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Hydrodynamic limit fluctuations of super-Brownian motion with a stable catalyst*

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Abstract

We consider the behaviour of a continuous super-Brownian motion catalysed by a random medium with infinite overall density under the hydrodynamic scaling of mass, time, and space. We show that, in supercritical dimensions, the scaled process converges to a macroscopic heat flow, and the appropriately rescaled random fluctuations around this macroscopic

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flow are asymptotically bounded, in the sense of log-Laplace transforms, by generalised stable Ornstein-Uhlenbeck processes. The most interesting new effect we observe is the occurrence of an index-jump from a 'Gaussian' situation to stable fluctuations of index $1 + \gamma$ where $\gamma \in (0, 1)$ is an index associated to the medium

Key words: Catalyst, reactant, superprocess, critical scaling, refined law of large numbers, catalytic branching, stable medium, random environment, supercritical dimension, generalised stable Ornstein-Uhlenbeck process, index jump, parabolic Anderson model with stable random potential, infinite overall density

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1 Introduction and main results

1.1 Motivation and background

In order to describe the long-term behaviour of infinite interacting spatial particle systems with mass preservation on average, limit theorems under mass-time-space rescaling are an established tool. A typical feature that can be captured by this means is the clumping behaviour of spatial branching processes in *low dimensions*: In some models, for a critical scaling one can observe convergence to a nontrivial field of isolated mass clumps. The spatial contraction allows to get hold of large mass clumps in remote locations, and the index of mass-rescaling serves as a measure of the strength of the clumping effect, quantifying the degree of intermittency. In some of these results a macroscopic time dependence can be retained, giving insight in the long-time developments of the clumps. For a recent result in this direction, see Dawson et al. (DFM02).

In *higher dimensions* one does not expect to observe clumping under mass-time-space rescaling, but convergence to a non-random mass flow, the *hydrodynamic limit*. In this case one can hope to get a deeper understanding from the investigation of fluctuations around this limit. Such fluctuations were studied by Holley and Stroock (HS78) and Dawson (Daw78), and their results were later refined and extended, e.g. by Dittrich (Dit87). There is also a large body of literature on hydrodynamic limits of interacting particle systems, see e.g. (DMP91; KL99; Spo91). Our main motivation behind this paper is to investigate the possible effects on fluctuations around the hydrodynamic limit if the original process is influenced by a *random medium*, which in our model acts as a catalyst for the local branching rates.

In Dawson et al. (DFG89), fluctuations under mass-time-space rescaling were derived for a class of spatial infinite branching particle systems in $\mathbb{R}^d$ (with symmetric α–stable motion and $(1+\beta)$–branching) in supercritical dimensions in a random medium with *finite* overall density. This leads to *generalized Ornstein-Uhlenbeck processes* which are the same as for the model in the constant (averaged) medium. In other words, for the log-Laplace equation the governing effect is homogenization: After rescaling, the equation approximates an equation with homogeneous branching rate, the medium is simply averaged out. The nature of the fluctuations for the case of a medium with *infinite* overall density remained unresolved over the years.

The purpose of the present paper is to get progress in this direction. Our main result shows that a medium with an *infinite* overall density can have a drastic effect on the fluctuation behaviour of the model under critical rescaling in supercritical dimensions, and homogenization is no longer the effect governing the macroscopic behaviour. In fact, despite the infinite overall density of the medium, we still have a law of large numbers under a certain mass-time-space rescaling. But under this scaling, the variances (given the medium) blow up, and the related fluctuations do *not* obey a central limit theorem. However, fluctuations can be described to some degree by a stable process.

To be more precise, we start with a branching system with *finite* variance given the medium, considered as a branching process with a random law, where this randomness of the laws comes from the randomness of the medium (quenched approach). Under a mass-time-space rescaling, the random laws of the fluctuations are asymptotically bounded from above and below by the laws of constant multiples of a generalized Ornstein-Uhlenbeck process with *infinite* variance. Here the ordering of random laws is defined in terms of the random Laplace transforms. The generalized Ornstein-Uhlenbeck process is the same as the fluctuation limit of a super-Brownian motion with infinite variance branching in the case of a constant medium. In fact, the branching mechanism is $(1+\gamma)$–branching, where $\gamma \in (0,1)$ is the index of the medium. Altogether, the present result is a big step towards an affirmative answer to the old open problem of understanding fluctuations in the case of a random medium with infinite overall density. It also leads to random medium effects which are in line with experiences concerning the clumping behaviour in subcritical dimensions as in (DFM02).

1.2 Preliminaries: notation

For $\lambda \in \mathbb{R}$, introduce the reference function

$$\phi_\lambda(x) \ := \ \mathrm{e}^{-\lambda|x|} \quad \text{for } \ x \in \mathbb{R}^d.$$

For $f : \mathbb{R}^d \to \mathbb{R}$, set

$$|f|_\lambda \ := \ \|f/\phi_\lambda\|_\infty$$

where $\|\cdot\|_\infty$ refers to the supremum norm. Denote by $\mathcal{C}_\lambda$ the separable Banach space of all continuous functions $f:\mathbb{R}^d\to\mathbb{R}$ such that $|f|_\lambda$ is finite and that $f(x)/\phi_\lambda(x)$ has a finite limit as $|x|\to\infty$. Introduce the space

$$\mathcal{C}_{\exp} \;=\; \mathcal{C}_{\exp}(\mathbb{R}^d) \;:=\; \bigcup_{\lambda>0}\mathcal{C}_\lambda$$

of (at least) *exponentially decreasing* continuous test functions on $\mathbb{R}^d$. An index $+$ as in $\mathbb{R}_+$ or $\mathcal{C}_{\exp}^+$ refers to the corresponding non-negative members.

Let $\mathcal{M}=\mathcal{M}(\mathbb{R}^d)$ denote the set of all (non-negative) Radon measures μ on $\mathbb{R}^d$ and d_0 a complete metric on $\mathcal{M}$ which induces the vague topology. Introduce the space $\mathcal{M}_{\mathrm{tem}}=\mathcal{M}_{\mathrm{tem}}(\mathbb{R}^d)$ of all measures μ in $\mathcal{M}$ such that $\langle\mu,\phi_\lambda\rangle := \int d\mu\;\phi_\lambda<\infty$, for all $\lambda>0$. We topologize this set $\mathcal{M}_{\mathrm{tem}}$ of *tempered* measures by the metric

$$\mathrm{d}_{\mathrm{tem}}(\mu,\nu) \;:=\; \mathrm{d}_0(\mu,\nu)+\sum_{n=1}^{\infty}2^{-n}\left(|\mu-\nu|_{1/n}\wedge 1\right)\quad\text{for }\mu,\nu\in\mathcal{M}_{\mathrm{tem}}\,.$$

Here $|\mu-\nu|_\lambda$ is an abbreviation for $\bigl|\langle\mu,\phi_\lambda\rangle-\langle\nu,\phi_\lambda\rangle\bigr|$. Note that $\mathcal{M}_{\mathrm{tem}}$ is a Polish space (that is, $(\mathcal{M}_{\mathrm{tem}}\,,\mathrm{d}_{\mathrm{tem}})$ is a complete separable metric space), and that $\mu_n\to\mu$ in $\mathcal{M}_{\mathrm{tem}}$ if and only if

$$\langle\mu_n\,,\varphi\rangle \underset{n\uparrow\infty}{\longrightarrow} \langle\mu,\varphi\rangle\quad\text{for }\varphi\in\mathcal{C}_{\exp}\,.$$

Probability measures will be denoted as $\mathsf{P},\mathbb{P},\mathcal{P}$, whereas $\mathsf{E},\mathbb{E},\mathcal{E}$ and $\mathsf{Var},\mathbb{V}\mathrm{ar},\mathcal{V}\mathrm{ar}$ refer to the corresponding expectation and variance symbols.

Let p denote the standard heat kernel in $\mathbb{R}^d$ given by

$$p_t(x) \;:=\; (2\pi t)^{-d/2}\,\exp\Bigl[-\frac{|x|^2}{2t}\Bigr]\quad\text{for }t>0,\;\;x\in\mathbb{R}^d.$$

Write $W=\bigl(W,\,(\mathcal{F}_t)_{t\ge0}\,,\,\mathcal{P}_x,\,x\in\mathbb{R}^d\bigr)$ for the corresponding (standard) Brownian motion in $\mathbb{R}^d$ with natural filtration, and $S=\{S_t:\,t\ge0\}$ for the related semigroup. W_t and S_t are formally set to W_0 and S_0, respectively, if $t<0$.

Let ℓ denote the Lebesgue measure on $\mathbb{R}^d$. Write $B(x,r)$ for the closed ball around $x\in\mathbb{R}^d$ with radius $r>0$. In this paper, G denotes the Gamma function.

With $c=c(q)$ we always denote a positive constant which (in the present case) might depend on a quantity q and might also change from place to place. Moreover, an index on c as $c_{(\#)}$ or $c_{\#}$ will indicate that this constant first occurred in formula line (#) or (for instance) Lemma #, respectively. We apply the same labelling rules also to parameters like λ and k.

1.3 Modelling of catalyst and reactant

Of course, there is some freedom in choosing the model we want to work with. To avoid unnecessary limit procedures, we work on $\mathbb{R}^d$ and with continuous-state branching as the branching system, namely with continuous super-Brownian motion, which is a spatial version of Feller's branching diffusion. The branching rate of an intrinsic 'particle' varies in space and in fact is selected from a random field to be specified. In this context, it is convenient to speak also of the random field as the *catalyst,* and of the branching system given the random medium as the *reactant.*

First we want to specify the catalyst. In our context, a very natural way is to start from a *stable random measure* Γ on $\mathbb{R}^d$ with index $\gamma \in (0,1)$ determined by its log-Laplace functional

$$-\log \mathsf{E} \exp \langle \Gamma, -\varphi \rangle \;=\; \int dz\; \varphi^{\gamma}(z) \quad \text{for } \varphi \in \mathcal{C}^{+}_{\exp}\,. \tag{1}$$

(The letter P always stands for the law of the catalyst, whereas $\mathbb{P}$ is reserved for the law of the reactant given the catalyst.) See, for instance, (DF92, Lemma 4.8) for background concerning Γ. Clearly, Γ is a spatially homogeneous random measure with independent increments and infinite expectation. Γ has a simple *scaling property,*

$$\Gamma(k\,dz) \;\stackrel{\mathcal{L}}{=}\; k^{d/\gamma}\,\Gamma(dz) \quad \text{for } k>0, \tag{2}$$

where $\stackrel{\mathcal{L}}{=}$ refers to equality in law. However, Γ is a purely atomic measure, hence, its atoms cannot be hit by a Brownian path or a super-Brownian motion in dimensions $d \geq 2$. Thus, Γ cannot serve directly as a catalyst for a non-degenerate reaction model based on Brownian particles in higher dimensions. Therefore we look at the density function after smearing out Γ by the (non-normalized) function ϑ_1, where $\vartheta_r := 1_{B(0,r)}$, $r>0$, that is,

$$\Gamma^1(x) \;:=\; \int \Gamma(dz)\; \vartheta_1(x-z) \quad \text{for } x \in \mathbb{R}^d. \tag{3}$$

In the sequel, the unbounded function Γ^1 with infinite overall density will play the rôle of the *random medium*: It will act as a catalyst that determines the spatially varying branching rate of the reactant. Once again, smoothing is needed, since otherwise the medium will not be hit by an intrinsic Brownian reactant particle. In our proofs, the independence and scaling properties of Γ will be advantageous, though one would expect analogous results to hold for quite general random media with infinite overall density.

Consider now the *continuous super-Brownian motion* $X = X[\Gamma^1]$ in $\mathbb{R}^d$, $d \geq 1$, *with random catalyst* Γ^1. More precisely, *for almost all samples* Γ^1, this is a continuous time-homogeneous Markov process $X = X[\Gamma^1] = (X, \mathbb{P}_\mu, \mu \in \mathcal{M}_{\text{tem}})$ with log-Laplace transition functional

$$-\log \mathbb{E}_\mu \exp \langle X_t\,, -\varphi \rangle \;=\; \big\langle \mu, u(t,\,\cdot\,)\big\rangle \quad \text{for } \varphi \in \mathcal{C}^{+}_{\exp}\,,\ \mu \in \mathcal{M}_{\text{tem}}\,, \tag{4}$$

where $u = u[\varphi, \Gamma^1] = \big\{u(t,x) : t \geq 0,\ x \in \mathbb{R}^d\big\}$ is the unique mild non-negative solution of the reaction diffusion equation

$$\frac{\partial}{\partial t} u(t,x) \;=\; \tfrac{1}{2}\Delta u(t,x) - \varrho\, \Gamma^1(x)\, u^2(t,x) \quad \text{for } t \geq 0,\ x \in \mathbb{R}^d, \tag{5}$$

with initial condition $u(0,\,\cdot\,) = \varphi$. Here $\varrho > 0$ is an additional parameter (for scaling purposes). For background on super-Brownian motion we recommend (Daw93), (Eth00), or (Per02), and for a survey on catalytic super-Brownian motion, see e.g. (DF02) or (Kle00).

From Dawson and Fleischmann (DF83; DF85) the following dichotomy concerning the long-term behaviour of X is basically known (although there the phase space is $\mathbb{Z}^d$ and the processes are in discrete time): Starting from the Lebesgue measure $X_0 = \ell$, the process X dies locally in law as $t \uparrow \infty$ if $d \leq 2/\gamma$ (recall that $0 < \gamma < 1$ is the index of the random medium Γ^1), whereas in all higher dimensions one has persistent convergence in law to a non-trivial limit state denoted by X_∞. *From now on, we restrict our attention to (supercritical) dimensions* $d > 2/\gamma$.

We are interested in the large scale behaviour of X.

1.4 Main results of the paper

Introduce the *scaled processes* X^k, $k > 0$, defined by

$$X_t^k(B) := k^{-d} X_{k^2 t}(kB) \quad \text{for } t \geq 0, \ B \subseteq \mathbb{R}^d \text{ Borel.} \tag{6}$$

This *hydrodynamic rescaling* leaves the underlying Brownian motions invariant (in law), and the expectation of the scaled process is the heat flow:

$$\mathbb{E}_\mu X_t^k = S_t \mu^k \quad \text{for } X_0 = \mu \in \mathcal{M}_{\text{tem}}.$$

In particular, if X is started with the Lebesgue measure ℓ, the expectation is preserved in time. We also define the *critical scaling index*

$$\varkappa_{\text{c}} := \frac{\gamma d - 2}{1 + \gamma} > 0. \tag{7}$$

Theorem 1 (Refined law of large numbers). *Suppose* $d > 2/\gamma$. *Start* X *with* k*–dependent initial states* $X_0 = \mu_k \in \mathcal{M}_{\text{tem}}$ *such that* $X_0^k = \mu \in \mathcal{M}_{\text{tem}}$ *for* $k > 0$. *Let* $t \geq 0$ *and* $\varkappa \in [0, \varkappa_{\text{c}})$. *Then*

(a) in P*-probability,* $k^\varkappa\big(X_t^k - S_t\mu\big) \underset{k\uparrow\infty}{\Longrightarrow} 0$ *in* $\mathbb{P}_{\mu_k}$*-law;*

(b) in $\mathsf{E}\mathbb{P}_{\mu_k}$*-law,* $k^\varkappa\big(X_t^k - S_t\mu\big) \underset{k\uparrow\infty}{\Longrightarrow} 0.$

The refined law of large numbers is actually a by-product of the proofs of our main result, Theorem 2, as will be explained immediately after Proposition 15. Convergence in law will be shown via Laplace transforms, which is in contrast to (DFG89), where Fourier transforms are used. This is possible although the fluctuations are *signed* objects. Indeed, these fluctuations themselves are deviations from non-negative X^k, and related stable limiting quantities have skewness parameter $\beta = 1$, for which Laplace transforms are meaningful. In our main result, Laplace transforms will enable us to use stochastic ordering (see also Remark 4).

For $x \in \mathbb{R}^d$ we put

$$\text{en}(x) := \begin{cases} \log^+\big(|x|^{-1}\big) & \text{if } d = 4, \\ |x|^{4-d} & \text{if } d \geq 5, \end{cases} \tag{8}$$

and for $\mu \in \mathcal{M}_{\text{tem}}$, and $\lambda > 0$,

$$\text{En}_\lambda(\mu) := \int \mu(dx)\, \phi_\lambda(x) \int \mu(dy)\, \phi_\lambda(y)\ \text{en}(x-y). \tag{9}$$

Note that $\text{En}_\lambda(\delta_x) \equiv \infty$ if $d > 3$.

Theorem 2 (Asymptotic fluctuations). *Suppose $d>2/\gamma$. Start X with k–dependent initial states $X_0=\mu_k\in\mathcal{M}_{\mathrm{tem}}$ such that $X_0^k=\mu\in\mathcal{M}_{\mathrm{tem}}$ for $k>0$. In the case $d>3$, suppose additionally that μ is a measure of finite energy in the sense that $\mathrm{En}_\lambda(\mu)<\infty$ for all $\lambda>0$. If $\varkappa=\varkappa_{\mathrm{c}}$, then there exists constants $\overline{c}>\underline{c}>0$ such that for any $\varphi_1,\ldots,\varphi_n\in\mathcal{C}_{\exp}^+$ and $0=:t_0\le t_1\le\cdots\le t_n$, in P–probability,*

$$\begin{aligned}\limsup_{k\uparrow\infty}\mathbb{E}_{\mu_k}\exp\Big[\sum_{i=1}^n k^{\varkappa}\big\langle X_{t_i}^k-S_{t_i}\mu,\,-\varphi_i\big\rangle\Big]\\ \le\ \exp\Bigg[\overline{c}\,\Bigg\langle\mu,\sum_{i=1}^n\int_{t_{i-1}}^{t_i}dr\ S_r\Big(\Big(\sum_{j=i}^n S_{t_j-r}\varphi_j\Big)^{1+\gamma}\Big)\Bigg\rangle\Bigg]\end{aligned}\tag{10}$$

and

$$\begin{aligned}\liminf_{k\uparrow\infty}\mathbb{E}_{\mu_k}\exp\Big[\sum_{i=1}^n k^{\varkappa}\big\langle X_{t_i}^k-S_{t_i}\mu,\,-\varphi_i\big\rangle\Big]\\ \ge\ \exp\Bigg[\underline{c}\,\Bigg\langle\mu,\sum_{i=1}^n\int_{t_{i-1}}^{t_i}dr\ S_r\Big(\Big(\sum_{j=i}^n S_{t_j-r}\varphi_j\Big)^{1+\gamma}\Big)\Bigg\rangle\Bigg].\end{aligned}\tag{11}$$

Explicit values of $\overline{c}$ and $\underline{c}$ are given in (37) and (94), respectively. Clearly, a statement of the form

$$\limsup_{k\uparrow\infty}\xi_k\le c\quad\text{in }\mathsf{P}\text{–probability}$$

means that

$$\mathsf{P}\Big(\sup_{k\ge n}\xi_k>c+\varepsilon\Big)\ \underset{n\uparrow\infty}{\longrightarrow}\ 0\quad\text{for all }\ \varepsilon>0.$$

Remark 3 (Generalized Ornstein-Uhlenbeck process). The right-hand sides of (10) and (11) are the Laplace transforms of the finite-dimensional distributions of different multiples of a process Y taking values in the Schwartz space of tempered distributions. This process Y can be called a *generalized Ornstein-Uhlenbeck process* as it solves the *generalized Langevin equation*,

$$dY_t\ =\ \tfrac12\Delta Y_t\,dt+dZ_t\quad\text{for }\ t\ge0,\ \ Y_0=0,$$

where dZ_t/dt is a $(1+\gamma)$–stable noise, i.e. Z is the process with independent increments with values in the Schwartz space such that, for $0\le s\le t$ and $\varphi\in\mathcal{C}_{\exp}^+$,

$$E\mathrm{e}^{-\langle Z_t-Z_s,\varphi\rangle}\ =\ \exp\Big[\int_s^t dr\ \big\langle S_r\mu,\varphi^{1+\gamma}\big\rangle\Big].$$

Y is described in detail in (DFG89, Section 4) in a Fourier setting, where it appeared as the hydrodynamic fluctuation limit process corresponding to super-Brownian motion with finite mean branching rate, but with infinite variance $(1+\gamma)$–branching. Recall that the Markov process Y has log-Laplace transition functional

$$-\log E\big\{\exp\langle Y_t,-\varphi\rangle\ \big|\ Y_0\big\}\ =\ \langle Y_0,S_t\varphi\rangle+\big\langle\mu,v(t,\,\cdot\,)\big\rangle\quad\text{for }\ \varphi\in\mathcal{C}_{\exp}^+,$$

where $v = v[\varphi] = \{v(t,x) : t \geq 0,\ x \in \mathbb{R}^d\}$ solves

$$\begin{aligned} \frac{\partial}{\partial t} v(t,x) \ &= \ \tfrac{1}{2}\Delta v(t,x) \ + \ (S_t\varphi)^{1+\gamma}(x) \\ &\text{with initial condition } \ v(0,\,\cdot\,) = 0. \end{aligned} \tag{12}$$

In particular, in our limit procedure the finite variance property of the original process given the medium is lost and, by a subtle averaging effect, an index jump of size $1-\gamma > 0$ occurs. $\diamond$

Remark 4 (**Stochastic ordering**)**.** The stochastic ordering of the random laws in our asymptotic bounds in (10) and (11) is well-known in queueing and risk theory, see (MS02) for background. $\diamond$

Remark 5 (**Existence of a fluctuation limit**)**.** Theorem 2 leaves *open*, whether a fluctuation limit exists in P–probability. Naturally, one would expect the limit to be an Ornstein-Uhlenbeck type process driven by a singular process, as the infinite mean fluctuations should produce singularities which get smoothed out by the rescaled Gaussian dynamics. Since the environment has independent increments one would expect the same for the singularities due to the high-dimensional setting. However, the spatial correlations make the random environment setting harder to study than the analogous infinite variance branching setting. Therefore this heuristic cannot explain the exact form of limiting fluctuations. $\diamond$

Remark 6 (**Variance considerations**)**.** In the case $\mu_k \equiv \ell$, for $\varphi \in \mathcal{C}_{\exp}$, the P–random variance

$$\begin{aligned} \mathbb{V}\mathrm{ar}_\ell\Big[k^{\varkappa}\langle X^k_t - S_t\mu, \varphi\rangle\Big] \ &= \ k^{2\varkappa}\,\mathbb{V}\mathrm{ar}_\ell\langle X^k_t, \varphi\rangle \\ &= \ 2\varrho\, k^{2\varkappa-2d}\int_0^{k^2 t} ds \int dx\ \Gamma^1(x)\,\big[S_{k^2t-s}\varphi\,(k^{-1}\cdot)\big]^2(x) \end{aligned} \tag{13}$$

equals (by scaling) approximately

$$2\varrho\, k^{2\varkappa-2d+2+d/\gamma}\int_0^t ds \int \Gamma(dz)\,[S_s\varphi]^2(z) \quad \text{as } \ k \uparrow \infty.$$

Hence, for $\varkappa$ satisfying

$$0 \ \leq \ \varkappa \ < \ \varkappa_{\mathrm{var}} \ := \ \frac{(2\gamma-1)d - 2\gamma}{2\gamma}\,, \tag{14}$$

implying $\gamma \in (\frac{1}{2}, 1)$ and $d > 2\gamma/(2\gamma-1)$, the random variances (13) converge to zero as $k \uparrow \infty$, yielding the refined law of large numbers, Theorem 1(a), whereas for $\varkappa > \varkappa_{\mathrm{var}}$ these variances explode. Note that $\varkappa_{\mathrm{var}} < \varkappa_{\mathrm{c}}$, since $(\gamma-1)(d-2\gamma) < 0$. Therefore a quenched variance consideration as in (13) can only imply a law of large numbers in the restricted case (14). Of course, *annealed* variances are infinite already for fixed k, which follows from (13). $\diamond$

1.5 Heuristics, concept of proof, and outline

For this discussion we first focus on the case $n = 1$ in Theorem 2. From (4), (5), and scaling,

$$\log \mathbb{E}_{\mu_k}\exp\Big[-k^{\varkappa}\langle X^k_t - S_t\mu, \varphi\rangle\Big] \ = \ \big\langle \mu,\, k^{\varkappa}S_t\varphi - u_k(t,\,\cdot\,)\big\rangle, \tag{15}$$

where u_k solves the (scaled) equation

$$\begin{gathered}\frac{\partial}{\partial t}u_k(t,x) \;=\; \tfrac{1}{2}\Delta u_k(t,x) \;-\; k^{2-d}\,\varrho\,\Gamma^1(kx)\,u_k^2(t,x)\\ \text{with initial condition } \ u_k(0,\,\cdot\,) = k^{\varkappa}\varphi.\end{gathered}\tag{16}$$

Since $v(t,x) := k^{\varkappa}S_t\varphi\,(x)$ is the solution of

$$\frac{\partial}{\partial t}v(t,x) \;=\; \tfrac{1}{2}\Delta v(t,x) \quad \text{with initial condition } \ v(0,\,\cdot\,) = k^{\varkappa}\varphi,$$

we see that $f_k(t,x) := k^{\varkappa}S_t\varphi\,(x) - u_k(t,x)$ solves

$$\begin{gathered}\frac{\partial}{\partial t}f_k(t,x) \;=\; \tfrac{1}{2}\Delta f_k(t,x) \;+\; k^{2-d}\,\varrho\,\Gamma^1(kx)\,\big[k^{\varkappa}S_t\varphi\,(x) - f_k(t,x)\big]^2\\ \text{with initial condition } \ f_k(0,\,\cdot\,) = 0.\end{gathered}\tag{17}$$

Consider now the critical scaling $\varkappa = \varkappa_{\rm c}$. By our claims in Theorem 2, f_k should be asymptotically bounded in P–law by solutions v of

$$\frac{\partial}{\partial t}v(t,x) \;=\; \tfrac{1}{2}\Delta v(t,x) \;+\; c\,(S_t\varphi)^{1+\gamma}\,(x) \tag{18}$$

for different constants c. Consequently, in a sense, we have to justify the transition from equation (17) to the log-Laplace equation (18) corresponding to the limiting fluctuations, recall (12). Here the $x \mapsto \Gamma^1(kx)$ entering into equation (17) are random homogeneous fields with infinite overall density, and the solutions f_k depend on Γ^1. But the most fascinating fact here seems to be the index jump from 2 to $1+\gamma$, which occurs when passing from (17) to (18). Unfortunately, we are unable to explain this from an individual ergodic theorem acting on the (ergodic) underlying random measure Γ.

We take another route. For the heuristic exposition, we simplify as follows. First of all, we restrict our attention to the case $\varphi(x) \equiv \theta$ corresponding to total mass process fluctuations. Clearly, we have the domination

$$0 \;\le\; u_k(t,x) \;\le\; k^{\varkappa}\,\theta.$$

Replacing one of the $u_k(t,x)$ factors in the non-linear term of (16) by $k^{\varkappa}S_t\varphi\,(x) \equiv k^{\varkappa}\,\theta$, and denoting the solution to the new equation with the same initial condition by w_k, then $u_k \ge w_k$, and we can explicitly calculate w_k by the Feynman-Kac formula,

$$w_k(t,x) \;=\; k^{\varkappa}\,\theta\,\mathcal{E}_x \exp\Big[-k^{2-d}\int_0^t ds\;\varrho\,\Gamma^1(kW_s)\,k^{\varkappa}\,\theta\Big]. \tag{19}$$

For the upper bound (10), we may work with w_k instead of u_k. It suffices to show that $\langle\mu,\,k^{\varkappa}S_t\varphi - w_k(t,\cdot)\rangle$ converges to $\langle\mu, v\rangle$ in $L^2(\mathsf{P})$, where v is the solution to (18) with constant $c = \overline{c}$. We therefore show that the P–expectations converge, and the P–variances go to 0. In this heuristics we concentrate on the convergence of E–expectations only, and we simplify by assuming $\mu = \delta_x$ (although formally excluded in the theorem by (9) if $d > 3$). We then have to show that

$$\mathsf{E}k^{\varkappa}\,\theta\,\mathcal{E}_x\Big(1 - \exp\Big[-k^{2-d+\varkappa}\,\theta\int_0^t ds\;\varrho\,\Gamma^1(kW_s)\Big]\Big) \;\underset{k\uparrow\infty}{\longrightarrow}\; t\,\overline{c}\,\theta^{1+\gamma}. \tag{20}$$

By definition (3) of Γ^1 and (1) of Γ, the left hand side of (20) can be rewritten as

$$k^{\varkappa}\,\theta\,\mathcal{E}_x\Big(1-\mathsf{E}\exp\Big[-\int\Gamma(dz)\;k^{2-d+\varkappa}\,\varrho\theta\int_0^t ds\;\vartheta_1(kW_s-z)\Big]\Big)$$
$$=\;k^{\varkappa}\,\theta\,\mathcal{E}_x\Big(1-\exp\Big[-k^{(2-d+\varkappa)\gamma+d}\,(\varrho\theta)^{\gamma}\int dz\,\Big(\int_0^t ds\;1_{B(z,\frac{1}{k})}(W_s)\Big)^{\gamma}\Big]\Big).$$

We may additionally introduce the indicator $1_{\{\tau\leq t\}}$ where $\tau=\tau_{1/k}^{z}[W]$ denotes the *first hitting time* of the ball $B(z,1/k)$ by the path W starting from x, and we continue with

$$=\;k^{\varkappa}\,\theta\,\mathcal{E}_x\Big(1-\exp\Big[-k^{(2-d+\varkappa)\gamma+d}\,(\varrho\theta)^{\gamma}\int dz\,1_{\{\tau\leq t\}}\Big(\int_0^t ds\;1_{B(z,\frac{1}{k})}(W_s)\Big)^{\gamma}\Big]\Big).$$

Now we look at the $\mathcal{E}_x$–expectation of the exponent term. As the probability of hitting the small ball $B(z,1/k)$ is of order k^{2-d} if $x\neq z$, and the time spent afterwards in the ball is of order k^{-2}, the expectation of the exponent term is of order $k^{(-d+\varkappa)\gamma+2}=k^{-\varkappa}$ converging to zero as $k\uparrow\infty$. Heuristically this justifies the use of the approximation $1-\mathrm{e}^{-x}\approx x$. Note that then the leading factor $k^{\varkappa}$ is cancelled, and we arrive at a constant multiple of $\theta^{1+\gamma}$.

According to this simplified calculation, the index jump has its origin in an averaging of exponential functionals of Γ [as in (1)], generating a transition from θ to θ^{γ}. Note that the smallness of the exponent is largely due to the presence of the indicator of $\{\tau\leq t\}$. This fact is also behind our estimates of variances in Section 3.3.

We recall that the simplification $u_k\rightsquigarrow w_k$ which we used in the upper bound is basically a *linearization* of the problem, that is we pass from the non-linear log-Laplace equation (16) to the linear equation

$$\frac{\partial}{\partial t}w_k(t,x)\;=\;\tfrac{1}{2}\Delta w_k(t,x)\;-\;k^{2-d}\,\varrho\,\Gamma^1(kx)\,k^{\varkappa}\theta\,w_k(t,x)$$
$$\text{with initial condition }\;u_k(0,\,\cdot\,)=k^{\varkappa}\theta.$$

In the case of a catalyst with finite expectation as in (DFG89), this linearization was a key step for deriving the limiting fluctuations. The difference between u_k and w_k was asymptotically negligible. But in the present model of a catalyst of infinite overall density, this is *no longer the case.* In fact, $u_k(t,x)-w_k(t,x)$ does not converge to 0 in P–probability. Therefore, our upper bound is not sharp.

For the lower bound, we replace u_k^2 in (16) by w_k^2, and denoting the solution to the new equation with the same initial condition by m_k. Then

$$k^{\varkappa}\theta-u_k(t,x)\;\geq\;k^{\varkappa}\theta-m_k(t,x)\;=\;k^{2-d}\varrho\,\mathcal{E}_x\int_0^t ds\;\Gamma^1(kW_s)\,w_k^2(t-s,W_s).$$

Inserting for w_k the Feynman-Kac representation (19) we arrive at an explicit expression. Similarly as above, we then show that $\big\langle\mu,\,k^{\varkappa}S_t\varphi-m_k(t,\cdot)\big\rangle$ converges to $\langle\mu,v\rangle$ in $L^2(\mathsf{P})$, where v is the solution to (18) with constant $c=\underline{c}$.

The *structure of the remaining paper* is as follows. After some basic preparations, in Section 3 we concentrate on the upper bound, whereas the lower bound follows in Section 4.

2 Preparation: Some basic estimates

In this section we provide some simple but useful tools for the main body of the proof. For basic facts on Brownian motion, see, for instance, (RY91) or (KS91).

2.1 Simple estimates for the Brownian semigroup

We frequently use the argument (based on the triangle inequality) that, for $\eta > 0$ and $t > 0$, there exists $c = c(\eta, t)$ such that for all x,

$$\sup_{0<s\le t} \int dy\ \phi_\eta(y)\, p_s(x-y) \ \le\ \phi_\eta(x) \sup_{0<s\le t} \int dy\ \mathrm{e}^{\eta|x-y|}\, p_s(x-y) \ =\ c\, \phi_\eta(x). \tag{21}$$

Let $\varphi \in \mathcal{C}^+_{\exp}$. Recall that $(s,x) \mapsto S_s\varphi\,(x)$ is uniformly continuous, hence for any $\varepsilon > 0$ one may choose $\delta > 0$ such that

$$\bigl|S_r\varphi\,(x) - S_s\varphi\,(y)\bigr| \ \le\ \varepsilon \quad \text{if} \ \ |r-s| \le \delta,\ \ |x-y| \le \delta. \tag{22}$$

For convenience we expose the following simple fact.

Lemma 7 (**Brownian semigroup estimate**). *For $t > 0$ and $\varphi \in \mathcal{C}^+_{\exp}$, there is a $\lambda_7 = \lambda_7(t,\varphi) > 0$ and a constant $c_7 = c_7(t,\varphi)$ such that, for every $x \in \mathbb{R}^d$,*

$$\tilde{\phi}(x) \ :=\ \sup_{s\le t}\ \sup_{y\in B(x,1)}\ S_s\varphi\,(y) \ \le\ c_7\, \phi_{\lambda_7}(x). \tag{23}$$

Note that in all dimensions, for each $\lambda > 0$,

$$\sup_{x\in\mathbb{R}^d} \int dz\ \phi_\lambda(z)\, |z-x|^{2-d} \ <\ \infty. \tag{24}$$

In fact, on the unit ball $B(x,1)$, use that $\int_{|z|\le 1} dz\, |z|^{2-d} < \infty$, and outside this ball, exploit $|z-x|^{2-d} \le 1$.

We continue with the following observation.

Lemma 8. *Let $d \ge 5$. Then, for some constant c_8 and all $x, y \in \mathbb{R}^d$,*

$$\int dz\ |z-x|^{2-d}\, |z-y|^{2-d} \ =\ c_8\, |x-y|^{4-d} \ =\ c_8\ \mathrm{en}(x-y).$$

Proof. Clearly, using the definition of the Green function as an integral of the transition densities,

$$\int dz\ |z-x|^{2-d}\, |z-y|^{2-d} \ =\ c \int dz \int_0^\infty ds\ p_s(z-x) \int_0^\infty dt\ p_t(z-y).$$

Interchanging integrations, using Chapman-Kolmogorov, substituting, and interchanging again gives

$$=\ c \int_0^\infty dt\ t\, p_t(x-y) \ =\ c\, |x-y|^{4-d} \int_0^\infty dt\ t\, p_t(\iota)$$

with ι any point on the unit sphere. The latter integral is finite since $d > 4$, finishing the proof. $\square$

In dimension four, the situation is slightly more involved.

Lemma 9. *Let* $d = 4$ *and* $\lambda > 0$. *Then, for some constant* $c_9 = c_9(\lambda)$ *and all* $x, y \in \mathbb{R}^4$,

$$\int dz\ \phi_\lambda(z)\, |z-x|^{-2}\, |z-y|^{-2} \ \leq\ c_9 \left[1 + \log^+\left(|x-y|^{-1}\right)\right]. \tag{25}$$

Proof. If $|x-y| > 2$, then the left hand side of (25) is bounded in x, y. In fact, for z in a unit sphere around a singularity, say x, we use $|z-y| \geq 1$ and estimate (24). Outside both unit spheres, the integrand is bounded by ϕ_λ.

Now suppose $|x-y| \leq 2$. We may also assume that $x \neq y$. As in the proof of Lemma 8, the left hand side of (25) leads to the integral

$$\int_0^\infty ds \int_0^\infty dt \int dz\ \phi_\lambda(z)\, p_s(z-x)\, p_t(z-y).$$

First we additionally restrict the integrals to $s, t \leq |x-y|^{-1}$. In this case, we drop $\phi_\lambda(z)$, use Chapman-Kolmogorov, substitute, and interchange the order of integration to get the bound

$$\int_0^{2\,|x-y|^{-1}} dt\ t\, p_t(x-y) \ \leq\ \int_0^{2\,|x-y|^{-3}} dt\ t\, p_t(\iota) \ \leq\ c\ \left[1 + \log\left(|x-y|^{-1}\right)\right].$$

To see the last step, split the integral at $t = 1$. To finish the proof, by symmetry in x, y, it suffices to consider

$$\int_0^\infty ds \int_{|x-y|^{-1}}^\infty dt \int dz\ \phi_\lambda(z)\, p_s(z-x)\, p_t(z-y). \tag{26}$$

Now, by a substitution,

$$\int_{|x-y|^{-1}}^\infty dt\ p_t(z-y) \ \leq\ |z-y|^{-2} \int_{|x-y|^{-1}\,|z-y|^{-2}}^\infty dt\ c\,t^{-2} \ =\ c\,|x-y| \ \leq\ 2c. \tag{27}$$

Plugging (27) into (26) and using the Green's function again gives the bound

$$c \sup_{x \in \mathbb{R}^4} \int dz\ \phi_\lambda(z)\, |z-x|^{-2},$$

which is finite by (24). $\square$

2.2 Brownian hitting and occupation time estimates

Further key tools are the asymptotics of the hitting times of small balls. Recall that $\tau = \tau_{1/k}^z[W]$ denotes the *first hitting time* of the closed ball $B(z, 1/k)$ by the Brownian motion W started in x. The following results are taken from (LG86), see formula (0a) and Lemma 2.1 there.

Lemma 10 (Hitting time asymptotics and bounds). *Suppose* $d \geq 3$ *and fix* $t > 0$. *Then the following results hold.*

(a) There is a constant $c_{(28)}$, *which depends only on the dimension d, such that*

$$\mathcal{P}_x(\tau < \infty) \ \leq\ c_{(28)}\, k^{2-d}\, |z-x|^{2-d} \quad \textit{for } x, z \in \mathbb{R}^d. \tag{28}$$

(b) *There are constants* $c_{(29)}$ *and* $\lambda_{(29)} > 0$*, depending on* d *and* t*, such that for* $x, z \in \mathbb{R}^d$*,*

$$k^{d-2}\,\mathcal{P}_x(\tau \le t) \;\le\; c_{(29)}\big[|z-x|^{2-d}+1\big]\exp\big[-\lambda_{(29)}|z-x|^2\big]. \tag{29}$$

(c) *The following convergence holds uniformly in* x, z *whenever* $|x-z|$ *is bounded away from zero,*

$$\lim_{k\uparrow\infty} k^{d-2}\,\mathcal{P}_x(\tau \le t) \;=\; c_{(30)}\int_0^t ds\; p_s(z-x), \tag{30}$$

where $c_{(30)} := \frac{(d-2)\pi^{d/2}}{G(d/2)}$ *(and* G *is the Gamma function).*

(d) *Finally, writing* $\tau^i := \tau_1^{z_i}[W]$ *for* $i=1,2$*, there are constants* $c_{(31)}$ *and* $\lambda_{(31)} > 0$*, depending on* d *and* t*, such that for* $x, z_1, z_2 \in \mathbb{R}^d$*,*

$$\begin{aligned}
&\mathcal{P}_x\big(\tau^1 < \tau^2 < k^2 t\big)\\
&\le\; c_{(31)}\,k^{4-2d}\Big(\big|(z_1-x)/k\big|^{2-d}+1\Big)\exp\Big[-\lambda_{(31)}\big|(z_1-x)/k\big|^2\Big]\\
&\qquad\qquad\times\;\Big(\big|(z_2-z_1)/k\big|^{2-d}+1\Big)\exp\Big[-\lambda_{(31)}\big|(z_2-z_1)/k\big|^2\Big].
\end{aligned} \tag{31}$$

The following two lemmas are consequences of Lemma 10.

Lemma 11. *Let* $d \ge 3$*. Fix* $\varphi \in \mathcal{C}^+_{\exp}$*,* $\eta \ge 0$*, and* $t > 0$*. Then there are constants* c_{11} *and* λ_{11} *such that for* $x, z \in \mathbb{R}^d$ *with* $|x-z| \ge \frac{1}{k}$*,*

$$\begin{aligned}
&\mathcal{E}_x\varphi(W_t)1_{\{\tau\le t\}}\Big(k^2\int_0^t ds\;\vartheta_1(kW_s-kz)\Big)^\eta\\
&\le\; c_{11}k^{2-d}\phi_{\lambda_7}(z)\,\big[|z-x|^{2-d}+1\big]\exp\big[-\lambda_{11}|z-x|^2\big].
\end{aligned} \tag{32}$$

Proof. Initially, let φ be any non-negative function. Using the strong Markov property at time τ,

$$\begin{aligned}
&\mathcal{E}_x\varphi(W_t)\;1_{\{\tau\le t\}}\Big(k^2\int_0^t ds\;\vartheta_1(kW_s-kz)\Big)^\eta\\
&\quad=\;\mathcal{E}_x\,\varphi(W_t)\,1_{\{\tau\le t\}}\,\mathcal{E}_x\bigg\{\Big(k^2\int_0^t ds\;\vartheta_1(kW_s-kz)\Big)^\eta\;\bigg|\;\mathcal{F}_\tau\bigg\}\\
&\quad=\;\mathcal{E}_x\,\varphi(W_t)\,1_{\{\tau\le t\}}\,g(\tau, W_\tau),
\end{aligned} \tag{33}$$

where

$$g(r,y) \;:=\; \mathcal{E}_y\Big(k^2\int_0^{t-r} ds\;\vartheta_1(kW_s-kz)\Big)^\eta$$

for $0 \le r \le t$ and $y \in \partial B(z, 1/k)$. But,

$$g(r,y) \le \mathcal{E}_y\Big(\int_0^\infty ds\;\vartheta_1(kW_{k^{-2}s}-kz)\Big)^\eta = \mathcal{E}_{ky}\Big(\int_0^\infty ds\;\vartheta_1(W_s-kz)\Big)^\eta.$$

Note that the right hand side is independent of k, z, y (in the considered range of y), and finite since in $d \geq 3$ all such moments are finite. Consequently, there is a constant c such that $g(r, y) \leq c$. If now $\varphi \in \mathcal{C}_{\exp}^{+}$, by the strong Markov property at time τ,

$$\mathcal{E}_x \varphi(W_t)\, 1_{\{\tau \leq t\}} \;=\; \mathcal{E}_x\, 1_{\{\tau \leq t\}}\, \mathcal{E}_{W_\tau} \varphi(\tilde{W}_{t-\tau}) \;\leq\; \mathcal{P}_x(\tau \leq t)\, \phi_{\lambda_7}(z), \tag{34}$$

using (23) in the second step. Here the Brownian variable $\tilde{W}$ is subject to the internal expectation operator $\mathcal{E}_{W_\tau}$. By (29),

$$\mathcal{P}_x(\tau \leq t) \;\leq\; c_{(29)} k^{2-d} \big[|z-x|^{2-d} + 1\big] \exp\!\big[-\lambda_{(29)} |z-x|^2\big]. \tag{35}$$

The result follows by combining (34) and (35). $\square$

Lemma 12. *Let $d \geq 3$. Fix $\eta \geq 0$, $\varphi \in \mathcal{C}_{\exp}^{+}$, and $t > 0$. Then there is a constant c_{12} such that*

(a) $\displaystyle \mathcal{E}_x \Big(k^2 \int_0^t ds\; \vartheta_1(kW_s - kz) \Big)^{\eta} \;\leq\; c_{12}\, k^{2-d}\, |z-x|^{2-d},$
for all $x, z \in \mathbb{R}^d$ and $k \geq 1$.

(b) $\displaystyle \int dz\; \mathcal{E}_x \varphi(W_t) 1_{\{\tau \leq t\}} \Big(k^2 \int_0^t ds\; \vartheta_1(kW_s - kz) \Big)^{\eta} \;\leq\; c_{12} k^{2-d} \phi_{\lambda_7}(x),$
for all $x \in \mathbb{R}^d$ and $k \geq 1$.

Proof. The proof of (a) follows from (33) for $\varphi \equiv 1$ and (28), the proof of (b) by integrating (32) and applying (21). $\square$

3 Upper bound: Proof of (10)

3.1 Parabolic Anderson model with stable random potential

As motivated in Section 1.5, for $\varkappa = \varkappa_{\mathrm{c}}$, $\varphi \in \mathcal{C}_{\exp}^{+}$, and $k > 0$, we look at the mild solution to the linear equation on $\mathbb{R}_+ \times \mathbb{R}^d$,

$$\begin{gathered} \frac{\partial}{\partial t} w_k(t,x) \;=\; \tfrac{1}{2} \Delta w_k(t,x) - k^{2-d}\, \varrho\, \Gamma^1(kx)\, k^{\varkappa} S_t \varphi\,(x)\, w_k(t,x) \\ \text{with initial condition } w_k(0,\cdot\,) = k^{\varkappa} \varphi. \end{gathered} \tag{36}$$

This is a *parabolic Anderson model* with the time-dependent scaled stable random potential $-k^{2-d}\, \varrho\, \Gamma^1(kx)\, k^{\varkappa} S_t \varphi\,(x)$. We study its fluctuation behaviour around the heat flow:

Proposition 13 (Limiting fluctuations of w_k). *Under the assumptions of Theorem 2, for any $\varphi \in \mathcal{C}_{\exp}^{+}$ and $t \geq 0$, in P–probability,*

$$\big\langle \mu,\, k^{\varkappa} S_t \varphi - w_k(t,\cdot) \big\rangle \;\underset{k\uparrow\infty}{\longrightarrow}\; \overline{c} \Big\langle \mu, \int_0^t dr\; S_r\big((S_{t-r}\varphi)^{1+\gamma}\big) \Big\rangle,$$

where the constant $\overline{c} = \overline{c}(\gamma, \varrho)$ is given by

$$\overline{c} \;:=\; \varrho^{\gamma}\, \frac{(d-2)\pi^{d/2}}{G(d/2)}\; \mathcal{E}_{\imath} \Big(\int_0^{\infty} ds\; \vartheta_1(W_s) \Big)^{\gamma}, \tag{37}$$

where $\imath$ is any point on the unit sphere of $\mathbb{R}^d$.

To see how the case $n = 1$ of (10) follows from Proposition 13, we fix a sample Γ. For $\varphi \in \mathcal{C}_{\exp}^{+}$, we use the abbreviation

$$\varphi_k(x) \ := \ \varphi(x/k) \quad \text{for} \ \ k>0, \ \ x\in\mathbb{R}^d. \tag{38}$$

Formulas (4) and (6) give

$$\begin{aligned}
&\log\,\mathbb{E}_{\mu_k}\exp\Big[k^{\varkappa}\big(\langle X_t^k,-\varphi\rangle - \langle S_t\mu,-\varphi\rangle\big)\Big]\\
&\quad = \ \log\mathbb{E}_{\mu_k}\exp\Big[\langle X_{k^2t}\,,-k^{\varkappa-d}\varphi_k\rangle + k^{\varkappa}\langle S_t\mu,\varphi\rangle\Big] \\
&\quad = \ -\big\langle \mu_k, v_k(k^2t)\big\rangle + k^{\varkappa}\langle S_t\mu,\varphi\rangle \ = \ \langle \mu,\,k^{\varkappa}S_t\varphi\rangle - \big\langle \mu_k^k,\,k^d\,v_k(k^2t,k\,\cdot\,)\big\rangle,
\end{aligned} \tag{39}$$

with v_k the mild solution to (5) with initial condition $v_k(0) = k^{\varkappa-d}\varphi_k$. Setting

$$u_k(t,x) \ := \ k^d\,v_k(k^2t,kx) \quad \text{for} \ \ t\geq 0, \ \ x\in\mathbb{R}^d, \tag{40}$$

u_k solves

$$u_k(t,x) \ = \ k^{\varkappa}S_t\varphi\,(x) \ - \ k^{2-d}\varrho\int_0^t ds\ S_s\big(\Gamma^1(k\,\cdot\,)\,u_k^2(t-s,\,\cdot\,)\big)(x). \tag{41}$$

Recall that this can be rewritten in Feynman-Kac form as

$$\begin{aligned}
&k^{\varkappa}S_t\varphi\,(x) - u_k(t,x)\\
&\quad = \ k^{\varkappa}\mathcal{E}_x\varphi(W_t)\bigg(1-\exp\Big[-k^{2-d}\varrho\int_0^t ds\ \Gamma^1(kW_s)\,u_k(t-s,W_s)\Big]\bigg).
\end{aligned} \tag{42}$$

Using $u_k(t-s,W_s) \leq k^{\varkappa}S_{t-s}\varphi\,(W_s)$ in (42), and the Feynman-Kac representation

$$w_k(t,x) \ := \ k^{\varkappa}\mathcal{E}_x\varphi(W_t)\exp\Big[-k^{2-d}\varrho\int_0^t ds\ \Gamma^1(kW_s)\,k^{\varkappa}S_{t-s}\varphi\,(W_s)\Big], \tag{43}$$

we arrive at

$$0 \ \leq \ k^{\varkappa}S_t\varphi\,(x) - u_k(t,x) \ \leq \ k^{\varkappa}S_t\varphi\,(x) - w_k(t,x). \tag{44}$$

Hence, the case $n=1$ of (10) follows from Proposition 13.

At this point we make the following easy observation on the right hand side of (44), which follows immediately from (43).

Lemma 14 (**Monotone dependence on** φ). *For the solution* w_k *of* (36) *we have that* $\varphi \mapsto k^{\varkappa}S_t\varphi\,(x) - w_k(t,x)$ *is non-decreasing.*

Proposition 13 is proved in two steps: In Section 3.2 we show that the expectations converge, and in Section 3.3 that the variances vanish asymptotically. For this we fix $t>0$ and $\varphi\in\mathcal{C}_{\exp}^{+}$.

3.2 Convergence of expectations

Proposition 15 (Convergence of expectations). *Let* $\varkappa = \varkappa_{\mathrm{c}}$. *There exists a* $\lambda_{15} > 0$ *such that for every* $\varepsilon > 0$ *there is a* $k_{15} = k_{15}(\varepsilon) > 0$ *with*

$$\Big|\mathsf{E}\big(k^{\varkappa} S_t\varphi(x) - w_k(t,x)\big) - \ \overline{c}\int_0^t dr\ S_r(S_{t-r}\varphi)^{1+\gamma}(x)\Big| \ \leq\ \varepsilon\phi_{\gamma\lambda_{15}}(x)$$

for $x \in \mathbb{R}^d$, $k \geq k_{15}$, *where the constant* $\overline{c}$ *is as in* (37).

We now show how Theorem 1 follows from this proposition. Turning back to the situation $\varkappa < \varkappa_{\mathrm{c}}$, for both parts of the theorem it suffices to show that in P-probability, for all $\varepsilon > 0$,

$$\mathbb{P}_{\mu_k}\Big(\ \Big|k^{\varkappa}\langle X_t^k - S_t\mu,\, -\varphi\rangle\Big| > \varepsilon\Big) \underset{k\uparrow\infty}{\longrightarrow} 0.$$

Abbreviate $\xi_k := \langle X_t^k - S_t\mu,\, -\varphi\rangle$. Given $\delta > 0$, we take $C > 1$ and estimate

$$\begin{aligned}&\mathsf{P}\Big(\mathbb{P}_{\mu_k}\big(|k^{\varkappa}\xi_k| > \varepsilon\big) > \delta\Big)\\ &\leq\ \mathsf{P}\Big(\mathbb{E}_{\mu_k}\mathrm{e}^{k^{\varkappa_{\mathrm{c}}}\xi_k} > C\Big) + \mathsf{P}\Big(\mathbb{P}_{\mu_k}\big(|k^{\varkappa}\xi_k| > \varepsilon\big) > \delta,\ \mathbb{E}_{\mu_k}\mathrm{e}^{k^{\varkappa_{\mathrm{c}}}\xi_k} \leq C\Big).\end{aligned} \tag{45}$$

We first show that C can be chosen such that the first term in (45) is small for all k. Note from (39) and (44) that

$$0\ \leq\ \log\,\mathbb{E}_{\mu_k}\mathrm{e}^{k^{\varkappa_{\mathrm{c}}}\xi_k}\ =\ \big\langle\mu,\, k^{\varkappa_{\mathrm{c}}}S_t\varphi - u_k(t,\,\cdot\,)\big\rangle\ \leq\ \big\langle\mu,\, k^{\varkappa_{\mathrm{c}}}S_t\varphi - w_k(t,\,\cdot\,)\big\rangle.$$

By Proposition 15, the E-expectation of the right hand term remains bounded in k. By Chebyshev's inequality,

$$\mathsf{P}\Big(\mathbb{E}_{\mu_k}\mathrm{e}^{k^{\varkappa_{\mathrm{c}}}\xi_k} > C\Big)\ \leq\ \frac{1}{\log C}\,\sup_k \mathsf{E}\log\,\mathbb{E}_{\mu_k}\mathrm{e}^{k^{\varkappa_{\mathrm{c}}}\xi_k},$$

hence the first term in (45) can be made arbitrarily small, uniformly in k. For the second term, we first observe

$$\mathbb{P}_{\mu_k}(k^{\varkappa}\xi_k > \varepsilon)\ =\ \mathbb{P}_{\mu_k}\big(k^{\varkappa_{\mathrm{c}}}\xi_k > \varepsilon k^{\varkappa_{\mathrm{c}}-\varkappa}\big)\ \leq\ \mathrm{e}^{-\varepsilon k^{\varkappa_{\mathrm{c}}-\varkappa}}\,\mathbb{E}_{\mu_k}\mathrm{e}^{k^{\varkappa_{\mathrm{c}}}\xi_k}, \tag{46}$$

and therefore, for sufficiently large k,

$$\mathsf{P}\Big(\mathbb{P}_{\mu_k}\big(k^{\varkappa}\xi_k > \varepsilon\big) > \tfrac{\delta}{2},\ \mathbb{E}_{\mu_k}\mathrm{e}^{k^{\varkappa_{\mathrm{c}}}\xi_k} \leq C\Big)\ \leq\ \mathsf{P}\Big(C\,\mathrm{e}^{-\varepsilon k^{\varkappa_{\mathrm{c}}-\varkappa}} > \tfrac{\delta}{2}\Big)\ =\ 0. \tag{47}$$

On the other hand, on the event $\mathbb{E}_{\mu_k}\mathrm{e}^{k^{\varkappa_{\mathrm{c}}}\xi_k} \leq C$, by (46) we have

$$\mathbb{E}_{\mu_k}\{k^{\varkappa}\xi_k;\ \xi_k \geq 0\}\ =\ \int_0^\infty \mathbb{P}_{\mu_k}(k^{\varkappa}\xi_k > y)\,\mathrm{d}y\ \leq\ C\int_0^\infty \mathrm{e}^{-yk^{\varkappa_{\mathrm{c}}-\varkappa}}\,\mathrm{d}y\ =\ C\,k^{-(\varkappa_{\mathrm{c}}-\varkappa)}.$$

Since $\mathbb{E}_{\mu_k}k^{\varkappa}\xi_k = 0$, the latter implies

$$-\,\mathbb{E}_{\mu_k}\{k^{\varkappa}\xi_k;\ \xi_k \leq 0\}\ =\ \mathbb{E}_{\mu_k}\{k^{\varkappa}\xi_k;\ \xi_k \geq 0\}\ \leq\ C\,k^{-(\varkappa_{\mathrm{c}}-\varkappa)},$$

and therefore, by Chebychev's inequality,

$$\mathbb{P}_{\mu_k}\big(k^{\varkappa}\xi_k < -\varepsilon\big) \;\le\; -\varepsilon^{-1}\,\mathbb{E}_{\mu_k}\{k^{\varkappa}\xi_k;\ \xi_k \le 0\} \;\le\; \varepsilon^{-1}\,C\,k^{-(\varkappa_{\rm c}-\varkappa)}.$$

From this we get, for sufficiently large k,

$$\mathsf{P}\Big(\mathbb{P}_{\mu_k}\big(k^{\varkappa}\xi_k < -\varepsilon\big) > \tfrac{\delta}{2}\,,\ \mathbb{E}_{\mu_k}\mathrm{e}^{k^{\varkappa_{\rm c}}\xi_k} \le C\Big) \;\le\; \mathsf{P}\Big(\varepsilon^{-1}\,C\,k^{-(\varkappa_{\rm c}-\varkappa)} > \tfrac{\delta}{2}\Big) \;=\; 0. \tag{48}$$

Combining (47) and (48), we see that the second term in (45) disappears. This completes the proof of Theorem 1.

The rest of this section is devoted to the proof of Proposition 15. From now on we assume $\varkappa = \varkappa_{\rm c}$, which is defined in (7). The proof is prepared by six lemmas. In all these lemmas, $\tau = \tau^{y}_{1/k}[W]$ denotes the first hitting time of the ball $B(y,1/k)$ by the Brownian motion W, and π_x the law of $\tau^{y}_{1/k}[W]$ if W is started in x.

Lemma 16. *There exists a constant $c_{16} > 0$ such that*

$$\begin{aligned} k^{d-2}\int\!dy\ \mathcal{E}_x 1_{\tau\le t}\,\mathcal{E}_{W_\tau}\varphi(\tilde W_{t-\tau})\Big(k^2\int_{M/k^2}^{\infty} ds\ \vartheta_{1/k}(\tilde W_s - y)\,\phi_{\lambda_7}(y)\Big)^{\gamma} \\ \le\ c_{16}M^{\gamma(1-d/2)}\phi_{\gamma\lambda_7}(x) \quad \textit{for}\ M>1,\ \ k>0,\ \ x\in\mathbb{R}^d. \end{aligned}$$

Proof. Note that, for any $\iota \in \partial B(0,1)$, by Brownian scaling,

$$\begin{aligned} \mathcal{E}_{\iota/k}\,k^2\int_{M/k^2}^{\infty} ds\ \vartheta_{1/k}(W_s) \;&=\; \mathcal{E}_{\iota}\int_{M}^{\infty} ds\ \vartheta_1(W_s) \\ &=\; \int_M^\infty ds\ \mathcal{P}_\iota\big(|W_s|\le 1\big) \;\le\; \int_{|y|\le 1} dy\int_M^\infty ds\ p_s(y) \;\le\; c\,M^{1-d/2}. \end{aligned} \tag{49}$$

We now use $\varphi \le c$, Jensen's inequality, (49), (29), and (21), to get

$$\begin{aligned} & k^{d-2}\int\!dy\ \mathcal{E}_x 1_{\tau\le t}\,\mathcal{E}_{W_\tau}\varphi(\tilde W_{t-\tau})\Big(k^2\int_{M/k^2}^{\infty} ds\ \vartheta_{1/k}(\tilde W_s - y)\,\phi_{\lambda_7}(y)\Big)^{\gamma} \\ &\le\ c\,k^{d-2}\int\!dy\ \phi_{\gamma\lambda_7}(y)\,\mathcal{E}_x 1_{\tau\le t}\,\mathcal{E}_{\iota/k}\Big(k^2\int_{M/k^2}^{\infty} ds\ \vartheta_{1/k}(\tilde W_s)\Big)^{\gamma} \\ &\le\ c\,M^{\gamma(1-d/2)}\int\!dy\ \phi_{\gamma\lambda_7}(y)\,\Big[|x-y|^{2-d}+1\Big]\exp\!\big[-\lambda_{(29)}|x-y|^2\big] \\ &\le\ c\,M^{\gamma(1-d/2)}\phi_{\gamma\lambda_7}(x). \end{aligned}$$

This is the required statement. ☐

Lemma 17. *For every $\delta > 0$, there exists a constant $c_{17} = c_{17}(\delta) > 0$ such that*

$$\mathcal{E}_x\varphi(W_t)\Big[\int\!dy\,\Big(\int_0^t ds\ \vartheta_1(kW_s - y)\,S_{t-s}\varphi\,(W_s)\Big)^{\gamma}\Big]^2 \;\le\; c_{17}\,k^{4-4\gamma+\delta}\phi_{\gamma\lambda_7}(x),$$

for all $x\in\mathbb{R}^d$ and $k\ge 1$.

Proof. Using Brownian scaling in the second, substitution and (23) in the last step, we estimate,

$$\begin{aligned}
&\mathcal{E}_x\varphi(W_t)\Big[\int dy\,\Big(\int_0^t ds\ \vartheta_1(kW_s-y)\,S_{t-s}\varphi\,(W_s)\Big)^\gamma\Big]^2\\
&\le\ \|\varphi\|_\infty\iint dy_1dy_2\ \mathcal{E}_x\prod_{i=1}^2\Big(\int_0^t ds\ \vartheta_1(kW_s-y_i)\,S_{t-s}\varphi\,(W_s)\Big)^\gamma\\
&=\ \|\varphi\|_\infty\iint dy_1dy_2\ \mathcal{E}_0\prod_{i=1}^2\Big(\int_0^t ds\ \vartheta_1(W_{k^2s}+kx-y_i)\,S_{t-s}\varphi\,(\tfrac{1}{k}W_{k^2s}+x)\Big)^\gamma\\
&\le\ k^{-4\gamma}\,\|\varphi\|_\infty\iint dy_1dy_2\ \mathcal{E}_0\prod_{i=1}^2\Big(\int_0^{k^2t} ds\ \vartheta_1(W_s-y_i)\,\tilde{\phi}(y_i/k+x)\Big)^\gamma. \qquad (50)
\end{aligned}$$

To study the double integral, denote by τ^1, τ^2 the first hitting times of the balls $B(y_1,1)$ respectively $B(y_2,1)$ by the Brownian path W. Pick $p>1$ such that $2d+2(2-d)/p<4+\delta$, and q such that $1/p+1/q=1$. By Hölder's inequality,

$$\begin{aligned}
\mathcal{E}_0\prod_{i=1}^2\Big(\int_0^{k^2t} ds\ \vartheta_1(W_s-y_i)\,\tilde{\phi}(y_i/k+x)\Big)^\gamma\ &\le\ \big[\mathcal{P}_0\big(\tau^1<k^2t,\ \tau^2<k^2t\big)\big]^{1/p}\\
&\quad\times\ \Big[\mathcal{E}_0\prod_{i=1}^2\Big(\int_0^\infty ds\ \vartheta_1(W_s-y_i)\,\tilde{\phi}(y_i/k+x)\Big)^{\gamma q}\Big]^{1/q}.
\end{aligned}$$

For the second factor on the right hand side we get, using Cauchy-Schwarz, and the maximum principle to pass from y_i to 0,

$$\begin{aligned}
&\Big[\mathcal{E}_0\prod_{i=1}^2\Big(\int_0^\infty ds\ \vartheta_1(W_s-y_i)\,\tilde{\phi}(y_i/k+x)\Big)^{\gamma q}\Big]^{1/q}\\
&\le\ \prod_{i=1}^2\Big(\mathcal{E}_0\Big(\int_0^\infty ds\ \vartheta_1(W_s-y_i)\,\tilde{\phi}(y_i/k+x)\Big)^{2\gamma q}\Big)^{1/2q}\\
&\le\ \tilde{\phi}^\gamma(y_1/k+x)\,\tilde{\phi}^\gamma(y_2/k+x)\,\Big(\mathcal{E}_0\Big(\int_0^\infty ds\ \vartheta_1(W_s)\Big)^{2\gamma q}\Big)^{1/q}.
\end{aligned}$$

Recall from Lemma 12(a) that the total occupation times of Brownian motion in the unit ball in $d\ge 3$ have moments of all orders. Hence, the latter expectation is finite.

By (31) using substitution in the y-variables,

$$\begin{aligned}
&\iint dy_1dy_2\ \tilde{\phi}^\gamma(y_1/k+x)\,\tilde{\phi}^\gamma(y_2/k+x)\ \big[\mathcal{P}_0\big(\tau^1<k^2t,\ \tau^2<k^2t\big)\big]^{1/p}\\
&\le\ c_{(31)}^{1/p}\,k^{2d+2(2-d)/p}\int dy_1\,\tilde{\phi}^\gamma(y_1+x)\big(|y_1|^{2-d}+1\big)^{1/p}\exp\big[-\lambda_{(31)}|y_1|^2/p\big]\\
&\qquad\qquad\times\int dy_2\,\tilde{\phi}^\gamma(y_2+x)\big(|y_2|^{2-d}+1\big)^{1/p}\exp\big[-\lambda_{(31)}|y_2|^2/p\big]\\
&\le\ c\,k^{4+\delta}\phi_{\gamma\lambda_7}(x), \qquad (51)
\end{aligned}$$

using (21) in the last step. Plugging (51) into (50) completes the proof. $\square$

Lemma 18. *For all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ and $k_{18} = k_{18}(\varepsilon) > 0$, such that*

$$k^{d-2} \int dy\ \mathcal{E}_x \mathbf{1}_{t-\delta\le\tau\le t}\, \mathcal{E}_{W_\tau}\Big(k^2 \int_0^{t-\tau} ds\ \vartheta_{1/k}(\tilde{W}_s - y)\, S_{t-\tau-s}\varphi\,(\tilde{W}_s)\Big)^\gamma$$
$$\le\ \varepsilon\phi_{\gamma\lambda_7}(x) \quad for\ \ k \ge k_{18}\ \ and\ \ x \in \mathbb{R}^d.$$

Proof. For any $\delta, M > 0$ we have,

$$k^{d-2} \int dy\ \mathcal{E}_x \mathbf{1}_{t-\delta\le\tau\le t}\, \mathcal{E}_{W_\tau}\Big(k^2 \int_0^{t-\tau} ds\ \vartheta_{1/k}(\tilde{W}_s - y)\, S_{t-\tau-s}\varphi\,(\tilde{W}_s)\Big)^\gamma$$
$$\le\ k^{d-2} \int dy\ \phi_{\gamma\lambda_7}(y)\, \mathcal{E}_x \mathbf{1}_{t-\delta\le\tau\le t}\, \mathcal{E}_{W_\tau}\Big(k^2 \int_0^{M/k^2} ds\ \vartheta_{1/k}(\tilde{W}_s - y)\Big)^\gamma \tag{52a}$$
$$+\ k^{d-2} \int dy\ \phi_{\gamma\lambda_7}(y)\, \mathcal{E}_x \mathbf{1}_{\tau\le t}\, \mathcal{E}_{W_\tau}\Big(k^2 \int_{M/k^2}^{\infty} ds\ \vartheta_{1/k}(\tilde{W}_s - y)\Big)^\gamma. \tag{52b}$$

We look at (52b) and choose M such that this term is small. Indeed, the inner expectation in (52b) can be made arbitrarily small (simultaneously for all k and y) by choice of M. Hence we can use (29) to see that this term can be bounded by $\varepsilon\phi_{\gamma\lambda_7}(x)$, for all sufficiently large k, by choice of M (and independently of δ).

We look at (52a) and choose $\delta > 0$ such that

$$c_{(30)} M^\gamma \int_{t-\delta}^t ds \int dy\ \phi_{\gamma\lambda_7}(y)\, p_s(y-x)\ <\ \varepsilon\phi_{\gamma\lambda_7}(x). \tag{53}$$

The term (52a) can be bounded from above by

$$M^\gamma k^{d-2} \int dy\ \phi_{\gamma\lambda_7}(y)\, \pi_x[t-\delta, t]. \tag{54}$$

By (30) there exists $A \subset \mathbb{R}^d$ and $k_{18} \ge 0$ such that, for all $x - y \in A$ and $k \ge k_{18}$,

$$k^{d-2}\pi_x[t-\delta, t]\ -\ c_{(30)} \int_{t-\delta}^t ds\, p_s(y-x)\ <\ \varepsilon \int_0^t ds\, p_s(y-x)$$

and

$$\int_{A^c} dz\, \big[|z|^{2-d} + 1\big] \exp\big[\lambda_7|z| - \lambda_{(29)}|z|^2\big]\ <\ \varepsilon. \tag{55}$$

We can thus bound (54), for all $k \ge k_{18}$ and $x \in \mathbb{R}^d$ by

$$M^\gamma k^{d-2} \int dy\ \phi_{\gamma\lambda_7}(y)\, \pi_x[t-\delta, t]\ \le\ c_{(30)} M^\gamma \int_{x+A} dy\ \phi_{\gamma\lambda_7}(y) \int_{t-\delta}^t ds\, p_s(y-x)$$
$$+\ \varepsilon\, M^\gamma \int dy\ \phi_{\gamma\lambda_7}(y) \int_0^t ds\, p_s(y-x)\ +\ M^\gamma \int_{x+A^c} dy\ \phi_{\gamma\lambda_7}(y)\ k^{d-2}\pi_x[0,t].$$

By (53) the first term is bounded by $\varepsilon\phi_{\gamma\lambda_7}(x)$, as is the second term. For the last term we use the upper bound (29) for $k^{d-2}\pi_x[0,t]$ and then (55) to see the upper bound of $\varepsilon\phi_{\gamma\lambda_7}(x)$. $\square$

Lemma 19. *For every $M > 1$ and $\varepsilon > 0$, there exists a $k_{19} = k_{19}(M, \varepsilon) > 0$ such that*

$$k^{d-2} \int dy\ \mathcal{E}_x 1_{\tau \le t}\, \mathcal{E}_{W_\tau} \varphi(\tilde{W}_{t-\tau}) \Big| \Big(k^2 \int_0^{M/k^2} ds\, \vartheta_{1/k}(\tilde{W}_s - y) S_{t-\tau-s}\varphi(\tilde{W}_s)\Big)^\gamma$$
$$- \Big(k^2 \int_0^{M/k^2} ds\, \vartheta_{1/k}(\tilde{W}_s - y) S_{t-\tau}\varphi(y)\Big)^\gamma \Big| \ \le\ \varepsilon \phi_{\gamma\lambda_7}(x) \quad \text{for } k \ge k_{19},\ \ x \in \mathbb{R}.$$

Proof. Recall that $|a^\gamma - b^\gamma| \le |a-b|^\gamma$. We use (22) to choose $k_{19} > 1/M$ such that

$$\big|S_r\varphi(x) - S_s\varphi(y)\big| \le \varepsilon^{1/\gamma} \quad \text{if } |r-s| \le M/k_{19}^2,\ \ |x-y| \le 1/k_{19}.$$

Hence, for all $k \ge k_{19}$ and $x \in \mathbb{R}^d$,

$$\begin{aligned}
& k^{d-2} \int dy\ \mathcal{E}_x 1_{\tau \le t} \mathcal{E}_{W_\tau} \varphi(\tilde{W}_{t-\tau}) \Big| \Big(k^2 \int_0^{M/k^2} ds\, \vartheta_{1/k}(\tilde{W}_s - y) S_{t-\tau-s}\varphi(\tilde{W}_s)\Big)^\gamma \\
&\qquad\qquad - \Big(k^2 \int_0^{M/k^2} ds\, \vartheta_{1/k}(\tilde{W}_s - y) S_{t-\tau}\varphi(y)\Big)^\gamma \Big| \\
&\le\ k^{d-2} \int dy\ \mathcal{E}_x 1_{\tau \le t}\, \mathcal{E}_{W_\tau} \varphi(\tilde{W}_{t-\tau}) \Big(k^2 \int_0^{M/k^2} ds\, \vartheta_{1/k}(\tilde{W}_s - y) \\
&\qquad\qquad \times \big|S_{t-\tau-s}\varphi(\tilde{W}_s) - S_{t-\tau}\varphi(y)\big|\Big)^\gamma \\
&\le\ \varepsilon k^{d-2} \int dy\ \mathcal{E}_x 1_{\tau \le t}\, \mathcal{E}_{W_\tau} \varphi(\tilde{W}_{t-\tau}) \Big(k^2 \int_0^{M/k^2} ds\, \vartheta_{1/k}(\tilde{W}_s - y)\Big)^\gamma.
\end{aligned}$$

To complete the proof use Cauchy-Schwarz, (23), (29), and (21), to get

$$\begin{aligned}
& k^{d-2} \int dy\ \mathcal{E}_x 1_{\tau \le t}\, \mathcal{E}_{W_\tau} \varphi(\tilde{W}_{t-\tau}) \Big(k^2 \int_0^{M/k^2} ds\ \vartheta_{1/k}(\tilde{W}_s - y)\Big)^\gamma \\
&\le\ k^{d-2} \int dy\ \mathcal{E}_x 1_{\tau \le t} \big(\mathcal{E}_{W_\tau} \varphi^2(\tilde{W}_{t-\tau})\big)^{1/2} \\
&\qquad\qquad \times \Big[\mathcal{E}_{W_\tau} \Big(k^2 \int_0^{M/k^2} ds\ \vartheta_{1/k}(\tilde{W}_s - y)\Big)^{2\gamma}\Big]^{1/2} \\
&\le\ c\, k^{d-2} \int dy\ \phi_{\lambda_7}(y)\, \mathcal{E}_x 1_{\tau \le t} \Big[\mathcal{E}_0 \Big(\int_0^\infty ds\ \vartheta_1(W_s)\Big)^{2\gamma}\Big]^{1/2} \\
&\le\ c \int dy\ \phi_{\lambda_7}(y) \Big[|x-y|^{2-d} + 1\Big] \exp\big[-\lambda_{(29)}|x-y|^2\big] \\
&\le\ c\, \phi_{\gamma\lambda_7}(x).
\end{aligned}$$

This gives the required statement (renaming ε). □

Lemma 20. *Let $M > 1$ and $c_{20} := \mathcal{E}_\iota \big\{(\int_0^M ds\ \vartheta_1(W_s))^\gamma\big\}$ for $\iota \in \partial B(0,1)$. For every $\varepsilon > 0$ there exists a $k_{20} = k_{20}(\varepsilon, M) > 0$ such that, for all $k \ge k_{20}$ and $x \in \mathbb{R}^d$,*

$$\begin{aligned}
& k^{d-2} \int dy\ \mathcal{E}_x 1_{\tau \le t} \big(S_{t-\tau}\varphi(y)\big)^\gamma \\
& \times \Big| \mathcal{E}_{W_\tau} \Big(k^2 \int_0^{M/k^2} ds\, \vartheta_{1/k}(\tilde{W}_s - y)\Big)^\gamma \mathcal{E}_{\tilde{W}_{M/k^2}} \varphi(\tilde{\tilde{W}}_{t-\tau-M/k^2}) - c_{20} S_{t-\tau}\varphi(y) \Big| \\
& \le\ \varepsilon \phi_{\gamma\lambda_7}(x).
\end{aligned}$$

Proof. In a first step we note that, by Brownian scaling, for fixed W_τ,

$$\begin{aligned}
&\mathcal{E}_{W_\tau}\Big(k^2\int_0^{M/k^2} ds\,\vartheta_{1/k}(\tilde W_s-y)\Big)^\gamma \mathcal{E}_{\tilde W_{M/k^2}}\varphi(\tilde{\tilde W}_{t-\tau-M/k^2})\\
&\quad=\ \mathcal{E}_0\Big(k^2\int_0^{M/k^2} ds\,\vartheta_{1/k}(\tfrac{1}{k}\tilde W_{sk^2}-y+W_\tau)\Big)^\gamma \mathcal{E}_{W_\tau+\frac1k\tilde W_M}\varphi(\tilde{\tilde W}_{t-\tau-M/k^2}).
\end{aligned}$$

The main contribution to the $\mathcal{E}_0$-expectation is coming from the paths $\tilde W$ with $|\tilde W_M|\le\sqrt{k}$. Indeed, the remaining part of the integral can be estimated by a constant multiple of $M^\gamma\mathcal{P}_0\big\{|\tilde W_M|>\sqrt{k}\big\}$. Therefore we can estimate, with constants c depending on M,

$$\begin{aligned}
&k^{d-2}\int dy\ \ \mathcal{E}_x 1_{\tau\le t}\left(S_{t-\tau}\varphi\,(y)\right)^\gamma\ M^\gamma\,\mathcal{P}_0\big(|\tilde W_M|>\sqrt{k}\big)\\
&\le\ c\,\mathrm{e}^{-k/2M}\,k^{d-2}\int dy\ \ \mathcal{P}_x(\tau\le t)\Big(\sup_{r\le t}S_r\varphi\,(y)\Big)^\gamma\\
&\le\ c\,\mathrm{e}^{-k/2M}\int dy\ \phi_{\gamma\lambda_7}(y)\big[|x-y|^{2-d}+1\big]\exp\big[-\lambda_{(29)}|x-y|^2\big]\\
&\le\ \varepsilon\phi_{\gamma\lambda_7}(x),
\end{aligned}$$

for sufficiently large values of k, recalling (29), (23), and (21).

Coming back to the main contribution, we use (22) to choose k large enough such that

$$\big|S_r\varphi\,(w+z)-S_s\varphi\,(y)\big|\le\varepsilon\quad\text{if }|r-s|\le M/k^2,\ |z|\le 1/\sqrt{k},\ |w-y|\le 1/k.$$

Using this with $z=\frac1k\tilde W_M$ and $w=W_\tau$, by (23),

$$\begin{aligned}
&k^{d-2}\int dy\ \ \mathcal{E}_x 1_{\tau\le t}\left(S_{t-\tau}\varphi\,(y)\right)^\gamma\mathcal{E}_0 1_{|\tilde W_M|<\sqrt{k}}\Big(k^2\int_0^{M/k^2} ds\,\vartheta_{1/k}(\tilde W_s-y)\Big)^\gamma\\
&\qquad\qquad\times\Big|\mathcal{E}_{W_\tau+\frac1k\tilde W_M}\varphi(\tilde{\tilde W}_{t-\tau-M/k^2})-\mathcal{E}_y\varphi(\tilde{\tilde W}_{t-\tau})\Big|\\
&\le\ \varepsilon k^{d-2}\int dy\ \ \phi_{\gamma\lambda_7}(y)\,\mathcal{E}_x 1_{\tau\le t}\,\mathcal{E}_{W_\tau}\Big(k^2\int_0^{M/k^2} ds\,\vartheta_{1/k}(\tilde W_s-y)\Big)^\gamma\\
&\le\ \varepsilon\int dy\ \ \phi_{\gamma\lambda_7}(y)k^{d-2}\mathcal{P}_x\left(\tau\le t\right)\ \mathcal{E}_0\Big(\int_0^\infty ds\,\vartheta_1(\tilde W_s)\Big)^\gamma.
\end{aligned}$$

The last line is $\le\varepsilon\,c\,\phi_{\gamma\lambda_7}(x)$ by (29) and (21).

Now it remains to observe that, by Brownian scaling,

$$\begin{aligned}
&k^{d-2}\int dy\ \ \mathcal{E}_x 1_{\tau\le t}\left(S_{t-\tau}\varphi\,(y)\right)^\gamma\mathcal{E}_{W_\tau}\Big(k^2\int_0^{M/k^2} ds\,\vartheta_{1/k}(\tilde W_s-y)\Big)^\gamma\mathcal{E}_y\varphi(\tilde{\tilde W}_{t-\tau})\\
&\quad=\ k^{d-2}\int dy\ \mathcal{E}_x 1_{\tau\le t}\left(S_{t-\tau}\varphi\,(y)\right)^{1+\gamma}\mathcal{E}_{kW_\tau}\Big(\int_0^M ds\,\vartheta_1(\tilde W_s-y)\Big)^\gamma.
\end{aligned}$$

For $y\notin B(x,1/k)$ the inner expectation is constant and equals c_{20}. The contribution coming from $y\in B(x,1/k)$ is very easily seen to be bounded by a constant multiple of $k^{-2}\phi_{\gamma\lambda_7}(x)$. This completes the proof. □

The following lemma is at the heart of our proof of Proposition 15. Recall that π_x denotes the law of $\tau = \tau^y_{1/k}[W]$ for W starting in x.

Lemma 21 (A hitting time statement). *For every $\varepsilon > 0$ there exists a $k_{21} = k_{21}(\varepsilon) > 0$ such that*

$$\Big| k^{d-2} \int dy \int_0^t \pi_x(ds) \big(S_{t-s}\varphi(y)\big)^{1+\gamma} - c_{(30)} \int_0^t ds\ S_s(S_{t-s}\varphi)^{1+\gamma}(x) \Big| \\ \leq\ \varepsilon \phi_{\lambda_7}(x) \quad \textit{for } x \in \mathbb{R}^d,\ \ k \geq k_{21}\,.$$

Proof. Fix $\varepsilon > 0$. Recall that $(s,y) \mapsto S_s\varphi(y)$ is uniformly continuous and bounded, and that there exists $R > 0$ (dependent on ε) such that $\big(S_s\varphi(y)\big)^{1+\gamma} \leq \varepsilon\phi_{\lambda_7}(y)$ for all $s \leq t$, $|y| > R$. We can therefore choose $0 = t_0 \leq \cdots \leq t_n = t$ such that, for all $t_j \leq r, s \leq t_{j+1}$ and $y \in \mathbb{R}^d$,

$$\Big| \big(S_{t-s}\varphi(y)\big)^{1+\gamma} - \big(S_{t-r}\varphi(y)\big)^{1+\gamma} \Big| \ \leq\ \varepsilon\phi_{\lambda_7}(y). \tag{57}$$

Using (30) we may find k_{21} such that, for all $k \geq k_{21}$,

$$\Big| k^{d-2}\pi_x[t_j, t_{j+1}] - c_{(30)} \int_{t_j}^{t_{j+1}} ds\, p_s(x,y) \Big| \ <\ \varepsilon\, k^{d-2}\pi_x[t_j, t_{j+1}] \tag{58}$$

for all $0 \leq j \leq n-1$ and all $x - y \in A$, where $A \subset \mathbb{R}^d$ is a set with

$$\int_{A^c} dz\, \big[|z|^{2-d} + 1\big] \exp\big[\lambda_7|z| - \lambda_{(29)}|z|^2\big] \ <\ \varepsilon. \tag{59}$$

Now we show that for all $x \in \mathbb{R}^d$, and $k \geq k_{21}$,

$$k^{d-2} \int dy \int_0^t \pi_x(ds)\big(S_{t-s}\varphi(y)\big)^{1+\gamma} \\ \leq\ c_{(30)} \int_0^t ds\ \big(S_s(S_{t-s}\varphi)^{1+\gamma}\big)(x) + \varepsilon\, \phi_{\lambda_7}(x). \tag{60}$$

Indeed, using (57) and (58), we can estimate

$$k^{d-2} \int dy \int_0^t \pi_x(ds)\big(S_{t-s}\varphi(y)\big)^{1+\gamma}$$

$$\leq\ k^{d-2} \int dy \sum_{j=0}^{n-1} \Big[\big(S_{t-t_j}\varphi(y)\big)^{1+\gamma} + \varepsilon\phi_{\lambda_7}(y)\Big] \pi_x[t_j, t_{j+1}]$$

$$\leq\ c_{(30)} \int dy \sum_{j=0}^{n-1} \Big[\big(S_{t-t_j}\varphi(y)\big)^{1+\gamma} + \varepsilon\phi_{\lambda_7}(y)\Big] \int_{t_j}^{t_{j+1}} ds\, p_s(x,y) \tag{61a}$$

$$+\ \varepsilon \int dy \sum_{j=0}^{n-1} \Big[\big(S_{t-t_j}\varphi(y)\big)^{1+\gamma} + \varepsilon\phi_{\lambda_7}(y)\Big]\, k^{d-2}\pi_x[t_j, t_{j+1}] \tag{61b}$$

$$+\ \int_{x+A^c} dy \sum_{j=0}^{n-1} \Big[\big(S_{t-t_j}\varphi(y)\big)^{1+\gamma} + \varepsilon\phi_{\lambda_7}(y)\Big]\, k^{d-2}\pi_x[t_j, t_{j+1}]. \tag{61c}$$

We give estimates for the two final summands, the error terms. (61b) can be estimated, using (29) and (21), by

$$\begin{aligned}
\varepsilon\!\int\! dy \sum_{j=0}^{n-1}\Big[\big(S_{t-t_j}\varphi\,(y)\big)^{1+\gamma}+\varepsilon\phi_{\lambda_7}(y)\Big]k^{d-2}\pi_x[t_j,t_{j+1}] \;&\le\; \varepsilon c\!\int\! dy\ \phi_{\lambda_7}(y)k^{d-2}\pi_x[0,t]\\
&\le\; \varepsilon c\int dy\ \phi_{\lambda_7}(y)\big[|x-y|^{2-d}+1\big]\exp\big[-\lambda_{(29)}|x-y|^2\big]\\
&\le\; \varepsilon c\,\phi_{\lambda_7}(x).
\end{aligned}$$

The error term (61c) can be estimated as follows,

$$\begin{aligned}
&\int_{x+A^c} dy \sum_{j=0}^{n-1}\Big[\big(S_{t-t_j}\varphi\,(y)\big)^{1+\gamma}+\varepsilon\phi_{\lambda_7}(y)\Big]\,k^{d-2}\pi_x[t_j,t_{j+1}]\\
&\quad\le\; c_{(29)}\int_{x+A^c} dy\ \phi_{\lambda_7}(y)\big[|x-y|^{2-d}+1\big]\exp\big[-\lambda_{(29)}|x-y|^2\big]\\
&\quad\le\; c\,\phi_{\lambda_7}(x)\int_{x+A^c} dy\ \big[|x-y|^{2-d}+1\big]\exp\big[\lambda_7|x-y|-\lambda_{(29)}|x-y|^2\big],
\end{aligned}$$

and the integral is smaller than ε by (59). For the first summand, the main term (61a), we argue that

$$\begin{aligned}
&\int\! dy \sum_{j=0}^{n}\Big[\big(S_{t-t_j}\varphi\,(y)\big)^{1+\gamma}+\varepsilon\phi_{\lambda_7}(y)\Big]\int_{t_j}^{t_{j+1}} ds\,p_s(x,y)\\
&\quad\le\; \int\! dy\int_0^t ds\ \big[(S_{t-s}\varphi\,(y))^{1+\gamma}+2\varepsilon\,\phi_{\lambda_7}(y)\big]p_s(x,y)\\
&\quad\le\; \int_0^t ds\,S_s\big((S_{t-s}\varphi\,(x))^{1+\gamma}\big)+2\varepsilon\int\! dy\ \phi_{\lambda_7}(y)\int_0^t ds\,p_s(x,y).
\end{aligned}$$

The last summand is again bounded by a constant multiple of $\varepsilon\phi_{\lambda_7}(x)$. Hence we have verified (60) and by the analogous argument one can see that, for all $k\ge k_{21}$ and $x\in\mathbb{R}^d$,

$$k^{d-2}\int dy\int_0^t \pi_x(ds)\big(S_{t-s}\varphi\,(y)\big)^{1+\gamma}\;\ge\; c_{(30)}\int_0^t ds\ \big(S_s(S_{t-s}\varphi)^{1+\gamma}\big)(x)-\varepsilon\,\phi_{\lambda_7}(x).$$

This completes the proof. $\square$

Proof of Proposition 15. Recall from (43) that

$$\begin{aligned}
&\mathsf{E}\big(k^{\varkappa}S_t\varphi\,(x)-w_k(t,x)\big)\\
&\quad=\; k^{\varkappa}\,\mathsf{E}\mathcal{E}_x\varphi(W_t)\bigg(1-\exp\Big[-k^{2-d+\varkappa}\varrho\int_0^t ds\ \Gamma^1(kW_s)\,S_{t-s}\varphi\,(W_s)\Big]\bigg).
\end{aligned}$$

We use (1) to evaluate the expectation with respect to the medium.

$$\begin{aligned}
&\mathsf{E}k^{\varkappa}\mathcal{E}_x\varphi(W_t)\bigg(1-\exp\Big[-k^{2-d+\varkappa}\varrho\int_0^t ds\ \Gamma^1(kW_s)\,S_{t-s}\varphi\,(W_s)\Big]\bigg) \qquad (62)\\
&=\; k^{\varkappa}\mathcal{E}_x\varphi(W_t)\\
&\quad\times\bigg(1-\exp\Big[-k^{(2+\varkappa-d)\gamma}\varrho^{\gamma}\int\! dy\Big(\int_0^t ds\ \vartheta_1(kW_s-y)\,S_{t-s}\varphi\,(W_s)\Big)^{\gamma}\Big]\bigg).
\end{aligned}$$

We now compare (62) to

$$k^{\varkappa}\mathcal{E}_x\varphi(W_t)k^{(2+\varkappa-d)\gamma}\varrho^{\gamma}\int dy\Big(\int_0^t ds\ \vartheta_1(kW_s-y)\,S_{t-s}\varphi\,(W_s)\Big)^{\gamma}. \tag{63}$$

Clearly,

$$x-x^2\ \leq\ 1-\mathrm{e}^{-x}\ \leq\ x\quad\text{for}\ x\geq 0.$$

By the second inequality, the term (63) is always an upper bound for (62). On the other hand, by the first inequality and Lemma 17, the difference is bounded from above by a constant multiple of

$$\begin{aligned}&k^{\varkappa}\mathcal{E}_x\varphi(W_t)k^{2(2+\varkappa-d)\gamma}\Big[\int dy\Big(\int_0^t ds\ \vartheta_1(kW_s-y)\,S_{t-s}\varphi\,(W_s)\Big)^{\gamma}\Big]^2\\ &\leq\ c_{17}k^{\varkappa+2(2+\varkappa-d)\gamma+4-4\gamma+\delta}\ \phi_{\gamma\lambda_7}(x).\end{aligned}$$

Note that the exponent is negative iff $d\gamma>2+\delta[1+\gamma]$, hence choosing $\delta>0$ sufficiently small justifies the approximation of (62) by (63).

Recall that $\tau=\tau^{y}_{1/k}[W]$ denotes the first hitting time of the ball $B(y,1/k)$ by our Brownian motion W started in x. Now note that (63) equals

$$k^{d-2}\varrho^{\gamma}\int dy\ \ \mathcal{E}_x 1_{\tau\leq t}\,\mathcal{E}_{W_\tau}\,\varphi(\tilde W_{t-\tau})\Big(k^2\int_0^{t-\tau} ds\ \vartheta_{1/k}(\tilde W_s-y)\,S_{t-\tau-s}\varphi\,(\tilde W_s)\Big)^{\gamma}, \tag{64}$$

where the strong Markov property was used and the value for $\varkappa$ was plugged in. By Lemma 16 we may choose (and henceforth fix) a value $M>1$ such that contributions to the innermost integral coming from $s>M/k^2$, can be bounded by $\varepsilon\phi^{\gamma}_{\lambda_7}(x)$, and additionally that

$$\mathcal{E}_\iota\Big(\int_0^{\infty} ds\,\vartheta_1(\tilde W_s)\Big)^{\gamma}-\mathcal{E}_\iota\Big(\int_0^{M} ds\,\vartheta_1(\tilde W_s)\Big)^{\gamma}\ <\ \varepsilon. \tag{65}$$

Denoted by $c_{(65)}$ the first expectation in (65). By Lemma 18, if $k\geq k_{18}$, the contribution to (64) coming from $t-\delta\leq\tau\leq t$ can be made smaller than $\varepsilon\phi_{\gamma\lambda_7}(x)$ by choice of $\delta>0$. Additionally, by Lemma 7, the δ can be chosen small enough such that

$$\overline{c}\int_{t-\delta}^{t} dr\ S_r(S_{t-r}\varphi)^{1+\gamma}(x)\ <\ \varepsilon\phi_{\lambda_7}(x)\quad\text{for}\ x\in\mathbb{R}^d.$$

Fix such a δ from now on.

We let

$$k_{(66)}\ :=\ \sqrt{M/\delta} \tag{66}$$

and note that $t-\tau\geq M/k^2$ whenever $t-\delta\geq\tau$ and $k\geq k_{(66)}$. Now let $k_{15}\ :=\ k_{18}\vee k_{19}\vee k_{20}\vee k_{21}\vee k_{(66)}$. It remains to show that

$$\begin{aligned}&\Big|k^{d-2}\varrho^{\gamma}\int dy\ \ \mathcal{E}_x 1_{\tau\leq t-\delta}\,\mathcal{E}_{W_\tau}\varphi(\tilde W_{t-\tau})\Big(k^2\int_0^{M/k^2} ds\,\vartheta_{1/k}(\tilde W_s-y)S_{t-\tau-s}\varphi\,(\tilde W_s)\Big)^{\gamma}\\ &\quad-\ \overline{c}\int_0^{t-\delta} dr\ S_r(S_{t-r}\varphi)^{1+\gamma}(x)\Big|\ <\ \varepsilon\phi_{\gamma\lambda_7}(x)\qquad\text{for}\ k\geq k_{15}\,,\ \ x\in\mathbb{R}^d.\end{aligned}$$

This will be done in three steps by the triangle inequality. The steps are prepared in Lemmas 19 to 21.

In the *first* step note that by Lemma 19 we have, for all $k \geq k_{15}$ and $x \in \mathbb{R}^d$,

$$k^{d-2}\varrho^\gamma \int dy \;\; \mathcal{E}_x 1_{\tau \leq t}\, \mathcal{E}_{W_\tau} \varphi(\tilde{W}_{t-\tau}) \bigg| \Big(k^2 \int_0^{M/k^2} ds\, \vartheta_{1/k}(\tilde{W}_s - y) S_{t-\tau-s}\varphi\,(\tilde{W}_s)\Big)^\gamma - \Big(k^2 \int_0^{M/k^2} ds\, \vartheta_{1/k}(\tilde{W}_s - y) S_{t-\tau}\varphi\,(y)\Big)^\gamma \bigg| \;\leq\; \varepsilon \phi_{\gamma\lambda_7}(x).$$

We may therefore continue, using the Markov property,

$$\begin{aligned} &k^{d-2}\varrho^\gamma \int dy \;\; \mathcal{E}_x 1_{\tau \leq t-\delta}\, \mathcal{E}_{W_\tau} \varphi(\tilde{W}_{t-\tau}) \Big(k^2 \int_0^{M/k^2} ds\, \vartheta_{1/k}(\tilde{W}_s - y) S_{t-\tau}\varphi\,(y)\Big)^\gamma \\ &= \; k^{d-2}\varrho^\gamma \int dy \;\; \mathcal{E}_x 1_{\tau \leq t-\delta} \left(S_{t-\tau}\varphi\,(y)\right)^\gamma \mathcal{E}_{W_\tau} \Big(k^2 \int_0^{M/k^2} ds\, \vartheta_{1/k}(\tilde{W}_s - y)\Big)^\gamma \\ &\qquad\qquad \times \mathcal{E}_{\tilde{W}_{M/k^2}} \varphi(\tilde{\tilde{W}}_{t-\tau-M/k^2}) \end{aligned}$$

As a *second* step, by Lemma 20 we have, for all $k \geq k_{15}$ and $x \in \mathbb{R}^d$,

$$\begin{aligned} &k^{d-2}\varrho^\gamma \int dy \;\; \mathcal{E}_x 1_{\tau \leq t} \left(S_{t-\tau}\varphi\,(y)\right)^\gamma \\ &\times \left| \mathcal{E}_{W_\tau} \Big(k^2 \int_0^{M/k^2} ds\, \vartheta_{1/k}(\tilde{W}_s - y)\Big)^\gamma \mathcal{E}_{\tilde{W}_{M/k^2}} \varphi(\tilde{\tilde{W}}_{t-\tau-M/k^2}) - c_{20} S_{t-\tau}\varphi\,(y) \right| \\ &\leq \; \varepsilon\, \phi_{\gamma\lambda_7}(x). \end{aligned}$$

By (65),

$$k^{d-2}\varrho^\gamma \int dy\, \mathcal{E}_x 1_{\tau \leq t} \left(S_{t-\tau}\varphi\,(y)\right)^\gamma \left| c_{20} S_{t-\tau}\varphi\,(y) - c_{(65)} S_{t-\tau}\varphi\,(y) \right| \;<\; \varepsilon \phi_{\gamma\lambda_7}(x).$$

In the *third* step we recall that, by Lemma 21, for all $k \geq k_{19}$, and $x \in \mathbb{R}^d$,

$$\left| \varrho^\gamma c_{(65)} k^{d-2} \int dy \int_0^{t-\delta} \pi_x(dr) \big(S_{t-r}\varphi\,(y)\big)^{1+\gamma} - \overline{c}\, \varrho^\gamma \int_0^{t-\delta} S_r (S_{t-r}\varphi)^{1+\gamma}(x) \right| < \varepsilon \phi_{\gamma\lambda_7}(x),$$

with $\overline{c} = c_{(30)} c_{(65)}$. This completes the proof of Proposition 15. $\square$

3.3 Convergence of variances

In this section we establish that the variances with respect to the medium for the solutions of the linearized integral equation vanish asymptotically. Recall that $t > 0$ and $\varphi \in \mathcal{C}^+_{\exp}$.

Proposition 22 (Convergence of variances). *For every $\mu \in \mathcal{M}_{\text{tem}}$ satisfying the assumption in Theorem 2,*

$$\lim_{k \uparrow \infty} \mathsf{Var}\, k^{\varkappa} \int \mu(dx)\, \mathcal{E}_x \varphi(W_t) \exp\Big[- k^{2-d+\varkappa} \varrho \int_0^t ds\; \Gamma^1(kW_s)\, S_{t-s}\varphi\,(W_s)\Big] \;=\; 0.$$

The remainder of this section is devoted to the proof of this proposition. Recalling the definition (3) of Γ^1, the variance expression in Proposition 22 equals

$$
\begin{aligned}
&k^{2\varkappa}\int\mu(dx)\int\mu(dy)\,\mathcal{E}_x\otimes\mathcal{E}_y\,\varphi(W_t^1)\varphi(W_t^2) \qquad (67)\\
&\times\Bigg(\mathsf{E}\exp\Big[-\int\Gamma(dz)\,k^{2-d+\varkappa}\varrho\sum_{i=1,2}\int_0^t ds\;\vartheta_1(kW_s^i-z)\,S_{t-s}\varphi\,(W_s^i)\Big]\\
&\qquad-\prod_{i=1,2}\mathsf{E}\exp\Big[-\int\Gamma(dz)\,k^{2-d+\varkappa}\varrho\int_0^t ds\;\vartheta_1(kW_s^i-z)\,S_{t-s}\varphi\,(W_s^i)\Big]\Bigg),
\end{aligned}
$$

where (W^1,W^2) is distributed according to $\mathcal{P}_x\otimes\mathcal{P}_y$. Exploiting the Laplace functional (1) of Γ, (67) can be rewritten as

$$
\begin{aligned}
&k^{2\varkappa}\int\mu(dx)\int\mu(dy)\,\mathcal{E}_x\otimes\mathcal{E}_y\,\varphi(W_t^1)\varphi(W_t^2) \qquad (68)\\
&\times\Bigg(\exp\Big[-\int dz\;k^{(\varkappa-d)\gamma}\varrho^\gamma\Big(\sum_{i=1,2}k^2\int_0^t ds\;\vartheta_1(kW_s^i-z)\,S_{t-s}\varphi\,(W_s^i)\Big)^\gamma\Big]\\
&\qquad-\exp\Big[-\int dz\;k^{(\varkappa-d)\gamma}\varrho^\gamma\sum_{i=1,2}\Big(k^2\int_0^t ds\;\vartheta_1(kW_s^i-z)\,S_{t-s}\varphi\,(W_s^i)\Big)^\gamma\Big]\Bigg).
\end{aligned}
$$

Note that by the elementary inequality

$$
(a+b)^\gamma\;\le\;a^\gamma+b^\gamma\quad\text{for }\;a,b\ge0, \qquad (69)
$$

the argument in the first exponential expression is not smaller than the argument in the second one. Therefore we may apply the elementary inequality

$$
\mathrm{e}^{-a}-\mathrm{e}^{-b}\;\le\;b-a\quad\text{for }\;0\le a\le b, \qquad (70)
$$

and a z substitution to get for the non-negative total expression in (68) the upper bound (we may drop from now on the factor ϱ^γ)

$$
\begin{aligned}
&k^{2\varkappa+(\varkappa-d)\gamma+d}\int dz\int\mu(dx)\int\mu(dy)\,\mathcal{E}_x\otimes\mathcal{E}_y\,\varphi(W_t^1)\varphi(W_t^2) \qquad (71)\\
&\times\Bigg[\sum_{i=1,2}\Big(k^2\int_0^t ds\;\vartheta_{1/k}(W_s^i-z)\,S_{t-s}\varphi\,(W_s^i)\Big)^\gamma\\
&\qquad-\Big(\sum_{i=1,2}k^2\int_0^t ds\;\vartheta_{1/k}(W_s^i-z)\,S_{t-s}\varphi\,(W_s^i)\Big)^\gamma\Bigg].
\end{aligned}
$$

It remains to show that (71) converges to zero as $k\uparrow\infty$. The proof rests solely on the fact that the square bracket expression vanishes if *one* of the motions does not hit the ball $B(z,1/k)$. For simplification, write now $\tau[W^i]$ for the first hitting time $\tau_{1/k}^z[W^i]$ of $B(z,1/k)$ by the Brownian motion W^i. Hence, we get the bound

$$
\begin{aligned}
&k^{2\varkappa+(\varkappa-d)\gamma+d}\int dz\int\mu(dx)\int\mu(dy)\,\mathcal{E}_x\otimes\mathcal{E}_y\,1_{\{\tau[W^1]\le t\}}\,1_{\{\tau[W^2]\le t\}}\\
&\times\varphi(W_t^1)\,\varphi(W_t^2)\sum_{i=1,2}\Big(k^2\int_0^t ds\;\vartheta_{1/k}(W_s^i-z)\,S_{t-s}\varphi\,(W_s^i)\Big)^\gamma,
\end{aligned}
$$

where we dropped the subtracted term. Interchanging expectation and summation, and using independence, we obtain

$$k^{2\varkappa+(\varkappa-d)\gamma+d}\int dz\int\mu(dx)\int\mu(dy)\sum_{i=1,2}\mathcal{E}_x\mathbf{1}_{\{\tau[W^i]\le t\}}\,\varphi(W_t^i) \tag{72}$$

$$\times\Big(k^2\int_0^t ds\;\vartheta_{1/k}(W_s^i-z)\,S_{t-s}\varphi\,(W_s^i)\Big)^{\gamma}\,\mathcal{E}_y\mathbf{1}_{\{\tau[W^j]\le t\}}\,\varphi(W_t^j),$$

where $j=3-i$. Then we may bound (72) by

$$c_7^{\gamma}k^{2\varkappa+(\varkappa-d)\gamma+d}\int dz\;\tilde{\phi}^{\gamma}(z)\int\mu(dx)\int\mu(dy)\sum_{i=1,2}\mathcal{E}_x\mathbf{1}_{\{\tau[W^i]\le t\}}\,\varphi(W_t^i) \tag{73}$$

$$\times\Big(k^2\int_0^t ds\;\vartheta_{1/k}(W_s^i-z)\Big)^{\gamma}\,\mathcal{E}_y\mathbf{1}_{\{\tau[W^j]\le t\}}\,\varphi(W_t^j).$$

By Lemma 11, for all $x,y\in\mathbb{R}^d$,

$$\mathcal{E}_y\mathbf{1}_{\{\tau[W^j]\le t\}}\,\varphi(W_t^j)\;\le\;c\,k^{2-d}\phi_{\lambda_7}(z)\,\big[|z-y|^{2-d}+1\big], \tag{74}$$

and

$$\mathcal{E}_x\mathbf{1}_{\{\tau[W^i]\le t\}}\Big(k^2\int_0^t ds\;\vartheta_{1/k}(W_s^i-z)\Big)^{\gamma}\varphi(W_t^i) \tag{75}$$

$$\le\;c\,\phi_{\lambda_7}(x)\big[1+|z-x|^{2-d}\big]\,k^{2-d}.$$

Assume for the moment that $d\ge 5$. Then, by (24) and Lemma 8, for each $\lambda>0$ there is a constant $c=c(\lambda)$, such that

$$\int dz\;\phi_\lambda(z)\,\big[1+|z-x|^{2-d}\big]\,\big[1+|z-y|^{2-d}\big]\;\le\;c\,\big[1+|x-y|^{4-d}\big]. \tag{76}$$

If $d=3$, the left hand side of (76) is even bounded in x,y. In fact by statement (24) and Cauchy-Schwarz, it suffices to consider the singularity $\int_{|z|\le 1}dz\,|z|^{2(2-d)}<\infty$. Finally, if $d=4$, by (24) and Lemma 9, estimate (76) holds if $|x-y|^{4-d}$ is replaced by $\log^+\big(|x-y|^{-1}\big)$. If we extend definition (8) by setting $\mathrm{en}(x):\equiv 1$ in the case $d=3$, then we can combine the last three steps to obtain that for each $\lambda>0$ there is a constant $c=c(\lambda)$, so that for all $d\ge 3$,

$$\int dz\;\phi_\lambda(z)\,\big[1+|z-x|^{2-d}\big]\,\big[1+|z-y|^{2-d}\big]\;\le\;c\,\big[1+\mathrm{en}(x-y)\big]. \tag{77}$$

Based on (74), (75) and (77), from (73) we get the upper bound

$$c\,k^{2\varkappa+(\varkappa-d)\gamma+d}\,k^{4-2d}\int\mu(dx)\,\phi_{\lambda_7}(x)\int\mu(dy)\,\phi_{\lambda_7}(y)\,\big[1+\mathrm{en}(x-y)\big].$$

By our condition on μ, the latter integral is finite. Moreover, $2\varkappa+(\varkappa-d)\gamma+d+4-2d<2-d+\varkappa$. But the last expression is negative, finishing the proof. □

3.4 Upper bound for finite-dimensional distributions

We use an induction argument to extend the result from the convergence of one-dimensional distributions to all finite dimensional distributions. Recall that we have to show that, for any $\varphi_1, \ldots, \varphi_n$ and $0 = t_0 < t_1 < \cdots < t_n$, in P–probability,

$$\begin{aligned}&\limsup_{k\uparrow\infty} \mathbb{E}_{\mu_k} \exp\Big[\sum_{i=1}^{n} k^{\varkappa}\big\langle X_{t_i}^k - S_{t_i}\mu,\, -\varphi_i\big\rangle\Big] \\ &\qquad \le\ \exp\Bigg[\overline{c}\,\Big\langle \mu, \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} dr\ S_r\Big(\big(\sum_{j=i}^{n} S_{t_j-r}\varphi_j\big)^{1+\gamma}\Big)\Big\rangle\Bigg].\end{aligned} \tag{78}$$

The case $n = 1$ was shown in the previous paragraphs, so we may assume that it holds for $n-1$ and show that it also holds for n. By conditioning on $\{X^k(t)\colon t \le t_{n-1}\}$ and applying the transition functional we get

$$\begin{aligned}&\mathbb{E}_{\mu_k} \exp\Big[\sum_{i=1}^{n} k^{\varkappa}\big\langle X_{t_i}^k - S_{t_i}\mu,\, -\varphi_i\big\rangle\Big] \\ &\quad = \ \mathbb{E}_{\mu_k} \exp\Bigg[\sum_{i=1}^{n-1} k^{\varkappa}\big\langle X_{t_i}^k - S_{t_i}\mu,\, -\varphi_i\big\rangle \\ &\qquad\qquad + \ k^{\varkappa}\big\langle S_{t_{n-1}}\mu, S_{t_n-t_{n-1}}\varphi_n\big\rangle - \big\langle X_{t_{n-1}}^k, u_k(t_n - t_{n-1})\big\rangle\Bigg],\end{aligned}$$

where u_k is the solution of (41) with φ replaced by φ_n. Separating the non-random terms yields

$$\begin{aligned}&= \ \exp\Big[\big\langle S_{t_{n-1}}\mu,\, k^{\varkappa}S_{t_n-t_{n-1}}\varphi_n - u_k(t_n-t_{n-1})\big\rangle\Big] \\ &\quad \times \mathbb{E}_{\mu_k}\exp\Bigg[\sum_{i=1}^{n-2} k^{\varkappa}\big\langle X_{t_i}^k - S_{t_i}\mu,\, -\varphi_i\big\rangle \\ &\qquad\qquad + \ k^{\varkappa}\Big\langle X_{t_{n-1}}^k - S_{t_{n-1}}\mu,\, -\varphi_{n-1} - k^{-\varkappa}u_k(t_n - t_{n-1})\Big\rangle\Bigg].\end{aligned}$$

By (44) and Proposition 13 with starting measure $S_{t_{n-1}}\mu$, in P–probability,

$$\begin{aligned}&\limsup_{k\uparrow\infty} \exp\Big[\big\langle S_{t_{n-1}}\mu,\, k^{\varkappa}S_{t_n-t_{n-1}}\varphi_n - u_k(t_n-t_{n-1})\big\rangle\Big] \\ &\qquad \le\ \exp\Bigg[\overline{c}\,\Big\langle S_{t_{n-1}}\mu, \int_0^{t_n-t_{n-1}} dr\ S_r(S_{t_n-t_{n-1}-r}\varphi_n)^{1+\gamma}\Big\rangle\Bigg] \\ &\qquad =\ \exp\Bigg[\overline{c}\,\Big\langle \mu, \int_{t_{n-1}}^{t_n} dr\ S_r(S_{t_n-r}\varphi_n)^{1+\gamma}\Big\rangle\Bigg].\end{aligned} \tag{79}$$

It remains to deal with

$$\begin{aligned}&\mathbb{E}_{\mu_k}\exp\Bigg[\sum_{i=1}^{n-2} k^{\varkappa}\big\langle X_{t_i}^k - S_{t_i}\mu,\, -\varphi_i\big\rangle \\ &\qquad + \ k^{\varkappa}\big\langle X_{t_{n-1}}^k - S_{t_{n-1}}\mu,\, -\varphi_{n-1} - k^{-\varkappa}u_k(t_n - t_{n-1})\big\rangle\Bigg].\end{aligned} \tag{80}$$

We first show that (80) is bounded in k in P-probability. Indeed, if non-negative $p_1,\dots,p_{n-1}$ satisfy $\frac{1}{p_1}+\cdots+\frac{1}{p_{n-1}}=1$, then by Hölder's inequality we get the upper bound

$$\begin{aligned}&\prod_{i=1}^{n-2}\Big(\mathbb{E}_{\mu_k}\exp\Big[p_i k^{\varkappa}\big\langle X^k_{t_i}-S_{t_i}\mu,\,-\varphi_i\big\rangle\Big]\Big)^{1/p_i}\\ &\times\bigg(\mathbb{E}_{\mu_k}\exp\Big[p_{n-1}k^{\varkappa}\big\langle X^k_{t_{n-1}}-S_{t_{n-1}}\mu,\,-\varphi_{n-1}-k^{-\varkappa}u_k(t_n-t_{n-1})\big\rangle\Big]\bigg)^{1/p_{n-1}}.\end{aligned}$$

By (10) in the case $n=1$, the first line is bounded in P-probability. To deal with the second line, denote $\psi:=p_{n-1}\big(\varphi_{n-1}+k^{-\varkappa}u_k(t_n-t_{n-1})\big)$. Calculating the Laplace transform and using (44), we get the upper bound $\exp\langle\mu,\,k^{\varkappa}S_{t_{n-1}}\psi-w_k(t_{n-1})\rangle$, where w_k is the solution of (43) with φ replaced by ψ. By the monotonicity in Lemma 14, we can replace ψ by the upper bound $p_{n-1}\big(\varphi_{n-1}+S_{t_n-t_{n-1}}\varphi_n\big)$, which does not depend on k. Hence, boundedness in P-probability follows from Proposition 15.

To identify the lim sup of (80), we rewrite it as

$$\begin{aligned}\mathbb{E}_{\mu_k}\exp\bigg[&\sum_{i=1}^{n-2}k^{\varkappa}\big\langle X^k_{t_i}-S_{t_i}\mu,\,-\varphi_i\big\rangle\\ &+\;k^{\varkappa}\big\langle X^k_{t_{n-1}}-S_{t_{n-1}}\mu,\,-\varphi_{n-1}-S_{t_n-t_{n-1}}\varphi_n\big\rangle\\ &+\;k^{\varkappa}\Big\langle X^k_{t_{n-1}}-S_{t_{n-1}}\mu,\,S_{t_n-t_{n-1}}\varphi_n-k^{-\varkappa}u_k(t_n-t_{n-1})\Big\rangle\bigg].\end{aligned}\tag{81}$$

Observe that, by the induction assumption, in P–probability,

$$\begin{aligned}&\limsup_{k\uparrow\infty}\mathbb{E}_{\mu_k}\exp\Big[\sum_{i=1}^{n-2}k^{\varkappa}\big\langle X^k_{t_i}-S_{t_i}\mu,\,-\varphi_i\big\rangle\\ &\qquad\qquad+\;k^{\varkappa}\big\langle X^k_{t_{n-1}}-S_{t_{n-1}}\mu,\,-\varphi_{n-1}-S_{t_n-t_{n-1}}\varphi_n\big\rangle\Big]\\ &\le\;\exp\bigg[\overline{c}\,\Big\langle\mu,\sum_{i=1}^{n-2}\int_{t_{i-1}}^{t_i}dr\;S_r\Big(\big(\sum_{j=i}^{n-1}S_{t_j-r}\varphi_j\big)^{1+\gamma}\Big)\\ &\qquad\qquad+\int_{t_{n-2}}^{t_{n-1}}dr\;S_r\Big(\big(S_{t_{n-1}-r}\varphi_{n-1}+S_{t_{n-1}-r}S_{t_n-t_{n-1}}\varphi_n\big)^{1+\gamma}\Big)\Big\rangle\bigg].\end{aligned}\tag{82}$$

We show below that, for any $q\ge1$,

$$\mathbb{E}_{\mu_k}\exp\Big[qk^{\varkappa}\Big\langle X^k_{t_{n-1}}-S_{t_{n-1}}\mu,\;-S_{t_n-t_{n-1}}\varphi_n+k^{-\varkappa}u_k(t_n-t_{n-1})\Big\rangle\Big]\underset{k\uparrow\infty}{\Longrightarrow}1.\tag{83}$$

Let us first see, how (78) follows from this. We denote by Z_k the term in the exponent on the left hand side of (82), by Z the exponent on the right hand side of (82), and by $-qY_k$ the exponent on the left hand side of (83). Note that (82) implies that, in P-probability,

$$\limsup_{k\uparrow\infty}\mathbb{E}_{\mu_k}\mathrm{e}^{pZ_k}\;\le\;\mathrm{e}^{p^{1+\gamma}Z}\quad\text{for all }\;p\ge1.\tag{84}$$

Applying Cauchy-Schwarz we get

$$\begin{aligned}\mathbb{E}_{\mu_k}\mathrm{e}^{Z_k+Y_k} \;&\le\; \mathbb{E}_{\mu_k}\mathrm{e}^{Z_k} + \mathbb{E}_{\mu_k}\mathrm{e}^{Z_k+Y_k}(1-\mathrm{e}^{-Y_k})\,1_{\{Y_k\ge 0\}}\\ &\le\; \mathbb{E}_{\mu_k}\mathrm{e}^{Z_k} + \Big(\mathbb{E}_{\mu_k}\mathrm{e}^{2(Z_k+Y_k)}\Big)^{1/2}\Big(\mathbb{E}_{\mu_k}(1-\mathrm{e}^{-Y_k})^2\Big)^{1/2}.\end{aligned}$$

By (84), the lim sup of the first term of the second line is bounded by e^{Z}. The second factor in the second term converges to zero by (83), while the first factor is bounded as seen after (80). Hence, in P-probability,

$$\limsup_{k\uparrow\infty}\mathbb{E}_{\mu_k}\mathrm{e}^{Z_k+Y_k} \;\le\; \mathrm{e}^{Z}, \tag{85}$$

in other words, (81) is asymptotically bounded from above by

$$\begin{aligned}\exp\Big[\overline{c}\,\Big\langle\mu, \sum_{i=1}^{n-2}\int_{t_{i-1}}^{t_i} dr\;S_r\Big(\big(\sum_{j=i}^{n-1}S_{t_j-r}\varphi_j\big)^{1+\gamma}\Big)\\ +\int_{t_{n-2}}^{t_{n-1}} dr\;S_r\Big(\big(S_{t_{n-1}-r}\varphi_{n-1}+S_{t_n-r}\varphi_n\big)^{1+\gamma}\Big)\Big\rangle\Big].\end{aligned} \tag{86}$$

Putting together (79) and (86) yields the claimed statement (78) subject to the proof of (83).

To prove (83), we use the Feynman-Kac representation (42). The expectation in (83) equals

$$\begin{aligned}\exp\Big\langle\mu, qk^{\varkappa}\mathcal{E}_x\big(S_{t_n-t_{n-1}}\varphi_n(W_{t_{n-1}}) - k^{-\varkappa}u_k(t_n-t_{n-1},W_{t_{n-1}})\big)\\ \times\Big(1-\exp\Big[-k^{2-d}\varrho\int_0^{t_{n-1}} dr\;\Gamma^1(kW_r)\,U_k(t_{n-1}-r,W_r)\Big]\Big)\Big\rangle,\end{aligned}$$

where U_k is the solution of (41) with φ replaced by $q\big(S_{t_n-t_{n-1}}\varphi_n - k^{-\varkappa}u_k(t_n-t_{n-1})\big)$. It therefore suffices to show that

$$\begin{aligned}\Big\langle\mu,\, qk^{\varkappa}\mathcal{E}_x\big(S_{t_n-t_{n-1}}\varphi_n(W_{t_{n-1}}) - k^{-\varkappa}u_k(t_n-t_{n-1},W_{t_{n-1}})\big)\\ \times\Big(1-\exp\Big[-k^{2-d}\varrho\int_0^{t_{n-1}} dr\;\Gamma^1(kW_r)\,U_k(t_{n-1}-r,W_r)\Big]\Big)\Big\rangle\end{aligned}$$

converges in $L^1(\mathsf{P})$ to zero. As this term is non-negative and as

$$U_k(t_{n-1}-r) \;\le\; qk^{\varkappa}S_{t_n-r}\Big(S_{t_n-t_{n-1}}\varphi_n - k^{-\varkappa}u_k(t_n-t_{n-1})\Big) \;\le\; qk^{\varkappa}S_{t_n-r}\varphi_n\,,$$

it finally suffices to show that

$$\begin{aligned}\mathsf{E}\Big\langle\mu,\, qk^{\varkappa}\mathcal{E}_x\big(S_{t_n-t_{n-1}}\varphi_n(W_{t_{n-1}}) - k^{-\varkappa}u_k(t_n-t_{n-1},W_{t_{n-1}})\big)\\ \times\Big(1-\exp\Big[-qk^{2-d+\varkappa}\varrho\int_0^{t_{n-1}} dr\;\Gamma^1(kW_r)S_{t_n-r}\varphi_n(W_r)\Big]\Big)\Big\rangle\end{aligned} \tag{87}$$

converges to zero. The first factor in the expectation can be expressed using the Feynman-Kac representation (42) of u_k, which gives

$$q\int\mu(dx)\,\mathsf{E}\mathcal{E}_x\Big(k^{\varkappa}\mathcal{E}_{W_{t_{n-1}}}\varphi_n(\tilde W_{t_n-t_{n-1}})\\ \times\Big(1-\exp\Big[-k^{2-d}\varrho\int_0^{t_n-t_{n-1}}dr\;\Gamma^1(k\tilde W_r)u_k(t_n-t_{n-1}-r,\tilde W_r)\Big]\Big)\\ \times\Big(1-\exp\Big[-qk^{2-d+\varkappa}\varrho\int_0^{t_{n-1}}dr\;\Gamma^1(kW_r)S_{t_n-r}\varphi_n(W_r)\Big]\Big)\Big),$$

which again is dominated by

$$q\int\mu(dx)\,\mathsf{E}\mathcal{E}_x\Big(k^{\varkappa}\mathcal{E}_{W_{t_{n-1}}}\varphi_n(\tilde W_{t_n-t_{n-1}})\\ \times\Big(1-\exp\Big[-qk^{2-d+\varkappa}\varrho\int_0^{t_n-t_{n-1}}dr\;\Gamma^1(k\tilde W_r)S_{t_n-t_{n-1}-r}\varphi_n(\tilde W_r)\Big]\Big)\\ \times\Big(1-\exp\Big[-qk^{2-d+\varkappa}\varrho\int_0^{t_{n-1}}dr\;\Gamma^1(kW_r)S_{t_n-r}\varphi_n(W_r)\Big]\Big)\Big).$$

We can now multiply the factors out and obtain

$$q\int\mu(dx)\,\mathsf{E}\mathcal{E}_x k^{\varkappa}\mathcal{E}_{W_{t_{n-1}}}\varphi_n(\tilde W_{t_n-t_{n-1}})\\ \times\Big[\Big(1-\exp\Big[-qk^{2-d+\varkappa}\varrho\int_0^{t_n-t_{n-1}}dr\;\Gamma^1(k\tilde W_r)S_{t_n-t_{n-1}-r}\varphi_n(\tilde W_r)\Big]\Big) \tag{88a}$$
$$+\Big(1-\exp\Big[-qk^{2-d+\varkappa}\varrho\int_0^{t_{n-1}}dr\;\Gamma^1(kW_r)S_{t_n-r}\varphi_n(W_r)\Big]\Big) \tag{88b}$$
$$-\Big(1-\exp\Big[-qk^{2-d+\varkappa}\varrho\int_0^{t_n-t_{n-1}}dr\;\Gamma^1(k\tilde W_r)S_{t_n-t_{n-1}-r}\varphi_n(\tilde W_r) \tag{88c}$$
$$-qk^{2-d+\varkappa}\varrho\int_0^{t_{n-1}}dr\;\Gamma^1(kW_r)S_{t_n-r}\varphi_n(W_r)\Big]\Big)\Big].$$

We can now determine the limit in each of the summands (88a) to (88c) separately. For the first one we obtain from Proposition 15, as $k\uparrow\infty$,

$$q\int\mu(dx)\,\mathsf{E}\mathcal{E}_x k^{\varkappa}\mathcal{E}_{W_{t_{n-1}}}\varphi_n(\tilde W_{t_n-t_{n-1}})\\ \times\Big(1-\exp\Big[-qk^{2-d+\varkappa}\varrho\int_0^{t_n-t_{n-1}}dr\;\Gamma^1(k\tilde W_r)S_{t_n-t_{n-1}-r}\varphi_n(\tilde W_r)\Big]\Big)\\ \underset{k\uparrow\infty}{\Longrightarrow}\;q\int\mu(dx)\,\mathcal{E}_x\,\overline{c}\int_0^{t_n-t_{n-1}}dr\;S_r\big(q\varrho S_{t_n-t_{n-1}-r}\varphi_n\big)^{1+\gamma}(W_{t_{n-1}})\\ =\;\overline{c}\,q^{2+\gamma}\varrho^{1+\gamma}\Big\langle\mu,\int_{t_{n-1}}^{t_n}dr\;S_r\big(S_{t_n-r}\varphi_n\big)^{1+\gamma}\Big\rangle. \tag{89}$$

Similarly, the second one, (88b), converges by Proposition 15, as $k\uparrow\infty$,

$$
\begin{aligned}
& q\int\mu(dx)\,\mathsf{E}\mathcal{E}_x k^{\varkappa}\mathcal{E}_{W_{t_{n-1}}}\varphi_n(\tilde{W}_{t_n-t_{n-1}}) \\
& \times\Big(1-\exp\Big[-qk^{2-d+\varkappa}\varrho\int_0^{t_{n-1}}dr\ \Gamma^1(kW_r)S_{t_{n-1}-r}\big(S_{t_n-t_{n-1}}\varphi_n\big)(W_r)\Big]\Big) \\
& \underset{k\uparrow\infty}{\Longrightarrow}\ q\int\mu(dx)\,\overline{c}\int_0^{t_{n-1}}dr\ S_r\big(S_{t_{n-1}-r}(q\varrho S_{t_n-t_{n-1}}\varphi_n)\big)^{1+\gamma}(x)\Big\} \\
& \quad =\ \overline{c}\,q^{2+\gamma}\varrho^{1+\gamma}\Big\langle\mu,\int_0^{t_{n-1}}dr\ S_r\big(S_{t_n-r}\varphi_n\big)^{1+\gamma}\Big\rangle.
\end{aligned}
\tag{90}
$$

Finally, the last expression (88c) equals

$$
\begin{aligned}
& -q\int\mu(dx)\,\mathsf{E}\mathcal{E}_x k^{\varkappa}\varphi_n(W_{t_n}) \\
& \qquad\times\ \Big(1-\exp\Big[-qk^{2-d+\varkappa}\varrho\int_0^{t_n}dr\ \Gamma^1(kW_r)S_{t_n-r}\varphi_n(W_r)\Big]\Big) \\
& \underset{k\uparrow\infty}{\Longrightarrow}\ -\,\overline{c}\,q^{2+\gamma}\varrho^{1+\gamma}\Big\langle\mu,\int_0^{t_n}dr\ S_r\big(S_{t_n-r}\varphi_n\big)^{1+\gamma}\Big\rangle,
\end{aligned}
\tag{91}
$$

where the limit statement follows from Proposition 15. Comparing the right hand sides of (89) to (91) shows that they cancel completely, which proves (87) and completes the argument.

4 Lower bound: Proof of (11)

4.1 A heat equation with random inhomogeneity

As motivated in Section 1.5, for $\varkappa=\varkappa_{\mathrm{c}}$, $\varphi\in\mathcal{C}_{\exp}^{+}$, and $k>0$, we look at the mild solution m_k to the linear equation on $\mathbb{R}_+\times\mathbb{R}^d$,

$$
\begin{gathered}
\frac{\partial}{\partial t}m_k(t,x)\ =\ \tfrac12\Delta m_k(t,x)-k^{2-d}\,\varrho\,\Gamma^1(kx)\,w_k^2(t,x) \\
\text{with initial condition }\ m_k(0,\,\cdot\,)=k^{\varkappa}\varphi.
\end{gathered}
\tag{92}
$$

This is a heat equation with the time-dependent scaled random inhomogeneity $-k^{2-d}\,\varrho\,\Gamma^1(kx)\,w_k^2(t,x)$. We study its asymptotic fluctuation behaviour around the heat flow:

Proposition 23 (Limiting fluctuations of m_k). *Under the assumptions of Theorem 2, for any $\varphi\in\mathcal{C}_{\exp}^{+}$ and $t\geq 0$, in P–probability,*

$$
\liminf_{k\uparrow\infty}\big\langle\mu,\,k^{\varkappa}S_t\varphi-m_k(t,\cdot)\big\rangle\ \geq\ \underline{c}\Big\langle\mu,\int_0^t dr\ S_r\big((S_{t-r}\varphi)^{1+\gamma}\big)\Big\rangle,
\tag{93}
$$

where the constant $\underline{c}=\underline{c}(\gamma,\varrho)$ is given by

$$
\underline{c}\ :=\ \gamma\,\varrho^{\gamma}\,\frac{2\,\pi^{d/2}}{d\,G(d/2)}\,\mathcal{E}_0\otimes\mathcal{E}_0\Big[\int_0^\infty dr\ \vartheta_2(W_r^1)+\int_0^\infty dr\ \vartheta_2(W_r^2)\Big]^{\gamma-1}.
\tag{94}
$$

To see how the case $n = 1$ of (11) follows from Proposition 23, we fix a sample Γ. Recall that

$$\log \mathbb{E}_{\mu_k} \exp\Big[k^{\varkappa}\big(\langle X_t^k, -\varphi\rangle - \langle S_t\mu, -\varphi\rangle\big)\Big] = \big\langle \mu,\, k^{\varkappa} S_t\varphi - u_k(t,\,\cdot\,)\big\rangle,$$

where u_k solves

$$k^{\varkappa} S_t\varphi\,(x) \;-\; u_k(t,x) \;=\; k^{2-d}\varrho \int_0^t ds\; S_s\big(\Gamma^1(k\,\cdot\,)\,u_k^2(t-s,\,\cdot\,)\big)(x).$$

As $u_k^2 \geq w_k^2$, we obtain from (92),

$$k^{\varkappa} S_t\varphi\,(x) - u_k(t,x) \;\geq\; k^{\varkappa} S_t\varphi\,(x) - m_k(t,x). \tag{95}$$

Hence, the case $n = 1$ of (11) follows from Proposition 23.

Proposition 23 is proved in two steps: In Section 4.2 we show that the right hand side of (93) is an asymptotic lower bound of the expectations of the left hand side, and in Section 4.3 that the variances vanish asymptotically.

4.2 Convergence of expectations

Fix again $t \geq 0$ and $\varphi \in \mathcal{C}_{\exp}^{+}$.

Proposition 24 (Convergence of expectations). *For $\mu \in \mathcal{M}_{\mathrm{tem}}$ and $\underline{c}$ as in* (94),

$$\liminf_{k\uparrow\infty} \mathsf{E}\,\big\langle \mu,\, k^{\varkappa} S_t\varphi - m_k(t,\cdot)\big\rangle \;\geq\; \underline{c}\Big\langle \mu, \int_0^t dr\; S_r\big((S_{t-r}\varphi)^{1+\gamma}\big)\Big\rangle.$$

The remainder of this section is devoted to the proof of this proposition. Set

$$M_1(x) \;:=\; \mathsf{E}\,\big(k^{\varkappa} S_t\varphi(x) - m_k(t,x)\big) \quad \text{for } x \in \mathbb{R}^d,$$

and for $y \in \mathbb{R}^d$, $0 \leq s \leq t$,

$$I_s(y,W) \;:=\; \int_0^{t-s} dr\; \vartheta_1(kW_r - y)\, S_{t-s-r}\varphi\,(W_r) \;\geq\; 0.$$

Lemma 25 (Dropping the exponential). *For each $\delta > 0$ and for c_{17} from Lemma* 17*,*

$$\bigg| M_1(x) \;-\; k^{2\gamma-2}\,\gamma\,\varrho^{\gamma} \int dz\; \mathcal{E}_x \int_0^t ds\; \vartheta_1(kW_s - z)\,\mathcal{E}_{W_s}\varphi(W_{t-s}^1)\,\mathcal{E}_{W_s}\varphi(W_{t-s}^2) \\ \times \big(I_s(z,W^1) + I_s(z,W^2)\big)^{\gamma-1}\bigg| \;\leq\; c_{17}\; 2\varrho^{2\gamma}\, k^{\delta-\varkappa}\, \phi_{\gamma\lambda_7}(x),$$

for all $x \in \mathbb{R}^d$ and $k \geq 1$.

Proof. By (92) and the Feynman-Kac representation (43),

$$\begin{aligned}
\mathsf{E}\big(k^{\varkappa}S_t\varphi\,(x) - m_k(t,x)\big) \;&=\; k^{2-d}\,\varrho\,\mathsf{E}\mathcal{E}_x\int_0^t ds\;\Gamma^1(kW_s)\,w_k^2(t-s,W_s)\\
=\; k^{2-d+2\varkappa}\,\varrho\,\mathsf{E}\mathcal{E}_x\int_0^t ds\;&\Gamma^1(kW_s)\,\mathcal{E}_{W_s}\varphi(W^1_{t-s})\,\mathcal{E}_{W_s}\varphi(W^2_{t-s})\\
\times\exp\Big[-k^{2-d+\varkappa}\varrho\int_0^{t-s}&dr\;\Gamma^1(kW^1_r)\,S_{t-s-r}\varphi\,(W^1_r)\\
-\,k^{2-d+\varkappa}\varrho\int_0^{t-s}&dr\;\Gamma^1(kW^2_r)\,S_{t-s-r}\varphi\,(W^2_r)\Big],
\end{aligned}$$

where W^1 and W^2 are independent Brownian motions starting from W_s. By the definition (3) of Γ^1 this equals

$$\begin{aligned}
k^{2-d+2\varkappa}\,\varrho\,\mathsf{E}\int\Gamma(dz)\,\mathcal{E}_x\int_0^t ds\;\vartheta_1(kW_s-z)\,\mathcal{E}_{W_s}\varphi(W^1_{t-s})\,\mathcal{E}_{W_s}\varphi(W^2_{t-s})\\
\times\exp\Big[-\int\Gamma(dy)\,k^{2-d+\varkappa}\varrho\,\Big(I_s(y,W^1)+I_s(y,W^2)\Big)\Big].
\end{aligned}\tag{96}$$

Recall that for measurable $\varphi,\psi\geq 0$,

$$\mathsf{E}\,\langle\Gamma,\varphi\rangle\;\mathrm{e}^{-\langle\Gamma,\psi\rangle}\;=\;\gamma\int dz\;\varphi(z)\,\psi^{\gamma-1}(z)\,\exp\Big[-\int dy\;\psi^{\gamma}(y)\Big]\tag{97}$$

(cf. (DF92, Section 4)) and $k^{2-d+2\varkappa}k^{(2-d+\varkappa)(\gamma-1)}=k^{2\gamma-2}$ for $\varkappa=\varkappa_{\mathrm{c}}$. Applying this to (96) yields

$$\begin{aligned}
&k^{2\gamma-2}\,\gamma\,\varrho^{\gamma}\int dz\;\mathcal{E}_x\int_0^t ds\;\vartheta_1(kW_s-z)\,\mathcal{E}_{W_s}\varphi(W^1_{t-s})\,\mathcal{E}_{W_s}\varphi(W^2_{t-s})\\
&\times\big(I_s(z,W^1)+I_s(z,W^2)\big)^{\gamma-1}\;\exp\Big[-\int dy\;k^{(2-d+\varkappa)\gamma}\varrho^{\gamma}\big(I_s(y,W^1)+I_s(y,W^2)\big)^{\gamma}\Big].
\end{aligned}$$

By the inequality $1-\mathrm{e}^{-a}\leq a$ we have

$$\begin{aligned}
\Big|M_1(x)\;-\;k^{2\gamma-2}\,\gamma\,\varrho^{\gamma}\int dz\;\mathcal{E}_x\int_0^t ds\;\vartheta_1(kW_s-z)\,\mathcal{E}_{W_s}\varphi(W^1_{t-s})\,\mathcal{E}_{W_s}\varphi(W^2_{t-s})\\
\times\big(I_s(z,W^1)+I_s(z,W^2)\big)^{\gamma-1}\Big|\\
\leq\;k^{2\gamma-2}k^{(2-d+\varkappa)\gamma}\,\gamma\,\varrho^{\gamma}\int dz\;\mathcal{E}_x\int_0^t ds\;\vartheta_1(kW_s-z)\,\mathcal{E}_{W_s}\varphi(W^1_{t-s})\,\mathcal{E}_{W_s}\varphi(W^2_{t-s})\\
\times\big(I_s(z,W^1)+I_s(z,W^2)\big)^{\gamma-1}\int dy\;\varrho^{\gamma}\big(I_s(y,W^1)+I_s(y,W^2)\big)^{\gamma}.
\end{aligned}$$

Applying (69) to the last integrand and using the symmetry in W^1,W^2, we see that the right hand side in the former display does not exceed

$$\begin{aligned}
2k^{2\gamma-2}k^{(2-d+\varkappa)\gamma}\,\gamma\,\varrho^{2\gamma}\int dz\int dy\;\;\mathcal{E}_x\int_0^t ds\;\vartheta_1(kW_s-z)\,\mathcal{E}_{W_s}\varphi(W^1_{t-s})\,\mathcal{E}_{W_s}\varphi(W^2_{t-s})\\
\times\big(I_s(z,W^1)+I_s(z,W^2)\big)^{\gamma-1}I_s(y,W^1)^{\gamma}.
\end{aligned}$$

We now drop $I_s(z,W^2)$ and evaluate the expectation with respect to W^2, obtaining the upper bound

$$2k^{2\gamma-2}k^{(2-d+\varkappa)\gamma}\,\gamma\,\varrho^{2\gamma}\int\!dz\int\!dy\;\;\mathcal{E}_x\int_0^t\!ds\;\vartheta_1(kW_s-z)\,S_{t-s}\varphi\,(W_s)$$
$$\times\,\mathcal{E}_{W_s}\varphi(W^1_{t-s})\,I_s(z,W^1)^{\gamma-1}I_s(y,W^1)^{\gamma}.$$

Applying the Markov property at time s and time-homogeneity, this equals

$$2k^{2\gamma-2}k^{(2-d+\varkappa)\gamma}\,\gamma\,\varrho^{2\gamma}\int\!dz\int\!dy\;\;\mathcal{E}_x\int_0^t\!ds\;\vartheta_1(kW_s-z)\,S_{t-s}\varphi\,(W_s)$$
$$\times\,\varphi(W_t)\Big(\int_s^t\!dr\;\vartheta_1(kW_r-z)\,S_{t-r}\varphi\,(W_r)\Big)^{\gamma-1}\Big(\int_s^t\!dr\;\vartheta_1(kW_r-y)\,S_{t-r}\varphi\,(W_r)\Big)^{\gamma}.$$

The last factor can be bounded by $\left(I_0(y,W)\right)^{\gamma}$. Then we integrate with respect to s and obtain

$$2k^{2\gamma-2}k^{(2-d+\varkappa)\gamma}\,\varrho^{2\gamma}\int\!dz\int\!dy\;\;\mathcal{E}_x\varphi(W_t)\Big(\int_0^t\!dr\;\vartheta_1(kW_r-z)\,S_{t-r}\varphi\,(W_r)\Big)^{\gamma}$$
$$\times\left(I_0(y,W)\right)^{\gamma}\;=\;2k^{2\gamma-2}k^{(2-d+\varkappa)\gamma}\,\varrho^{2\gamma}\mathcal{E}_x\varphi(W_t)\Big[\int\!dy\;\;I_0(y,W)^{\gamma}\Big]^2.$$

Using now Lemma 17, we arrive at

$$\bigg|M_1(x)\;-\;k^{2\gamma-2}\,\gamma\,\varrho^{\gamma}\int\!dz\;\mathcal{E}_x\int_0^t\!ds\;\vartheta_1(kW_s-z)\,\mathcal{E}_{W_s}\varphi(W^1_{t-s})\,\mathcal{E}_{W_s}\varphi(W^2_{t-s})$$
$$\times\left(I_s(z,W^1)+I_s(z,W^2)\right)^{\gamma-1}\bigg|$$
$$\le\;c_{17}\;2\varrho^{2\gamma}\,k^{2\gamma-2}k^{(2-d+\varkappa)\gamma}\,k^{4-4\gamma+\delta}\,\phi_{\gamma\lambda_7}(x)\;=\;c_{17}\;2\varrho^{2\gamma}\,k^{\delta-\varkappa}\,\phi_{\gamma\lambda_7}(x),$$

finishing the proof. $\square$

It remains to find the limit of

$$k^{2\gamma-2}\,\gamma\,\varrho^{\gamma}\int\!dz\;\mathcal{E}_x\int_0^t\!ds\;\vartheta_1(kW_s-z)\,\mathcal{E}_{W_s}\varphi(W^1_{t-s})\,\mathcal{E}_{W_s}\varphi(W^2_{t-s})$$
$$\times\left(I_s(z,W^1)+I_s(z,W^2)\right)^{\gamma-1}.$$

Substituting $z\rightsquigarrow kz$ gives

$$k^{2\gamma-2+d}\,\gamma\,\varrho^{\gamma}\int\!dz\;\mathcal{E}_x\int_0^t\!ds\;\vartheta_{1/k}(W_s-z)\,\mathcal{E}_{W_s}\varphi(W^1_{t-s})\,\mathcal{E}_{W_s}\varphi(W^2_{t-s})$$
$$\times\left(\int_0^{t-s}\!dr\,\Big[\vartheta_{1/k}(W^1_r-z)\,S_{t-s-r}\varphi\,(W^1_r)+\vartheta_{1/k}(W^2_r-z)\,S_{t-s-r}\varphi\,(W^2_r)\Big]\right)^{\gamma-1}.$$

Fix $x,z\in\mathbb{R}^d$ and $0<s<t$ for a while and consider

$$g_k(s,x,z)\;:=\;k^{2\gamma-2+d}\,\gamma\,\mathcal{E}_x\,\vartheta_{1/k}(W_s-z)\,\mathcal{E}_{W_s}\varphi(W^1_{t-s})\,\mathcal{E}_{W_s}\varphi(W^2_{t-s})$$
$$\times\left(\int_0^{t-s}\!dr\,\Big[\vartheta_{1/k}(W^1_r-z)\,S_{t-s-r}\varphi\,(W^1_r)+\vartheta_{1/k}(W^2_r-z)\,S_{t-s-r}\varphi\,(W^2_r)\Big]\right)^{\gamma-1}.$$

Lemma 26.

$$\liminf_{k\uparrow\infty}\, g_k(s,x,z) \;\geq\; \underline{c}\, p_s(x-z)\, \big(S_{t-s}\varphi\,(z)\big)^{1+\gamma}.$$

The lemma immediately implies Proposition 24. Indeed, applying Fatou's lemma we get

$$\begin{aligned}\liminf_{k\uparrow\infty} \int \mu(dx) \int dz \int_0^t ds\; g_k(s,x,z) \;&\geq\; \int \mu(dx) \int dz \int_0^t ds\; \liminf_{k\uparrow\infty} g_k(s,x,z) \\ &\geq\; \underline{c} \int_0^t ds\, \big\langle \mu,\; S_s(S_{t-s}\varphi)^{1+\gamma}\big\rangle.\end{aligned}$$

Proof of Lemma 26. Shifting the Brownian motions,

$$\begin{aligned} g_k(s,x,z) \;=\; & k^{2\gamma-2+d}\, \gamma\, \mathcal{E}_x \vartheta_{1/k}(W_s - z)\, \mathcal{E}_z \varphi(W^1_{t-s} + W_s - z)\, \mathcal{E}_z \varphi(W^2_{t-s} + W_s - z) \\ & \times \Big(\int_0^{t-s} dr\; \vartheta_{1/k}(W^1_r + W_s - 2z)\, S_{t-s-r}\varphi\,(W^1_r + W_s - z) \\ & + \int_0^{t-s} dr\; \vartheta_{1/k}(W^2_r + W_s - 2z)\, S_{t-s-r}\varphi\,(W^2_r + W_s - z) \Big)^{\gamma-1}, \end{aligned}$$

where the expectation operators $\mathcal{E}_x$, $\mathcal{E}_z$, $\mathcal{E}_z$ apply to W, W^1, W^2, respectively. By the uniform continuity of φ,

$$\lim_{k\uparrow\infty} \sup_{|W_s - z| \leq 1/k} \big|\varphi(W^i_{t-s} + W_s - z) - \varphi(W^i_{t-s})\big| \;=\; 0,$$

and by (22),

$$\lim_{k\uparrow\infty} \sup_{|W_s - z| \leq 1/k} \big|S_{t-s-r}\varphi(W^i_{t-s} + W_s - z) - S_{t-s-r}\varphi(W^i_{t-s})\big| \;=\; 0, \tag{98}$$

we get

$$\begin{aligned} g_k(s,x,z) \;=\; & k^{2\gamma-2+d}\, \big(\gamma + o(1)\big)\; \mathcal{E}_x\, \vartheta_{1/k}(W_s - z)\, \mathcal{E}_z \varphi(W^1_{t-s})\, \mathcal{E}_z \varphi(W^2_{t-s}) \\ & \times \Big(\int_0^{t-s} dr\; \vartheta_{1/k}(W^1_r + W_s - 2z)\, S_{t-s-r}\varphi\,(W^1_r) \\ & + \int_0^{t-s} dr\; \vartheta_{1/k}(W^2_r + W_s - 2z)\, S_{t-s-r}\varphi\,(W^2_r) \Big)^{\gamma-1}. \end{aligned}$$

By the triangle inequality, $\vartheta_{1/k}(W^i_r + W_s - 2z) \leq \vartheta_{2/k}(W^i_s - z)$. Hence,

$$\begin{aligned} g_k(s,x,z) \;\geq\; & k^{2\gamma-2+d}\, \big(\gamma + o(1)\big)\; \mathcal{E}_x\, \vartheta_{1/k}(W_s - z)\, \mathcal{E}_z \varphi(W^1_{t-s})\, \mathcal{E}_z \varphi(W^2_{t-s}) \\ & \times \Big(\int_0^{t-s} dr\; \vartheta_{2/k}(W^1_r - z)\, S_{t-s-r}\varphi\,(W^1_r) \\ & + \int_0^{t-s} dr\; \vartheta_{2/k}(W^2_r - z)\, S_{t-s-r}\varphi\,(W^2_r) \Big)^{\gamma-1}. \end{aligned}$$

Calculating the expectation with respect to W gives

$$\mathcal{E}_x\, \vartheta_{1/k}(W_s - z) \;=\; \frac{\pi^{d/2}}{G(1 + d/2)}\, k^{-d}\, p_s(x-z)\, \big(1 + o(1)\big).$$

Using (98) once more we obtain

$$
\begin{aligned}
g_k(s,x,z) \ \geq \ & k^{2\gamma-2}\,\bigl(\gamma+o(1)\bigr)\,\frac{\pi^{d/2}}{G(1+d/2)}\,p_s(x-z)\,\mathcal{E}_z\varphi(W^1_{t-s})\,\mathcal{E}_z\varphi(W^2_{t-s}) \\
& \times\Bigl(\int_0^{t-s} dr\ \bigl[\vartheta_{2/k}(W^1_r-z)+\vartheta_{2/k}(W^2_r-z)\bigr]\,S_{t-s-r}\varphi\,(z)\Bigr)^{\gamma-1}.
\end{aligned}
$$

Define events

$$
A^i_k(z) \ := \ \bigl\{|W^i_r-z|>1/k\ \forall r>1/k\bigr\}.
$$

Evidently,

$$
\begin{aligned}
&\mathcal{E}_z\varphi(W^1_{t-s})\,\mathcal{E}_z\varphi(W^2_{t-s})\Bigl(\int_0^{t-s} dr\ \bigl[\vartheta_{2/k}(W^1_r-z)+\vartheta_{2/k}(W^2_r-z)\bigr]\,S_{t-s-r}\varphi\,(z)\Bigr)^{\gamma-1} \\
&\geq \mathcal{E}_z\varphi(W^1_{t-s})\,\mathcal{E}_z\varphi(W^2_{t-s}) \\
&\quad\times\Bigl(\int_0^{1/k} dr\ \bigl[\vartheta_{2/k}(W^1_r-z)+\vartheta_{2/k}(W^2_r-z)\bigr]\,S_{t-s-r}\varphi\,(z)\Bigr)^{\gamma-1}\mathbf{1}_{A^1_k(z)}\mathbf{1}_{A^2_k(z)} \\
&\geq \mathcal{E}_z\varphi(W^1_{t-s})\,\mathcal{E}_z\varphi(W^2_{t-s})\Bigl(\int_0^{1/k} dr\,\bigl[\vartheta_{2/k}(W^1_r-z)+\vartheta_{2/k}(W^2_r-z)\bigr]\,S_{t-s-r}\varphi(z)\Bigr)^{\gamma-1} \\
&\qquad -2\mathcal{E}_z\varphi(W^1_{t-s})\,\mathcal{E}_z\varphi(W^2_{t-s})\Bigl(\int_0^{1/k} dr\ \vartheta_{2/k}(W^1_r-z)\,S_{t-s-r}\varphi\,(z)\Bigr)^{\gamma-1}(1-\mathbf{1}_{A^1_k(z)}).
\end{aligned}
$$

We calculate the expressions in the last two lines separately. For the first line we get, by the Markov property at time $1/k$,

$$
\begin{aligned}
&\mathcal{E}_z\varphi(W^1_{t-s})\,\mathcal{E}_z\varphi(W^2_{t-s})\Bigl(\int_0^{1/k} dr\ \bigl[\vartheta_{2/k}(W^1_r-z)+\vartheta_{2/k}(W^2_r-z)\bigr]\,S_{t-s-r}\varphi\,(z)\Bigr)^{\gamma-1} \\
&= \mathcal{E}_0\otimes\mathcal{E}_0\Bigl(\int_0^{1/k} dr\ \bigl[\vartheta_{2/k}(W^1_r)+\vartheta_{2/k}(W^2_r)\bigr]\,S_{t-s-r}\varphi\,(z)\Bigr)^{\gamma-1} \\
&\qquad\qquad\qquad\qquad\times S_{t-s-1/k}\varphi\,(z+W^1_{1/k})\,S_{t-s-1/k}\varphi\,(z+W^2_{1/k}).
\end{aligned}
$$

By (22), this equals asymptotically

$$
\begin{aligned}
&\bigl(S_{t-s}\varphi\,(z)\bigr)^{1+\gamma}\,\mathcal{E}_0\otimes\mathcal{E}_0\Bigl(\int_0^{1/k} dr\ \bigl[\vartheta_{2/k}(W^1_r)+\vartheta_{2/k}(W^2_r)\bigr]\,\Bigr)^{\gamma-1} \\
&= \ \bigl(S_{t-s}\varphi\,(z)\bigr)^{1+\gamma}\,k^{2-2\gamma}\,\mathcal{E}_0\otimes\mathcal{E}_0\Bigl(\int_0^{k} dr\ \bigl[\vartheta_{2}(W^1_r)+\vartheta_{2}(W^2_r)\bigr]\,\Bigr)^{\gamma-1},
\end{aligned}
$$

where in the last step Brownian scaling was used. Therefore the first line is asymptotically equivalent to

$$
\bigl(S_{t-s}\varphi\,(z)\bigr)^{1+\gamma}\,k^{2-2\gamma}\mathcal{E}_0\otimes\mathcal{E}_0\Bigl(\int_0^{\infty} dr\ \bigl[\vartheta_{2}(W^1_r)+\vartheta_{2}(W^2_r)\bigr]\,\Bigr)^{\gamma-1}. \tag{99}
$$

Turning now to the second line,

$$
\begin{aligned}
&2\mathcal{E}_z\varphi(W^1_{t-s})\,\mathcal{E}_z\varphi(W^2_{t-s})\Bigl(\int_0^{1/k} dr\ \vartheta_{2/k}(W^1_r-z)\,S_{t-s-r}\varphi\,(z)\Bigr)^{\gamma-1}(1-\mathbf{1}_{A^1_k(z)}) \\
&= \ 2\,\bigl(S_{t-s}\varphi\,(z)\bigr)^{\gamma}\,\bigl(1+o(1)\bigr)\,\mathcal{E}_z\varphi(W^1_{t-s})\Bigl(\int_0^{1/k} dr\ \vartheta_{2/k}(W^1_r-z)\Bigr)^{\gamma-1}(1-\mathbf{1}_{A^1_k(z)}),
\end{aligned}
$$

where the expectation with respect to W^2 was evaluated, and (22) was used. Recalling that φ is bounded and applying Cauchy-Schwarz we obtain an upper bound, which is a constant multiple of

$$\begin{aligned}
&\big(\mathcal{P}_0\big(A_k^1(0)^{\mathrm{c}}\big)\big)^{1/2}\Big[\mathcal{E}_0\Big(\int_0^{1/k} dr\ \vartheta_{2/k}(W_r^1)\Big)^{2\gamma-2}\Big]^{1/2} \qquad (100)\\
&= k^{2-2\gamma}\Big[\mathcal{P}_0\big(\exists r>k:\ |W_r|\leq 1\big)\Big]^{1/2}\Big[\mathcal{E}_0\Big(\int_0^{k} dr\ \vartheta_2(W_r^1)\Big)^{2\gamma-2}\Big]^{1/2}.
\end{aligned}$$

Since the expectation is bounded and the probability goes to zero, (100) is $o(k^{2-2\gamma})$. Together with (99) this proves the lemma. □

4.3 Convergence of variances

Proposition 27 (Convergence of variances). *For every $\mu\in\mathcal{M}_{\mathrm{tem}}$ satisfying the assumption in Theorem 2, for $\varphi\in\mathcal{C}_{\exp}^{+}$ and $t>0$,*

$$\lim_{k\uparrow\infty}\mathsf{Var}\int\mu(dx)\Big[k^{\varkappa}S_t\varphi(x)-m_k(t,x)\Big] \ =\ 0.$$

The remainder of this section is devoted to the proof of this proposition. For simplification, we set $\varrho=1$. Recall that

$$\begin{aligned}
k^{\varkappa}S_t\varphi(x)-m_k(t,x) \ &= \ k^{2-d+2\varkappa}\,\mathcal{E}_x\int_0^t ds\int\Gamma(dz)\ \vartheta_1(kW_s-z)\\
&\times\mathcal{E}_{W_s}\otimes\mathcal{E}_{W_s}\varphi(W_{t-s}^1)\,\varphi(W_{t-s}^2)\exp\Big[-k^{2-d+\varkappa}\int\Gamma(dz)\,\big(I_s(z,W^1)+I_s(z,W^2)\big)\Big].
\end{aligned}$$

Define

$$M_2(x,\tilde{x}) \ :=\ \mathsf{E}\,\big(k^{\varkappa}S_t\varphi(x)-m_k(t,x)\big)\,\big(k^{\varkappa}S_t\varphi(\tilde{x})-m_k(t,\tilde{x})\big).$$

Similarly to (97), for measurable $\varphi_1,\varphi_2,\psi\geq 0$,

$$\begin{aligned}
\mathsf{E}\,\langle\Gamma,\varphi_1\rangle\,\langle\Gamma,\varphi_2\rangle\ \mathrm{e}^{-\langle\Gamma,\psi\rangle} \ &=\ \gamma(1-\gamma)\int dz\ \varphi_1(z)\,\varphi_2(z)\,\psi^{\gamma-2}(z)\ \exp\Big[-\int dy\ \psi^{\gamma}(y)\Big]\\
&+\ \gamma^2\int dz_1\ \varphi_1(z_1)\,\psi^{\gamma-1}(z_1)\int dz_2\ \varphi_2(z_2)\,\psi^{\gamma-1}(z_2)\ \exp\Big[-\int dy\ \psi^{\gamma}(y)\Big].
\end{aligned}$$

Applying this formula, we get

$$M_2(x,\tilde{x}) \ =\ M_{21}(x,\tilde{x})+M_{22}(x,\tilde{x}),$$

where

$$\begin{aligned}
M_{21}(x,\tilde{x}) \ &:=\ \gamma(1-\gamma)\,k^{4-2d+4\varkappa}\,k^{(2-d+\varkappa)(\gamma-2)}\,\mathcal{E}_x\otimes\mathcal{E}_{\tilde{x}}\int_0^t ds\int_0^t d\tilde{s}\int dz\\
&\vartheta_1(kW_s-z)\,\vartheta_1(k\tilde{W}_{\tilde{s}}-z)\,\mathcal{E}_{W_s}\otimes\mathcal{E}_{W_s}\varphi(W_{t-s}^1)\varphi(W_{t-s}^2)\,\mathcal{E}_{\tilde{W}_{\tilde{s}}}\otimes\mathcal{E}_{\tilde{W}_{\tilde{s}}}\varphi(\tilde{W}_{t-\tilde{s}}^1)\varphi(\tilde{W}_{t-\tilde{s}}^2)\\
&\quad\times\Big(I_s(z,W^1)+I_s(z,W^2)+I_{\tilde{s}}(z,\tilde{W}^1)+I_{\tilde{s}}(z,\tilde{W}^2)\Big)^{\gamma-2}\\
&\quad\times\exp\Big[-k^{\gamma(2-d+\varkappa)}\int dy\ \Big(I_s(y,W^1)+I_s(y,W^2)+I_{\tilde{s}}(y,\tilde{W}^1)+I_{\tilde{s}}(y,\tilde{W}^2)\Big)^{\gamma}\Big]
\end{aligned}$$

and

$$\begin{aligned}
M_{22}(x,\tilde{x}) \ &:= \ \gamma^2\, k^{4-2d+4\varkappa}\, k^{(2-d+\varkappa)(2\gamma-2)}\, \mathcal{E}_x \otimes \mathcal{E}_{\tilde{x}} \int_0^t ds \int_0^t d\tilde{s} \int dz \int d\tilde{z} \\
&\vartheta_1(kW_s - z)\, \vartheta_1(k\tilde{W}_{\tilde{s}} - \tilde{z})\, \mathcal{E}_{W_s} \otimes \mathcal{E}_{W_s} \varphi(W^1_{t-s})\varphi(W^2_{t-s})\, \mathcal{E}_{\tilde{W}_{\tilde{s}}} \otimes \mathcal{E}_{\tilde{W}_{\tilde{s}}} \varphi(\tilde{W}^1_{t-\tilde{s}})\varphi(\tilde{W}^2_{t-\tilde{s}}) \\
&\quad \times \Big(I_s(z,W^1) + I_s(z,W^2) + I_{\tilde{s}}(z,\tilde{W}^1) + I_{\tilde{s}}(z,\tilde{W}^2) \Big)^{\gamma-1} \\
&\quad \times \Big(I_s(\tilde{z},W^1) + I_s(\tilde{z},W^2) + I_{\tilde{s}}(\tilde{z},\tilde{W}^1) + I_{\tilde{s}}(\tilde{z},\tilde{W}^2) \Big)^{\gamma-1} \\
&\quad \times \exp\Big[-k^{\gamma(2-d+\varkappa)} \int dy\ \Big(I_s(y,W^1) + I_s(y,W^2) + I_{\tilde{s}}(y,\tilde{W}^1) + I_{\tilde{s}}(y,\tilde{W}^2) \Big)^{\gamma} \Big].
\end{aligned}$$

The following Lemmas 28 and 29 together directly imply Proposition 27.

Lemma 28.

$$\lim_{k\uparrow\infty} \int \mu(dx) \int \mu(d\tilde{x})\, M_{21}(x,\tilde{x}) \ = \ 0$$

Lemma 29.

$$\limsup_{k\uparrow\infty} \int \mu(dx) \int \mu(d\tilde{x})\ [M_{22}(x,\tilde{x}) - M_1(x)M_1(\tilde{x})] \ \leq \ 0.$$

Proof of Lemma 28. By definition of the critical index $\varkappa = \varkappa_{\mathrm{c}}$,

$$4 - 2d + 4\varkappa + (\gamma-2)(2-d+\varkappa) \ = \ \varkappa - 2 + 2\gamma.$$

Dropping the exponential in $M_{21}(x,\tilde{x})$ and $I_s(z,W^2) + I_s(z,W^2)$ gives

$$\begin{aligned}
M_{21}(x,\tilde{x}) \ &\leq \ k^{\varkappa-2+2\gamma}\gamma(1-\gamma)\, \mathcal{E}_x \otimes \mathcal{E}_{\tilde{x}} \int_0^t ds \int_0^t d\tilde{s} \int dz\ \vartheta_1(kW_s - z)\, \vartheta_1(k\tilde{W}_{\tilde{s}} - z) \\
&\mathcal{E}_{W_s} \otimes \mathcal{E}_{W_s} \varphi(W^1_{t-s})\varphi(W^2_{t-s})\, \mathcal{E}_{\tilde{W}_{\tilde{s}}} \otimes \mathcal{E}_{\tilde{W}_{\tilde{s}}} \varphi(\tilde{W}^1_{t-\tilde{s}})\varphi(\tilde{W}^2_{t-\tilde{s}}) \big(I_s(z,W^1) + I_{\tilde{s}}(z,\tilde{W}^1) \big)^{\gamma-2}.
\end{aligned}$$

By independence of all Brownian paths, it follows that the expression in the second line in the previous formula is bounded by

$$S_{t-s}\varphi(W_s)\, S_{t-\tilde{s}}\varphi(\tilde{W}_{\tilde{s}}) \mathcal{E}_{W_s}\varphi(W^1_{t-s}) \mathcal{E}_{\tilde{W}_{\tilde{s}}}\varphi(\tilde{W}^1_{t-\tilde{s}})\ \Big(I_s(z,W^1) + I_{\tilde{s}}(z,\tilde{W}^1) \Big)^{\gamma-2}.$$

By the Markov property,

$$\begin{aligned}
M_{21}(x,\tilde{x}) \ &\leq \ k^{\varkappa-2+2\gamma}\gamma(1-\gamma)\, \mathcal{E}_x \otimes \mathcal{E}_{\tilde{x}} \varphi(W_t)\varphi(\tilde{W}_t) \\
&\times \int dz \int_0^t ds \int_0^t d\tilde{s}\ \vartheta_1(kW_s - z)\, \vartheta_1(k\tilde{W}_{\tilde{s}} - z)\, S_{t-s}\varphi(W_s)\, S_{t-\tilde{s}}\varphi(\tilde{W}_{\tilde{s}}) \\
&\times \Big(\int_s^t dr\ \vartheta_1(kW_r - z)\, S_{t-r}\varphi(W_r) + \int_{\tilde{s}}^t dr\ \vartheta_1(k\tilde{W}_r - z)\, S_{t-r}\varphi(\tilde{W}_r) \Big)^{\gamma-2}.
\end{aligned}$$

Carrying out the integration over s and $\tilde{s}$ gives

$$\begin{aligned}
&k^{\varkappa-2+2\gamma}\, \mathcal{E}_x \otimes \mathcal{E}_{\tilde{x}} \varphi(W_t)\varphi(\tilde{W}_t) \\
&\qquad \int dz \Big(I_0(z,W)^{\gamma} + I_0(z,\tilde{W})^{\gamma} - \big(I_0(z,W) + I_0(z,\tilde{W}) \big)^{\gamma} \Big).
\end{aligned}$$

Changing the integration variable $k \rightsquigarrow kz$, we obtain

$$\int\mu(dx)\int\mu(d\tilde{x})\,M_{21}(x,\tilde{x}) \;\le\; k^{\varkappa-2+d}\int dz\int\mu(dx)\int\mu(d\tilde{x})\;\mathcal{E}_x\otimes\mathcal{E}_{\tilde{x}}\varphi(W_t)\varphi(\tilde{W}_t)$$
$$\times\Bigg[\Big(k^2\int_0^t ds\;\vartheta_{1/k}(W_s-z)\,S_{t-s}\varphi\,(W_s)\Big)^{\gamma}+\Big(k^2\int_0^t ds\;\vartheta_{1/k}(\tilde{W}_s-z)\,S_{t-s}\varphi\,(\tilde{W}_s)\Big)^{\gamma}$$
$$-\Big(k^2\int_0^t ds\;\Big[\vartheta_{1/k}(W_s-z)\,S_{t-s}\varphi\,(W_s)+\vartheta_{1/k}(\tilde{W}_s-z)\,S_{t-s}\varphi\,(\tilde{W}_s)\Big]\Big)^{\gamma}\Bigg].$$

The right hand side of this inequality coincides with (71), since $2\varkappa+(\varkappa-d)\gamma+d=\varkappa-2+d$, hence converges to zero. $\square$

Proof of Lemma 29. Dropping some non-negative summands, we get

$$M_{22}(x,\tilde{x}) \;\le\; k^{4\gamma-4}\,\gamma^2\,\mathcal{E}_x\otimes\mathcal{E}_{\tilde{x}}\int_0^t ds\int_0^t d\tilde{s}\int dz\int d\tilde{z}$$
$$\vartheta_1(kW_s-z)\,\vartheta_1(k\tilde{W}_{\tilde{s}}-\tilde{z})\,\mathcal{E}_{W_s}\otimes\mathcal{E}_{W_s}\varphi(W^1_{t-s})\varphi(W^2_{t-s})\,\mathcal{E}_{\tilde{W}_{\tilde{s}}}\otimes\mathcal{E}_{\tilde{W}_{\tilde{s}}}\varphi(\tilde{W}^1_{t-\tilde{s}})\varphi(\tilde{W}^2_{t-\tilde{s}})$$
$$\times\big(I_s(z,W^1)+I_s(z,W^2)\big)^{\gamma-1}\big(I_{\tilde{s}}(\tilde{z},\tilde{W}^1)+I_{\tilde{s}}(\tilde{z},\tilde{W}^2)\big)^{\gamma-1}$$
$$\times\exp\Big[-k^{\gamma(2-d+\varkappa)}\int dy\;\Big(I_s(y,W^1)+I_s(y,W^2)+I_{\tilde{s}}(y,\tilde{W}^1)+I_{\tilde{s}}(y,\tilde{W}^2)\Big)^{\gamma}\Big].$$

On the other hand,

$$M_1(x) \;=\; k^{2\gamma-2}\,\gamma\,\mathcal{E}_x\int_0^t ds\int dz\;\vartheta_1(kW_s-z)\,\mathcal{E}_{W_s}\varphi(W^1_{t-s})\,\mathcal{E}_{W_s}\varphi(W^2_{t-s})$$
$$\times\big(I_s(z,W^1)+I_s(z,W^2)\big)^{\gamma-1}\exp\Big[-\int dy\;k^{(2-d+\varkappa)\gamma}\big(I_s(y,W^1)+I_s(y,W^2)\big)^{\gamma}\Big].$$

Taking the difference, applying inequality (70) and using symmetry, we get

$$M_{22}(x,\tilde{x})-M_1(x)M_1(\tilde{x}) \;\le\; k^{4\gamma-4}\,\gamma^2\,\mathcal{E}_x\otimes\mathcal{E}_{\tilde{x}}\int_0^t ds\int_0^t d\tilde{s}\int dz\int d\tilde{z}$$
$$\vartheta_1(kW_s-z)\,\vartheta_1(k\tilde{W}_{\tilde{s}}-\tilde{z})\,\mathcal{E}_{W_s}\otimes\mathcal{E}_{W_s}\varphi(W^1_{t-s})\varphi(W^2_{t-s})\,\mathcal{E}_{\tilde{W}_{\tilde{s}}}\otimes\mathcal{E}_{\tilde{W}_{\tilde{s}}}\varphi(\tilde{W}^1_{t-\tilde{s}})\varphi(\tilde{W}^2_{t-\tilde{s}})$$
$$\times\big(I_s(z,W^1)+I_s(z,W^2)\big)^{\gamma-1}\big(I_{\tilde{s}}(\tilde{z},\tilde{W}^1)+I_{\tilde{s}}(\tilde{z},\tilde{W}^2)\big)^{\gamma-1}$$
$$\times k^{(2-d+\varkappa)\gamma}\,4\int dy\;\big(I_s(y,W^1)\big)^{\gamma}.$$

Dropping further non-negative summands and using again the Markov property, we get the bound

$$4\,k^{4\gamma-4}k^{(2-d+\varkappa)\gamma}\,\gamma^2\,\mathcal{E}_x\otimes\mathcal{E}_{\tilde{x}}\int_0^t ds\int_0^t d\tilde{s}\int dz\int d\tilde{z}$$
$$\vartheta_1(kW_s-z)\,\vartheta_1(k\tilde{W}_{\tilde{s}}-\tilde{z})\,S_{t-s}\varphi(W_s)\;S_{t-\tilde{s}}\varphi(\tilde{W}_{\tilde{s}})\,\varphi(W_t)\,\varphi(\tilde{W}_t)$$
$$\times\Big(\int_s^t dr\;\vartheta_1(kW_r-z)\,S_{t-r}\varphi(W_r)\Big)^{\gamma-1}\Big(\int_{\tilde{s}}^t dr\;\vartheta_1(k\tilde{W}_r-\tilde{z})\,S_{t-r}\varphi(\tilde{W}_r)\Big)^{\gamma-1}$$
$$\times\int dy\;\Big(\int_0^t dr\;\vartheta_1(kW_r-y)\,S_{t-r}\varphi(W_r)\Big)^{\gamma}.$$

Carrying out the s and $\tilde{s}$ integration, we obtain

$$\begin{aligned}
&4\,k^{4\gamma-4}k^{(2-d+\varkappa)\gamma}\,\mathcal{E}_x\otimes\mathcal{E}_{\tilde{x}}\varphi(W_t)\,\varphi(\tilde{W}_t)\int dz\int d\tilde{z}\\
&\times\Big(\int_0^t dr\ \vartheta_1(kW_r-z)\,S_{t-r}\varphi(W_r)\Big)^\gamma\Big(\int_0^t dr\ \vartheta_1(k\tilde{W}_r-\tilde{z})\,S_{t-r}\varphi(\tilde{W}_r)\Big)^\gamma\\
&\qquad\qquad\qquad\qquad\times\int dy\ \Big(\int_0^t dr\ \vartheta_1(kW_r-y)\,S_{t-r}\varphi(W_r)\Big)^\gamma.
\end{aligned}$$

We collect identical terms, use the boundedness of φ, and obtain, up to a constant factor, the bound

$$\begin{aligned}
&k^{4\gamma-4}k^{(2-d+\varkappa)\gamma}\,\mathcal{E}_x\varphi(W_t)\Big[\int dz\ \Big(\int_0^t dr\ \vartheta_1(kW_r-z)S_{t-r}\varphi(W_r)\Big)^\gamma\Big]^2\\
&\qquad\qquad\qquad\qquad\times\mathcal{E}_{\tilde{x}}\,\varphi(\tilde{W}_t)\int d\tilde{z}\,\Big(\int_0^t dr\ \vartheta_1(k\tilde{W}_r-\tilde{z})\Big)^\gamma.
\end{aligned}$$

By Lemma 12(b) with $\eta=\gamma$,

$$\begin{aligned}
&\mathcal{E}_{\tilde{x}}\,\varphi(\tilde{W}_t)\int d\tilde{z}\,\Big(\int_0^t dr\ \vartheta_1(k\tilde{W}_r-\tilde{z})\Big)^\gamma\\
&=\ k^{d-2\gamma}\int d\tilde{z}\ \mathcal{E}_{\tilde{x}}\,\varphi(\tilde{W}_t)\Big(k^2\int_0^t dr\ \vartheta_1(k\tilde{W}_r-k\tilde{z})\Big)^\gamma\ \le\ c_{12}\,k^{2-2\gamma}\,\phi_{\lambda_7}(\tilde{x}),
\end{aligned}$$

and by Lemma 17,

$$\mathcal{E}_x\varphi(W_t)\Big[\int dz\ \Big(\int_0^t dr\ \vartheta_1(kW_r-z)S_{t-r}\varphi(W_r)\Big)^\gamma\Big]^2\ \le\ c_{17}\,k^{4-4\gamma+\delta}\,\phi_{\lambda_7}(x).$$

Noting that $4\gamma-4+(2-d+\varkappa)\gamma+6-6\gamma=-\varkappa$ and choosing $\delta<\varkappa$, we obtain, up to a constant factor, the upper bound $k^{\delta-\varkappa}\phi_{\lambda_7}(x)\,\phi_{\lambda_7}(\tilde{x})$. The proof is completed by integration. □

4.4 Lower bound for finite-dimensional distributions

The proof is analogous to the upper bound in Section 3.4. Again we use induction to show that, for any $\varphi_1,\dots,\varphi_n$ and $0=t_0<t_1<\cdots<t_n$, in P–probability,

$$\begin{aligned}
&\liminf_{k\uparrow\infty}\mathbb{E}_{\mu_k}\exp\Big[\sum_{i=1}^n k^\varkappa\big\langle X^k_{t_i}-S_{t_i}\mu,\,-\varphi_i\big\rangle\Big]\\
&\qquad\ge\ \exp\Big[\underline{c}\,\Big\langle\mu,\sum_{i=1}^n\int_{t_{i-1}}^{t_i}dr\ S_r\Big(\big(\sum_{j=i}^n S_{t_j-r}\varphi_j\big)^{1+\gamma}\Big)\Big\rangle\Big].
\end{aligned}\tag{101}$$

For the case $n=1$ this was shown in the previous paragraphs, so we may assume that it holds for $n-1$ and show that it also holds for n. By conditioning on $\{X^k(t)\colon t\le t_{n-1}\}$ and applying the transition functional we get

$$\begin{aligned}
&\mathbb{E}_{\mu_k}\exp\Big[\sum_{i=1}^{n} k^{\varkappa}\big\langle X^k_{t_i} - S_{t_i}\mu,\, -\varphi_i\big\rangle\Big]\\
&\quad = \exp\Big[\big\langle S_{t_{n-1}}\mu,\, k^{\varkappa}S_{t_n-t_{n-1}}\varphi_n - u_k(t_n-t_{n-1})\big\rangle\Big]\\
&\qquad \times \mathbb{E}_{\mu_k}\exp\Bigg[\sum_{i=1}^{n-2} k^{\varkappa}\big\langle X^k_{t_i} - S_{t_i}\mu,\, -\varphi_i\big\rangle\\
&\qquad\qquad + k^{\varkappa}\Big\langle X^k_{t_{n-1}} - S_{t_{n-1}}\mu,\, -\varphi_{n-1} - k^{-\varkappa}u_k(t_n-t_{n-1})\Big\rangle\Bigg],
\end{aligned}$$

where u_k is the solution of (41) with φ replaced by φ_n. By (95) and Proposition 23, in P–probability,

$$\begin{aligned}
&\liminf_{k\uparrow\infty}\exp\Big[\big\langle S_{t_{n-1}}\mu,\, k^{\varkappa}S_{t_n-t_{n-1}}\varphi_n - u_k(t_n-t_{n-1})\big\rangle\Big]\\
&\qquad \geq \exp\Big[\underline{c}\Big\langle \mu, \int_{t_{n-1}}^{t_n} dr\ S_r(S_{t_n-r}\varphi_n)^{1+\gamma}\Big\rangle\Big].
\end{aligned}\tag{102}$$

The remaining expectation can be written as

$$\begin{aligned}
&\mathbb{E}_{\mu_k}\exp\Bigg[\sum_{i=1}^{n-2} k^{\varkappa}\big\langle X^k_{t_i} - S_{t_i}\mu,\, -\varphi_i\big\rangle\\
&\qquad + k^{\varkappa}\big\langle X^k_{t_{n-1}} - S_{t_{n-1}}\mu,\, -\varphi_{n-1} - S_{t_n-t_{n-1}}\varphi_n\big\rangle\\
&\qquad + k^{\varkappa}\Big\langle X^k_{t_{n-1}} - S_{t_{n-1}}\mu,\, S_{t_n-t_{n-1}}\varphi_n - k^{-\varkappa}u_k(t_n-t_{n-1})\Big\rangle\Bigg].
\end{aligned}\tag{103}$$

Observe that, by the induction assumption, in P–probability,

$$\begin{aligned}
&\liminf_{k\uparrow\infty}\mathbb{E}_{\mu_k}\exp\Big[\sum_{i=1}^{n-2} k^{\varkappa}\big\langle X^k_{t_i} - S_{t_i}\mu,\, -\varphi_i\big\rangle\\
&\qquad\qquad + k^{\varkappa}\big\langle X^k_{t_{n-1}} - S_{t_{n-1}}\mu,\, -\varphi_{n-1} - S_{t_n-t_{n-1}}\varphi_n\big\rangle\Big]\\
&\quad \geq \exp\Bigg[\underline{c}\Big\langle \mu, \sum_{i=1}^{n-2}\int_{t_{i-1}}^{t_i} dr\ S_r\Big(\big(\sum_{j=i}^{n-1} S_{t_j-r}\varphi_j\big)^{1+\gamma}\Big)\\
&\qquad\qquad + \int_{t_{n-2}}^{t_{n-1}} dr\ S_r\Big(\big(S_{t_{n-1}-r}\varphi_{n-1} + S_{t_{n-1}-r}S_{t_n-t_{n-1}}\varphi_n\big)^{1+\gamma}\Big)\Big\rangle\Bigg].
\end{aligned}\tag{104}$$

Define again Z_k to be the exponent on the left hand side of (104), Z to be the exponent on the right hand side of (104), and Y_k as after (83). Note that Z_k+Y_k is the exponent in (103). Applying Hölder's inequality, we get

$$\mathbb{E}_{\mu_k}\mathrm{e}^{\frac{1}{p}Z_k} \ \leq\ \big(\mathbb{E}_{\mu_k}\mathrm{e}^{Z_k+Y_k}\big)^{1/p}\,\big(\mathbb{E}_{\mu_k}\mathrm{e}^{-\frac{q}{p}Y_k}\big)^{1/q},$$

whenever $\frac{1}{p}+\frac{1}{q}=1$. Hence, combining (83) and (104), we have in P-probability

$$\liminf_{k\uparrow\infty}\mathbb{E}_{\mu_k}\mathrm{e}^{Z_k+Y_k} \ \geq\ \liminf_{k\uparrow\infty}\big(\mathbb{E}_{\mu_k}\mathrm{e}^{\frac{1}{p}Z_k}\big)^{p}\big(\mathbb{E}_{\mu_k}\mathrm{e}^{-\frac{q}{p}Y_k}\big)^{-p/q} \ \geq\ \big(\mathrm{e}^{p^{-\gamma}Z}\big)^{p},$$

which converges to e^Z as $p \downarrow 1$. Combining this with (102) proves (101), and this completes the proof. $\square$

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