Electronic Journal of Probability

Vol. 10 (2005), Paper no. 42, pages 1398-1416.

Journal URL
http://www.math.washington.edu/∼ejpecp/

ON CAUCHY-DIRICHLET PROBLEM IN HALF-SPACE FOR LINEAR INTEGRO-DIFFERENTIAL EQUATIONS IN WEIGHTED HÖLDER SPACES

R. Mikulevicius and **H. Pragarauskas**

Department of Mathematics, USC
Los Angeles 90089-1113, USA
E-mail: mikulvcs@math.usc.edu

and

Institute of Mathematics and Informatics
Akademijos 4, Vilnius, Lithuania
E-mail: jurgita@ktl.mii.lt

Abstract: We study the Cauchy-Dirichlet problem in half-space for linear parabolic integro-differential equations. Sufficient conditions are derived under which the problem has a unique solution in weighted Hölder classes. The result can be used in the regularity analysis of certain functionals arising in the theory of Markov processes.

Keywords: Markov jump processes, parabolic integro-differential equations

AMS subject classification: 60J75, 35K20.

Submitted to EJP on November 3, 2004. Final version accepted on November 16, 2005.

1. Introduction

The paper is devoted to the Cauchy-Dirichlet problem for integro-differential equations associated with d-dimensional Markov process $X_t^{s,x},\ t \geq s$, defined by Ito stochastic differential equation

$$\begin{aligned} dX_t &= \sigma(t,X_t)\ dW_t + b(t,X_t)\ dt + \int_{U_1} c(t,X_{t-},z)\ q(dt,dz) \\ &\quad + \int_{U_0} c(t,X_{t-},z)\ p(dt,dz), \\ X_s &= x, \end{aligned} \tag{1.1}$$

where W_t is a standard d-dimensional Wiener process, $p(dt,dz)$ is a well measurable point random measure on a measurable space $([0,\infty)\times U, \mathcal{B}([0,\infty))\otimes\mathcal{U})$ ($\mathcal{B}([0,\infty))$ is the Borel σ-algebra, $U_1\in\mathcal{U}, U_0 = U_1^c = U\backslash U_1$) with the compensator $\pi(t,X_t,dz)dt$, and $q(dt,dz) = p(dt,dz) - \pi(t,X_t,dz)dt$ is the corresponding martingale measure. A very simple example is the process X_t satisfying the equation

$$\begin{aligned} dX_t &= \sigma(t,X_t)\ dW_t + b(t,X_t)\ dt + Z_t, \\ X_s &= x, \end{aligned} \tag{1.2}$$

where Z_t is α-stable process, $\alpha\in(0,2)$. In this case $U = \mathbf{R}^d, U_1 = \{z : |z| \leq 1\}, c(t,x,z) = z, \pi(t,x,dz) = dz/|z|^{d+\alpha}$.

In many problems arising in the theory of Markov processes it is important to study the smoothness properties of the functionals

$$v(s,x) = \mathbf{E}\int_0^{\tau^{s,x}\wedge T} f(t,X_t^{s,x})\ dt,$$

where $\tau^{s,x}$ is the first exit time of the process $X_t^{s,x}$ from a domain $D\subseteq\mathbf{R}^d$ and $\mathbf{E}$ denotes the mathematical expectation. If v is a sufficiently smooth function, it is a solution to Cauchy-Dirichlet problem

$$\begin{aligned} \partial_t u + Lu + f &= 0, \text{ in } (0,T)\times D, \\ u(T,x) &= 0,\ x\in\mathbf{R}^d, \\ u(t,x) &= 0,\ t\in[0,T],\ x\notin D, \end{aligned} \tag{1.3}$$

where

$$\begin{aligned} Lu(t,x) &= a^{ij}(t,x)\partial_{ij}^2 u(t,x) + b_i(t,x)\partial_i u(t,x) \\ &\quad + \int_U [u(t,x+c(t,x,z)) - u(t,x) - \partial_i u(t,x)c_i(t,x,z)\mathbf{1}_{U_1}(z)]\ \pi(t,x,dz), \end{aligned}$$

$a^{ij}(t,x) = (1/2)\sigma(t,x)\sigma^*(t,x)$, $\mathbf{1}_{U_1}$ is the indicator function of U_1 and the implicit summation convention over repeated indices is assumed.

The problem of such type (including the case of nonlinear equations) was considered by a number of authors (see e.g. [1], [4], [2], [5], [7] and references therein) in Sobolev and Hölder spaces under certain restrictive assumptions on the behavior of the function $c(t,x,z)$. We show for $D = \{x\in\mathbf{R}^d : x_d > 0\}$ that these assumptions can be relaxed by considering the problem in a suitably chosen weighted spaces of functions that are Hölder continuous with respect to the space variable x (no regularity in t is assumed). These spaces are deterministic versions of the spaces used in [6] for the analysis of stochastic partial differential equations. A time-independent version of such spaces was considered in [8]. In [2], the equation (1.3) was solved in similar weighted spaces of functions that are Hölder continuous not only in x but in t as well. Besides that, the conditions in [2] imposed on $\pi(t,x,dz)$ and $c(t,x,z)$ are rather restrictive: only finite number of jumps can occur outside of D in finite time, Theorem 3.9 of Chapter II in [2] covers only a bounded variation jump part. The results in [2] do not apply for the equation (1.2) which is obviously covered by our Theorem 1 (see Remark 1 below). We discuss more precisely the differences with [2] immediately after Theorem 1 in the next section.

The results of the paper can be extended to more general situations, including the case of random coefficients and additional stochastic terms, which correspond to the stochastic processes mentioned above in a random environment (see e. g. [6] for the case $c = 0$). The results can be also used in the analysis of non-linear equations arising in the optimal control theory of Markov processes (see e.g. [5] for the case of fully nonlinear integro-differential equations in Hölder spaces).

The main result of the paper is presented in Section 2 and proved in Section 3.

2. Notation and main result

Let $\mathbf{R}^d$ be a d-dimensional Euclidean space with points $x = (x_1, \ldots, x_d)$,

$$\mathbf{R}^d_+ = \{x = (x_1, \ldots, x_d) \in \mathbf{R}^d : x_d > 0\}, H = [0,T] \times \mathbf{R}^d_+.$$

Let $B_{loc}(H)$ be the space of locally bounded Borel functions on H, i.e.

$$|u|_{0;K} = \sup_{(t,x)\in K} |u(t,x)| < \infty$$

for every compact subset $K \subseteq H$. We denote

$$B(H) = \{u \in B_{loc}(H) : |u|_{0;H} = \sup_{(t,x)\in H} |u(t,x)| < \infty\}.$$

If u is a function on H, we denote its partial derivatives as follows:

$$\partial_i u = \frac{\partial u}{\partial x_i}, \partial^2_{ij} u = \frac{\partial^2 u}{\partial x_i \partial x_j},$$

$\partial u = (\partial_1 u, \ldots, \partial_d u)$ is the gradient of u with respect to x and $\partial^2 u = (\partial^2_{ij} u)$ is the matrix of the second order partial derivatives of u with respect to x.

Let $C^m(H)$ be the class of all functions $u \in B_{loc}(H)$ such that u is m-times continuously differentiable in x and its derivatives $\partial^k u \in B_{loc}(H)$ for all $k \le m$.

Let $C^m(\bar{H})$ be the class of all $u \in C^m(H)$ whose derivatives up to the order m have continuous extensions to $\bar{H}$, the closure of H, and the norm

$$|u|_{m;H} = \sum_{k\le m} |\partial^k u|_{0;H}$$

is finite.

Let $C^{m+\beta}(\bar{H})$, $\beta \in (0,1)$, be the class of all functions $u \in C^m(\bar{H})$ such that

$$|u|_{m+\beta;H} = |u|_{m;H} + [\partial^m u]_{\beta;H},$$

where

$$[g]_{\beta;H} = \sup_{\substack{(t,x),(t,y)\in H \\ x\neq y}} \frac{|g(t,x)-g(t,y)|}{|x-y|^{\beta}}.$$

Denote $d(x) = x_d \wedge 1, x \in \mathbf{R}^d_+$, and let us introduce weighted Hölder spaces. Let $N^{\beta}(H),\ \beta \in (0,1)$, be the space of all $u \in C^0(H)$ with the finite norm

$$|u|_{(\beta);H} = |d^{1-\beta}u|_{0;H} + [du]_{\beta;H}.$$

Let $S^{m+\beta}(H)$, $m = 1,2, \beta \in (0,1)$, be the space of all functions $u \in C^m(H) \cap C^0(\bar{H})$ with a finite norm

$$|u|_{(m+\beta);H} = |\partial^m u|_{(\beta);H} + |u|_{0;H}.$$

Denote $\bar{S}^{m+\beta}(H)$ the class of all functions $u \in S^{m+\beta}(H)$ which are continuous in $\bar{H}$ with respect to (t,x) and extended by zero to $[0,T] \times \mathbf{R}^d$, i.e. $u(t,x) = 0$, if $x \notin H, t \in [0,T]$. Let us introduce the following operators:

$$\begin{aligned} Lu(t,x) &= a^{ij}(t,x)\partial^2_{ij}u(t,x) + b^i(t,x)\partial_i u(t,x) - r(t,x)u(t,x), \\ Iu(t,x) &= \int \nabla^2_{c(t,x,z)}u(t,x)\pi(t,x,dz), \end{aligned}$$

where

$$\nabla^2_{c(t,x,z)} = u(t,x+c(t,x,z)) - u(t,x) - \partial_i u(t,x)c^i(t,x,z)\mathbf{1}_{U_1}(z).$$

The summation convention that repeated indices indicate summation from 1 to d is followed here as it will throughout. The functions $a^{ij}, b^i, r \in B(H),\ a^{ij}(t,x) = a^{ji}(t,x)$ for each $(t,x) \in H$ and $i,j = 1,\dots,d,\ r \geq 0,\ \pi(t,x,dy)$ is a measure on Borel subsets of $\mathbf{R}^d$ such that

$$\int_{U_1} |c(t,x,z)|^2\ \pi(t,x,dz) + \int_{U_0} |c(t,x,z)| \wedge 1\ \pi(t,x,dz) < \infty,\ (t,x) \in H,$$

and $\pi(\cdot,\cdot,\Gamma)$ is a Borel function for each $\Gamma \in \mathcal{U}$.

Throughout he paper $C = (\cdot,\dots,\cdot)$ denotes constants depending only on quantities appearing in parentheses. In a given context the same letter will be used to denote different constants depending on the same set of arguments.

Let us consider the Cauchy-Dirichlet problem

$$\begin{aligned} \partial_t u &= Lu + Iu + f \text{ in } H, \\ u(0,x) &= 0, x \in \mathbf{R}^d_+, \qquad (2.1) \\ u(t,x) &= 0,\ x \notin \mathbf{R}^d_+, \end{aligned}$$

where $f \in N^{\beta}(H)$. We say that (2.1) holds for $u \in \bar{S}^{2+\beta}(H)$ or u is a solution to the problem (2.1) if

$$u(t,x) = \int_0^t (Lu + Iu + f)(s,x)\ ds,\ (t,x) \in H,$$

and $u(t,x) = 0,\ (t,x) \notin H$. Hence the derivative $\partial_t u$ is defined at all Lebesgue points of the function $(Lu + Iu + f)(\cdot,x)$.

Let us introduce the following assumptions.

A1. (i) The functions $a^{ij}, b^i, r,\ i,j=1,\dots d$, belong to the space $C^{\beta}(\bar{H})$;
(ii) There is a constant $\kappa>0$ such that for each $(t,x)\in\bar{H}$ and $\xi\in\mathbf{R}^d$

$$a^{ij}(t,x)\xi_i\xi_j \geq \kappa|\xi|^2,$$

and for each $t\in(0,T)$

$$\lim_{s\to t}\sup_{x\in\mathbf{R}^d, x_d=0}|a^{ij}(t,x)-a^{ij}(s,x)|=0.$$

A2. (i) There is a constant K such that for all $\bar{x}=(t,x)\in H$

$$\int_{U_1\cap\{z:x_d+c_d(\bar{x},z)>0\}}|c(\bar{x},z)|^2\pi(\bar{x},dz)+\int_{U_1\cap\{z:x_d+c_d(\bar{x},z)\leq 0\}}|c(\bar{x},z)|^{2-\beta}\pi(\bar{x},dz)$$
$$+\int_{U_0}|c(\bar{x},z)|\wedge 1\ \pi(\bar{x},dz)\leq K.$$

(ii) There is a constant K_1 such that for all $\bar{x}=(t,x)\in H, \bar{\xi}=(t,\xi)\in H$

$$\begin{aligned}\int_{U_1}|c(\bar{x},z)-c(\bar{\xi},z)|^2\pi(\bar{x},dz) &\leq K_1|x-\xi|^{2\beta},\\ \int_{U_0}|c(\bar{x},z)-c(\bar{\xi},z)|\wedge 1\ \pi(\bar{x},dz) &\leq K_1|x-\xi|,\end{aligned}$$

and

$$\begin{aligned}&\int_{U_1}|c(\bar{x},z)|^2|\pi(\bar{x},dz)-\pi(\bar{\xi},dz)|\\ &+\int_{U_0}|c(\bar{x},z)|\wedge 1\ |\pi(\bar{x},dz)-\pi(\bar{\xi},dz)|\\ \leq\ & K_1|x-\xi|^{\beta},\end{aligned}$$

A3. There exists a sequence of measurable subsets $U_n\subseteq U_1, n\geq 2$, with the following properties:
(i) There are positive constants $\nu(n)$, $k(n)$ such that $\nu(n)\to 0$, as $n\to\infty$, and for all $\bar{x}=(t,x)\in H$

$$\begin{aligned}\int_{U_n\cap\{z:x_d+c_d(\bar{x},z)>0\}}|c(\bar{x},z)|^2\ \pi(\bar{x},dz) &\leq \nu(n),\\ \int_{U_1\setminus U_n}|c(\bar{x},z)|\ \pi(\bar{x},dz) &\leq k(n).\end{aligned}$$

(ii) For each $n\geq 1$ there are positive constant $\nu_1(n),k_1(n)$ such that $\nu_1(n)\to 0$, as $n\to\infty$, and for all $\bar{x}=(t,x)\in H, \bar{\xi}=(t,\xi)\in H$

$$\begin{aligned}\int_{U_n\cap\{z:x_d+c_d(\bar{x},z)>0\}}|c(\bar{x},z)|^2\ |\pi(\bar{x},dz)-\pi(\bar{\xi},dz)| &\leq \nu_1(n)|x-\xi|^{\beta},\\ \int_{U_1\setminus U_n}|c(\bar{x},z)|\ |\pi(\bar{x},dz)-\pi(\bar{\xi},dz)| &\leq k_1(n)|x-\xi|^{\beta}.\end{aligned}$$

Remark 1. *Typical measures* $\pi(t,x,dz)$ *arising in the theory of stable Markov processes are as follows:*

$$\pi(t,x,dz) = p(t,x,z)\frac{dz}{|z|^{d+\alpha}},$$

and

$$\pi(t,x,dz) = p(t,x,z)\frac{dz}{|z|^{d+\alpha}}\mathbf{1}_{\{x_d+z_d>0\}},$$

where $\alpha \in (0,2), z \in U = \mathbf{R}^d$. *As can be easily seen, A2 and A3 are satisfied with* $c(t,x,z) = z, U_1 = \{z \in \mathbf{R}^d : |z| \leq 1\}$, *if* $2-\beta-\alpha > 0$ *and the function* p *satisfies Hölder condition in* x *with exponent* β *uniformly with respect to* t *and* y *(in the case of (1.2),* $p(t,x,z) = 1$*). Assumptions A2 and A3 are also satisfied if the exponent* α *is a sufficiently smooth function* $\alpha = \alpha(t,x)$ *in* x, *the case corresponding to the stable measures of varying order.*

The main result of the paper is the following theorem.

Theorem 1. *Let* $\beta \in (0,1)$ *and let Assumptions A1-A3 be satisfied.*

Then for each $f \in N^{\beta}(H)$ *there exists a unique solution* $u \in \bar{S}^{2+\beta}(H)$ *to the problem (2.1). Moreover, there is a constant* C *not depending on* u *and* f *such that*

$$|u|_{(2+\beta);H} \leq C|f|_{(\beta);H}, \tag{2.2}$$

and for all $s \in (0,T)$

$$|u(\cdot+s,\cdot) - u(\cdot,\cdot)|_{(1+\beta);H_s} \leq Cs^{1/2}|f|_{(\beta);H}, \tag{2.3}$$

where $H_s = (0, T-s) \times \mathbf{R}^d_+$.

In [2], the existence and uniqueness in weighted spaces of functions that are Hölder continuous in space and time variables was proved. Theorem 3.9 in Chapter II of [2] (see Section 1 of Chapter II in [2] as well) requires the following assumptions (i)-(v) for π and c:

(i) The set $U_1 = \emptyset$, and there is a measure $\pi(dz)$ on U_0 such that

$$\pi(\bar{x}, dz) = m(\bar{x}, z)\ \pi(dz), \bar{x} \in H,$$

$0 \leq m(\bar{x}, z) \leq 1$;

(ii) For a fixed $\gamma \in [0,1]$ there is a measurable function $j_\gamma(z)$ and a constant $K > 0$ such that for all $\bar{x} = (t,x) \in H$, we have $|c(\bar{x}, z)| \leq j_\gamma(z)$ and

$$\int_{U_0} j_\gamma(z)^p\ \pi(\bar{x}, dz) \leq K$$

for all $p \in [\gamma, 1]$, if $\gamma > 0$, and $\pi(U_0) < \infty$, if $\gamma = 0$;

(iii) For all $\bar{x} = (t,x) \in H$

$$x_d + c_d(\bar{x}, z) > 0$$

$\pi(\bar{x}, dz)$-a.e. on U_0, if $\gamma > 0$;

(iv) There exists a constant $c_1 \in (0,1]$ such that the determinant

$$\det(I + \partial_x c(t,x,z)) \geq c_1 > 0,$$

where I is $d \times d$ identity matrix;

(v) There is a constant M such that for all $(t,x),(s,\xi)\in H$,

$$\begin{aligned}|m(t,x)-m(s,\xi)| &\leq |x-\xi|^{\beta}+|t-s|^{\beta/2},\\ |c(t,x,z)-c(s,\xi,z)| &\leq j_{\gamma}(z)(|x-\xi|^{\beta}+|t-s|^{\beta/2}),\\ |x-\xi+c(t,x,z)-c(s,\xi,z)| &\leq M(|x-\xi|+|t-s|^{1/2}).\end{aligned}$$

For an application of our Theorem 1 in a similar case, we need the following condition instead:

A. The set $U_1=\emptyset$, and there is a constant K such that

$$\int_{U_0}|c(\bar{x},z)|\wedge 1\ \pi(\bar{x},dz)\leq K$$

and

$$\int_{U_0}|c(\bar{x},z)-c(\bar{\xi},z)|\wedge 1\ \pi(\bar{x},dz)+\int_{U_0}|c(\bar{x},z)|\wedge 1\ |\pi(\bar{x},dz)-\pi(\bar{\xi},dz)|\leq K|x-\xi|^{\beta}$$

for all $\bar{x}=(t,x)\in H,\bar{\xi}=(t,\xi)\in H$.

It is claimed in Chapter 2 of [2] without a proof that the case $\gamma\in(1,2]$ can be considered under *additional* and very technical condition (see (3.25) in Chapter II of [2]). If $\gamma>0$, the jumps outside D are not allowed. Only finite number of them can occur in finite time when $\gamma=0$.

3. Proof of Theorem 1

In order to prove Theorem 1 we need the following auxiliary statements.

Lemma 1. *For each $\varepsilon>0$ there is a constant C_ε such that for all $u\in S^{2+\beta}(H)$ and $i=1,\dots,d$*

$$|\partial_i u|_{0;H}\leq\varepsilon|u|_{(2+\beta);H}+C_\varepsilon|u|_{0;H}.$$

Proof. From Lemma 1 ([8], Appendix 2) it follows that there is a constant C such that for all $u\in S^{2+\beta}(H)$

$$[\partial_i u]_{\beta;H}\leq C|u|_{(2+\beta);H},$$

$i=1,\dots,d$. This estimate, together with the well known inequality that for each $\varepsilon>0$ there is a constant C_ε such that for all $u\in S^{2+\beta}(H)$

$$|\partial_i u|_{0;H}\leq\varepsilon[\partial_i u]_{\beta;H}+C_\varepsilon|u|_{0;H},$$

$i=1,\dots,d$, yields the assertion of the lemma. □

Lemma 2. *Let Assumptions A2 and A3 be satisfied. Then for each $\varepsilon>0$ there is a constant C_ε such that for all $u\in\bar{S}^{2+\beta}(H)$*

$$|Iu|_{(\beta);H}\leq\varepsilon|u|_{(2+\beta);H}+C_\varepsilon|u|_{0;H}.$$

Proof. Let $\bar{x}=(t,x)\in H$ and $n\geq 1$. Denote

$$\begin{aligned}\Gamma(\bar{x},n) &= \{z: x_d+c_d(\bar{x},z)>0, z\in U_n\},\\ G(\bar{x},n) &= \{z: x_d+c_d(\bar{x},z)\leq 0, z\in U_n\}.\end{aligned}$$

a) First, we shall prove that for each $\varepsilon>0$ there is a constant C_ε such that for all $u\in\bar{S}^{2+\beta}(H)$

$$|d^{1-\beta}Iu|_{0;H}\leq\varepsilon|u|_{(2+\beta);H}+C_\varepsilon|u|_{0;H}. \tag{3.1}$$

For $\bar{x}=(t,x)$ we have

$$\begin{aligned} Iu(\bar{x}) &= \int_{\Gamma(\bar{x},n)} \nabla^2_{c(\bar{x},z)} u(\bar{x})\ \pi(\bar{x},dz) + \int_{G(\bar{x},n)} \nabla^2_{c(\bar{x},z)} u(\bar{x})\ \pi(\bar{x},dz) \\ &\quad + \int_{U_1\backslash U_n} \nabla^2_{c(\bar{x},z)} u(\bar{x})\ \pi(\bar{x},dz) + \int_{U_0} [u(t,x+c(\bar{x},z)) - u(\bar{x})]\ \pi(\bar{x},dz) \\ &= \sum_{i=1}^4 I_i u(\bar{x}). \end{aligned}$$

Let us estimate $I_1 u(\bar{x})$. The inclusion $z\in\Gamma(\bar{x},n)$ implies the inequalities

$$\begin{aligned} |\nabla^2_{c(\bar{x},z)} u(\bar{x})| &\le \left|\int_0^1 (1-\tau)\partial^2_{ij} u(t,x+\tau c(\bar{x},z))\ d\tau\ c_i(\bar{x},z)c_j(\bar{x},z)\right| \\ &\le |c(\bar{x},z)|^2 |d^{1-\beta}\partial^2 u|_{0;H} \int_0^1 (1-\tau) d^{-1+\beta}(x+\tau c(\bar{x},z))\ d\tau, \end{aligned}$$

and

$$d(x+\tau c(\bar{x},z)) = (x_d + \tau c_d(\bar{x},z))\wedge 1 \ge (1-\tau)d(x).$$

Therefore

$$\int_0^1 (1-\tau) d^{-1+\beta}(x+\tau c(\bar{x},z))\ d\tau \le C d^{-1+\beta}(x).$$

Thus, according to Assumption 2,

$$\begin{aligned} d^{1-\beta}(x)|I_1 u(\bar{x})| &\le C|d^{1-\beta}\partial^2 u|_{0;H} \int_{\Gamma(\bar{x},n)} |c(\bar{x},z)|^2\ \pi(\bar{x},dz) \\ &\le C\nu(n)|u|_{(2+\beta);H}, \end{aligned} \tag{3.2}$$

where $\nu(n)\to 0$, as $n\to\infty$.

Let us estimate $I_2 u(t,x)$. Since $u(t,\cdot)$ satisfies Lipschitz condition on $\mathbf{R}^d$ uniformly with respect to t, we have

$$|\nabla^2_{c(\bar{x},z)} u(\bar{x})| \le 2|c(\bar{x},z)||\partial u|_{0;H},\ \ z\in G(\bar{x},n).$$

Moreover, if $z\in G(\bar{x},n)$, then $|c(\bar{x},z)\wedge 1| \ge d(x)$. Hence, according to Assumption 2

$$\begin{aligned} d^{1-\beta}(x)|I_2 u(\bar{x})| &\le 2|\partial u|_{0;H} d^{1-\beta}(x) \int_{G(\bar{x},n)} |c(\bar{x},z)|\ \pi(\bar{x},dz) \\ &\le 2|\partial u|_{0;H} \int_{G(\bar{x},n)} |c(\bar{x},z)|^{2-\beta}\ \pi(\bar{x},dz) \le C|\partial u|_{0;H}. \end{aligned} \tag{3.3}$$

Let us estimate $I_3 u(\bar{x})$. We have for $z\in U_1\backslash U_n$

$$|\nabla^2_{c(\bar{x},z)} u(\bar{x})| \le 2|\partial u|_{0;H}|c(\bar{x},z)|\mathbf{1}_{U_1\backslash U_n}.$$

According to Assumption 2,

$$\begin{aligned} d^{1-\beta}(x)|I_3 u(\bar{x})| &\le 2|\partial u|_{0;H} \int_{U_1\backslash U_n} |c(\bar{x},z)|\ \pi(\bar{x},dz) \\ &\le 2k(n)|\partial u|_{0;H}. \end{aligned} \tag{3.4}$$

The estimates (3.2)-(3.4), together with Lemma 1, enable us to conclude that for each $\gamma > 0$ there is a constant C_γ such that

$$|d^{1-\beta} Iu|_{0;H} \leq C \left[\nu(n) + (1+k(n))\gamma)|u|_{(2+\beta);H} + C_\gamma(1+k(n))|u|_{0;H}\right].$$

By Assumption 2, $\nu(n) \to 0$, as $n \to \infty$. Hence, choosing first a sufficiently large $n \geq 2$ and then a sufficiently small $\gamma > 0$, we get assertion (3.1).

b) Let us prove that for each $\varepsilon > 0$ there is a constant C_ε such that for all $u \in \bar{S}^{2+\beta}(H)$

$$[dIu]_{\beta;H} \leq \varepsilon |u|_{(2+\beta);H} + C_\varepsilon |u|_{0;H}. \tag{3.5}$$

Fix $n \geq 2$, $\delta_0 \in (0,1)$ and points $\bar{x} = (t,x) \in H, \bar{\xi} = (t,\xi) \in H$ such that $x \neq \xi, x_d \geq \xi_d$.

If $|x-\xi| \geq \delta_0$, then

$$|x-\xi|^{-\beta}|d(x)Iu(t,x) - d(\xi)Iu(t,\xi)| \leq 2\delta_0^{-\beta}|d^{1-\beta}Iu|_{0;H}.$$

If $|x-\xi| \geq \xi_d$, then $x_d \leq \xi_d + |x-\xi| \leq 2|x-\xi|$ and

$$|x-\xi|^{-\beta}|d(x)Iu(t,x) - d(\xi)Iu(t,\xi)| \leq 2\delta_0^{-\beta}|d^{1-\beta}Iu|_{0;H}.$$

These estimates, together with assertion (3.1), enable us to conclude that in order to prove assertion (3.5)it is sufficient to consider the case

$$|x-\xi| < \xi_d \wedge \delta_0$$

for arbitrary small $\delta_0 > 0$, which implies that

$$\begin{aligned} \xi_d &\leq x_d \leq \xi_d + |x-\xi| \leq 2\xi_d, \\ d(\xi) &\leq d(x) \leq 2d(\xi). \end{aligned}$$

We have

$$\begin{aligned} |d(x)Iu(\bar{x}) - d(\xi)Iu(\bar{\xi})| &\leq \int |V(t,x,\xi,z)|\ \pi(\bar{\xi},dz) \\ &\quad + d(x)\int |\nabla^2_{c(\bar{x},z)}u(\bar{x})|\ |\pi(\bar{x},dz) - \pi(\bar{\xi},dz)| \\ &= J_1u(t,x,\xi) + J_2u(t,x,\xi), \end{aligned} \tag{3.6}$$

where

$$V(t,x,\xi,z) = d(x)\nabla^2_{c(\bar{x},z)}u(\bar{x}) - d(\xi)\nabla^2_{c(\bar{\xi},z)}u(\bar{\xi}).$$

We will show that

$$\begin{aligned} J_1u(t,x,\xi) &\leq C|x-\xi|^\beta[(\delta_0^{1-\beta} + \delta_0^{1-\beta}c(n) + \nu(n) + \nu(n)^{1/2} + \delta_0^{\beta/2})\ |u|_{(2+\beta);H} \\ &\quad + (1+c_1(n))|\partial u|_{0;H} + |u|_{0;H}]. \end{aligned} \tag{3.7}$$

and

$$J_2u(t,x,\xi) \leq C|x-\xi|^\beta[\nu_1(n)|u|_{(2+\beta);H} + |\partial u|_{0;H}(1+c_1(n)) + |u|_{0;H}]. \tag{3.8}$$

Obviously,

$$
\begin{aligned}
V(t,x,\xi,z) &= d(x)\nabla^2_{c(\bar{x},z)}u(\bar{x}) - d(\xi)\nabla^2_{c(\bar{\xi},z)}u(\bar{\xi}). \\
&= (d(x)-d(\xi))\,\nabla^2_{c(\bar{\xi},z)}u(\bar{\xi}) + d(x)(\nabla^2_{c(\bar{x},z)}u(\bar{x}) - \nabla^2_{c(\bar{\xi},z)}u(\bar{\xi})) \qquad (3.9) \\
&= A(t,x,\xi,z) + B(t,x,\xi,z).
\end{aligned}
$$

Let us estimate $J_1 u(t,x,\xi)$.

Obviously, for $z \in G(\bar{\xi},n)$, we have $\xi_d \le |c(\bar{\xi},z)|, x_d \le 2|c(\bar{\xi},z)|$. Since

$$
\begin{aligned}
|B(t,x,\xi,z)| &= d(x)|\nabla^2_{c(\bar{x},z)}u(\bar{x}) - \nabla^2_{c(\bar{\xi},z)}u(\bar{\xi})| \\
&\le d(x)(|u(t,x+c(\bar{x},z))| + |u(\bar{x}) - u(\bar{\xi})| \\
&\quad +|\partial_i u(\bar{x})c_i(\bar{x},z) - \partial_i u(\bar{\xi})c_i(\bar{\xi},z)|) \\
&= C_1 + C_2 + C_3,
\end{aligned}
$$

we estimate every term C_1, C_2, C_3. We have

$$
\begin{aligned}
&C_1 = d(x)|u(t,x+c(\bar{x},z))| \\
&\le d(x)|\int_0^1 \partial_d u(t,x'+c'(\bar{x},z),\tau x_d + \tau c_d(\bar{x},z))|\,d\tau(|x_d-\xi_d| \\
&\quad +|c_d(\bar{x},z) - c_d(\bar{\xi},z))| \\
&\le C|\partial u|_{0;H}(|x-\xi|^\beta |c(\bar{\xi},z)|^{2-\beta} + |c_d(\bar{\xi},z)|\;|c_d(\bar{x},z) - c_d(\bar{\xi},z)|),
\end{aligned}
$$

where for $y = (y_1,\ldots,y_{d-1},y_d) \in \mathbf{R}^d$ we write $y = (y',y_d),\ y' = (y_1,\ldots,y_{d-1})$. Then

$$
\begin{aligned}
&C_2 = d(x)|u(\bar{x}) - u(\bar{\xi})| \\
&\le d(x)|u(t,x',x_d) - u(x',\xi_d)| + |u(t,x',\xi_d) - u(\xi',\xi_d)|) \\
&\le Cd(x)[|\partial u|_{0;H}|x_d-\xi_d| + \int_0^1 |\partial_d u(t,x',\tau\xi_d) - \partial_d u(\xi',\tau\xi_d)|\xi_d d\tau] \\
&\le C|x-\xi|^\beta(|\partial u|_{0;H}|c(\bar{\xi},z)|^{2-\beta} + [\partial u]_{\beta;H}|c(\bar{\xi},z)|^2).
\end{aligned}
$$

and

$$
\begin{aligned}
&C_3 = d(x)|\partial_i u(\bar{x})c_i(\bar{x},z) - \partial_i u(\bar{\xi})c_i(\bar{\xi},z)| \\
&\le d(x)|\partial u(\bar{x}) - \partial u(\bar{\xi})|\;|c(\bar{\xi},z)| + |\partial u(\bar{x})|\;|c(\bar{\xi},z) - c(\bar{x},z)| \\
&\le C(|x-\xi|^\beta |c(\bar{\xi},z)|^2[\partial u]_\beta + |\partial u|_{0;H}|c(\bar{\xi},z)|\;|c(\bar{\xi},z) - c(\bar{x},z)|).
\end{aligned}
$$

Since for $z \in G(\bar{\xi}, n)$

$$\begin{aligned} |A(t,x,\xi,z)| &\leq 2|x_d - \xi_d|\ |\partial u|_{0;H} |c(\bar{\xi}, z)| \\ &\leq C|x_d - \xi_d|^\beta\ |\partial u|_{0;H} |c(\bar{\xi}, z)|^{2-\beta}, \end{aligned}$$

collecting all the estimates and applying Hölder inequality we obtain

$$\begin{aligned} &\int_{G(\bar{\xi},n)} |V(t,x,\xi,z)|\ \pi(t,\xi,dz) \\ \leq\ & C|x-\xi|^\beta[|\partial u|_{0;H} \int_{G(\bar{\xi},n)} |c(\bar{\xi},z)|^{2-\beta}\pi(t,\xi,dz) + [\partial u]_{\beta;H} \int_{G(\bar{\xi},n)} |c(\bar{\xi},z)|^2\pi(t,\xi,dz) \\ & +|\partial u|_{0;H}(\int_{G(\bar{\xi},n)} |c(\bar{\xi},z)|^2\ \pi(\bar{\xi},dz)^{1/2}(\int_{G(\bar{\xi},n)} |c(\bar{\xi},z) - c(\bar{x},z)|^2\ \pi(\bar{\xi},dz)^{1/2}]. \end{aligned}$$

By Assumption A3,

$$\int_{G(\bar{\xi},n)} |V(t,x,\xi,z)|\ \pi(t,\xi,dz) \leq C|x-\xi|^\beta[|\partial u|_{0;H} + [\partial u]_{\beta;H}\nu(n)]. \tag{3.10}$$

Now we estimate B on $\Gamma(\bar{\xi}, n)$. We split

$$\begin{aligned} |B(t,x,\xi)| &\leq d(x)|\nabla^2_{c(\bar{x},z)}u(\bar{x} - \nabla^2_{c(\bar{\xi},z)}u(\bar{x})| \\ &+d(x)|\nabla^2_{c(\bar{\xi},z)}u(\bar{x}) - \nabla^2_{c(\bar{\xi},z)}u(\bar{\xi})| \\ &= B_1 + B_2 \end{aligned} \tag{3.11}$$

and estimate B_1 and B_2. If $z \in \Gamma(\bar{\xi}, n) \cap \Gamma(\bar{x}, n)$, we have

$$\begin{aligned} B_1 &\leq d(x)|u(t, x + c(\bar{x},z)) - u(t, x + c(\bar{\xi},z)) \\ &-\partial_i u(\bar{x})(c_i(\bar{x},z) - c_i(\bar{\xi},z))| \\ &\leq d(x)\int_0^1\int_0^1 |\partial^2 u(t, x + s(1-\tau)c(\bar{\xi},z) + s\tau c(\bar{x},z))|\ |c(\bar{x},z) \\ &-c(\bar{\xi},z)|(|c(\bar{\xi},z)| + |c(\bar{x},z)|)\ dsd\tau. \end{aligned}$$

Since

$$\begin{aligned} & x_d + s(1-\tau)c_d(\bar{\xi},z) + s\tau c_d(\bar{x},z)) \\ =\ & \tau(x_d + sc_d(\bar{x},z)) + (1-\tau)(x_d + sc_d(\bar{\xi},z)) \\ \geq\ & (1-s)x_d \geq (1-s)d(x), \end{aligned}$$

it follows that

$$B_1 \leq Cd(x)^\beta|d^{1-\beta}\partial^2 u|_{0;H}|c(\bar{x},z) - c(\bar{\xi},z)|(|c(\bar{\xi},z)| + |c(\bar{x},z|).$$

If $z\in\Gamma(\bar{\xi},n)\cap G(\bar{x},n)$, then

$$\begin{aligned} d(x) &\leq |c(\bar{x},z)|,\\ 0 &< x_d+c_d(\bar{\xi},z)\leq |c(\bar{x},z)-c(\bar{\xi},z)|, \end{aligned}$$

and

$$B_1\leq C|\partial u|_{0;H}d(x)|c(\bar{x},z)-c(\bar{\xi},z)|\leq C|\partial u|_{0;H}|c(\bar{x},z)||c(\bar{x},z)-c(\bar{\xi},z)|.$$

Using Hölder inequality, we get

$$\begin{aligned} &\int_{\Gamma(\bar{\xi},n)}|B_1(t,x,\xi,z)|\ \pi(\bar{\xi},dz)\\ \leq\ & C|\partial^2 u|_{(2+\beta);H}[(\int_{U_n}|c(\bar{x},z)-c(\bar{\xi},z)|^2\ \pi(\bar{\xi},dz))^{1/2}(\int_{U_n}|c(\bar{\xi},z)|^2\ \pi(\bar{\xi},dz))^{1/2}\\ &+(\int_{U_n}|c(\bar{x},z)|^2\ |\pi(\bar{\xi},dz)-\pi(\bar{x},dz)|)^{1/2}(\int_{U_n}|c(\bar{x},z)-c(\bar{\xi},z)|^2\ \pi(\bar{\xi},dz))^{1/2}]. \end{aligned}$$

So,

$$\int_{\Gamma(\bar{\xi},n)}|B_1(t,x,\xi,z)|\ \pi(\bar{\xi},dz)\leq C|x-\xi|^{\beta}|\partial^2 u|_{(2+\beta);H}[\nu(n)^{1/2}+|x-\xi|^{\beta/2}]. \tag{3.12}$$

Now we estimate B_2 for $z\in\Gamma(\bar{\xi},n)$. If $z\in\Gamma\left(\bar{\xi},n\right)$ and $|c(\bar{\xi},z)|\leq|x-\xi|$, then $\xi_d+c_d(\bar{\xi},z)>0, x_d+c_d(\bar{\xi},z)>0$,and

$$\begin{aligned} &B_2=d(x)|\nabla^2_{c(\bar{\xi},z)}u(\bar{x})-\nabla^2_{c(\bar{\xi},z)}u(\bar{\xi})|\\ \leq\ & d(x)\int_0^1(1-\tau)|\partial^2 u(t,x+\tau c(\bar{\xi},z))-\partial^2 u(t,\xi+\tau c(\bar{\xi},z))|\ d\tau|c(\bar{\xi},z)|^2\\ \leq\ & \int_0^1(1-\tau)|d(x)-d(x+\tau c(\bar{\xi},z))|\ |\partial^2 u(t,x+\tau c(\bar{\xi},z))|\ d\tau|c(\bar{\xi},z)|^2\\ &+\int_0^1(1-\tau)|d(x)-d(\xi+\tau c(\bar{\xi},z))|\ |\partial^2 u(t,\xi+\tau c(\bar{\xi},z))|\ d\tau|c(\bar{\xi},z)|^2\\ &+\int_0^1(1-\tau)|(d\partial^2 u)(t,x+\tau c(\bar{\xi},z))-(d\partial^2 u)(t,\xi+\tau c(\bar{\xi},z))|\ d\tau|c(\bar{\xi},z)|^2. \end{aligned}$$

Since

$$\xi_d+\tau c_d(\bar{\xi},z)\geq(1-\tau)d(\xi),\ \ x_d+\tau c_d(\bar{\xi},z)\geq(1-\tau)d(x),$$

we obtain

$$|B_2(t,x,\xi,z)|\leq C|x-\xi|^{\beta}|u|_{(2+\beta)}|c(\bar{\xi},z)|^2.$$

If $z \in \Gamma\left(\bar{\xi}, n\right)$ and $|c(\bar{\xi}, z)| > |x - \xi|$, then

$$\begin{aligned} |B_2(t,x,\xi,z)| &\leq d(x)\int_0^1 |\partial u(t, x+\tau c(\bar{\xi},z)) - \partial u(t,\xi+\tau c(\bar{\xi},z)) \\ &\quad -(\partial u(t,x) - \partial u(t,\xi))|\ d\tau |c(\bar{\xi},z)| \\ &\leq d(x)|x-\xi|\ |c(\bar{\xi})| \int_0^1\int_0^1 (|\partial^2 u(t,\xi+\tau c(\bar{\xi},z) + s(x-\xi))| \\ &\quad +|\partial^2 u(t,\xi+s(x-\xi))|)\ d\tau ds. \end{aligned}$$

Since $d(\xi + \tau c(\bar{\xi}, z) + s(x - \xi)) \geq (1 - \tau)d(\xi), d(\xi + s(x - \xi)) \geq d(\xi)$, we get

$$|B_2(t,x,\xi,z)| \leq C|x-\xi|^\beta |d^{1-\beta}\partial^2 u|_{0;H}|c(\bar{\xi})|^2.$$

So,

$$\begin{aligned} \int_{\Gamma(\bar{\xi},n)} |B_2(t,x,\xi,z)|\ \pi(\bar{\xi},dz) &\leq C|x-\xi|^\beta |u|_{(2+\beta)} \int_{U_n} |c(\bar{\xi},z)|^2 \pi(\bar{\xi},dz) \\ &\leq C|x-\xi|^\beta |u|_{(2+\beta)}\nu(n). \end{aligned} \tag{3.13}$$

For $z \in \Gamma\left(\bar{\xi}, n\right)$ and $\tau \in (0, 1)$ we have $\xi_d + c_d(\bar{\xi}, z) > 0, \xi_d + \tau c_d(\bar{\xi}, z) \geq (1 - \tau)\xi_d$ and

$$\begin{aligned} &|A(t,x,\xi,z)| \\ &\leq |x-\xi| \int_0^1 (1-\tau)|\partial^2 u(t,\xi+\tau c(\bar{\xi},z))|\ d\tau |c(\bar{\xi},z)|^2 \\ &\leq |x-\xi| \int_0^1 \frac{(1-\tau)\ d\tau}{(1-\tau)^{1-\beta} d(\xi)^{1-\beta}} |d^{1-\beta}\partial^2 u|_{0;H} |c(\bar{\xi},z)|^2 \\ &\leq C|x-\xi|^\beta |d^{1-\beta}\partial^2 u|_{0;H}|c(\bar{\xi},z)|^2. \end{aligned}$$

So,

$$\int_{\Gamma(\bar{\xi},n)} |A(t,x,\xi,z)|\ \pi(\bar{\xi},dz) \leq C|x-\xi|^\beta |d^{1-\beta}\partial^2 u|_{0;H}\nu(n),$$

and by (3.11)-(3.13)

$$\int_{\Gamma(\bar{\xi},n)} |V(t,x,\xi,z)|\ \pi(\bar{\xi},dz) \leq C|x-\xi|^\beta |\partial^2 u|_{(2+\beta);H}[\nu(n) + \nu(n)^{1/2} + \delta_0^{\beta/2}]. \tag{3.14}$$

Assume $z \in U_1 \backslash U_n$. Then

$$\begin{aligned} |A(t,x,\xi,z)| &\leq C|x-\xi|\ |\partial u|_{0;H}|c(\bar{\xi},z)| \\ &\leq C|x-\xi|^\beta \delta_0^{1-\beta}\ |\partial u|_{0;H}|c(\bar{\xi},z)|, \end{aligned} \tag{3.15}$$

and

$$B_1 \leq C|\partial u|_{0;H}|c(\bar{x},z) - c(\bar{\xi},z)|. \tag{3.16}$$

If $\xi_d + c_d(\bar{\xi}, z) > 0$, then

$$
\begin{aligned}
|B_2| &\leq |x-\xi|\ |c(\bar{\xi}, z)| \int_0^1 \int_0^1 |\partial^2 t, \xi + \tau c(\bar{\xi}, z) + s(x-\xi))|\ d\tau ds \\
&\leq |x-\xi|^\beta \delta_0^{1-\beta} |d^{1-\beta}\partial^2 u|_{0;H} |c(\bar{\xi}, z)|.
\end{aligned}
$$

If $\xi_d + c_d(\bar{\xi}, z) \leq 0$, then $\xi_d \leq |c(\bar{\xi}, z)|$ and

$$
|B_2| \leq C|x-\xi|\ |\partial u|_{0;H} |c(\bar{\xi}, z)| \leq C|x-\xi|^\beta \delta_0^{1-\beta}\ |\partial u|_{0;H} |c(\bar{\xi}, z)|. \tag{3.17}
$$

So, by (3.15)-(3.17),

$$
\begin{aligned}
\int_{U_1 \backslash U_n} |V(t,x,\xi,z)|\ \pi(\bar{\xi}, dz) &\leq C|x-\xi|^\beta \delta_0^{1-\beta} |u|_{(2+\beta);H} \int_{U_1 \backslash U_n} |c(\bar{\xi}, z)|\ \pi(\bar{\xi}, dz) \\
&\quad + |\partial u|_{0;H} \int_{U_1 \backslash U_n} |c(\bar{x}, z) - c(\bar{\xi}, z)|\ \pi(\bar{\xi}, dz). \\
&\leq C|x-\xi|^\beta [\delta_0^{1-\beta} c(n) |u|_{(2+\beta);H} + c_1(n) |\partial u|_{0;H}].
\end{aligned}
$$

By Assumption A3,

$$
\int_{U_1 \backslash U_n} |V(t,x,\xi,z)|\ \pi(\bar{\xi}, dz) \leq C|x-\xi|^\beta [\delta_0^{1-\beta} k(n) |u|_{(2+\beta);H} + k_1(n) |\partial u|_{0;H}]. \tag{3.18}
$$

Now we derive the estimates on U_0. If $z \in U_0$, then

$$
\begin{aligned}
&|B(t,x,\xi,z)| \\
&\leq d(x) |u(t, x + c(\bar{x}, z)) - u(\bar{x}) - (u(t, \xi + c(\bar{\xi}, z)) - u(\bar{\xi}))| \\
&\leq d(x) |u(t, x + c(\bar{x}, z)) - u(t, x + c(\bar{\xi}, z))| \\
&\quad + d(x) |u(t, x + c(\bar{\xi}, z)) - u(\bar{x}) - (u(t, \xi + c(\bar{\xi}, z)) - u(\bar{\xi}))| \\
&= D + E,
\end{aligned} \tag{3.19}
$$

and we estimate D and E. Obviously,

$$
D \leq (|\partial u|_{0;H} + |u|_{0;H}) \left(|c(\bar{\xi}, z) - c(\bar{x}, z)| \wedge 1\right). \tag{3.20}
$$

If $z \in U_0, \xi_d + c_d(\bar{\xi}, z) > 0, |c(\bar{\xi}, z)| \leq 1$, then $x_d + c_d(\bar{\xi}, z) > 0$ and

$$
\begin{aligned}
E &\leq d(x) \int_0^1 |\partial u(t, x + \tau c(\bar{\xi}, z)) - \partial u(t, \xi + \tau c(\bar{\xi}, z))|\ d\tau |c(\bar{\xi}, z)| \\
&\leq d(x) \int_0^1 \int_0^1 |\partial^2 u(t, \xi + \tau c(\bar{\xi}, z) + s(x-\xi))|\ d\tau ds\ |c(\bar{\xi}, z)|\ |x-\xi|.
\end{aligned}
$$

So, for $z \in U_0, \xi_d + c_d(\bar{\xi}, z) > 0, |c(\bar{\xi}, z)| \leq 1$,

$$
E \leq C|x-\xi| |d^{1-\beta} \partial^2 u|_{0;H} |c(\bar{\xi}, z)|. \tag{3.21}
$$

If $z \in U_0, \xi_d + c_d(\bar{\xi}, z) > 0, |c(\bar{\xi}, z)| > 1$, then

$$
E \leq 2 |\partial u|_{0;H} |x-\xi|. \tag{3.22}
$$

Therefore, by (3.21), (3.22), for $z \in U_0$ and $\xi_d + c_d(\bar{\xi}, z) > 0$ we have

$$E \le C|x-\xi|\ |u|_{(2+\beta);H}(|c(\bar{\xi}, z)| \wedge 1). \tag{3.23}$$

If $z \in U_0, \xi_d + c_d(\bar{\xi}, z) \le 0$, then $\xi_d \le |c(\bar{\xi}, z)|, d(\xi) \le |c(\bar{\xi}, z)| \wedge 1, d(x) \le 2|c(\bar{\xi}, z)| \wedge 1$ and

$$E \le C\left(|c(\bar{\xi}, z)| \wedge 1\right)|\partial u|_{0;H}|x-\xi|.$$

So, for $z \in U_0$,

$$\begin{aligned} E &\le C|x-\xi|\ |u|_{(2+\beta);H}(|c(\bar{\xi}, z)| \wedge 1) \\ &\le C|x-\xi|^{\beta}\delta_0^{1-\beta}\ |u|_{(2+\beta);H}(|c(\bar{\xi}, z)| \wedge 1). \end{aligned} \tag{3.24}$$

By (3.19), (3.20) and (3.24), we obtain

$$\begin{aligned} \int_{U_0} |B(t,x,\xi,z)|\ \pi(\bar{\xi}, dz) &\le C[|x-\xi|^{\beta}\delta_0^{1-\beta}\ |u|_{(2+\beta);H}\int_{U_0}|c(\bar{\xi}, z)| \wedge 1\ \pi(\bar{\xi}, dz) \\ &\quad +(|\partial u|_{0;H} + |u|_{0;H})\int_{U_0}|c(\bar{\xi}, z) - c(\bar{x}, z)| \wedge 1\ \pi(\bar{\xi}, dz)]. \end{aligned}$$

Therefore, by Assumption A2,

$$\int_{U_0} |B(t,x,\xi,z)|\ \pi(\bar{\xi}, dz) \le C|x-\xi|^{\beta}[\delta_0^{1-\beta}\ |u|_{(2+\beta);H} + |\partial u|_{0;H} + |u|_{0;H}],$$

and, similarly,

$$\begin{aligned} \int_{U_0} |A(t,x,\xi,z)|\ \pi(\bar{\xi}, dz) &\le |x-\xi|\ (|\partial u|_{0;H} + |u|_{0;H})\int_{U_0}|c(\bar{\xi}, z)| \wedge 1\ \pi(\bar{\xi}, dz) \\ &\le C|x-\xi|^{\beta}\delta_0^{1-\beta}\ (|\partial u|_{0;H} + |u|_{0;H}). \end{aligned}$$

Therefore,

$$\int_{U_0} |V(t,x,\xi,z)|\ \pi(\bar{\xi}, dz) \le C|x-\xi|^{\beta}[\delta_0^{1-\beta}\ |u|_{(2+\beta);H} + |\partial u|_{0;H} + |u|_{0;H}] \tag{3.25}$$

and

$$\begin{aligned} J_1 u(t,x,\xi) &\le C|x-\xi|^{\beta}[(\delta_0^{1-\beta} + \delta_0^{1-\beta}c(n) + \nu(n) + \nu(n)^{1/2} + \delta_0^{\beta/2})\ |u|_{(2+\beta);H} \\ &\quad +(1+c_1(n))|\partial u|_{0;H} + |u|_{0;H}]. \end{aligned}$$

The inequality (3.7) follows by (3.10), (3.14), (3.18), (3.25).

Now we will estimate $J_2 u(t,x,\xi)$. We have

$$\begin{aligned} & d(x)\int_{U_0}|\nabla^2_{c(\bar{x},z)}u(\bar{x})|\ |\pi(\bar{x}, dz) - \pi(\bar{\xi}, dz)| \\ &\le (|u|_{0;H} + |\partial u|_{0;H})\int_{U_0}|c(\bar{x}, z)| \wedge 1\ |\pi(\bar{x}, dz) - \pi(\bar{\xi}, dz)| \\ &\le C|x-\xi|^{\beta}(|u|_{0;H} + |\partial u|_{0;H}). \end{aligned} \tag{3.26}$$

If $x_d + c_d(\bar{x}, z) \le 0, z \in U_1$, then $x_d \le |c(\bar{x}, z)|, d(x) \le |c(\bar{x}, z)| \wedge 1$ and

$$d(x)|\nabla^2_{c(\bar{x},z)}u(\bar{x})| \le 2|\partial u|_{0;H}|c(\bar{x}, z)|^2. \tag{3.27}$$

If $z \in \Gamma(\bar{x}, n)$, then

$$|\nabla^2_{c(\bar{x},z)} u(\bar{x})| \leq \int_0^1 (1-\tau)|\partial^2 u(t, x + \tau c(\bar{x}, z))| \ d\tau \ |c(\bar{x}, z)|^2. \tag{3.28}$$

Since $x_d + \tau c_d(\bar{x}, z) \geq (1-\tau)x_d$, we have

$$d(x)|\nabla^2_{c(\bar{x},z)} u(\bar{x})| \leq C|d^{1-\beta}\partial^2 u|_{0;H}|c(\bar{x}, z)|^2. \tag{3.29}$$

If $x_d + c_d(\bar{x}, z) > 0, z \in U_1 \backslash U_n$, then

$$d(x)|\nabla^2_{c(\bar{x},z)} u(\bar{x})| \leq C|\partial u|_{0;H}|c(\bar{x}, z)|. \tag{3.30}$$

By (3.27)-(3.30),

$$\begin{aligned} & d(x) \int_{U_n} |\nabla^2_{c(\bar{x},z)} u(\bar{x})| \ |\pi(\bar{x}, dz) - \pi(\bar{\xi}, dz)| \\ \leq \ & C|u|_{(2+\beta);H} \int_{U_n} |c(\bar{x}, z)|^2 \ |\pi(\bar{x}, dz) - \pi(\bar{\xi}, dz)| \\ \leq \ & C\nu_1(n)|x - \xi|^\beta |u|_{(2+\beta);H}, \end{aligned} \tag{3.31}$$

and

$$\begin{aligned} d(x) & \int_{U_1 \backslash U_n} |\nabla^2_{c(\bar{x},z)} u(\bar{x})| \ |\pi(\bar{x}, dz) - \pi(\bar{\xi}, dz)| \\ \leq \ & C|\partial u|_{0;H}[\int_{U_1} |c(\bar{x}, z)|^2 \ |\pi(\bar{x}, dz) - \pi(\bar{\xi}, dz)| \\ & + \int_{U_1 \backslash U_n} |c(\bar{x}, z)| \ |\pi(\bar{x}, dz) - \pi(\bar{\xi}, dz)|] \\ \leq \ & C|\partial u|_{0;H}(1 + k_1(n))|x - \xi|^\beta. \end{aligned} \tag{3.32}$$

The inequalities (3.26), (3.31), 3.32) imply the inequality (3.8).

Now, by (3.7) and (3.8),

$$\begin{aligned} & |d(x)Iu(\bar{x}) - d(\xi)Iu(\bar{\xi})| \\ \leq \ & C|x - \xi|^\beta[(\delta_0^{1-\beta} + \delta_0^{1-\beta}k(n) + \nu_1(n) + \nu(n) + \nu(n)^{1/2} + \delta_0^{\beta/2}) \ |u|_{(2+\beta);H} \\ & +(1 + k_1(n))|\partial u|_{0;H} + |u|_{0;H}]. \end{aligned}$$

Applying Lemma 1, we conclude that for each $\gamma > 0$ there is a constant C_γ such that

$$\begin{aligned} & |d(x)Iu(\bar{x}) - d(\xi)Iu(\bar{\xi})| \\ \leq \ & C|x - \xi|^\beta[(\delta_0^{1-\beta} + \delta_0^{1-\beta}k(n) + \nu_1(n) + \nu(n) + \nu(n)^{1/2} + \delta_0^{\beta/2}) \ |u|_{(2+\beta);H} \\ & +\gamma(1 + k_1(n))|u|_{(2+\beta);H} + C_\gamma|u|_{0;H}]. \end{aligned}$$

Hence choosing first a sufficiently large $n \geq 2$ and then a sufficiently small $\delta_0 > 0$ and $\gamma > 0$, we get assertion (3.5). The lemma is proved. □

The following statement is essentially a partial case of Theorem 2 in ([6]) for stochastic partial differential equations.

Theorem 2. *Let $\lambda > 0$ and let the assumption A1 be satisfied. Then for each $f \in N^{\beta}(H)$ the Cauchy -Dirichlet problem*

$$\begin{aligned} \partial_t u &= Lu - \lambda u + f \text{ in } H, \\ u(0,x) &= 0,\ x \in \mathbf{R}^d_+, \\ u(t,x) &= 0,\ t \in [0,T],\ x \in \partial\mathbf{R}^d_+ = \{y : y_d = 0\}, \end{aligned}$$

has a unique solution $u \in S^{2+\beta}(H)$. Moreover, there is a constant C not depending on u, f and λ and a bounded function $\delta(\lambda)$ not depending on u and f such that

$$|u|_{(2+\beta);H} \le C|f|_{(\beta);H}, \tag{3.33}$$

for all $s \in (0,T)$

$$|u(\cdot+s,\cdot) - u(\cdot,\cdot)|_{(1+\beta);H_s} \le Cs^{1/2}|f|_{(\beta);H}, \tag{3.34}$$

where $H_s = (0, T-s) \times \mathbf{R}^d_+$, and

$$|u|_{0;H} \le C\delta(\lambda)|f|_{(\beta);H}, \tag{3.35}$$

where $\delta(\lambda) \to 0$, as $\lambda \to \infty$.

Theorem 2 in [6] has a multiplier $s^{1/4}$ in (3.34), which appears because of the influence of stochastic terms in the problem. Analyzing the proof of Lemma 11 in [6] (the case $g = 0, h = 0$), we conclude that (3.34) holds with the multiplier $s^{1/2}$.

3.1. **Proof of Theorem 1.** Let $u \in \bar{S}^{2+\beta}(H)$ be a solution to problem (2.1). Then the function $u_\lambda = u_\lambda(t,x) = e^{-\lambda t}u(t,x)$ is a solution to the same problem with the coefficient r replaced by $r + \lambda$. By Theorem 2,

$$|u_\lambda|_{(2+\beta);H} \le C|Iu_\lambda + f_\lambda|_{(\beta);H}, \tag{3.36}$$

$$|u_\lambda|_{0;H} \le C\delta(\lambda)|Iu_\lambda + f_\lambda|_{(\beta);H}, \tag{3.37}$$

where $f_\lambda(t,x) = e^{-\lambda t}f(t,x)$, $\delta(\lambda) \to 0$, as $\lambda \to \infty$, and the constant C does not depend on $u_\lambda, \lambda, f_\lambda$.

Be Lemma 2, for each $\varepsilon > 0$ there is a constant C_ε not depending on λ and u_λ such that

$$|Iu_\lambda|_{(\beta);H} \le \varepsilon|u|_{(2+\beta);H} + C_\varepsilon|u_\lambda|_{0;H}. \tag{3.38}$$

Choosing $\varepsilon > 0$ such that $C\varepsilon < 1/2$ and using estimate (3.36), we have

$$|u_\lambda|_{(2+\beta);H} \le C_1(|u_\lambda|_{0;H} + |f_\lambda|_{(\beta);H}).$$

Now, choosing a sufficiently large λ such that $C_1C\delta(\lambda) < 1/2$, from the estimates (3.37) and (3.38) we derive the apriori estimate

$$|u_\lambda|_{(2+\beta);H} \le C_2|f_\lambda|_{(\beta);H},$$

which yields estimate (2.2) by the definition of u_λ and f_λ.

According to Theorem 2 with $\lambda = 0$, for each $v \in (0,T)$

$$|u(\cdot+v,\cdot) - u(\cdot,\cdot)|_{(1+\beta);H} \le C|Iu + f|_{(\beta);H}.$$

Using Lemma 2 and apriori estimate (2.2) we get estimate (2.3).

We finish the proof using apriori estimate (2.2) and applying a standard procedure of extension by a parameter. Let

$$L_\tau u = Lu + \tau I u,\ \ \tau \in [0,1].$$

Introduce the space $\tilde{S}^{2+\beta}(H)$ which consists of all functions $u \in \bar{S}^{2+\beta}(H)$ such that for each $(t,x) \in H$

$$u(t,x) = \int_0^t F(s,x)\ ds,$$

where $F \in N^\beta(H)$. It is a Banach space with respect to the norm

$$||u||_{(2+\beta);H} = |u|_{(2+\beta|;H} + |F|_{(\beta);H}.$$

Consider the mapping $T_\tau : \tilde{S}^{2+\beta}(H) \to N^\beta(H)$ defined by

$$u = u(t,x) = \int_0^t F(s,x)\ ds \mapsto F - L_\tau u.$$

Obviously, there is a constant C not depending on τ such that

$$|T_\tau u|_{(\beta);H} \leq C||u||_{(2+\beta);H}.$$

On the other hand, there is a constant C not depending on τ such that

$$||u||_{(2+\beta);H} \leq |T_\tau u|_{(\beta);H}.$$

Indeed,

$$u(t,x) = \int_0^t \left(L_\tau u + (F - L_\tau u)\right)(s,x)\ ds.$$

By apriori estimate (2.2) and Lemma 2, there is a constant C not depending on τ such that

$$|yu|_{(2+\beta);H} \leq C|F - L_\tau u|_{(\beta);H}.$$

Thus

$$\begin{aligned}
||u||_{(2+\beta);H} &= |u|_{(2+\beta);H} + |F|_{(\beta);H} \\
&\leq |u|_{(2+\beta);H} + |F - L_\tau u|_{(\beta);H} + |L_\tau u|_{(\beta);H} \\
&\leq C(|u|_{(2+\beta);H} + |F - L_\tau u|_{(\beta);H}) \\
&\leq C|F - L_\tau u|_{(\beta);H} = C|T_\tau u|_{(\beta);H}.
\end{aligned}$$

According to Theorem 2, T_0 is an onto map. By Theorem 5.2 in [3], all the T_τ are onto maps. The theorem is proved.

Acknowledgement. *We are very grateful to our referee for valuable comments.*

References

[1] Bony, J. M., Problem de Dirichlet et semi-groupe fortement fellerien associees a un operateur integrodifferentielle, C. R. Acad. Sci., **265** (1967), 362-364.

[2] Garroni, M. G. and Menaldi, J.-L., Green Functions for Second Order Parabolic Integro-Differential Problems, Longman Scientific, Essex, 1992.

[3] Gilbarg, D. and Trudinger, N. S., Elliptic Partial Differential Equations od Second Order, Springer, New York, 1983.

[4] Gimbert, G. and Lions, P. L., Existence and regularity results for solutions of second order elliptic integrodifferential equations, Richerce di Matematica, **33** (1984), 315-358.

[5] Mikulevicius, R. and Pragarauskas, H., On classical solutions of boundary value problems for certain nonlinear integro-differential equations, Lithuanina Mtah. J., **34** (1994, 275-287.

[6] Mikulevicius, R. and Pragarauskas, H., On Cauchy-Dirichlet problem in half-space for parabolic SPDEs in weighted Hölder spaces, Staoch. Proc. Appl., **106** (2003), 185-222.

[7] Stupelis, L., Solvability of the first boundary-value problem in $W^2_{p,0}(\Omega)$ for a class of integro-differential equations, Lithuanian Math. J., **14** (1974), 502-506.

[8] Stupelis, L., Navier-Stokes Equations in Irregular Domains, Kluwer Dordrecht, 1995.