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Gaussian Moving Averages and Semimartingales

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Abstract

In the present paper we study moving averages (also known as stochastic convolutions) driven by a Wiener process and with a deterministic kernel. Necessary and sufficient conditions on the kernel are provided for the moving average to be a semimartingale in its natural filtration. Our results are constructive - meaning that they provide a simple method to obtain kernels for which the moving average is a semimartingale or a Wiener process. Several examples are considered. In the last part of the paper we study general Gaussian processes with stationary increments. We provide necessary and sufficient conditions on spectral measure for the process to be a semimartingale.

Key words: semimartingales; Gaussian processes; stationary processes; moving averages; stochastic convolutions; non-canonical representations.

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1 Introduction

In this paper we study moving averages, that is processes $(X_t)_{t\in\mathbb{R}}$ on the form

$$X_t = \int (\varphi(t-s) - \psi(-s))\, dW_s, \qquad t \in \mathbb{R}, \tag{1.1}$$

where $(W_t)_{t\in\mathbb{R}}$ is a Wiener process and φ and ψ are two locally square integrable functions such that $s \mapsto \varphi(t-s) - \psi(-s) \in L^2_{\mathbb{R}}(\lambda)$ for all $t \in \mathbb{R}$ (λ denotes the Lebesgue measure). We are concerned with the semimartingale property of $(X_t)_{t\geq 0}$ in the filtration $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$, where $\mathcal{F}_t^{X,\infty} := \sigma(X_s : s \in (-\infty, t])$ for all $t \geq 0$.

The class of moving averages includes many interesting processes. By Doob [1990, page 533] the case $\psi = 0$ corresponds to the class of centered Gaussian $L^2(P)$-continuous stationary processes with absolutely continuous spectral measure. Moreover, (up to scaling constants) the fractional Brownian motion corresponds to $\varphi(t) = \psi(t) = (t\vee 0)^{H-1/2}$, and the Ornstein-Uhlenbeck process to $\varphi(t) = e^{-\beta t}1_{\mathbb{R}_+}(t)$ and $\psi = 0$. It is readily seen that all moving averages are Gaussian with stationary increments. Note however that in general we do not assume that φ and ψ are 0 on $(-\infty, 0)$. In fact, Karhunen [1950, Satz 5] shows that a centered Gaussian $L^2(P)$-continuous stationary process has the representation (1.1) with $\psi = 0$ and $\varphi = 0$ on $(-\infty, 0)$ if and only if it has an absolutely continuous spectral measure and the spectral density f satisfies

$$\int \frac{\log(f(u))}{1+u^2}\, du > -\infty.$$

In the case where $\psi = 0$ and φ is 0 on $(-\infty, 0)$, it follows from Knight [1992, Theorem 6.5] that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale if and only if

$$\varphi(t) = \alpha + \int_0^t h(s)\, ds, \qquad t \geq 0, \tag{1.2}$$

for some $\alpha \in \mathbb{R}$ and $h \in L^2_{\mathbb{R}}(\lambda)$. Related results, also concerning general ψ, are found in Cherny [2001] and Cheridito [2004]. Knight's result is extended to the case $X_t = \int_{-\infty}^t K_t(s)\, dW_s$ in Basse [2008b, Theorem 4.6].

The results mentioned above are all concerned with the semimartingale property in the $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-filtration. Much less is known when it comes to the $(\mathcal{F}_t^{X})_{t\geq 0}$-filtration or the $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-filtration ($\mathcal{F}_t^X := \sigma(X_s : 0 \leq s \leq t)$). In particular no simple necessary and sufficient conditions, as in (1.2), are available for the semimartingale property in these filtrations. Let $(X_t)_{t\geq 0}$ be given by (1.1) and assume it is $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-adapted; it is then easier for $(X_t)_{t\geq 0}$ to be an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale than an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale and harder than being an $(\mathcal{F}_t^{X})_{t\geq 0}$-semimartingale. It follows from Basse [2008a, Theorem 4.8, iii] that when ψ equals 0 or φ and $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X})_{t\geq 0}$-semimartingale with canonical decomposition $X_t = X_0 + M_t + A_t$, then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale as well if and only if $t \mapsto E[\mathrm{Var}_{[0,t]}(A)]$ is Lipschitz continuous on $\mathbb{R}_+$ ($\mathrm{Var}_{[0,t]}(A)$ denotes the total variation of $s \mapsto A_s$ on $[0, t]$). In the case $\psi = 0$, Jeulin and Yor [1993, Proposition 19] provides necessary and sufficient conditions on the Fourier transform of φ for $(X_t)_{t\geq 0}$ to be an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale.

In the present paper we provide necessary and sufficient conditions on φ and ψ for $(X_t)_{t\geq 0}$ to be an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale. The approach taken relies heavily on Fourier theory and

Hardy functions as in Jeulin and Yor [1993]. Our main result can be described as follows. Let S^1 denote the unit circle in the complex plane $\mathbb{C}$. For each measurable function $f\colon \mathbb{R} \to S^1$ satisfying $\overline{f} = f(-\cdot)$, define $\tilde{f}\colon \mathbb{R} \to \mathbb{R}$ by

$$\tilde{f}(t) := \lim_{a\to\infty} \int_{-a}^{a} \frac{e^{its} - 1_{[-1,1]}(s)}{is} f(s)\, ds,$$

where the limit is in λ-measure. For simplicity let us assume $\psi = \varphi$. We then show that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale if and only if φ can be decomposed as

$$\varphi(t) = \beta + \alpha \tilde{f}(t) + \int_0^t \widehat{f\hat{h}}(s)\, ds, \qquad \lambda\text{-a.a. } t \in \mathbb{R}, \tag{1.3}$$

where $\alpha, \beta \in \mathbb{R}$, $f\colon \mathbb{R} \to S^1$ such that $\overline{f} = f(-\cdot)$, and $h \in L^2_{\mathbb{R}}(\lambda)$ is 0 on $\mathbb{R}_+$ when $\alpha \neq 0$. In this case $(X_t)_{t\geq 0}$ is in fact a continuous $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale, where the martingale component is a Wiener process and the bounded variation component is an absolutely continuous Gaussian process. Several applications of (1.3) are provided.

In the last part of the paper we are concerned with the spectral measure of $(X_t)_{t\in\mathbb{R}}$, where $(X_t)_{t\in\mathbb{R}}$ is either a stationary Gaussian semimartingale or a Gaussian semimartingale with stationary increments and $X_0 = 0$. In both cases we provide necessary and sufficient conditions on the spectral measure of $(X_t)_{t\in\mathbb{R}}$ for $(X_t)_{t\geq 0}$ to be an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale.

2 Notation and Hardy functions

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. By a filtration we mean an increasing family $(\mathcal{F}_t)_{t\geq 0}$ of σ-algebras satisfying the usual conditions of right-continuity and completeness. For a stochastic process $(X_t)_{t\in\mathbb{R}}$ let $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$ denote the least filtration subject to $\sigma(X_s : s \in (-\infty, t]) \subseteq \mathcal{F}_t^{X,\infty}$ for all $t \geq 0$.

Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration. Recall that an $(\mathcal{F}_t)_{t\geq 0}$-adapted càdlàg process $(X_t)_{t\geq 0}$ is said to be an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale if there exists a decomposition of $(X_t)_{t\geq 0}$ such that

$$X_t = X_0 + M_t + A_t, \qquad t \geq 0,$$

where $(M_t)_{t\geq 0}$ is a càdlàg $(\mathcal{F}_t)_{t\geq 0}$-local martingale which starts at 0 and $(A_t)_{t\geq 0}$ is a càdlàg $(\mathcal{F}_t)_{t\geq 0}$-adapted process of finite variation which starts at 0.

A process $(W_t)_{t\in\mathbb{R}}$ is said to be a Wiener process if for all $n \geq 1$ and $t_0 < \cdots < t_n$

$$W_{t_1} - W_{t_0}, \ldots, W_{t_n} - W_{t_{n-1}}$$

are independent, for $-\infty < s < t < \infty$ $W_t - W_s$ follows a centered Gaussian distributed with variance $\sigma^2(t-s)$ for some $\sigma^2 > 0$, and $W_0 = 0$. If $\sigma^2 = 1$, $(W_t)_{t\in\mathbb{R}}$ is said to be a standard Wiener process.

Let $f\colon \mathbb{R} \to \mathbb{R}$. Then (unless explicitly stated otherwise) all integrability matters of f are with respect to the Lebesgue measure λ on $\mathbb{R}$. If f is a locally integrable function and $a < b$, then $\int_b^a f(s)\, ds$ should be interpreted as $-\int_a^b f(s)\, ds = -\int 1_{[a,b]}(s) f(s)\, ds$. For $t \in \mathbb{R}$ let $\tau_t f$ denote the function $s \mapsto f(t-s)$.

Remark 2.1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a locally square integrable function satisfying $\tau_t f - \tau_0 f \in L^2_{\mathbb{R}}(\lambda)$ for all $t \in \mathbb{R}$. Then $t \mapsto \tau_t f - \tau_0 f$ is a continuous mapping from $\mathbb{R}$ into $L^2_{\mathbb{R}}(\lambda)$.

A similar result is obtained in Cheridito [2004, Lemma 3.4]. However, a short proof is given as follows. By approximation with continuous functions with compact support it follows that $t \mapsto 1_{[a,b]}(\tau_t f - \tau_0 f)$ is continuous for all $a < b$. Moreover, since $\tau_t f - \tau_0 f = \lim_n 1_{[-n,n]}(\tau_t f - \tau_0 f)$ in $L^2_{\mathbb{R}}(\lambda)$, the Baire Characterization Theorem (or more precisely a generalization of it to functions with values in abstract spaces, see e.g. Reĭnov [1984] or Stegall [1991]) states that the set of continuity points C of $t \mapsto \tau_t f - \tau_0 f$ is dense in $\mathbb{R}$. Furthermore, since the Lebesgue measure is translation invariant we obtain $C = \mathbb{R}$ and it follows that $t \mapsto \tau_t f - \tau_0 f$ is continuous.

For measurable functions $f, g\colon \mathbb{R} \to \mathbb{R}$ satisfying $\int |f(t-s)g(s)|\,ds < \infty$ for $t \in \mathbb{R}$, we let $f * g$ denote the convolution between f and g, that is $f * g$ is the mapping

$$t \mapsto \int f(t-s)g(s)\,ds.$$

A locally square integrable function $f\colon \mathbb{R} \to \mathbb{R}$ is said to have orthogonal increments if $\tau_t f - \tau_0 f \in L^2_{\mathbb{R}}(\lambda)$ for all $t \in \mathbb{R}$ and for all $-\infty < t_0 < t_1 < t_2 < \infty$ we have that $\tau_{t_2} f - \tau_{t_1} f$ is orthogonal to $\tau_{t_1} f - \tau_{t_0} f$ in $L^2_{\mathbb{R}}(\lambda)$.

We now give a short survey of Fourier theory and Hardy functions. For a comprehensive survey see Dym and McKean [1976]. The Hardy functions will become an important tool in the construction of the canonical decomposition of a moving average. Let $L^2_{\mathbb{R}}(\lambda)$ and $L^2_{\mathbb{C}}(\lambda)$ denote the spaces of real and complex valued square integrable functions from $\mathbb{R}$. For $f, g \in L^2_{\mathbb{C}}(\lambda)$ define their inner product as $\langle f, g\rangle_{L^2_{\mathbb{C}}(\lambda)} := \int f\overline{g}\,d\lambda$, where $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. For $f \in L^2_{\mathbb{C}}(\lambda)$ define the Fourier transform of f as

$$\hat{f}(t) := \lim_{a\downarrow -\infty,\ b\uparrow\infty} \int_a^b f(x)e^{ixt}\,dx,$$

where the limit is in $L^2_{\mathbb{C}}(\lambda)$. The Plancherel identity shows that for all $f, g \in L^2_{\mathbb{C}}(\lambda)$ we have $\langle \hat{f}, \hat{g}\rangle_{L^2_{\mathbb{C}}(\lambda)} = 2\pi\langle f, g\rangle_{L^2_{\mathbb{C}}(\lambda)}$. Moreover, for $f \in L^2_{\mathbb{C}}(\lambda)$ we have that $\hat{\hat{f}} = 2\pi f(-\cdot)$. Thus, the mapping $f \mapsto \hat{f}$ is (up to the factor $\sqrt{2\pi}$) a linear isometry from $L^2_{\mathbb{C}}(\lambda)$ onto $L^2_{\mathbb{C}}(\lambda)$. Furthermore, if $f \in L^2_{\mathbb{C}}(\lambda)$, then f is real valued if and only if $\overline{\hat{f}} = \hat{f}(-\cdot)$.

Let $\mathbb{C}_+$ denote the open upper half plane of the complex plane $\mathbb{C}$, i.e. $\mathbb{C}_+ := \{z \in \mathbb{C} : \Im z > 0\}$. An analytic function $H\colon \mathbb{C}_+ \to \mathbb{C}$ is a Hardy function if

$$\sup_{b>0} \int |H(a+ib)|^2\,da < \infty.$$

Let $\mathbb{H}^2_+$ denote the space of all Hardy functions. It can be shown that a function $H\colon \mathbb{C}_+ \to \mathbb{C}$ is a Hardy function if and only if there exists a function $h \in L^2_{\mathbb{C}}(\lambda)$ which is 0 on $(-\infty, 0)$ and satisfies

$$H(z) = \int e^{izt}h(t)\,dt, \qquad z \in \mathbb{C}_+. \tag{2.1}$$

In this case $\lim_{b\downarrow 0} H(a+ib) = \hat{h}(a)$ for λ-a.a. $a \in \mathbb{R}$ and in $L^2_{\mathbb{C}}(\lambda)$.

Let $H \in \mathbb{H}^2_+$ with h given by (2.1). Then H is called an outer function if it is non-trivial and for all $a + ib \in \mathbb{C}_+$ we have

$$\log(|H(a+ib)|) = \frac{b}{\pi} \int \frac{\log(|\hat{h}(u)|)}{(u-a)^2 + b^2}\, du.$$

An analytic function $J\colon \mathbb{C}_+ \to \mathbb{C}$ is called an inner function if $|J| \leq 1$ on $\mathbb{C}_+$ and with $j(a) := \lim_{b\downarrow 0} J(a+ib)$ for λ-a.a. $a \in \mathbb{R}$ we have $|j| = 1$ λ-a.s. For $H \in \mathbb{H}^2_+$ (with h given by (2.1)) it is possible to factor H as a product of an outer function H^o and an inner function J. If h is a real function, J can be chosen such that $\overline{J(z)} = J(-\overline{z})$ for all $z \in \mathbb{C}_+$.

3 Main results

By S^1 we shall denote the unit circle in the complex field $\mathbb{C}$, i.e. $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. For each measurable function $f\colon \mathbb{R} \to S^1$ satisfying $\overline{f} = f(-\cdot)$ we define $\tilde{f}\colon \mathbb{R} \to \mathbb{R}$ by

$$\tilde{f}(t) := \lim_{a\to\infty} \int_{-a}^{a} \frac{e^{its} - 1_{[-1,1]}(s)}{is} f(s)\, ds,$$

where the limit is in λ-measure. The limit exists since for $a \geq 1$ we have

$$\int_{-a}^{a} \frac{e^{its} - 1_{[-1,1]}(s)}{is} f(s)\, ds = \int_{-1}^{1} \frac{e^{its} - 1}{is} f(s)\, ds + \int_{-a}^{a} e^{its} 1_{[-1,1]^c}(s) f(s) (is)^{-1}\, ds,$$

and the last term converges in $L^2_{\mathbb{R}}(\lambda)$ to the Fourier transform of

$$s \mapsto 1_{[-1,1]^c}(s) f(s) (is)^{-1}.$$

Moreover, $\tilde{f}$ takes real values since $\overline{f} = f(-\cdot)$. Note that $\tilde{f}(t)$ is defined by integrating $f(s)$ against the kernel $(e^{its} - 1_{[-1,1]}(s))/is$, whereas the Fourier transform $\hat{f}(t)$ occurs by integration of $f(s)$ against e^{its}.

For $u \leq t$ we have

$$\tilde{f}(t + \cdot) - \tilde{f}(u + \cdot) = \widehat{\hat{1}_{[u,t]} f}, \qquad \lambda\text{-a.s.} \tag{3.1}$$

Using this it follows that $\tilde{f}$ has orthogonal increments. To see this let $t_0 < t_1 < t_2 < t_3$ be given. Then

$$\begin{aligned} &\langle \tilde{f}(t_3 - \cdot) - \tilde{f}(t_2 - \cdot), \tilde{f}(t_1 - \cdot) - \tilde{f}(t_0 - \cdot)\rangle_{L^2_{\mathbb{C}}(\lambda)} \\ &= 2\pi \langle \hat{1}_{[t_2,t_3]} f, \hat{1}_{[t_0,t_1]} f\rangle_{L^2_{\mathbb{C}}(\lambda)} = \langle \hat{1}_{[t_2,t_3]}, \hat{1}_{[t_0,t_1]}\rangle_{L^2_{\mathbb{C}}(\lambda)} = \langle 1_{[t_2,t_3]}, 1_{[t_0,t_1]}\rangle_{L^2_{\mathbb{C}}(\lambda)} = 0, \end{aligned}$$

which shows the result.

In the following let $t \mapsto \operatorname{sgn}(t)$ denote the signum function defined by $\operatorname{sgn}(t) = -1_{(-\infty,0)}(t) + 1_{(0,\infty)}(t)$. Let us calculate $\tilde{f}$ in three simple cases.

Example 3.1. We have the following:

(i) if $f \equiv 1$ then $\tilde{f}(t) = \pi \operatorname{sgn}(t)$,

(ii) if $f(t) = (t+i)(t-i)^{-1}$ then $\tilde{f}(t) = 4\pi(e^{-t} - 1/2)1_{\mathbb{R}_+}(t)$,

(iii) if $f(t) = i\,\mathrm{sgn}(t)$ then $\tilde{f}(t) = -2(\gamma + \log|t|)$, where γ denotes Euler's constant.

(i) follows since $\int_0^x \frac{\sin(s)}{s}\,ds \to \pi/2$ as $x \to \infty$. Let f be given as in (ii). Then for all $t \in \mathbb{R}$ we have

$$\int_{-a}^{a} \frac{e^{its} - 1_{[-1,1]}(s)}{is} f(s)\,ds = 4\int_0^a \frac{\cos(ts) - 1_{[0,1]}(s)}{s^2+1}\,ds + 2\int_0^a \frac{\sin(ts)}{s}\frac{s^2-1}{s^2+1}\,ds,$$

which converges to

$$\begin{cases} 4\frac{\pi}{4}(2e^{-t}-1) + 2\frac{\pi}{2}(2e^{-t}-1) = 2\pi(2e^{-t}-1), & t > 0, \\ 4\frac{\pi}{4}(2e^{-t}-1) - 2\frac{\pi}{2}(2e^{-t}-1) = 0, & t < 0, \end{cases}$$

as $a \to \infty$. This shows (ii).
Finally let $f(t) = i\,\mathrm{sgn}(t)$. For $t > 0$ and $a \geq 1$,

$$\begin{aligned} &\int_{-a}^{a} \frac{e^{its} - 1_{[-1,1]}(s)}{is} f(s)\,ds = \int_{-a}^{a} \frac{\cos(ts) - 1_{[-1,1]}(s)}{is} f(s)\,ds \\ &= 2\int_0^{at} \frac{\cos(s) - 1_{[0,t]}(s)}{is} f(s/t)\,ds = 2\Big(\int_0^{at} \frac{\cos(s) - 1_{[0,1]}(s)}{s}\,ds - \log(t)\Big), \end{aligned}$$

which shows (iii) since $\tilde{f}(-t) = \tilde{f}(t)$. ◇

Let $(W_t)_{t\in\mathbb{R}}$ be a standard Wiener process and $\varphi, \psi\colon \mathbb{R} \to \mathbb{R}$ be two locally square integrable functions such that $\varphi(t - \cdot) - \psi(-\cdot) \in L^2_{\mathbb{R}}(\lambda)$ for all $t \in \mathbb{R}$. In the following we let $(X_t)_{t\in\mathbb{R}}$ be given by

$$X_t = \int (\varphi(t-s) - \psi(-s))\,dW_s, \qquad t \in \mathbb{R}. \tag{3.2}$$

Now we are ready to characterize the class of $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingales.

Theorem 3.2. *$(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale if and only if the following two conditions* (a) *and* (b) *are satisfied:*

(a) φ can be decomposed as

$$\varphi(t) = \beta + \alpha\tilde{f}(t) + \int_0^t \widehat{f\hat{h}}(s)\,ds, \qquad \lambda\text{-a.a. } t \in \mathbb{R}, \tag{3.3}$$

where $\alpha, \beta \in \mathbb{R}$, $f\colon \mathbb{R} \to S^1$ is a measurable function such that $\overline{f} = f(-\cdot)$, and $h \in L^2_{\mathbb{R}}(\lambda)$ is 0 on $\mathbb{R}_+$ when $\alpha \neq 0$.

(b) Let $\xi := f\widehat{(\widehat{\varphi - \psi})}$. If $\alpha \neq 0$ then

$$\int_0^r \left(\frac{|\xi(s)|}{\sqrt{\int_s^\infty \xi(u)^2\,du}} \right) ds < \infty, \qquad \forall\, r > 0, \tag{3.4}$$

where $\frac{0}{0} := 0$.

In this case $(X_t)_{t\geq 0}$ is a continuous $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale where the martingale component is a Wiener process with parameter $\sigma^2 = (2\pi\alpha)^2$ and the bounded variation component is an absolutely continuous Gaussian process. In the case $X_0 = 0$ we may choose α, β, h and f such that the $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-canonical decomposition of $(X_t)_{t\geq 0}$ is given by $X_t = M_t + A_t$, where

$$M_t = \alpha \int \Big(\tilde{f}(t-s) - \tilde{f}(-s)\Big)\, dW_s \quad \text{and} \quad A_t = \int_0^t \Big(\int \widehat{f\hat{h}}(s-u)\, dW_u\Big)\, ds.$$

Furthermore, when $\alpha \neq 0$ and $X_0 = 0$, the law of $(\frac{1}{2\pi\alpha}X_t)_{t\in[0,T]}$ is equivalent to the Wiener measure on $C([0,T])$ for all $T > 0$. $\diamond$

The proof is given in Section 5. Let us note the following:

Remark 3.3.

1. The case $X_0 = 0$ corresponds to $\psi = \varphi$. In this case condition (b) is always satisfied since we then have $\xi = 0$.
2. When $f \equiv 1$, (a) and (b) reduce to the conditions that φ is absolutely continuous on $\mathbb{R}_+$ with square integrable density and φ and ψ are constant on $(-\infty, 0)$. Hence by Cherny [2001, Theorem 3.1] an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale is an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale if and only if we may choose $f \equiv 1$.
3. The condition imposed on ξ in (b) is the condition for expansion of filtration in Chaleyat-Maurel and Jeulin [1983, Theoreme I.1.1].

Corollary 3.4. *Assume $X_0 = 0$. Then $(X_t)_{t\geq 0}$ is a Wiener process if and only if $\varphi = \beta + \alpha\tilde{f}$, for some measurable function $f\colon \mathbb{R} \to S^1$ satisfying $\overline{f} = f(-\cdot)$ and $\alpha, \beta \in \mathbb{R}$.*

The corollary shows that the mapping $f \mapsto \tilde{f}$ (up to affine transformations) is onto the space of functions with orthogonal increments (recall the definition on page 1143). Moreover, if $f, g\colon \mathbb{R} \to S^1$ are measurable functions satisfying $\overline{f} = f(-\cdot)$ and $\overline{g} = g(-\cdot)$ and $\tilde{f} = \tilde{g}$ λ-a.s. then (3.1) shows that for $u \leq t$ we have

$$\hat{1}_{[u,t]} f = \hat{1}_{[u,t]} g, \qquad \lambda\text{-a.s.}$$

which implies $f = g$ λ-a.s. Thus, we have shown:

Remark 3.5. The mapping $f \mapsto \tilde{f}$ is one to one and (up to affine transformations) onto the space of functions with orthogonal increments.

For each measurable function $f\colon \mathbb{R} \to S^1$ such that $\overline{f} = f(-\cdot)$ and for each $h \in L^2_{\mathbb{R}}(\lambda)$ we have

$$\begin{aligned}\int_0^t \widehat{f\hat{h}}(s)\, ds &= \langle 1_{[0,t]}, \widehat{f\hat{h}}\rangle_{L^2_{\mathbb{C}}(\lambda)} = \langle \hat{1}_{[0,t]}, (f\hat{h})(-\cdot)\rangle_{L^2_{\mathbb{C}}(\lambda)} \\ &= \langle \hat{1}_{[0,t]} f, \hat{h}(-\cdot)\rangle_{L^2_{\mathbb{C}}(\lambda)} = \langle \widehat{\hat{1}_{[0,t]} f}, h\rangle_{L^2_{\mathbb{C}}(\lambda)} = \int \big(\tilde{f}(t+s) - \tilde{f}(s)\big) h(s)\, ds,\end{aligned} \tag{3.5}$$

which gives an alternative way of writing the last term in (3.3).

In some cases it is of interest that $(X_t)_{t\geq 0}$ is $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-adapted. This situation is studied in the next result. We also study the case where $(X_t)_{t\geq 0}$ is a stationary process, which corresponds to $\psi = 0$.

Proposition 3.6. *We have*

(i) Assume $\psi = 0$. Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale if and only if φ satisfies (a) *of Theorem 3.2 and $t \mapsto \alpha + \int_0^t h(-s)\,ds$ is square integrable on $\mathbb{R}_+$ when $\alpha \neq 0$.*

(ii) Assume ψ equals 0 or φ and $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale. Then $(X_t)_{t\geq 0}$ is $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-adapted if and only if we may choose f and h of Theorem 3.2 (a) *such that $f(a) = \lim_{b\downarrow 0} J(-a+ib)$ for λ-a.a. $a \in \mathbb{R}$, for some inner function J, and h is 0 on $\mathbb{R}_+$. In this case there exists a constant $c \in \mathbb{R}$ such that*

$$\varphi = \beta + \alpha\tilde{f} + (\tilde{f} - c) * g, \qquad \lambda\text{-a.s.} \tag{3.6}$$

where $g = h(-\cdot)$.

According to Beurling [1948] (see also Dym and McKean [1976, page 53]), $J\colon \mathbb{C}_+ \to \mathbb{C}$ is an inner function if and only if it can be factorized as:

$$J(z) = Ce^{i\alpha z} \exp\Big(\frac{1}{\pi i}\int \frac{1+sz}{s-z}\,F(ds)\Big) \prod_{n\geq 1} \varepsilon_n \frac{z_n - z}{\overline{z}_n - z}\,, \tag{3.7}$$

where $C \in S^1$, $\alpha \geq 0$, $(z_n)_{n\geq 1} \subseteq \mathbb{C}_+$ satisfies $\sum_{n\geq 1} \Im(z_n)/(|z_n|^2+1) < \infty$ and $\varepsilon_n = z_n/\overline{z}_n$ or 1 according as $|z_n| \leq 1$ or not, and F is a nondecreasing bounded singular function. Thus, a measurable function $f\colon \mathbb{R} \to S^1$ with $\overline{f} = f(-\cdot)$ satisfies the condition in Proposition 3.6 (ii) if and only if

$$f(a) = \lim_{b\downarrow 0} J(-a+ib), \qquad \lambda\text{-a.a. } a \in \mathbb{R}, \tag{3.8}$$

for a function J given by (3.7). If $f\colon \mathbb{R} \to S^1$ is given by $f(t) = i\,\mathrm{sgn}(t)$, then according to Example 3.1, $\tilde{f}(t) = -2(\gamma + \log|t|)$. Thus this f does not satisfy the condition in Proposition 3.6 (ii).

In the next example we illustrate how to obtain (φ, ψ) for which $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale or a Wiener process (in its natural filtration). The idea is simply to pick a function $f\colon \mathbb{R} \to S^1$ satisfying $\overline{f} = f(-\cdot)$ and calculate $\tilde{f}$. Moreover, if one wants $(X_t)_{t\geq 0}$ to be $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-adapted one has to make sure that f is given as in (3.8).

Example 3.7. *Let $(X_t)_{t\in\mathbb{R}}$ be given by*

$$X_t = \int (\varphi(t-s) - \varphi(-s))\,dW_s, \qquad t \in \mathbb{R}.$$

(i) If φ is given by $\varphi(t) = (e^{-t} - 1/2)1_{\mathbb{R}_+}(t)$ or $\varphi(t) = \log|t|$ for all $t \in \mathbb{R}$, then $(X_t)_{t\geq 0}$ is a Wiener process (in its natural filtration).

(ii) If φ is given by

$$\varphi(t) = \log|t| + \int_0^t \log\Big|\frac{s-1}{s}\Big|\,ds, \qquad t \in \mathbb{R},$$

then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale.

(i) is a consequence of Corollary 3.4 and Example 3.1 (ii)-(iii). To show (ii) let $f(t) = i\,\mathrm{sgn}(t)$ as in Example 3.1 (iii). According to Theorem 3.2 it is enough to show

$$\widehat{f\hat{h}}(t) = \log\Big|\frac{t-1}{t}\Big|, \qquad t \in \mathbb{R}, \tag{3.9}$$

for some $h \in L^2_{\mathbb{R}}(\lambda)$ which is 0 on $\mathbb{R}_+$. Let $h(t) = 1_{[-1,0]}(t)$. Due to the fact that $\hat{h}(t) = \frac{1-\cos(t)}{it} + \frac{\sin(t)}{t}$, we have

$$\begin{aligned}
&\int_{-a}^{a} e^{its}\hat{h}(s)f(s)\,ds = 2\Big(\int_0^a \frac{\cos(ts) - (\cos(ts)\cos(s) + \sin(ts)\sin(s))}{s}\,ds\Big)\\
&= 2\int_0^a \frac{\cos(ts) - \cos((t-1)s)}{s}\,ds = 2\int_0^{ta} \frac{\cos(s) - \cos(s(t-1)/t)}{s}\,ds \to 2\log\Big|\frac{t-1}{t}\Big|
\end{aligned}$$

as $a \to \infty$, for all $t \in \mathbb{R} \setminus \{0,1\}$. This shows that $h/2$ satisfies (3.9) and the proof of (ii) is complete. $\diamond$

As a consequence of Example 3.7 (i) we have the following: Let $(X_t)_{t\geq 0}$ be the stationary Ornstein-Uhlenbeck process given by

$$X_t = X_0 - \int_0^t X_s\,ds + W_t, \qquad t \geq 0,$$

where $(W_t)_{t\geq 0}$ is a standard Wiener process and $X_0 \overset{\mathcal{D}}{=} N(0,1/2)$ is independent of $(W_t)_{t\geq 0}$. Then $(B_t)_{t\geq 0}$, given by

$$B_t := W_t - 2\int_0^t X_s\,ds, \qquad t \geq 0,$$

is a Wiener process (in its natural filtration). Representations of the Wiener process have been extensively studied by Lévy [1956], Cramér [1961], Hida [1961] and many others. One famous example of such a representation is

$$B_t = W_t - \int_0^t \frac{1}{s} W_s\,ds, \qquad t \geq 0,$$

see Jeulin and Yor [1990].

Let $X_t = \int (\varphi(t-s) - \varphi(-s))\,dW_s$ for $t \in \mathbb{R}$. Then φ has to be continuous on $[0,\infty)$ (in particular bounded on compacts of $\mathbb{R}$) for $(X_t)_{t\geq 0}$ to be an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale. This is not the case for the $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale property. Indeed, Example 3.7 shows that if $\varphi(t) = \log|t|$ then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-martingale, but φ is unbounded on $[0,1]$.

4 Functions with orthogonal increments

In the following we collect some properties of functions with orthogonal increments. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function with orthogonal increments. For $t \in \mathbb{R}$ we have

$$\begin{aligned}
\|\tau_t f - \tau_0 f\|^2_{L^2_{\mathbb{R}}(\lambda)} &= \|\tau_t f - \tau_{t/2} f\|^2_{L^2_{\mathbb{R}}(\lambda)} + \|\tau_{t/2} f - \tau_0 f\|^2_{L^2_{\mathbb{R}}(\lambda)} \\
&= 2\|\tau_{t/2} f - \tau_0 f\|^2_{L^2_{\mathbb{R}}(\lambda)}.
\end{aligned} \tag{4.1}$$

Moreover, since $t \mapsto \|\tau_t f - \tau_0 f\|^2_{L^2_{\mathbb{R}}(\lambda)}$ is continuous by Remark 2.1 (recall that f by definition is locally square integrable), equation (4.1) shows that $\|\tau_t f - \tau_0 f\|^2_{L^2_{\mathbb{R}}(\lambda)} = K|t|$, where $K := \|\tau_1 f - \tau_0 f\|^2_{L^2_{\mathbb{R}}(\lambda)}$. This implies that $\|\tau_t f - \tau_u f\|^2_{L^2_{\mathbb{R}}(\lambda)} = K|t-u|$ for $u, t \in \mathbb{R}$. For a step function $h = \sum_{j=1}^k a_j 1_{(t_{j-1}, t_j]}$ define the mapping

$$\int h(u)\, d\tau_u f := \sum_{j=1}^k a_j (\tau_{t_j} f - \tau_{t_{j-1}} f).$$

Then $v \mapsto (\int h(u)\, d\tau_u f)(v)$ is square integrable and

$$\sqrt{K} \|h\|_{L^2_{\mathbb{R}}(\lambda)} = \|\int h(u)\, d\tau_u f\|_{L^2_{\mathbb{R}}(\lambda)}.$$

Hence, by standard arguments we can define $\int h(u)\, d\tau_u f$ through the above isometry for all $h \in L^2_{\mathbb{R}}(\lambda)$ such that $h \mapsto \int h(u)\, d\tau_u f$ is a linear isometry from $L^2_{\mathbb{R}}(\lambda)$ into $L^2_{\mathbb{R}}(\lambda)$.
Assume that $g \colon \mathbb{R}^2 \to \mathbb{R}$ is a measurable function, and μ is a finite measure such that

$$\int \int g(u,v)^2 \, du \, \mu(dv) < \infty.$$

Then $(v,s) \mapsto (\int g(u,v)\, d\tau_u f)(s)$ can be chosen measurable and in this case we have

$$\int \Big(\int g(u,v)\, d\tau_u f \Big) \mu(dv) = \int \Big(\int g(u,v)\, \mu(dv) \Big) d\tau_u f. \tag{4.2}$$

Lemma 4.1. *Let $g \colon \mathbb{R} \to \mathbb{R}$ be given by*

$$g(t) = \begin{cases} \alpha + \int_0^t h(v)\, dv & t \geq 0 \\ 0 & t < 0, \end{cases}$$

where $\alpha \in \mathbb{R}$ and $h \in L^2_{\mathbb{R}}(\lambda)$. Then, $g(t - \cdot) - g(-\cdot) \in L^2_{\mathbb{R}}(\lambda)$ for all $t \in \mathbb{R}$.
Let f be a function with orthogonal increments.

(i) Let φ be a measurable function. Then there exists a constant $\beta \in \mathbb{R}$ such that

$$\varphi(t) = \beta + \alpha f(t) + \int_0^\infty \big(f(t-v) - f(-v) \big) h(v)\, dv, \quad \lambda\text{-a.a. } t \in \mathbb{R}, \tag{4.3}$$

if and only if for all $t \in \mathbb{R}$ we have

$$\tau_t \varphi - \tau_0 \varphi = \int (g(t-u) - g(-u))\, d\tau_u f, \qquad \lambda\text{-a.s.} \tag{4.4}$$

(ii) Assume g is square integrable. Then there exists a $\beta \in \mathbb{R}$ such that λ-a.s.

$$\int g(-u)\, d\tau_u f = \beta + \alpha f(-\cdot) + \int_0^\infty \big(f(-u-\cdot) - f(-u) \big) h(u)\, du. \tag{4.5}$$

Proof. From Jensen's inequality and Tonelli's Theorem it follows that

$$\int \Big(\int_{-s}^{t-s} h(u)\, du \Big)^2 ds \le t \int \left(\int_{-s}^{t-s} h(u)^2\, du \right) ds = t^2 \int h(s)^2\, du < \infty,$$

which shows $g(t-\cdot) - g(-\cdot) \in L^2_{\mathbb{R}}(\lambda)$.

(i): We may and do assume that h is 0 on $(-\infty, 0)$. For $t, u \in \mathbb{R}$ we have

$$g(t-u) - g(-u) = \begin{cases} \alpha 1_{(0,t]}(u) + \int_{-u}^{t-u} h(v)\, dv, & t \ge 0, \\ -\alpha 1_{(t,0]}(u) - \int_{t-u}^{-u} h(v)\, dv, & t < 0, \end{cases}$$

which by (4.2) implies that for $t \in \mathbb{R}$ we have λ-a.s.

$$\int (g(t-u) - g(-u))\, d\tau_u f = \alpha(\tau_t f - \tau_0 f) + \int (\tau_{t-v} f - \tau_{-v} f)\, h(v)\, dv. \tag{4.6}$$

First assume (4.4) is satisfied. For $t \in \mathbb{R}$ it follows from (4.6) that

$$\tau_t \varphi - \tau_0 \varphi = \alpha(\tau_t f - \tau_0 f) + \int (\tau_{t-v} f - \tau_{-v} f)\, h(v)\, dv, \qquad \lambda\text{-a.s.}$$

Hence, by Tonelli's Theorem there exists a sequence $(s_n)_{n\ge 1}$ such that $s_n \to 0$ and such that

$$\begin{aligned} \varphi(t-s_n) = {} & \varphi(-s_n) - \alpha f(s_n) + \alpha f(t-s_n) \\ & + \int (f(t-v-s_n) - f(-v-s_n))\, h(v)\, dv, \qquad \forall n \ge 1,\ \lambda\text{-a.a. } t \in \mathbb{R}. \end{aligned} \tag{4.7}$$

From Remark 2.1 it follows that $\varphi(\cdot - s_n) - \varphi(\cdot)$ and $f(\cdot - s_n) - f(\cdot)$ converge to 0 in $L^2_{\mathbb{R}}(\lambda)$ and

$$\int \big(f(t-v-s_n) - f(-v-s_n)\big) h(v)\, dv \to \int [f(t-v) - f(-v)] h(v)\, dv, \qquad t \in \mathbb{R}.$$

Thus we obtain (4.5) by letting n tend to infinity in (4.7).

Assume conversely (4.3) is satisfied. For $t \in \mathbb{R}$ we have

$$\tau_t \varphi - \tau_0 \varphi = \alpha(\tau_t f - \tau_0 f) + \int (\tau_{t-v} f - \tau_{-v} f)\, h(v)\, dv, \qquad \lambda\text{-a.s.}$$

and hence we obtain (4.4) from (4.6).

(ii): Assume in addition that $g \in L^2_{\mathbb{R}}(\lambda)$. By approximation we may assume h has compact support. Choose $T > 0$ such that h is 0 outside $(0, T)$. Since $g \in L^2_{\mathbb{R}}(\lambda)$, it follows that $\alpha = -\int_0^T h(s)\, ds$ and therefore g is on the form

$$g(t) = -1_{[0,T]}(t) \int_t^T h(s)\, ds, \qquad t \in \mathbb{R}.$$

From (4.2) it follows that

$$\begin{aligned} & \int g(-u)\, d\tau_u f = \int \left(\int -1_{(-u,T]}(s) 1_{[0,T]}(-u) h(s)\, ds \right) d\tau_u f \\ & = \int \left(\int -1_{(-u,T]}(s) 1_{[0,T]}(-u) h(s)\, d\tau_u f \right) ds = \int_0^T -h(s) \left(\int_{-s}^0 d\tau_u f \right) ds \\ & = \int_0^T -h(s)\, (\tau_0 f - \tau_{-s} f)\, ds = \alpha \tau_0 f + \int_0^T h(s) \tau_{-s} f\, ds. \end{aligned}$$

Thus, if we let $\beta := \int_0^T h(s)f(-s)\,ds$, then

$$\int g(-u)\,d\tau_u f = \beta + \alpha f(-\cdot) + \int h(s)\,(f(-s-\cdot) - f(-s))\,ds,$$

which completes the proof. $\square$

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function with orthogonal increments and let $(B_t)_{t\in\mathbb{R}}$ be given by

$$B_t = \int (f(t-s) - f(-s))\,dW_s, \qquad t \in \mathbb{R}.$$

Then it follows that $(B_t)_{t\in\mathbb{R}}$ is a Wiener process and

$$\int q(s)\,dB_s = \int \Big(\int q(u)\,d\tau_u f\Big)(s)\,dW_s, \qquad \forall\, q \in L^2_{\mathbb{R}}(\lambda). \tag{4.8}$$

This is obvious when q is a step function and hence by approximation it follows that (4.8) is true for all $q \in L^2_{\mathbb{R}}(\lambda)$.
Let $f\colon \mathbb{R} \to S^1$ denote a measurable function satisfying $\overline{f} = f(-\cdot)$. Then

$$\int q(u)\,d\tau_u \tilde{f} = (\widehat{\widehat{q}f})(-\cdot), \qquad \forall\, q \in L^2_{\mathbb{R}}(\lambda). \tag{4.9}$$

To see this assume first q is a step function on the form $\sum_{j=1}^k a_j 1_{(t_{j-1},t_j]}$. Then

$$\Big(\int q(u)\,d\tau_u \tilde{f}\Big)(s) = \sum_{j=1}^k a_j \Big(\tilde{f}(t_j - s) - \tilde{f}(t_{j-1} - s)\Big)$$

$$= \int \sum_{j=1}^k a_j \frac{e^{it_j u} - e^{it_{j-1}u}}{iu} f(u) e^{-isu}\,du = \int \widehat{q}(u) f(u) e^{-isu}\,du = (\widehat{\widehat{q}f})(-s),$$

which shows that (4.9) is valid for step functions and hence the result follows for general $q \in L^2_{\mathbb{R}}(\lambda)$ by approximation. Thus, if $(B_t)_{t\in\mathbb{R}}$ is given by $B_t = \int (\tilde{f}(t-s) - \tilde{f}(-s))\,dW_s$ for all $t \in \mathbb{R}$, then by combining (4.8) and (4.9) we have

$$\int q(s)\,dB_s = \int (\widehat{\widehat{q}f})(-s)\,dW_s, \qquad \forall\, q \in L^2_{\mathbb{R}}(\lambda). \tag{4.10}$$

Lemma 4.2. *Let $f\colon \mathbb{R} \to S^1$ be a measurable function such that $\overline{f} = f(-\cdot)$. Then $\tilde{f}$ is constant on $(-\infty, 0)$ if and only if there exists an inner function J such that*

$$f(a) = \lim_{b\downarrow 0} J(-a + ib), \qquad \lambda\text{-a.a. } a \in \mathbb{R}. \tag{4.11}$$

Proof. Assume $\tilde{f}$ is constant on $(-\infty, 0)$ and let $t \geq 0$ be given. We have $\widehat{\hat{1}_{[0,t]}\overline{f}}(-s) = 0$ for λ-a.a. $s \in (-\infty, 0)$ due to the fact that $\widehat{\hat{1}_{[0,t]}\overline{f}}(-s) = \tilde{f}(s) - \tilde{f}(-t+s)$ for λ-a.a. $s \in \mathbb{R}$ and hence $\hat{1}_{[0,t]}\overline{f} \in \mathbb{H}^2_+$. Moreover, since $\hat{1}_{[0,t]}\overline{f}$ has outer part $\hat{1}_{[0,t]}$ we conclude that $\overline{f}(a) = \lim_{b\downarrow 0} J(a+ib)$ for λ-a.a. $a \in \mathbb{R}$ and an inner function $J\colon \mathbb{C}_+ \to \mathbb{C}$.

Assume conversely (4.11) is satisfied and fix $t \geq 0$. Let $G \in \mathbb{H}^2_+$ be the Hardy function induced by $1_{[0,t]}$. Since J is an inner function, we obtain $GJ \in \mathbb{H}^2_+$ and thus

$$G(z)J(z) = \int e^{itz}\kappa(t)\,dt, \qquad z \in \mathbb{C}_+,$$

for some $\kappa \in L^2_{\mathbb{R}}(\lambda)$ which is 0 on $(-\infty, 0)$. The remark just below (2.1) shows

$$\widehat{1_{[0,t]}}(a)\overline{f}(a) = \lim_{b \downarrow 0} G(a+ib)J(a+ib) = \hat{\kappa}(a), \qquad \lambda\text{-a.a. } a \in \mathbb{R},$$

which implies

$$\tilde{f}(s) - \tilde{f}(-t+s) = \widehat{\hat{1}_{[0,t]}\overline{f}}(-s) = \hat{\hat{\kappa}}(-s) = 2\pi k(s),$$

for λ-a.a. $s \in \mathbb{R}$. Hence, we conclude that $\tilde{f}$ is constant on $(-\infty, 0)$ λ-a.s. $\square$

5 Proofs of main results

Let $(X_t)_{t\in\mathbb{R}}$ denote a stationary Gaussian process. Following Doob [1990], $(X_t)_{t\in\mathbb{R}}$ is called deterministic if $\overline{\mathrm{sp}}\{X_t : t \in \mathbb{R}\}$ equals $\overline{\mathrm{sp}}\{X_t : t \leq 0\}$ and when this is not the case $(X_t)_{t\in\mathbb{R}}$ is called regular. Let $\varphi \in L^2_{\mathbb{R}}(\lambda)$ and let $(X_t)_{t\in\mathbb{R}}$ be given by $X_t = \int \varphi(t-s)\,dW_s$ for all $t \in \mathbb{R}$. By the Plancherel identity $(X_t)_{t\in\mathbb{R}}$ has spectral measure given by $(2\pi)^{-1}|\hat{\varphi}|^2\,d\lambda$. Thus according to Szegö's Alternative (see Dym and McKean [1976, page 84]), $(X_t)_{t\in\mathbb{R}}$ is regular if and only if

$$\int \frac{\log|\hat{\varphi}|(u)}{1+u^2}\,du > -\infty. \tag{5.1}$$

In this case the remote past $\cap_{t<0}\sigma(X_s : s < t)$ is trivial and by Karhunen [1950, Satz 5] (or Doob [1990, Chapter XII, Theorem 5.3]) we have

$$X_t = \int_{-\infty}^t g(t-s)\,dB_s,\ t \in \mathbb{R} \qquad \text{and} \qquad (\mathcal{F}_t^{X,\infty})_{t\geq 0} = (\mathcal{F}_t^{B,\infty})_{t\geq 0},$$

for some Wiener process $(B_t)_{t\in\mathbb{R}}$ and some $g \in L^2_{\mathbb{R}}(\lambda)$. However, we need the following explicit construction of $(B_t)_{t\in\mathbb{R}}$.

Lemma 5.1 (Main Lemma)**.** *Let $\varphi \in L^2_{\mathbb{R}}(\lambda)$ and $(X_t)_{t\in\mathbb{R}}$ be given by $X_t = \int \varphi(t-s)\,dW_s$ for $t \in \mathbb{R}$, where $(W_t)_{t\in\mathbb{R}}$ is a Wiener process.*

(i) If $(X_t)_{t\in\mathbb{R}}$ is a regular process then there exist a measurable function $f\colon \mathbb{R} \to S^1$ with $\overline{f} = f(-\cdot)$, a function $g \in L^2_{\mathbb{R}}(\lambda)$ which is 0 on $(-\infty, 0)$ such that we have the following: First of all $(B_t)_{t\in\mathbb{R}}$ defined by

$$B_t = \int \Big(\tilde{f}(t-s) - \tilde{f}(-s)\Big)\,dW_s, \qquad t \in \mathbb{R}, \tag{5.2}$$

is a Wiener process. Moreover,

$$X_t = \int_{-\infty}^t g(t-s)\,dB_s, \qquad t \in \mathbb{R}, \tag{5.3}$$

and finally $(\mathcal{F}_t^{X,\infty})_{t\geq 0} = (\mathcal{F}_t^{B,\infty})_{t\geq 0}$.

(ii) If φ is 0 on $(-\infty, 0)$ and $\varphi \neq 0$, then $(X_t)_{t\in\mathbb{R}}$ is regular and the above f is given by $f(a) = \lim_{b\downarrow 0} J(-a+ib)$ for λ-a.a. $a \in \mathbb{R}$, where J is an inner function.

Proof. (i): Due to the fact that $|\hat{\varphi}|^2$ is a positive integrable function which satisfies (5.1), Dym and McKean [1976, Chapter 2, Section 7, Exercise 4] shows there is an outer Hardy function $H^o \in \mathbb{H}^2_+$ such that $|\hat{\varphi}|^2 = |\hat{h}^0|^2$ and $\overline{\hat{h}^o} = \hat{h}^o(-\cdot)$, where h^0 is given by (2.1). Additionally, H^o is given by

$$H^o(z) = \exp\Big(\frac{1}{\pi i}\int \frac{uz+1}{u-z}\frac{\log|\hat{\varphi}|(u)}{u^2+1}\,du\Big), \qquad z \in \mathbb{C}_+.$$

Define $f\colon \mathbb{R} \to S^1$ by $\overline{f} = \hat{\varphi}/\hat{h}^o$ and note that $\overline{f} = f(-\cdot)$. Let $(B_t)_{t\in\mathbb{R}}$ be given by (5.2), then $(B_t)_{t\in\mathbb{R}}$ is a Wiener process due to the fact that $\tilde{f}$ has orthogonal increments. Moreover, by definition of f we have $\widehat{\tau_t h^o} f = \widehat{\tau_t\varphi}$, which shows that

$$\widehat{(\widehat{\tau_t h^o} f)} = 2\pi\tau_t\varphi(-\cdot). \tag{5.4}$$

Thus if we let $g := (2\pi)^{-1}h^o$, then $g \in L^2_{\mathbb{R}}(\lambda)$ and (5.3) follows by (4.10) and (5.4). Furthermore, since H^o is an outer function we have $(\mathcal{F}_t^{X,\infty})_{t\geq 0} = (\mathcal{F}_t^{B,\infty})_{t\geq 0}$ according to page 95 in Dym and McKean [1976].

(ii): Assume $\varphi \in L^2_{\mathbb{R}}(\lambda)$ is 0 on $(-\infty, 0)$ and $\varphi \neq 0$. By definition $(X_t)_{t\in\mathbb{R}}$ is clearly regular. Let h^o, f and $(B_t)_{t\in\mathbb{R}}$ be given as above (recall that $\overline{f} = f(-\cdot)$). It follows by Dym and McKean [1976, page 37] that $J := H/H^o$ is an inner function and by definition of J, $f(-a) = \lim_{b\downarrow 0} J(a+ib)$ for λ-a.a. $a \in \mathbb{R}$, which completes the proof. $\square$

The following lemma is related to Hardy and Littlewood [1928, Theorem 24] and hence the proof is omitted.

Lemma 5.2. *Let κ be a locally integrable function and let $\Delta_t\kappa$ denote the function*

$$s \mapsto t^{-1}(\kappa(t+s) - \kappa(s)), \qquad t > 0.$$

Then $(\Delta_t\kappa)_{t>0}$ is bounded in $L^2_{\mathbb{R}}(\lambda)$ if and only if κ is absolutely continuous with square integrable density.

The following simple, but nevertheless useful, lemma is inspired by Masani [1972] and Cheridito [2004].

Lemma 5.3. *Let $(X_t)_{t\in\mathbb{R}}$ denote a continuous and centered Gaussian process with stationary increments. Then there exists a continuous, stationary and centered Gaussian process $(Y_t)_{t\in\mathbb{R}}$, satisfying*

$$Y_t = X_t - e^{-t}\int_{-\infty}^t e^s X_s\,ds \quad \text{and} \quad X_t - X_0 = Y_t - Y_0 + \int_0^t Y_s\,ds,$$

for all $t \in \mathbb{R}$, and $\mathcal{F}_t^{X,\infty} = \sigma(X_0) \vee \mathcal{F}_t^{Y,\infty}$ for all $t \geq 0$.
Furthermore, if $(X_t)_{t\in\mathbb{R}}$ is given by (3.2),

$$\kappa(t) := \int_{-\infty}^0 e^u\big(\varphi(t) - \varphi(u+t)\big)\,du, \qquad t \in \mathbb{R}, \tag{5.5}$$

is a well-defined square integrable function and $(Y_t)_{t\in\mathbb{R}}$ is given by $Y_t = \int \kappa(t-s)\,dW_s$ for $t \in \mathbb{R}$.

The proof is simple and hence omitted.

Remark 5.4. A càdlàg Gaussian process $(X_t)_{t\geq 0}$ with stationary increments has P-a.s. continuous sample paths. Indeed, this follows from Adler [1990, Theorem 3.6] since $P(\Delta X_t = 0) = 1$ for all $t \geq 0$ by the stationary increments.

Proof of Theorem 3.2. If: Assume (a) and (b) are satisfied. We show that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale.

(1): The case $\alpha \neq 0$. Let $(B_t)_{t\in\mathbb{R}}$ denote the Wiener process given by

$$B_t := \int \Big(\tilde{f}(t-s) - \tilde{f}(-s)\Big)\, dW_s, \qquad t \in \mathbb{R},$$

and let $g\colon \mathbb{R} \to \mathbb{R}$ be given by

$$g(t) = \begin{cases} \alpha + \int_0^t h(-u)\, du & t \geq 0 \\ 0 & t < 0. \end{cases}$$

Since φ satisfies (3.3) it follows by (3.5), Lemma 4.1 and (4.8) that

$$X_t - X_0 = \int (\tau_t\varphi(s) - \tau_0\varphi(s))\, dW_s = \int (g(t-s) - g(-s))\, dB_s, \qquad t \in \mathbb{R}.$$

From Cherny [2001, Theorem 3.1] it follows that $(X_t - X_0)_{t\geq 0}$ is an $(\mathcal{F}_t^{B,\infty})_{t\geq 0}$-semimartingale with martingale component $(\alpha B_t)_{t\geq 0}$. Let $k = (2\pi)^{-2}\xi \in L^2_{\mathbb{R}}(\lambda)$ (ξ is given in (b)). Since $\widehat{kf} = \varphi - \psi$ it follows by (4.10) that $X_0 = \int k(s)\, dB_s$. Moreover, since k satisfies (3.4) it follows from Chaleyat-Maurel and Jeulin [1983, Theoreme I.1.1] that $(B_t)_{t\geq 0}$ is an $(\mathcal{F}_t^B \vee \sigma(\int_0^\infty k(s)\, dB_s))_{t\geq 0}$-semimartingale and since $\mathcal{F}_t^B \vee \sigma(\int_0^\infty k(s)\, dB_s) \vee \sigma(B_u : u \leq 0) = \mathcal{F}_t^{B,\infty} \vee \sigma(X_0)$, $(B_t)_{t\geq 0}$ is also an $(\mathcal{F}_t^{B,\infty} \vee \sigma(X_0))_{t\geq 0}$-semimartingale. Thus we conclude that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{B,\infty} \vee \sigma(X_0))_{t\geq 0}$-semimartingale and hence also an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale, since $\mathcal{F}_t^{X,\infty} \subseteq \mathcal{F}_t^{B,\infty} \vee \sigma(X_0)$ for all $t \geq 0$.

(2): The case $\alpha = 0$. Let us argue as in Cherny [2001, page 8]. Since φ is absolutely continuous with square integrable density, Lemma 5.2 implies

$$E[(X_t - X_u)^2] = \int \big(\varphi(t-s) - \varphi(u-s)\big)^2 ds \leq K|t-u|^2, \qquad t, u \geq 0, \tag{5.6}$$

for some constant $K \in \mathbb{R}_+$. The Kolmogorov-Čentsov Theorem shows that $(X_t)_{t\geq 0}$ has a continuous modification and from (5.6) it follows that this modification is of integrable variation. Hence $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale.

Only if: Assume conversely that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale and hence continuous, according to Remark 5.4.

(3): First assume (in addition) that $(X_t)_{t\geq 0}$ is of unbounded variation. Let κ and $(Y_t)_{t\in\mathbb{R}}$ be given as in Lemma 5.3. Since

$$Y_t = X_t - e^{-t}\int_{-\infty}^t e^s X_s\, ds, \quad t \geq 0, \quad \text{and} \quad (\mathcal{F}_t^{Y,\infty} \vee \sigma(X_0))_{t\geq 0} = (\mathcal{F}_t^{X,\infty})_{t\geq 0}, \tag{5.7}$$

we deduce that $(Y_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Y,\infty})_{t\geq 0}$-semimartingale of unbounded variation. This implies that $\mathcal{F}_0^{Y,\infty} \neq \mathcal{F}_\infty^{Y,\infty}$ and we conclude that $(Y_t)_{t\in\mathbb{R}}$ is regular. Now choose f and g according to Lemma 5.1 (with (φ, X) replaced by (κ, Y)) and let $(B_t)_{t\in\mathbb{R}}$ be given as in the lemma such that

$$Y_t = \int_{-\infty}^t g(t-s)\, dB_s, \quad t \in \mathbb{R}, \quad \text{and} \quad (\mathcal{F}_t^{Y,\infty})_{t\geq 0} = (\mathcal{F}_t^{B,\infty})_{t\geq 0}.$$

Since $(Y_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{B,\infty})_{t\geq 0}$-semimartingale, Knight [1992, Theorem 6.5] shows that

$$g(t) = \alpha + \int_0^t \zeta(u)\, du, \qquad t \geq 0,$$

for some $\alpha \in \mathbb{R} \setminus \{0\}$ and some $\zeta \in L^2_{\mathbb{R}}(\lambda)$ and the $(\mathcal{F}_t^{B,\infty})_{t\geq 0}$-martingale component of $(Y_t)_{t\geq 0}$ is $(\alpha B_t)_{t\geq 0}$. Equation (5.7) actually shows that $(Y_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Y,\infty} \vee \sigma(X_0))_{t\geq 0}$-semimartingale, and since $(\mathcal{F}_t^{Y,\infty})_{t\geq 0} = (\mathcal{F}_t^{B,\infty})_{t\geq 0}$ it follows that $(Y_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{B,\infty} \vee \sigma(X_0))_{t\geq 0}$-semimartingale. Hence $(B_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{B,\infty} \vee \sigma(X_0))_{t\geq 0}$-semimartingale. As in (1) we have $X_0 = \int k(s)\, dB_s$ where $k := (2\pi)^{-2}\xi$. Since $(B_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{B,\infty} \vee \sigma(X_0))_{t\geq 0}$-semimartingale and $\mathcal{F}_t^B \vee \sigma(\int_0^\infty k(s)\, dB_s) \subseteq \mathcal{F}_t^{B,\infty} \vee \sigma(X_0)$, $(B_t)_{t\geq 0}$ is also an $(\mathcal{F}_t^B \vee \sigma(\int_0^\infty k(s)\, dB_s))_{t\geq 0}$-semimartingale. Thus according to Chaleyat-Maurel and Jeulin [1983, Theoreme I.1.1] k satisfies (3.4) which shows condition (b). From this theorem it follows that the bounded variation component is an absolutely continuous Gaussian process and the martingale component is a Wiener process with parameter $\sigma^2 = (2\pi\alpha)^2$. Let $\eta := \zeta + g$ and let ρ be given by

$$\rho(t) = \alpha + \int_0^t \eta(u)\, du, \quad t \geq 0, \quad \text{and} \quad \rho(t) = 0, \qquad t < 0.$$

For all $t \in \mathbb{R}$ we have

$$\begin{aligned} X_t - X_0 = Y_t - Y_0 - \int_0^t Y_u\, du &= Y_t - Y_0 - \int \left(\int_0^t g(u-s)\, du \right) dB_s \\ &= \int \left(g(t-s) - g(-s) + \int_{-s}^{t-s} g(u)\, du \right) dB_s = \int (\rho(t-s) - \rho(-s))\, dB_s, \end{aligned}$$

where the second equality follows from Protter [2004, Chapter IV, Theorem 65]. Thus from (4.8) we have

$$\tau_t\varphi - \tau_0\varphi = \int (\rho(t-u) - \rho(-u))\, d\tau_u \tilde{f}, \qquad \lambda\text{-a.s. } \forall t \in \mathbb{R},$$

which by Lemma 4.1 (i) implies

$$\varphi(t) = \beta + \alpha\tilde{f}(t) + \int_0^\infty \left(\tilde{f}(t-v) - \tilde{f}(-v) \right) \eta(v)\, dv, \qquad \lambda\text{-a.a. } t \in \mathbb{R},$$

for some $\beta \in \mathbb{R}$. We obtain (3.3) (with $h = \eta(-\cdot)$) by (3.5). This completes the proof of (a).

Let us study the canonical decomposition of $(X_t)_{t\geq 0}$ in the case $X_0 = 0$. For $t \geq 0$ we have

$$X_t - X_0 = \alpha B_t + \int \left(\int_{-s}^{t-s} \widehat{f\hat{h}}(u)\, du \right) dW_s = \alpha B_t + \int_0^t \left(\int \widehat{f\hat{h}}(s-u)\, dW_u \right) ds, \tag{5.8}$$

and by (4.10) we have

$$\int \widehat{f\hat{h}}(s-u)\, dW_u = \int h(u-s)\, dB_u. \tag{5.9}$$

Recall that $(\mathcal{F}_t^{X,\infty})_{t\geq 0} = (\mathcal{F}_t^{B,\infty})_{t\geq 0}$. From (5.9) it follows that the last term of (5.8) is $(\mathcal{F}_t^{B,\infty})_{t\geq 0}$-adapted and hence the canonical $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-decomposition of $(X_t)_{t\geq 0}$ is given by (5.8). Furthermore, by combining (5.8) and (5.9), Cheridito [2004, Proposition 3.7] shows that the law of $(\frac{1}{2\pi\alpha}X_t)_{t\in[0,T]}$ is equivalent to the Wiener measure on $C([0,T])$ for all $T>0$, when $X_0=0$.

(4) : Assume $(X_t)_{t\geq 0}$ is of bounded variation and therefore of integrable variation (see Stricker [1983]). By Lemma 5.2 we conclude that φ is absolutely continuous with square integrable density and hereby on the form (3.3) with $\alpha=0$ and $f\equiv 1$. This completes the proof. $\square$

Proof of Proposition 3.6. To prove (ii) assume ψ equals 0 or φ and $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale.

Only if: Assume $(X_t)_{t\geq 0}$ is $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-adapted. By studying $(X_t - X_0)_{t\geq 0}$ we may and do assume that $\psi=\varphi$. Furthermore, it follows that φ is constant on $(-\infty,0)$ since $(X_t)_{t\geq 0}$ is $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-adapted. Let us first assume that $(X_t)_{t\geq 0}$ is of bounded variation. By arguing as in (4) in the proof of Theorem 3.2 it follows that φ is on the form (3.3) where h is 0 on $\mathbb{R}_+$ and $f\equiv 1$ (these h and f satisfies the additional conditions in (ii)). Second assume $(X_t)_{t\geq 0}$ is of unbounded variation. Proceed as in (3) in the proof of Theorem 3.2. Since φ is constant on $(-\infty,0)$ it follows by (5.5) that κ is 0 on $(-\infty,0)$. Thus according to Lemma 5.1 (ii), f is given by $f(a)=\lim_{b\downarrow 0} J(-a+ib)$ for some inner function J and the proof of the *only if* part is complete.

If: According to Lemma 4.2, $\tilde{f}$ is constant on $(-\infty,0)$ λ-a.s. and from (3.5) it follows that (recall that h is 0 on $\mathbb{R}_+$)

$$\int_0^t \widehat{f\hat{h}}(s)\,ds = \int_{-\infty}^0 \Big(\tilde{f}(t+s)-\tilde{f}(s)\Big)\,h(s)\,ds, \qquad t\in\mathbb{R}.$$

This shows that φ is constant on $(-\infty,0)$ λ-a.s. and hence $(X_t)_{t\geq 0}$ is $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-adapted since ψ equals 0 or φ.

To prove (3.6) assume that φ is represented as in (3.3) with $f(a)=\lim_{b\downarrow 0} J(-a+b)$ for λ-a.a. $a\in\mathbb{R}$ for some inner function J and h is 0 on $\mathbb{R}_+$. Lemma 4.2 shows that there exists a constant $c\in\mathbb{R}$ such that $\tilde{f}=c$ λ-a.s. on $(-\infty,0)$. Let $g:=h(-\cdot)$. By (3.5) we have

$$\begin{aligned}\int_0^t \widehat{f\hat{h}}(s)\,ds &= \int \Big(\tilde{f}(t-s)-\tilde{f}(-s)\Big)\,g(s)\,ds \\ &= \int \Big(\tilde{f}(t-s)-c\Big)\,g(s)\,ds = \Big((\tilde{f}-c)*g\Big)(t),\end{aligned}$$

where the third equality follows from the fact that g only differs from 0 on $\mathbb{R}_+$ and on this set $\tilde{f}(-\cdot)$ equals c. This shows (3.6).

To show (i) assume $\psi=0$.

Only if: We may and do assume that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale of unbounded variation. We have to show that we can decompose φ as in (a) of Theorem 3.2 where $\alpha + \int_0^{\cdot} h(-s)\,ds$ is square integrable on $\mathbb{R}_+$. However, this follows as in (3) in the proof of Theorem 3.2 (without referring to Lemma 5.3).

If: Assume (a) of Theorem 3.2 is satisfied with α, β, h and f and that g defined by

$$g(t) = \begin{cases} \alpha + \int_0^t h(-v)\,dv & t \geq 0 \\ 0 & t < 0, \end{cases}$$

is square integrable. From Lemma 4.1 (ii) it follows that there exists a $\tilde{\beta} \in \mathbb{R}$ such that

$$\int g(-u)\,d\tau_u \tilde{f} = \tilde{\beta} + \alpha \tilde{f}(-\cdot) + \int \big(\tilde{f}(-v-\cdot) - \tilde{f}(-v)\big) h(-v)\,dv, \qquad \lambda\text{-a.s.}$$

which by (3.3) and (3.5) implies

$$\int g(-u)\,d\tau_u \tilde{f} = \tilde{\beta} - \beta + \varphi(-\cdot), \qquad \lambda\text{-a.s.}$$

The square integrability of φ shows $\tilde{\beta} = \beta$ and by (4.9) it follows that $\widehat{\varphi f} = (2\pi)^2 g(-\cdot)$. Since $g(-\cdot)$ is zero on $\mathbb{R}_+$ this shows that condition (b) in Theorem 3.2 is satisfied and hence it follows by Theorem 3.2 that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale.

□

6 The spectral measure of stationary semimartingales

For $t \in \mathbb{R}$, let $X_t = \int_{-\infty}^t \varphi(t-s)\,dW_s$ where $\varphi \in L^2_{\mathbb{R}}(\lambda)$. In this section we use Knight [1992, Theorem 6.5] to give a condition on the Fourier transform of φ for $(X_t)_{t\geq 0}$ to be an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale. In the case where $(X_t)_{t\geq 0}$ is a Markov process we use this to provide a simple condition on $\hat{\varphi}$ for $(X_t)_{t\geq 0}$ to be an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale. In the last part of this section we study a general stationary Gaussian process $(X_t)_{t\in\mathbb{R}}$. As in Jeulin and Yor [1993] we provide conditions on the spectral measure of $(X_t)_{t\in\mathbb{R}}$ for $(X_t)_{t\geq 0}$ to be an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale.

Proposition 6.1. *Let $(X_t)_{t\in\mathbb{R}}$ be given by $X_t = \int \varphi(t-s)\,dW_s$, where $\varphi \in L^2_{\mathbb{R}}(\lambda)$ and $(W_t)_{t\in\mathbb{R}}$ is a Wiener process. Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale if and only if*

$$\hat{\varphi}(t) = \frac{\alpha + \hat{h}(t)}{1 - it}, \qquad \lambda\text{-a.a. } t \in \mathbb{R},$$

for some $\alpha \in \mathbb{R}$ and some $h \in L^2_{\mathbb{R}}(\lambda)$ which is 0 on $(-\infty, 0)$.

The result follows directly from Knight [1992, Theorem 6.5], once we have shown the following technical result.

Lemma 6.2. *Let $\varphi \in L^2_{\mathbb{R}}(\lambda)$. Then φ is on the form*

$$\varphi(t) = \begin{cases} \alpha + \int_0^t h(s)\,ds & t \geq 0 \\ 0 & t < 0, \end{cases} \tag{6.1}$$

for some $\alpha \in \mathbb{R}$ and some $h \in L^2_{\mathbb{R}}(\lambda)$ if and only if

$$\hat{\varphi}(t) = \frac{c + \hat{k}(t)}{1 - it}, \tag{6.2}$$

for some $c \in \mathbb{R}$ and some $k \in L^2_{\mathbb{R}}(\lambda)$ which is 0 on $(-\infty, 0)$.

Proof. Assume φ satisfies (6.1). By square integrability of φ we can find a sequence $(a_n)_{n\geq 1}$ converging to infinity such that $\varphi(a_n)$ converges to 0. For all $n \geq 1$ we have

$$\begin{aligned}
\int_0^{a_n} \varphi(s) e^{its}\, ds &= \int_0^{a_n} \alpha e^{its} ds + \int_0^{a_n} \left(\int_0^s h(u)\, du \right) e^{its}\, ds \\
&= \frac{\alpha(e^{ia_n t} - 1)}{it} + \int_0^{a_n} h(u) \left(\int_u^{a_n} e^{its} ds \right) du \\
&= \frac{\alpha(e^{ia_n t} - 1)}{it} + \int_0^{a_n} h(u) \left(\frac{e^{ia_n t} - e^{iut}}{it} \right) du \\
&= \frac{1}{it} \left(e^{ia_n t} \left(\alpha + \int_0^{a_n} h(u) du \right) - \alpha - \int_0^{a_n} h(u) e^{itu}\, du \right) \\
&= \frac{1}{it} \left(e^{ia_n t} \varphi(a_n) - \alpha - \int_0^{a_n} h(u) e^{itu} du \right).
\end{aligned}$$

Hence by letting n tend to infinity it follows that $\hat{\varphi}(t) = -(it)^{-1}(\alpha + \hat{h}(t))$ and we obtain (6.2). Assume conversely that (6.2) is satisfied and let $e(t) := e^{-t} 1_{\mathbb{R}_+}(t)$ for $t \in \mathbb{R}$. We have

$$\hat{\varphi}(t) = \frac{c + \hat{k}}{1 - it} = c\hat{e}(t) + \hat{k}(t)\hat{e}(t). \tag{6.3}$$

Note that $k * e$ is square integrable and $\widehat{k * e} = \hat{k}\hat{e}$. Thus from (6.3) it follows that $\varphi = ce + k * e$ λ-a.s. This shows in particular that φ is 0 on $(-\infty, 0)$ and $k(t) - k * e(t) = ce(t) + k(t) - \varphi(t) =: f(t)$, which implies that

$$h(t) - h(0) = f(t) - f(0) - \int_0^t f(s)\, ds,$$

and hence

$$\varphi(t) = \begin{cases} \varphi(0) + \int_0^t (\varphi(s) - k(s))\, ds & t \geq 0 \\ 0 & t < 0. \end{cases}$$

This completes the proof of (6.1). □

Let $(X_t)_{t\in\mathbb{R}}$ be given by $X_t = \int_{-\infty}^t \varphi(t-s)\, dW_s$ for some $\varphi \in L^2_{\mathbb{R}}(\lambda)$. Below we characterize when $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^X)_{t\geq 0}$-Markov process by means of two constants and an inner function. Moreover, we provide a simple condition on the inner function for $(X_t)_{t\geq 0}$ to be an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale. Finally, this condition is used to construct a rather large class of φ's for which $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale but not an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale. Cherny [2001, Example 3.4] constructs a φ for which $(X_t)_{t\in\mathbb{R}}$ given by (3.2) (with $\psi = \varphi$) is an $(\mathcal{F}_t^X)_{t\geq 0}$-Wiener process but not an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale.

Proposition 6.3. *Let $(X_t)_{t\in\mathbb{R}}$ be given by $X_t = \int \varphi(t-s)\, dW_s$, for $t \in \mathbb{R}$, where $\varphi \in L^2_{\mathbb{R}}(\lambda)$ is non-trivial and 0 on $(-\infty, 0)$.*

(i) $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^X)_{t\geq 0}$-Markov process if and only if φ is given by

$$\hat{\varphi}(t) = \frac{cj(t)}{\theta - it}, \qquad t \in \mathbb{R}, \tag{6.4}$$

where J is an inner function satisfying $\overline{J(z)} = J(-\overline{z})$, $j(a) = \lim_{b\downarrow 0} J(a+ib)$ and $c, \theta > 0$. In this case $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale, and an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale if and only if $J - \alpha \in \mathbb{H}^2_+$ for some $\alpha \in \{-1, 1\}$.

(ii) In particular, let φ be given by (6.4), where J is a singular inner function, i.e. on the form

$$J(z) = \exp\Big(\frac{-1}{\pi i}\int \frac{sz+1}{s-z}\frac{1}{1+s^2}\,F(ds)\Big), \qquad z \in \mathbb{C}_+,$$

where F is a singular measure which integrates $s \mapsto (1+s^2)^{-1}$, and assume F is symmetric, concentrated on $\mathbb{Z}$, $(F(\{k\}))_{k\in\mathbb{Z}}$ is bounded and $\sum_{k\in\mathbb{Z}} F(\{k\})^2 = \infty$. Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^X)_{t\geq 0}$-Markov process, an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale and $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-adapted, but not an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale.

Proof. Assume $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^X)_{t\geq 0}$-Markov process and let J denote the inner part of the Hardy function induced by φ. Note that $\overline{J(z)} = J(-\overline{z})$. Since $(X_t)_{t\geq 0}$ is an $L^2(P)$-continuous, centered Gaussian $(\mathcal{F}_t^X)_{t\geq 0}$-Markov process it follows by Doob [1942, Theorem 1.1] that $(X_t)_{t\geq 0}$ is an Ornstein-Uhlenbeck process and hence

$$|\hat{\varphi}(t)|^2 = \frac{c}{\theta + t^2}, \qquad \lambda\text{-a.a. } t \in \mathbb{R},$$

for some $\theta, c > 0$. This implies that the outer part of $\hat{\varphi}$ is $z \mapsto c/(\theta - iz)$ and thus φ satisfies (6.4). Assume conversely that φ is given by (6.4). It is readily seen that φ is a real function which is 0 on $(-\infty, 0)$. Moreover, since $|\hat{\varphi}|^2 = c^2/(\theta^2 + t^2)$ it follows that $(X_t)_{t\geq 0}$ is an Ornstein-Uhlenbeck process and hence an $(\mathcal{F}_t^X)_{t\geq 0}$-Markov process and an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale. According to Proposition 6.1, $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale if and only if

$$\hat{\varphi}(t) = \frac{\alpha + \hat{h}(t)}{\theta - it}, \qquad \lambda\text{-a.a. } t \in \mathbb{R},$$

for some $\alpha \in \mathbb{R}$ and $h \in L^2_{\mathbb{R}}(\lambda)$ which is 0 on $(-\infty, 0)$, which by (6.4) is equivalent to $J - \alpha/c = H/c$, where H is the Hardy function induced by h. This completes the proof of (i).

To prove (ii), note first that $\overline{J(z)} = J(-\overline{z})$ since F is symmetric. Moreover,

$$|J(a+ib)| = \exp\Big(\int \frac{-b}{\pi((s-a)^2 + b^2)}\,F(ds)\Big).$$

If $f\colon \mathbb{R} \to \mathbb{R}$ is a bounded measurable function then $f \in L^2_{\mathbb{R}}(\lambda)$ if and only if $e^f - 1 \in L^2_{\mathbb{R}}(\lambda)$. We will use this on

$$f(a) := \int \frac{-b}{\pi((s-a)^2 + b^2)}\,F(ds), \qquad a \in \mathbb{R}.$$

The function f is bounded since $k \mapsto F(\{k\})$ is bounded. Moreover, $f \notin L^2_{\mathbb{R}}(\lambda)$ since

$$\begin{aligned}\int |f(a)|^2\,da &= \Big(\frac{b}{\pi}\Big)^2 \int \Big(\sum_{j\in\mathbb{Z}} \frac{F(\{j\})}{(j-a)^2+b^2}\Big)^2 da \\ &\geq \Big(\frac{b}{\pi}\Big)^2 \int \sum_{j\in\mathbb{Z}} \Big(\frac{F(\{j\})}{(j-a)^2+b^2}\Big)^2 da = \Big(\frac{b}{\pi}\Big)^2 \sum_{j\in\mathbb{Z}} \int \Big(\frac{F(\{j\})}{(j-a)^2+b^2}\Big)^2 da \\ &= \Big(\frac{b}{\pi}\Big)^2 \sum_{j\in\mathbb{Z}} \int \Big(\frac{F(\{j\})}{a^2+b^2}\Big)^2 da = \Big(\frac{b}{\pi}\Big)^2 \int \Big(\frac{1}{a^2+b^2}\Big)^2 da \sum_{j\in\mathbb{Z}} [F(\{j\})]^2 = \infty,\end{aligned}$$

where the first inequality follows from the fact that the terms in the sum are positive. It follows that $e^f - 1 \notin L^2_{\mathbb{R}}(\lambda)$. Let $\alpha \in \{-1, 1\}$. Then

$$|J(a+ib) - \alpha| \geq ||J(a+ib)| - 1| = e^{f(a)} - 1,$$

which shows that $J - \alpha \notin \mathbb{H}^2_+$ and hence $(X_t)_{t\geq 0}$ is not an $(\mathcal{F}^{W,\infty}_t)_{t\geq 0}$-semimartingale. □

Let $(X_t)_{t\in\mathbb{R}}$ denote an $L^2(P)$-continuous centered Gaussian process. Recall that the symmetric finite measure μ satisfying

$$E[X_t X_u] = \int e^{i(t-u)s}\,\mu(ds), \qquad \forall\, t, u \in \mathbb{R},$$

is called the spectral measure of $(X_t)_{t\in\mathbb{R}}$. The proof of the next result is quite similar to the proof of Jeulin and Yor [1993, Proposition 19].

Proposition 6.4. *Let $(X_t)_{t\in\mathbb{R}}$ be an $L^2(P)$-continuous stationary centered Gaussian process with spectral measure $\mu = \mu_s + f\,d\lambda$ (μ_s is the singular part of μ). Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^{X,\infty}_t)_{t\geq 0}$-semimartingale if and only if $\int t^2\,\mu_s(dt) < \infty$ and*

$$f(t) = \frac{|\alpha + \hat{h}(t)|^2}{1+t^2}, \qquad \lambda\text{-a.a. } t \in \mathbb{R},$$

for some $\alpha \in \mathbb{R}$ and some $h \in L^2_{\mathbb{R}}(\lambda)$ which is 0 on $(-\infty, 0)$ when $\alpha \neq 0$. Moreover, $(X_t)_{t\geq 0}$ is of bounded variation if and only if $\alpha = 0$.

Proposition 6.4 extends the well-known fact that an $L^2(P)$-continuous stationary Gaussian process is of bounded variation if and only if $\int t^2\,\mu(dt) < \infty$.

Proof of Proposition 6.4. Only if: If $(X_t)_{t\geq 0}$ is of bounded variation then $\int t^2\,\mu(dt) < \infty$ and therefore μ is on the stated form. Thus, we may and do assume $(X_t)_{t\geq 0}$ is an $(\mathcal{F}^{X,\infty}_t)_{t\geq 0}$-semimartingale of unbounded variation. It follows that $(X_t)_{t\in\mathbb{R}}$ is a regular process and hence it can be decomposed as (see e.g. Doob [1990])

$$X_t = V_t + \int_{-\infty}^t \varphi(t-s)\,dW_s, \qquad t \in \mathbb{R},$$

where $(W_t)_{t\in\mathbb{R}}$ is a Wiener process which is independent of $(V_t)_{t\in\mathbb{R}}$ and $W_r - W_s$ is $\mathcal{F}_t^{X,\infty}$-measurable for $s \le r \le t$. The process $(V_t)_{t\in\mathbb{R}}$ is stationary Gaussian and V_t is $\mathcal{F}_{-\infty}^{X,\infty}$-measurable for all $t \in \mathbb{R}$, where

$$\mathcal{F}_{-\infty}^{X,\infty} := \bigcap_{t\in\mathbb{R}} \mathcal{F}_t^{X,\infty}.$$

Moreover, $(V_t)_{t\in\mathbb{R}}$ respectively $(X_t - V_t)_{t\in\mathbb{R}}$ has spectral measure μ_s respectively $f\,d\lambda$. For $0 \le u \le t$ we have

$$\begin{aligned} E[|V_t - V_u|] &= E[|E[V_t - V_u|\mathcal{F}_u^{V,\infty}]|] = E[|E[X_t - X_u|\mathcal{F}_u^{V,\infty}]|] \\ &\le E[|E[X_t - X_u|\mathcal{F}_u^{X,\infty}]|], \end{aligned}$$

which shows that $(V_t)_{t\ge0}$ is of integrable variation and hence $\int t^2\,\mu_s(dt) < \infty$. The fact that $(V_t)_{t\ge0}$ is $(\mathcal{F}_t^{X,\infty})_{t\ge0}$-adapted and of bounded variation implies that

$$\Big(\int_{-\infty}^t \varphi(t-s)\,dW_s\Big)_{t\ge0}$$

is an $(\mathcal{F}_t^{X,\infty})_{t\ge0}$-semimartingale and therefore also an $(\mathcal{F}_t^{W,\infty})_{t\ge0}$-semimartingale. Thus, by Proposition 6.1 we conclude that

$$f(t) = |\hat\varphi(t)|^2 = \frac{|\alpha + \hat h(t)|^2}{1+t^2}, \qquad \lambda\text{-a.a. } t \in \mathbb{R},$$

for some $\alpha \in \mathbb{R}$ and some $h \in L^2_{\mathbb{R}}(\lambda)$ which is 0 on $(-\infty, 0)$.

If: If $\int t^2\,\mu(dt) < \infty$, then $(X_t)_{t\ge0}$ is of bounded variation and hence an $(\mathcal{F}_t^{X,\infty})_{t\ge0}$-semimartingale. Thus, we may and do assume $\int t^2 f(t)\,dt = \infty$. We show that $(X_t)_{t\ge0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\ge0}$-semimartingale by constructing a process $(Z_t)_{t\in\mathbb{R}}$ which equals $(X_t)_{t\in\mathbb{R}}$ in distribution and such that $(Z_t)_{t\ge0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\ge0}$-semimartingale. By Lemma 6.2 there exists a $\beta \in \mathbb{R}$ and a $g \in L^2_{\mathbb{R}}(\lambda)$ such that with $\varphi(t) = \beta + \int_0^t g(s)\,ds$ for $t \ge 0$ and $\varphi(t) = 0$ for $t < 0$, we have $|\hat\varphi|^2 = f$. Define $(Z_t)_{t\in\mathbb{R}}$ by

$$Z_t = V_t + \int_{-\infty}^t \varphi(t-s)\,dW_s, \qquad t \in \mathbb{R}, \tag{6.5}$$

where $(V_t)_{t\in\mathbb{R}}$ is a stationary Gaussian process with spectral measure μ_s and $(W_t)_{t\in\mathbb{R}}$ is a Wiener process which is independent of $(V_t)_{t\in\mathbb{R}}$. The processes $(X_t)_{t\in\mathbb{R}}$ and $(Z_t)_{t\in\mathbb{R}}$ are identical in distribution due to the fact that they are centered Gaussian processes with the same spectral measure and hence it is enough to show that $(Z_t)_{t\ge0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\ge0}$-semimartingale. It is well-known that $(V_t)_{t\ge0}$ is of bounded variation since $\int t^2\,\mu_s(dt) < \infty$ and by Knight [1992, Theorem 6.5] the second term on the right-hand side of (6.5) is an $(\mathcal{F}_t^{W,\infty})_{t\ge0}$-semimartingale. Thus we conclude that $(Z_t)_{t\ge0}$ is an $(\mathcal{F}_t^{Z,\infty})_{t\ge0}$-semimartingale. □

7 The spectral measure of semimartingales with stationary increments

Let $(X_t)_{t\in\mathbb{R}}$ be an $L^2(P)$-continuous Gaussian process with stationary increments such that $X_0 = 0$. Then there exists a unique positive symmetric measure μ on $\mathbb{R}$ which integrates $t \mapsto$

$(1+t^2)^{-1}$ and satisfies

$$E[X_tX_u] = \int \frac{(e^{its}-1)(e^{-ius}-1)}{s^2}\,\mu(ds), \qquad t,u\in\mathbb{R}.$$

This μ is called the spectral measure of $(X_t)_{t\in\mathbb{R}}$. The spectral measure of the fractional Brownian motion (fBm) with Hurst parameter $H\in(0,1)$ is

$$\mu(ds) = c_H|s|^{1-2H}\,ds, \tag{7.1}$$

where $c_H\in\mathbb{R}$ is a constant (see e.g. Yaglom [1987]). In particular the spectral measure of the Wiener process ($H=1/2$) equals the Lebesgue measure up to a scaling constant.

Theorem 7.1. *Let $(X_t)_{t\in\mathbb{R}}$ be an $L^2(P)$-continuous, centered Gaussian process with stationary increments such that $X_0=0$. Moreover, let $\mu=\mu_s+f\,d\lambda$ be the spectral measure of $(X_t)_{t\in\mathbb{R}}$. Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq0}$-semimartingale if and only if μ_s is a finite measure and*

$$f=|\alpha+\hat{h}|^2, \qquad \lambda\text{-a.s.}$$

for some $\alpha\in\mathbb{R}$ and some $h\in L^2_{\mathbb{R}}(\lambda)$ which is 0 on $(-\infty,0)$ when $\alpha\neq0$. Moreover, $(X_t)_{t\geq0}$ is of bounded variation if and only if $\alpha=0$.

Proof. Assume $(X_t)_{t\geq0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq0}$-semimartingale. Let $(Y_t)_{t\in\mathbb{R}}$ be the stationary centered Gaussian process given by Lemma 5.3, that is

$$X_t = Y_t - Y_0 + \int_0^t Y_s\,ds, \qquad t\in\mathbb{R}, \tag{7.2}$$

and let ν denote the spectral measure of $(Y_t)_{t\in\mathbb{R}}$, that is ν is a finite measure satisfying

$$E[Y_tY_u] = \int e^{i(t-u)a}\,\nu(da), \qquad t,u\in\mathbb{R}.$$

By using Fubini's Theorem it follows that

$$E[X_tX_u] = \int \frac{\big(e^{its}-1\big)\big(e^{-ius}-1\big)}{s^2}(1+s^2)\,\nu(ds), \qquad t,u\in\mathbb{R}. \tag{7.3}$$

Thus, by uniqueness of the spectral measure of $(X_t)_{t\in\mathbb{R}}$ we obtain $\mu(ds)=(1+s^2)\,\nu(ds)$. Since $(X_t)_{t\geq0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq0}$-semimartingale (7.2) implies that $(Y_t)_{t\geq0}$ is an $(\mathcal{F}_t^{Y,\infty})_{t\geq0}$-semimartingale and hence Proposition 6.4 shows that the singular part ν_s of ν satisfies $\int t^2\,\nu(dt)<\infty$ and the absolute continuous part is on the form

$$|\alpha+\hat{h}(s)|^2(1+s^2)^{-1}\,ds,$$

for some $\alpha\in\mathbb{R}$, and some $h\in L^2_{\mathbb{R}}(\lambda)$ which is 0 on $(-\infty,0)$ when $\alpha\neq0$. This completes the *only if* part of the proof.

Conversely assume that μ_s is a finite measure and $f=|\alpha+\hat{h}|^2$ for an $\alpha\in\mathbb{R}$ and an $h\in L^2_{\mathbb{R}}(\lambda)$ which is 0 on $(-\infty,0)$ when $\alpha\neq0$. Let $(Y_t)_{t\in\mathbb{R}}$ be a centered Gaussian process such that

$$E[Y_tY_u] = \int \frac{e^{i(t-u)a}f(a)}{1+a}\,da, \qquad t,u\in\mathbb{R}.$$

By Proposition 6.4 it follows that $(Y_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{Y,\infty})_{t\geq 0}$-semimartingale. Thus, $(Z_t)_{t\in\mathbb{R}}$ defined by

$$Z_t := Y_t - Y_0 + \int_0^t Y_s\,ds, \qquad t\in\mathbb{R},$$

is an $(\mathcal{F}_t^{Y,\infty})_{t\geq 0}$-semimartingale and therefore also an $(\mathcal{F}_t^{Z,\infty})_{t\geq 0}$-semimartingale. Moreover, by calculations as in (7.3) it follows that $(Z_t)_{t\in\mathbb{R}}$ is distributed as $(X_t)_{t\in\mathbb{R}}$, which shows that $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale. This completes the proof. $\square$

Let $(X_t)_{t\in\mathbb{R}}$ denote a fBm with Hurst parameter $H\in(0,1)$ (recall that the spectral measure of $(X_t)_{t\in\mathbb{R}}$ is given by (7.1)). If $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale then Theorem 7.1 shows that $c_H|s|^{1-2H} = |\alpha + \hat{h}(s)|$, for some $\alpha\in\mathbb{R}$ and some $h\in L^2_{\mathbb{R}}(\lambda)$ which is 0 on $(-\infty,0)$ when $\alpha\neq 0$. This implies $H=1/2$. It is well-known from Rogers [1997] that the fBm is not a semimartingale (even in the filtration $(\mathcal{F}_t^X)_{t\geq 0}$) when $H\neq 1/2$. However, the proof presented is new and illustrates the usefulness of the theorem. As a consequence of the above theorem we also have:

Corollary 7.2. *Let $(X_t)_{t\in\mathbb{R}}$ be a Gaussian process with stationary increments. Then $(X_t)_{t\geq 0}$ is of bounded variation if and only if $(X_t - X_0)_{t\in\mathbb{R}}$ has finite spectral measure.*

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