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LINEAR STOCHASTIC PARABOLIC EQUATIONS, DEGENERATING ON THE BOUNDARY OF A DOMAIN

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Abstract A class of linear degenerate second-order parabolic equations is considered in arbitrary domains. It is shown that these equations are solvable using special weighted Sobolev spaces in essentially the same way as the non-degenerate equations in $\mathbb{R}^d$ are solved using the usual Sobolev spaces. The main advantages of this Sobolev-space approach are less restrictive conditions on the coefficients of the equation and near-optimal space-time regularity of the solution. Unlike previous works on degenerate equations, the results cover both classical and distribution solutions and allow the domain to be bounded or unbounded without any smoothness assumptions about the boundary. An application to nonlinear filtering of diffusion processes is discussed.

Keywords L_p estimates, Weighted spaces, Nonlinear filtering.

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1 Introduction

Sobolev spaces H^γ_p are very convenient to study parabolic equations in all of $\mathbb{R}^d$:

$$\frac{\partial u}{\partial t} = a^{ij} D_i D_j u + b^i D_i u + cu + f, \tag{1.1}$$

where summation over the repeated indices is assumed from 1 to d. Roughly speaking, if the initial condition is in $H^{\gamma+1-2/p}_p$ and the right hand side is in $H^{\gamma-1}_p$, $p \geq 2$, then $u \in H^{\gamma+1}_p$ as long as the coefficients are bounded and sufficiently smooth, and the matrix (a^{ij}) is uniformly positive definite. It was shown in [4] that a similar result holds for the Ito stochastic parabolic equations

$$du = (a^{ij} D_i D_j u + b^i D_i u + cu + f)dt + (\sigma^{ik} D_i u + \nu^k u + g^k)dw_k \tag{1.2}$$

as long as the stochastic right hand side g^k is in H^γ_p and the matrix $(a^{ij} - (1/2)\sigma^{ik}\sigma^{jk})$ is uniformly positive definite.

The objective of this paper is to show that, if the Sobolev spaces H^γ_p are replaced with weighted spaces, then an analogous result holds for the Ito stochastic parabolic equations with quadratic degeneracy of the characteristic form. Let $\rho = \rho(x)$ be a smooth function so that $\rho(x) \sim dist(x, \partial G)$ near the boundary. Consider a linear stochastic parabolic equation

$$du = (\rho^2 a^{ij} D_i D_j u + \rho b^i D_i u + cu + f)dt + (\rho\sigma^{ik} D_i u + \nu^k u + g^k)dw_k. \tag{1.3}$$

Equations with operators of the type $\rho^\alpha \Delta$, $\alpha > 0$, have been studied by many authors in deterministic setting [9, 13, 14], and operators with quadratic degeneracy of the characteristic form ($\alpha = 2$) always required separate treatment. Therefore, in the stochastic setting, it is also natural to consider these operators separately.

To study equation (1.3), the Sobolev spaces H^γ_p are replaced with certain weighted Sobolev spaces. These weighted space $H^\gamma_{p,\theta}(G)$ were first introduced in [5] to study stochastic parabolic equations on the half-line, and further investigated in subsequent papers by both authors. In the notation $H^\gamma_{p,\theta}(G)$ of the space, the indices γ, p have the same meaning as in H^γ_p, and the index θ determines the boundary behavior of the functions from the space: the larger the value of θ, the faster the functions and their derivatives can blow up near the boundary of G. The advantage of solving equation (1.3) in the space $H^\gamma_{p,\theta}(G)$ is that existence and uniqueness results can be obtained for a wide class of linear and quasi-linear equations. Unlike many related works in the deterministic setting, these results, for different values of γ and θ, cover both classical and distribution solutions and, for $\gamma > 0$, go beyond abstract solvability. Indeed, for $\gamma > 0$, embedding theorems show that the solution is a continuous function of x and t and, for sufficiently large p, has almost equal number of classical and generalized derivatives.

Since the spaces $H^\gamma_{p,\theta}(G)$ have been used in the analysis of the Dirichlet boundary value problem for nondegenerate parabolic equations [6, 5, 7], let us recall the main result.

Consider the Dirichlet problem for equation (1.2) in a sufficiently regular domain. Suppose that the coefficients are sufficiently smooth, the matrix $(a^{ij} - (1/2)\sigma^{ik}\sigma^{jk})$ is uniformly positive definite inside the domain, and $f \in H^{\gamma-1}_{p,\theta}(G)$, $g^k \in H^{\gamma}_{p,\theta}(G)$, $u|_{t=0} \in H^{\gamma+1-2/p}_{p,\theta}(G)$, $p \geq 2$. Then, for certain values of θ, the solution will be in $H^{\gamma+1}_{p,\theta}(G)$. Note that θ of the solution space is different from the corresponding values for the initial condition and the right hand side, and if θ is too large or too small, then the corresponding solvability result does not hold.

The results are quite different, and completely analogous to the whole space, if we consider degenerate parabolic equations. Namely, for equation (1.3), the solution will be in $H^{\gamma+1}_{p,\theta}(G)$ as long as the coefficients are sufficiently smooth, the matrix $(a^{ij} - (1/2)\sigma^{ik}\sigma^{jk})$ is uniformly positive definite inside the domain, and $f \in H^{\gamma-1}_{p,\theta}(G)$, $g^k \in H^{\gamma}_{p,\theta}(G)$, $u|_{t=0} \in H^{\gamma+1-2/p}_{p,\theta}(G)$, $p \geq 2$. Now, the result holds for all real θ, and the function ρ can be chosen so that no restrictions are necessary about the domain G. The domain can be bounded or unbounded, without any smoothness of the boundary, and this generality makes the results new even in deterministic setting.

Recall [10, Chapter 5] that the stochastic characteristic for equation (1.3) is the diffusion process $x = x_t$ defined by

$$dx_t = -\rho(x_t)B(t, x_t)dt + \rho(x_t)r(t, x_t)dW(t) \tag{1.4}$$

with an appropriate choice of r, B, and W. Assuming that the functions ρ, B, r are globally Lipschitz continuous and bounded, the unique solvability of (1.4) implies that, if x_0 is in G, then x_t will never reach the boundary of G. This is why it is natural to expect that the solvability results for (1.3) will not involve any conditions about the boundary of G.

The construction and analysis of the spaces $H^{\gamma}_{p,\theta}(G)$ for general domains are in [8]. Section 2 presents a summary of results from [8] and the construction of the necessary stochastic parabolic spaces. The main result of the paper, the statement about solvability of a second-order degenerate semi-linear stochastic parabolic equation, is in Section 3. In Section 4, an application is given to the problem of nonlinear filtering of diffusion processes, when the unobserved process evolves in a bounded region.

In this paper, notation D^m is used for a generic partial derivative of order m with respect to the spatial variable $x = (x_1, \ldots, x_d)$; $D_i = \partial/\partial x_i$. Summation over repeated indices is assumed.

2 Definition of the spaces

Let $G \subset \mathbb{R}^d$ be a domain (open connected set) with non-empty boundary ∂G. Denote by $\rho_G(x)$, $x \in G$, the distance from x to ∂G. For $n \in \mathbb{Z}$ define the subsets G_n of G by

$$G_n = \{x \in G : 2^{-n-1} < \rho_G(x) < 2^{-n+1}\}. \tag{2.1}$$

Let $\{\zeta_n,\ n\in\mathbb{Z}\}$ be a collection of non-negative functions with the following properties:

$$\zeta_n \in C_0^\infty(G_n),\ |D^m\zeta_n(x)| \le N(m)2^{mn},\ \sum_{n\in\mathbb{Z}}\zeta_n(x)\ge\delta>0. \tag{2.2}$$

The function $\zeta_n(x)$ can be constructed by mollifying the characteristic (indicator) function of G_n. If G_n is an empty set, then the corresponding ζ_n is identical zero.

Recall [13, Section 2.3.3] that the space H_p^γ is defined for $\gamma\in\mathbb{R}$ and $p\ge 1$ as the completion of $C_0^\infty(\mathbb{R}^d)$ with respect to the norm $\|\cdot\|_{H_p^\gamma} = \|\Lambda^\gamma\cdot\|_{L_p(\mathbb{R}^d)}$, where $\Lambda^\gamma f = ((1+|\xi|^2)^{\gamma/2}\hat{f}\,)\check{}$, and $\hat{}$, $\check{}$ are the Fourier transform and its inverse. Similarly, $H_p^\gamma(l_2)$ is the set of sequences $g=\{g^k,\ k\ge 1\}$ for which

$$\|g\|_{H_p^\gamma(l_2)} := \|\,\|\Lambda^\gamma g\|_{l_2}\,\|_{L_p(\mathbb{R}^d)} < \infty, \tag{2.3}$$

where $\|g\|_{l_2} = \left(\sum_{k\ge1}|g^k|^2\right)^{1/2}$.

Definition 2.1 Let G be a domain in $\mathbb{R}^d$, θ and γ, real numbers, and $p\in[1,+\infty)$. Take a collection $\{\zeta_n,\ n\in\mathbb{Z}\}$ as above. Then

$$H_{p,\theta}^\gamma(G) := \left\{u\in\mathcal{D}'(G) : \|u\|^p_{H^\gamma_{p,\theta}(G)} := \sum_{n\in\mathbb{Z}} 2^{n\theta}\|\zeta_{-n}(2^n\cdot)u(2^n\cdot)\|^p_{H_p^\gamma} < \infty\right\}, \tag{2.4}$$

where $\mathcal{D}'(G)$ is the set of distribution on $C_0^\infty(G)$;

$$H_{p,\theta}^\gamma(G;l_2) := \left\{u\in\mathcal{D}'(G;l_2) : \|u\|^p_{H^\gamma_{p,\theta}(G;l_2)} := \sum_{n\in\mathbb{Z}} 2^{n\theta}\|\zeta_{-n}(2^n\cdot)u(2^n\cdot)\|^p_{H_p^\gamma(l_2)} < \infty\right\}. \tag{2.5}$$

A detailed analysis of the spaces $H^\gamma_{p,\theta}(G)$ is given in [8]. In particular, it is shown that $H^\gamma_{p,\theta}(G)$ does not depend on the particular choice of the system $\{\zeta_n\}$ and, for $p>1$, is a reflexive Banach space.

Remark 2.2 If G is a bounded domain, then summation in (2.4) is carried out over $n\le n_0$ for some integer n_0 depending on the domain. In particular, if G is bounded, then $H^\gamma_{p,\theta_2}(G)\subset H^\gamma_{p,\theta_1}(G)$ for $\theta_1>\theta_2$.

Definition 2.3 Let $\rho=\rho(x),\ x\in\mathbb{R}^d$, be a function so that

1. $\rho(x)=0, x\notin G$;
2. ρ is infinitely differentiable in G, and $|\rho_G^m(x)D^{m+1}\rho(x)|\le N(m)$ for all $x\in G$ and for every $m=0,1,\ldots$.
3. $N_1\rho_G(x)\le\rho(x)\le N_2\rho_G(x)$ for some $N_1,N_2>0$ and all $x\in G$.

Functions introduced in the above definition do exist; for example,

$$\rho(x) = \sum_{n\in\mathbb{Z}} 2^{-n}\zeta_n(x). \tag{2.6}$$

The conditions in the above definition imply that ρ is uniformly Lipschitz continuous in $\mathbb{R}^d$; in particular, this is true for the function defined by (2.6). On the other hand, if G is a bounded domain and the boundary ∂G is of class $C^{|\gamma|+2}$, $\gamma \in \mathbb{R}$, then, by Lemma 14.16 in [1], it is possible to choose the function ρ so that $\rho \in C^{|\gamma|+2}(\mathbb{R}^d)$.

For $\nu \geq 0$, define the space $A^\nu(G)$ as follows:

1. if $\nu = 0$, then $A^\nu(G) = L_\infty(G)$;
2. if $\nu = m = 1, 2, \ldots$, then

$$A^\nu(G) = \{a : a, \rho_G Da, \ldots, \rho_G^{m-1}D^{m-1}a \in L_\infty(G), \rho_G^m D^{m-1}a \in C^{0,1}(G)\}, \tag{2.7}$$

$$\|a\|_{A^\nu(G)} = \sum_{k=0}^{m-1} \|\rho_G^k\, D^k a\|_{L_\infty(G)} + \|\rho_G^m D^{m-1}a\|_{C^{0,1}(G)}; \tag{2.8}$$

3. if $\nu = m + \delta$, where $m = 0, 1, 2, \ldots$, $\delta \in (0, 1)$, then

$$A^\nu(G) = \{a : a, \rho_G Da, \ldots, \rho_G^m D^m a \in L_\infty(G),\ \rho_G^\nu D^m a \in C^\delta(G)\}, \tag{2.9}$$

$$\|a\|_{A^\nu(G)} = \sum_{k=0}^{m} \|\rho_G^k\, D^m a\|_{L_\infty(G)} + \|\rho_G^\nu D^m a\|_{C^\delta(G)}. \tag{2.10}$$

Theorem 2.4 *1. Assume that $\gamma - d/p = m + \nu$ for some $m = 0, 1, \ldots$ and $\nu \in (0, 1)$. If $u \in H^\gamma_{p,\theta}(G)$, then*

$$\sum_{k=0}^{m} \sup_{x\in G} |\rho^{k+\theta/p}(x) D^k u(x)| + [\rho^{m+\nu+\theta/p} D^m u]_{C^\nu(G)} \leq N(d, \gamma, p, \theta)\|u\|_{H^\gamma_{p,\theta}(G)}. \tag{2.11}$$

Recall that $[f]_{C^\nu(G)} = \sup_{x,y\in G} |x - y|^{-\nu}|f(x) - f(y)|$.

2. Given $\gamma \in \mathbb{R}$ define γ' so that $\gamma' = 0$ for integer γ and γ' is any number from $(0, 1)$ as long as $|\gamma| + \gamma'$ is not an integer for non-integer γ. Then, for every $u \in H^\gamma_{p,\theta}(G)$ and $a \in A^{|\gamma|+\gamma'}(G)$,

$$\|a\, u\|_{H^\gamma_{p,\theta}(G)} \leq N(\gamma, p, \theta, d)\|a\|_{A^{|\gamma|+\gamma'}(G)}\ \|u\|_{H^\gamma_{p,\theta}(G)}. \tag{2.12}$$

These and other properties of the spaces $H^\gamma_{p,\theta}(G)$ and $A^\gamma(G)$ can be found in [8].

Definition 2.5 Fix $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$, a stochastic basis with $\mathcal{F}$ and $\mathcal{F}_0$ containing all P-null subsets of Ω; τ, a stopping time, $(0, \tau]\!] = \{(\omega, t) \in \Omega \times \mathbb{R}_+ : 0 < t \leq \tau(\omega)\}$; $\mathcal{P}$, the σ-algebra of predictable sets; $\{w_k,\ k \geq 1\}$, independent standard Wiener processes. The Ito stochastic integral will be used.

The following Banach spaces were introduced in [4] to study stochastic parabolic equations on $\mathbb{R}^d$:

1. $\mathbb{H}_p^\gamma(\tau) = L_p((0,\tau]]; \mathcal{P}; H_p^\gamma)$, $\mathbb{H}_p^\gamma(\tau; l_2) = L_p((0,\tau]]; \mathcal{P}; H_p^\gamma(l_2))$;

2. $\mathcal{F}_p^\gamma(\tau) = \mathbb{H}_p^{\gamma-1}(\tau) \times \mathbb{H}_p^\gamma(\tau; l_2)$, $U_p^\gamma = L_p(\Omega; \mathcal{F}_0; H_p^{\gamma+1-2/p})$;

3. $\mathcal{H}_p^\gamma(\tau)$: the collection of processes from $\mathbb{H}_p^{\gamma+1}(\tau)$ that can be written, in the sense of distributions, as
$$u(t) = u_0 + \int_0^t f(s)ds + \int_0^t g^k(s)dw_k(s) \tag{2.13}$$
for some $u_0 \in U_p^\gamma$ and $(f,g) \in \mathcal{F}_p^\gamma(\tau)$;
$$\|u\|^p_{\mathcal{H}_p^\gamma(\tau)} = \|D^2 u\|^p_{\mathbb{H}_p^{\gamma-1}(\tau)} + \|(f,g)\|^p_{\mathcal{F}_p^\gamma(\tau)} + E\|u_0\|^p_{H_p^{\gamma+1-2/p}}. \tag{2.14}$$

For a positive real number $T > 0$, a stopping time $\tau \le T$, a real number $\delta \in (0,1]$, and a (Banach space) X -valued process u, we will use the following notation:

$$\|u\|^p_{C^\delta([0,\tau],X)} = \sup_{0\le t\le T} \|u(t\wedge\tau)\|^p_X + \sup_{0\le s<t\le T} \frac{\|u(t\wedge\tau) - u(s\wedge\tau)\|^p_X}{|t-s|^{p\delta}}. \tag{2.15}$$

It is proved in [4], Theorem 7.2, that if $u \in \mathcal{H}_p^\gamma(\tau)$, $p \ge 2$, and $\tau \le T$, then

$$E \sup_{0\le t\le T} \|u(t\wedge\tau,\cdot)\|^p_{H_p^\gamma} \le N(d,\gamma,p,T)\|u\|^p_{\mathcal{H}_p^\gamma(\tau)}, \tag{2.16}$$

and if in addition $1/p < \alpha < \beta < 1/2$, then

$$E\|u\|^p_{C^{\alpha-1/p}([0,\tau],H_p^{\gamma+1-2\beta})} \le N(\alpha,\beta,d,\gamma,p,T)\|u\|^p_{\mathcal{H}_p^\gamma(\tau)}. \tag{2.17}$$

Next, we define the similar spaces on G.

1. $\mathbb{H}_{p,\theta}^\gamma(\tau,G) = L_p((0,\tau]]; \mathcal{P}; H_{p,\theta}^\gamma(G))$, $\mathbb{H}_{p,\theta}^\gamma(\tau,G; l_2) = L_p((0,\tau]]; \mathcal{P}; H_{p,\theta}^\gamma(G; l_2))$;

2. $\mathcal{F}_{p,\theta}^\gamma(\tau,G) = \mathbb{H}_{p,\theta}^{\gamma-1}(\tau,G) \times \mathbb{H}_{p,\theta}^\gamma(\tau,G; l_2)$, $U_{p,\theta}^\gamma(G) = L_p(\Omega; \mathcal{F}_0; H_{p,\theta}^{\gamma+1-2/p}(G))$

3. $\mathcal{H}_{p,\theta}^\gamma(\tau,G)$: the collection of processes from $\mathbb{H}_{p,\theta}^{\gamma+1}(\tau,G)$ that can be written, in the sense of distributions, as
$$u(t) = u_0 + \int_0^t f(s)ds + \int_0^t g^k(s)dw_k(s) \tag{2.18}$$
for some $u_0 \in U_{p,\theta}^\gamma(G)$ and $(f,g) \in \mathcal{F}_{p,\theta}^\gamma(\tau,G)$;
$$\|u\|^p_{\mathcal{H}_{p,\theta}^\gamma(\tau,G)} = \|u\|^p_{\mathbb{H}_{p,\theta}^{\gamma+1}(\tau,G)} + \|(f,g)\|^p_{\mathcal{F}_{p,\theta}^\gamma(\tau,G)} + \|u_0\|^p_{U_{p,\theta}^\gamma(G)}. \tag{2.19}$$

It follows that

$$\|u\|^p_{\mathcal{H}^\gamma_{p,\theta}(\tau,G)} = \sum_{n\in\mathbb{Z}} 2^{n\theta} \|\zeta_{-n}(2^n\cdot)u(\cdot,2^n\cdot)\|^p_{\mathcal{H}^\gamma_p(\tau)} \tag{2.20}$$

with similar representations for $\mathcal{F}^\gamma_{p,\theta}(\tau,G)$ and $U^\gamma_{p,\theta}(G)$. In particular, all these are Banach spaces, and

$$E \sup_{0\le t\le T} \|u(t\wedge\tau,\cdot)\|^p_{H^\gamma_p(G)} \le N(d,\gamma,p,T)\|u\|^p_{\mathcal{H}^\gamma_{p,\theta}(\tau,G)}, \quad p\ge 2,\ \tau\le T; \tag{2.21}$$

$$E\|u\|^p_{C^{\alpha-1/p}([0,\tau],H^{\gamma+1-2\beta}_{p,\theta}(G))} \le N(\alpha,\beta,d,\gamma,p,T)\|u\|^p_{\mathcal{H}^\gamma_{p,\theta}(\tau,G)},\ 1/p<\alpha<\beta<1/2. \tag{2.22}$$

3 Main result

Take a function ρ from Definition 2.3 and consider the following equation:

$$\begin{aligned} du(t,x) &= (\rho^2(x)a^{ij}(t,x)D_iD_ju(t,x) + f(t,x,u,Du))dt \\ &\quad + (\rho(x)\sigma^{ik}(t,x)D_iu(t,x) + g^k(t,x,u))dw_k(t),\ 0<t\le T,\ x\in G \end{aligned} \tag{3.1}$$

with initial condition $u(0,x)=u_0(x)$. Summation over the repeated indices is assumed, and the Ito stochastic differential is used.

Assumption 3.1 (Coercivity.) There exist positive numbers κ_1 and κ_2 so that

$$\kappa_1|\xi|^2 \le \left(a^{ij} - \frac{1}{2}\sigma^{ik}\sigma^{jk}\right)\xi_i\xi_j \le \kappa_2|\xi|^2 \tag{3.2}$$

for all $(\omega,t)\in (\!(0,\tau]\!]$, $x\in G$, and $\xi\in\mathbb{R}^d$.

Assumption 3.2 (Regularity of a and σ.) For all $i,j=1,\dots,d$ and $k\ge1$, the functions a^{ij} and σ^{ik} are $\mathcal{P}\otimes\mathcal{B}(G)$ measurable,

$$\|a^{ij}(t,\cdot)\|_{A^{|\gamma-1|+\gamma'}(G)} + \|\sigma^{i\cdot}(t,\cdot)\|_{A^{|\gamma|+\gamma'}(G;l_2)} \le \kappa_2 \tag{3.3}$$

for all $(\omega,t)\in (\!(0,\tau]\!]$, and, for every $\varepsilon>0$, there exists $\delta_\varepsilon>0$ so that

$$|\rho_G(x)a^{ij}(t,x) - \rho_G(y)a^{ij}(t,y)| + \|\rho_G(x)\sigma^{i\cdot}(t,x) - \rho_G(y)\sigma^{i\cdot}(t,y)\|_{l_2} \le \varepsilon \tag{3.4}$$

for all $(\omega,t)\in (\!(0,\tau]\!]$ and $x,y\in G$ with $|x-y|<\delta_\varepsilon$. See Theorem 2.4(2) for the definition of γ'.

Assumption 3.3 (Regularity of the free terms.)

$$(f(\cdot,\cdot,0,0), g(\cdot,\cdot,0)) \in \mathcal{F}^\gamma_{p,\theta}(\tau,G), \tag{3.5}$$

and for every $\varepsilon>0$ there exists $\mu_\varepsilon>0$ so that

$$\begin{aligned} &\|(f(\cdot,\cdot,u,Du) - f(\cdot,\cdot,v,Dv), g(\cdot,\cdot,u) - g(\cdot,\cdot,v))\|_{\mathcal{F}^\gamma_{p,\theta}(\tau,G)} \\ &\qquad \le \varepsilon\|u-v\|_{\mathcal{H}^\gamma_{p,\theta}(\tau,G)} + \mu_\varepsilon\|u-v\|_{\mathbb{H}^{\gamma_1}_{p,\theta}(\tau,G)},\ \gamma_1<\gamma+1, \end{aligned} \tag{3.6}$$

for all $u,v\in\mathcal{H}^\gamma_{p,\theta}(\tau,G)$.

Definition 3.1 A process $u \in \mathcal{H}^{\gamma}_{p,\theta}(\tau, G)$ is a solution of (3.1) if and only if the equality

$$u(t,x) = u_0(x) + \int_0^t (\rho^2(x) a^{ij}(s,x) D_i D_j u(s,x) + f(s,x,u,Du))ds + \int_0^t (\rho(x)\sigma^{ik}(s,x) D_i u(s,x) + g^k(s,x,u))dw_k(s) \quad (3.7)$$

holds in $\mathcal{H}^{\gamma}_{p,\theta}(\tau, G)$.

Theorem 3.2 *If $p \geq 2$, $\tau \leq T$, and $u_0 \in U^{\gamma}_{p,\theta}(G)$, then, under Assumptions 3.1–3.3, there is a unique solution u of equation (3.1) and*

$$\|u\|^p_{\mathcal{H}^{\gamma}_{p,\theta}(\tau,G)} \leq N \cdot \left(\|(f(\cdot,\cdot,0,0)\|^p_{\mathbb{H}^{\gamma-1}_{p,\theta}(\tau,G)} + \|g(\cdot,\cdot,0)\|^p_{\mathbb{H}^{\gamma}_{p,\theta}(\tau,G;l_2)} + E\|u_0\|^p_{H^{\gamma+1-2/p}_{p,\theta}(G)} \right) \quad (3.8)$$

with the constant N depending only on $\gamma, \kappa_1, \kappa_2, p, T, \theta$, and the functions $\rho, \delta_\varepsilon, \mu_\varepsilon$.

Proof. The arguments are very similar to the proof of Theorem 3.2 in [7]. A more detailed description of the method can be found in [3, Sections 6.4,6.5].

To simplify the presentation, assume that $\tau = T$ and introduce the following notations. Define the operators

$$\mathcal{A}u(t,x) = \rho(x) a^{ij}(t,x) D_i D_j u(t,x), \ \mathcal{B}^k u(t,x) = \rho(x)\sigma^{ik}(t,x) D_i u(t,x) \quad (3.9)$$

and write $(\!|\mathcal{A},\mathcal{B}|\!)u = (f, g, u_0)$ for some $(f,g) \in \mathcal{F}^{\gamma}_{p,\theta}(T,G)$, $u_0 \in U^{\gamma}_{p,\theta}(G)$ if $u \in \mathcal{H}^{\gamma}_{p,\theta}(T,G)$ and $u = u_0 + \int_0^t (\mathcal{A}u + f)ds + \int_0^t (\mathcal{B}^k u + g^k)dw_k(s)$.

Let $\{\zeta_n, n \in \mathbb{Z}\}$ be the collection of functions used in Definition 2.1 and let $\{\eta_n, n \in \mathbb{Z}\}$ be a collection of functions so that $\eta_n \in C_0^\infty(G_n)$, $|D^m \eta_n(x)| \leq N(m)2^{mn}$, $\eta_n(x) = 1$ on the support of ζ_n. Using Theorem 5.1 in [4], for $(\varphi, \psi) \in \mathcal{F}^{\gamma}_p(T)$ and $v_0 \in U^{\gamma}_p$, define $S_n(\varphi, \psi, v_0) \in \mathcal{H}^{\gamma}_p(T)$ so that $v = S_n(\varphi,\psi,v_0)$ if and only if $v \in \mathcal{H}^{\gamma}_p(T)$, $v|_{t=0} = v_0$, and

$$dv = (\eta_{-n}\rho^2 a^{ij} D_i D_j v + 2^{2n}(1-\eta_{-n})\Delta v + \varphi)dt + (\eta_{-n}\rho\sigma^{ik} D_i v + \psi^k)dw_k(t). \quad (3.10)$$

Note that

$$\|v(\cdot, 2^n \cdot)\|_{\mathcal{H}^{\gamma}_p(T)} \leq N \cdot \left(\|(\varphi(\cdot, 2^n\cdot), \psi(\cdot, 2^n \cdot))\|_{\mathcal{F}^{\gamma}_p(T)} + \|v_0(2^n\cdot)\|_{U^{\gamma}_p} \right) \quad (3.11)$$

with N independent of n. Indeed, for a function $f = f(x)$ write $f_n(x) = f(2^n x)$. Then $v_n(t,x) = v(t, 2^n x)$ satisfies

$$dv_n = (2^{2n}\eta_{-n,n}\rho_n^2 a_n^{ij} D_i D_j v_n + (1-\eta_{-n,n})\Delta v_n + \varphi_n)dt + (2^n \eta_{-n,n}\rho_n \sigma_n^{ik} D_i v_n + \psi_n^k)dw_k(t), \quad (3.12)$$

where $\eta_{-n,n}(x) = \eta_{-n}(2^n x)$. Since $\rho \sim 2^n$ on the support of η_{-n}, inequality (3.11) follows from Theorem 5.1 in [4].

Note also that, for every $u \in \mathcal{H}^{\gamma}_{p,\theta}(T,G)$,

$$S_n((\!|\mathcal{A},\mathcal{B}|\!)(\zeta_{-n}u)) = \zeta_{-n}u. \tag{3.13}$$

Assume first that f and g depend only on t and x. Also, with no loss of generality, assume that $\sum_n \zeta_n^2(x) = 1$ for all $x \in G$. If $(\!|\mathcal{A},\mathcal{B}|\!)u = (f,g,u_0)$, then it follows from (3.13) that

$$u = \sum_n \zeta_{-n} S_n(\zeta_{-n}f - A_n u, \zeta_{-n}g - B_n u, \zeta_{-n}u_0), \tag{3.14}$$

where $A_n u = \mathcal{A}(u\zeta_{-n}) - \zeta_{-n}\mathcal{A}u$, $B_n^k u = \mathcal{B}^k(\zeta_{-n}u) - \zeta_{-n}\mathcal{B}^k u$.

Conversely, for every $(f,g) \in \mathcal{F}^{\gamma}_{p,\theta}(T,G)$ and every $u_0 \in U^{\gamma}_{p,\theta}(G)$, equation (3.14) has a unique solution $u \in \mathcal{H}^{\gamma}_{p,\theta}(T,G)$ so that u satisfies (3.8) and (3.1) (recall that so far we assume that f, g do not depend on u). Indeed, by inequalities (3.11) and (2.21), a sufficiently high power of the operator $u \mapsto \sum_n \zeta_{-n}S_n(A_n u, B_n u, 0)$ is a contraction in $\mathcal{H}^{\gamma}_{p,\theta}(T,G)$, which implies the existence of a unique solution of (3.14). This solution satisfies

$$\|u\|^p_{\mathcal{H}^{\gamma}_{p,\theta}(T,G)} \le N \cdot \left(\|(f,g)\|^p_{\mathcal{F}^{\gamma}_{p,\theta}(T,G)} + \|u_0\|^p_{U^{\gamma}_{p,\theta}(G)} + \int_0^T \|u\|^p_{\mathcal{H}^{\gamma}_{p,\theta}(t,G)} dt \right),$$

and (3.8) follows by the Gronwall inequality. Since $u \in \mathcal{H}^{\gamma}_{p,\theta}(T,G)$, we have $(\!|\mathcal{A},\mathcal{B}|\!)u = (f_0, g_0, u_0)$ for some $(f_0,g_0) \in \mathcal{F}^{\gamma}_{p,\theta}(T,G)$, and it follows from (3.14) that $\bar{f} = f - f_0, \bar{g} = g - g_0$ satisfy $\sum_n \zeta_{-n}S_n(\zeta_{-n}\bar{f}, \zeta_{-n}\bar{g}, 0) = 0$. Applying the operator $(\!|\mathcal{A},\mathcal{B}|\!)$ to the last equality and using (3.13), we conclude that

$$\bar{f} = \sum_n A_n S_n(\zeta_{-n}\bar{f}, \zeta_{-n}\bar{g}, 0), \quad \bar{g} = \sum_n B_n S_n(\zeta_{-n}\bar{f}, \zeta_{-n}\bar{g}, 0).$$

Once again, using inequalities (3.11) and (2.21), we conclude that a sufficiently high power of the operator

$$(f,g) \mapsto \left(\sum_n A_n S_n(\zeta_{-n}f, \zeta_{-n}g, 0), \sum_n A_n S_n(\zeta_{-n}f, \zeta_{-n}g, 0) \right)$$

is a contraction, which means that $(\bar{f}, \bar{g}) = (0,0)$ and u is a solution of (3.1) with f, g independent of u.

Now, for every $(f,g) \in \mathcal{F}^{\gamma}_{p,\theta}(T,G)$ and every $u_0 \in U^{\gamma}_{p,\theta}(G)$ we can define $u = \mathcal{R}(f,g,u_0) \in \mathcal{H}^{\gamma}_{p,\theta}(T,G)$ so that $(\!|\mathcal{A},\mathcal{B}|\!)u = (f,g,u_0)$. This means that every solution of (3.1) with general (f,g) satisfies $u = \mathcal{R}(f(u,Du), g(u), u_0)$. To conclude the proof of the theorem, we note that Assumption 3.3 implies that a sufficiently high power of the operator $u \mapsto \mathcal{R}(f(u,Du), g(u), u_0)$ is a contraction in $\mathcal{H}^{\gamma}_{p,\theta}(T,G)$. Theorem 3.2 is proved. □

Corollary 3.3 *Assume that $\tau = T$, the initial condition u_0 is compactly supported in G and belongs to $H_p^{\gamma+1-2/p}$, and Assumption 3.3 holds for all $\theta \in \mathbb{R}$. If $p > 2$ and $\gamma + 1 - (d+2)/p > m$ for some positive integer m, then the solution $u(t,x)$ of (3.1) has the following properties:*

1. *for almost all $\omega \in \Omega$, u is a continuous function of (t,x);*

2. *for almost all $\omega \in \Omega$ and all $t \in [0,T]$, u is m times continuously differentiable in $\bar{G}$ as a function of x;*

3. *for almost all $\omega \in \Omega$ and all $t \in [0,T]$, $u(t,x)$ and its spatial partial derivatives of order less than or equal to m vanish on the boundary of G.*

Proof. By assumption, $u_0 \in H^{2-2/p}_{p,\theta}(G)$ for every $\theta \in \mathbb{R}$, because compactness of support of u_0 means that the corresponding sum in (2.4) contains finitely many non-zero terms. Consequently, by Theorem 3.2, $u \in \mathcal{H}^{\gamma}_{p,\theta}(G,T)$ for all $\theta \in \mathbb{R}$. It remains to use (2.22) with β sufficiently close to $1/p$, and then apply Theorem 2.4(1). □

Remark 3.4 The linear equation

$$\begin{aligned} du(t,x) = (\rho^2(x)a^{ij}(t,x)D_iD_ju(t,x) + \rho(x)b^i(t,x)D_iu(t,x) + c(t,x)u(t,x) + f(t,x))dt \\ + (\rho(x)\sigma^{ik}(t,x)D_iu(t,x) + \nu^k(t,x)u(t,x) + g^k(t,x))dw_k(t) \end{aligned} \tag{3.15}$$

satisfies the hypotheses of the theorem if $(f,g) \in \mathcal{F}^{\gamma}_{p,\theta}(\tau,G)$, the functions b^i, c, ν^k are $\mathcal{P} \otimes \mathcal{B}(G)$ measurable, and

$$\|b^i(t,\cdot)\|_{A^{n_b}(G)} + \|c(t,\cdot)\|_{A^{n_c}(G)} + \|\nu(t,\cdot)\|_{A^{n_\nu}(G;l_2)} \leq \kappa_2. \tag{3.16}$$

As for the values of n_b, n_c, n_ν, we can always take $n_b = |\gamma| + \gamma'$, $n_\nu = n_c = |\gamma+1| + \gamma'$, but these conditions can be relaxed. For example (cf. [4, Remark 5.6]), if $\gamma \geq 1$, we can take $n_b = n_c = \gamma - 1 + \gamma'$, $n_\nu = \gamma + \gamma'$.

4 Application to nonlinear filtering of diffusion processes

The classical problem of nonlinear filtering is considered for a pair of diffusion processes (X_t, Y_t) defined in $\mathbb{R}^{d_1}$ by equations

$$\begin{aligned} dX_t &= b(t,X_t,Y_t)dt + r(t,X_t,Y_t)dW_t \\ dY_t &= B(t,X_t,Y_t)dt + R(t,Y_t)dW_t \end{aligned} \tag{4.1}$$

with some initial conditions X_0, Y_0. It is assumed that W_t is a d_1-dimensional Wiener process on a complete probability space $(\Omega, \mathcal{F}, P)$, X_t is d-dimensional state process, $d_1 > d$, and Y_t is $(d_1 - d)$-dimensional observation process. The coefficients are known functions of corresponding dimension and are smooth enough for a unique strong solution to exist. The filtering problem consists in computing the conditional density of X_t given the observations up to time t. It is known [10, Chapter 6] that under some natural regularity assumptions, this conditional density satisfies a nonlinear stochastic parabolic equation, also know as Kushner's equation. Alternatively, the density can be computed by normalizing the solution of the Zakai equation, which is a linear equation. Both analytical theory and numerical methods for the Kushner and Zakai equations have been studied by many authors, and these studies made (4.1) the standard model in filtering theory.

Nonetheless, for most applications, (4.1) is only an approximation. There are two main reasons for that. First, the actual process X_t usually evolves in a bounded region, for example, because of mechanical restrictions, and therefore equations (4.1) are a suitable model only when the process X_t is away from the boundary. Second, even if X_t evolves in the whole space, the corresponding filtering equations, when solved numerically, are considered in a bounded domain, which effectively restricts the range of X_t. Therefore, it seems natural to start with a filtering model in which the state process evolves in a bounded region. The model presented below is just one possible way to address the issue of replacing the whole of $\mathbb{R}^d$ with a bounded domain. The model can be easily analysed using the theory developed in the previous sections of the paper, but it is certainly not the most general filtering model in a bounded domain.

Let G be a bounded domain and ρ, a scalar function as in Definition 2.3. Consider the following modification of the classical filtering model:

$$\begin{aligned} dx_t &= \rho(x_t)b(t,x_t,y_t)dt + \rho(x_t)r(t,x_t,y_t)dW_t \\ dy_t &= B(t,x_t,y_t)dt + R(t,y_t)dW_t \end{aligned} \tag{4.2}$$

with some initial conditions x_0, y_0. Since ρ is Lipschitz continuous in $\mathbb{R}^n$, by uniqueness of the solution of (4.2), the process x_t can never cross the boundary of G. Note that G is *any* bounded domain. It is shown below (Lemma 4.1) that, if the domain G is sufficiently large, $x_0 \in G$, and the function ρ is chosen in a special way, then (x_t, y_t) are close to (X_t, Y_t).

For a matrix M denote by M^* its transpose. We make the following assumptions. For the discussion of these assumptions see Section 8 in [4].

Assumption 4.1 The functions b, B, r, R are bounded and Borel measurable in (t, x, y) and uniformly Lipschitz continuous in (x, y). The function $r = r(t, x, y)$ is continuously differentiable with respect to x and the derivatives are continuous in y and uniformly Lipschitz continuous in x.

Assumption 4.2 The matrix RR^* is invertible and $V = (RR^*)^{-1/2}$ is a bounded function of (t, y).

Assumption 4.3 There exists $\delta > 0$ so that, for all $(x, y) \in \mathbb{R}^d \times \mathbb{R}^{d_1 - d}$, $t > 0$, and all $\xi \in \mathbb{R}^d$,

$$(r(1 - R^*V^2R)r^*\xi, \xi) \geq \delta|\xi|^2. \tag{4.3}$$

Assumption 4.4 The initial condition (x_0, y_0) is independent of W_t, the conditional distribution of x_0 given y_0 has a density Π_0, and Π_0 is compactly supported in G.

Lemma 4.1 *Let $\{G_K,\ K>0\}$ be a collection of domains with smooth boundaries so that $B_K:=\{x:|x|<K\}\subset G_K$ and $dist(\partial G_K, B_K)>\delta$ for some fixed $\delta>0$. For each G_K, let ρ_K be a function as in Definition 2.3 so that $\rho_K(x)=1,\ x\in B_K$, and $N_1\rho_{G_K}(x)\le\rho_K(x)\le N_2\rho_{G_K}(x),\ x\in G_K,\ N_1,N_2$ are independent of K,x. Define the processes $x=x^K,\ y=y^K$ according to (4.2) with $\rho=\rho_K,\ x_0^K=X_0I(X_0\in B_K), y_0^K=Y_0$, and assume that $E(|X_0|^p+|Y_0|^p)<\infty$ for some $p>0$. Then, as $K\to\infty$, $\sup_{0\le t\le T}|X_t-x_t^K|+\sup_{0\le t\le T}|Y_t-y_t^K|$ converges to zero with probability one and in every $L_q(\Omega),\ q<p$.*

Proof. First of all, notice that $E\sup_{0\le t\le T}(|X_t|+|x_t^K|+|Y_t|+|y_t^K|)^p\le C$ with C independent of K. Indeed, for example,

$$\sup_{0\le t\le T}|y_t^K|\le|Y_0|+CT+\sup_{0\le t\le T}|\int_0^t R(s,y_s^K)dW_s|$$

and it remains to use Burkholder-Davis-Gundy inequality [2, Theorem IV.4.1].

Define random variable $\eta_K=\sup_{0\le t\le T}|X_t-x_t^K|+\sup_{0\le t\le T}|Y_t-y_t^K|$. Since $\rho(x)=1$ for $|x|\le K$, equations (4.1) and (4.2) imply

$$\{\omega:\lim_{K\to\infty}\eta_K>0\}\subseteq\bigcap_{N\ge1}\{\omega:\sup_{0\le t\le T}|X_t|>N\},$$

and then, by the Chebuchev inequality,

$$P(\bigcap_{N\ge1}\{\sup_{0\le t\le T}|X_t|>N\})=\lim_{N\to\infty}P(\{\sup_{0\le t\le T}|X_t|>N\})\le\lim_{N\to\infty}\frac{E\sup_{0\le t\le T}|X_t|^p}{N^p}=0.$$

After that, since the family $\{\eta_K^q, K>0\}$ is uniformly integrable for $q<p$ [11, Lemma II.6.3],

$$E\eta_K^q=E\eta_K^qI(\sup_{0\le t\le T}|X_t|>K)\to0,\ K\to\infty.$$

□

Introduce the following notations:

$$\begin{gathered}a(t,x)=(1/2)rr^*(t,x,y_t)\in\mathbb{R}^{d\times d},\quad\sigma(t,x)=rR^*V(t,x,y_t)\in\mathbb{R}^{d\times(d_1-d)},\\ h(t,x)=VB(t,x,y_t)\in\mathbb{R}^{d_1-d},\quad h_t=h(t,x_t),\\ L[v]=D_iD_j(\rho^2a^{ij}v)-D_i(\rho b^iv),\quad\Lambda^k[v]=h^kv-D_i(\rho\sigma^{ik}v).\end{gathered}\tag{4.4}$$

The filtering problem for (4.2) is to find conditional distribution of x_t given the observations $\{y_s,\ 0<s\le t\}$. Since the function ρ is zero outside of G, equations (4.2) can be considered in the whole $\mathbb{R}^d$. Then, by Theorem 8.1 in [4], the conditional expectation of $f(x_t)$ given $\{y_s,\ 0<s\le t\}$, for every bounded measurable function $f=f(x)$, can be written as $\int_{\mathbb{R}^d}f(x)\Pi(t,x)dx$. By the same theorem, the function $\Pi=\Pi(t,x)$ belongs to $\mathcal{H}_p^1(T)$, for very $T>0$, and is the solution of a nonlinear equation

$$d\Pi=L[\Pi]dt+\sum_{k\le d_1-d}(\Lambda^k[\Pi]-\bar h_t^k\Pi)(V^kRdW_t+(h_t^k-\bar h_t^k)dt)\tag{4.5}$$

with initial condition Π_0, where V^k is the kth row of the matrix V and $\bar{h}_t = \int_{\mathbb{R}^d} h(t,x)\Pi(t,x)dx$. It is often convenient to consider the un-normalized filtering density $u = u(t,x)$ given by a linear equation

$$du = L[u]dt + \sum_{k \le d_1 - d} \Lambda^k[u](V^k R dW_t + h_t^k dt) \tag{4.6}$$

with initial condition $u|_{t=0} = \Pi_0$, so that

$$\Pi(t,x) = \frac{u(t,x)}{\int_{\mathbb{R}^d} u(t,x)dx}. \tag{4.7}$$

Theorem 4.2 *Suppose that G is a bounded domain and Assumptions 4.1 –4.4 hold. If Π_0 belongs to $L_p(\Omega; H_p^{2-2/p})$, for some $p \ge 2$, then both u and Π belong to $\mathcal{H}^1_{p,\theta}(T,G)$ for all $T > 0$ and all $\theta \in \mathbb{R}$.*

Proof. By Assumption 4.4, $\Pi_0 \in H^{2-2/p}_{p,\theta}(G)$ for every $\theta \in \mathbb{R}$, because compactness of support of Π_0 means that the corresponding sum in (2.4) contains finitely many non-zero terms. Consequently, Theorem 3.2 and Remark 3.4 imply that $u \in \mathcal{H}^1_{p,\theta}(T,G)$.

To show that $\Pi \in \mathcal{H}^1_{p,\theta}(T,G)$, we first use Theorem 8.1 in [10], according to which the solution Π of (4.5), considered in the whole space, belongs to $\mathcal{H}^1_p(T)$. Then by Theorems 4.2.2 and 4.3.2 in [12] we find

$$\|\zeta_{-n}(2^n\cdot)\Pi(\cdot,2^n\cdot)\|^p_{\mathcal{H}^1_p(T)} \le N\|\Pi(\cdot,2^n\cdot)\|^p_{\mathcal{H}^1_p(T)} \le N2^{\beta|n|}\|\Pi\|^p_{\mathcal{H}^1_p(T)}, \tag{4.8}$$

where β and N are positive and independent of n. Remark 2.2 then implies that $\Pi \in \mathcal{H}^1_{p,\theta_1}(T,G)$ for sufficiently large θ_1. On the other hand, by treating $\bar{h}_t$ as a known process and applying Theorem 3.2 with $\gamma = 1$ and $\theta < \theta_1$, we conclude that (4.5) has a unique solution belonging to $\mathcal{H}^1_{p,\theta}(T,G)$. By uniqueness, this solution is Π. □

If the conditions of Theorem 4.2 hold for sufficiently large p, then, by Corollary 3.3, for all $t \in [0,T]$ and almost all $\omega \in \Omega$, the conditional density $\Pi(t,x)$ is continuously differentiable in $\bar{G}$ and both $\Pi(t,x)$ and its first-order partial derivatives with respect to x vanish on the boundary of G.

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