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Random walks in a Dirichlet environment

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Abstract

This paper states a law of large numbers for a random walk in a random iid environment on $\mathbb{Z}^d$, where the environment follows some Dirichlet distribution. Moreover, we give explicit bounds for the asymptotic velocity of the process and also an asymptotic expansion of this velocity at low disorder

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1 Introduction

After a first breakthrough of Kalikow (3), giving a transience criterion for non-reversible multidimensional Random Walks in Random Environment, Sznitman and Zerner proved, several years later, a law of large numbers in (12), followed by a central limit theorem proved by Sznitman in (10). A generalization to the case of mixing environments was proved afterwards by Comets and Zeitouni in (1) and also Rassoul-Agha (6) who used the point of view of the environment viewed from the particle (we refer to (13) for an overview of the subject). Despite these progresses, many important questions, concerning recurrence or explicit criteria for a ballistic behavior, remain largely open.

Among random walks in random environment, random walks in an iid Dirichlet environment take a special place, since their annealed law coincides with the law of some transition reinforced random walk having an affine reinforcement (see (2)). These reinforced walks are defined as follows. At time 0, we attribute, in a translation invariant way, a weight to each oriented edge of $\mathbb{Z}^d$, and each time the walk crosses an edge, the weight of this edge is increased by one. Finally, the walk is a nearest neighbour walk, which chooses, at each time, an outgoing edge with a probability which is proportionnal to its weight. Let us also mention that our interest in Dirichlet environments is reinforced by their link with the beautiful theory of hypergeometric integrals, as it is shown in (8).

The question of transience and recurrence for such walks, was answered by Keane and Rolles, in (4), in the case where the walk evolves on a graph which is a product of the integer line with a finite graph. In the context of trees, a correspondance between reinforced random walks and random walks in random environment was used before, by Pemantle, in (5). Our purpose is to give some first results in the case of $\mathbb{Z}^d$.

In this paper, we state a law of large numbers for such random walks, under a simple and explicit condition on the weights. Moreover, we give explicit bounds for the asymptotic velocity of these walks and also an asymptotic expansion of this velocity at low disorder. Low disorder corresponds, in the random environment model, to the case where the law of the transition probabilities is concentrated around its mean value, and, in its reinforcement interpretation, to the case where the initial weights of the transitions are large, so that these weights are not significantly affected during the life of the walk (at least, if the walk is transient).

Let us precise that these walks do not enter the class of walks considered in (3), (7) and in several other works, asking the law of the environment to satisfy a uniform ellipticity condition.

This ellipticity hypothesis is usually used in two ways :

_ in the definition of Kalikow's auxiliary Markov chain which involves the expection of the Green function of the walk killed when exiting a given set. The uniform ellipticity is then a comfortable assumption for checking the integrability of this Green function.

_ in the estimates of the drift of Kalikow's auxiliary Markov chain, the ellipticity condition often plays a key role. We overcome this difficulty by using an integration by part formula.

In section 2, we give the definition of random walks in Dirichlet environment, remind their connection with transition reinforced random walks and we present our main results. In section 3, we present an integration by part formula that will be the key analytic tool in the proof of our results. Indeed, in section 4, it is shown how one can take advantage of the special form of the law of the environment, in order to estimate, using the formula of Section 3, the drift of the killed Kalikow's auxiliary walk.

It turns out that this "integration by part technic" is specially well adapted to the check of Kalikow's criterion. Note that Sznitman introduced finer criteria, the so called T and T' (see (11)), which ensure ballisticity, but we don't think we could easily verify these criteria by our "integration by part technic".

In section 5, we study the integrability of the Green function of the walk which ensures the existence of the original (non killed) Kalikow's auxiliary walk and finish the proof of our first result by applying the law of large numbers of Sznitman and Zerner (12). In section 6, we prove precise estimates for the transition probabilities of Kalikow's auxiliary random walk in order to get an expansion of the asymptotic velocity at low disorder.

2 Definitions and statement of the results

We denote by $T_{2d} := \{(x_1, ..., x_{2d}) \in]0,1]^{2d},\ s.t., \sum_{i=1}^{2d} x_i = 1\}$, and by $(e_i)_{1\leq i\leq 2d}$ the family of unitary vectors of $\mathbb{Z}^d$, defined as follows: $(e_i)_{1\leq i\leq d}$ is the canonical basis of $\mathbb{R}^d$, and for all $j \in \{d+1, ..., 2d\}$ $e_j = -e_{j-d}$.

For all $\vec{\alpha} := (\alpha_1, ..., \alpha_{2d}) \in]0, +\infty[^{2d}$, we denote by $\lambda^{\vec{\alpha}}$ the Dirichlet probability measure on T_{2d} with parameters $(\alpha_1, ..., \alpha_{2d})$ i.e. the measure on T_{2d}:

$$\frac{\Gamma(\alpha_1 + ... + \alpha_{2d})}{\Gamma(\alpha_1)...\Gamma(\alpha_{2d})} x_1^{\alpha_1-1}...x_{2d}^{\alpha_{2d}-1} dx_1...dx_{2d-1}.$$

For a unit vector e of $\mathbb{Z}^d$, we will sometimes write, for reading conveniences, α_e for the weight α_i where i is such that $e_i = e$.

Let us now introduce random walks in an iid Dirichlet environment on $\mathbb{Z}^d$.

We define an environment as an element $\omega = (\omega(x))_{x\in\mathbb{Z}^d}$ where at any vertex x, $\omega(x) := (\omega(x, x + e_1), ..., \omega(x, x + e_{2d}))$ belongs to T_{2d}. We set $\mu := \underset{x\in\mathbb{Z}^d}{\otimes} \lambda^{\vec{\alpha}}$, so that μ is a probability measure on the environments such that $(\omega(x))_{x\in\mathbb{Z}^d}$ are independent random variables of law $\lambda^{\vec{\alpha}}$.

We denote by P^ω the law of the Markov chain in the environment ω starting at 0 defined by:

$$\forall x \in \mathbb{Z}^d,\ \ \forall k \in \mathbb{N},\ \ \forall i = 1, ..., 2d,\ \ P^\omega(X_{k+1} = x + e_i | X_k = x) = \omega(x, x + e_i).$$

The law of the random walk in random environment (or the so-called annealed measure) is the probability measure $P^\mu = \int P^\omega d\mu(\omega)$.

In (2), we show that random walks in iid environment have the law of some reinforced random walk. The following proposition states that the case of a Dirichlet environment corresponds to a quite natural law of reinforcement:

Proposition 1. *The measure P^μ satisfies that P^μ-almost everywhere,*

$$P^\mu(X_{n+1} = x + e_i | \sigma(X_k, k \leq n)) = \frac{\alpha_i + N_i(n, X_n)}{\sum_{k=1}^{2d} \alpha_k + N_k(n, X_n)}$$

where $\vec{N}(n, x) = (N_i(n, x))_{1\leq i\leq 2d}$ *and* $N_i(n, x) = \sum_{l=0}^{n-1} 1_{\{X_{l+1}-X_l=e_i, X_l=x\}}$.

We refer the reader to (2) for the proof.

2.1 Bounds of the asymptotic velocity

We can now state our first result:

Theorem 1. *Let* $\vec{\alpha} := (\alpha_1, ..., \alpha_{2d}) \in]0, +\infty[^{2d}$, *and* $\mu = \bigotimes_{x\in\mathbb{Z}^d} \lambda^{\vec{\alpha}}$ *a probability measure on the environment. Let us assume that there exists* $i \in \{1, ..., 2d\}$ *such that* $\alpha_{e_i} > 1 + \alpha_{-e_i}$.
The process X_n *is transient under* P^μ, *and*

$$\exists v \in \mathbb{R}^d \setminus \{0\},\ \textit{such that}\, P^\mu(\frac{X_n}{n} \to_{n\to\infty} v) = 1.$$

Moreover, for all $i \in \{1, ..., d\}$,

$$\frac{\alpha_{e_i} - \alpha_{-e_i} - 1}{(\sum_{k=1}^{2d} \alpha_k) - 1} \le v.e_i \le \frac{\alpha_{e_i} - \alpha_{-e_i} + 1}{(\sum_{k=1}^{2d} \alpha_k) - 1}$$

Remark 1: The assumption on α_i ensures that the set

$$\prod_{i=1}^{d} [\alpha_{e_i} - \alpha_{-e_i} - 1, \alpha_{e_i} - \alpha_{-e_i} + 1]$$

does not contain 0. It is a key ingredient in the check of Kalikow's transience condition.

Remark 2: When the α_i's are large, v becomes close to the vector $\frac{1}{\sum_{k=1}^{2d} \alpha_k} \sum_{i=1}^{d} (\alpha_{e_i} - \alpha_{-e_i}) e_i$. This is not surprising if one thinks at the corresponding reinforced walk: the initial weights of the transitions are large enough so that they are not significantly affected during the life of the walk, and the law of the walk becomes close to the law of the Markov chain with probability transition $\frac{\alpha_i}{\sum_{k=1}^{2d} \alpha_k}$ in the direction e_i.

Remark 3: In dimension 1, the condition of theorem 1 is actually optimal. Indeed from (9), we know that the asymptotic velocity is not null if and only if either $E^\mu[\frac{\omega(0,e_1)}{\omega(0,-e_1)}] > 1$ or $E^\mu[\frac{\omega(0,-e_1)}{\omega(0,e_1)}] > 1$, which corresponds exactly to $\alpha_{e_1} > 1 + \alpha_{-e_1}$ or $\alpha_{-e_1} > 1 + \alpha_{e_1}$. Moreover, the asymptotic velocity of the walk is equal to $\frac{\alpha_{e_1} - \alpha_{-e_1} - 1}{\alpha_{e_1} + \alpha_{-e_1} - 1}$. This shows the optimality of the lower bound in Theorem 1.

2.2 Expansion of the velocity in the limit of large parameters

We turn now to the second result of the paper, which gives the asymptotic velocity of the walk in the limit of large parameters α_k. Let us remind that, in the limit of large parameters α_k, the environment is concentrated around its mean value.

Let us fix some notations. We consider some *fixed* transition probabilities

$$(m_i) \in T_{2d},$$

and a parameter $\gamma > 0$ (aimed to tend to ∞). We consider the weights

$$\alpha_k = \gamma m_k,$$

so that the expectation of the transition probability, $E_\mu(\omega(x, x+e_i))$, is independent of γ and equal to m_i.

The mean environment (m_i) defines the transition probabilities of an homogeneous walk on $\mathbb{Z}^d$, which is ballistic with asymptotic velocity

$$d_m = \sum_{k=1}^{2d} m_k e_k,$$

when the mean drift d_m is not null. We denote by G^m its Green function.

The following result gives an estimate in $O(\frac{1}{\gamma^2})$ of the asymptotic velocity (in section 6 we give explicit bounds for this estimate).

Theorem 2. *Assume $d_m \neq 0$.*
For γ large enough, Theorem 1 applies, i.e. there exists $v \neq 0$ such that $\lim_{n\to\infty} \frac{X_n}{n} = v$, P^μ a.s.. Moreover, when γ is large, we have the following expansion for v:

$$v = d_m - \frac{d_m}{\gamma}(G^m(0,0) - 1) + O(\frac{1}{\gamma^2}).$$

Remark 1: Surprisingly, the second order of the expansion is colinear to the mean drift d_m. We see that $(G^m(0,0) - 1) > 0$, which means that there is a slowdown effect, since the second order term is directed in the opposite direction to the mean drift.

Remark 2: In (7), the second author gave an expansion of the asymptotic velocity in the case of a uniformly elliptic environment. In this work, several of the estimates relied strongly on the ellipticity condition, so that the proofs of (7) have here to be modified. Nevertheless, if we apply the formula of (7) to this case, we get the same expansion (many simplifications occur due to the particular expression of the covariance matrix). It is not surprising that the speed-up effect obtained in some cases of (7) is not observed in the case of a Dirichlet environment. The example of (7), section 2, was based, indeed, on some correlation between the transition probabilities in orthogonal directions. Here, there is a kind of independence of the transition probabilities in each direction, in the following sense: under $\mu = \underset{x\in\mathbb{Z}^d}{\otimes} \lambda^{\vec{\alpha}}$, the law of $\omega(z, z+e_i)$ is independent of the law of $(\frac{\omega(z,z+e_k)}{1-\omega(z,z+e_i)})_{k\neq i}$.

Remark 3: The Green function $G^m(0,0)$ has the following explicit Fourier expression

$$G^m(0,0) = \frac{1}{(2\pi)^d} \int_{[0,2\pi]^d} \frac{1}{1 - 2\sum_{i=1}^d \sqrt{m_{e_i} m_{-e_i}} \cos(\theta_i)} d\theta_1 \cdots d\theta_d.$$

(we refer to Step 2 of the proof of Proposition 3).

3 An integration by part formula

In this section, we present an integration by part formula on T_{2d} that will appear to be the key analytic tool in the estimation of the drift of Kalikow's auxiliary walk.

Lemma 1. *For all $\vec{\alpha} \in]0,+\infty[^{2d}$, and all differentiable function f on $\mathbb{R}^{2d}$,*

$$\int_{T_{2d}} f d\lambda^{\vec{\alpha}} = \frac{\alpha_1 + ... + \alpha_d}{\alpha_1} \int_{T_{2d}} x_1.f d\lambda^{\vec{\alpha}} + \frac{1}{\alpha_1} \int_{T_{2d}} x_1.((\sum_{k=1}^{2d} x_k \frac{\partial f}{\partial x_k}) - \frac{\partial f}{\partial x_1}) d\lambda^{\vec{\alpha}}.$$

Proof: We use the well known identity between the Dirichlet law $\lambda^{\vec{\alpha}}$ and the law of the vector $(\frac{Z_1}{\sum_{i=1}^{2d} Z_i}, ..., \frac{Z_{2d}}{\sum_{i=1}^{2d} Z_i})$ where the random variables Z_i are independent variables following the gamma distribution $\Gamma(\alpha_i, 1)$ of density $\frac{1}{\Gamma(\alpha_i)} z^{\alpha_i - 1} e^{-z}$ on $\mathbb{R}_+$.
This identity implies

$$\int_{T_{2d}} f d\lambda^{\vec{\alpha}} =$$

$$\frac{1}{\Gamma(\alpha_1)...\Gamma(\alpha_{2d})} \int_{\mathbb{R}_+^{2d}} f(\frac{z_1}{\sum_{i=1}^{2d} z_i}, ..., \frac{z_{2d}}{\sum_{i=1}^{2d} z_i}) e^{-\sum_{i=1}^{2d} z_i} z_1^{\alpha_1 - 1} ... z_{2d}^{\alpha_{2d}-1} dz_1 ... dz_{2d}.$$

Integrating by part with respect to z_1, we get

$$\int_{T_{2d}} f d\lambda^{\vec{\alpha}} = \frac{1}{\Gamma(\alpha_1 + 1)...\Gamma(\alpha_{2d})} \int_{\mathbb{R}_+^{2d}} (\tilde{f} - \frac{\partial \tilde{f}}{\partial z_1}) e^{-\sum_{i=1}^{2d} z_i} z_1^{\alpha_1} ... z_{2d}^{\alpha_{2d}-1} dz_1 ... dz_{2d}$$

where $\tilde{f}(z_1, ..., z_{2d}) := f(\frac{z_1}{\sum_{i=1}^{2d} z_i}, ..., \frac{z_{2d}}{\sum_{i=1}^{2d} z_i})$.
Now, we decompose this last integral into the $\tilde{f}$-part and the $\frac{\partial \tilde{f}}{\partial z_1}$-part.
Using, in the reverse sense, the "Gamma" interpretation of the Dirichlet law $\lambda^{(\alpha_1+1, \alpha_2, ..., \alpha_{2d})}$, the $\tilde{f}$-part becomes

$$\frac{\Gamma(\alpha_1 + ... + \alpha_{2d} + 1)}{\Gamma(\alpha_1 + 1)...\Gamma(\alpha_{2d})} \int_{T_{2d}} f.x_1^{\alpha_1} ... x_{2d}^{\alpha_{2d}-1} dx_1 ... dx_{2d-1} = \frac{\alpha_1 + ... + \alpha_{2d}}{\alpha_1} \int_{T_{2d}} x_1.f d\lambda^{\vec{\alpha}}.$$

Now, the $\frac{\partial \tilde{f}}{\partial z_1}$-part writes

$$-\frac{1}{\alpha_1 \Gamma(\alpha_1)...\Gamma(\alpha_{2d})} \int_{\mathbb{R}_+^d} (z_1.\frac{\partial \tilde{f}}{\partial z_1}) e^{-\sum_{i=1}^{2d} z_i} z_1^{\alpha_1 - 1} ... z_{2d}^{\alpha_{2d}-1} dz_1 ... dz_{2d}$$

and

$$\begin{aligned} z_1.\frac{\partial \tilde{f}}{\partial z_1} &= (\frac{z_1}{\sum_{i=1}^{2d} z_i} - \frac{z_1^2}{(\sum_{i=1}^{2d} z_i)^2}) \tilde{f}_1 - \frac{z_1 z_2}{(\sum_{i=1}^{2d} z_i)^2} \tilde{f}_2 - ... - \frac{z_1 z_{2d}}{(\sum_{i=1}^{2d} z_i)^2} \tilde{f}_d \\ &= (\frac{z_1}{\sum_{i=1}^{2d} z_i})(\tilde{f}_1 - \sum_{k=1}^{2d} (\frac{z_k}{\sum_{i=1}^{2d} z_i}) \tilde{f}_k) \end{aligned}$$

where $\tilde{f}_k(z_1, .., z_{2d}) = \frac{\partial f}{\partial x_k}(\frac{z_1}{\sum_{i=1}^{2d} z_i}, ..., \frac{z_{2d}}{\sum_{i=1}^{2d} z_i})$.
The "Gamma" interpretation of the Dirichlet law $\lambda^{(\alpha_1, ..., \alpha_{2d})}$ (used for the third time) allows to conclude. $\square$

4 Kalikow's auxiliary walk

We remind here the generalization of Kalikow's auxiliary walk (see (3)) which was already presented in (7).

Let U be a connected subset of $\mathbb{Z}^d$, and $\delta \in]0,1]$. We denote by ∂U the boundary set of U, i.e. $\partial U := \{z \in \mathbb{Z}^d \setminus U, \exists x \in U, |z-x| = 1\}$.

For all $z \in U$ and $z' \in U \cup \partial U$, and for all environment ω, we introduce the Green function of the random walk under the environment ω killed at rate δ and at the boundary of U:

$$G^{\omega}_{U,\delta}(z,z') = E^{\omega}_{z}\left(\sum_{k=0}^{T_U} \delta^k 1_{X_k = z'}\right)$$

where $T_U = inf\{k, X_k \in \mathbb{Z}^d \setminus U\}$.

In the sequel, we will drop the subscript δ when $\delta = 1$, and we will write $G^{\omega}_{U}(z,z')$ instead of $G^{\omega}_{U,1}(z,z')$.

We introduce now the generalized Kalikow's transition probabilities (originally, Kalikow's transition probabilities were introduced in the case $\delta = 1$):

$$\hat{\omega}_{U,\delta,z_0}(z, z+e_i) = \frac{E_{\mu}[G^{\omega}_{U,\delta}(z_0,z)\omega(z,z+e_i)]}{E_{\mu}[G^{\omega}_{U,\delta}(z_0,z)]}.$$

In order to give bounds for these transition probabilities, we will be led to apply the integration by part formula of the previous section to the functions $G^{\omega}_{U,\delta}(x,y)$, viewed as functions of the environment ω.

For this purpose, we need the following lemma which gives the expression of the derivatives of these functions. Before stating this lemma, we introduce the transition matrix Ω_U defined by $\Omega_U(x,y) = \omega(x,y)$ if $x \in U$, and $\Omega_U(x,y) = 0$ if $x \in \partial U$. Obviously,

$$G^{\omega}_{U,\delta}(x,y) = (I - \delta\Omega_U)^{-1}(x,y) = \sum_{n\geq 0} \delta^n (\Omega_U)^n_{(x,y)}$$

When $\delta < 1$, we notice that the two last expressions of $G^{\omega}_{U,\delta}(x,y)$ make it possible to extend its definition to more general ω's for which Ω_U is not necessarily stochastic (at least in the neighbourhood of a stochastic matrix). The following lemma is concerned with the partial derivative of this extension of $G^{\omega}_{U,\delta}(x,y)$.

Lemma 2. *For all connected subset U of $\mathbb{Z}^d$, for all $x_1, x_2, x_4 \in U$, $x_3 \in U \cup \partial U$, $|x_3 - x_2| = 1$, and for all $\delta \in]0,1[$,*

$$\frac{\partial G^{\omega}_{U,\delta}(x_1,x_4)}{\partial(\omega(x_2,x_3))} = \delta G^{\omega}_{U,\delta}(x_1,x_2) G^{\omega}_{U,\delta}(x_3,x_4)$$

Remark: When $x_3 \in \partial U$, the right-hand term vanishes since $G^{\omega}_{U,\delta}(x_3,x_4) = 0$.

Proof: This is a direct consequence of

$$G^{\omega}_{U,\delta}(x_1,x_4) = \sum_{n\geq 0} \delta^n (\Omega_U)^n_{(x_1,x_4)}$$

and
$$\frac{\partial(\Omega_U)^n_{(x_1,x_4)}}{\partial(\omega(x_2,x_3))} = \sum_{k_1+k_2=n-1} (\Omega_U)^{k_1}_{(x_1,x_2)}(\Omega_U)^{k_2}_{(x_3,x_4)}$$

so that, taking the derivatives term by term in the sum defining $G^{\omega}_{U,\delta}(x_1,x_4)$, we obtain the result. $\square$

We turn now to the estimation of the transition probabilities:

Proposition 2. *For all connected subset U of $\mathbb{Z}^d$, for all $z_0, z \in U$, for all $\delta \in]0,1[$ and all $i = 1, ..., 2d$,*

- *if* $(\sum_{k=1}^{2d} \alpha_k) > 1$, *then* $\quad \frac{\alpha_i - 1}{(\sum_{k=1}^{2d} \alpha_k) - 1} \leq \hat{\omega}_{U,\delta,z_0}(z, z+e_i) \leq \frac{\alpha_i}{(\sum_{k=1}^{2d} \alpha_k) - 1}$
- *if* $(\sum_{k=1}^{2d} \alpha_k) < 1$, *then* $\quad 0 \leq \hat{\omega}_{U,\delta,z_0}(z, z+e_i) \leq \frac{\alpha_i - 1}{(\sum_{k=1}^{2d} \alpha_k) - 1}$

Proof: For the clarity of notations we give the proof for $i = 1$.

Lemma 2 yields
$$\frac{\partial G^{\omega}_{U,\delta}(z_0,z)}{\partial(\omega(z,z+e_i))} = \delta G^{\omega}_{U,\delta}(z_0,z)G^{\omega}_{U,\delta}(z+e_i,z).$$

We now apply Lemma 1 with $f = G^{\omega}_{U,\delta}(z_0,z)$, viewed as a function of the only variables $x_i := \omega(z,z+e_i)$ for $i = 1, ..., 2d$, and we get

$$E_\mu[G^{\omega}_{U,\delta}(z_0,z)] = \frac{\alpha_1 + ... + \alpha_{2d}}{\alpha_1} E_\mu[G^{\omega}_{U,\delta}(z_0,z)\omega(z,z+e_1)]$$
$$+\frac{1}{\alpha_1} E_\mu\left[\omega(z,z+e_1).G^{\omega}_{U,\delta}(z_0,z)\left(\delta\sum_{k=1}^{2d}\omega(z,z+e_k)G^{\omega}_{U,\delta}(z+e_k,z) - \delta G^{\omega}_{U,\delta}(z+e_1,z)\right)\right] \quad (1)$$

We recall that
$$\delta\sum_{k=1}^{2d}\omega(z,z+e_k)G^{\omega}_{U,\delta}(z+e_k,z) = G^{\omega}_{U,\delta}(z,z) - 1$$

so that the second term in the right side of (1) writes

$$\frac{1}{\alpha_1} E_\mu\left[\omega(z,z+e_1).G^{\omega}_{U,\delta}(z_0,z)\left(G^{\omega}_{U,\delta}(z,z) - 1 - \delta G^{\omega}_{U,\delta}(z+e_1,z)\right)\right]$$

so that we get

$$\begin{aligned} E_\mu[G^{\omega}_{U,\delta}(z_0,z)] = \;& \frac{\alpha_1 + ... + \alpha_{2d}}{\alpha_1} E_\mu[G^{\omega}_{U,\delta}(z_0,z)\omega(z,z+e_1)] \\ & +\frac{1}{\alpha_1} E_\mu\left[\omega(z,z+e_1).G^{\omega}_{U,\delta}(z_0,z)\left(G^{\omega}_{U,\delta}(z,z) - 1 - \delta G^{\omega}_{U,\delta}(z+e_1,z)\right)\right] \end{aligned}$$

and for the ratio $\hat{\omega}_{U,\delta,z_0}(z,z+e_1) = \frac{E_\mu[\omega(z,z+e_1)G^{\omega}_{U,\delta}(z_0,z)]}{E_\mu[G^{\omega}_{U,\delta}(z_0,z)]}$,

$$\hat{\omega}_{U,\delta,z_0}(z,z+e_1) = \frac{\alpha_1}{(\sum_{k=1}^{2d}\alpha_k) - 1}$$

$$+\frac{1}{(\sum_{k=1}^{2d}\alpha_k)-1}\frac{E_\mu\left[\omega(z,z+e_1).G^\omega_{U,\delta}(z_0,z)\left(G^\omega_{U,\delta}(z,z)-\delta G^\omega_{U,\delta}(z+e_1,z)\right)\right]}{E_\mu[G^\omega_{U,\delta}(z_0,z)]} \tag{2}$$

But,

$$\sum_{k=1}^{2d}\omega(z,z+e_k)(G^\omega_{U,\delta}(z,z)-\delta G^\omega_{U,\delta}(z+e_k,z))=1$$

and therefore, for all $k=1,...,2d$,

$$0\le G^\omega_{U,\delta}(z,z)-\delta G^\omega_{U,\delta}(z+e_k,z)\le\frac{1}{\omega(z,z+e_k)}.$$

These inequalities allow to bound the ratio in the second term of the right side of (2), between 0 and 1, and this finishes the proof. □

5 Proof of Theorem 1

We gather now all the ingredients of the proof of Theorem 1. We want to apply Sznitman and Zerner's law of large numbers (12). From a careful reading of the proof of this law of large numbers, we can see that the only conditions that need to be fullfilled, are the integrability of the Green function $G^\omega_U(z,z)$ for all bounded U, and Kalikow's condition.

The integrability of the Green function is proved in the following lemma:

Lemma 3. *If there exists $i\in\{1,...,2d\}$, such that $\alpha_i>1$, then for all connected subset U of $\mathbb{Z}^d$ and all $z\in U$, $E_\mu[G^\omega_U(z,z)]$ is finite.*

Proof: For the clarity of notations, we suppose that $\alpha_1>1$.

Define now by N the least integer such that $z+Ne_1$ belongs to ∂U.

We have the following lower bound for the probability $P(\omega,z,U)$ to reach ∂U from z without returning to z_0 :

$$P(\omega,z,U)\ge\prod_{k=0}^{N-1}\omega(z+ke_1,z+(k+1)e_1).$$

The number of returns to z before hitting ∂U, being a geometric variable whose parameter is precisely $P(\omega,z,U)$, its expectation $G^\omega_U(z,z)$ is equal to $\dfrac{1}{P(\omega,z,U)}$.

We are now led to examine the integrability of $E_\mu\left[\left(\prod_{k=0}^{N-1}\omega(z+ke_1,z+(k+1)e_1)\right)^{-1}\right]$ which is equal to $(\int_{T_{2d}}\frac{1}{x_1}d\lambda^{\vec{\alpha}})^N$ which is finite since $\alpha_1>1$. □

We now have to check Kalikow's condition.

We notice first that, under the assumption of Theorem 1, Lemma 3 applies and Kalikow's auxiliary walk is well defined. Then, the monotone convergence theorem allows to make δ converge to 1 in the inequalities of Proposition 2.

We then deduce that the drift of Kalikow's walk belongs to

$$\frac{1}{(\sum_{k=1}^{2d} \alpha_k) - 1} \prod_{i=1}^{d} [\alpha_{e_i} - \alpha_{-e_i} - 1, \alpha_{e_i} - \alpha_{-e_i} + 1]$$

which does not contain 0, under the assumption of Theorem 1. This proves Kalikow's transience condition.

In order to estimate the asymptotic velocity of the process, we apply directly Proposition 3.2 of (7) which makes the link between v and the drift of Kalikow's walk.

Remark: in Lemma 3, we only got a sufficient condition for the integrability of the Green function to hold. A better result about this question would not have ameliorated the statement of Theorem 1 as far as our check of Kalikow's condition requires a stronger assumption.

6 Proof of Theorem 2

Theorem 2 of section 2 is actually a consequence of a more precise result, where the "O" in the expansion is replaced by an explicit upper bound.

Let us fix some notations: we set

$$\gamma = \sum_{i=1}^{2d} \alpha_i,$$

and

$$m_i = m_{e_i} = \frac{\alpha_i}{\gamma} = E^{\lambda^{(\alpha)}}(\omega(e_i)).$$

When γ is large, the environment $(\omega(x, e_i))$ tends to concentrate around its mean (m_i), what can be seen from the expression of the correlations

$$\mathrm{Cov}_\mu(\omega(x, x+e_i), \omega(x, x+e_j)) = \begin{cases} -\frac{m_i m_j}{\gamma+1}, \text{ if } i \neq j \\ \frac{m_i(1-\sum_{k\neq i} m_k)}{\gamma+1}, \text{ if } i = j, \end{cases}$$

The mean environment (m_i) defines the transition probabilities of an homogeneous walk on $\mathbb{Z}^d$, and we define

$$k_m = 2\sum_{i=1}^{d} \sqrt{m_{e_i} m_{-e_i}},$$

so that

$$1 - k_m = \sum_{i=1}^{d} (\sqrt{m_{e_i}} - \sqrt{m_{-e_i}})^2,$$

measures the non-symmetry of the walk. When $k_m < 1$, this walk is ballistic with asymptotic velocity

$$d_m = \sum_{i=1}^{2d} m_i e_i,$$

and we denote by $G^m(\cdot, \cdot)$ its Green function. Let us define

$$\eta_m = \frac{\max_i \sqrt{\frac{m_{e_i}}{m_{-e_i}}}}{1 - k_m}.$$

Proposition 3. *Assume we are in the condition of application of Theorem 1, and that*

$$\frac{2d}{\gamma}\eta_m \leq 1,$$

then we have the following estimate

$$\left| v - d_m\Big(1 - \frac{1}{\gamma-1}(G^m(0,0)-1)\Big)\right| \leq 16\left(\frac{d}{\gamma}\right)^2 \frac{\eta_m^2}{1-\frac{2d}{\gamma}\eta_m}.$$

Proof: Considering the domain $U = \mathbb{Z}^d$, a killing parameter $\delta < 1$ and $z_0 = 0$, we get from formula (2)

$$\hat{\omega}_\delta(z, z+e_i) = m_i + \frac{m_i}{\gamma-1} - \frac{1}{\gamma-1}\frac{E_\mu[G^\omega_\delta(0,z)\omega(z,z+e_i)(G^\omega_\delta(z,z) - \delta G^\omega_\delta(z+e_i,z))]}{E_\mu[G_\delta(0,z)]}.$$

In the sequel, we will sometimes forget the superscript ω in G^ω_δ, when there will be no ambiguity.
Let us introduce a new probability on the environments $\tilde{\mu}(d\omega)$ given by

$$\tilde{\mu}(d\omega) = \frac{G^\omega_\delta(0,z)}{E_\mu(G^\omega_\delta(0,z))}\mu(d\omega).$$

We see that

$$\frac{E_\mu[G_\delta(0,z)\omega(z,z+e_i)(G_\delta(z,z) - \delta G_\delta(z+e_i,z))]}{E_\mu[G_\delta(0,z)]} = E_{\tilde{\mu}}[(G_\delta(z,z) - \delta G_\delta(z+e_i,z))\omega(z,z+e_i)].$$

We proceed as in (7), and apply Kalikow's formula (cf. the generalized version in (7), Proposition 3.1) to the measure $\tilde{\mu}$.
It means that we have

$$E_{\tilde{\mu}}[G^\omega_\delta(z,z)\omega(z,z+e_i)] = G^{\tilde{\omega}^z}_\delta(z,z)\tilde{\omega}^z(z,z+e_i),$$

where $\tilde{\omega}^z$ is the auxiliary transition probability given by

$$\tilde{\omega}^z(y,y+e_j) = \frac{E_{\tilde{\mu}}[G^\omega_\delta(z,y)\omega(y,y+e_j)]}{E_{\tilde{\mu}}[G^\omega_\delta(z,y)]}.$$

Similarly,

$$E_{\tilde{\mu}}[G^\omega_\delta(z+e_i,z)\omega(z,z+e_i)] = G^{\tilde{\omega}^{z+e_i}}_\delta(z+e_i,z)\tilde{\omega}^{z+e_i}(z,z+e_i),$$

where $\tilde{\omega}^{z+e_i}$ is the auxiliary transition probability given by

$$\tilde{\omega}^{z+e_i}(y,y+e_j) = \frac{E_{\tilde{\mu}}[G^\omega_\delta(z+e_i,y)\omega(y,y+e_j)]}{E_{\tilde{\mu}}[G^\omega_\delta(z+e_i,y)]}.$$

Step 1: We want to estimate the transition probabilities $\tilde{\omega}^z$ and $\tilde{\omega}^{z+e_i}$.

Lemma 2 yields

$$(\frac{\partial}{\partial\omega(y,y+e_k)} - \frac{\partial}{\partial\omega(y,y+e_j)})G_\delta^\omega(\cdot,z) = \delta G_\delta^\omega(\cdot,y)(G_\delta^\omega(y+e_k,z) - G_\delta^\omega(y+e_j,z)),$$

moreover

$$\sum_{k=1}^{2d}\omega(y,y+e_k)(G_\delta^\omega(y,z) - \delta G_\delta^\omega(y+e_k,z)) = 1_{y=z}.$$

Using the integration by part formula given in Lemma 1, we get

$$\begin{aligned} m_{e_j}E_\mu[G_\delta(0,z)G_\delta(z,y)] = \quad & E_\mu[G_\delta(0,z)G_\delta(z,y)\omega(y,y+e_j)] \\ & + \frac{1}{\gamma}E_\mu\left[G_\delta(0,y)\left(G_\delta(y,z) - \delta G_\delta(y+e_j,z) - 1_{y=z}\right)G_\delta(z,y)\omega(y,y+e_j)\right] \\ & + \frac{1}{\gamma}E_\mu\left[G_\delta(0,z)G_\delta(z,y)\left(G_\delta(y,y) - \delta G_\delta(y+e_j,y) - 1\right)\omega(y,y+e_j)\right] \end{aligned}$$

But we have

$$0 \le \omega(y,y+e_j)(G_\delta(y,y) - \delta G_\delta(y+e_j,y)) \le 1, \quad (3)$$

and if $y \neq z$

$$|G_\delta(0,y)\omega(y,y+e_j)(G_\delta(y,z) - \delta G_\delta(y+e_j,z))| \le (2d-1)G_\delta(0,z). \quad (4)$$

Indeed, for all $k = 1,\cdots,2d$, we have

$$G_\delta^\omega(y+e_k,z) \ge E_{y+e_k}^\omega[\delta^{T_y}]G_\delta^\omega(y,z),$$

where T_y is the hitting time of y (equal to infinity if the random walk never hits y).
Since

$$\frac{1}{1 - \delta\sum_k \omega(y,y+e_k)E_{y+e_k}^\omega[\delta^{T_y}]} = G_\delta^\omega(y,y),$$

we get

$$\omega(y,y+e_k)(G_\delta^\omega(y,z) - \delta G_\delta^\omega(y+e_k,z)) \le \frac{G_\delta^\omega(y,z)}{G_\delta^\omega(y,y)}.$$

But, we also have

$$\sum_{k=1}^{2d}\omega(y,y+e_k)(G_\delta^\omega(y,z) - \delta G_\delta^\omega(y+e_k,z)) = 0, \quad \text{if } y \neq z,$$

so that we have

$$|\omega(y,y+e_j)(G_\delta^\omega(y,z) - \delta G_\delta^\omega(y+e_j,z))| \le (2d-1)\frac{G_\delta^\omega(y,z)}{G_\delta^\omega(y,y)},$$

which immediately implies the estimate (4).

The inequalities (3) and (4) imply that

$$\left|m_{e_j} E_\mu[G_\delta(0,z)G_\delta(z,y)] - E_\mu[G_\delta(0,z)G_\delta(z,y)\omega(y,y+e_j)]\right| \le \frac{2d}{\gamma} E_\mu[G_\delta(0,z)G_\delta(z,y)].$$

This gives the following estimate for $\tilde{\omega}^z$

$$|m_{e_j} - \tilde{\omega}^z(y,y+e_j)| \le \frac{2d}{\gamma}.$$

The same procedure gives the same estimate for $\tilde{\omega}^{z+e_i}$.

Hence, we see that

$$E_{\tilde{\mu}}[G_\delta^\omega(z,z)\omega(z,z+e_i)] = G_\delta^{m+\Delta m}(z,z)(m_i + \Delta m(z,z+e_i)),$$

where $\Delta m(z,z+e_i)$ is a correction to the homogeneous transition probability (m_i) uniformly bounded by

$$|\Delta m| \le \frac{2d}{\gamma}.$$

The same reasoning holds for

$$E_{\tilde{\mu}}[G_\delta^\omega(z+e_i,z)\omega(z,z+e_i)] = G_\delta^{m+\Delta m}(z+e_i,z)(m_i + \Delta m(z,z+e_i)),$$

even if the correction term Δm is not the same.

Step 2: We compare now the Green function $G_\delta^{m+\Delta m}$ with G_δ^m.

This is done in (7), but we reproduce the main lines of the proof, since we want to obtain explicit bounds. We first introduce the symmetrizing function

$$\phi^m(z) = \prod_{i=1}^{d} \sqrt{\frac{m_{e_i}}{m_{-e_i}}}^{z_i}.$$

The Green function G_δ^m is transformed into

$$G_\delta^m = M_{\phi^m}^{-1} G_{\delta k_m}^s M_{\phi^m}, \; (5)$$

where M_ϕ is the operator of multiplication by ϕ, and $G_{\delta k_m}^s$ is the Green function of the symmetric random walk with transition probability

$$s_{e_i} = s_{-e_i} = \frac{\sqrt{m_{e_i} m_{-e_i}}}{2\sum_{k=1}^{2d} \sqrt{m_{e_k} m_{-e_k}}}, \quad i = 1, \ldots, d,$$

with killing rates δk_m where

$$k_m = 2\sum_{k=1}^{2d} \sqrt{m_{e_k} m_{-e_k}}.$$

We refer to Step 2 of the proof of Lemma 4.3 in (7) for precisions about this fact.

Hence, we see that

$$\sum_y G^s_{\delta k_m}(0,y) \leq \frac{1}{1-\delta k_m},$$

which means that

$$\|G^s_{\delta k_m}\|_\infty \leq \frac{1}{1-\delta k_m}.$$

We also have

$$G^{m+\Delta m}_\delta - G^m_\delta = -G^m_\delta \left(I - (I - \delta \Delta P_m G^m_\delta)^{-1}\right),$$

where ΔP_m is the matrix $(\Delta P_m)_{x,x+e_i} = \Delta m(x, x+e_i)$ (and null anywhere else).
(Indeed, this is an application of the perturbation formula

$$(I - \delta(A+B))^{-1} = (I-\delta A)^{-1}(I - \delta B(I-\delta A)^{-1})^{-1}$$

where $A(x, x+e_i) = m_i$ and $B = \Delta P_m$.)
Thus, we get

$$G^{m+\Delta m}_\delta - G^m_\delta = \delta M_\phi^{-1} G^s_{\delta k_m} M_{\phi^m} \Delta P_m M^{-1}_{\phi^m} G^s_{\delta k_m}\left(I - \delta M_{\phi^m}\Delta P_m M^{-1}_{\phi^m} G^s_{\delta k_m}\right)^{-1} M_{\phi^m},$$

but

$$\|M_{\phi^m}\Delta P_m M^{-1}_{\phi^m}\|_\infty \leq (\max_i \phi^m(e_i))\frac{2d}{\gamma}$$

so that we get

$$(G^{m+\Delta m}_\delta - G^m_\delta)(x,y) \leq \phi^m(y-x)\frac{2d}{\gamma}\frac{1}{1-k_m}\eta_m \frac{1}{1-\frac{2d}{\gamma}\eta_m},$$

and

$$G^{m+\Delta m}_\delta(x,y) \leq \phi^m(y-x)\frac{1}{1-k_m}\frac{1}{1-\frac{2d}{\gamma}\eta_m}.$$

This implies that, for all $i \in \{1, ..., 2d\}$,

$$\begin{aligned}
&\left|\hat{\omega}_\delta(z, z+e_i) - m_i + \frac{m_i}{\gamma-1}(G^m_\delta(0,0) - \delta G^m_\delta(e_i,0) - 1)\right| \\
&\qquad \leq \frac{1}{\gamma-1}\left(2\frac{2d}{\gamma}\eta_m^2\frac{1}{1-\frac{2d}{\gamma}\eta_m} + 2\frac{2d}{\gamma}\eta_m\frac{1}{1-\frac{2d}{\gamma}\eta_m}\right) \\
&\qquad \leq 8\frac{d}{\gamma^2}\frac{\eta_m^2}{1-\frac{2d}{\gamma}\eta_m} \qquad (6)
\end{aligned}$$

(we used here $\eta_m \geq 1$).
The sum

$$\sum_{i=1}^{2d}\left(m_i - \frac{m_i}{\gamma-1}(G^m_\delta(0,0) - \delta G^m_\delta(e_i,0) - 1)\right).e_i$$

tends to $d_m(1 - \frac{1}{\gamma-1}(G^m(0,0)-1))$ when δ tends to 1. Indeed, the sum $\sum_{i=1}^{2d} m_i G^m(e_i,0).e_i$ cancels, due to the fact that for each $i \in \{1, ..., d\}$ $m_{e_i}G^m(e_i,0)$ and $m_{-e_i}G^m(-e_i,0)$ are both

equal to the common value $\sqrt{m_{e_i}m_{-e_i}}G^s_k(e_i,0)$ (cf formula (5) which implies that $G^m_\delta(e_i,0) = \phi^{-1}(e_i)G^s_{\delta k_m}(e_i,0)$).
The triangular inequality combined with the $2d$ inequalities (6) gives that, for all z,

$$\limsup_{\delta\to 1} \|d_{\hat{\omega}_\delta}(z) - d_m(1-\frac{1}{\gamma-1}(G^m(0,0)-1))\| \le 16\left(\frac{d}{\gamma}\right)^2 \frac{\eta_m^2}{1-\frac{2d}{\gamma}\eta_m},$$

where $d_{\hat{\omega}_\delta}(z) = \sum_{k=1}^{2d}\hat{\omega}_\delta(z,z+e_k)e_k$ is the local drift of the transition probability $\hat{\omega}_\delta$. Proposition 3.2 of (7) allows to conclude. $\square$

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