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BOUNDARY CONDITIONS FOR ONE-DIMENSIONAL BIHARMONIC PSEUDO PROCESS

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Abstract We study boundary conditions for a stochastic pseudo processes corresponding to the biharmonic operator. The biharmonic pseudo process (BPP for short). is composed, in a sense, of two different particles, a monopole and a dipole. We show how an initial-boundary problems for a 4-th order parabolic differential equation can be represented by BPP with various boundary conditions for the two particles: killing, reflection and stopping.

Keywords Boundary conditions for biharmonic pseudo process, killing, reflection, stopping

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1 Introduction

We will call $-\triangle^2 \equiv -\partial_x^4$ a *biharmonic operator*. It plays an important role in the theory of elasticity and fluid dynamics. For instance the parabolic differential equation

$$\partial_t u(t,x) = -\partial_x^4 u(t,x), \quad t>0, \quad x \in \mathbb{R}^1, \tag{1.1}$$

is closely related to the Kuramoto-Sivashinsky equation and the Cahne-Hilliard equation (see [17, Chapter III §4]). It is easy to see that the fundamental solution $p(t,x)$ to (1.1) is given by

$$p(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi \ \exp\{-i\xi x - \xi^4 t\}, \quad t>0, \quad x \in \mathbb{R}^1. \tag{1.2}$$

This $p(t,x)$ takes negative values (see Hochberg [8]), so no stochastic process corresponds to (1.1) in the usual sense.

Several attempts have been made to relate the biharmonic operator to random processes. Krylov [9] considered a stochastic pseudo process whose "transition probability density" was $p(t,x)$ as in (1.2), despite its taking negative values. We will call this pseudo process a *biharmonic pseudo process* or BPP for short (it is occasionally called the *Krylov motion*).

Following Krylov's idea, Hochberg [8] started a systematic study of BPP. His article included a definition of a stochastic integral with respect to BPP. Nishioka [14] calculated the joint distribution of the first hitting time and place for BPP hitting the boundary of a half-line. This was extended by Nakajima and Sato [10] to the space-time case. Some new developments in this direction can be found in a paper by Beghin, Hochberg and Orsingher [1].

Other attempts included a paper by Funaki [7] who introduced the concept of *iterated Brownian motion*. After his pioneering work, Burdzy [2] began to study path properties of iterated Brownian motion; a large number of related papers followed. Burdzy and Mądrecki [3, 4] modified Funaki's model in a different way; they defined a stochastic integral with respect to the their process.

This paper is devoted to the process BPP on a half-line. We will study various boundary conditions for the probabilistic model and relate them to their analytic counterparts. We will motivate our results by first reviewing the classical case of the standard Brownian motion on $(0,\infty)$.

Consider the following initial-boundary value problem for the heat equation:

$$\begin{aligned} &\partial_t u(t,x) = (1/2)\,\partial_x^2 u(t,x), \ t>0, \ x>0; \quad u(0,x)=f(x), \ x>0; \\ &\partial_x^{\ell} u(t,0) = 0, \quad t>0, \end{aligned} \tag{1.3}$$

where ℓ may take values $0, 1$ or 2. Solutions to this initial-boundary value problem may be represented using Brownian motion on $(0,\infty)$ which is killed, reflected, or stopped at the boundary point 0, depending on the value of ℓ. The relationship can be summarized as follows:

Table 1.1.

ℓ	Brownian boundary condition
0	killing
1	reflection
2	stopping

We note that there is no analytic difference between Brownian transition probabilities on the open interval $(0, \infty)$ in the cases when the Brownian particle is killed or stopped at the boundary but we make the distinction in Table 1.1 to emphasize the analogy with some results on BPP.

In this article, we will establish a relationship analogous to Table 1.1, between BPP and several initial-boundary value problems for (1.1).

We start with a few heuristic ideas. A big difference between a Brownian motion and BPP, proved in [13], is that the Brownian motion is a model for the motion of a single particle but when the BPP leaves the interval $(0, \infty)$, it appears to consist of two types of particles. We will call these particles a *monopole* and a *dipole*. The names are justified by a result on the conservation of charges given in the last section of this article. The particles behave independently at the boundary. Therefore we need two different boundary conditions in order to solve the initial-boundary value problem for (1.1). In a sense, each boundary condition controls the behavior of a monopole or a dipole, although the exact relationship, summarized in Table 1.2 below, is more complicated than that.

We proceed with a more formal presentation of the main results although the fully rigorous statements are postponed until later in the paper. Consider the following initial-boundary value problems for (1.1),

$$\begin{aligned} &\partial_t v(t,x) = -\partial_x^4 v(t,x), \quad t > 0, x > 0; \quad v(0,x) = f(x), \quad x > 0; \\ &\partial_x^\ell v(t,0) = 0 \;\text{ and }\; \partial_x^m v(t,0) = 0, \quad t > 0, \end{aligned} \tag{1.4}$$

where f is a given bounded smooth function on $[0, \infty)$ and ℓ and m can take the values $\ell = 0, \ldots, 4$ and $m = 0, \ldots, 5$. The following two tables summarize our main results.

Table 1.2.

	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$
$\ell=0$	$--$	00	$0r$	H	$\times$	$0s$
$\ell=1$	$--$	$--$	$r0$	H	$s0$	$\times$
$\ell=2$	$--$	$--$	$--$	H	sr	H
$\ell=3$	$--$	$--$	$--$	$--$	H	H
$\ell=4$	$--$	$--$	$--$	$--$	$--$	ss

For some values of ℓ and m, the initial boundary value problem has a probabilistic representation using BPP with killing, reflection or stopping for the monopole and dipole. The correspondence between the various values of ℓ and m and boundary conditions for BPP is coded in Tables 1.2 and 1.3 in the self-evident manner.

Table 1.3.

	killing for dipole	reflection for dipole	stopping for dipole
killing for monopole	00 (see §4)	$0r$ (see §6)	$0s$ (see §8.1)
reflection for monopole	$r0$ (see §5.1)	$--$ (see §5.3)	$--$ (see §8.2)
stopping for monopole	$s0$ (see §7.2)	sr (see §7.3)	ss (see §7.1)

In Table 1.2, the symbol "$\times$" means that (1.4) is not well-posed with such analytic boundary conditions (see Example 7.9). The symbol "H" means such analytic boundary conditions are not related to BPP with killing, reflection or stopping at the boundary. However, these analytic conditions can be related to BPP. This requires a new idea of a "higher order reflecting boundary" which will be discussed in a forthcoming article. The diagonal entries in Table 1.2 and those below the diagonal are irrelevant for obvious reasons and so they are marked with "$--$". The same symbol in Table 1.3 means that we have not found a relationship between this set of analytic conditions and BPP.

In §2, we review results from [13, 14] on the joint distribution of the first hitting time and place for BPP and the concepts of a monopole and dipole. In §3 through §8, we construct and study BPP's with various boundary conditions.

The construction of the BPP in the case when we have killing or stopping of the monopole at the boundary is similar in a sense to the Brownian motion case. Only slight modifications are needed when the same boundary conditions are prescribed to the dipole.

When we set the reflecting boundary for a monopole or dipole, we cannot apply a direct analogy with the reflecting Brownian motion (see §5.2). We will construct a BPP with the reflection as the limit of a suitable sequence of approximations.

In §9, we verify the law of the conservation of charges, and conclude that the total amount of charges is conservative if and only if there is no killing for a monopole.

We will use some basic results on solutions of 4-th order PDE's, Laplace and Fourier-Laplace transforms through the paper. These can be found in [5, 6, 16] for instance.

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2 Preliminaries

We start with a review of several function spaces used in the article.

Let X denote either the one dimensional Euclidean space $\mathbb{R}^1$ or the half-line $[0,\infty)$. $\mathbf{B}_b(X)$ will stand for the space of all bounded measurable functions defined on X. $\mathbf{C}(X)$ will be the space of all continuous functions on X while $\mathbf{C}_b(X)$ will be the space of all bounded functions in $\mathbf{C}(X)$. $\mathbf{C}^1(X)$ will denote the space of all continuously differentiable functions on X and $\mathbf{C}_b^1(X)$ will be the space of all bounded functions in $\mathbf{C}^1(X)$.

$\mathbf{D}[0,\infty)$ will denote the space of all right continuous functions defined on $[0,\infty)$ which have left hand limits.

$\mathcal{S}$ will be the space of all Schwartz class functions defined on $\mathbb{R}^1$.

Dirac's delta function will be denoted by $\delta(x)$ and $\delta_a(db)$ will be the delta measure with the unit mass at a point $\{a\}$, so we may use the convention $\delta_a(db) = \delta(a-b)\,db$.

For a fixed positive t, the function $p(t,x)$ of (1.2) is an even function of x and it belongs to $\mathcal{S}$.

For positive t and s, the following formulae hold

$$\begin{aligned} &\int_{-\infty}^{\infty} dx\ p(t,x) = 1, \quad p(t,x) = t^{-1/4}\ p\left(1, \frac{x}{t^{1/4}}\right), \\ \text{and} &\int_{-\infty}^{\infty} dy\ p(t, y-x)\ p(s,y) = p(t+s,x). \end{aligned} \tag{2.1}$$

Note that $p(t,x)$ can take negative values. Hochberg [8] proved that

$$p(1,|x|) = a|x|^{-1/3} \exp\{-b|x|^{4/3}\} \cos c|x|^{4/3} + \text{a lower order term}$$

for large $|x|$, where a, b, and c are positive constants. It follows that

$$\int_{-\infty}^{\infty} |p(t,x)|\, dx = \int_{-\infty}^{\infty} |p(1,x)|\, dx = \text{a constant} > 1 \text{ for any } t > 0. \tag{2.2}$$

Using this $p(t,x)$, Krylov [9] defined a finitely additive signed measure $\widetilde{\mathbf{P}}_x$ on cylinder sets in $\mathbb{R}^{[0,\infty)}$. *A cylinder set* in $\mathbb{R}^{[0,\infty)}$, say Γ, is a set such that

$$\Gamma = \{\omega \in \mathbb{R}^{[0,\infty)} : \omega(t_1) \in B_1, \cdots, \omega(t_n) \in B_n\} \tag{2.3}$$

where $0 \le t_1 < \cdots < t_n$ and B_k's are Borel sets in $\mathbb{R}^1$. We put

$$\begin{aligned} \widetilde{\mathbf{P}}_x[\Gamma] \equiv &\int_{B_1} dy_1 \cdots \int_{B_n} dy_n\ p(t_1, y_1 - x) \times \\ &\times p(t_2 - t_1, y_1 - y_2) \cdots p(t_n - t_{n-1}, y_n - y_{n-1}). \end{aligned} \tag{2.4}$$

If we fix $0 \le t_1 < \cdots < t_n$, then this $\widetilde{\mathbf{P}}_x$ is a σ–additive finite measure on $\mathbb{R}^n$, but it is not a σ–additive measure on the smallest σ–field which includes all cylinder sets in $\mathbb{R}^{[0,\infty)}$.
We say that a function f defined on $\mathbb{R}^{[0,\infty)}$ is *tame*, if

$$f(\omega) = g(\omega(t_1), \cdots, \omega(t_n)), \quad \omega \in \mathbb{R}^{[0,\infty)}, \tag{2.5}$$

for a Borel function g on $\mathbb{R}^n$ and $0 \le t_1 < \cdots < t_n$. For each tame function f, we define its expectation in the usual way—if f is as in (2.5), then we set

$$\widetilde{\mathbf{E}}_x[f(\omega)] = \int f(\omega)\ \widetilde{\mathbf{P}}_x[d\omega(t_1) \times \cdots \times d\omega(t_n)], \tag{2.6}$$

if the right hand side exists.
We extend the expectation to other functions as follows. Let n and N be natural numbers. For each $\omega \in \mathbb{R}^{[0,\infty)}$, we set

$$\omega_n^N(t) \equiv \begin{cases} \omega(k/2^n) & \text{if} \quad k/2^n \le t < (k+1)/2^n \ \text{ and } \ t < N \\ \omega(N) & \text{if} \quad t \ge N. \end{cases}$$

Definition 2.1. We say that a function F defined on $\mathbb{R}^{[0,\infty)}$ is *admissible* if F satisfies the following conditions,

(i) for each n and N, $F(\omega_n^N)$ is tame,

(ii) for each $\omega \in \mathbb{R}^{[0,\infty)}$, $\lim_{n\to\infty} \lim_{N\to\infty} F(\omega_n^N) = F(\omega)$,

(iii) there exists $\lim_{n\to\infty} \lim_{N\to\infty} \sum_{k=1}^N |\widetilde{\mathbf{E}}_x[F(\omega_n^k)] - \widetilde{\mathbf{E}}_x[F(\omega_n^{k-1})]|$. $\diamond$

We define the expectation of an admissible function $F(\omega)$ by

$$\mathbf{E}_x[F(\omega)] \equiv \lim_{n\to\infty} \lim_{N\to\infty} \widetilde{\mathbf{E}}_x[F(\omega_n^N)].$$

The expectation is unique if it exists, due to (iii) of Definition 2.1 (see [14]). Let A be a subset in $\mathbb{R}^{[0,\infty)}$. When the characteristic function $I_A(\omega)$ is admissible, we let

$$\mathbf{P}_x[A] \equiv \mathbf{E}_x[I_A(\omega)]. \tag{2.7}$$

For a positive number α, $\mathbf{H}^\alpha[0,\infty)$ will denote the space of all functions defined on $[0,\infty)$ which satisfy Hölder's condition of order α. According to Krylov [9],

$$\mathbf{P}_x\text{–total variation of } \mathbf{H}^\alpha[0,\infty)^{\,c} = 0 \quad \text{if} \quad \alpha < \tfrac{1}{4}.$$

This implies that the total mass of $|\mathbf{P}_x|$ is concentrated on $\mathbf{C}[0,\infty)$. However for a technical reason related to Definition 2.1 (we need to deal with ω_n^N), we will take a larger space $\mathbf{D}[0,\infty)$ as the *path space* of the pseudo process corresponding to $\mathbf{P}_x$. From now on, we will identify $\mathbf{P}_x$ and $\mathbf{E}_x$ with their restrictions to $\mathbf{D}[0,\infty)$.

Definition 2.2. *A biharmonic pseudo process*, or BPP in short, is the family of finitely additive signed measures $\{\mathbf{P}_x : x \in \mathbb{R}^1\}$, defined in (2.4) and (2.7). The domain of $\mathbf{P}_x$ is a finitely additive field in $\mathbf{D}[0,\infty)$ which includes all cylinder sets.

3 The first hitting time distribution, monopole and dipole

We will review results from [13, 14] in this section..

3.1 The distribution of the first hitting time and place

Given $\omega \in \mathbf{D}[0,\infty)$, the random time

$$\tau_0(\omega) \equiv \inf\{t > 0 : \omega(t) < 0\},$$

is called the *first hitting time of the interval* $(-\infty, 0)$. It can be proved that the function

$$\exp\{-\lambda\tau_0(\omega) + i\beta\omega(\tau_0)\} \tag{3.1}$$

is admissible, and we can calculate its expectation. For each $\lambda > 0$ and $\beta \in \mathbb{R}^1$,

$$\begin{aligned} &\mathbf{E}_x[\exp\{-\lambda\tau_0(\omega) + i\beta\omega(\tau_0)\}] \\ &\quad = \frac{1}{\sqrt{2}}\left[\overline{\theta_1}\exp\{\lambda^{1/4}\theta_2 x\} + \theta_1\exp\{\lambda^{1/4}\overline{\theta_2} x\}\right] \\ &\quad + \ \frac{i\beta}{\sqrt{2}\lambda^{1/4}}\left[-i\exp\{\lambda^{1/4}\theta_2 x\} + i\exp\{\lambda^{1/4}\overline{\theta_2} x\}\right], \end{aligned} \tag{3.2}$$

where $\theta_1 \equiv \exp\{\pi i/4\}$ and $\theta_2 \equiv \exp\{3\pi i/4\}$.

In the classical probability setting, one can derive the joint distribution of the first hitting time and place

$$\mathbf{P}_x[\tau_0(\omega) \in dt,\, \omega(\tau_0) \in da] \tag{3.3}$$

from (3.2), using Bochner's theorem. However the theorem cannot be applied to BPP, since (3.2) is not positive definite in β, and so we have to extend the notion of the "joint distribution" itself.

Given functions ϕ and φ in $\mathcal{S}$, it can be proved that the function

$$\exp\{-\lambda\tau_0(\omega)\}\,\phi(\tau_0(\omega))\,\varphi(\omega(\tau_0))$$

is admissible and its expectation defines a continuous bilinear functional on $\mathcal{S}\times\mathcal{S}$. This fact is directly implied by Schwartz's result on Fourier-Laplace transforms of his temperate distributions. The following lemma is a slight modification of his result.

Lemma 3.1 ([15]). *Let c be a positive number and $U(\lambda,\beta)$ be a complex-valued function of a real variable β and a complex variable λ with $\Re\lambda > c$. Assume that for each real β, U is a holomorphic function in λ with $\Re\lambda > c$ and it satisfies*

$$|U(\lambda,\beta)| \le C\big(1+|\lambda|+|\beta|\big)^k \quad \textit{for} \;\; \Re\lambda > c \;\; \textit{and} \;\; \beta \in \mathbb{R}^1, \tag{3.4}$$

where C and k are non-negative constants. Then U is the Fourier-Laplace transform of a Schwartz temperate distribution whose support lies in $[0,\infty)\times\mathbb{R}^1$.

We will always take the principal value for $\lambda^{1/4}$. If $\Re\lambda > c$ for any positive number c, then $-\pi/8 < \arg\lambda^{1/4} < \pi/8$. This implies that (3.2) satisfies (3.4). Hence there exists a Schwartz temperate distribution $q(t,a;x)$ such that

$$\begin{aligned} &\mathbf{E}_x[\exp\{-\lambda\tau_0(\omega)\}\,\phi(\tau_0(\omega))\,\varphi(\omega(\tau_0))] \\ &\quad = \int_0^\infty dt \int da\;\exp\{-\lambda t\}\,q(t,a;x)\;\phi(t)\;\varphi(a), \quad \lambda > c > 0, \end{aligned} \tag{3.5}$$

holds for all ϕ and φ in $\mathcal{S}$.

In the classical probability theory, distributions of real-valued random variables may be thought of as non-negative continuous linear functionals on the function space $\mathbf{C}_b(\mathbb{R}^1)$. We define distributions of functions of BPP to be continuous linear functionals on a function space which includes $\mathcal{S}$.

Definition 3.2. We call Schwartz's temperate distribution $q(t,a;x)$ in (3.5) the *density* of the distribution (3.3), and let

$$\mathbf{P}_x[\tau_0(\omega) \in dt, \omega(\tau_0) \in da] \equiv q(t,a;x)\;dt\;da,$$

which is a continuous bilinear functional on $\mathcal{S}\times\mathcal{S}$ for each $x \ge 0$.

Proposition 3.3 ([14]). *Let $x \ge 0$. Then in the distribution sense,*

$$\mathbf{P}_x[\tau_0(\omega) \in dt, \omega(\tau_0) \in da] = \big[K(t,x)\,\delta(a) - J(t,x)\,\delta'(a)\big]\,dt\,da \tag{3.6}$$

where $\delta'(a)$ is the first derivative of Dirac's delta function $\delta(a)$,

$$
\begin{aligned}
K(t,x) &\equiv \frac{2}{\pi}\int_0^\infty dy\ \exp\{-y^4 t\}\, y^3 \big(\sin yx - \cos yx + \exp\{-yx\}\big), \\
J(t,x) &\equiv \frac{2}{\pi}\int_0^\infty dy\ \exp\{-y^4 t\}\, y^2 \big(\sin yx - \cos yx + \exp\{-yx\}\big).
\end{aligned}
\tag{3.7}
$$

Moreover, the support of the distribution in (3.6) is $[0,\infty)\times\{0\}$.

Remark 3.4 ([14]). (i) For each $t>0$, $K(t,x)$ and $J(t,x)$ are $\mathbf{C}_b^\infty[0,\infty)$ functions in x. For each $x>0$, they are also $\mathbf{C}_b^\infty[0,\infty)$ functions in t.
(ii) For some positive constant C independent of t and x, we have

$$
\begin{aligned}
|K(t,x)| &\le \frac{C\,x^2}{t^{3/2}}\left(\frac{t^{1/4}}{x}\right)^n \quad \text{for} \quad n=0,\cdots,6; \\
|J(t,x)| &\le \frac{C\,x^2}{t^{5/4}}\left(\frac{t^{1/4}}{x}\right)^n \quad \text{for} \quad n=0,\cdots,5.
\end{aligned}
$$

(iii) For $x>0$,

$$
\int_0^\infty K(t,x)\,dt = 1 \quad \text{and} \quad \int_0^\infty J(t,x)\,dt = x.
$$

(iv) Let f be a bounded continuous function on $[0,\infty)$. For $t>0$,

$$
\lim_{x\to 0}\int_0^t ds\ K(s,x)\ f(s) = f(0) \quad \text{and} \quad \lim_{x\to 0}\int_0^t ds\ J(s,x)\ f(s) = 0.
$$

The explicit formula (3.6) makes it possible to extend (3.3) to a continuous bilinear functional on a larger space than $\mathcal{S}\times\mathcal{S}$.

Corollary 3.5 ([14]). *The distribution* (3.3) *can be extended to a continuous bilinear functional on* $\mathbf{B}_b[0,\infty)\times\mathbf{C}^1(\mathbb{R}^1)$.

The strong Markov property holds for BPP at time $\tau_0(\omega)$.

Proposition 3.6 (Strong Markov Property [14]). *Let $y<0<x$. The following holds in the sense of continuous linear functionals on* $\mathbf{B}_b[0,\infty)$,

$$
\mathbf{P}_x[\omega(t)\in dy] = \int_{s=0}^t \int_{a=-\infty}^\infty \mathbf{P}_x[\tau_0(\omega)\in ds,\, \omega(\tau_0)\in da]\ \mathbf{P}_a[\omega(t-s)\in dy]. \tag{3.8}
$$

3.2 Monopole and dipole

In physics a particle is called a *dipole* when it carries two charges of equal magnitude but opposite signs. A small magnet is a typical example of a dipole. Heuristically, one may represent a dipole by

$$
-\frac{1}{2\epsilon}\,\delta(a+\epsilon) + \frac{1}{2\epsilon}\,\delta(a-\epsilon). \tag{3.9}
$$

As $\epsilon \to 0$, (3.9) converges to $-\delta'(a)$ in the distribution sense. Therefore we will call $-\delta'(a)$ a dipole. For the Brownian motion $B(t)$, it is well known that

$$\mathbf{P}_x[\tau_0 \in dt, B(\tau_0) \in da] = \frac{x}{\sqrt{2\pi t^3}} \exp\{-x^2/2t\}\, \delta(a)\, dt\, da; \quad x > 0.$$

Comparing this with (3.6), we see that in a sense, BPP behaves as a mixture of two particles of different types when it hits an interval.

Informally speaking, a particle of the first type is represented by $\delta(a)$ and carries a charge of a single sign just as a Brownian particle does. The second particle is represented by $-\delta'(a)$ and corresponds to a dipole. The following informal definition will enable us to present our results using intuitive notation.

Definition 3.7. We will call a particle represented by $\delta(a)$ a *monopole* and a particle represented by $-\delta'(a)$ a *dipole*. We define distributions of the first hitting time and place for the monopole and dipole as follows.

$$\begin{aligned}
\mathbf{P}_x[\tau_0(\omega) \in dt,\ \omega(\tau_0) \text{ is a monopole and in } da] &\equiv K(t,x)\ \delta(a)\ dt\, da,\\
\mathbf{P}_x[\tau_0(\omega) \in dt,\ \omega(\tau_0) \text{ is a dipole and in } da] &\equiv J(t,x)\ (-\delta'(a))\ dt\, da,
\end{aligned}$$

where the first distribution is a continuous linear functional on $\mathbf{B}_b[0,\infty) \times \mathbf{C}(\mathbb{R}^1)$ and the latter is a functional on $\mathbf{B}_b[0,\infty) \times \mathbf{C}^1(\mathbb{R}^1)$.

Corollary 3.8. *In the sense of continuous linear functionals on* $\mathbf{B}_b[0,\infty) \times \mathbf{C}^1(\mathbb{R}^1)$,

$$\begin{aligned}
&\mathbf{P}_b[\tau_0(\omega) \in dt,\ \omega(\tau_0) \in da]\\
&\quad = \mathbf{P}_x[\tau_0(\omega) \in dt,\ \omega(\tau_0) \textit{ is a monopole and in } da]\\
&\quad + \ \mathbf{P}_x[\tau_0(\omega) \in dt,\ \omega(\tau_0) \textit{ is a dipole and in } da].
\end{aligned} \tag{3.10}$$

3.3 The initial particle

When BPP starts with an initial distribution $\mu(dy)$, its distribution at time t is given by

$$\int \mu(dy)\ \mathbf{P}_y[\omega(t) \in db]. \tag{3.11}$$

Recall from Definition 3.7 that $\delta(y)$ stands for the monopole and the dipole is represented by $-\delta'(y)$. Therefore when a monopole starts from a point x, the distribution of the process at time t is

$$\int dy\ \delta(y-x)\ \mathbf{P}_y[\omega(t) \in db] = p(t, b-x)\ db = \mathbf{P}_x[\omega(t) \in db].$$

On the other hand, when a dipole starts from a point x, the distribution at time t is

$$\int dy\ (-\delta'(y-x))\ \mathbf{P}_y[\omega(t) \in db] = \partial_x p(t, x-b)\ db.$$

We have a similar formula for the joint distribution of the first hitting time and place. When a monopole starts from a point x, the distribution is

$$\begin{aligned}
&\int dy\ \delta(y-x)\ \Big\{\mathbf{P}_y[\tau_0(\omega) \in dt,\ \omega(\tau_0) \text{ is a monopole and in } da]\\
&\qquad\qquad + \mathbf{P}_y[\tau_0(\omega) \in dt,\ \omega(\tau_0) \text{ is a dipole and in } da\ ]\Big\}\\
&\qquad\qquad = \text{same as the right hand side of (3.10)}.
\end{aligned} \tag{3.12}$$

On the other hand when a dipole starts from x, then the distribution is

$$\begin{aligned}
&\int dy\ (-\delta'(y-x))\ \Big\{\mathbf{P}_y[\tau_0(\omega)\in dt,\ \omega(\tau_0) \text{ is a monopole and in } da\ ] \\
&\quad + \mathbf{P}_y[\tau_0(\omega)\in dt,\ \omega(\tau_0) \text{ is a dipole and in } da\ ]\Big\} \\
&\quad = \Big\{\frac{\partial K}{\partial x}(t,x)\ \delta(a) - \frac{\partial J}{\partial x}(t,x)\ \delta'(a)\Big\} dt\, da.
\end{aligned} \tag{3.13}$$

Remark 3.9. Informally speaking, (3.13) shows that a dipole generates both a monopole and a dipole at the hitting time of $(-\infty, 0)$, just like a monopole does.

3.4 Some Laplace transforms

For future reference, we list some Laplace and Fourier-Laplace transforms of $K(t,x)$, $J(t,x)$ and $p(t,x)$. Let $\lambda > 0$ and $x \geq 0$. From Proposition 3.3, we have

$$\begin{aligned}
\hat{K}(\lambda,x) &\equiv \int_0^\infty dt\ \exp\{-\lambda t\}\ K(t,x) \\
&= \sqrt{2}\exp\{-\lambda^{1/4}x/\sqrt{2}\}\ \cos\left(\frac{\lambda^{1/4}x}{\sqrt{2}} - \frac{\pi}{4}\right),
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
\hat{J}(\lambda,x) &\equiv \int_0^\infty dt\ \exp\{-\lambda t\}\ J(t,x) \\
&= \frac{\sqrt{2}}{\lambda^{1/4}}\ \exp\{-\lambda^{1/4}x/\sqrt{2}\}\ \sin\frac{\lambda^{1/4}x}{\sqrt{2}},
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
\hat{G}(\lambda,x,\beta) &\equiv \int_0^\infty dt \int_{-\infty}^\infty dy\ \exp\{-\lambda t + i\beta y\}\, \mathbf{P}_x[\omega(t)\in dy] \\
&= \frac{\exp\{i\beta x\}}{\lambda+\beta^4}.
\end{aligned} \tag{3.16}$$

In addition, we define a new function

$$\begin{aligned}
&\hat{G}^{00}(\lambda,x,\beta) \equiv \int_0^\infty dt \int_{b\in\mathbf{R}} \exp\{-\lambda t + i\beta b\}\mathbf{P}_x[\omega(t)\in db] - \\
&\quad - \int_0^\infty dt \int_{b\in\mathbf{R}} \exp\{-\lambda t + i\beta b\}\times \\
&\quad \times \int_{s=0}^t \int_{a\in\mathbf{R}} \Big\{\mathbf{P}_x[\tau_0(\omega)\in ds,\ \omega(\tau_0) \text{ is a monopole and in } da] \\
&\qquad + \mathbf{P}_x[\tau_0(\omega)\in ds,\ \omega(\tau_0) \text{ is a dipole and in } da]\Big\}\ \mathbf{P}_a[\omega(t-s)\in db] \\
&= \hat{G}(\lambda,x,\beta) - \frac{1}{\lambda+\beta^4}\Big(\hat{K}(\lambda,x) + i\beta\hat{J}(\lambda,x)\Big).
\end{aligned} \tag{3.17}$$

It is elementary to check that

$$\begin{aligned}
&\hat{K}(\lambda,0) = 1, \quad \partial_x\hat{K}(\lambda,0) = 0, \quad \partial_x^2\hat{K}(\lambda,0) = -\sqrt{\lambda}, \\
&\partial_x^3\hat{K}(\lambda,0) = \sqrt{2}\lambda^{3/4}, \quad \partial_x^4\hat{K}(\lambda,0) = -\lambda, \quad \partial_x^5\hat{K}(\lambda,0) = 0;
\end{aligned} \tag{3.18}$$

$$
\begin{aligned}
&\hat{J}(\lambda,0)=0, \quad \partial_x\hat{J}(\lambda,0)=1, \quad \partial_x^2\hat{J}(\lambda,0)=-\sqrt{2}\lambda^{1/4},\\
&\partial_x^3\hat{J}(\lambda,0)=\sqrt{\lambda}, \quad \partial_x^4\hat{J}(\lambda,0)=0, \quad \partial_x^5\hat{J}(\lambda,0)=-\lambda;
\end{aligned} \tag{3.19}
$$

$$
\begin{aligned}
&\hat{G}^{00}(\lambda,0,\beta)=0, \quad \partial_x\hat{G}^{00}(\lambda,0,\beta)=0,\\
&\partial_x^2\hat{G}^{00}(\lambda,0,\beta)=\frac{-\beta^2+\sqrt{\lambda}+i\sqrt{2}\lambda^{1/4}\beta}{\lambda+\beta^4}\\
&\partial_x^3\hat{G}^{00}(\lambda,0,\beta)=\frac{-\sqrt{2}\lambda^{3/4}-i\beta^3-i\sqrt{\lambda}\beta}{\lambda+\beta^4},\\
&\partial_x^4\hat{G}^{00}(\lambda,0,\beta)=1, \quad \partial_x^5\hat{G}^{00}(\lambda,0,\beta)=i\beta.
\end{aligned} \tag{3.20}
$$

4 Killing boundaries for both particles

Definition 4.1. Let

$$
\begin{aligned}
\mathbf{P}_x^{00}[\omega(t)\in db] &= \mathbf{P}_x[\omega(t)\in db]-\\
&-\int_{s=0}^{t}\int_{a\in\mathbf{R}}\Big\{\mathbf{P}_x[\tau_0(\omega)\in ds,\ \omega(\tau_0) \text{ is a monopole and in } da]\\
&\quad+\mathbf{P}_x[\tau_0(\omega)\in ds,\ \omega(\tau_0) \text{ is a dipole and in } da]\Big\}\,\mathbf{P}_a[\omega(t-s)\in db].
\end{aligned} \tag{4.1}
$$

We will refer to $\{\mathbf{P}_x^{00}[\omega(t)\in db] : x\geq 0\}$ as a *BPP with killing boundaries for both particles.*

Remark 4.2. Here is an intuitive description of BPP with killing boundaries for both particles.

1. A monopole starts from a point $x\geq 0$, and moves according to the transition probability density $p(t,x)$ until it hits the interval $(-\infty,0)$.
2. After BPP hits $(-\infty,0)$, both the monopole and dipole are killed.

Theorem 4.3. (i) *For $x\geq 0$, $b\geq 0$, and $t>0$, we define a function*

$$
p^{00}(t,x,b)\equiv p(t,b-x)-\int_0^t ds\,\big\{K(s,x)\,p(t-s,b)+J(s,x)\,\partial_b p(t-s,b)\big\}. \tag{4.2}
$$

Then, we have

$$
\int_0^\infty dt\int_{-\infty}^{\infty} db\,\exp\{-\lambda t+i\beta b)\,p^{00}(t,x,b)=\hat{G}^{00}(\lambda,x,\beta).
$$

(ii) *The following linear functionals are identical and continuous on $\mathbf{B}_b[0,\infty)$,*

$$
\mathbf{P}_x^{00}[\omega(t)\in db]=p^{00}(t,x,b)\ db. \tag{4.3}
$$

Proof. We easily obtain (4.2) and (4.3) when we recall the definitions of the terms on the right hand side of (4.1) from Section 3.2. It is straightforward to check that (3.20) is the Fourier-Laplace transform of (4.2) in the usual sense. Part (ii) follows from the fact that $p(t,x)\in\mathcal{S}$ and from the estimates stated in Remark 3.4. $\square$

Remark 4.4. The following assertion easily follow from (4.2).

(i) If f is a bounded continuous function on $[0,\infty)$, then

$$\lim_{t\to 0}\int_0^\infty \mathbf{P}_x^{00}[\omega(t)\in db]\ f(b) = f(x).$$

(ii) If $x=0$, $\mathbf{P}_x^{00}[\omega(t)\in A]=0$ for any Borel set $A\in(0,\infty)$.

As expected, the distribution $\mathbf{P}_x^{00}[\omega(t)\in db]$ solves (1.4) with the Dirichlet boundary condition.

Theorem 4.5 ([14]). *Let f be a bounded smooth function on the interval $[0,\infty)$ such that $f(0)=0=f'(0)$. We put*

$$v(t,x)\equiv\int_0^\infty \mathbf{P}_x^{00}[\omega(t)\in db]\ f(b). \tag{4.4}$$

Then in the classical sense, this v satisfies (1.4) with $\ell=0$ and $m=1$.

Corollary 4.6. $\mathbf{P}_x^{00}[\omega(t)\in db]$ *satisfies the Chapman-Kolmogorov equations, that is*

$$\int_0^\infty db\ p^{00}(t,x,b)\ p^{00}(s,b,y) = p^{00}(t+s,x,y). \tag{4.5}$$

Proof. Fix $s>0$ and $y\geq 0$. Theorem 4.5 asserts that

$$v(t,x)\equiv\int_0^\infty db\ p^{00}(t,x,b)\ p^{00}(s,b,y)$$

is a solution of (1.4) where

$$f(x)\equiv p^{00}(s,x,y),$$

$\ell=0$ and $m=1$. The function $\tilde v(t,x)\equiv p^{00}(t+s,x,y)$ is also a solution of (1.4). It is well known that the solution is unique so it follows that $v=\tilde v$. The corollary is proved. □

Remark 4.7. Due to Corollary 4.6, we may consider $\mathbf{P}_x^{00}[\omega(t)\in db]$ of (4.3) as a finitely additive signed measure on cylinder sets in $\mathbf{D}[0,\infty)$ in the same way as in (2.4).

5 Reflecting boundary for the monopole

5.1 Reflection for the monopole and killing of the dipole

The construction of the process BPP with the boundary conditions specified above will use an approximating sequence. We will prove existence of a family $\{{}^\epsilon\mathbf{P}_x^{r0}[\omega(t)\in db] : x\geq 0\}$ satisfying

$$\begin{aligned}&{}^\epsilon\mathbf{P}_x^{r0}[\omega(t)\in db] = \mathbf{P}_x^{0}[\omega(t)\in db]+\\ &+\int_0^t\int_{-\infty}^\infty \mathbf{P}_x[\tau_0(\omega)\in ds,\ \omega(\tau_0) \text{ is a monopole and in } da]\ {}^\epsilon\mathbf{P}_{\epsilon+a}^{r0}[\omega(t-s)\in db].\end{aligned} \tag{5.1}$$

Remark 5.1. Heuristically, $\{{}^\epsilon\mathbf{P}_x^{r0}[\omega(t)\in db] : x\geq 0\}$ represents the following process. Fix $\epsilon>0$.

1. A monopole starts from a point $x \geq 0$, and moves according to the transition probability density $p(t, x)$ until it hits the interval $(-\infty, 0)$.
2. If $\omega(\tau_0)$ is a monopole when BPP hits $(-\infty, 0)$, then it restarts from the point $\epsilon + \omega(\tau_0)$.
3. If $\omega(\tau_0)$ is a dipole when BPP hits $(-\infty, 0)$, then it is killed.

Assume for the moment that ${}^{\epsilon}\mathbf{P}_x^{r0}[\omega(t) \in db]$ is a Schwartz temperate distribution and denote its Fourier-Laplace transform by

$$ {}^{\epsilon}U^{r0}(\lambda, x, \beta) \equiv \int_0^{\infty} dt \int {}^{\epsilon}\mathbf{P}_x^{r0}[\omega(t) \in db] \exp\{-\lambda t + i\beta b\}]. $$

Apply the Fourier-Laplace transform to the both sides of (5.1) to obtain

$$ {}^{\epsilon}U^{r0}(\lambda, x, \beta) = \hat{G}^{00}(\lambda, x, \beta) + \hat{K}(\lambda, x)\ {}^{\epsilon}U^{r0}(\lambda, \epsilon, \beta), \quad x > 0. \tag{5.2} $$

Lemma 5.2. (i) *When $\epsilon > 0$, the following is a solution of* (5.2)*:*

$$ {}^{\epsilon}U^{r0}(\lambda, x, \beta) = \hat{G}^{00}(\lambda, x, \beta) + \hat{K}(\lambda, x)\ \frac{\hat{G}^{00}(\lambda, \epsilon, \beta)}{1 - \hat{K}(\lambda, \epsilon)}. \tag{5.3} $$

(ii) *For small $\epsilon > 0$, this ${}^{\epsilon}U^{r0}$ satisfies* (3.4) *for any $c > 0$, and it is the Fourier-Laplace transform of a Schwartz temperate distribution ${}^{\epsilon}\mathbf{P}_x^{r0}[\omega(t) \in db]$.*

Proof. (i) Put $x = \epsilon$ in (5.2) to see that

$$ {}^{\epsilon}U^{r0}(\lambda, \epsilon, \beta) = \hat{G}^{00}(\lambda, \epsilon, \beta) + \hat{K}(\lambda, \epsilon)\ {}^{\epsilon}U^{r0}(\lambda, \epsilon, \beta). $$

This linear equation is easily solved with respect to ${}^{\epsilon}U^{r0}$. This and (5.2) yield (5.3).
(ii) Let $\Re\lambda > c > 0$. We take the principal value for $\lambda^{1/4}$ and recall that $-\pi/8 < \arg \lambda^{1/4} < \pi/8$. After applying this fact to (3.14) through (3.17), we see that (3.4) holds for $\hat{K}$, $\hat{J}$, and $\hat{G}^{00}$, and also for the fraction on the right hand side of (5.3) when $\epsilon > 0$ is small. Now Lemma 3.1 implies our assertion. $\square$

Lemma 5.3. *In the distribution sense, ${}^{\epsilon}\mathbf{P}_x^{r0}[\omega(t) \in db]$ converges to a limit as $\epsilon \to 0$.*

Definition 5.4. This above limit will be denoted by $\{\mathbf{P}_x^{r0}[\omega(t) \in db] : x \geq 0\}$ and called the *distribution of BPP with the reflecting boundary for the monopole and the killing boundary for the dipole.*

Proof of Lemma 5.3. Recalling (3.18) through (3.20), we let $\epsilon \to 0$ in (5.3), and obtain

$$ \begin{aligned} U^{r0}(\lambda, x, \beta) &\equiv \lim_{\epsilon \to 0} {}^{\epsilon}U^{r0}(\lambda, x, \beta) \\ &= \hat{G}^{00}(\lambda, x, \beta) - \hat{K}(\lambda, x) \frac{\partial_x^2 \hat{G}^{00}(\lambda, 0, \beta)}{\partial_x^2 \hat{K}(\lambda, 0)}. \end{aligned} \tag{5.4} $$

Now U^{r0} clearly satisfies (3.4), and so it is the Fourier-Laplace transform of a Schwartz temperate distribution, that is $\mathbf{P}_x^{r0}[\omega(t) \in db]$. $\square$

Assuming that $\Re\lambda > 0$, we calculate the usual inverse Fourier-Laplace transform of $\partial_x^2 \hat{G}^{00}(\lambda,0,\beta)/\partial_x^2 \hat{K}(\lambda,0)$. Let $b \geq 0$. By the residue theorem, we have

$$\begin{aligned} I^{r0}(\lambda,b) &\equiv \frac{1}{2\pi} \lim_{L\to\infty} \int_{-L}^{L} d\beta \ \exp\{-i\beta b\} \Big(-\frac{\partial_x^2 \hat{G}^{00}(\lambda,0,\beta)}{\partial_x^2 \hat{K}(\lambda,0)} \Big) \\ &= \frac{i}{\sqrt{2}\lambda^{3/4}} \Big(\exp\{\lambda^{1/4}\,\overline{\theta_2}\, b\} - \exp\{\lambda^{1/4}\,\theta_2\, b\} \Big), \end{aligned} \tag{5.5}$$

where $\theta_2 \equiv \exp\{i3\pi/4\}$ and $\overline{\theta_2}$ is its complex conjugate. Applying the usual inverse Laplace transform to this I^{r0}, we obtain

$$\begin{aligned} Q^{r0}(t,b) &\equiv \frac{1}{2\pi i} \lim_{M\to\infty} \int_{1-iM}^{1+iM} d\lambda \ \exp\{\lambda t\}\, I^{r0}(\lambda,b) \\ &= \frac{2}{\pi} \int_0^\infty dy \ e^{-y^4 t} \Big(\cos yb + \sin yb - \exp\{-yb\} \Big), \end{aligned} \tag{5.6}$$

which is a smooth bounded function in $b \geq 0$ if $t > 0$.

Theorem 5.5. *For $x \geq 0$, $b \geq 0$, and $t > 0$, we define a function*

$$p^{r0}(t,x,b) \equiv p^{00}(t,x,b) + \int_0^t ds \ K(s,x) \ Q^{r0}(t-s,b), \tag{5.7}$$

where p^{00} is the function in (4.2) *and Q^{r0} is defined in* (5.6). *Then we have*

$$\mathbf{P}_x^{r0}[\omega(t) \in db] = p^{r0}(t,x,b) \ db, \tag{5.8}$$

both linear functionals in (5.8) *are continuous on $\mathbf{B}_b[0,\infty)$ and*

$$\int_0^\infty \mathbf{P}_x^{r0}[\omega(t) \in db] = 1.$$

Proof. The first part of the theorem follows from (5.4), (5.6), and Lemma 5.3. From (5.4) with (3.18) through (3.20), we deduce that $U^{r0}(\lambda,x,0) = 1/\lambda$, and this justifies the last assertion. □

The distribution $\mathbf{P}_x^{r0}$ is related to the PDE in (1.4) with the Neumann boundary conditions.

Theorem 5.6. (i) *Let f be a bounded smooth function on the interval $[0,\infty)$ such that $f'(0) = 0 = f''(0)$. We put*

$$v(t,x) \equiv \int_0^\infty \mathbf{P}_x^{r0}[\omega(t) \in db] \ f(b). \tag{5.9}$$

Then in the classical sense, this v satisfies (1.4) *with $\ell = 1$ and $m = 2$.*
(ii) *$\mathbf{P}_x^{r0}[\omega(t) \in db]$ satisfies Chapman-Kolmogorov equations* (4.5), *and Remark* 4.7 *holds with $\mathbf{P}_x^{r0}[\omega(t) \in db]$ instead of $\mathbf{P}_x^{00}[\omega(t) \in db]$.*

Proof. Recall (3.18) through (3.20). From (5.4), we have

$$\partial_x^4 U^{r0}(\lambda,x,\beta) + \lambda U^{r0}(\lambda,x,\beta) = \exp\{i\beta x\} \ \text{ for } \forall x > 0, \tag{5.10}$$

and

$$\partial_x U^{r0}(\lambda,0,\beta) = 0 \quad \text{and} \quad \partial_x^2 U^{r0}(\lambda,0,\beta) = 0.$$

Since the Fourier-Laplace transform is unique for our p^{r0}, these prove the first part of the theorem. The second part is immediate from the same argument as in the proof of Corollary 4.6. □

5.2 An analogy to the classical reflecting Brownian motion

For the Brownian motion $B(t)$, it is well known that the following two process X_1 and X_2 have the same distributions. They are referred to as reflecting Brownian motions. Let τ_0 be the first hitting time of the point $x = 0$.

$$
\begin{aligned}
X_1(t) &\equiv |B(t)|, \\
X_2(t) &\equiv \begin{cases} B(t) & \text{if } t < \tau_0, \\ B(t) - \inf_{\tau_0 \le s \le t} B(s) & \text{if } t \ge \tau_0. \end{cases}
\end{aligned}
$$

It is not difficult to check that similarly defined transformations of BPP do not have identical distributions. Keeping this fact in mind, we define a new BPP as follows.

Definition 5.7. Let $\{\mathbf{P}_x^m[\omega(t) \in db] : x \ge 0\}$ be defined by

$$
\begin{aligned}
&\mathbf{P}_x^m[\omega(t) \in db] = \mathbf{P}_x^{00}[\omega(t) \in db] + \\
&+ \int_0^t \int_{-\infty}^{\infty} \mathbf{P}_x[\tau_0(\omega) \in ds,\, \omega(\tau_0) \text{ is a monopole and in } da] \times \\
&\times \mathbf{P}_a[\omega(t-s) - \inf_{u \le t-s} \omega(t) \in db].
\end{aligned}
\tag{5.11}
$$

Remark 5.8. We present an intuitive view of the last definition.

1. A monopole starts from a point $x \ge 0$, and moves according to the transition probability density $p(t, x)$ until it hits the interval $(-\infty, 0)$.
2. If $\omega(\tau_0)$ is a dipole when BPP hits $(-\infty, 0)$, then it is killed.
3. If $\omega(\tau_0)$ is a monopole when BPP hits $(-\infty, 0)$, then it restarts from the point $x = 0$, and moves according to the distribution of $[\omega(t + \tau_0) - \inf_{\tau_0 \le s \le t+\tau_0} \omega(s)]$.

In [14], the following was proved. For $b \ge 0$,

$$\mathbf{P}_0[\omega(t) - \inf_{0 \le s \le t} \omega(s) \in db] = Q^{r0}(t, b)\, db,$$

where Q^{r0} is given in (5.6). We record this partial analogy with the classical reflected Brownian motion in the next theorem.

Theorem 5.9. *The distributions* $\mathbf{P}_x^{r0}[\omega(t) \in db]$ *in Theorem* 5.5 *and* $\mathbf{P}_x^m[\omega(t) \in db]$ *in* (5.11) *are identical.*

Proof. The result follows easily from (5.11) and Theorem 5.5. $\square$

5.3 Reflecting boundaries for both particles

The construction of BPP with these boundary conditions requires an approximation procedure similar to that in the previous section. We will prove that there exists a family $\{{}^{\epsilon}\mathbf{P}_x^{rr}[\omega(t) \in$

$db] : x \geq 0\}$ satisfying the following equation.

$$
\begin{aligned}
{}^{\epsilon}\mathbf{P}_x^{rr}[\omega(t) \in db] &= \mathbf{P}_x^{00}[\omega(t) \in db] + \\
&+ \int_{s=0}^{t} \int_{a=-\infty}^{\infty} \Big\{ \mathbf{P}_x[\tau_0(\omega) \in ds,\ \omega(\tau_0) \text{ is a monopole and in } da] \\
&\quad + \ \mathbf{P}_x[\tau_0(\omega) \in ds,\ \omega(\tau_0) \text{ is a dipole and in } da\ ] \Big\}\, {}^{\epsilon}\mathbf{P}_{\epsilon+a}^{rr}[\omega(t-s) \in db].
\end{aligned} \tag{5.12}
$$

Remark 5.10. Heuristically, we are dealing with the following process. Let ϵ be a fixed positive number.

1. A monopole starts from a point $x \geq 0$, and moves according to the transition probability density $p(t,x)$ until it hits the interval $(-\infty, 0)$.
2. After BPP hits $(-\infty, 0)$, it restarts from the point $\epsilon + \omega(\tau_0)$ irrespective of whether $\omega(\tau_0)$ is a monopole or a dipole.

Assuming for the moment existence of ${}^{\epsilon}\mathbf{P}_x^{rr}[\omega(t) \in db]$, we denote by ${}^{\epsilon}U^{rr}$ its Fourier-Laplace transform. Recall (3.13) and note that (5.12) is transformed into the following equation. For $x > 0$,

$$
\begin{aligned}
{}^{\epsilon}U^{rr}(\lambda, x, \beta) &= \hat{G}^{00}(\lambda, x, \beta) + \\
&+ \hat{K}(\lambda, x)\ {}^{\epsilon}U^{rr}(\lambda, \epsilon, \beta) + \hat{J}(\lambda, x)\ \partial_x {}^{\epsilon}U^{rr}(\lambda, \epsilon, \beta).
\end{aligned} \tag{5.13}
$$

Lemma 5.11. (i) *For $\epsilon > 0$, the following is a solution of* (5.13)*:*

$$
\begin{aligned}
{}^{\epsilon}U^{rr}(\lambda, x, \beta) &= \hat{G}^{00}(\lambda, x, \beta) \\
&+ \hat{K}(\lambda, x)\, \frac{(1 - \partial_x \hat{J}(\epsilon, \lambda))\, \hat{G}^{00}(\lambda, \epsilon, \beta) + \hat{J}(\lambda, \epsilon)\, \partial_x \hat{G}^{00}(\lambda, \epsilon, \beta)}{(1 - \hat{K}(\lambda, \epsilon))(1 - \partial_x \hat{J}(\lambda, \epsilon)) - \partial_x \hat{K}(\epsilon, \lambda)\, \hat{J}(\lambda, \epsilon)} \\
&+ \hat{J}(\lambda, x)\, \frac{\partial_x \hat{K}(\lambda, \epsilon)\, \hat{G}^{00}(\lambda, \epsilon, \beta) - (1 - \hat{K}(\lambda, \epsilon))\, \partial_x \hat{G}^{00}(\lambda, \epsilon, \beta)}{(1 - \hat{K}(\lambda, \epsilon))(1 - \partial_x \hat{J}(\lambda, \epsilon)) - \partial_x \hat{K}(\lambda, \epsilon)\, \hat{J}(\lambda, \epsilon)}.
\end{aligned} \tag{5.14}
$$

(ii) *For small $\epsilon > 0$, this ${}^{\epsilon}U^{rr}$ satisfies* (3.4) *for any $c > 0$, and it is the Fourier-Laplace transform of a Schwartz temperate distribution ${}^{\epsilon}\mathbf{P}_x^{rr}[\omega(t) \in db]$.*

Proof. (i) Substitute ϵ for x in (5.13). Then differentiate both sides of (5.13) with respect to x and then again substitute ϵ for x. As a result, we obtain the following equations,

$$
\begin{aligned}
{}^{\epsilon}U^{rr}(\lambda, \epsilon, \beta) &= \hat{G}^{00}(\lambda, \epsilon, \beta) + \hat{K}(\lambda, \epsilon)\ {}^{\epsilon}U^{rr}(\lambda, \epsilon, \beta) \\
&\quad + \hat{J}(\lambda, \epsilon)\ \partial_x {}^{\epsilon}U^{rr}(\lambda, \epsilon, \beta),
\end{aligned}
$$

$$
\begin{aligned}
\partial_x {}^{\epsilon}U^{rr}(\lambda, \epsilon, \beta) &= \partial_x \hat{G}^{00}(\lambda, \epsilon, \beta) + \partial_x \hat{K}(\lambda, \epsilon)\ {}^{\epsilon}U^{rr}(\lambda, \epsilon, \beta) \\
&\quad + \partial_x \hat{J}(\lambda, \epsilon)\ \partial_x {}^{\epsilon}U^{rr}(\lambda, \epsilon, \beta).
\end{aligned}
$$

We can solve the equations for ${}^{\epsilon}U^{rr}(\lambda, \epsilon, \beta)$ and $\partial_x {}^{\epsilon}U^{rr}(\lambda, \epsilon, \beta)$ and substitute the results into (5.13). Then (5.14) easily follows. Part (ii) can be proved the same way as part (ii) of Lemma 5.2. $\square$

Theorem 5.12. *In the sense of convergence of distributions, ${}^{\epsilon}\mathbf{P}_x^{rr}[\omega(t) \in db]$ converges to $\mathbf{P}_x^{r0}[\omega(t) \in db]$ defined in Theorem 5.5, as $\epsilon \to 0$.*

Proof. If we let $\epsilon \to 0$ in (5.14) then we obtain

$$\lim_{\epsilon \to 0} {}^{\epsilon}U^{rr}(\lambda, x, \beta) = \hat{G}^0(\lambda, x, \beta) - \hat{K}(\lambda, x) \frac{\partial_x^2 \hat{G}^0(\lambda, 0, \beta)}{\partial_x^2 \hat{K}(\lambda, 0)}.$$

The right hand side equals to U^{r0} in the proof of Lemma 5.4, so it is the Fourier-Laplace transform of $\mathbf{P}_x^{r0}[\omega(t) \in db]$. $\square$

6 Killing of monopole and reflection for dipole

Once again, we start with an approximating sequence. We will find a solution $\{{}^{\epsilon}\mathbf{P}_x^{0r}[\omega(t) \in db] : x \geq 0, \}$ to the equation

$$\begin{aligned} {}^{\epsilon}\mathbf{P}_x^{0r}[\omega(t) \in db] &= \mathbf{P}_x^{00}[\omega(t) \in db] + \\ &+ \int_0^t \int_{-\infty}^{\infty} \mathbf{P}_x[\tau_0(\omega) \in ds,\, \omega(\tau_0) \text{ is a dipole and in } da]\, {}^{\epsilon}\mathbf{P}_{\epsilon+a}^{0r}[\omega(t-s) \in db]. \end{aligned} \tag{6.1}$$

Remark 6.1. Intuitively speaking, we have the following model. Consider fixed $\epsilon > 0$.

1. A monopole starts from a point $x \geq 0$, and moves according to the transition probability density $p(t, x)$ until it hits the interval $(-\infty, 0)$.
2. If $\omega(\tau_0)$ is a monopole when BPP hits $(-\infty, 0)$, then it is killed.
3. If $\omega(\tau_0)$ is a dipole when BPP hits $(-\infty, 0)$, then it restarts from the point $\epsilon + \omega(\tau_0)$.

Suppose $\{{}^{\epsilon}\mathbf{P}_x^{0r}[\omega(t) \in db] : x \geq 0, \}$ solving (6.1) exists. We denote the Fourier-Laplace transform of ${}^{\epsilon}\mathbf{P}_x^{0r}[\omega(t) \in db]$ by ${}^{\epsilon}U^{0r}(\lambda, x, \beta)$. Then we apply the Fourier-Laplace transform to (6.1). Using (3.13), we obtain for $x > 0$,

$${}^{\epsilon}U^{0r}(\lambda, x, \beta) = \hat{G}^{00}(\lambda, x, \beta) + \hat{J}(\lambda, x)\ \partial_x {}^{\epsilon}U^{r0}(\lambda, \epsilon, \beta). \tag{6.2}$$

The following lemma can be proved the same way as Lemma 5.11 so we omit its proof.

Lemma 6.2. (i) *For $\epsilon > 0$, the following is a solution of* (6.2)*:*

$${}^{\epsilon}U^{0r}(\lambda, x, \beta) = \hat{G}^{00}(\lambda, x, \beta) + \hat{J}(\lambda, x)\ \frac{\partial_x \hat{G}^0(\lambda, \epsilon, \beta)}{1 - \partial_x \hat{J}(\lambda, \epsilon)}. \tag{6.3}$$

(ii) *For small $\epsilon > 0$, this ${}^{\epsilon}U^{0r}$ satisfies* (3.4) *with any $c > 0$, and it is the Fourier-Laplace transform of a Schwartz temperate distribution ${}^{\epsilon}\mathbf{P}_x^{or}[\omega(t) \in db]$.*

Lemma 6.3. *In the sense of convergence of distributions, ${}^{\epsilon}\mathbf{P}_x^{0r}[\omega(t) \in db]$ converges to a limit as $\epsilon \to 0$.*

Definition 6.4. We denote this limit by $\{\mathbf{P}_x^{0r}[\omega(t) \in db] : x \geq 0\}$, and call it the distribution of *BPP with the killing boundary for the monopole and the reflecting boundary for the dipole.*

Proof of Lemma 6.3. We let ϵ tend to 0 in (6.3). Then (3.19) and (3.20) imply that

$$\begin{aligned} U^{0r}(\lambda, x, \beta) &\equiv \lim_{\epsilon \to \infty} {}^{\epsilon}U^{0r}(\lambda, x, \beta) \\ &= \hat{G}^{00}(\lambda, x, \beta) - \hat{J}(\lambda, x) \frac{\partial_x^2 \hat{G}^{00}(\lambda, 0, \beta)}{\partial_x^2 \hat{J}(\lambda, 0)}. \end{aligned} \tag{6.4}$$

When we take the principal value of $\lambda^{1/4}$, U^{0r} satisfies (3.4), and Lemma 3.1 implies our assertion. □

If $\Re\lambda > 0$, $\partial_x^2 \hat{G}^{00}(\lambda, 0, \beta)/\partial_x^2 \hat{J}(\lambda, 0)$ admits the usual inverse Fourier–Laplace transform. We present the calculations. Let $b \geq 0$. The residue theorem implies that

$$\begin{aligned} I^{0r}(\lambda, b) &\equiv \frac{1}{2\pi} \lim_{L \to \infty} \int_{-L}^{L} d\beta \ \exp\{-i\beta b\} \left(- \frac{\partial_x^2 \hat{G}^{00} \lambda, 0, \beta)}{\partial_x^2 \hat{J}(\lambda, 0)} \right) \\ &= \frac{i}{2\sqrt{\lambda}} \left(\exp\{\lambda^{1/4}\, b\, \overline{\theta_2}\} - \exp\{\lambda^{1/4}\, b\, \theta_2\} \right). \end{aligned} \tag{6.5}$$

For $t > 0$, we obtain that

$$\begin{aligned} Q^{0r}(t, b) &\equiv -\frac{1}{2\pi i} \lim_{M \to \infty} \int_{1-iM}^{1+iM} d\lambda \ \exp\{\lambda t\} \ I^{0r}(\lambda, b) \\ &= \frac{2}{\pi} \int_0^\infty dy \ \exp\{-y^4 t\} \ y \ \sin(yb). \end{aligned} \tag{6.6}$$

One can easily check that the density Q^{0r} is a smooth bounded function of non-negative b if $t > 0$, and it is integrable in t.

Theorem 6.5. *For $x \geq 0$, $b \geq 0$, and $t > 0$, we define a function*

$$p^{0r}(t, x, b) \equiv p^{00}(t, x, b) + \int_0^t ds \ J(s, x) \ Q^{0r}(t - s, b), \tag{6.7}$$

where Q^{0r} is the function in (6.6). *Let $\{\mathbf{P}_x^{0r}[\omega(t) \in db] : x \geq 0\}$ denote the distribution in Definition* 6.4. *Then we have*

$$\mathbf{P}_x^{0r}[\omega(t) \in db] = p^{0r}(t, x, b) \ db, \tag{6.8}$$

and these distributions are continuous linear functionals on $\mathbf{B}_b[0, \infty)$.

Proof. The theorem is immediate from Lemma 6.2 and (6.6). □

The following theorem is easily proved using (6.4) in the same way as Theorem 5.11 so we omit its proof.

Theorem 6.6. (i) *Let f be a bounded smooth function on the interval $[0, \infty)$ such that $f(0) = 0 = f''(0)$. We put*

$$v(t, x) \equiv \int_0^\infty \mathbf{P}_x^{0r}[\omega(t) \in db] \ f(b). \tag{6.9}$$

Then in the classical sense, this v satisfies (1.4) *with $\ell = 0$ and $m = 2$.*
(ii) $\mathbf{P}_x^{0r}[\omega(t) \in db]$ *satisfies Chapman-Kolmogorov equations* (4.5), *and the assertion in Remark 4.7 holds with $\mathbf{P}_x^{0r}[w(t) \in db]$ in place of $\mathbf{P}_x^{0}[\omega(t) \in db]$.*

7 Stopped monopole

7.1 Stopped monopole and dipole

Definition 7.1. Let $\{\mathbf{P}_x^{ss}[\omega(t) \in db] : x \geq 0\}$ be defined by

$$
\begin{aligned}
\mathbf{P}_x^{ss}[\omega(t) \in db] &= \mathbf{P}_x^{00}[\omega(t) \in db] + \\
&+ \int_0^t \int_{-\infty}^{\infty} \Big\{ \mathbf{P}_x[\tau_0(\omega) \in ds,\, \omega(\tau_0) \text{ is a monopole and in } da] \\
&\quad + \mathbf{P}_x[\tau_0(\omega) \in ds,\, \omega(\tau_0) \text{ is a dipole and in } da] \Big\}\ \delta(b-a)\, db.
\end{aligned} \tag{7.1}
$$

Remark 7.2. The family of distributions given in the last definition represents a BPP whose both particles are stopped after exiting $(0, \infty)$. An intuitive description of the process is the following.

1. A monopole starts from a point $x \geq 0$, and moves according to the transition probability density $p(t, x)$ until it hits the interval $(-\infty, 0)$.
2. After BPP hits the interval $(-\infty, 0)$, both particles are stopped.

Theorem 7.3. *For $x \geq 0$, $b \geq 0$, and $t > 0$, we define a linear functional on $\mathbf{C}_b^1[0, \infty)$,*

$$
p^{ss}(t, x, b) \equiv p^{00}(t, x, b) + \int_0^t ds\, \big\{ K(s, x)\, \delta(b) - J(s, x)\, \delta'(b) \big\}. \tag{7.2}
$$

Then we have

$$
\mathbf{P}_x^{ss}[\omega(t) \in db] = p^{ss}(t, x, b)\ db. \tag{7.3}
$$

Both sides of the last formula are continuous linear functionals on $\mathbf{C}_b^1[0, \infty)$.

Proof. The result follows easily from (3.13) and Definition 7.1. $\square$

Theorem 7.4. (i) *Let f be a bounded smooth function defined on the interval $[0, \infty)$ such that $f^{(4)}(0) = 0 = f^{(5)}(0)$. We define*

$$
v(t, x) \equiv \int_0^{\infty} \mathbf{P}_x^{ss}[\omega(t) \in db]\ f(b). \tag{7.4}
$$

Then in the classical sense, this v satisfies (1.4) with $\ell = 4$ and $m = 5$.
(ii) $\mathbf{P}_x^{ss}[\omega(t) \in db]$ *satisfies Chapman-Kolmogorov equations (4.5) in the sense of a linear functional on $\mathbf{C}_b^1[0, \infty)$.*

Proof. We apply the Fourier-Laplace transform to $\mathbf{P}_x^{ss}[\omega(t) \in db]$, and denote it by $U^{ss}(\lambda, x, \beta)$. Then (7.2) is transformed into

$$
U^{ss}(\lambda, x, \beta) = \hat{G}^{00}(\lambda, x, \beta) + \hat{K}(\lambda, x)\ \frac{1}{\lambda} + \hat{J}(\lambda, x)\ \frac{i\beta}{\lambda}, \tag{7.5}
$$

from which we see that (5.10) holds for this U^{ss}. Recalling (3.18), (3.19) and (3.20), we easily see that

$$
\partial_x^4 U^{ss}(\lambda, 0, \beta) = 0 \quad \text{and} \quad \partial_x^5 U^{ss}(\lambda, 0, \beta) = 0.
$$

This proves part (i) of the theorem, since the Fourier-Laplace transform is unique. For (ii), it is sufficient to note that $\partial_x p^{ss}(t, x, b)$ is a linear functional on $\mathbf{C}_b^1[0, \infty)$ and the left hand side of (4.5) is well-defined with $p^{00}(t, x, b)$ replaced by $\partial_x p^{ss}(t, x, b)$. $\square$

7.2 Stopped monopole and killed dipole

Definition 7.5. Let $\{\mathbf{P}_x^{s0}[\omega(t) \in db] : x \geq 0\}$ be given by

$$\begin{aligned} &\mathbf{P}_x^{s0}[\omega(t) \in db] = \mathbf{P}_x^{00}[\omega(t) \in db] + \\ &+ \int_{s=0}^{t} \int_{a=-\infty}^{\infty} \mathbf{P}_x[\tau_0(\omega) \in ds,\, \omega(\tau_0) \text{ is a monopole and in } da]\ \delta(a-b)\, db. \end{aligned} \tag{7.6}$$

Remark 7.6. A heuristic view of BPP with the boundary behavior specified in the section title is this.

1. A monopole starts from a point $x \geq 0$, and moves according to the transition probability density $p(t,x)$ until it hits the interval $(-\infty, 0)$.
2. If $\omega(\tau_0)$ is a monopole when BPP hits $(-\infty, 0)$, then it is stopped at the point $\omega(\tau_0)$.
3. If $\omega(\tau_0)$ is a dipole when BPP hits $(-\infty, 0)$, then it is killed.

Theorem 7.7. *For $x \geq 0$, $b \geq 0$, and $t > 0$, we define a linear functional on $\mathbf{B}_b[0,\infty)$,*

$$p^{s0}(t,x,b) \equiv p^0(t,x,b) + \int_0^t ds\ K(s,x)\ \delta(b). \tag{7.7}$$

We have

$$\mathbf{P}_x^{s0}[\omega(t) \in db] = p^{s0}(t,x,b)\ db, \tag{7.8}$$

where both expressions are continuous linear functionals on $\mathbf{B}_b[0,\infty)$.

Proof. The result follows easily from the definition (7.6). □

Theorem 7.8. (i) *Let f be a bounded smooth function on the interval $[0,\infty)$ such that $f'(0) = 0 = f^{(4)}(0)$. We put*

$$v(t,x) \equiv \int_0^\infty \mathbf{P}_x^{s0}[\omega(t) \in db]\ f(b). \tag{7.9}$$

Then in the classical sense, this v satisfies (1.4) with $\ell = 1$ and $m = 4$.
(ii) *$\mathbf{P}_x^{s0}[\omega(t) \in db]$ satisfies Chapman-Kolmogorov equations (4.5) in the sense of a linear functional on $\mathbf{B}_b[0,\infty)$.*

Proof. Let $U^{s0}(\lambda, x, \beta)$ be the Fourier-Laplace transform of $\mathbf{P}_x^{s0}$. Apply the Fourier-Laplace transform to (7.7). The equation is transformed to

$$U^{s0}(\lambda, x, \beta) = \hat{G}^{00}(\lambda, x, \beta) + \hat{K}(\lambda, x)\ \frac{1}{\lambda}. \tag{7.10}$$

From this, we see that (5.10) holds for this U^{s0} and that

$$\partial_x U^{s0}(\lambda, 0, \beta) = 0 \quad \text{and} \quad \partial_x^4 U^{s0}(\lambda, 0, \beta) = 0.$$

The first assertion of the theorem follows from this and from uniqueness of the Fourier-Laplace transform. The proof of the second is the same as in Theorem 7.3. □

We present an example of a "not well-posed problem" for (1.4):

Example 7.9. Let f be a bounded smooth function on the interval $[0,\infty)$ such that $f'(0) = 0, f''(0) = 0$, and $f^{(4)}(0) = 0$. Recalling Theorems 5.7 and 7.8, we define functions

$$v_1(t,x) \equiv \int_o^\infty \mathbf{P}^{r0}[\omega(t) \in db]\ f(b); \quad v_2(t,x) \equiv \int_o^\infty \mathbf{P}^{s0}[\omega(t) \in db]\ f(b).$$

These v_1 and v_2 are classical solutions of (1.4) with $\ell = 1$. We consider their Laplace transforms

$$V_j(\lambda,x) \equiv \int_0^\infty dt\ \exp\{-\lambda t\}\ v_j(t,x), \quad j = 1,2,$$

which satisfy

$$\partial_x^4 V_j(\lambda,x) = -\lambda V_j(\lambda,x) + f(x), \quad j = 1,2.$$

Since they are smooth and both satisfy $\partial_x V_j(\lambda,0) = 0$, we have

$$0 = \partial_x^5 V_j(\lambda,0) = -\lambda \partial_x V_j(\lambda,0) + f'(0) = 0, \quad j = 1,2.$$

This fact implies that solutions to (1.4) are not unique when $\ell = 1$ and $m = 5$. Note that an analogous argument also holds when $\ell = 0$ and $m = 4$.

7.3 Stopped monopole and reflected dipole

This case calls for an approximation procedure similar to those discussed in earlier sections. We are looking for a family $\{{}^\epsilon\mathbf{P}_x^{sr}[\omega(t) \in db] : x \geq 0\}$ satisfying

$$\begin{aligned}
&{}^\epsilon\mathbf{P}_x^{sr}[\omega(t) \in db] = \mathbf{P}_x^{00}[\omega(t) \in db] + \\
&+ \int_0^t \int_{-\infty}^\infty \mathbf{P}_x[\tau_0(\omega) \in ds,\ \omega(\tau_0) \text{ is a monopole and in } da]\, \delta(a-b)\, db \\
&+ \int_0^t \int_{-\infty}^\infty \mathbf{P}_x[\tau_0(\omega) \in ds,\ \omega(\tau_0) \text{ is a dipole and in } da]\, {}^\epsilon\mathbf{P}_{\epsilon+a}^{sr}[\omega(t-s) \in db].
\end{aligned} \tag{7.11}$$

Remark 7.10. An intuitive description of $\{{}^\epsilon\mathbf{P}_x^{sr}[\omega(t) \in db] : x \geq 0\}$ is the following. Let ϵ be a fixed positive number.

1. A monopole starts from a point $x \geq 0$, and moves according to the transition probability density $p(t,x)$ until it hits the interval $(-\infty, 0)$.
2. If $\omega(\tau_0)$ is a monopole when BPP hits $(-\infty,0)$, then it is stopped at the point $\omega(\tau_0)$.
3. If $\omega(\tau_0)$ is a dipole when BPP hits $(-\infty,0)$, then it restarts from the point $\epsilon + \omega(\tau_0)$.

Let ${}^\epsilon U^{sr}(\lambda,x,\beta)$ denote the Fourier-Laplace transform of ${}^\epsilon\mathbf{P}_x^{sr}$, if it exists. Recall (3.13) and note that (7.11) can be transformed to

$$\begin{aligned}
{}^\epsilon U^{sr}(\lambda,x,\beta) &= \hat{G}^{00}(\lambda,x,\beta) + \\
&+ \hat{K}(\lambda,x)\,\frac{1}{\lambda} + \hat{J}(\lambda,x)\,\partial_x {}^\epsilon U^{sr}(\lambda,\epsilon,\beta).
\end{aligned} \tag{7.12}$$

The following lemma is similar to several lemmas presented earlier in the paper so we omit its proof.

Lemma 7.11. (i) *For $\epsilon > 0$, the following is a solution of* (7.12)*:*

$$
\begin{aligned}
{}^{\epsilon}U^{sr}(\lambda, x, \beta) = \hat{G}^{00}(\lambda, x, \beta) + +\hat{K}(\lambda, \epsilon)\,\frac{1}{\lambda} + \\
+ \hat{J}(\lambda, \epsilon)\,\frac{\partial_x \hat{G}^{00}(\lambda, \epsilon, \beta) + \partial_x \hat{K}(\lambda, \epsilon)/\lambda}{1 - \partial_x \hat{J}(\lambda, \epsilon)}.
\end{aligned}
\tag{7.13}
$$

(ii) *When $\epsilon > 0$ is small, this ${}^{\epsilon}U^{sr}$ satisfies* (3.4) *for every $c > 0$ and it is the Fourier-Laplace transform of a Schwartz temperate distribution ${}^{\epsilon}\mathbf{P}_x^{sr}[\omega(t) \in db]$.*

Lemma 7.12. *${}^{\epsilon}\mathbf{P}_x^{sr}[\omega(t) \in db]$ converges to a limit as $\epsilon \to 0$, in the distribution sense.*

Definition 7.13. The above limit will be denoted by $\{\mathbf{P}_x^{sr}[\omega(t) \in db] : x \geq 0\}$, and called the distribution of *BPP with stopped monopole and reflecting dipole on the boundary.*

Proof. Recall (3.18) through (3.20), and let $\epsilon \to 0$ in (7.12). Then

$$
\begin{aligned}
U^{sr}(\lambda, x, \beta) \equiv \lim_{\epsilon \to 0} {}^{\epsilon}U^{sr}(\lambda, x, \beta) = \hat{G}^{00}(\lambda, x, \beta) + \hat{K}(\lambda, x)\,\frac{1}{\lambda} - \\
- \hat{J}(\lambda, x)\,\frac{\partial_x^2 \hat{G}^{00}(\lambda, 0, \beta) + \partial_x^2 \hat{K}(\lambda, 0)/\lambda}{\partial_x^2 \hat{J}(\lambda, 0)}.
\end{aligned}
\tag{7.14}
$$

This satisfies (3.4), so Lemma 3.1 implies our assertion. ☐

We will apply the inverse Fourier-Laplace transform to

$$
-\frac{\partial_x^2 \hat{G}^{00}(\lambda, 0, \beta) + \partial_x^2 \hat{K}(\lambda, 0)/\lambda}{\partial_x^2 \hat{J}(\lambda, 0)} = -\frac{\partial_x^2 \hat{G}^{00}(\lambda, 0, \beta)}{\partial_x^2 \hat{J}(\lambda, 0)} - \frac{\partial_x^2 \hat{K}(\lambda, 0)}{\lambda\ \partial_x^2 \hat{J}(\lambda, 0)}
\tag{7.15}
$$

in the distribution sense. We already know a formula for the first term of (7.15), which is the same as Q^{0r} in (6.6). As for the second term

$$
-\frac{\partial_x^2 \hat{K}(\lambda, 0)}{\lambda\ \partial_x^2 \hat{J}(\lambda, 0)} = \frac{1}{\sqrt{2}\,\lambda^{3/4}},
$$

we can apply the inverse Fourier-Laplace transform in the sense of Schwartz temperate distribution, to obtain

$$
-\frac{2}{\pi}\Big(\int_0^\infty dy\ \exp\{-y^4 t\}\Big)\,\delta(b).
$$

We conclude that the inverse Fourier-Laplace transform of (7.15) is

$$
Q^{sr}(t, b) \equiv Q^{0r}(t, b) - \frac{2}{\pi}\Big(\int_0^\infty dy\ \exp\{-y^4 t\}\Big)\,\delta(b).
\tag{7.16}
$$

This is a linear functional on $\mathbf{B}_b[0, \infty)$. The following theorem is immediate from Lemma 7.12 and the above arguments. Therefore we omit its proof.

Theorem 7.14. *For $x \geq 0$, $b \geq 0$, and $t > 0$, we define a linear functional on $\mathbf{B}_b[0,\infty)$,*

$$\begin{aligned} p^{sr}(t,x,b) \equiv p^{00}(t,x,b) + \int_0^t ds\, K(s,x)\, \delta(b) + \\ + \int_0^s ds\ \ J(s,x)\, Q^{sr}(t-s,b) \end{aligned} \tag{7.17}$$

where Q^{sr} is given in (7.16). We have

$$\mathbf{P}_x^{sr}[\omega(t) \in db] = p^{sr}(t,x,b)\ db. \tag{7.18}$$

Both sides are continuous linear functionals on $\mathbf{B}_b[0,\infty)$.

Theorem 7.15. (i) *Let f be a bounded smooth function on the interval $[0,\infty)$ such that $f''(0) = 0 = f^{(4)}(0)$. We put*

$$v(t,x) \equiv \int_0^\infty \mathbf{P}_x^{sr}[\omega(t) \in db]\ f(b). \tag{7.19}$$

Then in the classical sense, this v satisfies (1.4) with $\ell = 2$ and $m = 4$.
(ii) $\mathbf{P}_x^{sr}[\omega(t) \in db]$ *satisfies Chapman-Kolmogorov equations (4.5), and Remark 4.7 holds with $\mathbf{P}_x^{sr}[\omega(t) \in db]$ in place of $\mathbf{P}_x^{0}[\omega(t) \in db]$.*

Proof. Note that the Fourier-Laplace transform is unique. By (7.14), we see that (5.10) holds for U^{sr}. Moreover using (3.18) through (3.20), we have

$$\partial_x^2 U^{sr}(\lambda, 0, \beta) = 0 \quad \text{and} \quad \partial_x^4 U^{sr}(\lambda, 0, \beta) = 0.$$

These prove (i), and (ii) can be proved by the same arguments as similar results in earlier sections. $\square$

8 Stopped dipole

8.1 Stopped dipole and killed monopole

Definition 8.1. Let $\{\mathbf{P}_x^{0s}[\omega(t) \in db] : x \geq 0\}$ be given by

$$\begin{aligned} \mathbf{P}_x^{0s}[\omega(t) \in db] = \mathbf{P}_x^{00}[\omega(t) \in db] + \\ + \int_0^t \int_\infty^\infty \mathbf{P}_x[\tau_0(\omega) \in ds, \omega(\tau_0) \text{ is a dipole and in } da]\delta(a-b)\, db. \end{aligned} \tag{8.1}$$

Remark 8.2. Intuitively, we are dealing with the following model.

1. A monopole starts from a point $x \geq 0$, and moves according to the transition probability density $p(t,x)$ until it hits the interval $(-\infty, 0)$.

2. If $\omega(\tau_0)$ is a monopole when BPP hits $(-\infty, 0)$, then it is killed.

3. If $\omega(\tau_0)$ is a dipole when BPP hits $(-\infty, 0)$, then it is stopped at the point $\omega(\tau_0)$.

The following theorem can be easily derived from (3.13).

Theorem 8.3. *For $x \geq 0$, $b \geq 0$, and $t > 0$, we define a linear functional on $\mathbf{C}_b^1[0,\infty)$,*

$$p^{0s}(t,x,b) \equiv p^{00}(t,x,b) + \int_0^s ds\ J(s,x)\ \delta'(b). \tag{8.2}$$

Then we have

$$\mathbf{P}_x^{0s}[\omega(t) \in db] = p^{0s}(t,x,b)\, db, \tag{8.3}$$

where both expressions are continuous linear functionals on $\mathbf{C}_b^1[0,\infty)$.

Theorem 8.4. (i) *Let f be a bounded smooth function on the interval $[0,\infty)$ such that $f(0) = 0 = f^{(5)}(0)$. We put*

$$v(t,x) \equiv \int_0^\infty \mathbf{P}_x^{0s}[\omega(t) \in db]\ f(b). \tag{8.4}$$

Then in the classical sense, this v satisfies (1.4) *with $\ell = 0$ and $m = 5$.*
(ii) $\mathbf{P}_x^{0s}$ *satisfies Chapman-Kolmogorov equations* (4.5).

Proof. If we denote by $U^{0s}(x,\lambda,\beta)$ the Fourier-Laplace transform of $\mathbf{P}_x^{0s}$, then (8.3) is transformed into the following equality:

$$U^{0s}(x,\lambda,\beta) = \hat{G}^{00}(x,\lambda,\beta) + \hat{J}(x,\lambda,\beta)\ \frac{i\beta}{\lambda}, \tag{8.5}$$

from which we see that (5.10) holds for this U^{0s}. Using (3.18) through (3.20), we have

$$U^{0s}(\lambda,0,\beta) = 0 \quad \text{and} \quad \partial_x^5 U^{0s}(\lambda,0,\beta) = 0.$$

These prove (i). Note that $\partial_x p^{0s}(t,x,b)$ is a linear functional on $\mathbf{C}_b^1[0,\infty)$. The second part of the theorem can be proved by arguments used earlier in the paper. □

8.2 Stopped dipole and reflected monopole

For the last time in this paper we use an approximation procedure to construct a process. The first step is to find $\{{}^\epsilon\mathbf{P}_x^{rs}[\omega(t) \in db] : x \geq 0\}$ such that

$$\begin{aligned}
&{}^\epsilon\mathbf{P}_x^{rs}[\omega(t) \in db] = \mathbf{P}_x^0[\omega(t) \in db] + \\
&+ \int_0^t \int_{-\infty}^\infty \mathbf{P}_x[\tau_0(\omega) \in ds,\ \omega(\tau_0) \text{ is a monopole and in } da]\ {}^\epsilon\mathbf{P}_{\epsilon+\omega(\tau_0)}^{rs}[\omega(t) \in db] + \\
&+ \int_0^t \int_{-\infty}^\infty \mathbf{P}_x[\tau_0(\omega) \in ds,\ \omega(\tau_0) \text{ is a dipole and in } da]\ \delta(b-a)db.
\end{aligned} \tag{8.6}$$

Remark 8.5. The approximating process evolves according to the following rules. Fix $\epsilon > 0$.

1. A monopole starts from a point $x \geq 0$, and moves according to the transition probability density $p(t,x)$ until it hits the interval $(-\infty,0)$.
2. If $\omega(\tau_0)$ is a monopole when BPP hits $(-\infty,0)$, then it restarts from the point $\epsilon + \omega(\tau_0)$.

3. If $\omega(\tau_0)$ is a dipole when BPP hits $(-\infty, 0)$, then it is stopped at the point $\omega(\tau_0)$.

Assuming existence, we denote by ${}^{\epsilon}U^{rs}(\lambda, x, \beta)$ the Fourier-Laplace transform of ${}^{\epsilon}\mathbf{P}^{rs}_x$. Recall (3.13). We see that (8.6) is transformed into the following equation:

$$ {}^{\epsilon}U^{rs}(\lambda, x, \beta) = \hat{G}^{00}(\lambda, x, \beta) + \hat{K}(\lambda, x)\, {}^{\epsilon}U^{sr}(\lambda, \epsilon, \beta) + \hat{J}(\lambda, x)\, \frac{i\beta}{\lambda}. \tag{8.7} $$

We omit the proof of the following result.

Lemma 8.6. (i) *For $\epsilon > 0$, the following is a solution of* (8.7)*:*

$$ \begin{aligned} {}^{\epsilon}U^{rs}(\lambda, x, \beta) = \hat{G}^{00}(\lambda, x, \beta) + \hat{J}(\lambda, x)\, \frac{i\beta}{\lambda} + \\ + \hat{K}(\lambda, x)\, \frac{\hat{G}^{00}(\lambda, \epsilon, \beta) + i\beta\, \hat{J}(\lambda, \epsilon)/\lambda}{1 - \hat{K}(\lambda, \epsilon)}. \end{aligned} \tag{8.8} $$

(ii) *When $\epsilon > 0$ is small, this ${}^{\epsilon}U^{rs}$ satisfies* (3.4) *for every $c > 0$ and it is the Fourier-Laplace transform of a Schwartz temperate distribution ${}^{\epsilon}\mathbf{P}^{rs}_x[\omega(t) \in db]$.*

We have just proved that if $\epsilon > 0$, then there exist distributions ${}^{\epsilon}\mathbf{P}^{rs}_x[\omega(t) \in db]$ satisfying equations (8.6). However the following proposition asserts that we cannot pass to 0 with ϵ. Hence we cannot construct a BPP with reflection for the monopole and stopped dipole, at least not using our method.

Proposition 8.7. (8.8) *diverges as $\epsilon \to 0$.*

Remark 8.8. Proposition 8.7 is not very surprising if we consider the corresponding problem (1.4). Table 1.2 suggests that, if a BPP with reflection for the monopole and stopped dipole exists, then it corresponds to (1.4) with $\ell = 1$ and $m = 5$. But solutions to this boundary value problem (1.4) are not unique. So we cannot find a "fundamental solution" of such (1.4), which would serve as a transition density for the corresponding BPP.

Proof of Proposition 8.7. From (3.18) through (3.20), we see that

$$ \begin{aligned} 1 - \hat{K}(\lambda, \epsilon) &= \frac{\epsilon^2}{2}\, \partial_x^2 \hat{K}(\lambda, 0) + o(\epsilon^2) \\ &= -\frac{\epsilon^2 \sqrt{\lambda}}{2} + o(\epsilon^2), \end{aligned} \tag{8.9} $$

$$ \begin{aligned} \hat{G}^{00}(\lambda, \epsilon, \beta) + \frac{i\beta\, \hat{J}(\lambda, \epsilon)}{\lambda} &= \frac{i\beta}{\lambda} \times \epsilon\, \partial_x \hat{J}(\lambda, 0) + o(\epsilon) \\ &= \frac{\epsilon\, i\, \beta}{\lambda} + o(\epsilon). \end{aligned} \tag{8.10} $$

This shows that

$$ \frac{\hat{G}^{00}(\lambda, \epsilon, \beta) + i\beta\, \hat{J}(\lambda, \epsilon)/\lambda}{1 - \hat{K}(\lambda, \epsilon)} $$

diverges as $\epsilon \to 0$, and so (8.8) also diverges. $\square$

9 Conservation of charge

Let $\{\mathbf{Q}_x[\omega(t)] : x \geq 0\}$ be the generic notation for the distribution of BPP on $[0, \infty)$ with any boundary conditions for the monopole and dipole.

Definition 9.1. We say that $\mathbf{Q}_x[\omega(t) \in db]$ is *conservative* if

$$\rho(t, x) \equiv \int_{[0,\infty)} \mathbf{Q}_x[\omega(t) \in db] = 1 \quad \text{for all } t > 0 \text{ and } x > 0. \tag{9.1}$$

Theorem 9.2. *Let $\{\mathbf{Q}_x[\omega(t) \in db] : x \geq 0\}$ be the distribution of any BPP on $[0, \infty)$ constructed in this paper. Then $\mathbf{Q}_x[\omega(t) \in db]$ is conservative if and only if there is no killing for the monopole.*

Remark 9.3. This result intuitively supports for the names "monopole" and "dipole." Since a dipole carries two charges of equal magnitude but opposite signs, its loss does not change the total amount of charge. On the other hand, the monopole carries charge of single sign, and its loss changes the total amount of charge.

Proof of Theorem 9.2. Inspecting Table 1.2, we see that there is no killing for the monopole if and only if $\ell \geq 1$ and $m \geq 1$.

Assume that $\ell \geq 1$ and $m \geq 1$ in (1.4). Since $\rho(0, x) = 1$ in view of (9.1), we see that ρ is a solution of (1.4) with the constant initial function

$$f(x) \equiv 1.$$

The constant function $v(t, x) \equiv 1$ is a solution of (1.4) with $f \equiv 1$, if $\ell \geq 1$ and $m \geq 1$. Now uniqueness of the solution implies that $\rho(t, x) \equiv 1$, and $\mathbf{Q}_x[\omega(t) \in db]$ is conservative.

On the other hand, if $\ell = 0$, then the constant $v(t, x) \equiv 1$ is not a solution of (1.4) with the constant initial function $f \equiv 1$, and uniqueness of the solution implies that $\rho(t, x) \neq 1$ for some (t, x). In this case, $\mathbf{Q}_x[\omega(t) \in db]$ is not conservative. □

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