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GENERATION OF ONE-SIDED RANDOM DYNAMICAL SYSTEMS BY STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract: Let Z be an $\mathbb{R}^m$-valued semimartingale with stationary increments which is realized as a helix over a filtered metric dynamical system S. Consider a stochastic differential equation with Lipschitz coefficients which is driven by Z. We show that its solution semiflow ϕ has a version for which $\varphi(t,\omega) = \phi(0,t,\omega)$ is a cocycle and therefore (S,φ) is a random dynamical system. Our results generalize previous results which required Z to be continuous. We also address the case of local Lipschitz coefficients with possible explosions in finite time. Our abstract perfection theorems for semiflows are designed to cover also potential applications to infinite dimensional equations.

Keywords: stochastic differential equation, random dynamical system, cocycle, perfection

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1 Introduction

We start by defining the concept of a random dynamical system which has been introduced by L. Arnold and his school and which we adapt slightly according to our needs (cf. Arnold (1997), Arnold and Scheutzow (1995)).

Definition 1 *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space and* θ *a Borel* $\mathbb{R}$*-action on* Ω *i.e.* $\theta : \mathbb{R} \times \Omega \to \Omega$ *satisfies*

i) θ *is* $(\mathcal{B} \otimes \mathcal{F}, \mathcal{F})$*-measurable, where* $\mathcal{B}$ *is the Borel* σ*-algebra of* $\mathbb{R}$

ii) $\theta_0 = id_\Omega$

iii) $\theta_{t+s} = \theta_t \circ \theta_s$ *for all* $s, t \in \mathbb{R}$.

If in addition $N \in \mathcal{F}, \mathbb{P}(N) = 0$ *implies* $\mathbb{P}(\theta_t^{-1} N) = 0$ *for all* $t \in \mathbb{R}$ *i.e.* $\mathbb{P}$ *is quasi-invariant under* θ*, then* $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ *is called* **quasimetric dynamical system (QDS)**. *If even more* $\mathbb{P}(\theta_t^{-1} A) = \mathbb{P}(A)$ *for all* $A \in \mathcal{F}, t \in \mathbb{R}$ *i.e.* $\mathbb{P}$ *is invariant under* θ*, then* $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ *is called* **metric dynamical system (MDS)**.

Definition 2 *Let* $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ *be a QDS and* I *a subsemigroup of* $(\mathbb{R}, +)$, *(e.g.* $\mathbb{R}$, $[0, \infty)$, $\mathbb{N}$, $\mathbb{Z}$ *). Let* $(H, \circ)$ *be a semigroup. A map* $\varphi : I \times \Omega \to H$ *is called* **cocycle** *if*

$$\varphi(t+s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega) \qquad \textit{for all} \qquad t, s \in I,\, \omega \in \Omega.$$

$\varphi : I \times \Omega \to H$ *is called a* **crude cocycle** *if for every* $s \in I$ *there exists a* $\mathbb{P}$*-null set* N_s *such that*

$$\varphi(t+s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega) \qquad \textit{for all} \qquad t, s \in I,\, \omega \notin N_s.$$

$(\Omega, \mathcal{F}, \mathbb{P}, \theta, \varphi)$ *is called* **random dynamical system (RDS)**, *if* φ *is a cocycle over the MDS* $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$. *We call an RDS* **one-sided** *if* $I = [0, \infty)$ *or* $I = \mathbb{N}_0$.

We will always assume that θ is defined on $\mathbb{R} \times \Omega$ even if we consider cocycles which are defined on a proper subset of $\mathbb{R}$. This will be convenient and does not seem to be too restrictive provided one is willing to change the (e.g. one-sided) QDS on which the cocycle φ is defined without changing the law of φ (cf. (Arnold, Scheutzow (1995), Theorem 13) for a closely related question).

An important class of RDS arises via solution flows of stochastic differential equations (SDEs) driven by semimartingales Z with stationary increments. More precisely we assume that the m-dimensional semimartingale Z has the **helix** property on the MDS $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ i.e.

$$Z_{t+s}(\omega) - Z_s(\omega) = Z_t(\theta_s \omega), \qquad s, t \geq 0,\, \omega \in \Omega$$

i.e. Z is a cocycle with $(H, \circ) = (\mathbb{R}^m, +)$ and $I = [0, \infty)$. We can always extend Z to a helix with index set $\mathbb{R}$ in a unique way (Arnold, Scheutzow (1995)) but there will be no need to do this. Let $f : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ satisfy a global Lipschitz condition and let us assume that Z has continuous paths. Consider the n-dimensional SDE

$$\begin{aligned} dX_t &= f(X_{t-})dZ_t \,,\; t \geq s \\ X_s &= x \,, \qquad\qquad x \in \mathbb{R}^n \,,\; s \geq 0, \end{aligned} \tag{1}$$

By results of Kunita (Kunita (1984, 1990)) we know that (1) admits a solution flow of homeomorphisms (even without Z being a helix) i.e. a map $\phi : \Delta \times \Omega \to H$, where $\Delta = \{s, t \in \mathbb{R} : 0 \leq s \leq t < \infty\}$ and $(H, \circ)$ is the group of homeomorphisms on $\mathbb{R}^n$ with respect to composition such that

$$\phi_{s,u}(\omega) = \phi_{t,u}(\omega) \circ \phi_{s,t}(\omega) \qquad \text{for all} \qquad 0 \leq s \leq t \leq u,\, \omega \in \Omega \tag{2}$$

and such that $\phi_{s,t}(\omega)(x)$ solves (1) for each fixed $x \in \mathbb{R}^n$ and $s \geq 0$.

The helix property of Z can be shown to imply that for every $s \geq 0$ there exists a set M_s of measure zero such that

$$\phi_{s,s+t}(\omega) = \phi_{0,t}(\theta_s \omega) \tag{3}$$

for all $t \geq 0$ and all $\omega \notin M_s$ (Arnold, Scheutzow (1995)). If we define $\varphi(t, \omega) := \phi_{0,t}(\omega)$, $t \geq 0$, $\omega \in \Omega$, then φ is obviously a crude cocycle with $N_s = M_s$ (assuming that M_s and N_s are chosen as small as possible). Since one is interested in getting a cocycle φ rather than a crude cocycle and since ϕ is only uniquely defined up to sets of measure zero the question arises if there exists a version of ϕ (or of φ) such that $N_s = M_s = \emptyset$ for all $s \geq 0$ i.e. such that φ is a cocycle. This is usually referred to as the **perfection problem** of crude cocycles. Note that the union of all sets M_s need not be a set of measure zero in general. A positive solution of the perfection problem thus guarantees that (1) generates the RDS $(\Omega, \mathcal{F}, \mathbb{P}, \theta, \varphi)$.

The perfection problem in the above set-up (with continuous Z) was solved in (Arnold, Scheutzow (1995)) - even for more general equations than (1). In fact (Arnold, Scheutzow (1995)) and (Mohammed, Scheutzow (1996)) contain abstract perfection results for group-valued crude cocycles which enjoy continuity properties in the "time" variable (with different proofs). A more general perfection result for group-valued cocycles without continuity properties is given in (Scheutzow (1996)).

We will show that if Z is a semimartingale helix which is just cadlag (i.e. right continuous with left limits for all $\omega \in \Omega$) then there still exists a solution map ϕ with properties (2) and (3). It does not take values in the group of homeomorphisms in general but just in the semigroup $(C_n, \circ)$ of continuous functions from $\mathbb{R}^n$ to $\mathbb{R}^n$. If we define $\varphi(t, \omega) := \phi_{0,t}(\omega)$ as before then φ is still a crude cocycle but the perfection theorems above can not be applied. In fact all proofs of the perfection theorems mentioned above heavily use the existence of inverse elements in the group H.

Let us point out the difference between the perfection problem in the group-valued and the semigroup-valued case. We call ϕ a **crude semiflow** if it satisfies (2) and (3) and **semiflow** or **perfect semiflow** if it satisfies (2) and (3) without exceptional sets.

There is a one-to-one correspondence between H-valued semiflows $\widetilde{\phi}$ and H-valued cocycles $\widetilde{\varphi}$ given by $\widetilde{\varphi}(t,\omega) := \widetilde{\phi}_{0,t}(\omega)$ and $\widetilde{\phi}_{s,t}(\omega) := \widetilde{\varphi}(t-s,\theta_s\omega)$ which holds also if H is just a semigroup. For the corresponding crude objects ϕ and φ and the perfection problems there is still a close correspondence if H is a group but no longer if H is just a semigroup. Indeed if we start with a group-valued crude semiflow ϕ, define $\varphi(t,\omega) := \phi_{0,t}(\omega)$, find a perfection $\widetilde{\varphi}$ of φ and define $\widetilde{\phi}_{s,t}(\omega) := \widetilde{\varphi}(t-s,\theta_s\omega)$, then $\widetilde{\phi}$ is a semiflow which is indistinguishable from ϕ i.e. $\widetilde{\phi}$ and ϕ agree identically up to a set of measure zero. We call $\widetilde{\phi}$ a perfection of ϕ. Conversely if we start with a group-valued crude cocycle φ and define $\phi_{s,t}(\omega) := \varphi(t,\omega)\circ\varphi^{-1}(s,\omega)$, then ϕ is a crude semiflow (observe that $\varphi(0,\omega)$ equals the identity e_H of H almost surely). Note that $\phi_{s,t}(\omega) := \varphi(t-s,\theta_s\omega)$ will not be a crude semiflow in general! If $\widetilde{\phi}$ is a perfection of ϕ then $\widetilde{\varphi}(t,\omega) := \widetilde{\phi}_{0,t}(\omega)$ is a perfection of φ. So the perfection problems for φ and ϕ are equivalent in the group-valued case. This is no longer true in case H is just a semigroup.

If we start with a semigroup-valued crude semiflow ϕ, then $\widetilde{\phi}$ as defined above is still a semiflow but it need not be indistinguishable from ϕ (see Example 10). In general the best we can say is that for every fixed $s \geq 0$ there exists a $\mathbb{P}$-null set A_s such that $\phi_{s,t}(\omega) = \widetilde{\phi}_{s,t}(\omega)$ for all $t \geq s$ and all $\omega \notin A_s$. If we start with a semigroup-valued crude cocycle then we can not even define ϕ as above due to the possible nonexistence of inverses. So the perfection problems for crude cocycles and crude semiflows are different problems in the general semigroup-valued case.

Here we will be interested in obtaining perfect versions for solution semiflows of SDEs. Therefore we will concentrate on perfection results for crude semiflows (Theorems 3 and 4) which we will apply to solutions of SDEs in Chapter 3. In Theorem 5 we provide the existence of a nice solution semiflow of (1) without requiring Z to have stationary increments. Even though Theorem 5 and its counterpart Proposition 8 for the case of locally Lipschitz coefficients look somewhat "classical" we could not find them in the literature. Corollaries 6 and 9 then show (in particular) that if Z is a helix then (1) generates a RDS in case f is Lipschitz resp. locally Lipschitz.

The perfection problem for crude semigroup-valued cocycles has been solved in certain cases in (Kager (1996)) - even without continuity assumptions. We mention that a number of authors have treated the related perfection problem for multiplicative functionals of Markov processes which are by definition cocycles which take values in the semigroup $[0,\infty)$ with respect to multiplication (Meyer (1972), Walsh (1972), Sharpe (1988), Getoor (1990)). The restriction to the semigroup $[0,\infty)$ simplifies the problem. On the other hand it is complicated by the fact that in the Markov process literature θ is usually only one-sided.

2 Perfection of a crude semiflow

We will use the following notation:

$$
\begin{array}{lcl}
\Delta & = & \{(s,t) \in \mathbb{R}^2 : 0 \le s \le t\} \\
\mathcal{B}(T) & = & \text{Borel } \sigma\text{-algebra on the topological space T} \\
\lambda & = & \text{Lebesgue measure on } \mathbb{R} \text{ (or its restriction to a subset of } \mathbb{R}) \\
\mathbb{P}_* & = & \text{inner measure associated with } \mathbb{P} \\
\mathbb{Q}^+ & = & \text{set of nonnegative rationals} \\
\mathbb{D}_n & = & \text{space of } \mathbb{R}^n\text{-valued cadlag functions on } [0,\infty) \\
 & & \text{equipped with the topology of uniform convergence on compacts.}
\end{array}
$$

Theorem 3 *Let $(H, \circ)$ be a second countable Hausdorff topological semigroup with identity element e_H and $\mathcal{H} := \mathcal{B}(H)$. Let $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ be a QDS and assume that $\phi : \Delta \times \Omega \to H$ satisfies*

(i) $\phi(s,t,\omega) = \phi(u,t,\omega) \circ \phi(s,u,\omega)$ *for all* $\omega \in \Omega$, $0 \le s \le u \le t$,

(ii) *for every $s \ge 0$ there exists a $\mathbb{P}$-null set N_s such that*

$$\phi(s,t,\omega) = \phi(0, t-s, \theta_s\omega) \text{ for all } \omega \notin N_s, t \ge s,$$

(iii) *ϕ is $(\mathcal{B}(\Delta) \otimes \mathcal{F}, \mathcal{H})$-measurable,*

(iv) *$s \mapsto \phi(s,t,\omega)$ is right continuous on $[0,t]$ for every $t > 0$, $\omega \in \Omega$,*

(v) *$t \mapsto \phi(s,t,\omega)$ is right continuous on $[s,\infty)$ for every $s \ge 0$, $\omega \in \Omega$.*

Then there exists a map $\tilde{\phi} : \Delta \times \Omega \to H$ which satisfies (i), (iii), (iv), (v) and

(ii') $\tilde{\phi}(s,t,\omega) = \tilde{\phi}(0, t-s, \theta_s\omega)$ *for all* $\omega \in \Omega$, $0 \le s \le t$,

(vi) $\mathbb{P}_*(\{\omega : \tilde{\phi}(s,t,\omega) = \phi(s,t,\omega) \text{ for all } (s,t) \in \Delta\}) = 1$,

(vii) *for every $u \ge 0$*

$$\{\tilde{\phi}(s, s+u, \omega); s \ge 0, \omega \in \Omega\} \subseteq \{\phi(s, s+u, \omega); s \ge 0, \omega \in \Omega\} \cup \{e_H\}.$$

In particular $\phi(s,s,\omega) = e_H$ for all $s \ge 0, \omega \in \Omega$ implies $\tilde{\phi}(s,s,\omega) = e_H$ for all $s \ge 0, \omega \in \Omega$,

(viii) *if $(s,t) \mapsto \phi(s,t,\omega)$ is continuous resp. cadlag for all $\omega \in \Omega$, then the same is true for $\tilde{\phi}$.*

Remark: It is no restriction to assume in (ii) that the exceptional null sets do not depend on t because if they do then by assumption (v) and the Hausdorff property of H

$$N_s := \bigcup_{t\geq 0} N_{s,t} = \bigcup_{t\in \mathbb{Q}^+} N_{s,t}$$

has also measure zero for every $s \geq 0$.

Proof of Theorem 3:
Define

$$M := \{(s,\omega) \in [0,\infty) \times \Omega : \phi(s,s+t,\omega) = \phi(0,t,\theta_s\omega) \,\text{for all}\; t \geq 0\}$$

and

$$\tilde{\Omega} := \{\omega \in \Omega : (s+r, \theta_{-r}\omega) \in M \,\text{for}\;\; \lambda \otimes \lambda - \text{a.a.}\,(s,r) \in \mathbb{R}^2 \,\text{such that}\; s+r \geq 0\}.$$

Due to assumption (v) and the fact that H is Hausdorff we have

$$M = \bigcap_{t\in \mathbb{Q}^+} \{(s,\omega) : \phi(s,s+t,\omega) = \phi(0,t,\theta_s\omega)\}.$$

Using (iii) and the fact that the diagonal in $H \times H$ is closed (since H is Hausdorff) and hence in $\mathcal{B}(H \times H)$ and that $\mathcal{B}(H \times H) = \mathcal{B}(H) \otimes \mathcal{B}(H) = \mathcal{H} \otimes \mathcal{H}$ (since H is second countable) we see that $M \in \mathcal{B}([0,\infty)) \otimes \mathcal{F}$. Further (ii) implies $\lambda \otimes \mathbb{P}(([0,\infty) \times \Omega) \setminus M) = 0$. Quasi-invariance of $\mathbb{P}$ and Fubini's theorem then imply $\tilde{\Omega} \in \mathcal{F}$ and $\mathbb{P}(\tilde{\Omega}) = 1$. Note that $\tilde{\Omega}$ is invariant under θ_u for every $u \in \mathbb{R}$.

Define for $0 \leq s < t$

$$\tilde{\phi}(s,t,\omega) := \begin{cases} \operatorname{ess\,lim}_{h\downarrow 0} \quad \phi(0, t-s-h, \theta_{s+h}\omega), & \omega \in \tilde{\Omega} \\ e_H, & \omega \notin \tilde{\Omega} \end{cases}$$

See (Dellacherie, Meyer (1978), p. 105) for the definition of the essential limit. To see that the esslim exists, observe that

$$\phi(0, t-s-h, \theta_{s+h}\omega) = \phi(s+h+r, t+r, \theta_{-r}\omega) \tag{4}$$

provided $t \geq s+h$ and $(s+h+r, \theta_{-r}\omega) \in M$, so for fixed $0 \leq s < t$ (4) holds for all $\omega \in \tilde{\Omega}$ and $\lambda \otimes \lambda$-a.a. $(r,h) \in [0,\infty] \times [0, t-s]$. Now assumption (iv) implies the existence of the esslim and also that

$$\tilde{\phi}(s,t,\omega) = \phi(s+r, t+r, \theta_{-r}\omega) \tag{5}$$

for $0 \leq s < t$, $\omega \in \tilde{\Omega}$ and a.a. $r \geq 0$ where the exceptional null set may depend on s and ω. Finally define

$$\tilde{\phi}(s,s,\omega) = \begin{cases} \lim_{h\downarrow 0} \quad \widetilde{\phi}(s,s+h,\omega), & \omega \in \widetilde{\Omega} \\ e_H, & \omega \notin \widetilde{\Omega} \end{cases} \tag{6}$$

It is clear from (5) and (v) that the limit exists and also that (5) still holds if $s = t, \omega \in \widetilde{\Omega}$ for a.a. $r \geq 0$.

We claim that $\widetilde{\phi}$ has all the properties stated in the theorem.

(i) is obvious from assumption (i) and (5).

(ii') follows from (5) and the fact that $\widetilde{\Omega}$ is invariant under θ.

(iii) Since $\mathcal{H}$ is separating and countably generated we can embed $(H,\mathcal{H})$ in $([0,1],\mathcal{B}[0,1])$ as a measurable space (Zimmer (1984), p. 194). Then

$$\widetilde{\phi}(s,t,\omega) := \begin{cases} \int_0^1 \quad \phi(s+r,t+r,\,\theta_{-r}\omega)dr, & \omega \in \widetilde{\Omega} \\ e_H, & \omega \notin \widetilde{\Omega} \end{cases}$$

Now the fact that $\widetilde{\Omega} \in \mathcal{F}$, assumption (iii) and Fubini's theorem imply (iii).

(iv) follows from (5) and assumption (iv).

(v) follows from (5) and assumption (v).

(vi) If $\omega \in \widetilde{\Omega}$, then $\widetilde{\phi}(s,t,\omega) = \phi(s+r,t+r,\theta_{-r}\omega) = \phi(0,t-s,\theta_s\omega)$ for a.a. $r \geq 0, s \geq 0$ and all $t \geq s$. Further $\phi(0,t-s,\theta_s\omega) = \phi(s,t,\omega)$ for a.a. $\omega \in \Omega$, so for a.a. $s \geq 0$ and a.a. $\omega \in \Omega$ we have $\widetilde{\phi}(s,t,\omega) = \phi(s,t,\omega)$ for all $t \geq s$. Now (iv) implies (vi).

(vii) follows from (5) and the definition of $\widetilde{\phi}$.

(viii) follows from (5) and the definition of $\widetilde{\phi}$. ■

Remark: Contrary to the group-valued case for which a perfection theorem without continuity conditions in the "time" variable was proved in (Scheutzow (1996)) Theorem 3 is wrong if we drop (v) in both the assumption and conclusion. We will give an example in Chapter 4. Note that we made use of (v) twice in the proof: the first time to show $M \in \mathcal{B}([0,\infty)) \otimes \mathcal{F}$ and then in (6). The crucial step in which we can not avoid (v) completely is (6) - even if we define $\widetilde{\phi}(s,s,\omega)$ differently (see Example 10).

Let us show that (v) can be avoided to prove (4) and (5) because a similar argument will be needed in the proof of Theorem 4: the complement of M in $[0,\infty) \times \Omega$ is the projection of the set

$$A := \{(s,\omega,t) : \phi(s,s+t,\omega) \neq \phi(0,t,\theta_s\omega)\} \in \mathcal{B}([0,\infty)) \otimes \mathcal{F} \otimes \mathcal{B}([0,\infty))$$

onto the first two components. By the projection theorem (Cohn (1980), Prop. 8.4.4) M is measurable w.r.t. the completion of $\mathcal{B}([0,\infty)) \otimes \mathcal{F}$ w.r.t. $\lambda \otimes \mathbb{P}$. So we can find $M_1 \subset M \subset M_2$ such that $M_1, M_2 \in \mathcal{B}([0,\infty)) \otimes \mathcal{F}$ and $\lambda \otimes \mathbb{P}(M_2 \setminus M_1) = 0$. Clearly M_2 has full measure and therefore M_1 as well. If we define $\widetilde{\Omega}$ as before but with M_1 instead of M then (4) and (5) follow as before.

In a number of cases some of the assumptions in Theorem 3 will not be satisfied. In particular the continuity assumptions are rather strong in infinite dimensional cases if one uses the usual (strong) topologies on H (even in the deterministic case). We will therefore formulate a version of Theorem 3 which assumes only pathwise right continuity and which may well be applicable for certain infinite dimensional equations.

Theorem 4 *Let E be a Hausdorff second countable topological space with $\mathcal{E} = \mathcal{B}(E)$ and $(H, \circ)$ a semigroup of maps from E to E such that either E is Polish (i.e. second countable and complete metric) or there exists a countable set $D \subseteq E$ such that $h, \tilde{h} \in H$, $h|_D = \tilde{h}|_D$ implies $h = \tilde{h}$. Let $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ be a QDS and assume that $\phi : \Delta \times \Omega \to H$ satisfies (i), (ii) of Theorem 3 and*

(iii') *ϕ is $(\mathcal{B}(\Delta) \otimes \mathcal{F} \otimes \mathcal{E}, \mathcal{E})$-measurable.*

(iv') *$s \mapsto \phi(s,t,\omega,x)$ is right continuous for every $t > 0, \omega \in \Omega, x \in E$ on $[0,t)$.*

(v') *$t \mapsto \phi(s,t,\omega,x)$ is right continuous for every $s \geq 0, \omega \in \Omega, x \in E$ on $[s,\infty)$.*

Then there exists a map $\widetilde{\phi} : \Delta \times \Omega \to H$ which satisfies (i), (ii'), (iii'), (iv'), (vi), (vii) and

(viii') *If $(s,t) \mapsto \phi(s,t,\omega,x)$ is continuous resp. cadlag for every $\omega \in \Omega$ and $x \in E$, then the same is true for $\widetilde{\phi}$.*

Proof:
Define M as in the proof of Theorem 1. If a set D as in Theorem 4 exists, then

$$M = \bigcap_{t \in \mathbb{Q}^+} \bigcap_{x \in D} \{(s,\omega) : \phi(s, s+t, \omega, x) = \phi(0, t, \theta_s\omega, x)\}.$$

So $M \in \mathcal{B}([0,\infty)) \otimes \mathcal{F}$ as before. If E is Polish then define M_1 as in the previous remark with t replaced by the pair $(t,x) \in [0,\infty) \times E$. The rest of the proof is completely analogous to that of Theorem 3 except that we always add an additional argument $x \in E$ of ϕ and limits and the integral are taken for fixed x. ∎

3 Application to stochastic differential equations

Let $m, n \in \mathbb{N}$ and let $(Z_t)_{t\geq 0}$ be an m-dimensional semimartingale w.r.t. the stochastic basis $(\Omega, \overline{\mathcal{F}}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ which we assume to satisfy the usual conditions (Protter (1992), p. 3). Let $\mathcal{F}$ be a σ-algebra such that $Z : [0,\infty) \times \Omega \to \mathbb{R}^m$ is $(\mathcal{B}([0,\infty)) \otimes \mathcal{F}, \mathcal{B}(\mathbb{R}^m))$-measurable and such that $\overline{\mathcal{F}}$ is the $\mathbb{P}$-completion of $\mathcal{F}$. We could take $\mathcal{F} = \overline{\mathcal{F}}$ here but as soon as we will introduce θ and require that θ be $(\mathcal{B}([0,\infty)) \otimes \mathcal{F}, \mathcal{F})$- measurable then it is a severe restriction of generality to assume that $\mathcal{F}$ is complete.

Let $f : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ satisfy a global Lipschitz condition. We will first show that there exists a version of the solution map $\phi : \Delta \times \Omega \to H$ of the SDE

$$\begin{aligned} dX_t &= f(X_{t-})dZ_t\,,\ t \geq s \\ X_s &= x\,, \qquad\qquad s \geq 0, x \in \mathbb{R}^n \end{aligned} \tag{7}$$

which satisfies (i), (iii), (iv) and (v) of Theorem 3 if we take $H = C_n = C(\mathbb{R}^n, \mathbb{R}^n)$ equipped with the compact-open topology (which is the same as the one generated by uniform convergence on compact sets). It is well-known that H is a second countable and metrizable topological semigroup (Dugundji (1966), Chap. XII, Th. 2.2, Th. 5.2, 8.5). Then we will assume in addition that Z is a helix and we will show that ϕ automatically satisfies (ii). By Theorem 3 we get a version $\widetilde{\phi}$ of ϕ which is in particular a perfect semiflow. If we define $\varphi(t,\omega) := \widetilde{\phi}(0,t,\omega)$, then $(\Omega, \mathcal{F}, \mathbb{P}, \theta, \varphi)$ is an RDS which is generated by (7).

Theorem 5 *Let Z be an $\mathbb{R}^m$-valued semimartingale and let $f : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ satisfy a global Lipschitz condition. Then there exists a map $\phi : \Delta \times \Omega \to H(= C_n)$ which satisfies (i) and (iii) of Theorem 3 and also*

(ix) *$(s,t) \mapsto \phi(s,t,\omega)$ is cadlag for every $\omega \in \Omega$,*

(x) *$\phi(s,t,\omega)(x)$ solves (7) for every $s \geq 0, x \in \mathbb{R}^n$,*

(xi) *$\phi(s,s,\omega) = e_H$ for every $s \geq 0$, $\omega \in \Omega$,*

(xii) *$x \mapsto \phi(s, s+\cdot, \omega)(x)$ is continuous from $\mathbb{R}^n$ to $\mathbb{D}_n$ for every $s \geq 0$, $\omega \in \Omega$.*

(xiii) *If Z has continuous paths then $(s,t) \mapsto \phi(s,t,\omega)$ is continuous for every $\omega \in \Omega$ and $\phi(s,t,\omega)$ is a homeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$ for every $(s,t) \in \Delta$ and $\omega \in \Omega$.*

Remark: We assumed that the stochastic basis satisfies the usual conditions, so ϕ will automatically be progressively measurable for fixed s and x.

Proof:
It is known that for a given Lipschitz map f there exists $\epsilon > 0$ such that for every semimartingale Z satisfying $\|Z\|_{H^\infty} < \epsilon$ there exists a map $\bar{\phi} : [0,\infty) \times \Omega \to H$ such that

a) $\overline{\phi}(t,\omega)(x)$ solves (7) for $s=0$ and every $x \in \mathbb{R}^n$.

b) for every $t \geq 0$, $\omega \in \Omega$, $\overline{\phi}(t,\omega)$ is a homeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$,

c) $x \to \overline{\phi}(\cdot,\omega)(x)$ is continuous from $\mathbb{R}^n$ to $\mathbb{D}_n$ for every $\omega \in \Omega$.
(Meyer (1981), p.114), see also (Protter (1992), p.189) for the definition of $\|\cdot\|_{H^\infty}$.

A straightforward contradiction argument shows that b) and c) imply d) and e):

d) $t \mapsto \overline{\phi}(t,\omega)$ is cadlag (w.r.t. the topology on H),

e) $s \mapsto \overline{\phi}^{-1}(s,\omega)$ is cadlag.

Let us further show

f) $\overline{\phi}$ is $(\mathcal{B}([0,\infty)) \otimes \overline{\mathcal{F}}, \mathcal{H})$-measurable.
Fix $t \geq 0$, a compact set $K \subseteq \mathbb{R}^n$ and an open set $G \subseteq \mathbb{R}^n$. Then a) and c) imply

$$\bigcap_{x \in K} \{\omega : \overline{\phi}(t,\omega)(x) \in G\} \in \overline{\mathcal{F}}$$

i.e. $\omega \mapsto \overline{\phi}(t,\omega)$ is $(\overline{\mathcal{F}}, \mathcal{H})$-measurable for every $t \geq 0$. Using d) and the fact that H is metrizable we get f).

For the given semimartingale Z (which need not satisfy $\|Z\|_{H^\infty} < \epsilon$) we can find a sequence of finite stopping times $0 = T_0 < T_1 < T_2 < \cdots$ such that $\mathbb{P}(T_k \uparrow \infty) = 1$ and $\|Z^{T_k-} - Z^{T_{k-1}}\|_{H^\infty} < \epsilon$ for all $k \in \mathbb{N}$, where ϵ is chosen as before (depending on f). Here Z^T and Z^{T-} stand for the semimartingales $Z_t^T := Z_{T\wedge t}$ and $Z_t^{T-} := Z_t \mathbb{1}_{\{T>t\}} + Z_{T-}\mathbb{1}_{\{T\leq t\}}$ resp. (Protter (1992), p. 192/193). We can now apply the previous results to the semimartingales $\left(Z_t^{T_k-} - Z_t^{T_{k-1}}\right)_{t\geq 0}$. Denote the corresponding maps by $\bar{\phi}_k$. Define $\bar{\varphi}_k : \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ by

$$\overline{\varphi}_k(x,\omega) := x + f(x)\left(Z_{T_k(\omega)} - Z_{T_k(\omega)-}\right).$$

For $(s,t) \in \Delta, \omega \in \Omega, x \in \mathbb{R}^n$ define

$$\hat{\phi}(s,t,\omega)(x) := \overline{\phi}_l(t,\omega) \circ \prod_{j=k}^{l-1} (\overline{\varphi}_j(\cdot,\omega) \circ \overline{\phi}_j\,(T_{j}-(\omega),\omega)) \circ \overline{\phi}_k^{-1}(s,\omega)(x),$$

where $T_{k-1} \leq s < T_k$ and $T_{l-1} \leq t < T_l$. On the null set on which T_k does not converge to ∞ we let $\hat{\phi}(s,t,\omega) = e_H$ for all $(s,t) \in \Delta$. Obviously $\hat{\phi}$ takes values in H.

By f) $\hat{\phi}$ is $(\mathcal{B}(\Delta) \otimes \overline{\mathcal{F}}, \mathcal{H})$-measurable. Since $\mathcal{H}$ is separating and countably generated we can find a $\mathbb{P}$-null set $N \in \mathcal{F}$ such that ϕ defined as

$$\phi(s,t,\omega) := \begin{cases} \hat{\phi}(s,t,\omega), & \omega \notin N \\ e_H, & \omega \in N \end{cases}$$

is $(\mathcal{B}(\Delta) \otimes \overline{\mathcal{F}}, \mathcal{H})$-measurable (Scheutzow (1996), Lemma 2.7) i.e. ϕ satisfies (iii). Let us verify that ϕ has all other asserted properties.

(i) and **(xi)** follow from the definition of ϕ.

(ix) follows from d), e) and the definition of ϕ.

(x) From the definition of $\hat{\phi}$, it is clear that $\hat{\phi}$ (and hence ϕ) solves (7) for every $s \geq 0, x \in \mathbb{R}^n$.

(xii) is a consequence of (ix). In fact for a function $\psi : [0,\infty) \mapsto C(\mathbb{R}^n, \mathbb{R}^n)$ the following properties are equivalent:

- $t \mapsto \psi(t)$ is cadlag
- $x \mapsto \psi(\cdot)(x)$ is continuous from $\mathbb{R}^n$ to $\mathbb{D}_n$.

(xiii) By (Kunita (1990), Theorem 4.5.1) we know that if Z has continuous paths there exists a map $\phi : \Delta \times \Omega \to H$ with values in the group of homeomorphisms which satisfies (i), (x), (xi) and such that $(s,t,x) \mapsto \phi(s,t,\omega)(x)$ is continuous for every $\omega \in \Omega$. The last property implies that $(s,t) \mapsto \phi(s,t,\omega)$ is continuous - even in the stronger topology of the group of homeomorphisms (Kunita (1990), p. 115), so (xii) and (xiii) follow. To show (iii) we may have to change ϕ on a $\mathbb{P}$-null set as above (without destroying any of the other properties). ■

Let us now consider the particular case when Z has stationary increments. More precisely, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}, \theta)$ be a filtered MDS (FMDS) in the sense that $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is an MDS, $(\Omega, \overline{\mathcal{F}}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is a stochastic basis (satisfying the usual conditions) and θ_s is $(\mathcal{F}_{t+s}, \mathcal{F}_t)$- measurable for every $s, t \geq 0$. As before $\overline{\mathcal{F}}$ denotes the $\mathbb{P}$-completion of $\mathcal{F}$.

Corollary 6 *Let Z be an $\mathbb{R}^m$-valued semimartingale helix over the FMDS $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}, \theta)$ and assume that $f : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ satisfies a global Lipschitz condition. Then the map ϕ in Theorem 5 satisfies all assumptions of Theorem 3. In particular there exists a solution semiflow $\tilde{\phi}$ of (7) i.e. (7) generates an RDS.*

Proof:
We only need to check assumption (ii) of Theorem 3. Fix $s \geq 0$ and $x \in \mathbb{R}^n$. We have a.s.

$$\phi(0, t-s, \omega, x) = x + \left(\int_{(0,t-s]} f(\phi(0, u^-, \cdot, x)) dZ_u \right)(\omega), t \geq 0.$$

We want so show that almost surely

$$\phi(0,t-s,\theta_s\omega,x) = x + \left(\int_{(s,t]} f(\phi(0,u^- - s,\theta_s(\cdot),x))dZ_u\right)(\omega), t \geq s,$$

because this implies that for almost all ω we have $\phi(0,t-s,\theta_s\omega,x) = \phi(s,t,\omega,x)$ for all $t \geq s$ due to the fact that (7) has a unique solution.

Therefore we need to show that

$$\int_{(s,t]} g(u-s,\theta_s\omega)dZ_u = \left(\int_{(0,t-s]} g(u,\cdot)dZ_u\right)(\theta_s\omega) \quad a.s. \tag{8}$$

where $g(u,\omega) := f(\phi(0,u^-,\omega,x))$. Observe that the integral on the left side of (8) is well-defined because the integrand is caglad and adapted due to the measurability assumptions on θ. The fact that (8) holds follows by first checking (8) in case g is simple predictable (which follows immediately from the helix property of Z) and then approximating the given g by simple predictable processes uniformly on $[0,t-s]$ in probability. Alternatively we could derive (8) from (Protter (1986), Theorem 3.1 (vi)) which is more general (but has the disadvantage that his set-up is slightly different). ■

Remark (about the linear case):
Let us briefly specialize to the linear case i.e. $f_{ij}(x) = a_{ij}x$, where $a_{ij}^T \in \mathbb{R}^n$, $i = 1,\cdots,n$, $j = 1,\cdots,m$. Then the solution of (7) for $s = 0$ is given by $X_t = A_t x$, where $A_t, t \geq 0$ is an $\mathbb{R}^{n\times n}$-valued solution of

$$A_t = I + \int_{(0,t]} (d\tilde{Z}_u)A_{u-}, \tag{9}$$

where $\widetilde{Z}$ is an $\mathbb{R}^{n\times n}$-valued semimartingale whose components are linear combinations of the components of Z and where I is the $n\times n$- identity matrix. Any solution of (9) is called stochastic exponential of $\widetilde{Z}$ and is usually denoted by $\mathcal{E}(\widetilde{Z})_t, t \geq 0$. A glance at the proof of Theorem 5 shows that we can choose ϕ in such a way that $\phi(s,t,\omega)$ is linear for all $(s,t) \in \Delta, \omega \in \Omega$. If Z is a helix then Theorem 5 and (vii) of Theorem 3 show that there exists a perfection $\tilde{\phi}$ of ϕ which takes values in the subsemigroup of linear maps.

We will now consider the case in which $f : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is only locally Lipschitz continuous. Until further notice Z is an $\mathbb{R}^m$-valued semimartingale which need not be a helix. Let ∂ be an element not contained in $\mathbb{R}^n$. We follow the idea of the proof of Theorem V.38 in (Protter (1992)) and choose global Lipschitz functions $f_k, k \in \mathbb{N}$ such that $f_k(x) = f(x)$ for all $|x| \leq k$. Let $\phi^{(k)} : \Delta\times\Omega \to H(= C_n)$ be the corresponding maps of Theorem 5. Then there exists a $\mathbb{P}$-null set N such that

$$\phi^{(k)}(s,t,\omega)(x) = \phi^{(l)}(s,t,\omega)(x)$$

for all $k,l \in \mathbb{N}, x \in \mathbb{R}^n, (s,t) \in \Delta, \omega \notin N$ as long as $\sup_{s \le u < t} \mid \phi^{(k)}(s,u,\omega)(x) \mid \le k \wedge l$. Changing the $\phi^{(k)}$ on a global set of measure zero if necessary, we can and will assume that N is empty. Now define

$$\phi(s,t,\omega)(x) := \begin{cases} \lim_{k\to\infty} \phi^{(k)}(s,t,\omega)(x), & \text{if} \quad \sup_{k\in\mathbb{N}} \sup_{s\le u\le t} \mid \phi^{(k)}(s,u,\omega)(x) \mid < \infty \\ \partial, & \text{otherwise} \end{cases} \tag{10}$$

Note that the limit exists in $\mathbb{R}^n$ and that we could extend the sup over $s \le u < t$ instead of $s \le u \le t$ without changing ϕ. ϕ is called **strictly conservative** if there exists a $\mathbb{P}$-null set $\overline{N}$ such that $\phi(s,t,\omega)(x) \neq \partial$ for all $(s,t) \in \Delta, x \in \mathbb{R}^n, \omega \notin \overline{N}$. In this case we again can and will assume that $\overline{N}$ is empty.

Proposition 7 *If ϕ is strictly conservative then it satisfies all properties of Theorem 5 with the possible exception of the second part of (xiii). Instead we have*
(xiii') *If Z has continuous paths then $(s,t) \mapsto \phi(s,t,\omega)$ is continuous for every $\omega \in \Omega$ and $\phi(s,t,\omega) : \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one for every $(s,t) \in \Delta$ and $\omega \in \Omega$.*

Proof:
Fix $\omega \in \Omega$ and $N \in \mathbb{N}$. The definition of ϕ together with (ix) of Theorem 5 (applied to the $\phi^{(k)}$) imply that for every $\mid x \mid \le N$ and $0 \le s \le N$ there exists an open neighborhood $U_{s,x}$ of (s,x) such that $\sup_{\tilde{s}\le t\le N} \mid \phi(\tilde{s},t,\omega)(\tilde{x}) - \phi(s,t,\omega)(x) \mid \le 1$ for all $(\tilde{s},\tilde{x}) \in U_{s,x}$. Using the compactness of $[0,N] \times \{x \in \mathbb{R}^n : |x| \le N\}$ we get

$$\sup_{|x|\le N} \sup_{0\le s\le t\le N} \mid \phi(s,t,\omega)(x) \mid < \infty. \tag{11}$$

Property (iii) holds since $\phi(s,t,\omega) = \lim_{k\to\infty} \phi^{(k)}(s,t,\omega)$ in $H = C_n$, the $\phi^{(k)}$ satisfy (iii) and H is metrizable. All other properties follow from (11) and the corresponding properties for the $\phi^{(k)}$ except for the second part of (xiii) which need not hold in general. The one-to-one property again follows from (11) and property (xiii) applied to the $\phi^{(k)}$. ■

Remark: It follows from Proposition 7 that Corollary 6 still holds if $f : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is only locally Lipschitz continuous and if ϕ defined by (10) is strictly conservative. Observe that if Z has continuous paths, then the one-to-one property of ϕ is preserved by $\widetilde{\phi}$ due to (vii) of Theorem 3.

Now we consider the general case in which ϕ is not necessarily strictly conservative. Unfortunately the map ϕ defined by (10) is not necessarily right continuous in the first variable for fixed $x \in \mathbb{R}^n$ if we give $E = \mathbb{R}^n \cup \{\partial\}$ the topology of the Alexandrov one point compactification of $\mathbb{R}^n$, so Theorem 4 cannot be applied. It seems that Theorem 3 cannot be directly applied either. We will therefore construct a perfection $\widetilde{\phi}$ of ϕ via approximation by the $\widetilde{\phi}^{(k)}$ which we get from Theorem 3. In spite of the fact that we do not need the following proposition later we state it - together with a sketch of its proof - for completeness.

Proposition 8 *Let $f : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ be locally Lipschitz continuous and let Z be a semimartingale. Define $E := \mathbb{R}^n \cup \{\partial\}$ equipped with the topology of the one-point compactification of $\mathbb{R}^n$. Let $(H, \circ)$ be the semigroup of maps $h : E \to E$ which map ∂ to ∂ and which are continuous at x whenever $x \in \mathbb{R}^n$ and $h(x) \in \mathbb{R}^n$. Then the map ϕ defined by (10) is H-valued and satisfies (i), (iii') of Theorem 4 and also*

(xiv) *$t \mapsto \phi(s, s+t, \omega, x)$ is cadlag for every $\omega \in \Omega$, $x \in \mathbb{R}^n$, $s \geq 0$,*

(xv) *$\phi(s, s, \omega, x) = x$ for all $s \geq 0$, $\omega \in \Omega$, $x \in \mathbb{R}^n$.*

(xvi) *If $\phi(s, t, \omega, x) = \partial$ and $t > s$, then $\lim_{h\downarrow 0}$, $\phi(s, t-h, \omega, x) = \partial$,*

(xvii) *ϕ is a local solution of (7) i.e.*

$$\phi(s, t, \omega, x) = x + \int_s^t f(\phi(s, u^-, \omega, x))dZ_u \quad a.s.$$

for all $0 \leq s \leq t$, $x \in \mathbb{R}^n$ such that $\phi(s, t, \omega, x) \neq \partial$.

(xviii) *If $\phi(\overline{s}, \overline{t}, \omega, \overline{x}) \neq \partial$, then for every $\epsilon > 0$ there exist $\delta = \delta(\epsilon) > 0$ such that $| \phi(s, t, \omega, x) - \phi(\overline{s}, \overline{t}, \omega, \overline{x}) | < \epsilon$ for all $| x - \overline{x} | < \delta$, $\overline{t} \leq t < \overline{t} + \delta$, $\overline{s} \leq s < \overline{s} + \delta$ and $| \phi(s, t, \omega, x) - \phi(\overline{s}, \overline{t}^-, \omega, \overline{x}) | < \epsilon$ for all $| x - \overline{x} | < \delta$, $\overline{t} - \delta < t \leq \overline{t}$, $\overline{s} \leq s < \overline{s} + \delta$. An analogous statement holds for $\overline{s}^-$.*

(xix) *If Z has continuous paths then $\phi(s, t, \omega, x) = \phi(s, t, \omega, y) \neq \partial$ implies $x = y$.*

Proof: It suffices to prove (xvi) since all other properties then follow from the definition of ϕ. Fix $s \geq 0$, $x \in \mathbb{R}^n$, $\omega \in \Omega$ and assume that

$$\tau := \inf\{u > s : \phi(s, u, \omega, x) = \partial\} \in (s, \infty)$$

and $\phi(s, \tau, \omega, x) = \partial$. Further assume that $\lim_{h\downarrow 0} \phi(s, \tau - h, \omega, x)$ either does not exist or is unequal to ∂. Then there exists $N \in \mathbb{N}$ and a sequence $h_n \downarrow 0$ such that for all $n \in \mathbb{N}$ $| \phi(s, \tau - h_n, \omega, x) | \leq N$. By Theorem 5 (ix) there exists $h > 0$ such that $\sup_{|y|\leq N} |\phi^{(N+1)}(u, t, \omega, y)| \leq N + 1$ for all $\tau - h \leq u \leq t < \tau$. Further $\phi^{(N+1)}(u, t, \omega, y) = \phi(u, t, \omega, y)$ for all $\tau - h \leq u \leq t \leq \tau$ and all $|y| \leq N$. In particular $|\phi(u, \tau, \omega, y)| \neq \partial$ for all $\tau - h \leq u \leq \tau$ and all $|y| \leq N$. Inserting $y_n = \phi(s, \tau - h_n, \omega, x)$ we see that $\phi(s, \tau, \omega, x) \neq \partial$ which is a contradiction. ∎

Remark: $\phi(s, t, \omega)$ will in general not be continuous from E to E - in fact not even from $\mathbb{R}^n$ to E - because solutions which get (or start) close to ∂ do not necessarily stay close to ∂ (even in the strictly conservative case). For the same reason $s \mapsto \phi(s, t, \omega, x)$ need not be right continuous.

Corollary 9 *In addition to the assumptions of Proposition 8 we require that Z be an m-dimensional semimartingale helix on the FMDS $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}, \theta)$. Then there exist maps $\widetilde{\phi} : \Delta \times \Omega \to H$ and $\zeta : [0,\infty) \times \mathbb{R}^n \times \Omega \to (0,\infty]$ (the "residual life time") such that*

a) $\zeta(s,x,\omega) = \inf\{t \geq 0 : \widetilde{\phi}(s, s+t, \omega, x) = \partial\}$ *for all* $s \geq 0$, $x \in \mathbb{R}^n$, $\omega \in \Omega$,

b) $\widetilde{\phi}(s,\cdot,\cdot)(x)$ *solves*

$$X_t = x + \int_{(s,t]} f(X_{u^-})\, dZ_u$$

on $[s, s+\zeta(s,x,\omega))$ *for every* $s \geq 0$, $x \in \mathbb{R}^n$,

c) $\widetilde{\phi}(s,t,\omega) = \widetilde{\phi}(u,t,\omega) \circ \widetilde{\phi}(s,u,\omega)$ *for all* $\omega \in \Omega$, $0 \leq s \leq u \leq t$,

d) $\widetilde{\phi}(s,t,\omega) = \widetilde{\phi}(0, t-s, \theta_s\omega)$ *for all* $\omega \in \Omega$, $0 \leq s \leq t$,

e) $\zeta(s,x,\omega) = \zeta(0,x,\theta_s\omega) = \zeta(t, \widetilde{\phi}(s,t,\omega,x),\omega)) - t + s$ *for* $\omega \in \Omega$, $0 \leq s \leq t, x \in \mathbb{R}^n$ *such that* $\widetilde{\phi}(s,t,\omega,x) \in \mathbb{R}^n$,

f) $\widetilde{\phi}(s,s,\omega,x) = x$ *for all* $\omega \in \Omega$, $0 \leq s$, $x \in E$,

g) $(s,t,\omega,x) \mapsto \widetilde{\phi}(s,t,\omega,x)$ *is* $(\mathcal{B}(\Delta) \otimes \mathcal{F} \otimes \mathcal{E}, \mathcal{E})$*-measurable,*

h) *if* $\widetilde{\phi}(s,t,\omega,x) = \partial$, *then* $\lim_{h\downarrow 0} \widetilde{\phi}(s, t-h, \omega, x) = \partial$,

i) $t \mapsto \widetilde{\phi}(s, s+t, \omega, x)$ *is cadlag,*

j) *(xviii) of Proposition 8 holds for* $\widetilde{\phi}$,

k) $x \mapsto \zeta(s,x,\omega)$ *is lower semi-continuous.*

l) *If Z has continuous paths then* $\widetilde{\phi}(s,t,\omega,x) = \widetilde{\phi}(s,t,\omega,y) \neq \partial$ *implies* $x = y$.

Proof: From Theorem 5 we know that the $\phi^{(k)}$ defined before satisfy the assumptions of Theorem 3. We show how a slight modification of the proof of Theorem 3 provides us with a semiflow $\widetilde{\phi}$ as in the statement of the corollary. Define M as the intersection of all $M^{(k)}$, $k \in \mathbb{N}$, where $M^{(k)}$ is defined like M in the proof of Theorem 3 but with $\phi^{(k)}$ instead of ϕ. Proceeding as in the proof of Theorem 3 we get

$$\widetilde{\phi}^{(k)}(s,t,\omega) = \phi^{(k)}(s+r, t+r, \theta_{-r}\omega)$$

for all $k \in \mathbb{N}, (s,t) \in \Delta$, $\omega \in \widetilde{\Omega}$ and $r \notin N_{s,\omega}$ where $\lambda(N_{s,\omega}) = 0$. Now we define

$$\widetilde{\phi}(s,t,\omega,x) = \begin{cases} \lim_{k\to\infty} \widetilde{\phi}^{(k)}(s,t,\omega,x), & \text{if} \quad \sup_{k\in\mathbb{N}} \sup_{s\leq u\leq t} \mid \widetilde{\phi}^{(k)}(s,u,\omega,x) \mid < \infty \\ \partial, & \text{otherwise} \end{cases}$$

Clearly $\widetilde{\phi}$ is well-defined (i.e. the limit exists). Obviously $\widetilde{\phi}$ is indistinguishable from ϕ, so b) follows. All other statements either follow from the corresponding properties for the $\widetilde{\phi}^{(k)}$ or we have proved them already. ∎

4 Complements

We give an example which shows that in Theorem 3 one can not drop (v) in the assumption and conclusion.

Example 10 Let $(H, \circ) = ([0, \infty), \times), (\Omega, \mathcal{F}, \mathbb{P}) = ([0, 1), \mathcal{B}([0, 1)), \lambda\mid_{[0,1)})$, $\theta_s \omega := \omega + s \bmod 1$, $s \in \mathbb{R}$. Further define

$$\phi(s, t, \omega) = \begin{cases} 1, & \text{if } s = t = \omega \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to check that ϕ satisfies (i) - (iv) of Theorem 3. Assume that $\widetilde{\phi}$ satisfies (ii') of Theorem 3 and that ϕ and $\widetilde{\phi}$ agree identically on $\overline{\Omega} \subseteq \Omega$. Then $\widetilde{\phi}(s, s, s) = 1$ for $s \in \overline{\Omega} \backslash \{0\}$ and therefore $\widetilde{\phi}(0, 0, 2s \bmod 1) = 1$ for $s \in \overline{\Omega} \backslash \{0\}$. On the other hand $\widetilde{\phi}(0, 0, 2s \bmod 1) = \phi(0, 0, 2s \bmod 1) = 0$ if $2s \bmod 1 \in \overline{\Omega}$. Therefore $\mathbb{P}_*(\overline{\Omega}) \leq 1/2$ i.e. $\widetilde{\phi}$ can never satisfy (vi) of Theorem 3.

Remark on a generalization of Theorems 3 and 4:
The group-valued perfection results in (Arnold, Scheutzow (1995)) and (Scheutzow (1996)) are formulated for more general groups than $\mathbb{R}$ as the "time" variable of the underlying MDS, namely for locally compact second countable Hausdorff (LCCB) groups G. Therefore one might ask whether Theorems 3 and 4 admit a similar generalization.

This is indeed possible provided that the index set $I \subseteq G$ of the crude semiflow is a measurable subsemigroup of the LCCB group G and that I has strictly positive Haar measure. Of course we need to say what we mean by "right continuous" in this case. If we assume for simplicity that ϕ is jointly continuous in the first two variables, then the proof of Theorem 3 goes through with a few obvious modifications (replace λ by Haar measure on G and $t \geq 0$ by $t \in I$).

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