $\sum_{k=1}^{\infty} k^3 v_k(t) < +\infty$ then the sequence of probability measures $\mathbb{P}_n$ converges weakly to the Dirac measure concentrated on the unique solution of the critical forest fire equations* (12)+(13) *with initial condition* $\mathbf{v}(0)$*. In particular*

$$\forall \varepsilon > 0,\, t \geq 0 \quad \lim_{n\to\infty} \mathbf{P}\left(\left| v_{n,k}(t) - v_k(t) \right| \geq \varepsilon \right) = 0$$

2 Coagulation and fragmentation

2.1 Forest fire flows

In this section we investigate the underlying structure of forest fire evolutions arising from the coagulation-fragmentation dynamics of our model on n vertices.

We define auxiliary objects called forest fire flows: let $q_{n,k,l}(t)$ denote n^{-1} times the number of (k,l)-coagulation events (a component of size k merges with a component of size l) up to time t. Let $r_{n,k}(t)$ denote $n^{-1} \cdot k$ times the number of k-burning events (a component of size k burns) up to time t. For the precise definitions see (27), (28), (30) and (31).

In Subsection 2.1 and Subsection 2.2 we precisely formulate and prove lemmas based on the following heuristic ideas:

- The state $\mathbf{v}_n(t)$ of the forest fire process on n vertices (see (4)) can be recovered if we know the initial state $\mathbf{v}_n(0)$, and the flow: $q_{n,k,l}(t)$ for all $k, l \in \mathbb{N}$ and $r_{n,k}(t)$ for all k. The precise formula is (19).
- (19) is similar to the equations (16). This will help us proving Theorem 2: if $1 \ll n$ and $n^{-1} \ll \lambda(n) \ll 1$ then the random forest fire evolution $\mathbf{v}_n(t)$ "almost" satisfies the equations (12)+(13) that uniquely determine the deterministic limiting object $\mathbf{v}(t)$. We essentially prove that (12) is satisfied in the $n \to \infty$ limit in Proposition 1 of Subsection 2.2. We prove that (13) is satisfied in the limit in Subsection 3.3.

We define the moments of $\mathbf{v} \in \mathscr{V}$ as

$$m_0 = \sum_{k=1}^{\infty} v_k, \quad m_1 = \sum_{k=1}^{\infty} k \cdot v_k, \quad m_2 = \sum_{k=1}^{\infty} k^2 \cdot v_k, \quad m_3 = \sum_{k=1}^{\infty} k^3 \cdot v_k$$

By (1) and (2) $m_0 = 1$ if and only if $\mathbf{v} \in \mathscr{V}_1$.

Fix $T \in (0, \infty)$. A map $[0, T] \ni t \mapsto \mathbf{v}(t) \in \mathscr{V}$ is a *a forest fire evolution (FFE)* on $[0, T]$ if $v_k(\cdot)$, $k \in \mathbb{N}$ is of bounded variation and continuous from the left in $(0, T]$. Denote the space of FFE-s on $[0, T]$ by $\mathscr{E}[0, T]$ and the space of FFE-s with initial condition $\mathbf{v}(0) = \mathbf{v} \in \mathscr{V}$ on $[0, T]$ by $\mathscr{E}_{\mathbf{v}}[0, T]$. Note that a priori $\theta(\cdot) = 1 - \sum_{k\in\mathbb{N}} v_k(\cdot)$ need not be of bounded variation.

If $\mathbf{v}_n(\cdot) \in \mathscr{E}[0, T]$ is a sequence of FFE-s then we say that $\mathbf{v}_n(\cdot) \to \mathbf{v}(\cdot)$ if $v_{n,k}(\cdot) \Rightarrow v_k(\cdot)$ for all $k \in \mathbb{N}$ where "$\Rightarrow$" denotes weak convergence of the finite signed measures on $[0, T]$ corresponding to the functions $v_{n,k}(\cdot)$ and $v_k(\cdot)$. Note that we did not require the convergence of $\theta_n(\cdot)$ to $\theta(\cdot)$.

This topology is metrizable and the spaces $\mathscr{E}[0,T]$ and $\mathscr{E}_{\mathbf{v}}[0,T]$ endowed with this topology are separable and complete (by Fatou's lemma, $\lim_{n\to\infty}\mathbf{v}_n(t)$ stays in $\mathscr{V}$).

Denote $\mathbb{N} := \{1,2,\dots\}$ and $\bar{\mathbb{N}} := \mathbb{N}\cup\{\infty\}$.

A *forest fire flow (FFF)* is a collection of maps $[0,T]\ni t\mapsto \big(\mathbf{q}(t),\mathbf{r}(t)\big)$ where for $0\le s\le t\le T$

$$0=q_{k,l}(0)\le q_{k,l}(s)\le q_{k,l}(t), \qquad \mathbf{q}(t)=\big(q_{k,l}(t)\big)_{k,l\in\bar{\mathbb{N}}}, \qquad q_{k,l}(t)=q_{l,k}(t),$$

$$0=r_k(0)\le r_k(s)\le r_k(t), \qquad \mathbf{r}(t)=(r_k(t))_{k\in\bar{\mathbb{N}}}, \qquad r_1(t)\equiv 0$$

We define

$$q_k(t):=\sum_{l\in\bar{\mathbb{N}}} q_{k,l}(t), \qquad q(t):=\sum_{k\in\bar{\mathbb{N}}} q_k(t), \qquad r(t):=\sum_{k\in\bar{\mathbb{N}}} r_k(t) \tag{18}$$

and assume the *finiteness conditions* $q(T)<+\infty$, $r(T)<+\infty$. All functions involved are continuous from the left in $(0,T]$. This is why we have chosen to consider the left-continuous versions of these functions rather than the traditional c.à.d.l.à.g.: the supremum of increasing left-continuous functions is itself left-continuous, thus the left-continuity of q_k, q and r automatically follows from the left-continuity of $q_{k,l}$ and r_k.

We say that the FFF $[0,T]\ni t\mapsto\big(\mathbf{q}(t),\mathbf{r}(t)\big)$ is *consistent with the initial condition* $\mathbf{v}(0)=\mathbf{v}\in\mathscr{V}$ if $t\mapsto\mathbf{v}(t)$ defined by

$$v_k(t)=v_k(0)+\frac{k}{2}\sum_{l=1}^{k-1} q_{l,k-l}(t)-kq_k(t)-r_k(t)+\mathbb{1}_{\{k=1\}}r(t), \qquad k\in\mathbb{N}. \tag{19}$$

is in $\mathscr{E}_{\mathbf{v}}[0,T]$. That is: for all $t\in[0,T]$ and $k\in\mathbb{N}$ $v_k(t)\ge 0$ and $\sum_{k\in\mathbb{N}} v_k(t)\le 1$ holds. In this case we say that the FFF $\big(\mathbf{q}(\cdot),\mathbf{r}(\cdot)\big)$ *generates* the FFE $\mathbf{v}(\cdot)$.

We denote by $\mathscr{F}_{\mathbf{v}}[0,T]$ the space of FFF-s consistent with the initial condition $\mathbf{v}(0)=\mathbf{v}\in\mathscr{V}$. For any $\mathbf{v}\in\mathscr{V}$, $\mathscr{F}_{\mathbf{v}}[0,T]\neq\emptyset$, since the zero flow is consistent with any initial condition.

At this point we mention that later we are going to obtain a FFF $\big(\mathbf{q}_n(\cdot),\mathbf{r}_n(\cdot)\big)$ from a realization of our model on n vertices by (27), (28), (30) and (31). There is a FFF corresponding to the limit object as well: for the solution of the critical forest fire equations (12)+(13) (the uniqueness of the solution is stated in Theorem 1) we define $\big(\mathbf{q}(\cdot),\mathbf{r}(\cdot)\big)$ by

$$\dot{q}_{k,l}(t)=v_k(t)v_l(t), \quad q_{\infty,k}(t)\equiv q_{\infty,\infty}(t)\equiv 0, \quad r_k(t)\equiv 0, \quad \dot{r}_\infty(t)=\varphi(t) \tag{20}$$

with the $\varphi(t)$ of (17). In Definition 1 we define a topology on the space of FFFs. In later sections we are going to prove that

$$\big(\mathbf{q}_n(\cdot),\mathbf{r}_n(\cdot)\big) \xrightarrow{\mathbf{P}} \big(\mathbf{q}(\cdot),\mathbf{r}(\cdot)\big)$$

from which Theorem 2 will follow.

Summing (19) for $k\in\mathbb{N}$ we obtain a formula for the evolution of $\theta(\cdot)$ defined in (1): for $s\le t$

$$\theta(t)=\theta(s)+\lim_{K\to\infty}\sum_{k=1}^{K}\sum_{l=K-k+1}^{\infty} k\cdot\big(q_{k,l}(t)-q_{k,l}(s)\big)+ \\ \sum_{k=1}^{\infty} k\cdot\big(q_{k,\infty}(t)-q_{k,\infty}(s)\big)-\big(r_\infty(t)-r_\infty(s)\big) \tag{21}$$

Later we will see that the term $\lim_{K\to\infty}\sum_{k=1}^{K}\sum_{l=K-k+1}^{\infty} k\cdot\left(q_{k,l}(t)-q_{k,l}(s)\right)$ does not vanish for the FFF defined by (20) for the unique solution $\mathbf{v}(t)$ of (12)+(13) if $T_{\mathrm{gel}} \le s < t$: this phenomenon is a sign of self-organized criticality.

If $\left(\mathbf{q}(\cdot),\mathbf{r}(\cdot)\right)$ is a FFF then the functions $q_{k,l}$, q_k, q, r_k and r (where $k,l \in \bar{\mathbb{N}}$) are continuous from the left and increasing with initial condition 0: such functions are the distribution functions of nonnegative measures on $[0,T]$. By $q(T) < +\infty$ and $r(T) < +\infty$ these measures are finite. We denote by "$\Rightarrow$" the weak convergence of measures on $[0,T]$, which can alternatively be defined by point-wise convergence of the distribution functions at the continuity points of the limiting function.

Definition 1. *Let* $\left(\mathbf{q}_n(\cdot),\mathbf{r}_n(\cdot)\right) = \left(\left(q_{n,k,l}(\cdot)\right)_{k,l\in\bar{\mathbb{N}}}, \left(r_{n,k}(\cdot)\right)_{k\in\bar{\mathbb{N}}}\right)$, $n=1,2,\dots$ *be a sequence of FFFs. Define* $q_{n,k}(\cdot)$, $q_n(\cdot)$ *and* $r_n(\cdot)$ *for all* n *by* (18).

We say that $\left(\mathbf{q}_n(\cdot),\mathbf{r}_n(\cdot)\right) \to \left(\mathbf{q}(\cdot),\mathbf{r}(\cdot)\right)$ *as* $n\to\infty$ *if*

$$\begin{aligned} \forall\, k,l \in \mathbb{N} \quad & q_{n,k,l}(\cdot) \Rightarrow q_{k,l}(\cdot) \\ \forall\, k \in \mathbb{N} \quad & q_{n,k}(\cdot) \Rightarrow q_k(\cdot) \\ & q_n(\cdot) \Rightarrow q(\cdot) \\ \forall\, k \in \mathbb{N} \quad & r_{n,k}(\cdot) \Rightarrow r_k(\cdot) \\ & r_n(\cdot) \Rightarrow r(\cdot) \end{aligned}$$

Note that we do not require $r_{n,\infty}(\cdot) \Rightarrow r_\infty(\cdot)$ and $q_{n,k,\infty}(\cdot) \Rightarrow q_{k,\infty}$ for $k \in \bar{\mathbb{N}}$. Nevertheless these "missing" ingredients of the limit flow $\left(\mathbf{q}(\cdot),\mathbf{r}(\cdot)\right)$ of convergent flows are uniquely determined by the convergent ones if we rearrange the relations (18):

$$q_{k,\infty}(t) := q_k(t) - \sum_{l\in\mathbb{N}} q_{k,l}(t), \tag{22}$$

$$r_\infty(t) := r(t) - \sum_{k\in\mathbb{N}} r_k(t), \tag{23}$$

$$q_{\infty,\infty}(t) := q(t) - 2\sum_{k\in\mathbb{N}} q_k(t) + \sum_{k,l\in\mathbb{N}} q_{k,l}(t). \tag{24}$$

In fact, $r_{n,\infty}(\cdot) \not\Rightarrow r_\infty(\cdot)$ and $q_{n,k,\infty}(\cdot) \not\Rightarrow q_{k,\infty}$ have a physical meaning in the forest fire model if $\left(\mathbf{q}_n(\cdot),\mathbf{r}_n(\cdot)\right)$ is defined by (30) and (31):

- In the $\lambda(n) = \mathscr{O}(n^{-1})$ regime $0 \equiv q_{n,k,\infty}(\cdot) \not\Rightarrow q_{k,\infty}(\cdot) \not\equiv 0$ indicates the presence of a giant component. The precise formulation of this fact for the Erdős-Rényi model is (36).
- If $\lambda(n) \ll 1$ then only "large" components burn. Indeed in Proposition 1 we are going to prove that for all $k \in \mathbb{N}$ $r_{n,k}(\cdot)$ converges to 0 in probability an $n \to \infty$. Thus by (23) we have $r(\cdot) = r_\infty(\cdot)$ in the limit. But Theorem 1, Theorem 2 and (20) imply that $0 = r_{n,\infty}(t) \not\Rightarrow r_\infty(t) = \int_0^t \varphi(s)\,ds > 0$ for $t > T_{\mathrm{gel}}$.

$\mathscr{F}_{\mathbf{v}}[0,T]$ endowed with the topology of Definition 1 is a complete separable metric space:

Lemma 1. *If* $\left(\mathbf{q}_n(\cdot),\mathbf{r}_n(\cdot)\right) \in \mathscr{F}_{\mathbf{v}}[0,T]$ *for all* $n \in \mathbb{N}$ *and* $\left(\mathbf{q}_n(\cdot),\mathbf{r}_n(\cdot)\right) \to \left(\mathbf{q}(\cdot),\mathbf{r}(\cdot)\right)$, *then* $\left(\mathbf{q}(\cdot),\mathbf{r}(\cdot)\right) \in \mathscr{F}_{\mathbf{v}}[0,T]$.

Proof. By the definition of weak convergence, $q_{k,l}, q_k, q, r_k, r$ are increasing left-continuous functions with initial value 0. We need to check that the functions r_∞, $q_{k,\infty}$, and $q_{\infty,\infty}$ (defined by (23), (22) and (24), respectively) are increasing. We may assume that $0 \le s \le t \le T$ are continuity points of $q_{k,l}$, q_k, q, r_k and r for all $k, l \in \bar{\mathbb{N}}$.

By Fatou's lemma we get

$$\begin{aligned} r_\infty(t) - r_\infty(s) &= \lim_{n\to\infty} \big(r_n(t) - r_n(s)\big) - \sum_{k\in\mathbb{N}} \lim_{n\to\infty} \Big(r_{n,k}(t) - r_{n,k}(s)\Big) \\ &\ge \limsup_{n\to\infty} \left(r_n(t) - r_n(s) - \sum_{k\in\mathbb{N}} \Big(r_{n,k}(t) - r_{n,k}(s)\Big) \right) \\ &= \limsup_{n\to\infty} \Big(r_{n,\infty}(t) - r_{n,\infty}(s)\Big) \ge 0. \end{aligned}$$

One can prove similarly that $q_{k,\infty}$ is increasing for $k \in \mathbb{N}$. In order to prove that

$$q_{\infty,\infty}(t) - q_{\infty,\infty}(s) \ge \limsup_{n\to\infty} \Big(q_{n,\infty,\infty}(t) - q_{n,\infty,\infty}(s)\Big)$$

let $\alpha_{n,k,l} := q_{n,k,l}(t) - q_{n,k,l}(s)$ for $k, l \in \bar{\mathbb{N}}$. By (24) we only need to check

$$\lim_{n\to\infty} \sum_{k,l\in\bar{\mathbb{N}}} \alpha_{n,k,l} - \limsup_{n\to\infty} \alpha_{n,\infty,\infty} \ge 2 \sum_{k\in\mathbb{N}} \lim_{n\to\infty} \sum_{l\in\bar{\mathbb{N}}} \alpha_{n,k,l} - \sum_{k,l\in\mathbb{N}} \lim_{n\to\infty} \alpha_{n,k,l}. \tag{25}$$

Let

$$K_m := \{(k,l) : (k \ge m \text{ and } l = m) \text{ or } (l \ge m \text{ and } k = m)\} \cup \{(m,\infty)\} \cup \{(\infty,m)\}.$$

The left hand side of (25) is $\liminf_{n\to\infty} \sum_{m\in\mathbb{N}} \beta_{n,m}$, the right hand side is $\sum_{m\in\mathbb{N}} \lim_{n\to\infty} \beta_{n,m}$, where $\beta_{n,m} := \sum_{(k,l)\in K_m} \alpha_{n,k,l}$, and the inequality follows from Fatou's lemma.

Now that we have proved that the limit of convergent flows is itself a flow, we only need to check that the limit flow is consistent with the initial condition $\mathbf{v}$, but this follows from the facts that $\mathscr{E}_{\mathbf{v}}[0,T]$ is a closed metric space and the mapping from $\mathscr{F}_{\mathbf{v}}[0,T]$ to $\mathscr{E}_{\mathbf{v}}[0,T]$ defined by (19) is continuous with respect to the corresponding topologies.

$\square$

Finally we define the space of all FFF-s as follows:

$$\mathscr{D}[0,T] := \big\{ \big(\mathbf{v}, \mathbf{q}(\cdot), \mathbf{r}(\cdot)\big) : \mathbf{v} \in \mathscr{V},\ \big(\mathbf{q}(\cdot), \mathbf{r}(\cdot)\big) \in \mathscr{F}_{\mathbf{v}}[0,T] \big\}.$$

This space is again a complete and separable metric space if we define $\big(\mathbf{v}_n, \mathbf{q}_n(\cdot), \mathbf{r}_n(\cdot)\big) \to \big(\mathbf{v}, \mathbf{q}(\cdot), \mathbf{r}(\cdot)\big)$ by requiring $\mathbf{v}_n \to \mathbf{v}$ (coordinate-wise) and $\big(\mathbf{q}_n(\cdot), \mathbf{r}_n(\cdot)\big) \to \big(\mathbf{q}(\cdot), \mathbf{r}(\cdot)\big)$.

Lemma 2. *For any $C < \infty$ the subset*

$$\mathscr{K}_C[0,T] := \big\{ \big(\mathbf{v}, \mathbf{q}(\cdot), \mathbf{r}(\cdot)\big) \in \mathscr{D}[0,T] : q(T) \le C \big\}$$

is compact in $\mathscr{D}[0,T]$.

Proof.

$$\lim_{K\to\infty}\left[\frac{1}{2}\sum_{k=1}^{K}\sum_{l=1}^{k-1}q_{l,k-l}(T)-\sum_{k=1}^{K}q_k(T)\right]=-\frac{1}{2}q(T)+\frac{1}{2}q_{\infty,\infty}(T)$$

by $q(T)\le C$, dominated convergence and $q_{k,l}=q_{l,k}$. Thus summing the equations (19) with coefficients $\frac{1}{k}$ we get

$$\sum_{k=1}^{\infty}\frac{1}{k}v_k(T)-\sum_{k=1}^{\infty}\frac{1}{k}v_k(0)+\frac{1}{2}q(T)=\sum_{k=2}^{\infty}\frac{k-1}{k}r_k(T)+r_\infty(T)+\frac{1}{2}q_{\infty,\infty}(T).$$

The inequalities

$$r(T)\le 2+C,\quad r_\infty(T)\le 1+\frac{1}{2}C,\quad r_k(T)\le(1+\frac{C}{2})\frac{k}{k-1} \tag{26}$$

follow from $\mathbf{v}(T)\in\mathscr{V}$ and $q(T)\le C$.

By Helly's selection theorem and a diagonal argument we can choose a convergent subsequence from any sequence of elements of $\mathscr{K}_C[0,T]$ with the limiting FFF itself being an element of $\mathscr{K}_C[0,T]$.

□

2.2 The Markov process

It is easy to see that in order to prove Theorem 2 we do not need to know anything about the graph structure of the connected components: by the mean field property of the dynamics the stochastic process $\mathbf{v}_n(t)$ defined by (3) and (4) is itself a Markov chain.

The state space of the Markov chain $t\mapsto\mathbf{V}_n(t)$ is:

$$\Omega_n:=\{\mathbf{V}=(V_k)_{k\in\mathbb{N}}:V_k\in\{0,k,2k,\dots\},\ \sum_{k\ge1}V_k=n\}$$

The allowed jumps of the Markov chain are described by the following jump transformations for $i\le j$:

$$\sigma_{i,j}:\{\mathbf{V}\in\Omega_n:V_i\big(V_j-j1\!\!1_{\{i=j\}}\big)>0\}\to\Omega_n,$$
$$\big(\sigma_{i,j}\mathbf{V}\big)_k:=V_k-i1\!\!1_{\{k=i\}}-j1\!\!1_{\{k=j\}}+(i+j)1\!\!1_{\{k=i+j\}},$$

$$\tau_i:\{\mathbf{V}\in\Omega_n:V_i>0\}\to\Omega_n,\qquad\big(\tau_i\mathbf{V}\big)_k:=V_k+i1\!\!1_{\{k=1\}}-i1\!\!1_{\{k=i\}}$$

The corresponding jump rates are $a_{n,i,j},b_{n,i}:\Omega_n\to\mathbb{R}_+$:

$$a_{n,i,j}(\mathbf{V}):=\big((1+1\!\!1_{\{i=j\}})n\big)^{-1}V_i\big(V_j-j1\!\!1_{\{i=j\}}\big),\qquad b_{n,i}(\mathbf{V}):=\lambda(n)V_i.$$

The infinitesimal generator of the chain is :

$$L_nf(\mathbf{V})=\sum_{i\le j}a_{n,i,j}(\mathbf{V})\big(f(\sigma_{i,j}\mathbf{V})-f(\mathbf{V})\big)+\sum_i b_{n,i}(\mathbf{V})\big(f(\tau_i\mathbf{V})-f(\mathbf{V})\big).$$

We denote by $Q_{n,k,l}(t)$ and by $R_{n,k}(t)$ the number of $\sigma_{k,l}$-jumps, respectively k-times the number of τ_k-jumps occurred in the time interval $[0,t]$:

$$Q_{n,k,l}(t) := \big(1 + 1\!\!1_{\{k=l\}}\big) \cdot \Big|\big\{s \in [0,t] \,:\, \mathbf{V}_n(s+0) = (\sigma_{k,l}\mathbf{V}_n)(s-0)\big\}\Big|, \tag{27}$$

$$R_{n,k}(t) := 1\!\!1_{\{k\neq 1\}} k \cdot \Big|\big\{s \in [0,t] \,:\, \mathbf{V}_n(s+0) = \big(\tau_k \mathbf{V}_n\big)(s-0)\big\}\Big|. \tag{28}$$

Finally, the scaled objects are

$$v_{n,k}(t) := n^{-1} V_{n,k}(t), \qquad\qquad \mathbf{v}_n(t) := \big(v_{n,k}(t)\big)_{k\in\mathbb{N}}, \tag{29}$$

$$q_{n,k,l}(t) := n^{-1} Q_{n,k,l}(t), \qquad q_{n,k,\infty}(t) \equiv 0, \qquad \mathbf{q}_n(t) := \big(q_{n,k,l}(t)\big)_{k,l\in\bar{\mathbb{N}}}, \tag{30}$$

$$r_{n,k}(t) := n^{-1} R_{n,k}(t), \qquad r_{n,\infty}(t) \equiv 0, \qquad \mathbf{r}_n(t) := \big(r_{n,k}(t)\big)_{k\in\bar{\mathbb{N}}} \tag{31}$$

Now, given $T \in (0,\infty)$ and some initial conditions $\mathbf{v}_n(0) = \mathbf{v}_n \in \mathscr{V}_1$, clearly $t \mapsto \mathbf{v}_n(t) \in \mathscr{V}_1$ is a conservative FFE, generated by the FFF $\big(\mathbf{v}_n, \mathbf{q}_n(\cdot), \mathbf{r}_n(\cdot)\big) \in \mathscr{D}[0,T]$ through (19). We denote by $\mathbb{P}_n$ the probability distribution of this process on $\mathscr{D}[0,T]$. We will always assume that the initial conditions converge, as $n \to \infty$, to a deterministic element of $\mathscr{V}_1$:

$$\lim_{n\to\infty} v_{n,k}(0) = v_k, \qquad \mathbf{v} := (v_k)_{k\in\mathbb{N}} \in \mathscr{V}_1. \tag{32}$$

Proposition 1. *The sequence of probability measures $\mathbb{P}_n$ is tight on $\mathscr{D}[0,T]$. If $\lambda(n) \ll 1$, then any weak limit point $\mathbb{P}$ of the sequence $\mathbb{P}_n$ is concentrated on that subset of $\mathscr{D}[0,T]$ for which the following hold for $k,l \in \mathbb{N}$:*

$$q_{k,l}(t) = \int_0^t v_k(s) v_l(s) ds, \quad q_k(t) = \int_0^t v_k(s) ds, \quad q(t) \le t, \quad r_k(t) \equiv 0 \tag{33}$$

$$\mathbf{v}(0) = \mathbf{v}. \tag{34}$$

Proof. There is nothing to prove about the initial condition (34): it was a priori assumed in (32).

In order to prove the validity of the integral equations (33), note first that it is straightforward that the processes $\widetilde{q}_{n,k,l}(t)$, $\langle\widetilde{q}_{n,k,l}\rangle(t)$, $\widetilde{q}_{n,k}(t)$, $\langle\widetilde{q}_{n,k}\rangle(t)$, $\widetilde{r}_{n,k}(t)$, $\langle\widetilde{r}_{n,k}\rangle(t)$, defined below are martingales:

$$\widetilde{q}_{n,k,l}(t) := q_{n,k,l}(t) - \int_0^t v_{n,k}(s)v_{n,l}(s)ds + \frac{k\mathbb{1}_{\{k=l\}}}{n}\int_0^t v_{n,k}(s)ds,$$

$$\langle\widetilde{q}_{n,k,l}\rangle(t) := \widetilde{q}_{n,k,l}(t)^2 - \frac{\mathbb{1}_{\{k\neq l\}} + 2\mathbb{1}_{\{k=l\}}}{n}\int_0^t v_{n,k}(s)v_{n,l}(s)ds + \frac{2k\mathbb{1}_{\{k=l\}}}{n^2}\int_0^t v_{n,k}(s)ds,$$

$$\widetilde{q}_{n,k}(t) := q_{n,k}(t) - \int_0^t v_{n,k}(s)ds + \frac{k}{n}\int_0^t v_{n,k}(s)ds,$$

$$\langle\widetilde{q}_{n,k}\rangle(t) := \widetilde{q}_{n,k}(t)^2 - \frac{1}{n}\int_0^t \big(v_{n,k}(s)^2 + v_{n,k}(s)\big)ds + \frac{2k}{n^2}\int_0^t v_{n,k}(s)ds,$$

$$\widetilde{q}_n(t) := q_n(t) - t + \frac{1}{n}\int_0^t m_{n,1}(s)ds,$$

$$\langle\widetilde{q}_n\rangle(t) := \widetilde{q}_n(t)^2 - \frac{1}{n}\left(t + \int_0^t \sum_{k=1}^n v_{n,k}(s)^2 ds\right) + \frac{2}{n^2}\int_0^t m_{n,1}(s)ds,$$

$$\widetilde{r}_{n,k}(t) := r_{n,k}(t) - \lambda(n)k\int_0^t v_{n,k}(s)ds,$$

$$\langle\widetilde{r}_{n,k}\rangle(t) := \widetilde{r}_{n,k}(t)^2 - \frac{\lambda(n)k^2}{n}\int_0^t v_{n,k}(s)ds.$$

From Doob's maximal inequality it readily follows that for any $k, l \in \mathbb{N}$ and $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbf{P}\big(\sup_{0\le t\le T}\Big|q_{n,k,l}(t) - \int_0^t v_{n,k}(s)v_{n,l}(s)ds\Big| > \varepsilon\big) = 0,$$

$$\lim_{n\to\infty} \mathbf{P}\big(\sup_{0\le t\le T}\Big|q_{n,k}(t) - \int_0^t v_{n,k}(s)ds\Big| > \varepsilon\big) = 0,$$

$$\lim_{n\to\infty} \mathbf{P}\big(\sup_{0\le t\le T} q_n(t) - t > \varepsilon\big) = 0,$$

$$\lim_{n\to\infty} \mathbf{P}\big(\sup_{0\le t\le T}\big|r_{n,k}(t)\big| > \varepsilon\big) = 0.$$

Hence (33). Tightness follows from

$$\mathbf{E}\big(q_n(T)\big) \le T, \tag{35}$$

Markov's inequality and Lemma 2. □

If we consider the case $\lambda(n) \equiv 0$ (this is the dynamical Erdős-Rényi model) then (5)+(6) follows from Proposition 1 since (19) becomes

$$v_k(t) = v_k(0) + \frac{k}{2}\sum_{l=1}^{k-1} q_{l,k-l}(t) - kq_k(t) = v_k(0) + \int_0^t \frac{k}{2}\sum_{l=1}^{k-1} v_l(s)v_{k-l}(s) - kv_k(s)\,ds$$

which is the integral form of (6). Plugging (33) into (22) we get for $t > T_{\mathrm{gel}}$

$$q_{k,\infty}(t) = \int_0^t v_k(s)\theta(s)\,ds > 0. \tag{36}$$

2.3 The integrated Burgers control problem

If $\mathbf{v}(\cdot) \in \mathscr{E}_{\mathbf{v}_0}[0,T]$ is generated by a FFF satisfying (33) through (19), then

$$r(\cdot) = \sum_{k\in\bar{\mathbb{N}}} r_k(\cdot) = \sum_{k=1}^{\infty} r_k(\cdot) + r_\infty(\cdot) = \sum_{k=1}^{\infty} 0 + r_\infty(\cdot) = r_\infty(\cdot)$$

and $\mathbf{v}(\cdot)$ is a solution of the *controlled Smoluchowski integral equations* with control function $r(\cdot)$:

$$v_k(t) = v_k(0) + \frac{k}{2}\sum_{l=1}^{k-1}\int_0^t v_l(s)v_{k-l}(s)ds - k\int_0^t v_k(s)ds + \mathbb{1}_{\{k=1\}}r(t), \qquad k \in \mathbb{N} \tag{37}$$

$$v_k(t) \geq 0, \quad \sum_{k=1}^{\infty} v_k(t) \leq 1 \tag{38}$$

$$\mathbf{v}(0) = \mathbf{v}_0 \in \mathscr{V}_1. \tag{39}$$

By $q(T) \leq T$, $r_\infty(\cdot) = r(\cdot)$ and (26) we get

$$0 = r(0) \leq r(s) \leq r(t) \quad \text{for} \quad 0 \leq s \leq t \leq T, \quad r(T) \leq 1 + \frac{T}{2}. \tag{40}$$

Using induction on k one can see that the initial condition $\mathbf{v}_0$ and the control function $r(\cdot)$ determines the solution of (37), (39) uniquely.

For $\mathbf{v} \in \mathscr{V}$ we introduce the generating function

$$V : [0,\infty) \to [-1,0], \qquad V(x) := \sum_{k=1}^{\infty} v_k e^{-kx} - 1. \tag{41}$$

$x \mapsto V(x)$ is analytic on $(0,\infty)$ and has the following straightforward properties:

$$\lim_{x\to\infty} V(x) = -1, \qquad V'(x) \leq 0, \qquad V''(x) \geq 0. \tag{42}$$

It is easy to see that if $t \mapsto \mathbf{v}(t)$ is a solution of (37), (38), (39) then the corresponding generating functions $t \mapsto V(t,\cdot)$ will solve the *integrated Burgers control problem*

$$V(t,x) - V(0,x) + \int_0^t V(s,x)V'(s,x)ds = e^{-x}r(t), \tag{43}$$

$$-1 \leq V(t,0) \leq 0 \tag{44}$$

$$V(0,x) = V_0(x). \tag{45}$$

The control function $r(\cdot)$ was defined to be continuous from the left in (18), but it need not be continuous: when $\lambda(n) = n^{-1}\lambda$ then the FFE obtained as the $n \to \infty$ limit satisfies (37), (38), (39), but the control function $r(\cdot)$ evolves randomly according to the rules (10), (11):

$$\mathbf{P}\big(r(t+dt) = r(t) + \theta(t) \,\big|\, \mathscr{F}(t) \big) = \lambda\theta(t)dt + o(dt)$$

Thus $r(\cdot)$ is a random step function in this case.

In order to rewrite (43) as a differential equation we introduce a new time variable τ:

$$t(\tau) := \max\{t : t + r(t) \leq \tau\} \tag{46}$$

It is easily seen that $t(\tau)$ is increasing and Lipschitz-continuous:

$$t(\tau) = \int_0^\tau \alpha(s)\, ds \qquad 0 \leq \alpha(\cdot) \leq 1 \tag{47}$$

Given a solution $V(t,x)$ of (43), (44), (45) define

$$\mathbf{V}(\tau,x) := V(t(\tau),x) + (\tau - t(\tau) - r(t(\tau)))\, e^{-x} \tag{48}$$

Then by (43) we have

$$\mathbf{V}(\tau,x) = V(0,x) - \int_0^{t(\tau)} V(s,x)V'(s,x)\, ds + (\tau - t(\tau))e^{-x}. \tag{49}$$

Now we show that for all $\tau \geq 0$, $x > 0$ and $t \geq 0$ we have

$$\partial_\tau \mathbf{V}(\tau,x) = -\mathbf{V}(\tau,x)\mathbf{V}'(\tau,x)\alpha(\tau) + (1-\alpha(\tau))e^{-x} \tag{50}$$

$$-1 \leq \mathbf{V}(\tau,0) \leq 0 \tag{51}$$

$$\mathbf{V}(0,x) = V_0(x) \tag{52}$$

$$\mathbf{V}(t + r(t),x) = V(t,x) \tag{53}$$

First note that the fact

$$\mathbf{V}(\tau,x) \neq V(t(\tau),x) \qquad \Longrightarrow \qquad \alpha(\tau) = 0 \tag{54}$$

follows directly from (46), (47) and (48): if $r(t_+) \neq r(t)$, then $\alpha(\tau) = 0$ for all $t + r(t) < \tau \leq t + r(t_+)$. The differential equation (50) follows from (47), (49) and (54). The boundary inequality (51) follows from

$$-1 \leq V(t(\tau),x) \leq \mathbf{V}(\tau,x) \leq V(t(\tau)_+,x) \leq 0.$$

The initial conditions (45) and (52) are equivalent, and (53) follows from (48) and (46).

From the definition of Lebesgue-Stieltjes integration it follows that for all $t_1 \leq t_2$ we have

$$\int_{t_1+r(t_1)}^{t_2+r(t_2)} f(t(\tau))(1-\alpha(\tau))\, d\tau = \int_{t_1}^{t_2} f(t)\, dr(t) \tag{55}$$

3 Boundary behavior

3.1 Elementary facts about generating functions

In this subsection we collect some elementary facts about generating functions, which will be used along the proof of Theorem 1 and Theorem 2. For $\mathbf{v} \in \mathscr{V}$ we introduce the generating function $V(x)$ defined in (41) which has the straightforward properties listed in (42). It is also easy to see that for any $\mathbf{v} \in \mathscr{V}$ and any $x > 0$

$$|V'(x)| \leq \frac{1}{e} x^{-1}, \qquad V''(x) \leq \left(\frac{2}{e}\right)^2 x^{-2}, \qquad |V'''(x)| \leq \left(\frac{3}{e}\right)^3 x^{-3}. \tag{56}$$

We define the functions $E : (0,\infty) \to (0,\infty)$, $E^* : [0,\infty) \to (0,\infty]$, $E_* : [0,\infty) \to [0,\infty)$ as follows:

$$E(x) := -\frac{V'(x)^3}{V''(x)}, \quad E^*(x) := \sup_{0<y\leq x} E(y), \quad E_*(x) := \inf_{0<y\leq x} E(y) \tag{57}$$

Note that these functions are continuous on their domain of definition.

Lemma 3. *Let* $\mathbf{v} \in \mathscr{V}_1$.

1. *For any* $x > 0$

$$0 < \quad V(x)V'(x) \quad \leq E^*(x). \tag{58}$$

2. *If in addition*

$$V'(0) := \lim_{x\to 0} V'(x) = -\infty \tag{59}$$

then the following bounds hold

$$2^{1/2} E_*(x)^{1/2} x^{1/2} \leq \quad -V(x) \quad \leq 2^{1/2} E^*(x)^{1/2} x^{1/2} \tag{60}$$

$$2^{-1/2} E_*(x) E^*(x)^{-1/2} x^{-1/2} \leq \quad -V'(x) \quad \leq 2^{-1/2} E^*(x) E_*(x)^{-1/2} x^{-1/2} \tag{61}$$

$$2^{-3/2} E_*(x)^3 E^*(x)^{-5/2} x^{-3/2} \leq \quad V''(x) \quad \leq 2^{-3/2} E^*(x)^3 E_*(x)^{-5/2} x^{-3/2}$$

$$E_*(x) \leq \quad V(x)V'(x) \quad \leq E^*(x). \tag{62}$$

Proof. Since $\mathbf{v} \in \mathscr{V}_1$ we have $V(0) = 0$. Denote the inverse function of $-V(x)$ by $X(u)$: $X(-V(x)) = x$. Note that

$$E(x) = \frac{1}{X''(-V(x))}, \tag{63}$$

and thus

$$X(0) = 0, \qquad X'(0) = -V'(0)^{-1}, \qquad X''(u) = E(X(u))^{-1}.$$

It follows that for $u \in [0, -V(x)]$:

$$-V'(0)^{-1} + E^*(x)^{-1} u \leq \quad X'(u) \quad \leq -V'(0)^{-1} + E_*(x)^{-1} u,$$

$$-V'(0)^{-1} u + E^*(x)^{-1} \frac{u^2}{2} \leq \quad X(u) \quad \leq -V'(0)^{-1} u + E_*(x)^{-1} \frac{u^2}{2}.$$

Hence, all the bounds of the Lemma follow directly. ◻

3.2 Bounds on E

We assume given a solution of the *integrated Burgers control problem*: (43), (44), (45) with a control function $r(\cdot)$ satisfying (40).

We fix $\overline{t} \in (0, \infty)$, $\overline{x} \in (0, \infty)$. All estimates will be valid uniformly in the domain $(t, x) \in [0, \overline{t}] \times [0, \overline{x}]$. The various constants appearing in the forthcoming estimates will depend only on the initial conditions $V(0, x)$ and on the choice of $(\overline{t}, \overline{x})$. The notation

$$A(t, x) \asymp B(t, x)$$

means that there exists a constant $1 < C < \infty$ which depends only on the initial conditions (45) and the choice of $(\overline{t}, \overline{x})$, such that for any $(t, x) \in [0, \overline{t}] \times [0, \overline{x}]$

$$C^{-1} B(t, x) \leq A(t, x) \leq C B(t, x). \tag{64}$$

The notation $A(t, x) = \mathcal{O}(B(t, x))$ means that the upper bound of (64) holds.

In the sequel we denote the derivative of functions $f(t, x)$ with respect to the time and space variables by $\dot{f}(t, x)$ and $f'(t, x)$, respectively.

First we define the *characteristics* given a solution of (43), (45), (44): for $t \geq 0, x > 0$ let $[0, t] \ni s \mapsto \xi_{t,x}(s)$ be the *unique solution* of the integral equation

$$\xi_{t,x}(s) = x - V(t, x)(t - s) + \int_s^t (u - s) e^{-\xi_{t,x}(u)} dr(u). \tag{65}$$

Existence and uniqueness of the solution of (65) follow from a simple fixed point argument. Now we prove that (given (t, x) fixed) $s \mapsto \xi_{t,x}(s)$ is also solution of the initial value problem

$$\frac{d}{ds} \xi_{t,x}(s) =: \dot{\xi}_{t,x}(s) = V(s, \xi_{t,x}(s)), \qquad \xi_{t,x}(t) = x. \tag{66}$$

In order to prove this we define $\mathbf{V}(\tau, x)$ by (48). Thus from (54) it follows that that the solution of (66) satisfies

$$\frac{d}{d\tau} \xi_{t,x}(t(\tau)) = V(t(\tau), \xi_{t,x}(t(\tau))) \alpha(\tau) = \mathbf{V}(\tau, \xi_{t,x}(t(\tau))) \alpha(\tau) \tag{67}$$

From this and (50) we get that

$$\frac{d}{d\tau} \mathbf{V}(\tau, \xi_{t,x}(t(\tau))) = \dot{\mathbf{V}}(\tau, \xi_{t,x}(t(\tau))) + \mathbf{V}'(\tau, \xi_{t,x}(t(\tau))) \cdot \frac{d}{d\tau} \xi_{t,x}(t(\tau)) = (1 - \alpha(\tau)) e^{-\xi_{t,x}(t(\tau))}$$

Integrating this and using $\xi_{t,x}(t) = x$ and (53) we get for all $\tau_1 \leq t + r(t)$

$$\mathbf{V}(\tau_1, \xi_{t,x}(t(\tau_1))) = V(t, x) - \int_{\tau_1}^{t+r(t)} (1 - \alpha(\tau)) e^{-\xi_{t,x}(t(\tau))} d\tau$$

Substituting this into the r.h.s. of (67), integrating and using (47) we get for all $\tau_2 \leq t + r(t)$

$$\xi_{t,x}(t(\tau_2)) = x - V(t, x)(t - t(\tau_2)) + \int_{\tau_2}^{t+r(t)} (t(\tau) - t(\tau_2)) e^{-\xi_{t,x}(t(\tau))} (1 - \alpha(\tau)) d\tau$$

Now (65) follows from this by substituting $\tau_2 = s + r(s)$ and using (55).

We define (similarly to (57))

$$E(t,x) := -\frac{\partial_x V(t,x)^3}{\partial_x^2 V(t,x)}, \qquad E^*(t,x) := \sup_{0<y\le x} E(t,y), \qquad E_*(t,x) := \inf_{0<y\le x} E(t,y),$$
$$\mathbf{E}(\tau,x) := -\frac{\partial_x \mathbf{V}(\tau,x)^3}{\partial_x^2 \mathbf{V}(\tau,x)}, \qquad \mathbf{E}^*(\tau,x) := \sup_{0<y\le x} \mathbf{E}(\tau,y), \qquad \mathbf{E}_*(\tau,x) := \inf_{0<y\le x} \mathbf{E}(\tau,y).$$

Differentiating (50) with respect to x we get

$$\dot{\mathbf{V}}'(\tau,x) = -\mathbf{V}'(\tau,x)^2\alpha(\tau) - \mathbf{V}(\tau,x)\mathbf{V}''(\tau,x)\alpha(\tau) - (1-\alpha(\tau))e^{-x} \tag{68}$$
$$\dot{\mathbf{V}}''(\tau,x) = -3\mathbf{V}'(\tau,x)\mathbf{V}''(\tau,x)\alpha(\tau) - \mathbf{V}(\tau,x)\mathbf{V}'''(\tau,x)\alpha(\tau) + (1-\alpha(\tau))e^{-x} \tag{69}$$

Using this and (67) we obtain

$$\frac{d}{d\tau}\mathbf{E}(\tau,\xi_{t,x}(t(\tau))) = \left(3\frac{\mathbf{V}'(\tau,\xi_{t,x}(t(\tau)))^2}{\mathbf{V}''(\tau,\xi_{t,x}(t(\tau)))} + \frac{\mathbf{V}'(\tau,\xi_{t,x}(t(\tau)))^3}{\mathbf{V}''(\tau,\xi_{t,x}(t(\tau)))^2}\right) e^{-\xi_{t,x}(t(\tau))}(1-\alpha(\tau))d\tau \tag{70}$$

Lemma 4. *If $m_2(0) = \sum_{k=1}^\infty k^2 \cdot v_k(0) < +\infty$, then for any solution of the integrated Burgers control problem* (43), (45), (44) *with a control function satisfying* (40) *and for* $(t,x) \in [0,\overline{t}] \times (0,\overline{x}]$ *we have*

$$E(t,x) \asymp 1 \tag{71}$$

Proof. $E(0,x) = \mathbf{E}(0,x) \asymp 1$ follows from $m_2(0) < +\infty$. For $t \ge 0$ we use the formula (70) to show that $0 \le \frac{d}{d\tau}\mathbf{E}(\tau,\xi_{t,x}(t(\tau))) \le 3$. Since $0 \le e^{-\xi_{t,x}(t(\tau))}(1-\alpha(\tau)) \le 1$ by (47) we only need to show

$$0 \le \frac{V'(x)^2}{V''(x)^2}(3V''(x) + V'(x)) = 3\frac{V'(x)^2}{V''(x)} + \frac{V'(x)^3}{V''(x)^2} \le 3\frac{V'(x)^2}{V''(x)} \le 3. \tag{72}$$

The lower bound follows from $3V''(x) + V'(x) = \sum_{k=1}^\infty (3k^2 - k)v_k e^{-kx} > 0$.

The upper bound follows from Schwarz's inequality:

$$\frac{V'(x)^2}{V''(x)} = \frac{\left(\sum_{k=1}^\infty k \cdot v_k e^{-kx}\right)^2}{\sum_{k=1}^\infty k^2 \cdot v_k e^{-kx}} \le \sum_{k=1}^\infty v_k e^{-kx} \le m_0 \le 1.$$

Integrating (70), using $0 \le \frac{d}{d\tau}\mathbf{E}(\tau,\xi_{t,x}(t(\tau))) \le 3$, (53), (55) and the last inequality in (40) we obtain

$$E(0,\xi_{t,x}(0)) \le E(t,x) \le E(0,\xi_{t,x}(0)) + 3(t/2+1).$$

Next we observe that $x \le \xi_{t,x}(0) \le x + t$ by (66) and $-1 < V(t,x) \le 0$.

The last two bounds yield for $(t,x) \in [0,\overline{t}] \times (0,\overline{x}]$

$$0 < E_*(0,\overline{x}+\overline{t}) \le E(t,x) \le E^*(0,\overline{x}+\overline{t}) + 3(\overline{t}/2+1) < \infty.$$

□

Lemma 5. *If $m_2(0) < +\infty$, then for any solution of the integrated Burgers control problem* (43)*,* (44)*,* (45) *with a control function satisfying* (40) *there is a constant C^* which depends only on the initial conditions and T such that for $T_{gel} \le t_1 \le t_2 \le T$ we have*

$$\theta(t_2) - \theta(t_1) \le C^* \cdot (t_2 - t_1) \tag{73}$$

Proof. $\theta(t) = -V(t, 0_+)$. Since $V(t,x)$ arises from (41), we assume $-1 < V(t,x) \le 0$, $V'(t,x) < 0$ for all $x > 0$.

Let us pick an arbitrary $\overline{x} > 0$. Let C be a constant such that $E(t,x) \le C$ for $(t,x) \in [0,T] \times (0, \overline{x}]$.

First we are going to show that

$$\forall\, 0 \le t \le T\,, 0 < x \le \overline{x} \qquad V'V(t,x) := V'(t,x)V(t,x) \le C^* := \max\{1, 2C\} \tag{74}$$

Note that we cannot use (58) here since that bound uses $V(t,0) = 0$. But $V(0,0) = 0$ holds, thus (74) holds for $t = 0$. From (50) and (68) we get

$$\begin{aligned}
&\frac{d}{d\tau}\left(\mathbf{V}'\mathbf{V}(\tau,x)\right) = \\
&\quad \left(-2\mathbf{V}(\tau,x)\mathbf{V}'(\tau,x)^2 - \mathbf{V}(\tau,x)^2\mathbf{V}''(\tau,x)\right)\alpha(\tau) + \left(\mathbf{V}'(\tau,x) - \mathbf{V}(\tau,x)\right)e^{-x}(1-\alpha(\tau)) \le \\
&\qquad\qquad -\mathbf{V}(\tau,x)\mathbf{V}'(\tau,x)^2\left(2 - \frac{1}{C}\mathbf{V}'\mathbf{V}(\tau,x)\right)\alpha(\tau) + \left(\mathbf{V}'(\tau,x) - \mathbf{V}(\tau,x)\right)e^{-x}(1-\alpha(\tau))
\end{aligned}$$

From (51) we get

$$\mathbf{V}'\mathbf{V}(\tau,x) \ge 1 \quad \Longrightarrow \quad \mathbf{V}'(\tau,x) \le \frac{1}{\mathbf{V}(\tau,x)} \le -1 \le \mathbf{V}(\tau,x)$$

Thus by (47) we get

$$\begin{aligned}
\mathbf{V}'\mathbf{V}(\tau,x) \ge 1 \quad &\Longrightarrow \quad \left(\mathbf{V}'(\tau,x) - \mathbf{V}(\tau,x)\right)e^{-x}(1-\alpha(\tau)) \le 0 \\
\mathbf{V}'\mathbf{V}(\tau,x) \ge 2C \quad &\Longrightarrow \quad -\mathbf{V}(\tau,x)\mathbf{V}'(\tau,x)^2\left(2 - \frac{1}{C}\mathbf{V}'\mathbf{V}(\tau,x)\right)\alpha(\tau) \le 0 \\
\mathbf{V}'\mathbf{V}(\tau,x) \ge C^* \quad &\Longrightarrow \quad \frac{d}{d\tau}\left(\mathbf{V}'\mathbf{V}(\tau,x)\right) \le 0
\end{aligned}$$

From $\mathbf{V}'\mathbf{V}(0,x) \le C^*$ and the last differential inequality it easily follows by a “forbidden region”-argument that $\mathbf{V}'\mathbf{V}(\tau,x) \le C^*$ for all $0 < x < \overline{x}$ and $0 \le \tau \le T + r(T)$. This and (53) implies (74).

By (43) and (74) we have

$$V(t_1,x) - V(t_2,x) \le \int_{t_1}^{t_2} V(s,x)V'(s,x)ds \le C^* \cdot (t_2 - t_1)$$

for every $0 < x < \bar{x}$. Letting $x \to 0_+$ implies the claim of the Lemma. ☐

3.3 No giant component in the limit

The aim of this subsection is to prove the following proposition:

Proposition 2. *If $n^{-1} \ll \lambda(n) \ll 1$ and $m_2(0) < +\infty$ holds for $\mathbf{v}(0)$ on the right-hand side of* (32) *then any weak limit point $\mathbb{P}$ of the sequence of probability measures $\mathbb{P}_n$ is concentrated on the set of conservative forest fire evolutions:*

$$\mathbb{P}\big(\sum_{k=1}^{\infty} v_k(t) \equiv 1\big) = 1 \tag{75}$$

We are going to prove Proposition 2 by contradiction: in Lemma 6 we show that if $\theta(\cdot) \not\equiv 0$ in the limit, then there is a positive time interval such that $\theta(t)$ has a positive lower bound, and that this implies that even in the convergent sequence of finite-volume models, a lot of mass is contained in arbitrarily big components on this interval. Than in subsequent Lemmas we prove that these big components indeed burn, which produces such a big increase in the value of the burnt mass $r(\cdot)$ that is in contradiction with $\mathbf{E}\big(r(T)\big) \leq 2 + \mathbf{E}\big(q(T)\big) \leq 2 + T$.

By Proposition 1 the random FFE obtained as a weak limit point is almost deterministic: (37) holds with a possibly random control function $r(\cdot)$. Also, by (33) we $\mathbb{P}$-almost surely have $q(t) \leq t$ from which (40) follows. Thus (71) and (73) hold $\mathbb{P}$-almost surely for the random flow obtained as a weak limit point with a deterministic constant C^*.

Lemma 6. *If $\mathbb{P}_n \Rightarrow \mathbb{P}$ where $\mathbb{P}$ does not satisfy* (75) *on $[0, T]$, then there exist ε_1, ε_2, $\varepsilon_3 > 0$ and a deterministic $t^* \in [\varepsilon_1, T]$ such that for every $K < +\infty$, every $m < +\infty$ and every sequence*

$$t^* - \varepsilon_1 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_m < \beta_m < t^*$$

there exists an $n_0 < +\infty$ such that for every $n \geq n_0$ and $1 \leq i \leq m$ we have

$$\mathbb{P}_n\left(\max_{\alpha_i \leq t \leq \beta_i} 1 - \sum_{k=1}^{K-1} v_{n,k}(t) > \varepsilon_2\right) > \varepsilon_3. \tag{76}$$

Proof. First we prove that if $\mathbb{P}$ does not satisfy (75) then there exist $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ and $\varepsilon_1 \leq t^* \leq T$ such that

$$\mathbb{P}\big(\inf_{t^*-\varepsilon_1 \leq t \leq t^*} \theta(t) > \varepsilon_2\big) > \varepsilon_3. \tag{77}$$

Since (75) is violated, we have $\mathbb{P}\big(\sup_{0 \leq t \leq T} \theta(t) > \varepsilon\big) > \varepsilon$ for some $\varepsilon > 0$.

Let $L := \lfloor \frac{2C^* T}{\varepsilon} \rfloor$ and $t_i := \frac{\varepsilon i}{2C^*}$ for $1 \leq i \leq L$ where C^* is the constant in (73). Since $\theta(0) = 0$ we have

$$\{\sup_{0 \leq t \leq T} \theta(t) > \varepsilon\} \subseteq \bigcup_{i=1}^{L} \{\theta(t_i) > \frac{\varepsilon}{2}\}$$

almost surely with respect to $\mathbb{P}$. Thus $\mathbb{P}\big(\theta(t^*) > \frac{\varepsilon}{2}\big) > \frac{\varepsilon}{L}$ for some $t^* \in \{t_1, \dots t_L\}$. Using (73) again (77) follows with $\varepsilon_1 := \frac{\varepsilon}{4C^*}$, $\varepsilon_2 := \frac{\varepsilon}{4}$, $\varepsilon_3 = \frac{\varepsilon}{L}$.

Now given K and the intervals $[\alpha_i, \beta_i]$, $1 \le i \le m$ we define the continuous functionals $f_i : \mathscr{D}[0,T] \to \mathbb{R}$ by

$$f_i\left(\mathbf{v}(0), \mathbf{q}(\cdot), \mathbf{r}(\cdot)\right) := \frac{1}{\beta_i - \alpha_i} \int_{\alpha_i}^{\beta_i} \left(1 - \sum_{k=1}^{K} v_k(t)\right) dt$$

where $v_k(t)$ is defined by (19). Thus for all i

$$H_i := \{\left(\mathbf{v}(0), \mathbf{q}(\cdot), \mathbf{r}(\cdot)\right) \in \mathscr{D}[0,T] : f_i\left(\mathbf{v}(0), \mathbf{q}(\cdot), \mathbf{r}(\cdot)\right) > \varepsilon_2\}$$

is an open subset of $\mathscr{D}[0,T]$ with respect to the topology of Definition 1. Thus by the definition of weak convergence of probability measures we have

$$\lim_{n\to\infty} \mathbb{P}_n(H_i) \ge \mathbb{P}(H_i) \ge \mathbb{P}\left(\inf_{t^*-\varepsilon_1 \le t \le t^*} \theta(t) > \varepsilon_2\right) > \varepsilon_3$$

from which the claim of the lemma easily follows. ☐

Lemma 7. *If $n^{-1} \ll \lambda(n)$ then for every $\varepsilon_2 > 0$ there is a $\varepsilon_4 > 0$ such that for every $\tilde{t} > 0$ there is a K and an n_1 such that for all $n \ge n_1$ $1 - \sum_{k=1}^{K-1} v_{n,k}(0) \ge \varepsilon_2$ implies*

$$\mathbb{E}_n\left(r_n(\tilde{t})\right) \ge \varepsilon_4 \tag{78}$$

The proof of Lemma 7 will follow as a consequence of the Lemmas 8 and 9.

Proof of Proposition 2. We are going to show that if there is a sequence $\mathbb{P}_n$ such that the weak limit point $\mathbb{P}$ violates (75) then for some n we have

$$\mathbb{E}_n\left(r_n(T)\right) > T + 2 \tag{79}$$

which is in contradiction with (35) and (26). In fact, $T+2$ could be replaced with any finite constant in (79), but $T+2$ is big enough to have a contradiction.

We define ε_1, ε_2, $\varepsilon_3 > 0$ and t^* using Lemma 6. Next, we define ε_4 using this ε_2 and Lemma 7. Given these, we choose $\tilde{t}$ be so small that

$$\left\lfloor \frac{\varepsilon_1}{2\tilde{t}} \right\rfloor \varepsilon_3 \varepsilon_4 > T + 2.$$

We choose K and n_1 big enough so that (78) holds. Further on, we fix the intervals $[\alpha_i, \beta_i]$, $1 \le i \le m = \lfloor \frac{\varepsilon_1}{2\tilde{t}} \rfloor$ so that $\alpha_{i+1} - \beta_i > \tilde{t}$ holds for all i and also $T - \beta_m > \tilde{t}$ holds. We choose n_0 such that (76) holds and let $n := \max\{n_0, n_1\}$.

Finally, we define the stopping times $\tau_1, \tau_2, \ldots, \tau_m$ by

$$\tau_i := \beta_i \wedge \min\{t : t \ge \alpha_i \text{ and } 1 - \sum_{k=1}^{K-1} v_{n,k}(t) \ge \varepsilon_2\}.$$

We have $\tau_i + t^* \le \beta_i + t^* < \alpha_{i+1} \le \tau_{i+1}$.

Using the strong Markov property, (78) and (76), the inequality (79) follows:

$$\mathbf{E}\left(r_n(T)\right) \ge \sum_{i=1}^{m} \mathbf{E}\left(r_n(\tau_i + \tilde{t}) - r_n(\tau_i) \,\middle|\, \tau_i < \beta_i\right) \mathbf{P}\left(\tau_i < \beta_i\right) \ge m\varepsilon_4\varepsilon_3.$$

☐

Lemma 7 stated that if initially a lot of mass is contained in big components, then in a short time a lot of mass burns. We prove this statement in two steps: in Lemma 8 we prove that if we start with a lot of mass contained in big components, then in a short time either a lot of this mass is burnt or the big components coagulate, so a lot of mass is contained in components of size $n^{1/3}$ (the same proof works if we replace the exponent $\alpha = 1/3$ by any $0 < \alpha < 1/2$). Then in Lemma 9 we prove that if we start with a lot of components of size $n^{1/3}$ then in a short time a lot of mass burns.

We will make use of the following generating function estimates in the proof of Lemma 8. If $V(x)$ is defined as in (41) and if $\mathbf{v} \in \mathcal{V}_1$ then for $\varepsilon \leq \frac{1}{2}$

$$1 - \sum_{k=1}^{K-1} v_k \geq \varepsilon \quad \Longrightarrow \quad V(1/K) \leq (e^{-1} - 1)\varepsilon \tag{80}$$

$$V(1/K) \leq -\varepsilon \quad \Longrightarrow \quad 1 - \sum_{k=1}^{\varepsilon K/2} v_k \geq \varepsilon/4. \tag{81}$$

Lemma 8. *There are constants $C_1 < +\infty$, $C_2 > 0$, $C_3 > 0$ such that if*

$$1 - \sum_{k=1}^{K-1} v_{n,k}(0) \geq \varepsilon_2 \tag{82}$$

for all n then

$$\lim_{n\to\infty} \mathbf{P}\Big(\sum_{k=C_3\varepsilon_2 n^{1/3}}^{n} v_{n,k}(\bar{t}) + r_n(\bar{t}) \geq C_2\varepsilon_2 \Big) = 1 \tag{83}$$

Where $\bar{t} = \frac{C_1}{K\varepsilon_2}$.

Sketch proof. If we let $n \to \infty$ immediately, we get that the limiting functions $v_1(t), v_2(t), \dots$ solve (37), (38), (39) with a possibly random control function $r(t) \equiv r_\infty(t)$.

The $n \to \infty$ limit of (83) is

$$\theta(\bar{t}) + r(\bar{t}) \geq C_2\varepsilon_2 \tag{84}$$

Now we prove that if $\mathbf{v}(\cdot)$ is a solution of (37), (38), (39) then $1 - \sum_{k=1}^{K-1} v_k(0) \geq \varepsilon_2$ implies (84) with $C_1 = 4$ and $C_2 = \frac{1}{4}$. This proof will also serve as an outline of the proof of Lemma 8.

In order to prove (84) define $V(t,x)$ by (41). Thus $V(t,x)$ solves the integrated Burgers control problem (43), (44), (45).

Define $U(t,x) := V(t,x) - r(t)e^{-x}$. Thus $U'(t,x) = V'(t,x) + r(t)e^{-x}$ and by (43) we have $\dot{U}(t,x) = -V(t,x)V'(t,x)$. Define the characteristic curve $\xi(\cdot)$ by

$$\dot{\xi}(t) = V(t, \xi(t)) \qquad \xi(0) = \frac{1}{K} \tag{85}$$

Let $u(t) := U(t, \xi(t)) - V(0, \frac{1}{K})$. Thus $u(0) = 0$, and

$$\begin{aligned} \dot{u}(t) = \dot{U}(t,\xi(t)) + U'(t,\xi(t))\dot{\xi}(t) = -V(t,\xi(t))V'(t,\xi(t)) + \\ \Big(V'(t,\xi(t)) + r(t)e^{-\xi(t)}\Big) V(t,\xi(t)) = r(t)e^{-\xi(t)}V(t,\xi(t)) \leq 0. \end{aligned} \tag{86}$$

Thus $u(t) \leq 0$, moreover

$$V(t, \xi(t)) = V(0, \frac{1}{K}) + r(t)e^{-\xi(t)} + u(t) \leq V(0, \frac{1}{K}) + r(t), \tag{87}$$

$$\xi(t) = \frac{1}{K} + \int_0^t u(s)\,ds + \int_0^t r(s)e^{-\xi(s)}ds + tV(0, \frac{1}{K}) \leq \frac{1}{K} + t \cdot r(t) + tV(0, \frac{1}{K}). \tag{88}$$

By (80) we have $V(0, \frac{1}{K}) \leq -\frac{1}{2}\varepsilon_2$. In order to prove that $\theta(\bar{t}) + r(\bar{t}) \geq \frac{1}{4}\varepsilon_2$ with $\bar{t} = \frac{4}{K\varepsilon_2}$ we consider two cases:

If $r(\bar{t}) \geq \frac{1}{4}\varepsilon_2$ then we are done. If $r(\bar{t}) < \frac{1}{4}\varepsilon_2$ define $\tau := \min\{t : \xi(t) = 0\}$. By (88) we have

$$\xi(\bar{t}) \leq \frac{1}{K} + \bar{t} \cdot r(\bar{t}) + \bar{t} \cdot \left(-\frac{1}{2}\varepsilon_2\right) < \frac{1}{K} + \frac{1}{K} - \frac{2}{K} = 0$$

Thus $\tau \leq \bar{t}$. By (87) we get

$$-\theta(\tau) = V(\tau, 0) = V(\tau, \xi(\tau)) \leq -\frac{1}{2}\varepsilon_2 + \frac{1}{4}\varepsilon_2 = -\frac{1}{4}\varepsilon_2$$

Thus $\frac{1}{4}\varepsilon_2 \leq \theta(\tau) \leq \theta(\tau) + r(\tau) \leq \theta(\bar{t}) + r(\bar{t})$ because by (21) the function $\theta(t) + r(t)$ is increasing. $\square$

To make this proof work for Lemma 8 we have to deal with the fluctuations caused by randomness, combinatorial error terms and the fact that $\lambda(n)$ only disappears in the limit.

Proof of Lemma 8. Given a FFF obtained from a forest fire Markov process by (29),(30) and (31), define

$$U_n(t, x) := \sum_{k=1}^{n} \left[v_{n,k}(0) + \frac{k}{2}\sum_{l=1}^{k-1} q_{n,l,k-l}(t) - kq_{n,k}(t) - r_{n,k}(t) \right] e^{-kx} - 1 - \lambda(n)$$

By (19) we have

$$U_n(t, x) + r_n(t)e^{-x} = \sum_{k=1}^{n} v_{n,k}(t)e^{-kx} - 1 - \lambda(n) =: V_n(t, x) - \lambda(n) =: W_n(t, x).$$

$$W'(t, x) = -\sum_{k \geq 1} k \cdot v_{n,k}(t)e^{-kx}$$

$$-\frac{1}{2}\partial_x \left(W(t, x) + 1 + \lambda(n)\right)^2 = \sum_{k \geq 1} \frac{k}{2} \sum_{l=1}^{k-1} v_{n,l}(t)v_{n,k-l}(t)e^{-kx}$$

$$W''(t, x) = \sum_{k \geq 1} k^2 \cdot v_{n,k}(t)e^{-kx}$$

$$W''(t, 2x) = \sum_{k \geq 1} \left(\frac{k}{2}\right)^2 \cdot 1\!\!1[2 \mid k] \cdot v_{n,\frac{k}{2}}(t)e^{-kx}$$

If $X(t)$ is a process adapted to the filtration $\mathscr{F}(t)$, let

$$LX(t) := \lim_{dt\to 0} \frac{1}{dt}\mathbf{E}\big(X(t+dt)-X(t)\,\big|\,\mathscr{F}_t\big)$$

Using the martingales of Proposition 1 we get

$$\begin{aligned}
L\,U_n(t,x) &= \sum_{k\geq 1}\left[\frac{k}{2}\sum_{l=1}^{k-1} L\,q_{n,l,k-l}(t) - k\cdot L\,q_{n,k}(t) - L\,r_{n,k}(t)\right]e^{-kx} = \\
&\sum_{k\geq 1}\left[\frac{k}{2}\sum_{l=1}^{k-1}\left(v_{n,l}(t)v_{n,k-l}(t) - \frac{l\cdot \mathbb{1}[2l=k]}{n}v_{n,l}(t)\right) - \right.\\
&\left. k\cdot\left(v_{n,k}(t)-\frac{k}{n}v_{n,k}(t)\right) - \Big(\lambda(n)\cdot k\cdot v_{n,k}(t)\Big)\right]e^{-kx} = \\
&-\frac{1}{2}\partial_x\left(W(t,x)+1+\lambda(n)\right)^2 - \frac{1}{n}W''(t,2x) + \\
&W'(t,x)+\frac{1}{n}W''(t,x)+\lambda(n)W'(t,x) = \\
&-W_n'(t,x)W_n(t,x)+\frac{1}{n}\left(W_n''(t,x)-W_n''(t,2x)\right)
\end{aligned}\tag{89}$$

Given the random function $W_n(t,x)$ we define the random characteristic curve $\xi_n(t)$ similarly to (85):

$$\dot{\xi}_n(t) = W_n(t,\xi_n(t)), \qquad \xi_n(0) := \frac{1}{K}\tag{90}$$

This ODE is well-defined although $W_n(t,x)$ is not continuous in t, but almost surely it is a step function with finitely many steps which is a sufficient condition to have well-posedness for the solution of (90). Define $u_n(t) := U_n(t,\xi_n(t)) - W_n(0,\frac{1}{K})$. Thus $u_n(0)=0$ and

$$u_n(t) = W_n(t,\xi_n(t)) - W_n(0,\frac{1}{K}) - r_n(t)e^{-\xi_n(t)} = V_n(t,\xi_n(t)) - V_n(0,\frac{1}{K}) - r_n(t)e^{-\xi_n(t)}\tag{91}$$

The solution of (90) is

$$\xi_n(t) = \frac{1}{K} + \int_0^t u_n(s)\,ds + \int_0^t r_n(s)e^{-\xi_n(s)}ds + tW_n(0,\frac{1}{K})\tag{92}$$

Putting together (89) and (90) similarly to (86) and using (56) we get

$$L\,u_n(t) \leq \frac{1}{n}\left(W_n''(t,\xi_n(t)) - W_n''(t,2\xi_n(t))\right) \leq n^{-1}\cdot\xi_n(t)^{-2}\tag{93}$$

Now $\widetilde{u}_n(t) = u_n(t) - \int_0^t L\,u_n(s)ds$ is a martingale and

$$\begin{aligned}
L\,\widetilde{u}_n(t)^2 &= \lim_{h\to 0_+}\frac{1}{h}\mathbf{E}\big(\left(U_n(t+h,\xi_n(t)) - U_n(t,\xi_n(t))\right)^2\,\big|\,\mathscr{F}_t\big) \leq \\
&\frac{1}{2}\sum_{k,l=1}^{n}\left(\frac{k+l}{n}e^{-(k+l)\xi_n(t)} - \frac{k}{n}e^{-k\xi_n(t)} - \frac{l}{n}e^{-l\xi_n(t)}\right)^2 v_{n,k}(t)v_{n,l}(t)n \\
&+\sum_{l=1}^{n}\left(\frac{l}{n}e^{-l\xi_n(t)}\right)^2\lambda(n)v_{n,l}(t)n = \mathscr{O}\left(\frac{1}{n}W_n''(t,\xi_n(t))\right) = \mathscr{O}\left(n^{-1}\cdot\xi_n(t)^{-2}\right)
\end{aligned}\tag{94}$$

Define the stopping time

$$\tau_n := \min\{t : \xi_n(t) = n^{-\alpha}\} \quad \alpha = 1/3.$$

In fact any $0 < \alpha < 1/2$ would be just as good to make the right-hand side of (93) and (94) disappear when $t \le \tau_n$ and $n \to \infty$.

It follows from (94) and Doob's maximal inequality that

$$\sup_t \left|\widetilde{u}_n(t \wedge \tau_n \wedge T)\right| \Rightarrow 0 \quad \text{as} \quad n \to \infty$$

By (93) we have $\widetilde{u}_n(t) + \int_0^t n^{-1} \cdot \xi_n(s)^{-2}\, ds \ge u_n(t)$ thus

$$\sup_t u_n(t \wedge \tau_n \wedge T) \Rightarrow 0 \quad \text{as} \quad n \to \infty \tag{95}$$

By (80) and (82) we have

$$V_n(0, \frac{1}{K}) \le (e^{-1} - 1)\varepsilon_2 =: -\varepsilon_5 \tag{96}$$

Define the events A_n, B_n and the time $\bar{t}_n$ by

$$A_n := \{ \sup_{t \le \tau_n \wedge T} \int_0^t u_n(s) ds \le \frac{1}{K}\} \cap \{u_n(\tau_n \wedge T) \le \varepsilon_5/3\},$$

$$B_n := \{r_n(\tau_n) \le \varepsilon_5/3\},$$

$$\bar{t}_n := \frac{3}{K\left|W_n(0, \xi_n(0))\right|} \le \frac{3}{K\varepsilon_5},$$

We are going to show that that there are constants $C_2, C_3 < +\infty$ such that

$$A_n \subseteq \{ \sum_{k=C_3\varepsilon_2 n^{1/3}}^{n} v_{n,k}(\bar{t}) + r_n(\bar{t}) \ge C_2\varepsilon_2\} \tag{97}$$

which, since (95) implies that $\lim_{n\to\infty} \mathbf{P}(A_n) = 1$, gives (83).

First we show that

$$A_n \cap B_n \subseteq \{\tau_n \le \bar{t}_n\}. \tag{98}$$

If we assume indirectly that A_n, B_n and $\tau_n > \bar{t}_n$ hold then $\int_0^{\bar{t}_n} u_n(s)ds \le \frac{1}{K}$, so by (92) we get

$$\xi_n(\bar{t}_n) \le \frac{1}{K} + \frac{1}{K} + \int_0^{\bar{t}_n} r_n(s) e^{-\xi_n(s)} ds + \bar{t}_n W_n(0, \xi_n(0)) \le -\frac{1}{K} + \bar{t}_n \cdot r_n(\tau_n) \le 0.$$

But $\xi_n(\bar{t}_n) \le 0$ is in contradiction with $\tau_n > \bar{t}_n$, thus (98) holds.

Now, by (91) we have $V_n(\tau_n, n^{-1/3}) = u_n(\tau_n) + V_n(0, \frac{1}{K}) + r_n(\tau_n) e^{-n^{-1/3}}$. Thus by (96), the definition of A_n and B_n and (81) we get

$$A_n \cap B_n \subseteq \{u_n(\tau_n) \le \frac{\varepsilon_5}{3}\} \cap \{V_n(0, \frac{1}{K}) \le -\varepsilon_5\} \cap \{r_n(\tau_n) e^{-n^{-1/3}} \le \frac{\varepsilon_5}{3}\} \subseteq$$
$$\{V_n(\tau_n, n^{-1/3}) \le \frac{-\varepsilon_5}{3}\} \subseteq \{ \sum_{k=n^{1/3}\varepsilon_5/6}^{n} v_{n,k}(\tau_n) \ge \varepsilon_5/12\}$$

Thus we have

$$A_n \subseteq (A_n \cap B_n) \cup B_n^c \subseteq \{ \sum_{k=n^{1/3}\varepsilon_5/6}^{n} v_{n,k}(\tau_n) \geq \varepsilon_5/12\} \cup \{r_n(\tau_n) > \varepsilon_5/3\} \subseteq$$

$$\{ \sum_{k=C_3\varepsilon_2 n^{1/3}}^{n} v_{n,k}(\tau_n) + r_n(\tau_n) \geq C_2\varepsilon_2\}$$

with $C_3 = (1-e^{-1})/6$ and $C_2 = (1-e^{-1})/12$. But $\sum_{k=C_3\varepsilon_2 n^{1/3}}^{n} v_{n,k}(t) + r_n(t)$ increases with time, from which (97) follows. $\square$

Lemma 9. *There are constants $C_4 < +\infty$, $C_5 > 0$ such that if*

$$\sum_{k=C_3\varepsilon_2 n^{1/3}}^{n} v_{n,k}(0) \geq C_2\varepsilon_2/2$$

for all n then with

$$\bar{t}_n := C_4\varepsilon_2^{-2}\left(n^{-1/3}\log(n) + (n\lambda(n))^{-1}\right) \tag{99}$$

we have

$$\lim_{n\to\infty} \mathbf{E}\left(r_n(\bar{t}_n)\right) \geq C_5\varepsilon_2. \tag{100}$$

Remark. *The upper bound* (99) *is technical: on one hand it is not optimal, on the other hand, for the proof of Lemma 7 we only need $\bar{t}_n \ll 1$ as $n \to \infty$.*

Proof. If v is a vertex of the graph $G(n,t)$ let $\mathscr{C}_n(v,t)$ denote the connected component of v at time t. Denote by $\tau_b(v)$ the first burning time of v:

$$\tau_b(v) := \inf\{t \, : \, \left|\mathscr{C}_n(v,t_+)\right| < \left|\mathscr{C}_n(v,t_-)\right|\}$$

Of course $\left|\mathscr{C}_n(v,\tau_b(v)_+)\right| = 1$. Define $\bar{n} := C_3\varepsilon_2 n^{1/3}$ and

$$\mathscr{H}_n(t) := \{v \, : \, \left|\mathscr{C}_n(v,0)\right| \geq \bar{n} \text{ and } \tau_b(v) > t\}$$

Fix a vertex $v \in \mathscr{H}_n(0)$.

$$\begin{aligned}
c_n(t) &:= \frac{1}{n}\left|\mathscr{C}_n(v,(t\wedge\tau_b(v))_-)\right| \\
w_n(t) &:= \frac{1}{n}\left|\mathscr{H}_n(t)\right| \\
z_n(t) &:= \frac{1}{n}\sum_{w\in\mathscr{H}_n(0)} \mathbb{1}_{\{\tau_b(w)\leq t\}} = w_n(0) - w_n(t)
\end{aligned}$$

Thus $c_n(t)$ is an increasing process (we "freeze" $c_n(t)$ when it burns). We consider the right-continuous versions of the processes $c_n(t), w_n(t), z_n(t)$.

$$w_n(0) \geq C_2\varepsilon_2/2 =: \varepsilon_6.$$

We are going to prove that there are constants $C_4 < +\infty$, $C_5 > 0$ such that

$$\lim_{n\to\infty} \mathbf{E}\big(z_n(\bar{t}_n) \big) \geq C_5 \varepsilon_2 \tag{101}$$

which implies (100).

Define the stopping times

$$\begin{aligned}\tau_w &:= \inf\{t : w_n(t) < \varepsilon_6/2\} \\ \tau_g &:= \inf\{t : c_n(t) > \varepsilon_6/4\} \\ \tau &:= \tau_b(v) \wedge \tau_w \wedge \tau_g\end{aligned}$$

Since $v \in \mathscr{H}_n(0)$ we have

$$c_n(t) \geq c_n(0) = \frac{\left|\mathscr{C}_n(v,0)\right|}{n} \geq \frac{\bar{n}}{n}$$

If $\mathscr{C}_n(v,t)$ is connected to a vertex in $\mathscr{H}_n(t)$ by a new edge at time t then

$$c_n(t_+) - c_n(t_-) \geq \frac{\bar{n}}{n}, \quad \log(c_n(t_+)) - \log(c_n(t_-)) \geq \log\left(1 + \frac{\bar{n}}{nc_n(t_-)}\right) \geq \frac{\log(2)\bar{n}}{nc_n(t_-)}$$

$$\begin{aligned}&L\,\log(c_n(t)) \geq \frac{\log(2)\bar{n}}{nc_n(t)} \lim_{dt\to 0} \frac{1}{dt}\mathbf{P}\Big(c_n(t+dt) - c_n(t) \geq \frac{\bar{n}}{n} \,\Big|\, \mathscr{F}_t \Big) \geq \\ &\frac{\log(2)\bar{n}}{nc_n(t)} \cdot \frac{1}{n} \left|\mathscr{C}_n(v,t)\right| \Big(\left|\mathscr{H}_n(t)\right| - \left|\mathscr{C}_n(v,t)\right| \Big) \mathbb{1}_{\{t\ \leq \tau_b(v)\}} \geq \log(2)\bar{n} \cdot \big(w_n(t) - c_n(t) \big) \mathbb{1}_{\{t\ \leq \tau_b(v)\}} \geq \\ &\qquad \log(2)\bar{n}\frac{\varepsilon_6}{4}\mathbb{1}_{\{t\leq\tau\}} = n^{1/3}\frac{\log(2)}{8} \cdot C_2 \cdot C_3 \cdot (\varepsilon_2)^2 \cdot \mathbb{1}_{\{t\leq\tau\}} =: n^{1/3}\varepsilon_7\mathbb{1}_{\{t\leq\tau\}}\end{aligned}$$

Thus $\log(c_n(t)) - \varepsilon_7 \cdot n^{1/3}(t \wedge \tau)$ is a submartingale. Using the optional sampling theorem we get

$$-\varepsilon_7 \cdot n^{1/3}\mathbf{E}\big(\tau \big) \geq \mathbf{E}\big(\log(c_n(\tau)) \big) - \varepsilon_7 \cdot n^{1/3}\mathbf{E}\big(\tau \big) \geq \log(c_n(0)) \geq -\log(n)$$

By Markov's inequality we obtain that for some constant $C < +\infty$

$$\mathbf{P}\big(\tau \leq Cn^{-1/3}\varepsilon_2^{-2}\log(n) \big) \geq \frac{1}{2}$$

If $\tau_g \leq \tau_b(v) \wedge \tau_w$, then $\mathscr{C}_n(v,\tau_g) > \frac{\varepsilon_6}{4}n$, so $\mathbf{E}\big(\tau_b(v) - \tau_g \big) \leq (n\lambda(n))^{-1}\frac{4}{\varepsilon_6}$, which implies

$$\mathbf{P}\big(\tau_w \wedge \tau_b \leq Cn^{-1/3}\varepsilon_2^{-2}\log(n) + C'(n\lambda(n))^{-1}\varepsilon_2^{-1} \big) \geq \frac{1}{4}.$$

for some constant C'. We define $\bar{t}$ of (99) with $C_4 := \max\{C, C'\}$. Using the linearity of expectation we get

$$\mathbf{E}\big(z_n(\bar{t}) \big) = \mathbf{E}\Big(\frac{1}{n} \sum_{w\in\mathscr{H}_n(0)} \mathbb{1}_{\{\tau_b(w)\leq\bar{t}\}} \Big) \geq \varepsilon_6\mathbf{P}\big(\tau_b(v) \leq \bar{t} \big).$$

The inequality $\mathbb{1}_{\{\tau_w\leq\bar{t}\}}\frac{\varepsilon_6}{2} \leq z_n(\bar{t})$ follows from the definition of τ_w.

$$\frac{1}{4} \leq \mathbf{P}\big(\tau_w \wedge \tau_b \leq \bar{t} \big) \leq \mathbf{P}\big(\tau_w \leq \bar{t} \big) + \mathbf{P}\big(\tau_b \leq \bar{t} \big) \leq \mathbf{E}\big(z_n(\bar{t}) \big)\frac{2}{\varepsilon_6} + \mathbf{E}\big(z_n(\bar{t}) \big)\frac{1}{\varepsilon_6}$$

From this (101) follows. □

4 The critical equation

4.1 Elementary properties

Existence to the solutions of (37), (39) with initial condition satisfying $m_2(0) < +\infty$ and boundary condition

$$\sum_{k=1}^{\infty} v_k(t) \equiv 1 \tag{102}$$

follows as corollary to Propositions 1 and 2: indeed for any initial condition $\mathbf{v}_0 \in \mathcal{V}_1$ we can prepare a sequence of initial conditions of the random graph problem such that (32) holds as $n \to \infty$ (we do not need to assume convergence of $m_{n,2}(0)$ to $m_2(0)$). If $n^{-1} \ll \lambda(n) \ll 1$ then any weak limit of the probability measures $\mathbb{P}_n$ is concentrated on a subset of FFFs which generate a FFE satisfying (37), (102).

Moreover it is easily seen that (102) implies that $r(\cdot)$ must be continuous, and for $k \geq 2$, the functions $t \mapsto v_k(t)$ solving (37) are differentiable. Thus $\mathbf{v}(\cdot)$ solves (12), (13).

Note that assuming that $\mathbf{v}(\cdot) \in \mathscr{E}_{\mathbf{v}_0}[0,T]$ is a solution of (12),(13) one can deduce only from these equations that (37) holds with a control function $r(\cdot)$ satisfying (40): one has to define a FFF using (33) and $q_{k,\infty}(\cdot) \equiv 0$: plugging $\theta(t) \equiv 0$ into (21) we can see that the function $r(\cdot)$ is increasing.

Taking the generating function of a solution of (37), (39), (102) with initial condition satisfying $m_2(0) < +\infty$ we get a solution of (43), (45) satisfying the boundary condition $V(t,0) \equiv 0$.

In this case the increasing function $t \mapsto r(t)$ is absolutely continuous with respect to Lebesgue measure: its Radon-Nykodim derivative $\dot{r}(t) = \varphi(t)$ is a.e. bounded in compact domains:

Taking the limit $x \to 0$ in (43) and using (71), (58) (which holds because $V(t,0) \equiv 0$) we find

$$r(t_2) - r(t_1) = \lim_{x\to 0} \frac{1}{2} \int_{t_1}^{t_2} V(s,x)V'(s,x)ds \leq C \cdot (t_2 - t_1). \tag{103}$$

Thus in the sequel we assume given a solution of the *critical Burgers control problem*

$$\dot{V}(t,x) = -V'(t,x)V(t,x) + e^{-x}\varphi(t), \tag{104}$$

$$V(t,0) \equiv 0 \tag{105}$$

$$V(0,x) = V_0(x) \tag{106}$$

where $\varphi(t)$ is nonnegative and bounded on $[0,T]$, and $V(t,x)$ is of the form (41).

Lemma 10. *For any solution of* (104), (106), (105) *with* $V''(0) < +\infty$ *and for any* $t \geq T_{gel}$ *(see* (8)*) we have* $V'(t,0) := \lim_{x\to 0} V'(t,x) = -\infty$.

Proof. We actually prove that for any $\overline{t} < \infty$, $\overline{x} < \infty$ there exists a constant $C = C(\overline{t}, \overline{x}) > 0$ such that for any $(t,x) \in [T_{\text{gel}}, \overline{t}] \times (0, \overline{x}]$, $-V'(t,x) \geq C/\sqrt{x}$.

One can prove the upper bound of (60) for all $V(x)$ satisfying $V(0) = 0$ without the assumption (59) (the same proof works).

From (71) and the upper bound of (60) it follows that there exists a constant $\widetilde{C} < \infty$ such that for $(t,x) \in [T_{\text{gel}}, \overline{t}] \times (0, \overline{x}]$

$$E(t,x)^{-1} \leq \widetilde{C}, \qquad\qquad -V(t,x) \leq \widetilde{C} x^{1/2}.$$

Differentiating with respect to x in (104) we get

$$\begin{aligned} \frac{d}{dt}\left(-V'(t,x)\right) &= V'(t,x)^2 + V(t,x)V''(t,x) + e^{-x}\varphi(t) = \\ V'(t,x)^2 \cdot \left(1 - \frac{V(t,x)V'(t,x)}{E(t,x)}\right) &+ e^{-x}\varphi(t) \geq V'(t,x)^2 \left(1 - \widetilde{C}^2 x^{1/2} \cdot \left(-V'(t,x)\right)\right) \end{aligned} \tag{107}$$

There exists a $0 < \widehat{C}$ such that for $x \in (0, \overline{x}]$ we have

$$-V'(T_{\text{gel}}, x) \geq \widehat{C}/\sqrt{x} \tag{108}$$

by (61) and (71), since $V'(T_{\text{gel}}, 0) = -\infty \iff m_1(T_{\text{gel}}) = +\infty$ follows from the fact that for $t \leq T_{\text{gel}}$ the solutions of (6) and (12)+(13) coincide, and it is well-known from the theory of the Smoluchowski coagulation equations that we have (9) for the solution of (6).

From the differential inequality (107) it follows that

$$-V'(t,x) \leq \frac{1}{\widetilde{C}} x^{-1/2} \qquad \Longrightarrow \qquad \frac{d}{dt}\left(-V'(t,x)\right) \geq 0 \tag{109}$$

Let $C := \min\{\widehat{C}, \widetilde{C}^{-1}\}$. For $(t,x) \in [T_{\text{gel}}, \overline{t}] \times (0, \overline{x}]$ the inequality

$$-V'(t,x) \geq C/\sqrt{x}.$$

follows from (108) and (109) by a "forbidden region"-argument.

□

Summarizing: from Lemmas 3, 4, 10 and (103) it follows

Lemma 11. *For $(t,x) \in [T_{gel}, \overline{t}] \times (0, \overline{x}]$*

$$-V(t,x) \asymp x^{1/2}, \tag{110}$$

$$-V'(t,x) \asymp x^{-1/2}, \tag{111}$$

$$V''(t,x) \asymp x^{-3/2}, \tag{112}$$

$$V(t,x)V'(t,x) \asymp 1, \tag{113}$$

$$\varphi(t) \asymp 1. \tag{114}$$

4.2 Bounds on E'

In this subsection we assume given a solution of (104), (105), (106) satisfying $\left|V'''(0,0)\right| < +\infty$. All of the results of the previous subsection are valid for $V(t,x)$.

Lemma 12.

$$E'(T_{gel}, x) = \mathcal{O}(x^{-1/2}) \tag{115}$$

Proof. We consider the function $X(t,u)$ defined for every t as in the proof of Lemma 3. $X'''(0,u) = \mathcal{O}(1)$ for $u \in [0,\bar{u}]$ by $m_1(0) > 0$ and $m_3(0) < +\infty$. For $t \leq T_{\rm gel}$ we have $\varphi(t) \equiv 0$ thus $V(t,x)$ satisfies the Burgers equation

$$\dot{V}(t,x) + V(t,x)V'(t,x) = 0$$

from which

$$X(t,u) = X(0,u) - tu$$

follows. Differentiating (63) with respect to x we get

$$E'(T_{\rm gel}, x) = E(T_{\rm gel}, x)^2 X'''(0, -V(T_{\rm gel}, x))V'(T_{\rm gel}, x).$$

Now (115) follows from (71) and (61). ☐

From now on, we consider the solution of (104), (105), (106) for $t \geq T_{\rm gel}$, that is we assume that $T_{\rm gel} = 0$.

Since the function $r(t)$ is continuous we get that $t(\tau)$ defined by (46) is the inverse function of $t + r(t)$ which by (48) implies $\mathbf{V}(\tau, x) \equiv V(t(\tau), x)$. Integrating (70) and using (53), (55) we get for $0 \leq t_1 \leq t_2 < \infty$

$$E(t_2,x) = E(t_1,\xi_{t_2,x}(t_1)) + \int_{t_1}^{t_2} \{3\frac{V'(s,\xi_{t_2,x}(s))^2}{V''(s,\xi_{t_2,x}(s))} + \frac{V'(s,\xi_{t_2,x}(s))^3}{V''(s,\xi_{t_2,x}(s))^2}\}e^{-\xi_{t_2,x}(s)}\varphi(s)\,ds \tag{116}$$

$$= E(t_1,\xi_{t_2,x}(t_1)) + \int_{t_1}^{t_2} \{-3\frac{E(s,\xi_{t_2,x}(s))}{V'(s,\xi_{t_2,x}(s))} + \frac{E(s,\xi_{t_2,x}(s))^2}{V'(s,\xi_{t_2,x}(s))^3}\}e^{-\xi_{t_2,x}(s)}\varphi(s)\,ds. \tag{117}$$

Lemma 13. *The function $(t,x) \mapsto E(t,x)$ is continuous on the domain $(t,x) \in [0,\overline{t}] \times [0,\overline{x}]$, and*

$$\varphi(t) = \lim_{x \to 0} V'(t,x)V(t,x) = E(t,0). \tag{118}$$

Proof. From (114) and (65) it follows that the characteristic curves $\xi_{t,x}(s)$ are jointly continuous in the variables $\{(t,x,s) : 0 \leq s \leq t,\ 0 \leq x\}$. And hence, further on, from (116) and (72), by dominated convergence it follows that $(t,x) \mapsto E(t,x)$ is jointly continuous in $\{(t,x) : 0 \leq t,\ 0 \leq x\}$. Further, from (62) it follows that

$$\lim_{x \to 0} V(t,x)V'(t,x) = \lim_{x \to 0} E(t,x) =: E(t,0)$$

Hence, (118) follows from (103) again by dominated convergence. ☐

Lemma 14.

(i) The function $x \mapsto E(t,x)$ is Hölder-1/2 at $x \to 0$:

$$E(t,x) = \varphi(t)\big(1 + \mathcal{O}(x^{1/2})\big). \tag{119}$$

(ii) The function $t \mapsto \varphi(t)$ is Lipschitz continuous: there exists a constant $C < \infty$ (which depends only on the initial conditions (106) *and the choice of $\bar{t}$ such that for any $t_1, t_2 \in [0, \bar{t}]$*

$$|\varphi(t_1) - \varphi(t_2)| \le C|t_1 - t_2|. \tag{120}$$

Proof. (i) We prove $|E'(t,x)| = \mathcal{O}(x^{-1/2})$. In this order we shall use the following a priori estimates

$$\xi_{t,x}(s) \asymp \big(x^{1/2} + (t-s)\big)^2 \tag{121}$$

$$\xi'_{t,x}(s) := \partial_x \xi_{t,x}(s) = \mathcal{O}\left(\big(x^{1/2} + (t-s)\big)x^{-1/2}\right). \tag{122}$$

Indeed: (121) follows from (65), (110) and (114), and we get (122) from (111) and from the fact that characteristics do not intersect (thus $0 \le \xi'_{t,x}(s)$) by differentiating (65) w.r.t. x:

$$0 \le \xi'_{t,x}(s) \le 1 - V'(t,x)(t-s)$$

The a priori bound

$$|E'(t,x)| = \mathcal{O}(x^{-1}). \tag{123}$$

follows from

$$E'(t,x) = -3V'(t,x)^2 + E(t,x)\frac{-V'''(t,x)}{V''(t,x)} = \mathcal{O}((x^{-1/2})^2) + \mathcal{O}(x^{-1})$$

by (111), (71) and

$$-\frac{x}{2}V'''(t,x) \le \int_{\frac{x}{2}}^{x} V'''(y)dy \le V''(\frac{x}{2}) = \mathcal{O}(x^{-3/2})$$

using both the upper and lower bounds of (112).

Differentiating with respect to x in (117) yields

$$\begin{aligned} E'(t,x) = {} & E'(0,\xi_{t,x}(0))\xi'_{t,x}(0) + \\ & + \int_0^t \Big\{ -3\frac{E'(s,\xi_{t,x}(s))}{V'(s,\xi_{t,x}(s))} + 3\frac{E(s,\xi_{t,x}(s))V''(s,\xi_{t,x}(s))}{V'(s,\xi_{t,x}(s))^2} \\ & + 2\frac{E(s,\xi_{t,x}(s))E'(s,\xi_{t,x}(s))}{V'(s,\xi_{t,x}(s))^3} - 3\frac{E(s,\xi_{t,x}(s))^2V''(s,\xi_{t,x}(s))}{V'(s,\xi_{t,x}(s))^4} \\ & + 3\frac{E(s,\xi_{t,x}(s))}{V'(s,\xi_{t,x}(s))} - \frac{E(s,\xi_{t,x}(s))^2}{V'(s,\xi_{t,x}(s))^3}\Big\}\xi'_{t,x}(s)e^{-\xi_{t,x}(s)}\varphi(s)ds. \end{aligned} \tag{124}$$

Next using (123) bound we estimate the expression of $E'(t,x)$ given in (124). Using (71), (111), (112), (115), (121), and (122) we conclude that if (123) holds then actually

$$|E'(t,x)| = \mathcal{O}(x^{-1/2}). \tag{125}$$

The dominating order is given by the first term (outside the integral) and the first two terms under the integral on the right hand side of (124).

Finally, (119) follows from (118) and (125).

(ii) In order to prove (120) we note that from (116) and (118) it follows that for $0 \leq t_1 \leq t_2 \leq \overline{t}$

$$\varphi(t_1) - \varphi(t_2) = E(t_1, 0) - E(t_1, \xi_{t_2,0}(t_1)) \\ - \int_{t_1}^{t_2} \{3\frac{V'(s,\xi_{t_2,0}(s))^2}{V''(s,\xi_{t_2,0}(s))} + \frac{V'(s,\xi_{t_2,0}(s))^3}{V''(s,\xi_{t_2,0}(s))^2}\} e^{-\xi_{t_2,0}(s)} \varphi(s) ds$$

Hence, by (119), (121) and (72) we obtain directly (120). □

Summarizing again, from Lemmas 3, 4, 10, 13 and 14 it follows

Proposition 3. *For a solution of* (104)*,* (106)*,* (105) *with initial condition satisfying* $T_{gel} = 0$*,* (71) *and* (115) *and for* $(t,x) \in [0,\overline{t}] \times (0,\overline{x}]$

$$-V(t,x) = \sqrt{2\varphi(t)}x^{1/2}\big(1 + \mathcal{O}(x^{1/2})\big), \tag{126}$$

$$-V'(t,x) = \sqrt{\frac{\varphi(t)}{2}}x^{-1/2}\big(1 + \mathcal{O}(x^{1/2})\big), \tag{127}$$

$$V''(t,x) = \sqrt{\frac{\varphi(t)}{8}}x^{-3/2}\big(1 + \mathcal{O}(x^{1/2})\big), \tag{128}$$

$$V(t,x)V'(t,x) = \varphi(t)\big(1 + \mathcal{O}(x^{1/2})\big). \tag{129}$$

$$\dot{V}(t,x) = \mathcal{O}(x^{1/2}), \tag{130}$$

$$\dot{V}'(t,x) = \mathcal{O}(x^{-1/2}), \tag{131}$$

$$\varphi(t) \asymp 1, \qquad |\varphi(t_1) - \varphi(t_2)| \leq C|t_1 - t_2|. \tag{132}$$

In order to prove (14) we need Example (c) of Theorem 4. of chapter XIII.5 of [7]. With our notations each of the relations

$$-V(t,x) \sim x^{1-1/2}\sqrt{2\varphi(t)} \quad \text{and} \quad \sum_{l=k}^{\infty} v_l(t) \sim \frac{1}{\Gamma(\frac{1}{2})}k^{1/2-1}\sqrt{2\varphi(t)}$$

implies the other.

4.3 Uniqueness

We are going to prove Theorem 1. by proving the uniqueness of (104), (106), (105).

Proof of Theorem 1. Assume that $V(t,x)$ and $U(t,x)$ are two solutions of the critical Burgers control problem with the same initial conditions and with the control functions $\varphi(t)$ and $\psi(t)$, respectively. Denote

$$S(t,x) := \frac{V(t,x)+U(t,x)}{2}, \qquad \sigma(t) := \frac{\varphi(t)+\psi(t)}{2}, \qquad \sqrt{\varrho(t)} := \frac{\sqrt{\varphi(t)}+\sqrt{\psi(t)}}{2} \tag{133}$$

$$W(t,x) := \frac{V(t,x)-U(t,x)}{2}, \qquad \delta(t) := \frac{\varphi(t)-\psi(t)}{2}. \tag{134}$$

Then, it is easily seen that that (given $S(t,x)$) $W(t,x)$, $\delta(t)$ will solve the linear control problem

$$\dot{W}(t,x) + \big(S(t,x)W(t,x)\big)' = e^{-x}\delta(t), \tag{135}$$

$$W(0,x) \equiv 0, \tag{136}$$

$$W(t,0) \equiv 0. \tag{137}$$

We assume $S(t,x)$ and $\rho(t)$ given, with the regularity properties inherited from Proposition 3:

$$-S(t,x) = \sqrt{2\rho(t)}x^{1/2}\big(1+\mathcal{O}(x^{1/2})\big), \tag{138}$$

$$-S'(t,x) = \sqrt{\frac{\rho(t)}{2}}x^{-1/2}\big(1+\mathcal{O}(x^{1/2})\big), \tag{139}$$

$$S''(t,x) = \sqrt{\frac{\rho(t)}{8}}x^{-3/2}\big(1+\mathcal{O}(x^{1/2})\big), \tag{140}$$

$$S(t,x)S'(t,x) = \rho(t)\big(1+\mathcal{O}(x^{1/2})\big). \tag{141}$$

$$\dot{S}(t,x) = \mathcal{O}(x^{1/2}), \tag{142}$$

$$\dot{S}'(t,x) = \mathcal{O}(x^{-1/2}), \tag{143}$$

$$\rho(t) \asymp 1, \qquad |\rho(t_1)-\rho(t_2)| \leq C|t_1-t_2|. \tag{144}$$

We will prove that under these conditions, the unique solution of the problem (135), (136), (137) is $W(t,x)\equiv 0$, $\delta(t)\equiv 0$.

First we define the characteristics of the equation (135): these are the curves $[0,t] \ni s \mapsto \zeta_t(s)$ defined by the ODE

$$\dot{\zeta}_t(s) = S(s,\zeta_t(s)), \qquad \zeta_t(t) = 0, \qquad \zeta_t(s) > 0 \text{ for } s < t. \tag{145}$$

Next we define the functions $[0,t] \ni s \mapsto \beta_t(s)$

$$\beta_t(s) := S'(s,\zeta_t(s)).$$

The functions $[0,t] \ni s \mapsto \zeta_t(s)$ and $[0,t] \ni s \mapsto \beta_t(s)$ are directly determined by $S(t,x)$ and from (138), (139), (140) and (144) inherit the following regularity properties to be used later:

$$\zeta_t(s) = \quad \frac{\rho(t)}{2}(t-s)^2\big(1+\mathcal{O}(t-s)\big), \tag{146}$$

$$\dot{\zeta}_t(s) = -\rho(t)(t-s)\big(1+\mathcal{O}(t-s)\big), \tag{147}$$

$$\ddot{\zeta}_t(s) = \quad \rho(t)\big(1+\mathcal{O}(t-s)\big), \tag{148}$$

$$\beta_t(s) = -(t-s)^{-1}\big(1+\mathcal{O}(t-s)\big), \tag{149}$$

$$\dot{\beta}_t(s) = -(t-s)^{-2}\big(1+\mathcal{O}(t-s)\big). \tag{150}$$

We define $[0,t] \ni s \mapsto \eta_t(s)$ as

$$\eta_t(s) := W(s, \zeta_t(s)),$$

with $W(t,x)$ given in (134) being solution of (135), (136), (137). Then, for any $t \geq 0$, $\delta(s), \eta_t(s)$, $s \in [0,t]$ solves the ODE (boundary value) control problem

$$\dot{\eta}_t(s) + \beta_t(s)\eta_t(s) = e^{-\zeta_t(s)}\delta(s), \qquad \eta_t(0) = 0 = \eta_t(t) \tag{151}$$

We will prove that this implies $\delta(t) \equiv 0$. Hence it follows that $W(t,x) \equiv 0$.

On the domain $\{(t,s) : 0 \leq s \leq t < \infty\}$ we define the integral kernel

$$\mathcal{K}(t,s) := \exp\Big\{\int_0^s \beta_t(u)du - \zeta_t(s)\Big\} = \frac{t-s}{t}\mathcal{L}(t,s),$$

defined on the same domain $\{(t,s) : 0 \leq s \leq t < \infty\}$, where

$$\mathcal{L}(t,s) := \exp\Big\{\int_0^s \big(\beta_t(u) + (t-u)^{-1}\big)du - \zeta_t(s)\Big\}.$$

The ODE control problem (151) is equivalent to

$$\int_0^t \mathcal{K}(t,s)\delta(s)ds = 0. \tag{152}$$

It is handy to introduce the function

$$\gamma(t) := \int_0^t \delta(s)(t-s)ds.$$

Then, after two integrations by parts the identity (152) is transformed into the eigenvalue problem

$$\int_0^t \widehat{\mathcal{K}}(t,s)\gamma(s)ds = \gamma(t), \tag{153}$$

where

$$\widehat{\mathcal{K}}(t,s) := \big(\partial_s\mathcal{K}(t,t)\big)^{-1}\partial^2_{ss}\mathcal{K}(t,s) = \frac{2\partial_s\mathcal{L}(t,s) - (t-s)\partial^2_{ss}\mathcal{L}(t,s)}{\mathcal{L}(t,t)}.$$

Using the regularity properties (146), (147), (148), (149), (150) it follows that

$$\sup_{0\le s<t\le \bar{t}} \left|\widehat{\mathscr{K}}(t,s)\right| < \infty. \tag{154}$$

From (153) and (154), by a Grönwall argument we get $\gamma(t) \equiv 0$ and hence $\delta(t) \equiv 0 \equiv W(t,x)$, which proves uniqueness of the solution of (104), (106), (105).

□

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