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QUASIDERIVATIVES AND INTERIOR SMOOTHNESS OF HARMONIC FUNCTIONS ASSOCIATED WITH DEGENERATE DIFFUSION PROCESSES

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ABSTRACT. Proofs and two applications of two general results are given concerning the problem of establishing interior smoothness of probabilistic solutions of elliptic degenerate equations.

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1. Introduction

The main goal of this article is to give proofs and two applications of two general results concerning the problem of establishing interior smoothness of probabilistic solutions of elliptic degenerate equations.

More specifically, we are dealing with diffusion processes $x_t \in \mathbb{R}^d$ given as solutions of the Itô equation

$$dx_t = \sigma^k(x_t)\, dw_t^k + b(x_t)\, dt, \tag{1.1}$$

where w_t^k are independent one-dimensional Brownian motions, $k = 1, ..., d_1$, σ^k and b are $\mathbb{R}^d$-valued functions. We introduce σ as the matrix composed of the column-vectors σ^k: $\sigma = (\sigma^1, ..., \sigma^k)$, and define $a = (1/2)\sigma\sigma^*$, $\|\sigma\|^2 = \text{Tr}\ \sigma\sigma^*$,

$$Lu = a^{ij}u_{x^ix^j} + b^iu_{x^i}.$$

We assume that there is a constant $K < \infty$ such that, for all $x, y \in \mathbb{R}^d$,

$$\|\sigma(x)\| + |b(x)| \le K, \quad \|\sigma(x) - \sigma(y)\| + |b(x) - b(y)| \le K|x - y|. \tag{1.2}$$

Under this assumption, for any nonrandom initial data $x \in \mathbb{R}^d$, equation (1.1) has a unique solution $x_t(x)$. As is common we use the symbols E_x and P_x for the expectation of random variables and the probability of events defined in terms of $x_t(x)$ and drop the argument x behind the expectation and probability signs. For a domain $D \subset \mathbb{R}^d$ we denote by $\tau = \tau(x) = \tau_D(x)$ the first exit time of $x_t(x)$ from D.

Take a sufficiently smooth domain $D \subset \mathbb{R}^d$, a bounded $g \in C_b^1(\bar{D})$, assume that $P_x(\tau < \infty) = 1$ for all $x \in D$ and define

$$u(x) = E_x g(x_\tau). \tag{1.3}$$

The function u is known as a probabilistic solution of the equation $Lu = 0$ in D with boundary data $u = g$ on ∂D. However generally, u does not have two derivatives needed in L and the equation is understood in a generalized sense.

Considerable effort was applied to understand under which conditions u is twice differentiable and does satisfy the equation. The first probabilistic results were obtained by Freidlin in [3]. The techniques based on methods of the theory of partial differential equations is used in the basic sources of information about degenerate elliptic equations: [4] and [11]. In [4] and [11] in contrast with [3] the solutions are looked for in Sobolev classes. In that framework for special domains and operators one can get additional information from [1] and [5].

As long as the usual derivatives of u are concerned the best general results are presented in [6], where even controlled processes are considered. Their specification for the particular case in which L is the heat operator is given in [9] in multidimensional case. In [2] necessary and sufficient conditions are found in one space dimension for the heat equation under which the solution is k times continuously differentiable up to the boundary. However in these papers one assumes that g is at least four times continuously differentiable.

Generally, this assumption is necessary if we want to estimate the second-order derivatives of u up to the boundary.

Naturally, the question arises as to what happens if g is only once differentiable. In that case even if $L = \Delta$, one cannot assert that the first-order derivatives of u are bounded up to the boundary and one only can hope to prove that inside D the derivatives of u exist. The fact that under various conditions on the process the first-order derivatives of u can be indeed estimated inside D was proved in [7] and [8]. The trouble with these various conditions is that each particular case was treated by its own method. Furthermore, if $d = 2$, $D = \{|x| < 1\}$ and $Lu = u_{x^1x^1}$, the methods of [7] and [8] allowed estimating u_{x^2} only for $x^1 = 0$. This is particularly disturbing because in that case one can find u explicitly and the estimates are then straightforward.

In this article we present a unified probabilistic method which allows us to treat all cases of constant σ and b simultaneously. There is certain hope that the methods of this article can be applied to variable σ and b (cf. Remark 3.2). However, it is unlikely that controlled processes can be treated in the same way.

As in all known probabilistic approaches to proving smoothness of u, we differentiate formula (1.3) with respect to x. That $x_t(x)$ is differentiable with respect to x is a standard fact (trivial if σ and b are constant). The main difficulty is that $\tau = \tau(x)$ is not only non smooth in x but even discontinuous. To overcome this difficulty we use the so-called quasiderivatives of $x_t(x)$ with respect to x. This notion was explicitly introduced in [7] although the name appeared somewhat earlier.

We write $v \in \mathcal{M} = \mathcal{M}(D)$ if v is a real-valued continuously differentiable function given on D such that the process $v(x_t(x))$ is a local martingale on $[0, \tau(x))$ relative to $\mathcal{F}_t := \sigma(w_s, s \le t)$ for any $x \in D$. If u is continuously differentiable, then the strong Markov property of $x_t(x)$ implies that $u \in \mathcal{M}$. Next use the notation

$$v_{(\xi)}(x) = v_{x^i}(x)\xi^i,$$

where $\xi \in \mathbb{R}^d$ and v is sufficiently smooth.

Let $x \in D, \xi \in \mathbb{R}^d$, ξ_t and ξ_t^0 be adapted continuous processes defined on $[0, \tau(x))$ with values in $\mathbb{R}^d$ and $\mathbb{R}$, respectively, and such that $\xi_0 = \xi$. We say that ξ_t is a quasiderivative of $x_t(y)$ in the direction of ξ at point x if for any $v \in \mathcal{M}$ the following process

$$v_{(\xi_t)}(x_t(x)) + \xi_t^0 v(x_t(x)) \tag{1.4}$$

is a local martingale on $[0, \tau(x))$. In this case the process ξ_t^0 is called an adjoint process for ξ_t.

From [7] and [8] we know few examples of quasiderivatives. For instance, if processes $r_t \in \mathbb{R}$, $\pi_t \in \mathbb{R}^{d_1}$ are measurable, $\mathcal{F}_t$-adapted and such that

$$\int_0^T (|r_t|^2 + |\pi_t|^2)\, dt < \infty$$

on $\{T < \tau(x)\}$ (a.s.) for any $T \in [0,\infty)$, then, for any $\xi \in \mathbb{R}^d$, the solution ξ_t of the equation

$$\xi_t = \xi + \int_0^t [\sigma_{(\xi_s)}(x_s) + r_s \sigma(x_s)]\, dw_s$$
$$+ \int_0^t [b_{(\xi_s)}(x_s) + 2r_s b(x_s) + \sigma(x_s)\pi_s]\, ds \tag{1.5}$$

is a quasiderivative of $x_t(y)$ in the direction of ξ at point x with adjoint process given by

$$\xi_t^0 = -\int_0^t \pi_s\, dw_s.$$

According to the way equation (1.5) is derived we call the quasiderivative time-change related if $\pi_t \equiv 0$ and measure-change related if $r_t \equiv 0$. The same quasiderivatives are also available if σ and b depend not only on x but on time variable as well.

Now, an obvious idea of using quasiderivatives is to choose r_t and π_t in such a way that ξ_τ becomes tangent to ∂D at x_τ. If this is the case, then by taking the expectation of process (1.4) at time τ, we will express $v_{(\xi)}(x)$ through the derivatives of v along ∂D and the values of v itself on ∂D. Exactly this idea was used in [7] and [8] and different quasiderivatives were used in different cases, since we did not and still do not know how to steer ξ_t into the tangent plane.

In this article a different idea is exploited. Observe that the linear combination of quasiderivatives with nonrandom coefficients is also a quasiderivative. Our idea is based on the fact that sometimes one can allow the coefficients of the linear combination to be random and depend on the future.

The article is organized as follows. Our main result Theorem 4.4 is proved in Section 4 and its application to equation (1.1) with constant σ and b is given in Section 5. Since the argument in Section 5 is quite involved, in Section 3 we consider the case when $b \equiv 0$ and before that in Section 2 we prove a particular case of Theorem 4.4.

In conclusion we introduce some notation. Above we have already used $C_b^1(\bar{D})$ for the space of bounded continuous and once continuously differentiable functions on $\bar{D}$ with norm given by

$$|g|_{1,D} = |g|_{0,D} + |g_x|_{0,D}, \quad |g|_{0,D} = \sup_{x\in D} |g(x)|,$$

where g_x is the gradient of g. If D is bounded we drop the subscript b in $C_b^1(\bar{D})$ and use similar notation for spaces of functions with higher order derivatives. For $\alpha \in (0,1]$ we introduce $C^{0,\alpha}(\Gamma)$ as the set of functions g defined on a set Γ with finite norm

$$|g|'_{\alpha,\Gamma} = |g|_{0,\Gamma} + [g]_{\alpha,\Gamma}, \quad [g]_{\alpha,\Gamma} = \sup_{x,y\in\Gamma} \frac{|g(x)-g(y)|}{|x-y|^\alpha}.$$

We also use the summation convention over repeated indices and introduce more notation in Section 4.

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2. A general result

In this section D is a bounded C^1 domain in $\mathbb{R}^d$. We assume that in D

$$E_x\tau \le K, \quad E_x\tau \le K\operatorname{dist}(x,\partial D). \tag{2.1}$$

Lemma 2.1. *Let $p \in C^2(\bar{D})$, $g \in C^1(\bar{D})$. Introduce*

$$u(x) = E_x g(x_\tau), \quad v(x) = E_x(pg)(x_\tau)$$

and assume that $u, v \in C^1(\bar{D})$. Let $n(y)$ be the unit inward normal vector defined for $y \in \partial D$. Then for $y \in \partial D$ and $\eta \in \mathbb{R}^d$ we have

$$|v_{(\eta)}(y) - (pu)_{(\eta)}(y)| \le N|g|'_{1,D}(|p_x|_{0,D} + |Lp|_{0,D})|(n(y),\eta)|, \tag{2.2}$$

where N depends only on K.

Proof. First observe the following standard computations using the Markov property: for any integer $n \ge 2$

$$E_x\tau^n = nE_x\int_0^\infty (\tau - t)^{n-1}I_{\tau>t}\,dt = nE_x\int_0^\infty I_{\tau>t}E_{x_t}\tau^{n-1}\,dt$$
$$\le n\sup_{y\in D}E_y\tau^{n-1}E_x\int_0^\infty I_{\tau>t}\,dt = nE_x\tau\sup_{y\in D}E_y\tau^{n-1}$$

which by induction implies that

$$E_x\tau^n \le n!\sup_{y\in D}(E_y\tau)^n, \quad E_x\tau^n \le NE_x\tau \le N\operatorname{dist}(x,\partial D). \tag{2.3}$$

Next, notice that $v_{(\eta)}(y) - (pu)_{(\eta)}(y)$ is linear in η and vanishes if η is tangential to ∂D at y, since $v = pg = pu$ on ∂D. Therefore, it suffices to concentrate on $\eta = n = n(y)$. Fix a $y \in \partial D$, and choose $\varepsilon_0 > 0$ so that $y + \varepsilon n \in D$ as long as $0 < \varepsilon \le \varepsilon_0$. Also fix a $\varepsilon \in (0, \varepsilon_0]$ and for $x_0 = x := y + \varepsilon n$ write

$$p(x_\tau) = p(x_0) + \int_0^\tau p_{(\sigma^k)}(x_s)\,dw_s^k + \int_0^\tau Lp(x_s)\,ds.$$

Furthermore,

$$I(x) := E_x g(x_\tau)\int_0^\tau p_{(\sigma^k)}(x_s)\,dw_s^k = E_x[g(x_\tau) - g(x)]\int_0^\tau p_{(\sigma^k)}(x_s)\,dw_s^k$$
$$\le [g]_{1,D}\big(E_x|x_\tau - x|^2\big)^{1/2}\big(E_x\int_0^\tau |\sigma^* p_x(x_s)|^2\,ds\big)^{1/2}.$$

Here

$$E_x|x_\tau - x|^2 \le 2E_x|\int_0^\tau \sigma(x_t)\,dw_t|^2 + 2K^2E_x\tau^2 \le N\operatorname{dist}(x,\partial D),$$

so that

$$I(x) \le N[g]_{1,D}|\sigma^* p_x|_{0,D}\operatorname{dist}(x,\partial D).$$

Now,

$$v(x) = E_x(pg)(x_\tau) = p(x)E_xg(x_\tau) + I(x) + E_xg(x_\tau)\int_0^\tau Lp(x_s)\,ds. \quad (2.4)$$

Here $E_xg(x_\tau) = u(x)$ and the last term on the right in (2.4) is less than $K|g|_{0,D}|Lp|_{0,D}\text{dist}(x,\partial D)$. Hence we infer from (2.4) that

$$v(y+\varepsilon n) \le pu(y+\varepsilon n) + N([g]_{1,D}|p_x|_{0,D} + |g|_{0,D}|Lp|_{0,D})\text{dist}(x,\partial D). \quad (2.5)$$

Upon subtracting from this inequality the equality $v(y) = p(y)u(y)$, dividing through the result by ε, and letting $\varepsilon \downarrow 0$, we arrive at

$$v_{(n)}(y) - (pu)_{(n)}(y) \le N([g]_{1,D}|p_x|_{0,D} + |g|_{0,D}|Lp|_{0,D}).$$

Replacing g with $-g$ yields an estimate of $v_{(n)} - (pu)_{(n)}$ from below, which being combined with the above result leads to (2.2) and proves the lemma.

Corollary 2.2. *If $g \equiv 1$, then $u \equiv 1$ so that $u_{(n)} = 0$ and $|v_{(n)}| \le N|p|_{2,D}$.*

Remark 2.3. For any $\beta \ge 1$ we have $\tau^\beta \le \tau + \tau^n$, where $n = [\beta+1]$. Hence (2.3) implies that $E_x\tau^\beta \le N\text{dist}(x,\partial D)$, where N depends only on K and β.

Corollary 2.4. *Let $\xi \in \mathbb{R}^d$, $x_0 \in D$. Under the assumptions of the lemma let ξ_t be a quasiderivative of $x_t(x)$ at x_0 with adjoint process ξ^0_t and $\xi_0 = \xi$. Assume that $\xi_{t\wedge\tau(x_0)}, \xi^0_{t\wedge\tau(x_0)}$ are uniformly integrable. Then*

$$|v_{(\xi)}(x_0) - E_{x_0}\xi^0_\tau pg(x_\tau) - E_{x_0}pu_{(\xi_\tau)}(x_\tau)|$$
$$\le N|g|'_{1,D}\big(|p_x|_{0,D}E_{x_0}|\xi_\tau| + |Lp|_{0,D}E_{x_0}|(n(x_\tau),\xi_\tau)|\big). \quad (2.6)$$

Indeed, by definition

$$v_{(\xi)}(x_0) = E_{x_0}v_{(\xi_\tau)}(x_\tau) + E_{x_0}\xi^0_\tau v(x_\tau),$$

where $v(x_\tau) = pu(x_\tau) = pg(x_\tau)$ and $v_{(\xi_\tau)}(x_\tau) = (pu)_{(\xi_\tau)}(x_\tau) + I$ with

$$|I| \le N|g|'_{1,D}\big(|p_x|_{0,D} + |Lp|_{0,D}\big)|(n(x_\tau),\xi_\tau)|$$

and $|(pu)_{(\xi_\tau)}(x_\tau) - pu_{(\xi_\tau)}(x_\tau)| = |up_{(\xi_\tau)}(x_\tau)| \le |g|_{0,D}|p_x|_{0,D}|\xi_\tau|$.

Theorem 2.5. *Let some functions $p^{(1)},...,p^{(m)} \in C^2(\bar D)$, $g,q \in C^1(\bar D)$, a vector $\xi \in \mathbb{R}^d$ and a point $x_0 \in D$. Assume that $q > 0$ in $\bar D$ and on ∂D we have*

$$q(x) = \sum_{k=1}^m p^{(k)}(x).$$

Introduce

$$\bar g = g/q, \quad u(x) = E_xg(x_\tau), \quad \bar u = E_x\bar g(x_\tau), \quad v^{(k)}(x) = E_xp^{(k)}\bar g(x_\tau)$$

and assume that $u, \bar u, v^{(1)},...,v^{(m)} \in C^1(\bar D)$.

Let $\xi_t^{(k)}$, $k = 1, ..., m$, be the first quasiderivatives of $x_t(x)$ at point x_0 with adjoint processes $\xi_t^{(0k)}$ and $\xi_0^{(k)} = \xi$. Assume that, for $\tau = \tau(x_0)$, the processes $\xi_{t\wedge\tau}^{(k)}, \xi_{t\wedge\tau}^{(0k)}$ are uniformly integrable and (a.s.)

$$\bar{\xi}_\tau := \sum_{k=1}^{m} p^{(k)}(x_\tau)\xi_\tau^{(k)} \perp n(x_\tau). \tag{2.7}$$

Then we have

$$|u_{(\xi)}(x_0)| \leq \sum_{k=1}^{m} \big(|p^{(k)}\bar{g}|_{0,D}E_{x_0}|\xi_\tau^{(0k)}| + |p^{(k)}\bar{g}_x|_{0,D}E_{x_0}|\xi_\tau^{(k)}|\big)$$

$$+N|\bar{g}|'_{1,D}\sum_{k=1}^{m}\big(|p_x^{(k)}|_{0,D} + |Lp^{(k)}|_{0,D}\big)E_{x_0}|\xi_\tau^{(k)}|, \tag{2.8}$$

where N depends only on K.

Proof. By Corollary 2.4

$$|v_{(\xi)}^{(k)}(x_0) - E_{x_0}p^{(k)}\bar{g}(x_\tau)\xi_\tau^{(0k)} - E_{x_0}p^{(k)}\bar{u}_{\xi_\tau^{(k)}}(x_\tau)|$$

$$\leq N|\bar{g}|'_{1,D}\big(|p_x^{(k)}|_{0,D} + |Lp^{(k)}|_{0,D}\big)E_{x_0}|\xi_\tau^{(k)}|.$$

We sum up these inequalities with respect to i and observe that

$$\sum_{k=1}^{m} v^{(k)}(x) = E_x\bar{g}\sum_{k=1}^{m} p^{(k)}(x_\tau) \equiv u(x)$$

and owing to (2.7)

$$\sum_{k=1}^{m} p^{(k)}(x_\tau)\bar{u}_{(\xi_\tau^{(k)})}(x_\tau) = \bar{u}_{(\bar{\xi}_\tau)}(x_\tau)$$

$$= \bar{g}_{(\bar{\xi}_\tau)}(x_\tau) = \sum_{k=1}^{m} p^{(k)}(x_\tau)\bar{g}_{(\xi_\tau^{(k)})}(x_\tau).$$

Then we immediately get (2.8) and the theorem is proved.

Remark 2.6. The most natural choice for q is $q \equiv 1$. However in our applications this restriction leads to slightly worse results.

3. Equations without drift in strictly convex domains

Here we consider equation (1.1) with constant σ and $b = 0$ assuming that $\operatorname{tr}\sigma\sigma^* = 1$. Let $D \in C^3$ be a uniformly convex domain in $\mathbb{R}^d$. Then there exists a concave function $\psi \in C^3(\bar{D})$ such that $\psi > 0$ in D, $\psi = 0$ on ∂D and $\psi_{(l)(l)} \leq -2$ in D for any unit $l \in \mathbb{R}^d$. For such a function we have $L\psi \leq -1$, which along with Itô's formula imply that $E_x\tau \leq \psi(x)$, so that assumption (2.1) is satisfied with a constant depending only on $|\psi|_{2,D}$. By the way, observe that the diameter of D also can be easily estimated through $|\psi|_{2,D}$.

Theorem 3.1. *Let $g \in C^{0,1}(\bar{D})$. Then $u \in C^{0,1}_{loc}(D)$ and there is a constant N depending only on $|\psi|_{3,D}$ and d such that $|u_x| \leq N\psi^{-2}|g|_{1,D}$.*

Proof. It suffices to prove the theorem for nondegenerate a because as long as N is independent of a, one can always pass to the limit approximating a degenerate a with nondegenerate ones. For a similar reason we may assume that g and D are infinitely differentiable. In that case u is infinitely differentiable and we fix an $x_0 \in D$ and $\xi \in \mathbb{R}^d$ with the goal in mind to estimate $u_{(\xi)}(x_0)$.

We are going to only use quasiderivatives based on time change associated with parameter r. Take a constant $r > 0$ and let

$$d\xi^{(1)}_t = r\sigma^k\, dw^k_t = r\,dx_t, \quad d\xi^{(2)}_t = -r\sigma^k\, dw^k_t = -r\,dx_t,$$

$$\xi^{(1)}_t = \xi + r(x_t - x_0), \quad \xi^{(2)}_t = \xi - r(x_t - x_0), \quad \xi^{(0i)}_t \equiv 0.$$

Clearly, a vector $y \in \mathbb{R}^d$ is tangential to ∂D at a point $x \in \partial D$ if and only if $\psi_{(y)}(x) = 0$. Therefore, to satisfy (2.7) with $m = 2$ we need to find two functions $p^{(1)}$ and $p^{(2)}$ in $\bar{D}$ such that, for $x \in \partial D$,

$$\begin{aligned} 0 &= p^{(1)}\psi_{(\xi + r(x - x_0))}(x) + p^{(2)}\psi_{(\xi - r(x - x_0))}(x) \\ &= (p^{(1)} + p^{(2)})\psi_{(\xi)}(x) + r(p^{(1)} - p^{(2)})\psi_{(x - x_0)}(x). \end{aligned} \tag{3.1}$$

Let us first find $p^{(i)}$ such that $q \equiv 1$ in Theorem 2.5. Then we want to have $q = p^{(1)} + p^{(2)} = 1$ on ∂D. Hence on ∂D we find

$$p^{(1)}(x) = \frac{1}{2} - \frac{\psi_{(\xi)}(x)}{2r\psi_{(x - x_0)}(x)}, \quad p^{(2)}(x) = \frac{1}{2} + \frac{\psi_{(\xi)}(x)}{2r\psi_{(x - x_0)}(x)}.$$

Observe that since D is convex and ψ is concave, for $x \in \partial D$, we have

$$\psi_{(x - x_0)}(x) = -\psi(x_0) + \frac{1}{2}\psi_{(x - x_0)(x - x_0)}(\theta) \leq -\psi(x_0) < 0, \tag{3.2}$$

where θ is a point between x and x_0 on the straight line passing through these points.

The assumption about the smoothness of D allows us to continue $p^{(i)}$ from the boundary inside D in such a way that for thus obtained functions for which we keep the same notation we have $p^{(i)} \in C^2(\bar{D})$. Furthermore, owing to (3.2) we can do the continuation so that

$$|p^{(i)}|_{1,D} \leq N|\xi|r^{-1}\psi^{-2}(x_0), \quad |p^{(i)}|_{2,D} \leq N|\xi|r^{-1}\psi^{-3}(x_0).$$

Now

$$|p^{(i)}| \leq N(1 + |\xi|r^{-1}\psi^{-1}(x_0)), \quad E_{x_0}|\xi^{(i)}_\tau| \leq N(|\xi| + r\psi^{1/2}(x_0)),$$

so that (2.8) implies

$$\begin{aligned} |u_{(\xi)}(x_0)| &\leq N|g|_{1,D}(1 + |\xi|r^{-1}\psi^{-1}(x_0))(|\xi| + r\psi^{1/2}(x_0)) \\ &\quad + N|g|_{1,D}|\xi|r^{-1}\psi^{-3}(x_0)(|\xi| + r\psi^{1/2}(x_0)). \end{aligned}$$

By taking $r = |\xi|/\sqrt{\psi(x_0)}$ we come to

$$|u_{(\xi)}(x_0)| \leq N|g|_{1,D}|\xi|\psi^{-5/2}(x_0). \tag{3.3}$$

We have obtained a result which is slightly weaker than our claim by using a "natural" q. Now we improve it. To satisfy (2.7), which is (3.1) in our case, this time we choose

$$p^{(1)}(x) = -r\psi_{(x-x_0)}(x) + \psi_{(\xi)}(x), \quad p^{(2)}(x) = -r\psi_{(x-x_0)}(x) - \psi_{(\xi)}(x) \tag{3.4}$$

for all $x \in \bar{D}$. Obviously, $p^{(i)} \in C^2(\bar{D})$. Then due to (3.2) on ∂D we have

$$q := p^{(1)} + p^{(2)} = -2r\psi_{(x-x_0)}(x) \geq 2r\psi(x_0). \tag{3.5}$$

Using the smoothness of D we continue q from the boundary inside D and get a function $q(x), x \in \bar{D}$, such that $q \in C^1(\bar{D})$. Again (3.2) allows us to do the continuation so that

$$|q^{-1}|_{1,D} \leq Nr^{-1}\psi^{-2}(x_0), \quad |g/q|_{1,D} \leq Nr^{-1}|g|_{1,D}\psi^{-2}(x_0).$$

Finally,

$$|p^{(i)}|_{2,D} \leq N(r + |\xi|), \quad E_{x_0}|\xi^{(i)}_\tau| \leq N(|\xi| + r\psi^{1/2}(x_0)),$$

which along with (2.8) implies

$$|u_{(\xi)}(x_0)| \leq Nr^{-1}|g|_{1,D}\psi^{-2}(x_0)(r + |\xi|)(|\xi| + r\psi^{1/2}(x_0)).$$

By taking $r = |\xi|$ we come to $|u_{(\xi)}(x_0)| \leq N|g|_{1,D}|\xi|\psi^{-2}(x_0)$. The theorem is proved.

Remark 3.2. The above proof can be adjusted to cover the case when $0 \in D$ and we want to only estimate $u_{(\xi)}(0)$ under the additional assumption that there is a real-valued bounded function $\lambda(x)$ such that $\sigma_{(x)}(x) = \lambda(x)\sigma(x)$. If $\lambda \equiv 0$ this condition means that σ is constant in radial direction.

Indeed, if we define quasiderivatives $\xi^{(i)}_t$, $i = 1, 2$, on the basis of (1.5) again with $\pi \equiv 0$ and $r^{(i)}_t = (-1)^i r(1 - \lambda(x_t))$, then (1.5) becomes

$$\xi^{(i)}_t = \xi + \int_0^t [\sigma_{(\xi^{(i)}_s)}(x_s) + (-1)^i r(1 - \lambda(x_s))\sigma(x_s)]\, dw_s,$$

which has the solution $\xi^{(i)}_t = \eta_t + (-1)^i r x_t$, where η_t is the solution of the above equation with $r = 0$. However, to satisfy (2.7) which is

$$p^{(1)}\psi_{(\eta_\tau - r x_\tau)} + p^{(2)}\psi_{(\eta_\tau + r x_\tau)} = 0$$

we have to take functions $p^{(i)}$ depending not only on x but also on the extra coordinate η. We will see later that such extra coordinates also appear if $b \neq 0$. Of course, one also has to impose a condition on σ guaranteeing that $\eta_{t\wedge\tau}$ is uniformly integrable.

4. Main results

If in the situation of Section 3 we allow constant $b \neq 0$, then only using time change based quasiderivatives seems to be not enough. Indeed, in that case for constant r we have

$$\xi_t = \xi + r\sigma^k w_t^k + 2rbt = \xi + 2r(x_t - x_0) - r\sigma^k w_t^k.$$

In Section 3 by taking two different r and using linear combinations of $\xi + 2r(x_t - x_0)$ with weights $p^{(i)}$ we were able to steer the linear combination into the tangent plane to ∂D at x_τ. However this time we also have to deal with terms like $\sigma^k w_\tau^k$. We decided to make them disappear in the end by involving the parameters π associated with change of measure. These parameters contribute terms $\sigma^k \tau$ to the first quasiderivatives. By taking linear combination of such terms with weights w^k/τ we balance out the new terms. Important point to notice here is that we need to use weights depending not only on x_τ by also w_τ and τ and the weights now are not bounded and do not have bounded derivatives.

This is why we need the following generalization of Lemma 2.1. For $s \in \mathbb{R}$ and $x \in \mathbb{R}^d$ consider the following equation

$$dx_t = \sigma^k(s+t, x_t)\, dw_t^k + b(s+t, x_t)\, dt, \quad t \geq 0, x_0 = x. \tag{4.1}$$

where w_t^k and Borel functions σ^k, $k = 1, ..., d_1$, and b have the same meaning as before. As before we assume that there is a finite constant K such that (1.2), with $\sigma(s,\cdot), b(s,\cdot)$ in place of σ, b, holds for all $x, y \in \mathbb{R}^d$ and $s \in \mathbb{R}$. Then equation (4.1) has a unique solution $x_t(s,x)$. Similarly to E_x we introduce $E_{s,x}$ for expectations of certain quantities defined in terms of $x_t(s,x)$.

In this section D is a possibly unbounded domain in $\mathbb{R}^d$, $\tau = \tau(s,x) = \tau_D(s,x)$ is the first exit time of $x_t(s,x)$ from D. We assume that for all $s \in \mathbb{R}$ and $x \in D$

$$E_{s,x}\tau \leq K.$$

Define

$$Lu(s,x) = \partial u(s,x)/\partial s + a^{ij}(s,x)u_{x^i x^j}(s,x) + b^i(s,x)u_{x^i}(s,x). \tag{4.2}$$

As usual in the parabolic setting, for a function g given in $\bar{Q}_T$ with $Q_T := (T,\infty) \times D$ we denote

$$[g]_{1,\bar{Q}_T} = \sup_{t \geq T}[g(t,\cdot)]_{1,\bar{D}} + \sup_{x \in D}[g(\cdot,x)]_{1/2,[T,\infty)}, \quad |g|_{1,\bar{Q}_T} = |g|_{0,\bar{Q}_T} + [g]_{1,\bar{Q}_T}.$$

Lemma 4.1. *Let $y \in \partial D$, $s \in \mathbb{R}$, $n \in \mathbb{R}^d$, $|n| = 1$, $\varepsilon_0 > 0$. Assume that the straight segment $\Lambda := \{y + \varepsilon n : 0 < \varepsilon \leq \varepsilon_0\}$ lies in D and*

$$E_{s,y+\varepsilon n}\tau \leq K\varepsilon \quad \forall 0 < \varepsilon \leq \varepsilon_0. \tag{4.3}$$

Let p be a continuous function on $\bar{Q}_s$ such that the derivatives p_t, p_x, p_{xx} are continuous in Q_s and $p(s,\cdot) \in C^1(\bar{\Lambda})$. Assume that there are (finite) *constants* $|\sigma^* p_x|_{s,y}$ *and* $|Lp|_{s,y}$ *such that for* $0 < \varepsilon \leq \varepsilon_0$ *we have*

$$E_{s,y+\varepsilon n} \int_0^\tau |\sigma^* p_x(s+t, x_t)|^2 \, dt \leq \varepsilon |\sigma^* p_x|^2_{s,y},$$

$$E_{s,y+\varepsilon n} \int_0^\tau |Lp(s+t, x_t)| \, dt \leq \varepsilon |Lp|_{s,y}. \tag{4.4}$$

Let g *be a function with* $|g|_{1,\bar{Q}_s} < \infty$. *Introduce*

$$u(s,x) = E_{s,x} g(s+\tau, x_\tau), \quad v(s,x) = E_{s,x} pg(s+\tau, x_\tau) \tag{4.5}$$

and assume that $u(s,\cdot), v(s,\cdot) \in C^1(\bar{\Lambda})$. *Then*

$$|v_{(n)}(s,y) - pu_{(n)}(s,y)| \leq |p_{(n)} g(s,y)|$$
$$+ N[g]_{1,\bar{Q}_s} |\sigma^* p_x|_{s,y} + |g|_{0,\bar{Q}_s} |Lp|_{s,y}, \tag{4.6}$$

where N *depends only on* K.

Proof. By repeating the proof of Lemma 2.1 with obvious changes, instead of (2.5) we find that

$$v(s, y+\varepsilon n) \leq pu(s, y+\varepsilon n) + \varepsilon (N[g]_{1,\bar{Q}_s} |\sigma^* p_x|_{s,y} + |g|_{0,Q_s} |Lp|_{s,y}).$$

After that repeating the rest of the proof of Lemma 2.1 immediately leads to (4.6) and proves our lemma.

Remark 4.2. Obviously one can take $K^{1/2} |\sigma^* p_x|_{0,\bar{Q}_s}$ and $K|Lp|_{0,\bar{Q}_s}$ in place of $K|\sigma^* p_x|_{s,y}$ and $K|Lp|_{s,y}$, respectively, in (4.4).

The following lemma is proved in exactly the same way as Corollary 2.4 with only minor additional observation that, under the conditions of Lemma 4.3 the directional derivatives of v and pu in x along ∂D coincide, so that (4.6) yields estimates for $v_{(\eta)}(s,y) - pu_{(\eta)}(s,y)$ for all η.

Lemma 4.3. *(i) Let* ∂D *be once continuously differentiable,* $\xi \in \mathbb{R}^d, x_0 \in D$, $s_0 \in \mathbb{R}$.

(ii) For each $s > s_0$ *and* $y \in \partial D$ *let there exist an* $\varepsilon_0 > 0$ *such that condition* (4.3) *is fulfilled with* $n = n(y)$ *being the inward unit normal to* ∂D *at* y.

(iii) Let p *be a continuous function which is defined in* $(s_0, \infty) \times \bar{D}$ *such that the derivatives* p_t, p_x, p_{xx} *are continuous in* $(s_0, \infty) \times D$ *and* p_x *is continuous in* $(s_0, \infty) \times \bar{D}$. *Also assume that there are Borel functions* $|\sigma^* p_x|_{s,y}$ *and* $|Lp|_{s,y}$ *defined on* $(s_0, \infty) \times \partial D$ *such that for any* $s > s_0$, $y \in \partial D$, *and* $0 < \varepsilon \leq \varepsilon_0$ *condition* (4.4) *is satisfied.*

(iv) Let g *be a function with* $|g|_{1,\bar{Q}_{s_0}} < \infty$ *such that the functions* u *and* v *introduced in* (4.5) *are well defined in* $\bar{Q}_{s_0} \setminus (\{s_0\} \times \partial D)$ *and are continuously differentiable in* x *on this set.*

(v) Let an $\mathbb{R}^d$*-valued process* ξ_t *be a quasiderivative of* $x_t(s_0, x)$ *at* x_0 *with adjoint process* ξ^0_t *and* $\xi_0 = \xi$. *Assume that the process*

$$v_{x^i}(x_{t\wedge\tau}) \xi^i_{t\wedge\tau} + v(x_{t\wedge\tau}) \xi^0_{t\wedge\tau}$$

is uniformly integrable, where $x_t = x_t(s_0, x_0)$ *and* $\tau = \tau(s_0, x_0)$.

Then

$$|v_{(\xi)}(s_0, x_0) - E_{s_0,x_0}\xi^0_\tau pg(s_0+\tau, x_\tau) - E_{s_0,x_0} pu_{(\xi_\tau)}(s_0+\tau, x_\tau)|$$

$$\leq 2|g|_{0,\bar{Q}_{s_0}} E_{s_0,x_0}|p_x(s_0+\tau, x_\tau)|\,|\xi_\tau|$$

$$+N|g|_{1,\bar{Q}_{s_0}} E_{s_0,x_0}|(n(x_\tau), \xi_\tau)|(|\sigma^* p_x|_{s_0+\tau,x_\tau} + |Lp|_{s_0+\tau,x_\tau}). \tag{4.7}$$

Now follows the main result of the paper. Its proof is obtained on the basis of Lemma 4.3 in the same way as Theorem 2.5 is derived from Corollary 2.4.

Theorem 4.4. *Let assumptions (i) and (ii) of Lemma 4.3 be satisfied. Suppose that for* $k = 1, ..., m$ *we are given some objects* $p^{(k)}$, $|\sigma^* p^{(k)}_x|_{s,y}$, $|Lp^{(k)}|_{s,y}$ *having the same meaning as in Lemma 4.3 and satisfying assumption (iii) of Lemma 4.3 for each* k. *Let* q *be a function bounded away from zero with* $|q|_{1,\bar{Q}_{s_0}} < \infty$ *and such that*

$$q(t,x) = \sum_{k=1}^m p^{(k)}(t,x) \quad on \quad (s_0, \infty) \times \partial D.$$

Let g *be a function with* $|g|_{1,\bar{Q}_{s_0}} < \infty$ *such that, for* $\bar{g} := g/q$, *the functions*

$$u(s,x) = E_{s,x} g(s+\tau, x_\tau), \quad \bar{u}(s,x) = E_{s,x}\bar{g}(s+\tau, x_\tau),$$

$$v^{(k)}(s,x) = E_{s,x} p^{(k)} \bar{g}(s+\tau, x_\tau), \quad k = 1, ..., m$$

are well defined in $\bar{Q}_{s_0} \setminus (\{s_0\} \times \partial D)$ *and are continuously differentiable in* x *on this set.*

For $k = 1, ..., m$, *let* $\xi^{(k)}_t$ *be quasiderivatives of* $x_t(s_0, x)$ *at point* x_0 *with adjoint processes* $\xi^{(0k)}_t$ *and* $\xi^{(k)}_0 = \xi$. *Assume that, for* $k = 1, ..., m$, *the processes (there is no summation in* k *below)*

$$v^{(k)}_{x^i}(s_0 + t\wedge\tau, x_{t\wedge\tau})\xi^{(k)i}_{t\wedge\tau} + v^{(k)}(s_0 + t\wedge\tau, x_{t\wedge\tau})\xi^{(0k)}_{t\wedge\tau}, \tag{4.8}$$

are uniformly integrable, where $x_t = x_t(s_0, x_0)$ *and* $\tau = \tau(s_0, x_0)$.

Then

$$\left|u_{(\xi)}(s_0, x_0) - E_{s_0,x_0}\big[\sum_{k=1}^m p^{(k)}\bar{g}(s_0+\tau, x_\tau)\xi^{(0k)}_\tau + \bar{u}_{(\bar{\xi}_\tau)}(s_0+\tau, x_\tau)\big]\right|$$

$$\leq N|\bar{g}|_{1,D}\sum_{k=1}^m E_{s_0,x_0}|\xi^{(k)}_\tau|\big(|p^{(k)}_x(s_0+\tau, x_\tau)| + |\sigma^* p^{(k)}_x|_{s_0+\tau,x_\tau} + |Lp^{(k)}|_{s_0+\tau,x_\tau}\big),$$

where

$$\bar{\xi}_\tau = \sum_{k=1}^m p^{(k)}(s_0+\tau, x_\tau)\xi^{(k)}_\tau.$$

In the next section we need the following.

Corollary 4.5. *If*

$$\bar{\xi}_\tau \perp n(x_\tau) \quad (a.s.), \tag{4.9}$$

where $x_t = x_t(s_0, x_0), \tau = \tau(s_0, x_0)$, *then*

$$|u_{(\xi)}(s_0,x_0)| \leq \left|E_{s_0,x_0}\sum_{k=1}^{m}\left[p^{(k)}\bar{g}(s_0+\tau,x_\tau)\xi^{(0k)}_\tau + p^{(k)}\bar{g}_{(\xi^{(k)}_\tau)}(s_0+\tau,x_\tau)\right]\right|$$

$$+N|\bar{g}|_{1,D}\sum_{k=1}^{m}E_{s_0,x_0}|\xi^{(k)}_\tau|\big(|p^{(k)}_x(s_0+\tau,x_\tau)| + |\sigma^* p^{(k)}_x|_{s_0+\tau,x_\tau} + |Lp^{(k)}|_{s_0+\tau,x_\tau}\big). \tag{4.10}$$

5. General case of constant coefficients in uniformly convex domains

Here as in Section 3 we consider equation (1.1) with constant σ and b but without assuming that b is zero. Assume that

$$\operatorname{tr}\sigma\sigma^* + |b| = 1.$$

Again, we take $D \in C^3$ as a uniformly convex domain in $\mathbb{R}^d$.

It is trivial that $E_x\tau \leq N$ where N depends only on the diameter of D. However, in contrast with Section 3 now there is no guarantee that $E_x\tau$ goes to zero as $D \ni x \to \partial D$ not slower than $\operatorname{dist}(x,\partial D)$. This happens because $|b|$ can be large in comparison with $\operatorname{tr}\sigma\sigma^*$. If $\operatorname{tr}\sigma\sigma^* = 0$, then at some points on ∂D the function $E_x\tau$ will not even go to zero. Assuming that $\operatorname{tr}\sigma\sigma^* \geq \varepsilon^2 > 0$ does not change much the situation. For instance, it is not hard to show that, if $D \subset \mathbb{R}^2$ is the unit disk $\{|x| \leq 1\}$, $d_1 = 1$, σ^1 is the first basis vector times ε, $b = (0, 1-\varepsilon^2)$, then for any $\alpha \in (0,1)$ there is a $\varepsilon > 0$ such that $E_{(0,y)}\tau \geq (y+1)^\alpha$ for all $y \in (-1,0)$ sufficiently close to -1. This example can be treated by our methods just by enlarging the domain for $y > y_0$ so as to have a rather sharp curvature of the boundary near the south pole if we are only interested in the estimates at points (x,y) with $y > y_0$. Here b is orthogonal to the direction in which the diffusion is moving. On the other hand the situation in which $d \geq 1$ and $b = \sigma^k c_k$ with some constants c_k can be easily reduced to to the one with $b = 0$ just by using Girsanov's theorem.

However, we do not know how to treat more general situations and therefore we assume that there is a function $\phi \in C^2(\bar{D})$ such that $\phi = 0$ on ∂D, $\phi > 0$ in D and $L\phi \leq -1$ in D. Then by Itô's formula $E_x\tau \leq \phi(x) \leq N\operatorname{dist}(x,\partial D)$, where N is a constant. In that case our estimates are independent of any further relations between σ and b.

Theorem 5.1. *Under the above assumptions take a $g \in C^1(\bar{D})$ and introduce $u(x) = E_x g(x_\tau)$. Then u is locally Lipschitz continuous in D and (a.e.)*

$$|u_x(x)| \leq N|g|_{1,D}\operatorname{dist}^{-11/2}(x,\partial D), \tag{5.1}$$

where N depends only on D and ϕ.

Proof. As in the proof of Theorem 3.1, without losing generality we assume that a is uniformly nondegenerate and g and D are infinitely differentiable. Consider the two-component process $z_t = (x_t, y_t)$ given by

$$dx_t = \sigma\, dw_t + b\, dt, \quad dy_t = dw_t. \tag{5.2}$$

in the domain $D' = D \times \mathbb{R}^{d_1}$. Notice that the first exit time of (x_t, y_t) from D' is just the first exit time of x_t from D, so that in the notation of Section 4

$$u(x) = E_{s,x,y} g(x_\tau).$$

We represent the quasiderivatives of z_t as $\zeta_t = (\xi_t, \eta_t)$, where $\xi_t \in \mathbb{R}^d$ and $\eta_t \in \mathbb{R}^{d_1}$ and since we are only interested in the derivatives of u in x, we take

$$\zeta_0 = (\xi, 0), \quad x_0 \in D, \quad y_0 = 0, \quad s_0 = 0.$$

First as before we take two time change related quasiderivatives. Let $\zeta_t^{0i} \equiv 0$, $i = 1, 2$,

$$d\xi_t^{(1)} = \sigma^k\, dw_t^k + 2b\, dt, \quad d\eta_t^{(1)} = dw_t,$$

$$d\xi_t^{(2)} = -\sigma^k\, dw_t^k - 2b\, dt, \quad d\eta_t^{(1)} = -\, dw_t.$$

Then

$$\xi_t^{(1)} = \xi + 2(x_t - x_0) - \bar{\xi}_t, \quad \eta_t^{(1)} = w_t$$

$$\xi_t^{(2)} = \xi - 2(x_t - x_0) + \bar{\xi}_t, \quad \eta_t^{(2)} = -w_t,$$

where

$$\bar{\xi}_t = \int_0^t \sigma\, dw_s = \sigma^k w_t^k.$$

Take a convex $\psi \in C^3(\bar{D})$ such that $\psi = 0$ on ∂D and $\psi > 0$ in D, so that a vector κ is tangential to ∂D at a point $y \in \partial D$ if and only if $\psi_{(\kappa)}(y) = 0$.

Take the same $p^{(i)}(t, x, y) = p^{(i)}(x)$ as in (3.4) but with 2 in place of r. Then with $t = \tau$,$x = x_\tau = x_0 + \sigma w_\tau + b\tau$, $y = y_\tau = w_\tau$, and $\bar{\xi} = \bar{\xi}_\tau$ we have

$$p^{(1)}(t, x, y)\psi_{(\xi + 2(x - x_0) - \bar{\xi})}(x) + p^{(2)}(t, x, y)\psi_{(\xi - 2(x - x_0) + \bar{\xi})}(x)$$

$$= -2\psi_{(\bar{\xi})}(x)\psi_{(\xi)}(x) = -2y^k\psi_{(\sigma^k)}(x)\psi_{(\xi)}(x). \tag{5.3}$$

To make this vanish we use measure change related quasiderivatives and for $k = 1, ..., d_1$ introduce $\xi_t^{(01k)} = -\xi_t^{(02k)} = w_t^k$,

$$\xi_t^{(1k)} = \xi + \int_0^t \sigma^k\, dt = \xi + \sigma^k t, \quad \eta_t^{(1,k)} = t,$$

$$\xi_t^{(2k)} = \xi - \int_0^t \sigma^k\, dt = \xi - \sigma^k t, \quad \eta_t^{(2,k)} = -t,$$

$$p^{(1,k)}(t, x, y) := \psi_{(\xi)}(x)\frac{y^k}{t} =: -p^{(2,k)}(t, x, y).$$

Then with the same t, x, y as in (5.3)

$$\sum_{k=1}^{d_1} p^{(1,k)}(t,x,y)\psi_{(\zeta_t^{(1k)})}(x) + \sum_{k=1}^{d_1} p^{(2,k)}(t,x,y)\psi_{(\zeta_t^{(2k)})}(x)$$

$$= \sum_{k=1}^{d_1} p^{(1,k)}(t,x,y)(\psi_{(\xi_t^{(1k)})}(x) - \psi_{(\xi_t^{(2k)})}(x))$$

$$= 2t\sum_{k=1}^{d_1} p^{(1,k)}(t,x,y)\psi_{(\sigma^k)}(x) = 2y^k\psi_{(\sigma^k)}(x)\psi_{(\xi)}(x).$$

It follows that

$$\sum_{k=1}^{d_1} p^{(1,k)}(t,x,y)\psi_{(\zeta_t^{(1k)})}(x) + \sum_{k=1}^{d_1} p^{(2,k)}(t,x,y)\psi_{(\zeta_t^{(2k)})}(x)$$
$$+p^{(1)}(t,x,y)\psi_{(\zeta_t^{(1)})}(x) + p^{(2)}(t,x,y)\psi_{(\zeta_t^{(1)})}(x) = 0$$

and the vector

$$\sum_{k=1}^{d_1} p^{(1,k)}(\tau,x_\tau,y_\tau)\zeta_\tau^{(1k)} + \sum_{k=1}^{d_1} p^{(2,k)}(\tau,x_\tau,y_\tau)\zeta_\tau^{(2k)}$$
$$+p^{(1)}(\tau,x_\tau,y_\tau)\zeta_\tau^{(1)} + p^{(2)}(\tau,x_\tau,y_\tau)\zeta_\tau^{(2)}$$

is tangential to $\partial D'$ at (x_τ, y_τ).

Obviously, the same function q from (3.5) equals the sum of all $p^{(i)}, p^{(jk)}$ over $i, j = 1, 2$, $k = 1, ..., d_1$ in $(0,\infty) \times \partial D'$ (notice that the value $t = 0$ is excluded).

To check other assumptions of Theorem 4.4 we need the following.

Lemma 5.2. *Let $\alpha \geq 0, \beta \geq 1$ be some constants. Then there is a constant N depending only on D and α, β such that, for $s \geq 0$ and $x \in D$,*

$$E_{s,x,y}\int_0^\tau \frac{1}{(s+t)^\alpha}\,dt \leq N\frac{1}{s^\alpha}\psi(x), \tag{5.4}$$

$$E_{s,x,y}\int_0^\tau \frac{|y_t|^\beta}{(s+t)^\alpha}\,dt \leq N\frac{|y|^\beta+1}{s^\alpha}\psi(x). \tag{5.5}$$

$$E_x\frac{1}{(s+\tau)^\alpha} \leq N\frac{1}{s^\alpha+\psi^{2\alpha}(x)}, \tag{5.6}$$

$$E_x\frac{|w_\tau|^\beta}{(s+\tau)^\alpha} \leq N\frac{\psi^{1/2}(x)}{s^\alpha+\psi^{2\alpha}(x)}. \tag{5.7}$$

Proof. Estimate (5.4) follows from the fact that $(s+t)^{-\alpha} \leq s^{-\alpha}$. We use the same observation and also notice that since $y_t = y + w_t$ is a martingale, $|y_t|^\beta$ is a submartingale. Then the left-hand side of (5.5) times s^α turns out to be less than

$$\int_0^\infty E_x|y+w_t|^\beta I_{\tau>t}\,dt \leq \int_0^\infty E_x|y+w_\tau|^\beta I_{\tau>t}\,dt$$

$$= E_x|y + w_\tau|^\beta \tau \le |y|^\beta E_x\tau + (E_x|w_\tau|^{2\beta})^{1/2}(E_x\tau^2)^{1/2}.$$

Here by the Burkholder-Davis-Gundy inequalities and Remark 2.3

$$E_x|w_\tau|^{2\beta} \le NE_x\tau^\beta \le NE_x\tau.$$

This yields (5.5).

While proving (5.6) it suffices to concentrate on $s = 0$ since $(s+\tau)^{-\alpha}$ is less than both $s^{-\alpha}$ and $\tau^{-\alpha}$ and in addition $a^{-\alpha} \wedge b^{-\alpha} \le 2^\alpha (a+b)^{-\alpha}$.

Denote $\rho = \operatorname{dist}(x, \partial D)$ and observe that $\rho \le |x_\tau - x_0| \le |w_\tau| + \tau$. Therefore, by exponential estimates, for any $\varepsilon \in (0, \varepsilon_0)$, where $\varepsilon_0 := \rho/2$, we have

$$P_x(\tau < \varepsilon) \le P_x(\sup_{t\le\varepsilon} |w_t| \ge \rho - \varepsilon)$$

$$\le P_x(\sup_{t\le\varepsilon} |w_t| \ge \rho/2) \le Ne^{-\rho^2/(8\varepsilon)}.$$

Hence,

$$E_x\tau^{-\alpha} = \int_0^\infty P_x(\tau < t^{-1/\alpha})\,dt \le \varepsilon_0^{-\alpha} + N\int_{\varepsilon_0^{-\alpha}}^\infty e^{-\rho^2 t^{1/\alpha}/8}\,dt$$

$$\le N\rho^{-\alpha} + N\int_0^\infty e^{-\rho^2 t^{1/\alpha}/8}\,dt = N\rho^{-\alpha} + N\rho^{-2\alpha} \le N\rho^{-2\alpha}.$$

This proves (5.6). To prove (5.7) it suffices to use Cauchy's inequality along with the fact that $E_x|w_\tau|^{2\beta} \le NE_x\tau^\beta \le N\psi(x)$. The lemma is proved.

Lemma 5.2 implies, in particular, that for $s \ge 0$ the function

$$v^{(ik)}(s, x, y) = E_{s,x,y}p^{(ik)}(s+\tau, x_\tau, y_\tau)\bar{g}(x_\tau)$$

are well defined and (with N depending on x_0)

$$|v^{(ik)}(s, x, y)| \le N|\bar{g}|_{0,D}|\xi|E_x\frac{|y + w_\tau|}{s+\tau} \le N\frac{|y| + \psi^{1/2}(x)}{s + \psi^2(x)}.$$

To prove that $v^{(ik)}$ are smooth in $\bar{Q}_0 \setminus (\{0\} \times \partial D')$, where $Q_0 = (0,\infty) \times D'$, it is convenient to assume that $\sigma = \sqrt{2a}$. Obviously this assumption does not restrict generality and allows us to write $y_t = y_t(s,x,y) = w_t + y = \sigma^{-1}(x_t - x - bt) + y$. Hence,

$$v^{(ik)}(s, x, y) = E_x p^{(ik)}(s+\tau, x_\tau, \sigma^{-1}(x_\tau - b\tau))\bar{g}(x_\tau)$$

$$+(y - \sigma^{-1}x)^k E_x p^{(ik)}(s+\tau, x_\tau, e_k)\bar{g}(x_\tau)$$

$$=: \bar{v}^{(ik)}(s,x) + (y - \sigma^{-1}x)^k \tilde{v}^{(ik)}(s,x),$$

where e_k is the kth basis vector. This formula expresses $v^{(ik)}$ through the solutions $\bar{v}^{(ik)}$ and $\tilde{v}^{(ik)}$ of the equation $\partial v/\partial t + Lv = 0$ in $[0,\infty) \times D$. Since the equation is nondegenerate, local regularity results (see, for instance, Chapter 4 in [10]) show that $v^{(ik)}$ is indeed infinitely differentiable in $\bar{Q}_0 \setminus (\{0\} \times \partial D')$ and

$$|v^{(ik)}_{x,y}(s,x,y)| \le N\frac{|y|+1}{(s+\psi^2(x))\psi(x)} \le N\frac{|y|+1}{\psi^3(x)} \le N(1+|y|^2+\psi^{-6}(x)). \tag{5.8}$$

Furthermore, if $s_0>0$ and $x_0\in\partial D$, then in the intersection of $(s_0/2, 3s_0/2)\times\{x : |x-x_0|<\sqrt{s_0/2}\}$ with $(0,\infty)\times\partial D$ the C^3-norms of the boundary data of $\bar{v}^{(ik)}$ and $\tilde{v}^{(ik)}$ are bounded by a constant times $1+s_0^{-3}$. From boundary and interior regularity results for parabolic equations we now get that, for each $s>0$, $|\bar{v}^{(ik)}_x(s,x)|$ and $|\tilde{v}^{(ik)}_x(s,x)|$ are dominated by a constant times $1+s^{-7/2}$. It follows that

$$|v^{(ik)}_{x,y}(s,x,y)| \le N(1+|y|^2+s^{-7}),$$

which along with (5.8) yields

$$\begin{aligned}|v^{(ik)}_{x,y}(s,x,y)| &\le N(1+|y|^2+s^{-7}\wedge\psi^{-6}(x))\\ &\le N(1+|y|^2)+N\frac{1}{s^7+\psi^6(x)}.\end{aligned}$$

Now fix an $x\in D$, define $D_1=\{y\in D : \psi(y)>\psi(x)/2\}$ and let γ be the first exit time of $x_t=x_t(x)$ from D_1. Notice that, if $t\le\gamma$, then $\psi(x_t)>\psi(x)/2$. However, if $t\ge\gamma$, then $t^7\ge\gamma^7$. Since by Lemma 5.2 applied to D_1 we have $E_x\gamma^{-7}\le\psi^{-14}(x)$, the above argument shows that, for any $x\in D$,

$$E_x\sup_{t\le\tau}\frac{1}{t^7+\psi^6(x_t)} \le E_x\sup_{t\le\gamma}\frac{1}{t^7+\psi^6(x_t)} + E_x\sup_{\gamma\le t\le\tau}\frac{1}{t^7+\psi^6(x_t)} < \infty.$$

One can also easily estimate $E_x\sup_{t\le\tau}|\xi^{(ik)}_t|^2$ and make other necessary computations to see that the requirement in Theorem 4.4 about processes corresponding to (4.8) is satisfied in our particular situation.

To apply Theorem 4.4 we also need to check condition (iii) in Lemma 4.3 for $p^{(i)}, p^{(jk)}$. Denote by p any one of these functions and notice that for the operator L as in (4.2) associated with z_t we have

$$|\sigma^* p_z(t,x,y)| \le N(1+|\xi|)+N|\xi|\big(1+\frac{|y|+1}{t}\big),$$

$$|Lp(t,x,y)| \le N(1+|\xi|)+N|\xi|\big(1+\frac{|y|+1}{t}+\frac{|y|+1}{t^2}\big).$$

Lemma 5.2 shows that as $|\sigma^* p_z|_{s,x,y}$ and $|Lp|_{s,x,y}$ for $p=p^{(i)}, p^{(jk)}$ and $(s,x,y)\in(0,\infty)\times D'$ we can take

$$N+N|\xi|(|y|+1)(1+t^{-2})$$

and then

$$E_{0,x_0,0}(|p_z(\tau,z_\tau)|^2+|\sigma^* p_z|^2_{\tau,x_\tau,y_\tau}+|Lp|^2_{\tau,x_\tau,y_\tau}) \le N(1+|\xi|^2\psi^{-8}(x_0)). \tag{5.9}$$

We can finally use (4.10). Observe that the summation there is to include all terms corresponding to $p^{(i)}, p^{(jk)}$. First we are dealing with $p^{(jk)}$. We have

$$I_1 := \sum_{i=1}^{2}\sum_{k=1}^{d_1} p^{(ik)}\bar{g}(\tau, x_\tau)\xi_\tau^{(0ik)} = 2\bar{g}(\tau, x_\tau)\psi_{(\xi)}(x_\tau)|w_\tau|^2/\tau,$$

$$I_2 := \sum_{i=1}^{2}\sum_{k=1}^{d_1} p^{(ik)}\bar{g}_{(\xi_\tau^{(ik)})}(\tau, x_\tau) = 2\bar{g}_{(\sigma^k)}w_\tau^k\psi_{(\xi)}(x_\tau).$$

By Lemma 5.2

$$E_{0,x_0,0}(|I_1| + |I_2|) \le N|\xi|\psi^{-1-2+1/2}(x_0) + N|\xi|\psi^{-2+1/2}(x_0).$$

Owing to (5.9) and the estimate

$$E_{0,x_0,0}|\xi_\tau^{(ik)}|^2 \le N(|\xi|^2 + \psi(x_0)),$$

we see that the last term on the right in (4.10) corresponding to $p^{(jk)}$ is less than

$$N\psi^{-2}(x_0)(1 + |\xi|\psi^{-4}(x_0))(|\xi| + \psi^{1/2}(x_0)).$$

Similarly and somewhat easier on estimates the terms corresponding to $p^{(1)}, p^{(2)}$ and then according to (4.10) we conclude

$$|u_{(\xi)}(x_0)| \le N|\xi|\psi^{-5/2}(x_0) + N\psi^{-2}(x_0)(1 + |\xi|\psi^{-4}(x_0))(|\xi| + \psi^{1/2}(x_0)).$$

We substitute here $\psi^{1/2}(x_0)\xi/|\xi|$ in place of ξ and obtain (5.1) thus proving the theorem.

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