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Wiener Process with Reflection in Non-Smooth Narrow Tubes

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Abstract

Wiener process with instantaneous reflection in narrow tubes of width $\epsilon \ll 1$ around axis x is considered in this paper. The tube is assumed to be (asymptotically) non-smooth in the following sense. Let $V^{\epsilon}(x)$ be the volume of the cross-section of the tube. We assume that $\frac{1}{\epsilon}V^{\epsilon}(x)$ converges in an appropriate sense to a non-smooth function as $\epsilon \downarrow 0$. This limiting function can be composed by smooth functions, step functions and also the Dirac delta distribution. Under this assumption we prove that the x-component of the Wiener process converges weakly to a Markov process that behaves like a standard diffusion process away from the points of discontinuity and has to satisfy certain gluing conditions at the points of discontinuity.

Key words: Narrow Tubes, Wiener Process, Reflection, Non-smooth Boundary, Gluing Conditions, Delay.

AMS 2000 Subject Classification: Primary 60J60, 60J99, 37A50.

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1 Introduction

For each $x \in \mathbb{R}$ and $0 < \epsilon << 1$, let D_x^ϵ be a bounded interval in $\mathbb{R}$ that contains 0. To be more specific, let $D_x^\epsilon = [-V^{l,\epsilon}(x), V^{u,\epsilon}(x)]$, where $V^{l,\epsilon}(x), V^{u,\epsilon}(x)$ are sufficiently smooth, nonnegative functions, where at least one of the two is a strictly positive function. Consider the state space $D^\epsilon = \{(x,y) : x \in \mathbb{R}, y \in D_x^\epsilon\} \subset \mathbb{R}^2$. Assume that the boundary ∂D^ϵ of D^ϵ is smooth enough and denote by $\gamma^\epsilon(x,y)$ the inward unit normal to ∂D^ϵ. Assume that $\gamma^\epsilon(x,y)$ is not parallel to the x-axis.

Denote by $V^\epsilon(x) = V^{l,\epsilon}(x) + V^{u,\epsilon}(x)$ the length of the cross-section D_x^ϵ of the stripe. We assume that D^ϵ is a narrow stripe for $0 < \epsilon << 1$, i.e. $V^\epsilon(x) \downarrow 0$ as $\epsilon \downarrow 0$. In addition, we assume that $\frac{1}{\epsilon}V^\epsilon(x)$ converges in an appropriate sense to a non-smooth function, $V(x)$, as $\epsilon \downarrow 0$. The limiting function can be composed for example by smooth functions, step functions and also the Dirac delta distribution. Next, we state the problem and we rigorously introduce the assumptions on $V^\epsilon(x)$ and $V(x)$. At the end of this introduction we formulate the main result.

Consider the Wiener process $(X_t^\epsilon, Y_t^\epsilon)$ in D^ϵ with instantaneous normal reflection on the boundary of D^ϵ. Its trajectories can be described by the stochastic differential equations:

$$\begin{aligned} X_t^\epsilon &= x + W_t^1 + \int_0^t \gamma_1^\epsilon(X_s^\epsilon, Y_s^\epsilon) dL_s^\epsilon \\ Y_t^\epsilon &= y + W_t^2 + \int_0^t \gamma_2^\epsilon(X_s^\epsilon, Y_s^\epsilon) dL_s^\epsilon. \end{aligned} \tag{1}$$

Here W_t^1 and W_t^2 are independent Wiener processes in $\mathbb{R}$ and (x,y) is a point inside D^ϵ; γ_1^ϵ and γ_2^ϵ are both projections of the unit inward normal vector to ∂D^ϵ on the axis x and y respectively. Furthermore, L_t^ϵ is the local time for the process $(X_t^\epsilon, Y_t^\epsilon)$ on ∂D^ϵ, i.e. it is a continuous, non-decreasing process that increases only when $(X_t^\epsilon, Y_t^\epsilon) \in \partial D^\epsilon$ such that the Lebesgue measure $\Lambda\{t > 0 : (X_t^\epsilon, Y_t^\epsilon) \in \partial D^\epsilon\} = 0$ (eg. see [12]).

Our goal is to study the weak convergence of the $x-$component of the solution to (1) as $\epsilon \downarrow 0$. The $y-$component clearly converges to 0 as $\epsilon \downarrow 0$. The problem for narrow stripes with a smooth boundary was considered in [4] and in [5]. There, the authors consider the case $\frac{1}{\epsilon}V^\epsilon(x) = V(x)$, where $V(x)$ is a smooth function. It is proven that X_t^ϵ converges to a standard diffusion process X_t, as $\epsilon \downarrow 0$. More precisely, it is shown that for any $T > 0$

$$\sup_{0\le t\le T} E_x|X_t^\epsilon - X_t|^2 \to 0 \text{ as } \epsilon \to 0, \tag{2}$$

where X_t is the solution of the stochastic differential equation

$$X_t = x + W_t^1 + \int_0^t \frac{1}{2}\frac{V_x(X_s)}{V(X_s)} ds \tag{3}$$

and $V_x(x) = \frac{dV(x)}{dx}$.

In this paper we assume that $\frac{1}{\epsilon}V^\epsilon(x)$ converges to a non-smooth function as described by (5)-(9) below, as $\epsilon \downarrow 0$. Owing to the non smoothness of the limiting function, one cannot hope to obtain a limit in mean square sense to a standard diffusion process as before. In particular, as we will see, the non smoothness of the limiting function leads to the effect that the limiting diffusion may have points where the scale function is not differentiable (skew diffusion) and also points with positive speed measure (points with delay).

For any $\epsilon > 0$, we introduce the functions

$$u^\epsilon(x) := \int_0^x 2\frac{\epsilon}{V^\epsilon(y)}dy \quad \text{and} \quad v^\epsilon(x) := \int_0^x \frac{V^\epsilon(y)}{\epsilon}dy. \tag{4}$$

Now, we are in position to describe the limiting behavior of $\frac{1}{\epsilon}V^\epsilon(x)$.

1. We assume that $V^{l,\epsilon}, V^{u,\epsilon} \in \mathscr{C}^3(\mathbb{R})$ for every fixed $\epsilon > 0$ and that $V^\epsilon(x) \downarrow 0$ as $\epsilon \downarrow 0$ (in particular $V^0(x) = 0$). Moreover, there exists a universal positive constant ζ such that

$$\frac{1}{\epsilon}V^\epsilon(x) > \zeta > 0 \text{ for every } x \in \mathbb{R} \text{ and for every } \epsilon > 0. \tag{5}$$

2. We assume that the functions

$$\begin{aligned} u(x) &:= \lim_{\epsilon\downarrow 0} u^\epsilon(x),\ x \in \mathbb{R} \\ v(x) &:= \lim_{\epsilon\downarrow 0} v^\epsilon(x),\ x \in \mathbb{R}\setminus\{0\}, \end{aligned} \tag{6}$$

are well defined and the limiting function $u(x)$ is continuous and strictly increasing whereas the limiting function $v(x)$ is right continuous and strictly increasing. In general, the function $u(x)$ can have countable many points where it is not differentiable and the function $v(x)$ can have countable many points where it is not continuous or not differentiable. However, here we assume for brevity that the only non smoothness point is $x = 0$. In other words, we assume that for $x \in \mathbb{R}\setminus\{0\}$

$$V(x) = \frac{\partial V^\epsilon(x)}{\partial \epsilon}|_{\epsilon=0} > 0, \tag{7}$$

and that the function $V(x)$ is smooth for $x \in \mathbb{R}\setminus\{0\}$.

In addition, we assume that the first three derivatives of $V^{l,\epsilon}(x)$ and $V^{u,\epsilon}(x)$ (and in consequence of $V^\epsilon(x)$ as well) behave nicely for $|x| > 0$ and for ϵ small. In particular, we assume that for any connected subset K of $\mathbb{R}$ that is away from an arbitrarily small neighborhood of $x = 0$ and for ϵ sufficiently small

$$|V_x^{l,\epsilon}(x)| + |V_{xx}^{l,\epsilon}(x)| + |V_{xxx}^{l,\epsilon}(x)| + |V_x^{u,\epsilon}(x)| + |V_{xx}^{u,\epsilon}(x)| + |V_{xxx}^{u,\epsilon}(x)| \le C_0\epsilon \tag{8}$$

uniformly in $x \in K$. Here, C_0 is a constant.

After the proof of the main theorem (at the end of section 2), we mention the result for the case where there exist more than one non smoothness point.

3. Let $g^\epsilon(x)$ be a smooth function and let us define the quantity

$$\xi^\epsilon(g^\epsilon) := \sup_{|x|\le 1}[|\frac{1}{\epsilon}[g_x^\epsilon(x)]^3| + |g_x^\epsilon(x)g_{xx}^\epsilon(x)| + |\epsilon g_{xxx}^\epsilon(x)|]$$

We assume the following growth condition

$$\xi^\epsilon := \xi^\epsilon(V^\epsilon) + \xi^\epsilon(V^{l,\epsilon}) + \xi^\epsilon(V^{u,\epsilon}) \downarrow 0, \text{ as } \epsilon \downarrow 0. \tag{9}$$

Remark 1.1. *Condition (9), i.e.* $\xi^{\epsilon} \downarrow 0$, *basically says that the behavior of* $V^{l,\epsilon}(x)$ *and* $V^{u,\epsilon}(x)$ *in the neighborhood of* $x=0$ *can be at most equally bad as described by* $\xi^{\epsilon}(\cdot)$ *for* ϵ *small. This condition will be used in the proof of Lemma 2.4 in section 4. Lemma 2.4 is essential for the proof of our main result. In particular, it provides us with the estimate of the expectation of the time it takes for the solution to (1) to leave the neighborhood of the point* 0, *as* $\epsilon \downarrow 0$. *At the present moment, we do not know if this condition can be improved and this is subject to further research.*

In this paper we prove that under assumptions (5)-(9), the X_t^{ϵ} component of the process $(X_t^{\epsilon}, Y_t^{\epsilon})$ converges weakly to a one-dimensional strong Markov process, continuous with probability one. It behaves like a standard diffusion process away from 0 and has to satisfy a gluing condition at the point of discontinuity 0 as $\epsilon \downarrow 0$. More precisely, we prove the following Theorem:

Theorem 1.2. *Let us assume that (5)-(9) hold. Let* X *be the solution to the martingale problem for*

$$A = \{(f, Lf) : f \in \mathscr{D}(A)\} \tag{10}$$

with

$$Lf(x) = D_v D_u f(x) \tag{11}$$

and

$$\begin{aligned} \mathscr{D}(A) = \{ \quad f : \quad & f \in \mathscr{C}_c(\mathbb{R}), \text{ with } f_x, f_{xx} \in \mathscr{C}(\mathbb{R} \setminus \{0\}), \\ & [u'(0+)]^{-1} f_x(0+) - [u'(0-)]^{-1} f_x(0-) = [v(0+) - v(0-)] Lf(0) \\ & \text{and } Lf(0) = \lim_{x \to 0^+} Lf(x) = \lim_{x \to 0^-} Lf(x)\}. \end{aligned} \tag{12}$$

Then we have

$$X_{\cdot}^{\epsilon} \longrightarrow X_{\cdot} \text{ weakly in } \mathscr{C}_{0T}, \text{ for any } T < \infty, \text{ as } \epsilon \downarrow 0, \tag{13}$$

where $\mathscr{C}_{0T}$ *is the space of continuous functions in* $[0, T]$.

□

As proved in Feller [2] the martingale problem for A, (10), has a unique solution X. It is an asymmetric Markov process with delay at the point of discontinuity 0. In particular, the asymmetry is due to the possibility of having $u'(0+) \neq u'(0-)$ (see Lemma 2.5) whereas the delay is because of the possibility of having $v(0+) \neq v(0-)$ (see Lemma 2.4).

For the convenience of the reader, we briefly recall the Feller characterization of all one-dimensional Markov processes, that are continuous with probability one (for more details see [2]; also [13]). All one-dimensional strong Markov processes, that are continuous with probability one, can be characterized (under some minimal regularity conditions) by a generalized second order differential operator $D_v D_u f$ with respect to two increasing functions $u(x)$ and $v(x)$; $u(x)$ is continuous, $v(x)$ is right continuous. In addition, D_u, D_v are differentiation operators with respect to $u(x)$ and $v(x)$ respectively, which are defined as follows:

$D_u f(x)$ exists if $D_u f(x+) = D_u f(x-)$, where the left derivative of f with respect to u is defined as follows:

$$D_u f(x-) = \lim_{h \downarrow 0} \frac{f(x-h) - f(x)}{u(x-h) - u(x)} \quad \text{provided the limit exists.}$$

The right derivative $D_u f(x+)$ is defined similarly. If v is discontinuous at y then

$$D_v f(y) = \lim_{h \downarrow 0} \frac{f(y+h) - f(y-h)}{v(y+h) - v(y-h)}.$$

A more detailed description of these Markov processes can be found in [2] and [13].

Remark 1.3. *Notice that if the limit of $\frac{1}{\epsilon}V^{\epsilon}(x)$, as $\epsilon \downarrow 0$, is a smooth function then the limiting process X described by Theorem 1.2 coincides with (3).*

We conclude the introduction with a useful example. Let us assume that $V^{\epsilon}(x)$ can be decomposed in three terms

$$V^{\epsilon}(x) = V_1^{\epsilon}(x) + V_2^{\epsilon}(x) + V_3^{\epsilon}(x), \tag{14}$$

where the functions $V_i^{\epsilon}(x)$, for $i = 1, 2, 3$, satisfy the following conditions:

1. There exists a strictly positive, smooth function $V_1(x) > 0$ such that

$$\frac{1}{\epsilon} V_1^{\epsilon}(x) \to V_1(x), \text{ as } \epsilon \downarrow 0, \tag{15}$$

   uniformly in $x \in \mathbb{R}$.

2. There exists a nonnegative constant $\beta \geq 0$ such that

$$\frac{1}{\epsilon} V_2^{\epsilon}(x) \to \beta \chi_{\{x>0\}}, \text{ as } \epsilon \downarrow 0, \tag{16}$$

   uniformly for every connected subset of $\mathbb{R}$ that is away from an arbitrary small neighborhood of 0 and weakly within a neighborhood of 0. Here χ_A is the indicator function of the set A.

3. $$\frac{1}{\epsilon} V_3^{\epsilon}(x) \to \mu \delta_0(x), \text{ as } \epsilon \downarrow 0, \tag{17}$$

   in the weak sense. Here μ is a nonnegative constant and $\delta_0(x)$ is the Dirac delta distribution at 0.

4. Condition (9) holds.

Let us define $\alpha = V_1(0)$. In this case the operator (11) and its domain of definition (12) for the limiting process X become

$$Lf(x) = \begin{cases} \frac{1}{2} f_{xx}(x) + \frac{1}{2}\frac{d}{dx}[\ln(V_1(x))] f_x(x), & \text{x<0} \\ \frac{1}{2} f_{xx}(x) + \frac{1}{2}\frac{d}{dx}[\ln(V_1(x) + \beta)] f_x(x), & \text{x>0,} \end{cases} \tag{18}$$

and

$$\begin{aligned} \mathscr{D}(A) = \{ \quad f : \quad & f \in \mathscr{C}_c(\mathbb{R}), \text{ with } f_x, f_{xx} \in \mathscr{C}(\mathbb{R} \setminus \{0\}), \\ & [\alpha + \beta] f_x(0+) - \alpha f_x(0-) = [2\mu] Lf(0) \\ & \text{and } Lf(0) = \lim_{x \to 0^+} Lf(x) = \lim_{x \to 0^-} Lf(x) \}. \end{aligned} \tag{19}$$

For instance, consider $0 < \delta = \delta(\epsilon) \ll 1$ a small $\epsilon-$dependent positive number and assume that:

1. $V^{l,\epsilon}(x)=0$ and $V^{u,\epsilon}(x)=V^{\epsilon}(x)=V_1^{\epsilon}(x)+V_2^{\epsilon}(x)+V_3^{\epsilon}(x)$ where
2. $V_1^{\epsilon}(x)=\epsilon V_1(x)$, where $V_1(x)$ is any smooth, strictly positive function,
3. $V_2^{\epsilon}(x)=\epsilon V_2(\frac{x}{\delta})$, such that $V_2(\frac{x}{\delta})\to\beta\chi_{\{x>0\}}$, as $\epsilon\downarrow 0$
4. $V_3^{\epsilon}(x)=\frac{\epsilon}{\delta}V_3(\frac{x}{\delta})$, such that $\frac{1}{\delta}V_3(\frac{x}{\delta})\to\mu\delta_0(x)$, as $\epsilon\downarrow 0$
5. and with δ chosen such that $\frac{\epsilon}{\delta^3}\downarrow 0$ as $\epsilon\downarrow 0$.

Then, it can be easily verified that (15)-(17) and (9) are satisfied. Moreover, in this case, we have $\mu=\int_{-\infty}^{\infty}V_3(x)dx$.

In section 2 we prove our main result assuming that we have all needed estimates. After the proof of Theorem 1.2, we state the result in the case that $\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}V^{\epsilon}(x)$ has more than one point of discontinuity (Theorem 2.6). In section 3 we prove relative compactness of X_t^{ϵ} (this follows basically from [8]) and we consider what happens outside a small neighborhood of $x=0$. In section 4 we estimate the expectation of the time it takes for the solution to (1) to leave the neighborhood of the point 0. The derivation of this estimate uses assumption (9). In section 5 we: (a) prove that the behavior of the process after it reaches $x=0$ does not depend on where it came from, and (b) calculate the limiting exit probabilities of $(X_t^{\epsilon},Y_t^{\epsilon})$, from the left and from the right, of a small neighborhood of $x=0$. The derivation of these estimates is composed of two main ingredients. The first one is the characterization of all one-dimensional Markov processes, that are continuous with probability one, by generalized second order operators introduced by Feller (see [2]; also [13]). The second one is a result of Khasminskii on invariant measures [11].

Lastly, we would like to mention here that one can similarly consider narrow tubes, i.e. $y\in D_x^{\epsilon}\subset\mathbb{R}^n$ for $n>1$, and prove a result similar to Theorem 1.2.

2 Proof of the Main Theorem

Before proving Theorem 1.2 we introduce some notation and formulate the necessary lemmas. The lemmas are proved in sections 3 to 5.

In this and the following sections we will denote by C_0 any unimportant constants that do not depend on any small parameter. The constants may change from place to place though, but they will always be denoted by the same C_0.

For any $B\subset\mathbb{R}$, we define the Markov time $\tau(B)=\tau_{x,y}^{\epsilon}(B)$ to be:

$$\tau_{x,y}^{\epsilon}(B)=\inf\{t>0:X_t^{\epsilon,x,y}\notin B\}. \tag{20}$$

Moreover, for $\kappa>0$, the term $\tau_{x,y}^{\epsilon}(\pm\kappa)$ will denote the Markov time $\tau_{x,y}^{\epsilon}(-\kappa,\kappa)$. In addition, $\mathbb{E}_{x,y}^{\epsilon}$ will denote the expected value associated with the probability measure $\mathbb{P}_{x,y}^{\epsilon}$ that is induced by the process $(X_t^{\epsilon,x,y},Y_t^{\epsilon,x,y})$.

For the sake of notational convenience we define the operators

$$\begin{aligned} L_-f(x) &= D_vD_uf(x) \text{ for } x<0 \\ L_+f(x) &= D_vD_uf(x) \text{ for } x>0 \end{aligned} \tag{21}$$

Furthermore, when we write $(x,y)\in A\times B$ we will mean $(x,y)\in\{(x,y): x\in A, y\in B\}$.

Most of the processes, Markov times and sets that will be mentioned below will depend on ϵ. For notational convenience however, we shall not always incorporate this dependence into the notation. So the reader should be careful to distinguish between objects that depend and do not depend on ϵ.

Throughout this paper $0<\kappa_0<\kappa$ will be small positive constants. We may not always mention the relation between these parameters but we will always assume it. Moreover κ_η will denote a small positive number that depends on another small positive number η.

Moreover, one can write down the normal vector $\gamma^\epsilon(x,y)$ explicitly:

$$\gamma^\epsilon(x,y)=\begin{cases}\frac{1}{\sqrt{1+[V_x^{u,\epsilon}(x)]^2}}(V_x^{u,\epsilon}(x),-1), & y=V^{u,\epsilon}(x)\\ \frac{1}{\sqrt{1+[V_x^{l,\epsilon}(x)]^2}}(V_x^{l,\epsilon}(x),1), & y=-V^{l,\epsilon}(x).\end{cases}$$

Lemma 2.1. *For any $(x,y)\in D^\epsilon$, let $\mathbb{Q}^\epsilon_{x,y}$ be the family of distributions of $X^\epsilon_\cdot$ in the space $\mathscr{C}[0,\infty)$ of continuous functions $[0,\infty)\to\mathbb{R}$ that correspond to the probabilities $\mathbb{P}^\epsilon_{x,y}$. Assume that for any $(x,y)\in D^\epsilon$ the family of distributions $\mathbb{Q}^\epsilon_{x,y}$ for all $\epsilon\in(0,1)$ is tight. Moreover suppose that for any compact set $K\subseteq\mathbb{R}$, any function $f\in\mathscr{D}(A)$ and for every $\lambda>0$ we have:*

$$\mathbb{E}^\epsilon_{x,y}\int_0^\infty e^{-\lambda t}[\lambda f(X^\epsilon_t)-Lf(X^\epsilon_t)]dt-f(x)\to 0, \tag{22}$$

as $\epsilon\downarrow 0$, uniformly in $(x,y)\in K\times D^\epsilon_x$.

The measures $\mathbb{Q}^\epsilon_{x,y}$ corresponding to $\mathbb{P}^\epsilon_{x,y}$ converge weakly to the probability measure $\mathbb{P}_x$ that is induced by $X_\cdot$ as $\epsilon\downarrow 0$.

□

Lemma 2.2. *The family of distributions $\mathbb{Q}^\epsilon_{x,y}$ in the space of continuous functions $[0,\infty)\to\mathbb{R}$ corresponding to $\mathbb{P}^\epsilon_{x,y}$ for small nonzero ϵ is tight.*

□

Lemma 2.3. *Let $0<x_1<x_2$ be fixed positive numbers and f be a three times continuously differentiable function in $[x_1,x_2]$. Then for every $\lambda>0$:*

$$\mathbb{E}^\epsilon_{x,y}[e^{-\lambda\tau^\epsilon(x_1,x_2)}f(X^\epsilon_{\tau^\epsilon(x_1,x_2)})+\int_0^{\tau^\epsilon(x_1,x_2)}e^{-\lambda t}[\lambda f(X^\epsilon_t)-Lf(X^\epsilon_t)]dt]\to f(x),$$

as $\epsilon\downarrow 0$, uniformly in (x,y) such that $(x,y)\in[x_1,x_2]\times D^\epsilon_x$. The statement holds true for $\tau^\epsilon(-x_2,-x_1)$ in place of $\tau^\epsilon(x_1,x_2)$ as well.

□

Lemma 2.4. *Define* $\theta = \frac{v(0+)-v(0-)}{[u'(0+)]^{-1}+[u'(0-)]^{-1}}$. *For every* $\eta > 0$ *there exists a* $\kappa_\eta > 0$ *such that for every* $0 < \kappa < \kappa_\eta$ *and for sufficiently small* ϵ

$$|\mathbb{E}^\epsilon_{x,y}\tau^\epsilon(\pm\kappa) - \kappa\theta| \leq \kappa\eta,$$

for all $(x,y) \in [-\kappa,\kappa] \times D^\epsilon_x$. *Here,* $\tau^\epsilon(\pm\kappa) = \tau^\epsilon(-\kappa,\kappa)$ *is the exit time from the interval* $(-\kappa,\kappa)$.

□

Lemma 2.5. *Define* $p_+ = \frac{[u'(0+)]^{-1}}{[u'(0+)]^{-1}+[u'(0-)]^{-1}}$ *and* $p_- = \frac{[u'(0-)]^{-1}}{[u'(0+)]^{-1}+[u'(0-)]^{-1}}$. *For every* $\eta > 0$ *there exists a* $\kappa_\eta > 0$ *such that for every* $0 < \kappa < \kappa_\eta$ *there exists a positive* $\kappa_0 = \kappa_0(\kappa)$ *such that for sufficiently small* ϵ

$$\begin{aligned} |\mathbb{P}^\epsilon_{x,y}\{X^\epsilon_{\tau^\epsilon(\pm\kappa)} = \kappa\} - p_+| &\leq \eta, \\ |\mathbb{P}^\epsilon_{x,y}\{X^\epsilon_{\tau^\epsilon(\pm\kappa)} = -\kappa\} - p_-| &\leq \eta, \end{aligned}$$

for all (x,y) *such that* $(x,y) \in [-\kappa_0,\kappa_0] \times D^\epsilon_x$.

□

Proof of Theorem 1.2. We will make use of Lemma 2.1. The tightness required in Lemma 2.1 is the statement of Lemma 2.2. Thus it remains to prove that (22) holds.

Let $\lambda > 0$, $(x,y) \in D^\epsilon$ and $f \in \mathscr{D}(A)$ be fixed. In addition let $\eta > 0$ be an arbitrary positive number. Choose $0 < x_* < \infty$ so that

$$\mathbb{E}^\epsilon_{x,y} e^{-\lambda\tilde{\tau}^\epsilon} < \frac{\eta}{\|f\| + \lambda^{-1}\|\lambda f - Lf\|} \tag{23}$$

for sufficiently small ϵ, where $\tilde{\tau}^\epsilon = \inf\{t > 0 : |X^\epsilon_t| \geq x_*\}$. It is Lemma 2.2 that makes such a choice possible. We assume that $x_* > |x|$.

To prove (22) it is enough to show that for every $\eta > 0$ there exists an $\epsilon_0 > 0$, independent of (x,y), such that for every $0 < \epsilon < \epsilon_0$:

$$|\mathbb{E}^\epsilon_{x,y}[e^{-\lambda\tilde{\tau}} f(X^\epsilon_{\tilde{\tau}}) - f(x) + \int_0^{\tilde{\tau}} e^{-\lambda t}[\lambda f(X^\epsilon_t) - Lf(X^\epsilon_t)]dt]| < \eta. \tag{24}$$

Choose ϵ small and $0 < \kappa_0 < \kappa$ small positive numbers. We consider two cycles of Markov times $\{\sigma_n\}$ and $\{\tau_n\}$ such that:

$$0 = \sigma_0 \leq \tau_1 \leq \sigma_1 \leq \tau_2 \leq \ldots$$

where:

$$\begin{aligned} \tau_n &= \tilde{\tau} \wedge \inf\{t > \sigma_{n-1} : |X^\epsilon_t| \geq \kappa\} \\ \sigma_n &= \tilde{\tau} \wedge \inf\{t > \tau_n : |X^\epsilon_t| \in \{\kappa_0, x_*\}\} \end{aligned} \tag{25}$$

In figure 1 we see a trajectory of the process $Z^\epsilon_t = (X^\epsilon_t, Y^\epsilon_t)$ along with its associated Markov chains $\{Z^\epsilon_{\sigma_n}\}$ and $\{Z^\epsilon_{\tau_n}\}$. We will also write $z = (x,y)$ for the initial point.

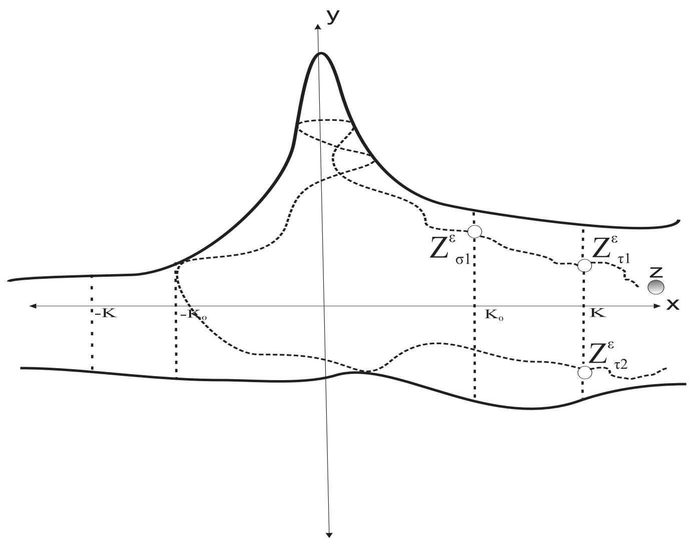

Figure 1: $z = (x, y)$ is the initial point and $Z_t^\epsilon = (X_t^\epsilon, Y_t^\epsilon)$.

We denote by χ_A the indicator function of the set A. The difference in (24) can be represented as the sum over time intervals from σ_n to τ_{n+1} and from τ_n to σ_n. It is equal to:

$$\mathbb{E}_z^\epsilon[e^{-\lambda\tilde{\tau}} f(X^\epsilon_{\tilde{\tau}}) - f(x) + \int_0^{\tilde{\tau}} e^{-\lambda t}[\lambda f(X_t^\epsilon) - Lf(X_t^\epsilon)]dt] = \tag{26}$$
$$= \mathbb{E}_z^\epsilon \sum_{n=0}^{\infty} [e^{-\lambda\tau^\epsilon_{n+1}} f(X^\epsilon_{\tau^\epsilon_{n+1}}) - e^{-\lambda\sigma^\epsilon_n} f(X_{\sigma^\epsilon_n}) + \int_{\sigma^\epsilon_n}^{\tau^\epsilon_{n+1}} e^{-\lambda t}[\lambda f(X_t^\epsilon) - Lf(X_t^\epsilon)]dt]] +$$
$$+ \mathbb{E}_z^\epsilon \sum_{n=1}^{\infty} [e^{-\lambda\sigma^\epsilon_n} f(X^\epsilon_{\sigma^\epsilon_n}) - e^{-\lambda\tau^\epsilon_n} f(X_{\tau^\epsilon_n}) + \int_{\tau^\epsilon_n}^{\sigma^\epsilon_n} e^{-\lambda t}[\lambda f(X_t^\epsilon) - Lf(X_t^\epsilon)]dt]$$

The formally infinite sums are finite for every trajectory for which $\tilde{\tau} < \infty$. Assuming that we can write the expectation of the infinite sums as the infinite sum of the expectations, the latter equality becomes

$$\mathbb{E}_z^\epsilon[e^{-\lambda\tilde{\tau}} f(X^\epsilon_{\tilde{\tau}}) - f(x) + \int_0^{\tilde{\tau}} e^{-\lambda t}[\lambda f(X_t^\epsilon) - Lf(X_t^\epsilon)]dt] = \tag{27}$$
$$= \sum_{n=0}^{\infty} \mathbb{E}_z^\epsilon [e^{-\lambda\tau^\epsilon_{n+1}} f(X^\epsilon_{\tau^\epsilon_{n+1}}) - e^{-\lambda\sigma^\epsilon_n} f(X_{\sigma^\epsilon_n}) + \int_{\sigma^\epsilon_n}^{\tau^\epsilon_{n+1}} e^{-\lambda t}[\lambda f(X_t^\epsilon) - Lf(X_t^\epsilon)]dt]] +$$
$$+ \sum_{n=1}^{\infty} \mathbb{E}_z^\epsilon [e^{-\lambda\sigma^\epsilon_n} f(X^\epsilon_{\sigma^\epsilon_n}) - e^{-\lambda\tau^\epsilon_n} f(X_{\tau^\epsilon_n}) + \int_{\tau^\epsilon_n}^{\sigma^\epsilon_n} e^{-\lambda t}[\lambda f(X_t^\epsilon) - Lf(X_t^\epsilon)]dt]$$

The aforementioned calculation can be done if

$$\sum_{n=0}^{\infty} \mathbb{E}_z^\epsilon[e^{-\lambda\sigma_n}\chi_{\sigma_n<\tilde{\tau}}], \sum_{n=1}^{\infty} \mathbb{E}_z^\epsilon[e^{-\lambda\tau_n}\chi_{\tau_n<\tilde{\tau}}] < \infty.$$

Indeed, by Markov property we have:

$$\begin{aligned}\mathbb{E}^{\epsilon}_{z}[e^{-\lambda\sigma_n}\chi_{\sigma_n<\tilde{\tau}}] &\leq \mathbb{E}^{\epsilon}_{z}[e^{-\lambda\tau_n}\chi_{\tau_n<\tilde{\tau}}]\max_{|x|=\kappa, y\in D^{\epsilon}_{x}}\phi^{\epsilon}_{1}(x,y)\leq\\ &\leq \mathbb{E}^{\epsilon}_{z}[e^{-\lambda\sigma_{n-1}}\chi_{\sigma_{n-1}<\tilde{\tau}}]\max_{|x|=\kappa, y\in D^{\epsilon}_{x}}\phi^{\epsilon}_{1}(x,y),\end{aligned} \tag{28}$$

where $\phi^{\epsilon}_{1}(x,y)=\mathbb{E}^{\epsilon}_{x,y}[e^{-\lambda\sigma_1}\chi_{|X^{\epsilon}_{\sigma_1}|=\kappa_0}]$. So by induction we have

$$\sum_{n=1}^{\infty}\mathbb{E}^{\epsilon}_{z}[e^{-\lambda\tau_n}\chi_{\tau_n<\tilde{\tau}}]\leq\sum_{n=0}^{\infty}\mathbb{E}^{\epsilon}_{z}[e^{-\lambda\sigma_n}\chi_{\sigma_n<\tilde{\tau}}]\leq\frac{1}{1-\max_{|x|=\kappa, y\in D^{\epsilon}_{x}}\phi^{\epsilon}_{1}(x,y)}. \tag{29}$$

Clearly $\max_{|x|=\kappa, y\in D^{\epsilon}_{x}}\phi^{\epsilon}_{1}(x,y)<1$ for $\kappa\in(\kappa_0,x_*)$. Therefore equality (27) is valid.
However we need to know how the sums in (29) behave in terms of κ. To this end we apply Lemma 2.3 to the function g that is the solution to

$$\begin{aligned}\lambda g - L_{\pm}g &= 0 \quad \text{in } x\in(\pm\kappa_0,\pm x_*)\\ g(\pm\kappa_0) &= 1\\ g(\pm x_*) &= 0\end{aligned} \tag{30}$$

By Lemma 2.3 we know that $g(x)$ approximates $\phi^{\epsilon}_{1}(x,y)$ for $|x|\in[\kappa_0,x_*]$ as $\epsilon\downarrow 0$. The idea is to bound $\phi^{\epsilon}_{1}(x,y)$ using $g(x)$. It follows by (30) (for more details see the related discussion in section 8.3 of [6], page 306) that there exists a positive constant C_0 that is independent of ϵ and a positive constant $\kappa^{'}$ such that for every $\kappa<\kappa^{'}$ and for all $\kappa_0<\kappa^{'}_{0}(\kappa)$ we have $g(\pm\kappa)\leq 1-C_0\kappa$.
So we conclude for ϵ and κ sufficiently small that

$$\sum_{n=1}^{\infty}\mathbb{E}^{\epsilon}_{z}[e^{-\lambda\tau_n}\chi_{\tau_n<\tilde{\tau}}]\leq\sum_{n=0}^{\infty}\mathbb{E}^{\epsilon}_{z}[e^{-\lambda\sigma_n}\chi_{\sigma_n<\tilde{\tau}}]\leq\frac{C_0}{\kappa}. \tag{31}$$

By the strong Markov property with respect to the Markov times τ_n and σ_n equality (27) becomes

$$\begin{aligned}&\mathbb{E}^{\epsilon}_{z}[e^{-\lambda\tilde{\tau}}f(X^{\epsilon}_{\tilde{\tau}})-f(x)+\int_0^{\tilde{\tau}}e^{-\lambda t}[\lambda f(X^{\epsilon}_{t})-Lf(X^{\epsilon}_{t})]dt]=\\ =\;&\sum_{n=0}^{\infty}\mathbb{E}^{\epsilon}_{z}[e^{-\lambda\sigma_n}\chi_{\sigma_n<\tilde{\tau}}\mathbb{E}^{\epsilon}_{Z^{\epsilon}_{\sigma^{\epsilon}_{n}}}[e^{-\lambda\tau^{\epsilon}_{1}}f(X^{\epsilon}_{\tau^{\epsilon}_{1}})-f(X_{\sigma^{\epsilon}_{n}})+\int_0^{\tau^{\epsilon}_{1}}e^{-\lambda t}[\lambda f(X^{\epsilon}_{t})-Lf(X^{\epsilon}_{t})]dt]]+\\ +\;&\sum_{n=1}^{\infty}\mathbb{E}^{\epsilon}_{z}[e^{-\lambda\tau_n}\chi_{\tau_n<\tilde{\tau}}\mathbb{E}^{\epsilon}_{Z^{\epsilon}_{\tau_n}}[e^{-\lambda\sigma^{\epsilon}_{1}}f(X^{\epsilon}_{\sigma^{\epsilon}_{1}})-f(X_{\tau^{\epsilon}_{n}})+\int_0^{\sigma^{\epsilon}_{1}}e^{-\lambda t}[\lambda f(X^{\epsilon}_{t})-Lf(X^{\epsilon}_{t})]dt]]\\ =\;&\sum_{n=0}^{\infty}\mathbb{E}^{\epsilon}_{z}[e^{-\lambda\sigma_n}\chi_{\sigma_n<\tilde{\tau}}\phi^{\epsilon}_{2}(Z^{\epsilon}_{\sigma^{\epsilon}_{n}})]+\sum_{n=1}^{\infty}\mathbb{E}^{\epsilon}_{z}[e^{-\lambda\tau_n}\chi_{\tau_n<\tilde{\tau}}\phi^{\epsilon}_{3}(Z^{\epsilon}_{\tau_n})]\end{aligned} \tag{32}$$

where

$$\phi^{\epsilon}_{2}(x,y)=\mathbb{E}^{\epsilon}_{x,y}[e^{-\lambda\tau_1}f(X^{\epsilon}_{\tau_1})-f(x)+\int_0^{\tau_1}e^{-\lambda t}[\lambda f(X^{\epsilon}_{t})-Lf(X^{\epsilon}_{t})]dt] \tag{33}$$

and

$$\phi^{\epsilon}_{3}(x,y)=\mathbb{E}^{\epsilon}_{x,y}[e^{-\lambda\sigma_1}f(X^{\epsilon}_{\sigma_1})-f(x)+\int_0^{\sigma_1}e^{-\lambda t}[\lambda f(X^{\epsilon}_{t})-Lf(X^{\epsilon}_{t})]dt] \tag{34}$$

Because of (31), equality (32) becomes

$$
\begin{aligned}
&|\mathbb{E}^{\epsilon}_{x,y}[e^{-\lambda\tilde{\tau}}f(X^{\epsilon}_{\tilde{\tau}})-f(x)+\int_0^{\tilde{\tau}}e^{-\lambda t}[\lambda f(X^{\epsilon}_t)-Lf(X^{\epsilon}_t)]dt]|\leq\\
\leq\ & |\phi^{\epsilon}_2(x,y)|+\frac{C_0}{\kappa}\left[\max_{|x|=\kappa_0,y\in D^{\epsilon}_x}|\phi^{\epsilon}_2(x,y)|+\max_{|x|=\kappa,y\in D^{\epsilon}_x}|\phi^{\epsilon}_3(x,y)|\right] \qquad (35)
\end{aligned}
$$

By Lemma 2.3 we get that $\max_{|x|=\kappa,y\in D^{\epsilon}_x}|\phi^{\epsilon}_3(x,y)|$ is arbitrarily small for sufficiently small ϵ, so

$$\frac{C_0}{\kappa}\max_{|x|=\kappa,y\in D^{\epsilon}_x}|\phi^{\epsilon}_3(x,y)|\leq\frac{\eta}{3} \qquad (36)$$

Therefore, it remains to consider the terms $|\phi^{\epsilon}_2(x,y)|$, where (x,y) is the initial point, and $\frac{1}{\kappa}\max_{|x|=\kappa_0,y\in D^{\epsilon}_x}|\phi^{\epsilon}_2(x,y)|$.

Firstly, we consider the term $|\phi^{\epsilon}_2(x,y)|$, where (x,y) is the initial point. Clearly, if $|x|>\kappa$, then Lemma 2.3 implies that $|\phi^{\epsilon}_2(x,y)|$ is arbitrarily small for sufficiently small ϵ, so

$$|\phi^{\epsilon}_2(x,y)|\leq\frac{\eta}{3}. \qquad (37)$$

We consider now the case $|x|\leq\kappa$. Clearly, in this case Lemma 2.3 does not apply. However, one can use the continuity of f and Lemma 2.4, as the following calculations show. We have:

$$|\phi^{\epsilon}_2(x,y)|\leq\mathbb{E}^{\epsilon}_{x,y}|f(X^{\epsilon}_{\tau_1})-f(x)|+|\lambda|\|f\|+\|\lambda f-Lf\||\mathbb{E}^{\epsilon}_{x,y}\int_0^{\tau_1}e^{-\lambda t}dt.$$

Choose now a positive κ' so that

$$|f(x)-f(0)|<\frac{\eta}{6},\text{ for all }|x|\leq\kappa'$$

and that

$$\mathbb{E}^{\epsilon}_{x,y}\int_0^{\tau_1}e^{-\lambda t}dt\leq\frac{\eta}{6[\lambda\|f\|+\|\lambda f-Lf\|]}$$

for sufficiently small ϵ and for all $|x|\leq\kappa'$. Therefore, for $\kappa\leq\kappa'$ and for sufficiently small ϵ we have

$$|\phi^{\epsilon}_2(x,y)|\leq\frac{\eta}{3},\text{ for all }(x,y)\in[-\kappa,\kappa]\times D^{\epsilon}_x. \qquad (38)$$

Secondly, we consider the term $\frac{1}{\kappa}\max_{|x|=\kappa_0,y\in D^{\epsilon}_x}|\phi^{\epsilon}_2(x,y)|$. Here, we need a sharper estimate because of the factor $\frac{1}{\kappa}$. We will prove that for $(x,y)\in\{\pm\kappa_0\}\times D^{\epsilon}_{\pm\kappa_0}$ and for ϵ sufficiently small

$$|\phi^{\epsilon}_2(x,y)|\leq\kappa\frac{\eta}{3C_0}. \qquad (39)$$

For $(x,y)\in\{\pm\kappa_0\}\times D^{\epsilon}_{\pm\kappa_0}$ we have

$$\begin{aligned}
|\phi^{\epsilon}_2(x,y)| &= |\mathbb{E}^{\epsilon}_{x,y}[e^{-\lambda\tau_1}f(X^{\epsilon}_{\tau_1})-f(x)+\int_0^{\tau_1}e^{-\lambda t}[\lambda f(X^{\epsilon}_t)-Lf(X^{\epsilon}_t)]dt]| \\
&\leq |\mathbb{E}^{\epsilon}_{x,y}[f(X^{\epsilon}_{\tau_1})-f(x)-\kappa\theta Lf(0)]|+ \\
&\quad +|\mathbb{E}^{\epsilon}_{x,y}[\tau_1 Lf(0)-\int_0^{\tau_1}e^{-\lambda t}Lf(X^{\epsilon}_t)dt]|+ \\
&\quad +|\mathbb{E}^{\epsilon}_{x,y}[\kappa\theta Lf(0)-\tau_1 Lf(0)|+ \\
&\quad +|\mathbb{E}^{\epsilon}_{x,y}[e^{-\lambda\tau_1}f(X^{\epsilon}_{\tau_1})-f(X^{\epsilon}_{\tau_1})+\int_0^{\tau_1}e^{-\lambda t}\lambda f(X^{\epsilon}_t)dt]|, \qquad (40)
\end{aligned}$$

where $\theta=\frac{v(0+)-v(0-)}{[u'(0+)]^{-1}+[u'(0-)]^{-1}}$.

Since the one-sided derivatives of f exist, we may choose, a positive κ_η such that for every $0<\kappa_1\leq\kappa_\eta$

$$|\frac{f(w)-f(0)}{w}-f_x(0+)|,|\frac{f(-w)-f(0)}{-w}-f_x(0-)|\leq\frac{\eta}{C_0}, \qquad (41)$$

for all $w\in(0,\kappa_1)$.

Furthermore, by Lemma 2.5 we can choose for sufficiently small $\kappa_2>0$, a $\kappa_0(\kappa_2)\in(0,\kappa_2)$ such that for sufficiently small ϵ

$$|\mathbb{P}^{\epsilon}_{x,y}\{X^{\epsilon}_{\tau^{\epsilon}_1(\pm\kappa_2)}=\pm\kappa_2\}-p_{\pm}|\leq\frac{\eta}{C_0} \qquad (42)$$

for all (x,y) such that $(x,y)\in[-\kappa_0,\kappa_0]\times D^{\epsilon}_x$.

In addition, by Lemma 2.4 we can choose for sufficiently small $\kappa_\eta>0$, a $\kappa_3\in(0,\kappa_\eta)$ such that for sufficiently small ϵ

$$|\mathbb{E}^{\epsilon}_{x,y}\tau^{\epsilon}_1(\pm\kappa_3)-\kappa_3\theta|\leq\kappa_3\frac{\eta}{C_0} \qquad (43)$$

for all $(x,y)\in[-\kappa_3,\kappa_3]\times D^{\epsilon}_x$.

Choose now $0<\kappa\leq\min\{\kappa_1,\kappa_2,\kappa_3\}$ and $0<\kappa_0<\min\{\kappa_0(\kappa_2),\kappa\}$.

For sufficiently small ϵ and for all $(x,y)\in\{\pm\kappa_0\}\times D^{\epsilon}_{\pm\kappa_0}$ we have

$$\begin{aligned}
& |\mathbb{E}^{\epsilon}_{x,y}f(X^{\epsilon}_{\tau_1})-f(x)-\kappa\theta Lf(0)|\leq \\
\leq\ & |p_+[f(\kappa)-f(0)]+p_-[f(-\kappa)-f(0)]-\kappa\theta Lf(0)|+ \\
+\ & |\mathbb{P}^{\epsilon}_{x,y}\{X^{\epsilon}_{\tau^{\epsilon}_1(\pm\kappa)}=\kappa\}-p_+||f(\kappa)-f(0)|+ \\
+\ & |\mathbb{P}^{\epsilon}_{x,y}\{X^{\epsilon}_{\tau^{\epsilon}_1(\pm\kappa)}=-\kappa\}-p_-||f(-\kappa)-f(0)|+ \\
+\ & |f(0)-f(x)| \qquad (44)
\end{aligned}$$

Because of (41) and the gluing condition $p_+f_x(0+)-p_-f_x(0-)=\theta Lf(0)$, the first summand on the right hand side of (44) satisfies

$$\begin{aligned}
& |p_+[f(\kappa)-f(0)]+p_-[f(-\kappa)-f(0)]-\kappa\theta Lf(0)|\leq \\
\leq\ & |p_+\kappa f_x(0+)-p_-\kappa f_x(0-)-\kappa\theta Lf(0)|+\kappa\frac{\eta}{C_0}= \\
=\ & \kappa\frac{\eta}{C_0}. \qquad (45)
\end{aligned}$$

Moreover, for small enough $x \in \{\pm\kappa_0, \pm\kappa\}$ we also have that

$$|f(x) - f(0)| \leq |x||f_x(0\pm)| + |x|\eta. \tag{46}$$

The latter together with (42) imply that for sufficiently small ϵ the second summand on the right hand side of (44) satisfies

$$|P^{\epsilon}_{x,y}\{X^{\epsilon}_{\tau^{\epsilon}_1(\pm\kappa)} = \kappa\} - p_+||f(\kappa) - f(0)| \leq \kappa\frac{\eta}{C_0}. \tag{47}$$

A similar expression holds for the third summand on the right hand side of (44) as well. Therefore (45)-(47) and the fact that we take κ_0 to be much smaller than κ imply that for all $(x, y) \in \{\pm\kappa_0\} \times D^{\epsilon}_{\pm\kappa_0}$ and for ϵ sufficiently small, we have

$$|\mathbb{E}^{\epsilon}_{x,y} f(X^{\epsilon}_{\tau_1}) - f(x) - \kappa\theta Lf(0)| \leq \kappa\frac{\eta}{C_0}. \tag{48}$$

The second term of the right hand side of (40) can also be bounded by $\kappa\frac{\eta}{C_0}$ for κ and ϵ sufficiently small, as the following calculations show. For $(x, y) \in \{\pm\kappa_0\} \times D^{\epsilon}_{\pm\kappa_0}$ we have

$$\begin{aligned}
& |\mathbb{E}^{\epsilon}_{x,y}[\tau_1 Lf(0) - \int_0^{\tau_1} e^{-\lambda t} Lf(X^{\epsilon}_t)dt]| \leq \\
\leq \; & |Lf(0)||\mathbb{E}^{\epsilon}_{x,y}[\tau_1 - \int_0^{\tau_1} e^{-\lambda t} dt]| + \sup_{|x|\leq\kappa} |Lf(x) - Lf(0)|\mathbb{E}^{\epsilon}_{x,y}\tau_1 \leq \\
\leq \; & \lambda|Lf(0)|\mathbb{E}^{\epsilon}_{x,y}\tau_1 \left[\sup_{(x,y)\in\{\pm\kappa_0\}\times D^{\epsilon}_{\pm\kappa_0}} \mathbb{E}^{\epsilon}_{x,y}\tau_1\right] + \sup_{|x|\leq\kappa} |Lf(x) - Lf(0)|\mathbb{E}^{\epsilon}_{x,y}\tau_1
\end{aligned} \tag{49}$$

Therefore, Lemma 2.4 (in particular (43)) and the continuity of the function Lf give us for κ and ϵ sufficiently small that

$$|\mathbb{E}^{\epsilon}_{x,y}[\tau_1 Lf(0) - \int_0^{\tau_1} e^{-\lambda t} Lf(X^{\epsilon}_t)dt]| \leq \kappa\frac{\eta}{C_0}. \tag{50}$$

The third term of the right hand side of (40) is clearly bounded by $\kappa\frac{\eta}{C_0}$ for ϵ sufficiently small by Lemma 2.4. As far as the fourth term of the right hand side of (40) is concerned, one can use the continuity of f together with Lemma 2.4.

The latter, (48), (50) and (40) finally give us that

$$\frac{1}{\kappa} \max_{|x|=\kappa_0, y\in D^{\epsilon}_x} |\phi^{\epsilon}_2(x, y)]| \leq \frac{\eta}{3C_0}. \tag{51}$$

Of course, the constants C_0 that appear in the relations above are not the same, but for notational convenience they are all denoted by the same symbol C_0.

So, we finally get by (51), (36), (37), (38) and (35) that

$$|\mathbb{E}^{\epsilon}_{x,y}[e^{-\lambda\tilde{\tau}} f(X^{\epsilon}_{\tilde{\tau}}) - f(x) + \int_0^{\tilde{\tau}} e^{-\lambda t}[\lambda f(X^{\epsilon}_t) - Lf(X^{\epsilon}_t)]dt]| \leq \eta. \tag{52}$$

This concludes the proof of Theorem 1.2. $\square$

In case $\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}V^\epsilon(x)$ has more than one points of discontinuity, one can similarly prove the following theorem. Hence, the limiting Markov process X may be asymmetric at some point x_1, have delay at some other point x_2 or have both irregularities at another point x_3.

Theorem 2.6. *Let us assume that $\frac{1}{\epsilon}V^\epsilon(x)$ has a finite number of discontinuities, as described by (5)-(9), at x_i for $i\in\{1,\cdots,m\}$. Let X be the solution to the martingale problem for*

$$A=\{(f,Lf):f\in\mathscr{D}(A)\}$$

with

$$Lf(x)=D_vD_uf(x)$$

and

$$\begin{aligned}\mathscr{D}(A)=\{\quad f:\quad & f\in\mathscr{C}_c(\mathbb{R}),\text{ with } f_x,f_{xx}\in\mathscr{C}(\mathbb{R}\setminus\{x_1,\cdots,x_m\})\\ & [u'(x_i+)]^{-1}f_x(x_i+)-[u'(x_i-)]^{-1}f_x(x_i-)=[v(x_i+)-v(x_i-)]Lf(x_i)\\ & \text{and } Lf(x_i)=\lim_{x\to x_i^+}Lf(x)=\lim_{x\to x_i^-}Lf(x)\text{ for } i\in\{1,\cdots,m\}\},\end{aligned}$$

Then we have

$$X^\epsilon_\cdot\longrightarrow X_\cdot \text{ weakly in } \mathscr{C}_{0T},\text{ for any } T<\infty,\text{ as } \epsilon\downarrow 0.$$

□

3 Proof of Lemmata 2.1, 2.2 and 2.3

Proof of Lemma 2.1. The proof is very similar to the proof of Lemma 8.3.1 in [6], so it will not be repeated here. □

Proof of Lemma 2.2. The tool that is used to establish tightness of $\mathbb{P}^\epsilon$ is the martingale-problem approach of Stroock-Varadhan [15]. In particular we can apply Theorem 2.1 of [8]. The proof is almost identical to the part of the proof of Theorem 6.1 in [8] where pre-compactness is proven for the Wiener process with reflection in narrow-branching tubes. □

Before proving Lemma 2.3 we introduce the following diffusion process. Let $\hat{X}^\epsilon_t$ be the one-dimensional process that is the solution to:

$$\hat{X}^\epsilon_t=x+W^1_t+\int_0^t\frac{1}{2}\frac{V^\epsilon_x(\hat{X}^\epsilon_s)}{V^\epsilon(\hat{X}^\epsilon_s)}ds,\tag{53}$$

where $V^\epsilon_x(x)=\frac{dV^\epsilon(x)}{dx}$. The process $\hat{X}^\epsilon$ is solution to the martingale problem for $\hat{A}^\epsilon=\{(f,\hat{L}^\epsilon f):f\in\mathscr{D}(\hat{A}^\epsilon)\}$ with

$$\hat{L}^\epsilon=\frac{1}{2}\frac{d^2}{dx^2}+\frac{1}{2}\frac{V^\epsilon_x(\cdot)}{V^\epsilon(\cdot)}\frac{d}{dx}.\tag{54}$$

and

$$\mathscr{D}(\hat{A}^\epsilon)=\{f:f\in\mathscr{C}^2_c(\mathbb{R})\}\tag{55}$$

A simple calculation shows that

$$\hat{L}^\epsilon f(x) = D_{v^\epsilon} D_{u^\epsilon} f(x).$$

where the $u^\epsilon(x)$ and $v^\epsilon(x)$ functions are defined by (4). This representation of $u^\epsilon(x)$ and $v^\epsilon(x)$ is unique up to multiplicative and additive constants. In fact one can multiply one of these functions by some constant and divide the other function by the same constant or add a constant to either of them.

Using the results in [9] one can show (see Theorem 4.4 in [10]) that

$$\hat{X}^\epsilon_\cdot \longrightarrow X_\cdot \text{ weakly in } \mathscr{C}_{0T}, \text{ for any } T < \infty, \text{ as } \epsilon \downarrow 0, \tag{56}$$

where X is the limiting process with operator defined by (10).

Proof of Lemma 2.3. We prove the lemma just for $x \in [x_1, x_2]$. Clearly, the proof for $x \in [-x_2, -x_1]$ is the same.

We claim that it is sufficient to prove that

$$|\mathbb{E}^\epsilon_{x,y}[e^{-\lambda\tau^\epsilon(x_1,x_2)} f(X^\epsilon_{\tau^\epsilon(x_1,x_2)}) + \int_0^{\tau^\epsilon(x_1,x_2)} e^{-\lambda t}[\lambda f(X^\epsilon_t) - \hat{L}^\epsilon f(X^\epsilon_t)]dt] - f(x)| \to 0, \tag{57}$$

as $\epsilon \downarrow 0$, where $\hat{L}^\epsilon$ is defined in (54). The left hand side of (57) is meaningful since f is sufficiently smooth for $x \in [x_1, x_2]$.

We observe that:

$$\begin{aligned}
& |\mathbb{E}^\epsilon_{x,y}[e^{-\lambda\tau^\epsilon(x_1,x_2)} f(X^\epsilon_{\tau^\epsilon(x_1,x_2)}) + \int_0^{\tau^\epsilon(x_1,x_2)} e^{-\lambda t}[\lambda f(X^\epsilon_t) - Lf(X^\epsilon_t)]dt] - f(x)| \\
\leq \; & |\mathbb{E}^\epsilon_{x,y}[e^{-\lambda\tau^\epsilon(x_1,x_2)} f(X^\epsilon_{\tau^\epsilon(x_1,x_2)}) + \int_0^{\tau^\epsilon(x_1,x_2)} e^{-\lambda t}[\lambda f(X^\epsilon_t) - \hat{L}^\epsilon f(X^\epsilon_t)]dt] - f(x)| \\
& + |\mathbb{E}^\epsilon_{x,y} \int_0^{\tau^\epsilon(x_1,x_2)} e^{-\lambda t}[Lf(X^\epsilon_t) - \hat{L}^\epsilon f(X^\epsilon_t)]dt|
\end{aligned} \tag{58}$$

Now we claim that

$$\|\hat{L}^\epsilon f - Lf\|_{[x_1,x_2]} \to 0, \text{ as } \epsilon \downarrow 0, \tag{59}$$

where for any function g we define $\|g\|_{[x_1,x_2]} = \sup_{x \in [x_1,x_2]} |g(x)|$. This follows directly by our assumptions on the function $V^\epsilon(x)$. Therefore, it is indeed enough to prove (57).

By the Itô formula applied to the function $e^{-\lambda t} f(x)$ we immediately get that (57) is equivalent to

$$|\mathbb{E}^\epsilon_{x,y}[\int_0^{\tau^\epsilon(x_1,x_2)} e^{-\lambda u} f_x(X^\epsilon_u)\gamma^\epsilon_1(X^\epsilon_u, Y^\epsilon_u) dL^\epsilon_u - \int_0^{\tau^\epsilon(x_1,x_2)} \frac{1}{2} e^{-\lambda u}[f_x \frac{V^\epsilon_x}{V^\epsilon}](X^\epsilon_u) du]| \to 0, \tag{60}$$

as $\epsilon \downarrow 0$. We can estimate the left hand side of (60) as in Lemma 2.1 of [5]:

Consider the auxiliary function

$$v^\epsilon(x,y) = \frac{1}{2} y^2 f_x(x) \frac{V^\epsilon_x(x)}{V^\epsilon(x)} + y f_x(x) \frac{V^{u,\epsilon}_x(x) V^{l,\epsilon}(x) - V^{l,\epsilon}_x(x) V^{u,\epsilon}(x)}{V^\epsilon(x)}. \tag{61}$$

It is easy to see that v^ϵ is a solution to the P.D.E.

$$\begin{aligned} v^\epsilon_{yy}(x,y) &= f_x(x)\frac{V^\epsilon_x(x)}{V^\epsilon(x)}, \;\; y\in D^\epsilon_x \\ \frac{\partial_y v^\epsilon(x,y)}{\partial n^\epsilon(x,y)} &= -f_x(x)\frac{\gamma^\epsilon_1(x,y)}{|\gamma^\epsilon_2(x,y)|}, \;\; y\in\partial D^\epsilon_x, \end{aligned} \tag{62}$$

where $n^\epsilon(x,y)=\frac{\gamma^\epsilon_2(x,y)}{|\gamma^\epsilon_2(x,y)|}$ and $x\in\mathbb{R}$ is a parameter.

If we apply Itô formula to the function $e^{-\lambda t}v^\epsilon(x,y)$ we get that $e^{-\lambda t}v^\epsilon(X^\epsilon_t,Y^\epsilon_t)$ satisfies with probability one:

$$\begin{aligned} e^{-\lambda t}v^\epsilon(X^\epsilon_t,Y^\epsilon_t) &= v^\epsilon(x,y)+\int_0^t e^{-\lambda s}[\frac{1}{2}\triangle v^\epsilon(X^\epsilon_s,Y^\epsilon_s)-\lambda v^\epsilon(X^\epsilon_s,Y^\epsilon_s)]ds \\ &+ \int_0^t e^{-\lambda s}v^\epsilon_x(X^\epsilon_s,Y^\epsilon_s)dW^1_s+\int_0^t e^{-\lambda s}v^\epsilon_y(X^\epsilon_s,Y^\epsilon_s)dW^2_s \\ &+ \int_0^t e^{-\lambda s}v^\epsilon_x(X^\epsilon_s,Y^\epsilon_s)\gamma^\epsilon_1(X^\epsilon_s,Y^\epsilon_s)dL^\epsilon_s \\ &+ \int_0^t e^{-\lambda s}v^\epsilon_y(X^\epsilon_s,Y^\epsilon_s)\gamma^\epsilon_2(X^\epsilon_s,Y^\epsilon_s)dL^\epsilon_s \end{aligned} \tag{63}$$

Since $v^\epsilon(x,y)$ satisfies (62) we have:

$$\begin{aligned} &|\mathbb{E}^\epsilon_{x,y}[\int_0^{\tau^\epsilon(x_1,x_2)} e^{-\lambda u}f_x(X^\epsilon_u)\gamma^\epsilon_1(X^\epsilon_u,Y^\epsilon_u)dL^\epsilon_u-\int_0^{\tau^\epsilon(x_1,x_2)}\frac{1}{2}e^{-\lambda u}[f_x\frac{V^\epsilon_x}{V^\epsilon}](X^\epsilon_u)du]|\leq \\ \leq\;& |\mathbb{E}^\epsilon_{x,y}e^{-\lambda\tau^\epsilon(x_1,x_2)}v^\epsilon(X^\epsilon_{\tau^\epsilon(x_1,x_2)},Y^\epsilon_{\tau^\epsilon(x_1,x_2)})|+|v^\epsilon(x,y)|+ \\ +\;& |\mathbb{E}^\epsilon_{x,y}[\int_0^{\tau^\epsilon(x_1,x_2)} e^{-\lambda s}[\frac{1}{2}v^\epsilon_{xx}(X^\epsilon_s,Y^\epsilon_s)-\lambda v^\epsilon(X^\epsilon_s,Y^\epsilon_s)]ds]|+ \\ +\;& |\mathbb{E}^\epsilon_{x,y}[\int_0^{\tau^\epsilon(x_1,x_2)} e^{-\lambda s}v^\epsilon_x(X^\epsilon_s,Y^\epsilon_s)\gamma^\epsilon_1(X^\epsilon_s,Y^\epsilon_s)dL^\epsilon_s]| \end{aligned} \tag{64}$$

For any time $t\in[0,\tau^\epsilon(x_1,x_2)]$ the x-component of the process $(X^\epsilon_t,Y^\epsilon_t)$ satisfies $x_1\leq X^\epsilon_t\leq x_2$, i.e. it is far away from the point of discontinuity. Taking into account the latter and the definition of $v^\epsilon(x,y)$ by (61) we get that the first three terms in the right hand side of (64) are bounded by $\epsilon^2 C_0$ for ϵ small enough. So, it remains to consider the last term, i.e. the integral in local time. First of all it is easy to see that there exists a $C_0>0$ such that

$$|\mathbb{E}^\epsilon_{x,y}[\int_0^{\tau^\epsilon(x_1,x_2)} e^{-\lambda s}v^\epsilon_x(X^\epsilon_s,Y^\epsilon_s)\gamma^\epsilon_1(X^\epsilon_s,Y^\epsilon_s)dL^\epsilon_s]|\leq\epsilon^2 C_0\mathbb{E}^\epsilon_{x,y}\int_0^{\tau^\epsilon(x_1,x_2)} e^{-\lambda t}\epsilon dL^\epsilon_t \tag{65}$$

As far as the integral in local time on the right hand side of (65) is concerned, we claim that there exists an $\epsilon_0>0$ and a $C_0>0$ such that for all $\epsilon<\epsilon_0$

$$\mathbb{E}^\epsilon_{x,y}\int_0^{\tau^\epsilon(x_1,x_2)} e^{-\lambda t}\epsilon dL^\epsilon_t\leq C_0. \tag{66}$$

Hence, taking into account (65) and (66) we get that the right hand side of (64) converges to zero. So the convergence (60) holds.

It remains to prove the claim (66). This can be done as in Lemma 2.2 of [5]. In particular, one considers the auxiliary function

$$w^{\epsilon}(x,y)=y^2\frac{1}{V^{\epsilon}(x)}+y\frac{V^{l,\epsilon}(x)-V^{u,\epsilon}(x)}{V^{\epsilon}(x)}.$$

It is easy to see that w^{ϵ} is a solution to the P.D.E.

$$\begin{aligned} w^{\epsilon}_{yy}(x,y) &= \frac{2}{V^{\epsilon}(x)}, \quad y\in D^{\epsilon}_x \\ \frac{\partial_y w^{\epsilon}(x,y)}{\partial n^{\epsilon}(x,y)} &= -1, \quad y\in\partial D^{\epsilon}_x, \end{aligned} \tag{67}$$

where $n^{\epsilon}(x,y)=\frac{\gamma^{\epsilon}_2(x,y)}{|\gamma^{\epsilon}_2(x,y)|}$ and $x\in\mathbb{R}$ is a parameter.

The claim follows now directly by applying Itô formula to the function $e^{-\lambda t}w^{\epsilon}(x,y)$. □

4 Proof of Lemma 2.4

Proof of Lemma 2.4. Let $\hat{\tau}^{\epsilon}=\hat{\tau}^{\epsilon}(\pm\kappa)$ be the exit time of $\hat{X}^{\epsilon}_t$ (see (53)) from the interval $(-\kappa,\kappa)$. Denote also by $\hat{\mathbb{E}}_x$ the mathematical expectation related to the probability law induced by $\hat{X}^{\epsilon,x}_t$.

Let η be a positive number, $\kappa>0$ be a small positive number and consider x_0 with $|x_0|<\kappa$ and $y_0\in D^{\epsilon}_{x_0}$. We want to estimate the difference

$$|\mathbb{E}^{\epsilon}_{x_0,y_0}\tau^{\epsilon}(\pm\kappa)-\hat{\mathbb{E}}^{\epsilon}_{x_0}\hat{\tau}^{\epsilon}(\pm\kappa)|.$$

As it can be derived by Theorem 2.5.1 of [3] the function $\phi^{\epsilon}(x,y)=\mathbb{E}^{\epsilon}_{x,y}\tau^{\epsilon}(\pm\kappa)$ is solution to

$$\begin{aligned} \frac{1}{2}\triangle\phi^{\epsilon}(x,y) &= -1 \quad \text{in } (x,y)\in(-\kappa,\kappa)\times D^{\epsilon}_x \\ \phi^{\epsilon}(\pm\kappa,y) &= 0 \\ \frac{\partial\phi^{\epsilon}}{\partial\gamma^{\epsilon}}(x,y) &= 0 \quad \text{on } (x,y)\in(-\kappa,\kappa)\times\partial D^{\epsilon}_x \end{aligned} \tag{68}$$

Moreover, $\hat{\phi}^{\epsilon}(x)=\hat{\mathbb{E}}^{\epsilon}_x\hat{\tau}^{\epsilon}(\pm\kappa)$ is solution to

$$\begin{aligned} \frac{1}{2}\hat{\phi}^{\epsilon}_{xx}(x)+\frac{1}{2}\frac{V^{\epsilon}_x(x)}{V^{\epsilon}(x)}\hat{\phi}^{\epsilon}_x(x) &= -1 \quad \text{in } x\in(-\kappa,\kappa) \\ \hat{\phi}^{\epsilon}(\pm\kappa) &= 0 \end{aligned} \tag{69}$$

Let $f^{\epsilon}(x,y)=\phi^{\epsilon}(x,y)-\hat{\phi}^{\epsilon}(x)$. Then f^{ϵ} will satisfy

$$\begin{aligned} \frac{1}{2}\triangle f^{\epsilon}(x,y) &= \frac{1}{2}\frac{V^{\epsilon}_x(x)}{V^{\epsilon}(x)}\hat{\phi}^{\epsilon}_x(x) \quad \text{in } (x,y)\in(-\kappa,\kappa)\times D^{\epsilon}_x \\ f^{\epsilon}(\pm\kappa,y) &= 0 \\ \frac{\partial f^{\epsilon}}{\partial\gamma^{\epsilon}}(x,y) &= -\hat{\phi}^{\epsilon}_x(x)\gamma^{\epsilon}_1(x,y) \quad \text{on } (x,y)\in(-\kappa,\kappa)\times\partial D^{\epsilon}_x \end{aligned} \tag{70}$$

By applying Itô formula to the function f^{ϵ} and recalling that f^{ϵ} satisfies (70) we get that

$$|f^{\epsilon}(x,y)| \leq |\mathbb{E}^{\epsilon}_{x,y}[\int_0^{\tau^{\epsilon}(\pm\kappa)} \hat{\phi}^{\epsilon}_x(X^{\epsilon}_u)\gamma^{\epsilon}_1(X^{\epsilon}_u,Y^{\epsilon}_u)dL^{\epsilon}_u - \int_0^{\tau^{\epsilon}(\pm\kappa)} \frac{1}{2}[\hat{\phi}^{\epsilon}_x\frac{V^{\epsilon}_x}{V^{\epsilon}}](X^{\epsilon}_u)du]| \tag{71}$$

We can estimate the right hand side of (71) similarly to the left hand side of (60) of Lemma 2.3 (see also Lemma 2.1 in [5]). Consider the auxiliary function

$$w^{\epsilon}(x,y) = \frac{1}{2}y^2\hat{\phi}^{\epsilon}_x(x)\frac{V^{\epsilon}_x(x)}{V^{\epsilon}(x)} + y\hat{\phi}^{\epsilon}_x(x)\frac{V^{u,\epsilon}_x(x)V^{l,\epsilon}(x) - V^{l,\epsilon}_x(x)V^{u,\epsilon}(x)}{V^{\epsilon}(x)} \tag{72}$$

It is easy to see that w is a solution to the P.D.E.

$$\begin{aligned} w^{\epsilon}_{yy}(x,y) &= \hat{\phi}^{\epsilon}_x(x)\frac{V^{\epsilon}_x(x)}{V^{\epsilon}(x)}, \quad y \in D^{\epsilon}_x \\ \frac{\partial_y w^{\epsilon}(x,y)}{\partial n^{\epsilon}(x,y)} &= -\hat{\phi}^{\epsilon}_x(x)\frac{\gamma^{\epsilon}_1(x,y)}{|\gamma^{\epsilon}_2(x,y)|}, \quad y \in \partial D^{\epsilon}_x, \end{aligned} \tag{73}$$

where $n^{\epsilon}(x,y) = \frac{\gamma^{\epsilon}_2(x,y)}{|\gamma^{\epsilon}_2(x,y)|}$ and $x \in \mathbb{R}$ is a parameter.

Then if we apply Itô formula to the function $w^{\epsilon}(x,y)$ and recall that w^{ϵ} satisfies (73), we get an upper bound for the right hand side of (71) that is the same to the right hand side of (64) with $\lambda = 0$, v^{ϵ} replaced by w^{ϵ} and $\tau^{\epsilon}(x_1,x_2)$ replaced by $\tau^{\epsilon}(\pm\kappa)$. Namely, for $(x,y) = (x_0,y_0)$, we have the following

$$\begin{aligned} & |\mathbb{E}^{\epsilon}_{x_0,y_0}[\int_0^{\tau^{\epsilon}(\pm\kappa)} \hat{\phi}^{\epsilon}_x(X^{\epsilon}_u)\gamma^{\epsilon}_1(X^{\epsilon}_u,Y^{\epsilon}_u)dL^{\epsilon}_u - \int_0^{\tau^{\epsilon}(\pm\kappa)} \frac{1}{2}[\hat{\phi}^{\epsilon}_x\frac{V^{\epsilon}_x}{V^{\epsilon}}](X^{\epsilon}_u)du]| \leq \\ \leq\ & \sup_{(x,y)\in\{\pm\kappa\}\times D^{\epsilon}_{\pm\kappa}} |w^{\epsilon}(x,y)| + \sup_{(x,y)\in\{x_0\}\times D^{\epsilon}_{x_0}} |w^{\epsilon}(x,y)| + \\ +\ & |\mathbb{E}^{\epsilon}_{x_0,y_0}\int_0^{\tau^{\epsilon}(\pm\kappa)} \frac{1}{2}w^{\epsilon}_{xx}(X^{\epsilon}_s,Y^{\epsilon}_s)ds| + |\mathbb{E}^{\epsilon}_{x_0,y_0}\int_0^{\tau^{\epsilon}(\pm\kappa)} w^{\epsilon}_x(X^{\epsilon}_s,Y^{\epsilon}_s)\gamma^{\epsilon}_1(X^{\epsilon}_s,Y^{\epsilon}_s)dL^{\epsilon}_s| \end{aligned} \tag{74}$$

Now one can solve (69) explicitly and get that for $x \in [-\kappa,\kappa]$

$$\hat{\phi}^{\epsilon}(x) = \int_{-\kappa}^{x} \frac{-2}{V^{\epsilon}(y)}\int_{-\kappa}^{y} V^{\epsilon}(z)dzdy + [\frac{\int_{-\kappa}^{\kappa}\frac{2}{V^{\epsilon}(y)}\int_{-\kappa}^{y}V^{\epsilon}(z)dzdy}{\int_{-\kappa}^{\kappa}\frac{1}{V^{\epsilon}(y)}dy}]\int_{-\kappa}^{x}\frac{1}{V^{\epsilon}(y)}dy. \tag{75}$$

Using (75) and the form of $V^{\epsilon}(x)$ as described by (5)-(9) we get that the first two terms of the right hand side of (74) can be made arbitrarily small for ϵ sufficiently small. For ϵ small enough the two integral terms of the right hand side of (74) are bounded by $C_0\xi^{\epsilon}\mathbb{E}^{\epsilon}_{x_0,y_0}\tau^{\epsilon}(\pm\kappa)$, where ξ^{ϵ} is defined in (9). The local time integral can be treated as in Lemma 2.2 of [5], so it will not be repeated here (see also the end of the proof of Lemma 2.3). In reference to the latter, we mention that the singularity at the point $x = 0$ complicates a bit the situation. However, assumption (9) allows one to follow the procedure mentioned and derive the aforementioned estimate for the local time integral. Hence, we have the following upper bound for $f^{\epsilon}(x_0,y_0)$

$$|f^{\epsilon}(x_0,y_0)| \leq C_0[\kappa\eta + \xi^{\epsilon}\mathbb{E}^{\epsilon}_{x_0,y_0}\tau^{\epsilon}(\pm\kappa)]. \tag{76}$$

Moreover, it follows from (75) (see also [10]) that for $\eta > 0$ there exists a $\kappa_\eta > 0$ such that for every $0 < \kappa < \kappa_\eta$, for sufficiently small ϵ and for all x with $|x| \leq \kappa$

$$\frac{1}{\kappa}|\hat{\mathbb{E}}_x^\epsilon \hat{\tau}^\epsilon(\pm\kappa) - \kappa\theta| \leq \eta \tag{77}$$

,where $\theta = \frac{v(0+)-v(0-)}{[u'(0+)]^{-1}+[u'(0-)]^{-1}}$.

Therefore, since $\xi^\epsilon \downarrow 0$ (by assumption (9)), (76) and (77) give us for sufficiently small ϵ that

$$\frac{1}{\kappa}|\mathbb{E}_{x_0,y_0}^\epsilon \tau^\epsilon(\pm\kappa) - \hat{\mathbb{E}}_{x_0}^\epsilon \hat{\tau}^\epsilon(\pm\kappa)| \leq C_0\eta.$$

The latter and (77) conclude the proof of the lemma. □

5 Proof of Lemma 2.5

In order to prove Lemma 2.5 we will make use of a result regarding the invariant measures of the associated Markov chains (see Lemma 5.2) and a result regarding the strong Markov character of the limiting process (see Lemma 5.4 and the beginning of the proof of Lemma 2.5).

Of course, since the gluing conditions at 0 are of local character, it is sufficient to consider not the whole domain D^ϵ, but just the part of D^ϵ that is in the neighborhood of $x = 0$. Thus, we consider the process $(X_t^\epsilon, Y_t^\epsilon)$ in

$$\Xi^\epsilon = \{(x, y) : |x| \leq 1, y \in D_x^\epsilon\}$$

that reflects normally on $\partial\Xi^\epsilon$.

Recall that $0 < \kappa_0 < \kappa$. Define the set

$$\Gamma_\kappa = \{(x, y) : x = \kappa, y \in D_\kappa^\epsilon\} \tag{78}$$

For notational convenience we will write $\Gamma_{\pm\kappa} = \Gamma_\kappa \cup \Gamma_{-\kappa}$.

Define $\Delta_\kappa = \Gamma_{\pm\kappa} \cup \Gamma_{\pm(1-\kappa)}$ and $\Delta_{\kappa_0} = \Gamma_{\pm\kappa_0} \cup \Gamma_{\pm(1-\kappa_0)}$. We consider two cycles of Markov times $\{\tau_n\}$ and $\{\sigma_n\}$ such that:

$$0 = \sigma_0 \leq \tau_0 < \sigma_1 < \tau_1 < \sigma_2 < \tau_2 < \dots$$

where:

$$\begin{aligned} \tau_n &= \inf\{t \geq \sigma_n : (X_t^\epsilon, Y_t^\epsilon) \in \Delta_\kappa\} \\ \sigma_n &= \inf\{t \geq \tau_{n-1} : (X_t^\epsilon, Y_t^\epsilon) \in \Delta_{\kappa_0}\} \end{aligned} \tag{79}$$

We will use the relations

$$\begin{aligned} \mu_\epsilon(A) &= \int_{\Delta_\kappa} \nu_\epsilon^\kappa(dx, dy)\mathbb{E}_{x,y}^\epsilon \int_0^{\tau_1} \chi_{[(X_t^\epsilon, Y_t^\epsilon)\in A]}dt \\ &= \int_{\Delta_{\kappa_0}} \nu_\epsilon^{\kappa_0}(dx, dy)\mathbb{E}_{x,y}^\epsilon \int_0^{\sigma_1} \chi_{[(X_t^\epsilon, Y_t^\epsilon)\in A]}dt \end{aligned} \tag{80}$$

and

$$\begin{aligned} \nu_\epsilon^\kappa(B) &= \int_{\Delta_{\kappa_0}} \nu_\epsilon^{\kappa_0}(dx,dy)\mathbb{P}^\epsilon_{x,y}[(X^\epsilon_{\tau_0},Y^\epsilon_{\tau_0})\in B] \\ \nu_\epsilon^{\kappa_0}(C) &= \int_{\Delta_\kappa} \nu_\epsilon^{\kappa}(dx,dy)\mathbb{P}^\epsilon_{x,y}[(X^\epsilon_{\sigma_1},Y^\epsilon_{\sigma_1})\in C] \end{aligned} \tag{81}$$

between the invariant measures μ_ϵ of the process $(X_t^\epsilon, Y_t^\epsilon)$ in D^ϵ, ν_ϵ^κ and $\nu_\epsilon^{\kappa_0}$ of the Markov chains $(X^\epsilon_{\tau_n}, Y^\epsilon_{\tau_n})$ and $(X^\epsilon_{\sigma_n}, Y^\epsilon_{\sigma_n})$ on Δ_κ and Δ_{κ_0} respectively (see [11]; also [6]). It is clear that the first invariant measure is, up to a constant, the Lebesgue measure on D^ϵ.

Formula (80) implies the corresponding equality for the integrals with respect to the invariant measure μ_ϵ. This means that if $\Psi(x,y)$ is an integrable function, then

$$\begin{aligned} \int\int_{\mathbb{R}^2} \Psi(x,y)\mu_\epsilon(dx,dy) &= \int\int_{\Xi^\epsilon} \Psi(x,y)dxdy \\ &= \int_{\Delta_\kappa} \nu_\epsilon^\kappa(dx,dy)\mathbb{E}^\epsilon_{x,y}\int_0^{\tau_1} \Psi(X_t^\epsilon,Y_t^\epsilon)dt \\ &= \int_{\Delta_{\kappa_0}} \nu_\epsilon^{\kappa_0}(dx,dy)\mathbb{E}^\epsilon_{x,y}\int_0^{\sigma_1} \Psi(X_t^\epsilon,Y_t^\epsilon)dt \end{aligned} \tag{82}$$

The following simple lemma will be used in the proof of Lemmata 5.2 and 5.4.

Lemma 5.1. *Let $0 < x_1 < x_2$, ψ be a function defined in $[x_1,x_2]$ and ϕ be a function defined on x_1 and x_2. Then*

$$\lim_{\epsilon\downarrow 0}\mathbb{E}^\epsilon_{x,y}[\phi(X^\epsilon_{\tau(x_1,x_2)}) + \int_0^{\tau(x_1,x_2)} \psi(X_t^\epsilon)dt] = g(x),$$

uniformly in $(x,y)\in[x_1,x_2]\times D_x^\epsilon$ and

$$\begin{aligned} g(x) &= \frac{u(x_2)-u(x)}{u(x_2)-u(x_1)}[\phi(x_1) + \int_{x_1}^x (u(y)-u(x_1))\psi(y)dv(y)] \\ &+ \frac{u(x)-u(x_1)}{u(x_2)-u(x_1)}[\phi(x_2) + \int_x^{x_2} (u(x_2)-u(y))\psi(y)dv(y)] \end{aligned}$$

A similar result holds for $(x,y)\in[-x_2,-x_1]\times D_x^\epsilon$.

Proof. This lemma is similar to Lemma 8.4.6 of [6], so we briefly outline its proof. First one proves that $\mathbb{E}^\epsilon_{x,y}\tau(x_1,x_2)$ is bounded in ϵ for ϵ small enough and for all $(x,y)\in[x_1,x_2]\times D_x^\epsilon$. The latter and the proof of Lemma 2.3 show that in this case we can take $\lambda = 0$ in Lemma (2.3) and apply it to the function g that is the solution to

$$\begin{aligned} D_v D_u g(x) &= -\psi(x), \;\; x_1 < x < x_2 \\ g(x_i) &= \phi(x_i), \;\; i = 1,2. \end{aligned}$$

This gives the desired result. □

Lemma 5.2 below characterizes the asymptotics of the invariant measures ν_ϵ^κ and $\nu_\epsilon^{\kappa_0}$.

Lemma 5.2. *Let v be the function defined by (6). The following statements hold*

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\nu_\epsilon^\kappa(\Gamma_\kappa)[u(\kappa)-u(\kappa_0)] &= 1\\ \lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\nu_\epsilon^\kappa(\Gamma_{1-\kappa})[u(1-\kappa_0)-u(1-\kappa)] &= 1\end{aligned} \tag{83}$$

Similar statements are true for $\nu_\epsilon^\kappa(\Gamma_{-\kappa}), \nu_\epsilon^\kappa(\Gamma_{-(1-\kappa)})$ and $\nu_\epsilon^{\kappa_0}$.

Proof. We calculate the asymptotics of the invariant measures ν_ϵ^κ and $\nu_\epsilon^{\kappa_0}$ by selecting Ψ properly. For this purpose let us choose $\Psi(x,y)=\Psi(x)$ (i.e. it is a function only of x) that is bounded, continuous and is 0 outside $\kappa<x<1-\kappa$.

With this choice for Ψ the left hand side of (82) becomes:

$$\int\int_{\Xi^\epsilon}\Psi(x,y)dxdy=\int_\kappa^{1-\kappa}\Psi(x)V^\epsilon(x)dx=\epsilon\int_\kappa^{1-\kappa}\Psi(x)dv^\epsilon(x), \tag{84}$$

where we recall that $v^\epsilon(x)=\int_0^x\frac{V^\epsilon(y)}{\epsilon}dy$ (see (4)).

Moreover, the particular choice if Ψ, also implies that $\int_{\sigma_1}^{\tau_1}\Psi(X_t^\epsilon)dt=0$. Then, the right hand side of (82) becomes:

$$\begin{aligned}\int_{\Delta_\kappa}\nu_\epsilon^\kappa(dx,dy)\mathbb{E}_{x,y}^\epsilon\int_0^{\tau_1}\Psi(X_t^\epsilon)dt &= \int_{\Delta_\kappa}\nu_\epsilon^\kappa(dx,dy)\mathbb{E}_{x,y}^\epsilon\int_0^{\sigma_1}\Psi(X_t^\epsilon)dt \qquad (85)\\ = \int_{D_\kappa^\epsilon}\nu_\epsilon^\kappa(\kappa,dy)\mathbb{E}_{\kappa,y}^\epsilon\int_0^{\sigma_1}\Psi(X_t^\epsilon)dt &+ \int_{D_{1-\kappa}^\epsilon}\nu_\epsilon^\kappa(1-\kappa,dy)\mathbb{E}_{1-\kappa,y}^\epsilon\int_0^{\sigma_1}\Psi(X_t^\epsilon)dt\end{aligned}$$

Next, we express the right hand side of (85) through the v and u functions (defined by (6)) using Lemma 5.1. We start with the term $\mathbb{E}_{\kappa,y}^\epsilon\int_0^{\sigma_1}\Psi(X_t^\epsilon)dt$. Use Lemma 5.1 with $\phi=0$ and $\psi(x)=\Psi(x)$. For sufficiently small ϵ we have

$$\begin{aligned}\mathbb{E}_{\kappa,y}^\epsilon\int_0^{\sigma_1}\Psi(X_t^\epsilon)dt &= \frac{u(1-\kappa_0)-u(\kappa)}{u(1-\kappa_0)-u(\kappa_0)}\int_{\kappa_0}^{\kappa}(u(y)-u(\kappa_0))\Psi(y)dv(y)\\ &+\frac{u(\kappa)-u(\kappa_0)}{u(1-\kappa_0)-u(\kappa_0)}\int_\kappa^{1-\kappa_0}(u(1-\kappa_0)-u(y))\Psi(y)dv(y)+o(1)\\ &=\frac{u(\kappa)-u(\kappa_0)}{u(1-\kappa_0)-u(\kappa_0)}\int_\kappa^{1-\kappa}(u(1-\kappa_0)-u(y))\Psi(y)dv(y)+o(1)\end{aligned} \tag{86}$$

where the term $o(1)\downarrow 0$ as $\epsilon\downarrow 0$. Similarly for the term $\mathbb{E}_{1-\kappa,y}^\epsilon\int_0^{\sigma_1}\Psi(X_t^\epsilon)dt$ we have

$$\begin{aligned}\mathbb{E}_{1-\kappa,y}^\epsilon\int_0^{\sigma_1}\Psi(X_t^\epsilon)dt &= \frac{u(1-\kappa_0)-u(1-\kappa)}{u(1-\kappa_0)-u(\kappa_0)}\int_{\kappa_0}^{1-\kappa}(u(y)-u(\kappa_0))\Psi(y)dv(y)\\ &+\frac{u(1-\kappa)-u(\kappa_0)}{u(1-\kappa_0)-u(\kappa_0)}\int_{1-\kappa}^{1-\kappa_0}(u(1-\kappa_0)-u(y))\Psi(y)dv(y)+o(1)\\ &=\frac{u(1-\kappa_0)-u(1-\kappa)}{u(1-\kappa_0)-u(\kappa_0)}\int_\kappa^{1-\kappa}(u(y)-u(\kappa_0))\Psi(y)dv(y)+o(1)\end{aligned} \tag{87}$$

Taking into account relations (82) and (84)-(87) and the particular choice of the function Ψ we get the following relation for sufficiently small ϵ

$$\begin{aligned} & \epsilon\int_{\kappa}^{1-\kappa}\Psi(x)d\nu^{\epsilon}(x)= \qquad (88)\\ = \quad & \nu_{\epsilon}^{\kappa}(\Gamma_{\kappa})\left[\frac{u(\kappa)-u(\kappa_0)}{u(1-\kappa_0)-u(\kappa_0)}\int_{\kappa}^{1-\kappa}(u(1-\kappa_0)-u(y))\Psi(y)d\nu(y)+o(1)\right]\\ + \quad & \nu_{\epsilon}^{\kappa}(\Gamma_{1-\kappa})\left[\frac{u(1-\kappa_0)-u(1-\kappa)}{u(1-\kappa_0)-u(\kappa_0)}\int_{\kappa}^{1-\kappa}(u(y)-u(\kappa_0))\Psi(y)d\nu(y)+o(1)\right]\end{aligned}$$

At each continuity point of $\nu(x)$ we have $\nu(x)=\lim_{\epsilon\downarrow 0}\nu^{\epsilon}(x)$, so the equality above is true for an arbitrary continuous function Ψ if the following hold:

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\nu_{\epsilon}^{\kappa}(\Gamma_{\kappa})[u(\kappa)-u(\kappa_0)] &= 1\\ \lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\nu_{\epsilon}^{\kappa}(\Gamma_{1-\kappa})[u(1-\kappa_0)-u(1-\kappa)] &= 1\end{aligned}$$

Thus, the proof of Lemma 5.2 is complete. □

Remark 5.3. *Equalities (81) immediately give us that* $\nu_{\epsilon}^{\kappa}(\Gamma_{\pm\kappa})=\nu_{\epsilon}^{\kappa_0}(\Gamma_{\pm\kappa_0})$ *and* $\nu_{\epsilon}^{\kappa}(\Gamma_{\pm(1-\kappa)})=\nu_{\epsilon}^{\kappa_0}(\Gamma_{\pm(1-\kappa_0)})$ *(see also the related discussion in chapter* 8 *of [6]). Finally, it is easy to see that* $\nu_{\epsilon}^{\kappa}(\Gamma_{\pm\kappa})=\nu_{\epsilon}^{\kappa}(\Gamma_{\kappa})+\nu_{\epsilon}^{\kappa}(\Gamma_{-\kappa})$.

□

Lemma 5.4. *Let us consider fixed numbers* $0<x_1<x_2$ *and let* $x_0\in(x_1,x_2)$. *For every* x_0 *we have*

$$\lim_{\epsilon\downarrow 0}\max_{f:\|f\|\leq 1}|\mathbb{E}^{\epsilon}_{x_0,y_{01}}f(X^{\epsilon}_{\tau(x_1,x_2)},Y^{\epsilon}_{\tau(x_1,x_2)})-\mathbb{E}^{\epsilon}_{x_0,y_{02}}f(X^{\epsilon}_{\tau(x_1,x_2)},Y^{\epsilon}_{\tau(x_1,x_2)})|=0,$$

uniformly in $y_{01},y_{02}\in D^{\epsilon}_{x_0}$ *and for functions* $f(x,y)$ *that are well defined on* $(x,y)\in\{x_1,x_2\}\times D^{\epsilon}_{x}$.

Proof. We only need to observe that (a): Lemma 5.1 applied to $\phi(x)=f(x,0)$ and $\psi(x)=0$ immediately gives us that

$$\lim_{\epsilon\downarrow 0}\max_{f:\|f\|\leq 1}|\mathbb{E}^{\epsilon}_{x_0,y_{01}}f(X^{\epsilon}_{\tau(x_1,x_2)},0)-\mathbb{E}^{\epsilon}_{x_0,y_{02}}f(X^{\epsilon}_{\tau(x_1,x_2)},0)|=0,$$

and (b): the $y-$component of the process converges to zero. □

Proof of Lemma 2.5. Firstly, we prove (using Lemma 5.4) that $\mathbb{P}^{\epsilon}_{x,y}(X^{\epsilon}_{\tau(\pm\kappa)}=\kappa)$ has approximately the same value for all x that belong to a small neighborhood of zero for ϵ small enough. Secondly, we identify (using (80) and Lemma 5.2) that this value is $\frac{[u'(0+)]^{-1}}{[u'(0+)]^{-1}+[u'(0-)]^{-1}}$.

We begin by showing that for any $(x_i,y_i)\in\Gamma_{\pm\kappa_0}$ with $i=1,2$

$$|\mathbb{E}^{\epsilon}_{x_1,y_1}\chi_{(X^{\epsilon}_{\tau(\pm\kappa)}=\kappa)}-\mathbb{E}^{\epsilon}_{x_2,y_2}\chi_{(X^{\epsilon}_{\tau(\pm\kappa)}=\kappa)}|\to 0 \text{ as } \epsilon\downarrow 0, \qquad (89)$$

uniformly in y_1, y_2.

Let us define for notational convenience

$$F^\epsilon(x,y) = \mathbb{E}^\epsilon_{x,y}[\chi_{(X^\epsilon_{\tau(\pm\kappa)}=\kappa)}]$$

Firstly, we prove that (89) holds for $(\kappa_0, y_1), (\kappa_0, y_2) \in \Gamma_{\kappa_0}$. Let κ_1, κ_2 be such that $0 < \kappa_1 < \kappa_0 < \kappa_2 < \kappa$. Using strong Markov property with respect to $\tau(\kappa_1, \kappa_2) < \tau(\pm\kappa)$ we get

$$\mathbb{E}^\epsilon_{\kappa_0,y}\chi_{(X^\epsilon_{\tau(\pm\kappa)}=\kappa)} = \mathbb{E}^\epsilon_{\kappa_0,y}\mathbb{E}^\epsilon_{X_{\tau(\kappa_1,\kappa_2)},Y_{\tau(\kappa_1,\kappa_2)}}[\chi_{(X^\epsilon_{\tau(\pm\kappa)}=\kappa)}] \tag{90}$$

Equation (90) and Lemma 5.4 imply that (89) holds for $(\kappa_0, y_1), (\kappa_0, y_2) \in \Gamma_{\kappa_0}$ uniformly in $y_1, y_2 \in D^\epsilon_{\kappa_0}$.

Hence, we have

$$|F^\epsilon(\kappa_0, y_1) - F^\epsilon(\kappa_0, y_2)| \to 0 \text{ as } \epsilon \downarrow 0. \tag{91}$$

Similarly, it can be shown that (91) holds for $(-\kappa_0, y_1), (-\kappa_0, y_2) \in \Gamma_{-\kappa_0}$ as well.

Secondly, we observe that if (89) holds for $(x_1, y_1) = (\kappa_0, 0)$ and $(x_2, y_2) = (-\kappa_0, 0)$ then it will hold for any $(x_i, y_i) \in \Gamma_{\pm\kappa_0}$ with $i = 1, 2$. Indeed, we have

$$\begin{aligned}|F^\epsilon(\kappa_0, y_1) - F^\epsilon(-\kappa_0, y_2)| \quad &\leq \quad |F^\epsilon(\kappa_0, 0) - F^\epsilon(-\kappa_0, 0)| + |F^\epsilon(\kappa_0, y_1) - F^\epsilon(\kappa_0, 0)| \\ &\quad + |F^\epsilon(-\kappa_0, y_2) - F^\epsilon(-\kappa_0, 0)|\end{aligned}$$

The last two terms in the right hand side of the inequality above converge to zero as $\epsilon \downarrow 0$ by the discussion above. Hence, it remains to prove that

$$|F^\epsilon(\kappa_0, 0) - F^\epsilon(-\kappa_0, 0)| \to 0 \text{ as } \epsilon \downarrow 0. \tag{92}$$

Let us choose some $\kappa_0^{'}$ such that $0 < \kappa_0 < \kappa_0^{'} < \kappa$. Obviously, if the process starts from some point on Γ_x with $x \in [-\kappa, \kappa_0]$ then $\tau(-\kappa, \kappa_0^{'}) \leq \tau(\pm\kappa)$. So, by applying the strong Markov property with respect to $\tau(-\kappa, \kappa_0^{'}) \leq \tau(\pm\kappa)$ we have

$$\begin{aligned}\inf_{y \in D^\epsilon_{\kappa_0^{'}}} F^\epsilon(\kappa_0^{'}, y)\mathbb{P}^\epsilon_{x,0}[(X^\epsilon_{\tau(-\kappa,\kappa_0^{'})}, Y^\epsilon_{\tau(-\kappa,\kappa_0^{'})}) \in \Gamma_{\kappa_0^{'}}] \quad &\leq \quad F^\epsilon(x, 0) \leq \\ \leq \sup_{y \in D^\epsilon_{\kappa_0^{'}}} F^\epsilon(\kappa_0^{'}, y)\mathbb{P}^\epsilon_{x,0}[(X^\epsilon_{\tau(-\kappa,\kappa_0^{'})}, Y^\epsilon_{\tau(-\kappa,\kappa_0^{'})}) \in \Gamma_{\kappa_0^{'}}]\end{aligned} \tag{93}$$

Using the latter, we have

$$\begin{aligned}&|F^\epsilon(\kappa_0, 0) - F^\epsilon(-\kappa_0, 0)| \leq \\ \leq \quad &|\sup_{y \in D^\epsilon_{\kappa_0^{'}}} F^\epsilon(\kappa_0^{'}, y) - \inf_{y \in D^\epsilon_{\kappa_0^{'}}} F^\epsilon(\kappa_0^{'}, y)| \max_{x=\kappa_0,-\kappa_0}\{\mathbb{P}^\epsilon_{x,0}[(X^\epsilon_{\tau(-\kappa,\kappa_0^{'})}, Y^\epsilon_{\tau(-\kappa,\kappa_0^{'})}) \in \Gamma_{\kappa_0^{'}}]\} \\ + \quad &|\mathbb{P}^\epsilon_{\kappa_0,0}[(X^\epsilon_{\tau(-\kappa,\kappa_0^{'})}, Y^\epsilon_{\tau(-\kappa,\kappa_0^{'})}) \in \Gamma_{\kappa_0^{'}}] - \mathbb{P}^\epsilon_{-\kappa_0,0}[(X^\epsilon_{\tau(-\kappa,\kappa_0^{'})}, Y^\epsilon_{\tau(-\kappa,\kappa_0^{'})}) \in \Gamma_{\kappa_0^{'}}]| \times \\ &\times \max\{\sup_{y \in D^\epsilon_{\kappa_0^{'}}} F^\epsilon(\kappa_0^{'}, y), \inf_{y \in D^\epsilon_{\kappa_0^{'}}} F^\epsilon(\kappa_0^{'}, y)\} \\ \leq \quad &|\sup_{y \in D^\epsilon_{\kappa_0^{'}}} F^\epsilon(\kappa_0^{'}, y) - \inf_{y \in D^\epsilon_{\kappa_0^{'}}} F^\epsilon(\kappa_0^{'}, y)| + \\ + \quad &|\mathbb{P}^\epsilon_{\kappa_0,0}[(X^\epsilon_{\tau(-\kappa,\kappa_0^{'})}, Y^\epsilon_{\tau(-\kappa,\kappa_0^{'})}) \in \Gamma_{\kappa_0^{'}}] - \mathbb{P}^\epsilon_{-\kappa_0,0}[(X^\epsilon_{\tau(-\kappa,\kappa_0^{'})}, Y^\epsilon_{\tau(-\kappa,\kappa_0^{'})}) \in \Gamma_{\kappa_0^{'}}]|\end{aligned} \tag{94}$$

Relation (91) holds with $\kappa_0^{'}$ in place of κ_0 as well. Namely, for $(\kappa_0^{'}, y_1), (\kappa_0^{'}, y_2) \in \Gamma_{\kappa_0^{'}}$ we have

$$|F^\epsilon(\kappa_0^{'}, y_1) - F^\epsilon(\kappa_0^{'}, y_2)| \to 0 \text{ as } \epsilon \downarrow 0,$$

uniformly in $y_1, y_2 \in D^\epsilon_{\kappa_0^{'}}$. This implies that the first term on the right hand side of (94) can be made arbitrarily small.

We show now how the second term on the right hand side of (94) can be made arbitrarily small. For ϵ sufficiently small and for $\kappa_0^{'}$ much smaller than a small κ, it can be shown that $\mathbb{P}^\epsilon_{\kappa_0,0}[(X^\epsilon_{\tau(-\kappa,\kappa_0^{'})}, Y^\epsilon_{\tau(-\kappa,\kappa_0^{'})}) \in \Gamma_{\kappa_0^{'}}]$ and $\mathbb{P}^\epsilon_{-\kappa_0,0}[(X^\epsilon_{\tau(-\kappa,\kappa_0^{'})}, Y^\epsilon_{\tau(-\kappa,\kappa_0^{'})}) \in \Gamma_{\kappa_0^{'}}]$ are arbitrarily close to 1. This can be done by an argument similar to the one that was used to prove Lemma 2.4. Similarly to there, one estimates the difference

$$|\mathbb{P}^\epsilon_{\kappa_0,0}[(X^\epsilon_{\tau(-\kappa,\kappa_0^{'})}, Y^\epsilon_{\tau(-\kappa,\kappa_0^{'})}) \in \Gamma_{\kappa_0^{'}}] - \hat{\mathbb{P}}^\epsilon_{\kappa_0}[\hat{X}^\epsilon_{\hat{\tau}(-\kappa,\kappa_0^{'})} = \kappa_0^{'}]|$$

and uses the corresponding estimate for $\hat{\mathbb{P}}^\epsilon_{\kappa_0}[\hat{X}^\epsilon_{\hat{\tau}(-\kappa,\kappa_0^{'})} = \kappa_0^{'}]$ for ϵ sufficiently small, where $\hat{X}^\epsilon_t$ is the process defined by (53). One also needs to use Lemma 2.4. The treatment of $\mathbb{P}^\epsilon_{-\kappa_0,0}[(X^\epsilon_{\tau(-\kappa,\kappa_0^{'})}, Y^\epsilon_{\tau(-\kappa,\kappa_0^{'})}) \in \Gamma_{\kappa_0^{'}}]$ is almost identical with the obvious changes. We will not repeat the lengthy, but straightforward calculations here. Hence, the second term on the right hand side of (94) can be made arbitrarily small.

Using the above we finally get that

$$|F^\epsilon(\kappa_0, 0) - F^\epsilon(-\kappa_0, 0)| \to 0 \text{ as } \epsilon \downarrow 0. \tag{95}$$

Therefore, we have established that (89) holds for any $(x_1, y_1), (x_2, y_2) \in \Gamma_{\pm\kappa_0}$.

Next, we prove that

$$\max_{(x,y)\in\Omega_0} \mathbb{P}^\epsilon_{x,y}(X^\epsilon_{\tau(\pm\kappa)} = \kappa) - \min_{(x,y)\in\Omega_0} \mathbb{P}^\epsilon_{x,y}(X^\epsilon_{\tau(\pm\kappa)} = \kappa) \to 0 \text{ as } \epsilon \downarrow 0, \tag{96}$$

where $\Omega_0 = \{(x, y) : x \in [-\kappa_0, \kappa_0], y \in D^\epsilon_x\}$. We use again the strong Markov property. Let us choose some (x, y) such that $|x| < \kappa_0$ and $y \in D^\epsilon_x$. By strong Markov property with respect to the exit time from $\{(x, y) : |x| < \kappa_0, y \in D^\epsilon_x\}$ we have:

$$\inf_{(x,y)\in\Gamma_{\pm\kappa_0}} \mathbb{E}^\epsilon_{x,y}\chi_{(X^\epsilon_{\tau(\pm\kappa)}=\kappa)} \le \mathbb{P}^\epsilon_{x,y}(X^\epsilon_{\tau(\pm\kappa)} = \kappa) \le \sup_{(x,y)\in\Gamma_{\pm\kappa_0}} \mathbb{E}^\epsilon_{x,y}\chi_{(X^\epsilon_{\tau(\pm\kappa)}=\kappa)}$$

The latter implies that for any $(x_1, y_1), (x_2, y_2) \in \Omega_0$:

$$\begin{aligned} |\mathbb{P}^\epsilon_{x_1,y_1}(X^\epsilon_{\tau(\pm\kappa)} = \kappa) \quad &- \quad \mathbb{P}^\epsilon_{x_2,y_2}(X^\epsilon_{\tau(\pm\kappa)} = \kappa)| \le \\ &\le \quad |\sup_{(x,y)\in\Gamma_{\pm\kappa_0}} \mathbb{E}^\epsilon_{x,y}\chi_{(X^\epsilon_{\tau(\pm\kappa)}=\kappa)} - \inf_{(x,y)\in\Gamma_{\pm\kappa_0}} \mathbb{E}^\epsilon_{x,y}\chi_{(X^\epsilon_{\tau(\pm\kappa)})}| \end{aligned}$$

The latter inequality and (89) imply (96). Therefore $\mathbb{P}^\epsilon_{x,y}(X^\epsilon_{\tau(\pm\kappa)} = \kappa)$ has approximately the same value for all $(x, y) \in \Omega_0$ for ϵ small enough. Let us now identify this value. In order to do this we use the machinery with the invariant measures. In particular, we consider again the two cycles of Markov times $\{\sigma_n\}$ and $\{\tau_n\}$ that are defined by (79).

The numbers $\nu_\epsilon^{\kappa_0}(\Gamma_{\pm\kappa_0})$ and $\nu_\epsilon^{\kappa}(\Gamma_{\pm\kappa})$ are strictly positive. This is because, starting from each of these sets, there is a positive probability to reach every other $\Gamma_{\pm\kappa_0}$,$\Gamma_{\pm\kappa}$ in a finite number of cycles. We introduce the averages:

$$p_+(\epsilon) = \frac{\int_{\Delta_{\kappa_0}} \nu_\epsilon^{\kappa_0}(dx,dy)\mathbb{P}^\epsilon_{x,y}[(X^\epsilon_{\tau_1}, Y^\epsilon_{\tau_1}) \in \Gamma_\kappa]}{\nu_\epsilon^{\kappa}(\Gamma_{\pm\kappa})}. \tag{97}$$

By relation (96) we know that

$$|\mathbb{P}^\epsilon_{x,y}(X^\epsilon_{\tau(\pm\kappa)} = \kappa) - p_+(\epsilon)| \leq \eta \tag{98}$$

for all (x,y) such that $(x,y) \in [-\kappa_0,\kappa_0] \times D^\epsilon_x$ for ϵ sufficiently small.
Moreover using the first of equations (81) we see that (97) can be written as

$$p_+(\epsilon) = \frac{\nu_\epsilon^{\kappa}(\Gamma_\kappa)}{\nu_\epsilon^{\kappa}(\Gamma_{\pm\kappa})}. \tag{99}$$

Furthermore, we for $0 < \kappa_0 < \kappa$ sufficiently small we have

$$u(\kappa) - u(\kappa_0) \sim u'(0+)(\kappa - \kappa_0).$$

The latter and Lemma 5.2 imply for sufficiently small $0 < \kappa_0 < \kappa$ and sufficiently small ϵ that

$$\nu_\epsilon^{\kappa}(\Gamma_\kappa) \sim \frac{1}{u'(0+)}\frac{\epsilon}{\kappa - \kappa_0}. \tag{100}$$

Similarly, we have for sufficiently small $0 < \kappa_0 < \kappa$ and sufficiently small ϵ that

$$\nu_\epsilon^{\kappa}(\Gamma_{-\kappa}) \sim \frac{1}{u'(0-)}\frac{\epsilon}{\kappa - \kappa_0}. \tag{101}$$

Therefore, we finally get for sufficiently small $0 < \kappa_0 < \kappa$ and sufficiently small ϵ that

$$\nu_\epsilon^{\kappa}(\Gamma_{\pm\kappa}) \sim [\frac{1}{u'(0+)} + \frac{1}{u'(0-)}]\frac{\epsilon}{\kappa - \kappa_0}. \tag{102}$$

Hence, by equations (99)-(102) we get for sufficiently small $0 < \kappa_0 < \kappa$ and sufficiently small ϵ that

$$|p_+(\epsilon) - \frac{[u'(0+)]^{-1}}{[u'(0+)]^{-1} + [u'(0-)]^{-1}}| < \eta$$

Similarly, we can prove that

$$\max_{(x,y)\in\Omega_0} \mathbb{P}^\epsilon_{x,y}(X^\epsilon_{\tau(\pm\kappa)} = -\kappa) - \min_{(x,y)\in\Omega_0} \mathbb{P}^\epsilon_{x,y}(X^\epsilon_{\tau(\pm\kappa)} = -\kappa) \to 0 \text{ as } \epsilon \downarrow 0.$$

Then for

$$p_-(\epsilon) = \frac{\nu_\epsilon^{\kappa}(\Gamma_{-\kappa})}{\nu_\epsilon^{\kappa}(\Gamma_{\pm\kappa})},$$

we can obtain that

$$|p_-(\epsilon) - \frac{[u'(0-)]^{-1}}{[u'(0+)]^{-1} + [u'(0-)]^{-1}}| < \eta$$

and that the corresponding relation (98) holds, i.e.

$$|\mathbb{P}^\epsilon_{x,y}(X^\epsilon_{\tau(\pm\kappa)} = -\kappa) - p_-(\epsilon)| \leq \eta$$

for all (x,y) such that $(x,y) \in [-\kappa_0,\kappa_0] \times D^\epsilon_x$ for ϵ sufficiently small.
The latter concludes the proof of the Lemma. $\square$

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References

[1] S.N. Ethier, T.G. Kurtz, Markov processes: Characterization and Convergence, Wiley, New York, (1986). MR0838085

[2] W. Feller, Generalized second-order differential operators and their lateral conditions, Illinois Journal of Math. 1(1957) pp. 459-504. MR0092046

[3] M. Freidlin, Functional integration and partial diffferential equations, Princeton University Press, Princeton, NJ, (1985). MR0833742

[4] M. Freidlin, Markov processes and differential equations: Asymptotic Problems, Birkh$\ddot{a}$user-Verlag, Basel, Boston, Berlin, (1996). MR1399081

[5] M.I. Freidlin, K. Spiliopoulos, Reaction-diffusion equations with nonlinear boundary conditions in narrow domains, Journal of Asymptotic Analysis, Vol. 59, Number 3-4, (2008), pp.227-249. MR2450360

[6] M.I. Freidlin, A.D. Wentzell, Random perturbations of dynamical systems, Second Edition, Springer-Verlag, New York, (1998). MR1652127

[7] M.I. Freidlin, A.D. Wentzell, Random perturbations of Hamiltonian systems, Memoirs of American Mathematical Society, No. 523, (1994). MR1201269

[8] M.I. Freidlin, A.D. Wentzell, Diffusion processes on graphs and the averaging principle, The Annals of Probability, Vol. 21, No. 4, (1993),pp. 2215-2245. MR1245308

[9] M.I. Freidlin, A.D. Wentzell, Necessary and sufficient conditions for weak convergence of one-dimensional Markov process, The Dynkin Festschift: Markov Processes and their Applications, Birkh$\ddot{a}$user, (1994),pp. 95-109. MR1311713

[10] H. Kim, On continuous dependence of solutions to parabolic equations on coefficents, Asymptotic Analysis, Vol. 62, No. 3, (2009), pp. 147-162. MR2521761

[11] R.Z. Khasminskii , Ergodic properties of recurrent diffusion processes and stabilization of the solution of the Cauchy problem for parabolic equations, Teor. Veroyatnost. i. Primenen. 5, (1960),pp. 179-196. MR0133871

[12] I.Karatzas, S.E.Shreve, Brownian motion and stochastic calculus, Second edition, Springer-Verlag, (1994). MR1121940

[13] P. Mandl, Analytical treatment of one-dimensional Markov processes, Springer: Prague, Academia, (1968). MR0247667

[14] W. Rosenkrantz, Limit theorems for solutions to a class of stochastic differential equations, Indiana Univ. Math. J. 24, (1975), pp. 613-625. MR0368143

[15] D.W. Stroock, S.R.S. Varadhan, Multidimensional diffusion processes, Springer-Verlag, Berlin, Heidelberg, New York, (1979). MR0532498