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Continuity of the percolation threshold in randomly grown graphs*

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Abstract

We consider various models of randomly grown graphs. In these models the vertices and the edges accumulate within time according to certain rules. We study a phase transition in these models along a parameter which refers to the mean life-time of an edge. Although deleting old edges in the uniformly grown graph changes abruptly the properties of the model, we show that some of the macro-characteristics of the graph vary continuously. In particular, our results yield a lower bound for the size of the largest connected component of the uniformly grown graph.

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1 Introduction.

We begin with the definition of a *uniformly* grown graph given in (4). There is one single vertex at time $n = 1$. Then at each time step $n \geq 2$ a new vertex is added, and with probability δ a new edge is added between two vertices chosen uniformly at random out of the existing vertices. Let us denote the resulting graph at time $n \geq 1$ by $F_n(\delta)$. The remarkable feature of this model is that the corresponding bond percolation has an infinite order phase transition. (Recall that in the classical random graph model the phase transition is of second order.) For the proof of this fact and a detailed analysis of this model we refer to (2). The authors of (2) (see also (5)) showed that as far as the limiting structure of $F_n(\delta)$, $n \to \infty$, is concerned, this model is equivalent to the random graph $G_n(2\delta)$ on $\{1, \ldots, n\}$ where any possible edge (i, j) is present with probability $2\delta/\max\{i, j\}$ independently of the others. The percolation threshold of these models is at $\delta = 1/8$. In particular, this implies that the critical ratio of the mean value of the number of edges to the number of vertices, in the regime when the graph $F_n(\delta)$ has a giant component, is $1/8$, which is 4 times less than for the classical random graph with the corresponding ratio $1/2$. This gave a reason for the authors of (4) to argue in favor of the uniformly grown graph for applications.

Taking into consideration other important features of realistic networks, here we shall study the effect of deleting old edges in uniformly grown graphs. For this matter we consider another model of a randomly grown graph, which has an additional parameter of deletion of edges. This is the so-called dynamical random graph with memory, whose definition we recall from (11). Assume, at time $t = 0$ there is a single vertex with no edges. The number of vertices at $t > 0$ is described by the Yule process (see, e.g., (1)) defined as follows. With every vertex in the graph we associate a Poisson process with intensity γ, every occurrence of which corresponds to the appearance of a new vertex. As soon as there are at least two vertices in the graph, from each vertex we draw with intensity λ a new edge to a vertex which we choose with equal probabilities among the rest of the existing vertices. Every edge of the graph is deleted with rate μ. This means that the life time of any edge is exponentially distributed with mean value $1/\mu$. We assume that all the processes of appending and deleting are independent. The resulting graph may have multiple edges, but we shall consider the corresponding simple (i.e., no multiple edges) graph, where an edge between two vertices is present if there is at least one edge between these vertices in the original multi-graph. We shall denote this non-directed simple graph by $\mathcal{G}_t(\gamma, \mu, \lambda)$, $t \geq 0$.

This model has been studied in (11), (12), (13), and it is a subgraph of a more general structure introduced already in (**?**). Other related models were introduced and analyzed in (8), (9), (10).

Calling our model a graph *with memory* we refer to the parameter μ which shows how long the graph keeps or "remembers" an old connection, i.e., edge. Setting $\mu = 0$ one gets a graph model, call it $\mathcal{G}_t(\gamma, 0, \lambda)$, closely related to $G_n(c)$ or $F_n(\delta)$. In particular, it was already shown in (12) that as time (parameters t and n) goes to infinity the limiting degree distribution in $\mathcal{G}_t(\gamma, 0, \lambda)$ is equivalent to the one in $F_n(\delta)$, if $\delta = \lambda/\gamma$. This allows us to say that the graph $\mathcal{G}_t(\gamma, 0, \lambda)$ is also *uniformly* grown. Moreover, all values $\lambda/\gamma > 0$ make sense here, contrary to the case $0 < \delta \leq 1$ for $F_n(\delta)$.

2 Results.

It was already mentioned by the authors of (2) that the analysis they had provided for the model $G_n(c)$ should be also valid for $\mathcal{G}_t(\gamma, 0, \lambda)$. We shall not go into the detailed proof of this conjecture, for it appears convincing enough in view of the following observation. Let us compare the probabilities of edges in these models. First we introduce some notations. Let V_t denote the set of vertices at time t in a grown graph $\mathcal{G}_t(\gamma, \mu, \lambda)$, $\mu \geq 0$, or in $G_t(c)$. (Which model we talk about will be clear from the context.) Further let $s_1 < s_2 < \ldots$ be the consecutive moments when the first, the second, and so on, vertices are added to the graph. In the case of $G_n(c)$ we have $s_i = i$, $i \geq 1$. In the case of $\mathcal{G}_t(\gamma, \mu, \lambda)$ we set $s_1 = 0$, while the moments s_i, $i \geq 2$, are random: here $|V_t|$ is a Yule process. Then we can enumerate the vertices of a graph by the moments when they were added to the graph, and write for $t \geq 0$

$$V_t = \{s_1, \ldots, s_{|V_t|}\}.$$

For $G_n(c)$ one has $s_i = i$, $1 \leq i \leq n$, $|V_n| = n$, and the probability of an edge between any two vertices s and τ with $\max\{s, \tau\} \leq n$ is

$$\mathbf{P}\{s \sim \tau \text{ in } G_n(c)\} = \frac{c}{|V_{\max\{s,\tau\}}|}. \tag{1}$$

Here we write $u \sim v$ if there is an edge between the vertices u and v.

Consider now the probabilities of edges in $\mathcal{G}_t(\gamma, 0, \lambda)$. According to (11) there are subsets $\mathcal{A}(t)$, $t \geq 0$, of the trajectories $\{V_s, 0 \leq s \leq t\}$ such that

$$\mathbf{P}\{\mathcal{A}(t)\} = 1 - o(1), \quad t \to \infty, \tag{2}$$

and conditional on

$$\{V_s, 0 \leq s \leq t\} = \bar{V} := \{\bar{V}_s, 0 \leq s \leq t\} \in \mathcal{A}(t) \tag{3}$$

the probability of an edge between two vertices s and τ in $\mathcal{G}_t(\gamma, 0, \lambda)$ is

$$p_t(s, \tau \mid \bar{V}) = \frac{2\lambda}{\gamma} \frac{1}{|\bar{V}_{\max\{s,\tau\}}|} \left(1 - e^{-\gamma(t - \max\{s,\tau\})}\right)(1 + o(1)), \tag{4}$$

as $\max\{s, \tau\} \geq \sqrt{t}$ and $t \to \infty$ (see (11) for the proof of this formula). Comparing (4) with (1) one observes that the probabilities of edges in $\mathcal{G}_t(\gamma, 0, \lambda)$ are asymptotically equivalent to the ones in $G_n(2\lambda/\gamma)$. Thus, $\mathcal{G}_t(\gamma, 0, \lambda)$ is equivalent in some sense to $G_n(2\lambda/\gamma)$. In particular, here we shall prove that also in $\mathcal{G}_t(\gamma, 0, \lambda)$ the phase transition happens at $2\lambda/\gamma = 1/4$, just as in $G_n(c)$ where the corresponding critical value is $c = 1/4$.

We say that a time-dependent event happens with high probability (**whp**) if the probability of this event is tending to one as $t \to \infty$. If a connected component of a graph has size proportional to the size of the entire graph, we call it a giant component. Let $\lambda_{cr}(0, \gamma)$ be the smallest value such that for all $\lambda > \lambda_{cr}(0, \gamma)$ **whp** there is a giant component in the graph $\mathcal{G}_t(\gamma, 0, \lambda)$ as $t \to \infty$. First we get a lower bound for $\lambda_{cr}(0, \gamma)$.

Proposition 2.1. *For any $\gamma > 0$ if*

$$\lambda < \gamma/8 \tag{5}$$

whp *the size of the largest component in $\mathcal{G}_t(\gamma, 0, \lambda)$ is $o(|V_t|)$ as $t \to \infty$.*

Proof. Assume that $\lambda < \gamma/8$. Then (4) implies that there is a constant $c < 1/4$ such that conditional on (3) the probability of an edge between two vertices s and τ in $\mathcal{G}_t(\gamma, 0, \lambda)$ is

$$p_t(s, \tau \mid \bar{V}) = \frac{2\lambda}{\gamma} \frac{1}{|\bar{V}_{\max\{s,\tau\}}|} \left(1 - e^{-\gamma(t - \max\{s,\tau\})}\right) (1 + o(1)) < \frac{c}{|\bar{V}_{\max\{s,\tau\}}|} \tag{6}$$

for all sufficiently large t and $\max\{s, \tau\} > \sqrt{t}$. Then according to (1) the probability of the existence of a giant component in $\mathcal{G}_t(\gamma, 0, \lambda)$ is bounded by the probability of a giant component in $G_{[t]}(c')$ for some $c' < 1/4$, which is known (see (2)) to be $o(1)$ as $[t] \to \infty$. Therefore Proposition 2.1 follows. □

Proposition 2.1 immediately implies

$$\lambda_{cr}(0, \gamma) \geq \gamma/8. \tag{7}$$

Consider now $\mathcal{G}_t(\gamma, \mu, \lambda)$ with $\mu > 0$. We know from (13) that a phase transition in this model is similar to the one in the classical random graph model $G_{n,p}$. If for $G_{n,p}$ with $p = c/n$ the corresponding critical value is $c = c_{cr} = 1$, its counterpart for our model is a certain function $\lambda_{cr}(\mu, \gamma)$. To formulate this result precisely let us define for any $\gamma > 0$ and $\mu \geq 0$ the following functions

$$g(t, \gamma, \mu) \;=\; \begin{cases} \frac{e^{\left(1 - \frac{\mu}{\gamma}\right)t} - 1}{\gamma - \mu}, & \text{if } \mu \neq \gamma, \\ t/\gamma, & \text{if } \mu = \gamma, \end{cases}$$

and set for all $\mu > 0$

$$\lambda_{cr}(\gamma, \mu) = \frac{1}{2} \sup\{x > 0 : \sum_{k=2}^{\infty} x^k \; \mathbf{E} \prod_{i=1}^{k-1} g(\eta_i \wedge \eta_{i+1}, \gamma, \mu) < \infty\}, \tag{8}$$

where $\eta_1, \ldots, \eta_k$ are independent random variables with a common Exp (1)-distribution. Let us also recall (see (11)) that conditional on (3) the probability of an edge between two vertices s and τ in $\mathcal{G}_t(\gamma, \mu, \lambda)$ is

$$2\lambda \, \frac{g(\gamma(t - s \vee \tau), \gamma, \mu)}{|\bar{V}_{\max\{s,\tau\}}|} \, (1 + o(1)), \tag{9}$$

when $s \vee \tau \geq t^{1/2}$ and $t \to \infty$.

Let further $X(G)$ denote for a graph G the size of its largest connected component.

Theorem ((13)) For any $\gamma > 0$ and $\mu > 0$

I) if $\lambda < \lambda_{cr}(\gamma, \mu)$ then there exists a constant $c = c(\lambda, \gamma, \mu)$ such that

$$\mathbf{P}\left\{X\left(\mathcal{G}_t(\gamma, \mu, \lambda)\right) > c \log|V(t)|\right\} \to 0 \quad \text{as } t \to \infty;$$

II) if $\lambda > \lambda_{cr}(\gamma, \mu)$ then for any $\varepsilon > 0$

$$\mathbf{P}\left\{\left|\frac{X\left(\mathcal{G}_t(\gamma, \mu, \lambda)\right)}{|V(t)|} - \beta\right| < \varepsilon\right\} \to 1 \quad \text{as} \quad t \to \infty, \tag{10}$$

where

$$\beta = \int_0^\infty \widetilde{\beta}(s) e^{-s} ds \tag{11}$$

and the function $\widetilde{\beta}(s), s > 0$, is defined as the largest nonnegative solution of the following equation

$$1 = \widetilde{\beta}(s) + \exp\left\{-2\lambda \int_0^\infty g(s \wedge \tau, \gamma, \mu)\widetilde{\beta}(\tau)e^{-\tau} d\tau\right\}. \tag{12}$$

Clearly, the variation of the parameter μ from zero to any positive value changes essentially the properties of the model. For example, the limiting degree distribution is geometric in $\mathcal{G}_t(\gamma, 0, \lambda)$, and it is generalized Poisson in $\mathcal{G}_t(\gamma, \mu, \lambda)$ when $\mu > 0$ (see (12) for the details). The phase transition in $\mathcal{G}_t(\gamma, 0, \lambda)$ must be of the infinite order as in (2), while it is of the second order in $\mathcal{G}_t(\gamma, \mu, \lambda)$ for any $\mu > 0$. Now we shall consider $\lambda_{cr}(\mu, \gamma)$ as $\mu \to 0$. Write for all $\mu \geq 0$

$$F_k(\mu) = \mathbf{E} \prod_{i=1}^{k-1} g(\eta_i \wedge \eta_{i+1}, \gamma, \mu). \tag{13}$$

Theorem 2.1. *For any $\gamma > 0$ and $\mu > 0$*

$$\lambda_{cr}(\mu, \gamma) = \frac{1}{2} \lim_{k\to\infty} F_k(\mu)^{-1/k} \tag{14}$$

and

$$\frac{1}{2} \lim_{k\to\infty} F_k(0)^{-1/k} = \gamma/8. \tag{15}$$

Observe that for any $\mu > 0$ the value $(2\lambda)^k F_k(\mu)/k$ is equal to the expected number of k-cycles in $\mathcal{G}_t(\gamma, \mu, \lambda)$ as $t \to \infty$ (see (11)). However, this interpretation fails when $\mu = 0$ (see Chapter 17 in (3)).

One could derive the continuity of the function $\lim_{k\to\infty} F_k(\mu)^{-1/k}$ at $\mu = 0$ directly from its definition (13). Then Theorem 2.1 would imply

$$\lim_{\mu\to 0} \lambda_{cr}(\mu, \gamma) = \gamma/8. \tag{16}$$

Clearly, $\lambda_{cr}(0, \gamma) \leq \lambda_{cr}(\mu, \gamma)$ for any $\mu > 0$, which together with (16) yields

$$\lambda_{cr}(0, \gamma) \leq \gamma/8. \tag{17}$$

Then combining the last inequality with (7) we would get the following continuity result (see also (3)).

Theorem 2.2. *For all $\gamma > 0$*

$$\lim_{\mu\to 0} \lambda_{cr}(\mu, \gamma) = \gamma/8 = \lambda_{cr}(0, \gamma). \tag{18}$$

On the other hand, equality

$$\lim_{\mu\to 0} \lambda_{cr}(\mu,\gamma) = \lambda_{cr}(0,\gamma) \tag{19}$$

follows by the general Theorem 6.4 from (3), conditions of which are satisfied here (note however, that the actual value of $\lambda_{cr}(0,\gamma)$ is not given explicitly by the cited result). Then (19) and Theorem 2.1 prove in turn continuity of $\lim_{k\to\infty} F_k(\mu)^{-1/k}$ at $\mu = 0$, which immediately yields the first equality in (18).

Notice that our computation of the critical constant $\gamma/8$ is different from the one in (3), in particular we shall disclose below a curious relation between $F_k(0)$ and the Catalan numbers.

Theorem 2.2 shows that the transition from the model $\mathcal{G}_t(\gamma,\mu,\lambda)$, $\mu > 0$, where all the old edges are being deleted, to the uniformly grown graph $\mathcal{G}_t(\gamma,0,\lambda)$ along the parameter μ is not as drastic as it could be expected. On the contrary: one can approach arbitrarily close the critical value 1/8 for the ratio of edges and vertices even abandoning the old edges. Indeed, it is easy to compute (see also (12)) that for any positive γ and μ one has

$$R(\mu,\gamma) := \lim_{t\to\infty} \frac{\mathbf{E}\{\text{number of edges in } \mathcal{G}_t(\gamma,\mu,\lambda_{cr}(\mu,\gamma))\}}{\mathbf{E}\{\text{number of vertices in } \mathcal{G}_t(\gamma,\mu,\lambda_{cr}(\mu,\gamma))\}} = \frac{\lambda_{cr}(\mu,\gamma)}{\mu+\gamma}.$$

Then it follows by (18) that $R(\mu,\gamma)$ is also continuous at $\mu = 0$, and $R(0,\gamma) = 1/8$ for all $\gamma > 0$.

An open question remains about the continuity at $\mu = 0$ of the size of the giant component in the supercritical area. Recall that due to (2) there is a function $\phi(c)$, equal to zero when $c \le 1/4$ and positive otherwise, such that **whp**

$$X(G_n(c)) = (\phi(c) + o(1))n. \tag{20}$$

Correspondingly, according to (18) there is a function $\beta(\gamma,0,\lambda)$ which is zero when $\lambda/\gamma \le 1/8$ and is positive otherwise, such that the largest connected component of $\mathcal{G}_t(\gamma,0,\lambda)$ **whp** is

$$X(\mathcal{G}_t(\gamma,0,\lambda)) = (\beta(\gamma,0,\lambda) + o(1))|V_t|.$$

Now consider the function $\beta(\gamma,\mu,\lambda)$ defined in the second part of Theorem (13) cited above. Obviously, for any fixed $\gamma > 0$ this function is monotone in both arguments μ and λ, so that

$$\beta(\gamma,\mu,\lambda) \le \beta(\gamma,\mu',\lambda'), \text{ if } \mu' \le \mu, \quad \lambda' \ge \lambda > \lambda_{cr}(\mu,\gamma).$$

This gives us immediately a lower bound for $\beta(\gamma,0,\lambda)$ when $\lambda > \lambda_{cr}(0,\gamma) = \gamma/8$:

$$\beta(\gamma,0,\lambda) \ge \lim_{\mu\to 0} \beta(\gamma,\mu,\lambda). \tag{21}$$

It follows from the definition (11) that whenever $2\lambda/\gamma \ge 1/4$

$$\lim_{\mu\to 0} \beta(\gamma,\mu,\lambda) = b\left(\frac{2\lambda}{\gamma}\right), \tag{22}$$

where

$$b(c) = \int_0^\infty b_s(c)e^{-s}ds,$$

and the function $b_s(c)$, $s \geq 0$, is the largest nonnegative solution to

$$1 = b_s(c) + \exp\left\{ -c \int_0^\infty \left(e^{s\wedge\tau} - 1\right) b_\tau(c) e^{-\tau} d\tau \right\}. \tag{23}$$

This together with (21)-(22) and (18) gives a lower bound for the function $\phi(c)$ in (20):

$$\phi(c) \geq b(c), \;\; c > 1/4. \tag{24}$$

Notice, that equation (23) has at $c = 1/4$ only the zero solution, i.e., $b_s(1/4) \equiv 0$. This yields equality in (24) at least in one point, namely $\phi(1/4) = b(1/4)$. It remains a challenge to find out whether strict inequality ever holds in (24).

Remark. While completing this work the author became aware of (3) (which was a manuscript in that time). The results in (3) concern a more general model, and as a particular case also yield (18). However, Theorem 2.1 and relation (24) did not seem to appear explicitly elsewhere.

3 Proof of Theorem 2.1.

First we shall prove (14). By the definitions (8) and (13) we have

$$\lambda_{cr}(\gamma, \mu) = \frac{1}{2} \sup\{x > 0 : \sum_{k=3}^{\infty} x^k \; F_k(\mu) < \infty\} \; = \frac{1}{2} \liminf_{k\to\infty} \; F_k(\mu)^{-1/k}. \tag{25}$$

We shall prove that for any $\mu > 0$ there exists

$$\lim_{k\to\infty} F_k(\mu)^{-1/k}. \tag{26}$$

This together with (25) gives us statement (14).

We will show that there exists a constant $C = C(\mu, \gamma)$ such that for all $k, n \geq 3$

$$F_{k+n}(\mu) \leq C F_k(\mu) F_n(\mu). \tag{27}$$

Then statement (26) will follow from (27) by the standard use of the sub-additivity property. More exactly, (27) implies

$$\log F_{k+n}(\mu) \leq \log F_k(\mu) + \log F_n(\mu) + \log C.$$

Then by the results (6) the limit

$$\mathcal{L}(\mu) = \lim_{k\to\infty} \frac{\log F_k(\mu)}{k}$$

exists and satisfies $-\infty \leq \mathcal{L}(\mu) < \infty$. This clearly implies the existence of the limit in (26).

To prove (27) fix $\mu > 0$ arbitrarily and write shortly $F_k = F_k(\mu)$. Define also for all $s \geq 0$

$$f_k(s) = \mathbf{E}\, g(s \wedge \eta_2, \gamma, \mu) \prod_{i=2}^{k-1} g(\eta_i \wedge \eta_{i+1}, \gamma, \mu). \tag{28}$$

Then

$$F_{k+n} = \int_0^\infty f_{k+n}(s)\, e^{-s} \; ds = \int_0^\infty f_k(s) f_n(s)\, e^{-s} \; ds. \tag{29}$$

Now we need the following result.

Proposition 3.1. *There exists some $C = C(\mu, \gamma)$ such that for all $k \geq 2$*

$$f_k(s) \leq C \int_0^\infty f_k(s)\, e^{-s}\, ds = CF_k \tag{30}$$

uniformly in $s \geq 0$.

Proof. First we compute for all $\mu \geq 0$ and $\gamma > 0$

$$F_2 = \frac{1}{\gamma + \mu}. \tag{31}$$

Next we notice that

$$g(t, \gamma, \mu) \leq \begin{cases} \frac{e^{(1-\frac{\mu}{\gamma})t}}{\gamma - \mu}, & \text{if } 0 < \mu < \gamma, \\ t/\gamma, & \text{if } \mu = \gamma, \\ \frac{1}{\mu - \gamma}, & \text{if } \mu > \gamma. \end{cases}$$

We may assume without loss of generality that $0 < \mu < \gamma$; the two other cases are even simpler and can be treated in a similar way. It is easy to show that for any $m \geq 2$ there is some constant $A(m) = A(m, \mu, \gamma)$ such that

$$\mathbf{E}\, g(s \wedge \eta_2, \gamma, \mu) \left(\prod_{i=2}^{m-1} g(\eta_i \wedge \eta_{i+1}, \gamma, \mu) \right) g(\eta_m \wedge t, \gamma, \mu) \leq A(m) \left(e^{(1-m\frac{\mu}{\gamma})t} + 1 \right) \tag{32}$$

for all $s, t \geq 0$. Now let us fix a constant

$$m_0 = \left[\frac{\gamma}{\mu} \right] + 1,$$

so that the expectation in (32) with $m = m_0$ is uniformly bounded in $t \geq 0$. Then (32) implies the following bounds

$$f_k(s) \leq \begin{cases} B, & \text{if } 2 \leq k < m_0 + 2, \\ BF_{k-m_0}, & \text{if } k \geq m_0 + 2, \end{cases} \tag{33}$$

for some constant $B = B(m_0, \mu, \gamma)$. Straightforward computations yield for all $k \geq 3$

$$F_k = \int_0^\infty \left(\int_0^\infty g(s' \wedge s, \gamma, \mu) e^{-s'}\, ds' \right) f_{k-1}(s)\, e^{-s}\, ds \tag{34}$$

$$= \frac{1}{\mu} \int_0^\infty \left(1 - e^{-\frac{\mu}{\gamma} s} \right) f_{k-1}(s)\, e^{-s}\, ds = \frac{1}{\mu} \left(F_{k-1} - \int_0^\infty e^{-(1+\frac{\mu}{\gamma})s} f_{k-1}(s)\, ds \right).$$

Making a change of variables in the last integral and using the fact that for any $k \geq 3$ the function $f_k(s)$ is monotone increasing in $s \geq 0$, we derive from (34)

$$F_k = \frac{1}{\mu} \left(F_{k-1} - \frac{\gamma}{\gamma + \mu} \int_0^\infty e^{-y} f_{k-1}\left(\frac{\gamma}{\gamma + \mu} y\right)\, dy \right) \tag{35}$$

$$\geq \frac{1}{\mu} \left(F_{k-1} - \frac{\gamma}{\gamma + \mu} \int_0^\infty e^{-y} f_{k-1}(y)\, dy \right)$$

$$= \frac{1}{\mu}\left(F_{k-1} - \frac{\gamma}{\gamma+\mu}F_{k-1}\right) = \frac{1}{\gamma+\mu}F_{k-1}.$$

This together with (31) give us the lower bound

$$F_k \geq \left(\frac{1}{\gamma+\mu}\right)^k \tag{36}$$

for all $k \geq 2$. The bounds (36) and (35) allow us to derive from (33)

$$f_k(s) \leq B\,(1+\gamma+\mu)^{m_0+1}\,F_k \tag{37}$$

for all $k \geq 2$ and $s \geq 0$, and (30) follows. □

Inserting now (30) into (29) we readily get

$$F_{k+n} \leq \int_0^\infty CF_k f_n(s)\,e^{-s}\,ds = CF_kF_n, \tag{38}$$

which proves property (27), and therefore statement (14) follows.

Next we prove (15). Consider

$$F_k(0) = \mathbf{E}\prod_{i=1}^{k-1} g(\eta_i \wedge \eta_{i+1}, \gamma, 0) = \left(\frac{1}{\gamma}\right)^k \mathbf{E}\prod_{i=1}^{k-1}\left(e^{\eta_i\wedge\eta_{i+1}} - 1\right). \tag{39}$$

We shall find a useful representation for the last expectation. Let T_n, $n \geq 1$, denote the subset of vertices in $\mathbf{Z}^2$ within a triangle (including the vertices on its sides) with corners at $(0,0), (n,0), (n,n)$, i.e.,

$$T_n = \{z = (x,y) \in \mathbf{Z}^2 : 0 \leq y \leq x \leq n\}.$$

Let us call a path $z_0z_1\dots z_N$ with $z_i = (z_i^1, z_i^2) \in \mathbf{Z}^2$ monotone increasing, if at each step it goes "right" or "up", i.e., for all $1 \leq i \leq N$

$$z_i^j - z_{i-1}^j \geq 0, \quad j = 1, 2,$$

and

$$\left(z_i^1 - z_{i-1}^1\right) + \left(z_i^2 - z_{i-1}^2\right) = 1.$$

Lemma 3.1. *For all $n \geq 3$*

$$\mathbf{E}\prod_{i=1}^{n-1}\left(e^{\eta_i\wedge\eta_{i+1}} - 1\right) = \sum_{i_{n-2}=1}^{2}\ \sum_{i_{n-3}=1}^{i_{n-2}+1} \dots \sum_{i_1=1}^{i_2+1} 1\,, \tag{40}$$

which equals the number of monotone increasing paths within T_{n-1} from $(0,0)$ to $(n-1, n-1)$.

Proof. Consider

$$I_k := \mathbf{E}\prod_{i=1}^{k-1}\left(e^{\eta_i\wedge\eta_{i+1}} - 1\right). \tag{41}$$

Let us define a linear operator ϕ on the space of polynomials $f(x), x \in \mathbf{R}_+$ as follows

$$\phi(f(x)) = [\phi(f)](x) = \mathbf{E}\,\left(e^{x\wedge\eta} - 1\right)\, f(\eta).$$

It is straightforward to check that

$$\phi(1) = \mathbf{E}\,\left(e^{x\wedge\eta} - 1\right) = x, \quad \phi(x) = \mathbf{E}\,\left(e^{x\wedge\eta} - 1\right)\eta = \frac{x^2}{2} + x, \tag{42}$$

and for any $n \geq 2$

$$\phi(x^n) = \mathbf{E}\,\left(e^{x\wedge\eta} - 1\right)\eta^n = \frac{x^{n+1}}{n+1} + n\,\phi(x^{n-1}). \tag{43}$$

Also we shall write $\phi^1(x) = \phi(x)$ and define recursively

$$\phi^k(x) = \mathbf{E}\,\left(e^{x\wedge\eta} - 1\right)\phi^{k-1}(\eta) = \phi\left(\phi^{k-1}(x)\right), \qquad k \geq 2.$$

Now we have the following representation

$$I_2 = \int_0^\infty e^{-x}\, x dx,$$

and for any $k \geq 1$

$$I_{k+2} = \int_0^\infty e^{-x}\,\phi^k(x) dx. \tag{44}$$

It follows from (42) and (43) that

$$\phi(x^n) = n! \sum_{l=1}^{n+1} \frac{x^l}{l!}, \quad n \geq 1. \tag{45}$$

Having (43) and (45) we can write for all $k \geq 1$

$$\phi^k(x) = \sum_{i=1}^{k+1} a_{ki} x^i,$$

where the coefficients satisfy the following recursive relations:

$$a_{11} = 1, \qquad a_{12} = 1/2,$$

and

$$a_{(k+1)l} = \frac{1}{l!} \sum_{i=l-1}^{k+1} i! a_{ki}, \qquad 1 \leq l \leq k+2,$$

with

$$a_{k0} \equiv 0, \quad k \geq 1.$$

Now we have by (44)

$$I_{k+2} = \mathbf{E}\phi^k(\eta_1) = \sum_{i=1}^{k+1} i! a_{ki} = \sum_{i=1}^{k} \left(\sum_{l=1}^{i+1} 1\right) i! a_{(k-1)i}. \tag{46}$$

For all $i \geq 0$ set $\psi_0(i) = 1$, and then

$$\psi_k(i) = \sum_{l=1}^{i+1} \psi_{k-1}(l), \quad k \geq 1.$$

Notice that for any finite function ψ

$$\sum_{l=1}^{k} \psi(l) l! a_{(k-1)l} = \sum_{l=1}^{k} \psi(l) \sum_{i=l-1}^{k-1} i! a_{(k-2)i} = \sum_{i=1}^{k-1} \left(\sum_{l=1}^{i+1} \psi(l) \right) i! a_{(k-2)i}$$

holds. This helps us to compute the expression in (46) for all $k \geq 2$

$$I_{k+2} = \sum_{i=1}^{k} \psi_1(i) i! a_{(k-1)i} = \sum_{i=1}^{k-1} \psi_2(i) i! a_{(k-2)i} = \ldots = \sum_{i=1}^{2} \psi_{k-1}(i) i! a_{1i} = \sum_{i=1}^{2} \psi_{k-1}(i)$$

$$= \psi_k(1) = \sum_{i_k=1}^{2} \sum_{i_{k-1}=1}^{i_k+1} \ldots \sum_{i_1=1}^{i_2+1} 1 \, . \tag{47}$$

The statement of Lemma 3.1 follows. □

Corollary 3.1.

$$\mathbf{E} \prod_{i=1}^{n-1} \left(e^{\eta_i \wedge \eta_{i+1}} - 1 \right) = \frac{1}{n} \binom{2(n-1)}{n-1}, \tag{48}$$

i.e., the Catalan number C_{n-1}.

Proof. The number of the lattice paths described in Lemma 3.1 is exactly one of the representations of the Catalan numbers in (7), p. 221. Hence formula (48) follows. □

Making use of Stirling's formula we derive from (48)

$$\lim_{n \to \infty} \left(\mathbf{E} \prod_{i=1}^{n-1} \left(e^{\eta_i \wedge \eta_{i+1}} - 1 \right) \right)^{1/n} = 4. \tag{49}$$

Combining this with formula (39) we get the statement (15). This completes the proof of Theorem 2.1. □

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