Electronic Journal of Probability

Vol. 14 (2009), Paper no. 83, pages 2391–2417.

Journal URL
http://www.math.washington.edu/~ejpecp/

CLT for Linear Spectral Statistics of Wigner matrices*

Zhidong Bai† Xiaoying Wang‡ Wang Zhou§

Abstract

In this paper, we prove that the spectral empirical process of Wigner matrices under sixth-moment conditions, which is indexed by a set of functions with continuous fourth-order derivatives on an open interval including the support of the semicircle law, converges weakly in finite dimensions to a Gaussian process.

Key words: Bernstein polynomial, central limit theorem, Stieltjes transform, Wigner matrices.

AMS 2000 Subject Classification: Primary 15B52, 60F15; Secondary: 62H99.

Submitted to EJP on July 17, 2009, final version accepted October 14, 2009.

*The first two authors were partially supported by CNSF 10871036 and NUS grant R-155-000-079-112 and the third author was partially supported by grant R-155-000-076-112 at the National University of Singapore

†School of Mathematics and Statistics, Northeast Normal University, Changchun, 130024, P.R.C. Email: baizd@nenu.edu.cn

‡School of Mathematics and Statistics, Northeast Normal University, Changchun, 130024, P.R.C. Email: wangxy022@126.com

§Department of Statistics and Applied Probability, National University of Singapore,117546, Singapore. Email: stazw@nus.edu.sg

1 Introduction and Main result

The random matrices theory originates from the development of quantum mechanics in the 1950's. In quantum mechanics, the energy levels of a system are described by eigenvalues of a Hermitian operator on a Hilbert space. Hence physicists working on quantum mechanics are interested in the asymptotic behavior of the eigenvalues and from then on, the random matrices theory becomes a very popular topic among mathematicians, probabilitists and statisticians. The leading work, the famous semi-circle law for Wigner matrices was found in [19].

A real Wigner matrix of size n is a real symmetric matrix $W_n = (x_{ij})_{1\le i,j\le n}$ whose upper-triangle entries $(x_{ij})_{1\le i\le j\le n}$ are independent, zero-mean real-valued random variables satisfying the following moment conditions:

$$(1).\ \forall i,\ \mathbb{E}|x_{ii}|^2 = \sigma^2 > 0;\ \ (2).\ \forall i < j,\ \mathbb{E}|x_{ij}|^2 = 1.$$

The set of these real Wigner matrices is called the Real Wigner Ensemble (RWE).

A complex Wigner matrix of size n is a Hermitian matrix W_n whose upper-triangle entries $(x_{ij})_{1\le i\le j\le n}$ are independent, zero-mean complex-valued random variables satisfying the following moment conditions:

$$(1).\ \forall i,\ \mathbb{E}|x_{ii}|^2 = \sigma^2 > 0;\ \ (2).\ \forall i < j,\ \mathbb{E}|x_{ij}|^2 = 1,\ \text{and}\ \mathbb{E}x_{ij}^2 = 0.$$

The set of these complex Wigner matrices is called the Complex Wigner Ensemble (CWE).

The empirical distribution F_n generated by the n eigenvalues of the normalized Wigner matrix $n^{-1/2}W_n$ is called the empirical spectral distribution (ESD) of Wigner matrix. The semi-circle law states that F_n a.s. converges to the distribution F with the density

$$p(x) = \frac{1}{2\pi}\sqrt{4 - x^2}, \quad x \in [-2, 2].$$

Its various versions of convergence were later investigated. See, for example, [1], [2].

Clearly, one method of refining the above approximation is to establish the rate of convergence, which was studied in [3], [10], [12], [13], [18], [5] and [8]. Although the exact convergence rate remains unknown for Wigner matrices, Bai and Yao [6] proved that the spectral empirical process of Wigner matrices indexed by a set of functions analytic on an open domain of the complex plane including the support of the semi-circle law converges to a Gaussian process under fourth moment conditions.

To investigate the convergence rate of the ESD of Wigner matrix, one needs to use f as step functions. However, many evidences show that the empirical process associated with a step function can not converge in any metric space, see Chapter 9 of [8]. Naturally, one may ask whether it is possible to derive the convergence of the spectral empirical process of Wigner matrices indexed by a class of functions under as less assumption on the smoothness as possible. This may help us to have deeper understanding on the exact convergence rate of ESD to semi-circle law.

In this paper, we consider the empirical process of Wigner matrices, which is indexed by a set of functions with continuous fourth-order derivatives on an open interval including the support of the semicircle law. More precisely, let $C^4(\mathscr{U})$ denote the set of functions $f : \mathscr{U} \to \mathbb{C}$ that has continuous fourth-order derivatives, where $\mathscr{U}$ is an open interval including the interval $[-2,2]$, the support of $F(x)$. The empirical process $G_n \triangleq \{G_n(f)\}$ indexed by $C^4(\mathscr{U})$ is given by

$$G_n(f) \triangleq n \int_{-\infty}^{\infty} f(x)[F_n - F](dx), \quad f \in C^4(\mathscr{U}) \tag{1.1}$$

In order to give a unified treatment for Wigner matrices, we define the parameter κ with values 1 and 2 for the complex and real Wigner matrices respectively. Also set $\beta = \mathbb{E}(|x_{12}|^2 - 1)^2 - \kappa$. Our main result is as follows.

Theorem 1.1. *Suppose*

$$\mathbb{E}|x_{ij}|^6 \le M \text{ for all } i,j. \tag{1.2}$$

Then the spectral empirical process $G_n = \{G_n(f) : f \in C^4(\mathscr{U})\}$ converges weakly in finite dimensions to a Gaussian process $G := \{G(f) : f \in C^4(\mathscr{U})\}$ with mean function

$$\mathbb{E}G(f) = \frac{\kappa - 1}{4}[f(2) + f(-2)] - \frac{\kappa - 1}{2}\tau_0(f) + (\sigma^2 - \kappa)\tau_2(f) + \beta\tau_4(f).$$

and covariance function

$$\begin{aligned} c(f,g) &\triangleq \mathbb{E}[\{G(f) - \mathbb{E}G(f)\}\{G(g) - \mathbb{E}G(g)\}] \\ &= \frac{1}{4\pi^2}\int_{-2}^{2}\int_{-2}^{2} f'(t)g'(s)V(t,s)dtds \end{aligned}$$

where

$$\begin{aligned} V(t.s) &= \left(\sigma^2 - \kappa + \frac{1}{2}\beta ts\right)\sqrt{(4-t^2)(4-s^2)} \\ &\quad + \kappa \log\left(\frac{4 - ts + \sqrt{(4-t^2)(4-s^2)}}{4 - ts - \sqrt{(4-t^2)(4-s^2)}}\right) \end{aligned}$$

and

$$\begin{aligned} \tau_l(f) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(2cos\theta)e^{il\theta}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(2cos\theta)cos(l\theta)d\theta \\ &= \frac{1}{\pi}\int_{-1}^{1} f(2t)T_l(t)\frac{1}{\sqrt{1-t^2}}dt. \end{aligned}$$

Here $\{T_l, l \ge 0\}$ is the family of Chebyshev polynomials.

Remark 1.2. In the definition of $G_n(f)$, $\theta = \int f(x)dF(x)$ can be regarded as a population parameter. The linear spectral statistic $\hat{\theta} = \int f(x)dF_n(x)$ is then an estimator of θ. We remind the reader that the center $\theta = \int f(x)dF(x)$, instead of $\mathbb{E}\int f(x)dF_n(x)$, has its strong statistical meaning in the application of Theorem 1.1. For example, in quantum mechanics, Wigner matrix is a discretized version of a random linear transformation in a Hilbert space, and semicircular law is derived under ideal assumptions. Therefore, the quantum physicists may want to test the validity of the ideal assumptions. Therefore, they may suitably select one or several f's so that θ's may characterize the semicircular law. Using the limiting distribution of $G_n(f) = n(\hat{\theta} - \theta)$, one may perform statistical test of the ideal hypothesis. Obviously, one can not apply the limiting distribution of $n(\hat{\theta} - \mathbb{E}\hat{\theta})$ to the above test.

Remark 1.3. Checking the proof of Theorem 1.1, one finds that the proof still holds under the assumption of bounded fourth moments of the off-diagonal entries is finite, if the approximation $G_n(f_m)$ of $G_n(f)$ is of the desired order. Thus, the assumption of 6-th moment is only needed in deriving the convergence rate of $\|F_n - F\|$. Furthermore, the assumption of the fourth derivative of f is also related to the convergence rate of $\|F_n - F\|$. If the convergence rate of $\|F_n - F\|$ is improved and/or proved under weaker conditions, then the result of Theorem 1.1 would hold under the weakened conditions. We conjecture that the result would be true if we only assume the fourth moments of the off-diagonal elements of Wigner matrices are bounded and f have the first continuous derivative.

Pastur and Lytova in [17] studied asymptotic distributions of $n \int f(x)d(F_n(x) - \mathbb{E}F_n(x))$ under the conditions that the fourth cumulant of off-diagonal elements of Wigner matrices being zero and the Fourier transform of $f(x)$ having 5th moment, which implies that f has the fifth continuous derivative. Moreover, they assume f is defined on the whole real line. These are stronger than our conditions.

Remark 1.4. The strategy of the proof is to use Bernstein polynomials to approximate functions in $C^4(\mathscr{U})$. This will be done in Section 2. Then the problem is reduced to the analytic case, which has been intensively discussed in Bai and Yao [6]. But the functions in [6] are independent of n and thus one can choose fixed contour and then prove that the Stieltjes transforms tend to a limiting process. In the present case, the Berstein polynomials depend on n through increasing degrees and thus they are not uniformly bounded on any fixed contour. Therefore, we cannot simply use the results of Bai and Yao [6] and thus we have to choose a sequence of contours approaching to the real axes so that the approximating polynomials are uniformly bounded on the corresponding contours. On this sequence of contours, the Stieltjes transforms don't have a limiting process. Our Theorem 1.1 cannot follow directly from [6]. We have to find alternative ways to prove the CLT.

Remark 1.5. It has been also found in literature that the so-called *secondary freeness* is proposed and investigated in *free probability*. Readers of interest are referred to Mingo [14; 15; 16]. We shall not be much involved in this direction in the present paper. As a matter, we only comment here that both the freeness and the secondary freeness are defined on sequences of random matrices (or ensembles) for which the limits of expectations of the normalized traces of all powers of the random matrices and the limits of covariances of unnormalized traces of powers of the random matrices exist. Therefore, for a single sequence of random matrices, the existence of these limits has to be verified and the verification is basically equivalent to moment convergence method. Results obtained in [14] is in some sense equivalent to those of [17].

The paper is organized as follows. The truncation and re-normalization step is in Section 3. We derive the mean function of the limiting process in Section 4. The convergence of the empirical processes is proved in Section 5.

2 Bernstein polynomial approximation

It is well-known that if $\tilde{f}(y)$ is a continuous function on the interval $[0,1]$, the Bernstein polynomials

$$\tilde{f}_m(y) = \sum_{k=0}^{m} \binom{m}{k} y^k (1-y)^{m-k} \tilde{f}\left(\frac{k}{m}\right)$$

converge to $\tilde{f}(y)$ uniformly on $[0,1]$ as $m \to \infty$. Suppose $\tilde{f}(y) \in C^4[0,1]$. Taylor expansion gives

$$\begin{aligned} \tilde{f}\left(\frac{k}{m}\right) &= \tilde{f}(y) + \left(\frac{k}{m} - y\right)\tilde{f}'(y) + \frac{1}{2}\left(\frac{k}{m} - y\right)^2 \tilde{f}''(y) \\ &\quad + \frac{1}{3!}\left(\frac{k}{m} - y\right)^3 \tilde{f}^{(3)}(y) + \frac{1}{4!}\left(\frac{k}{m} - y\right)^4 \tilde{f}^{(4)}(\xi_y). \end{aligned}$$

Hence

$$\tilde{f}_m(y) - \tilde{f}(y) = \frac{y(1-y)\tilde{f}''(y)}{2m} + O\left(\frac{1}{m^2}\right). \tag{2.1}$$

For the function $f \in C^4(\mathscr{U})$, there exists $a > 2$ such that $[-a,a] \subset \mathscr{U}$. Make a linear transformation $y = Kx + \frac{1}{2}$, $\epsilon \in (0,1/2)$, where $K = (1-2\epsilon)/(2a)$, then $y \in [\epsilon, 1-\epsilon]$ if $x \in [-a,a]$. Define $\tilde{f}(y) \triangleq f(K^{-1}(y-1/2)) = f(x)$, $y \in [\epsilon, 1-\epsilon]$, and define

$$f_m(x) \triangleq \tilde{f}_m(y) = \sum_{k=1}^{m} \binom{m}{k} y^k (1-y)^{m-k} \tilde{f}\left(\frac{k}{m}\right).$$

From (2.1), we have

$$f_m(x) - f(x) = \tilde{f}_m(y) - \tilde{f}(y) = \frac{y(1-y)\tilde{f}''(y)}{2m} + O\left(\frac{1}{m^2}\right).$$

Since $\tilde{h}(y) \triangleq y(1-y)\tilde{f}''(y)$ has a second-order derivative, we can use Bernstein polynomial approximations once again to get

$$\begin{aligned} \tilde{h}_m(y) - \tilde{h}(y) &= \sum_{k=1}^{m} \binom{m}{k} y^k (1-y)^{m-k} \tilde{h}(\frac{k}{m}) - \tilde{h}(y) \\ &= O\left(\frac{1}{m}\right). \end{aligned}$$

So, with $h_m(x) = \tilde{h}_m(Kx + \frac{1}{2})$,

$$f(x) = f_m(x) - \frac{1}{2m} h_m(x) + O\left(\frac{1}{m^2}\right).$$

Therefore, $G_n(f)$ can be split into three parts:

$$\begin{aligned}
G_n(f) &= n\int_{-\infty}^{\infty} f(x)[F_n-F](dx) \\
&= n\int f_m(x)[F_n-F](dx) - \frac{n}{2m}\int h_m(x)[F_n-F](dx) \\
&\qquad +n\int \big(f(x)-f_m(x)+\frac{1}{2m}h_m(x)\big)[F_n-F](dx) \\
&= \Delta_1+\Delta_2+\Delta_3.
\end{aligned}$$

For Δ_3, by Lemma 6.1 given in Appendix,

$$\|F_n - F\| = O_p(n^{-2/5}).$$

Here and in the sequel, the notation $Z_n = O_p(c_n)$ means that for any $\epsilon > 0$, there exists an $M > 0$ such that $\sup_n P(|Z_n| \le Mc_n) < \epsilon$. Similarly, $Z_n = o_p(c_n)$ means that for any $\epsilon > 0$, $\lim_n P(|Z_n| \le \epsilon c_n) = 0$.

Taking $m^2 = [n^{3/5+\epsilon_0}]$, for some $\epsilon_0 > 0$ and using integration by parts, we have that

$$\begin{aligned}
\Delta_3 &= -n\int \big(f(x)-f_m(x)+\frac{1}{2m}h_m(x)\big)'\big(F_n(x)-F(x)\big)\,dx \\
&= O_p(n^{-\epsilon_0})
\end{aligned}$$

since $\big(f(x)-f_m(x)+\frac{1}{2m}h_m(x)\big)' = O(m^{-2})$. From now on we choose $\epsilon_0 = 1/20$, and then $m = n^{13/40}$.

Note that $f_m(x)$ and $h_m(x)$ are both analytic. Following from the result proved in Section 5, replacing f_m by h_m, we obtain that

$$\Delta_2 = \frac{O(\Delta_1)}{m} = o_p(1).$$

It suffices to consider $\Delta_1 = G_n(f_m)$. In the above, $f_m(x)$ and $g_m(y)$ are only defined on the real line. Clearly, the two polynomials can be considered as analytic functions on the complex regions $[-a,a]\times[-\xi,\xi]$ and $[\epsilon,1-\epsilon]\times[-K\xi,K\xi]$, respectively.

Since $g\in C^4[0,1]$, there is a constant M, such that $|g(y)|<M,\quad \forall y\in[\epsilon,1-\epsilon]$. Noting that for $(u,v)\in[\epsilon,1-\epsilon]\times[-K\xi,K\xi]$,

$$\begin{aligned}
|u+iv|+|1-u-iv| &= \sqrt{u^2+v^2}+\sqrt{(1-u)^2+v^2} \\
&\le u\left[1+\frac{v^2}{2u^2}\right]+(1-u)\left[1+\frac{v^2}{2(1-u)^2}\right] \\
&\le 1+\frac{v^2}{\epsilon}
\end{aligned}$$

we have, for $y = Kx + 1/2 = u + iv$,

$$\begin{aligned}
|f_m(x)| &= |\tilde{f}_m(y)| = \left|\sum_{k=1}^{n}\binom{m}{k}y^k(1-y)^{m-k}\tilde{f}\left(\frac{k}{m}\right)\right| \\
&\le M\left(1+\frac{v^2}{\epsilon}\right)^m.
\end{aligned}$$

If we take $|\xi| \le K/\sqrt{m}$, then $|\tilde{f}_m(y)| \le M\left(1+K^2/(m\epsilon)\right)^m \to Me^{K^2/\epsilon}$, as $m\to\infty$. So $\tilde{f}_m(y)$ is bounded when $y \in [\epsilon, 1-\epsilon]\times[-K/\sqrt{m}, K/\sqrt{m}]$. In other words $f_m(x)$ is bounded when $x \in [-a,a]\times[-1/\sqrt{m},1/\sqrt{m}]$. Let γ_m be the contour formed by the boundary of the rectangle with vertices $(\pm a \pm i/\sqrt{m})$. Similarly, we can show that $h_m(x)$, $f_m'(x)$ and $h_m'(x)$ are bounded on γ_m.

3 Simplification by truncation and Preliminary formulae

3.1 Truncation

As proposed in [9], to control the fluctuations around the extreme eigenvalues under condition (1.2), we will truncate the variables at a convenient level without changing their weak limit.

Condition (1.2) implies the existence of a sequence $\eta_n \downarrow 0$ such that

$$\frac{1}{n^2\eta_n^4}\sum_{i,j}\mathbb{E}\left[|x_{ij}|^4\mathbb{I}_{|x_{ij}|\ge\eta_n\sqrt{n}}\right]=o(1).$$

We first truncate the variables as $\hat{x}_{ij}=x_{ij}\mathbb{I}_{|x_{ij}|\le\eta_n\sqrt{n}}$ and normalize them to $\tilde{x}_{ij}=(\hat{x}_{ij}-\mathbb{E}\hat{x}_{ij})/s_{ij}$, where s_{ij} is the standard deviation of $\hat{x}_{ij}$.

Let $\hat{F}_n$ and $\tilde{F}_n$ denote the ESDs of the Wigner matrices $n^{-1/2}(\hat{x}_{ij})$ and $n^{-1/2}(\tilde{x}_{ij})$, and $\hat{G}_n$ and $\tilde{G}_n$ the corresponding empirical process, respectively. First of all, by (1.2),

$$\begin{aligned}
P(G_n\neq\hat{G}_n) &\leqslant P(F_n\neq\hat{F}_n)\leqslant P(\hat{x}_{ij}\neq\tilde{x}_{ij})\\
&\leqslant \sum_{i,j}P(|x_{ij}|\ge\eta_n\sqrt{n})\\
&\leqslant (\eta_n\sqrt{n})^{-4}\sum_{i,j}\mathbb{E}|x_{ij}|^4\mathbb{I}_{|x_{ij}|\ge\eta_n\sqrt{n}}=o(1).
\end{aligned}$$

Second, we compare $\tilde{G}_n(f_m)$ and $\hat{G}_n(f_m)$. Note that

$$\begin{aligned}
\max_{i,j}|1-s_{ij}| &\leqslant \max_{i,j}|1-s_{ij}^2|\\
&= \max_{i,j}\left[\mathbb{E}(|x_{ij}|^2\mathbb{I}_{|x_{ij}|\ge\eta_n\sqrt{n}})+|\mathbb{E}(x_{ij}\mathbb{I}_{|x_{ij}|\ge\eta_n\sqrt{n}})|^2\right]\\
&\le (n^{-1}\eta_n^{-2}+M\eta_n^{-6}n^{-3})\max_{i,j}[\mathbb{E}(|x_{ij}|^4\mathbb{I}_{|x_{ij}|\ge\eta_n\sqrt{n}})]\to 0.
\end{aligned}$$

Therefore, there exist positive constants M_1 and M_2 so that

$$\sum_{i,j}\mathbb{E}(|x_{ij}|^2|1-s_{ij}^{-1}|^2)\le M_1\sum_{i,j}(1-s_{ij}^2)^2\le\frac{M_2}{\eta_n^4n^2}\sum_{i,j}\mathbb{E}(x_{ij}^4\mathbb{I}_{|x_{ij}|\ge\eta_n\sqrt{n}})\to 0.$$

Since $f(x) \in C^4(\mathscr{U})$ implies that $f_m(x)$ are uniformly bounded in $x \in [-a, a]$ and m, we obtain

$$\begin{aligned}
\mathbb{E}|\tilde{G}_n(f_m) - \hat{G}_n(f_m)|^2 &\leq M\mathbb{E}\left(\sum_{j=1}^{n}|\tilde{\lambda}_{nj} - \hat{\lambda}_{nj}|\right)^2 \leq Mn\mathbb{E}\sum_{j=1}^{n}|\tilde{\lambda}_{nj} - \hat{\lambda}_{nj}|^2 \\
&= Mn\mathbb{E}\sum_{i,j}|n^{-1/2}(\tilde{x}_{ij} - \hat{x}_{ij})|^2 \\
&\leq M\left[\sum_{i,j}(\mathbb{E}|x_{ij}|^2)|1 - s_{ij}^{-1}|^2 + \sum_{i,j}|\mathbb{E}(\hat{x}_{ij})|^2 s_{ij}^{-2}\right] \\
&= o(1),
\end{aligned}$$

where $\tilde{\lambda}_{nj}$ and $\hat{\lambda}_{nj}$ are the jth largest eigenvalues of the Wigner matrices $n^{-1/2}(\tilde{x}_{ij})$ and $n^{-1/2}(\hat{x}_{ij})$, respectively.

Therefore, the weak limit of the variables $(G_n(f_m))$ is not affected if we substitute the normalized truncated variables $\tilde{x}_{ij}$ for the original x_{ij}.

From the normalization, the variables $\tilde{x}_{ij}$ all have mean 0 and the same absolute second moments as the original variables. But for the CWE, the condition $\mathbb{E}x_{ij}^2 = 0$ does no longer remain true after these simplifications. Fortunately, we have the estimate $\mathbb{E}\tilde{x}_{ij}^2 = o\left(n^{-1}\eta_n^2\right)$, which is good enough for our purposes.

For brevity, in the sequel we still use x_{ij} to denote the truncated and normalized variables $\tilde{x}_{ij}$.

3.2 Preliminary formulae

Recall that for a distribution function H, its Stieltjes transform $s_H(z)$ is defined by

$$s_H(z) = \int \frac{1}{x - z} dH(x), \qquad z \in \mathbb{C}.$$

The Stieltjes transform $s(z)$ of the semicircle law F is given by $s(z) = -\frac{1}{2}(z - \mathrm{sgn}(\mathfrak{Im}(z))\sqrt{z^2 - 4})$ for z with $\mathfrak{Im}(z) \neq 0$ which satisfies the equation $s(z)^2 + zs(z) + 1 = 0$. Here and throughout the paper, $\sqrt{z}$ of a complex number z denotes the square root of z with positive imaginary part.

Define $D = (n^{-1/2}W_n - zI_n)^{-1}$. Let α_k be the kth column of W_n with x_{kk} removed and $W_n(k)$ the submatrix extracted from W_n by removing its kth row and kth column. Define $D_k = (n^{-1/2}W_n(k) - zI_{n-1})^{-1}$. Let A^* denote the complex conjugate and transpose of matrix or vector A. We shall use the following notations:

$$\begin{aligned}
\beta_k(z) &= -\frac{1}{\sqrt{n}}x_{kk} + z + n^{-1}\alpha_k^* D_k \alpha_k, \\
\delta_n(z) &= -\frac{1}{n}\sum_{j=1}^{n}\frac{\varepsilon_k(z)}{\beta_k(z)}, \\
\varepsilon_k(z) &= \frac{1}{\sqrt{n}}x_{kk} - \frac{1}{n}\alpha_k^* D_k \alpha_k + \mathbb{E}s_n(z), \\
q_k(z) &= \frac{1}{\sqrt{n}}x_{kk} - \frac{1}{n}\left[\alpha_k^* D_k(z)\alpha_k - trD_k(z)\right], \\
\omega_k(z) &= -\frac{1}{\sqrt{n}}x_{kk} + n^{-1}\alpha_k^* D_k \alpha_k - s(z), \\
b_n(z) &= n[\mathbb{E}s_n(z) - s(z)].
\end{aligned}$$

The Stieltjes transform $s_n(z)$ of F_n has the representation

$$\begin{aligned}
s_n(z) &= \frac{1}{n}trD = \frac{1}{n}tr\left(\frac{W_n}{\sqrt{n}} - zI_n\right)^{-1} \\
&= -\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\beta_k(z)} = -\frac{1}{z+\mathbb{E}s_n(z)} + \frac{\delta_n(z)}{(z+\mathbb{E}s_n(z))}. \qquad (3.2)
\end{aligned}$$

Throughout this paper, M may denote different constants on different occassions and ϵ_n a sequence of numbers which converges to 0 as n goes to infinity.

4 The mean function

Let $\lambda_{ext}(n^{-1/2}W_n)$ denote both the smallest and the largest eigenvalue of the matrix $n^{-1/2}W_n$ (defined by the truncated and normalized variables). For $\eta_0 < (a-2)/2$, define

$$A_n = \{|\lambda_{ext}(n^{-1/2}W_n)| \le 2 + \eta_0\}.$$

Then, by Cauchy integral we have

$$\begin{aligned}
\Delta_1 = G_n(f_m) = \frac{1}{2\pi i}\int\oint_{\gamma_m}\frac{f_m(z)}{z-x}n[F_n - F](dx)dz\ \mathbb{I}_{A_n} \\
+\int f_m(x)n[F_n - F](dx)\ \mathbb{I}_{A_n^c}.
\end{aligned}$$

Since $P(A_n^c) = o(n^{-t})$ for any $t > 0$ (see the proof of Theorem 2.12 in [4]),

$$\int f_m(x)n[F_n - F](dx)\ \mathbb{I}_{A_n^c} \to 0$$

in probability. It suffices to consider

$$\Delta \;=\; -\frac{1}{2\pi i}\oint_{\gamma_m} f_m(z)n[s_n(z)-s(z)]\mathbb{I}_{A_n}dz. \tag{4.3}$$

In the remainder of this section, we will handle the asymptotic mean function of Δ. The convergence of random part of Δ will be given in Section 5.

To this end, write $b_n(z)=n[\mathbb{E}s_n(z)-s(z)]$ and $b(z)\triangleq[1+s'(z)]s^3(z)[\sigma^2-1+(\kappa-1)s'(z)+\beta s^2(z)]$. Then we prove

$$\begin{aligned}
\mathbb{E}\Delta_h &= -\frac{1}{2\pi i}\oint_{\gamma_{mh}} f_m(z)b(z)dz \\
&\quad -\frac{1}{2\pi i}\oint_{\gamma_{mh}} f_m(z)[n(\mathbb{E}s_n(z)\mathbb{I}_{A_n}-s(z))-b(z)]dz+o(1) \\
&\triangleq R_1+R_2+o(1) && (4.4)\\
\Delta_v &= -\frac{1}{2\pi i}\oint_{\gamma_{mv}} f_m(z)n[s_n(z)-s(z)]\mathbb{I}_{A_n}dz=o_p(1), && (4.5)
\end{aligned}$$

where and in the sequel γ_{mh} denotes the union of the two horizontal parts of γ_m and γ_{mv} the union of the two vertical parts. The limit of R_1 is given in the following proposition.

Proposition 4.1. *R_1 tends to*

$$\mathbb{E}G(f)=\frac{\kappa-1}{4}[f(2)+f(-2)]-\frac{\kappa-1}{2}\tau_0(f)+(\sigma^2-\kappa)\tau_2(f)+\beta\tau_4(f)$$

for both the CWE and RWE.

Proof. Since $f_m(z)$ are analytic functions, by [6], we have

$$\begin{aligned}
R_1 &= -\frac{1}{2\pi i}\oint_{\gamma_{mh}} f_m(z)b(z)dz \\
&\simeq \frac{\kappa-1}{4}[f_m(2)+f_m(-2)]-\frac{\kappa-1}{2}\tau_0(f_m)+(\sigma^2-\kappa)\tau_2(f_m)+\beta\tau_4(f_m),
\end{aligned}$$

where $a\simeq b$ stands for $a/b\to 1$ as $n\to\infty$,

$$\tau_l(f_m)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f_m(2cos\theta)e^{il\theta}d\theta.$$

As $f_m(t)\to f(t)$ uniformly on $t\in[-a,a]$ as $m\to\infty$, it follows that

$$R_1\longrightarrow\frac{\kappa-1}{4}[f(2)+f(-2)]-\frac{\kappa-1}{2}\tau_0(f)+(\sigma^2-\kappa)\tau_2(f)+\beta\tau_4(f).$$

□

Since $f_m(z)$ is bounded on and in γ_m, in order to prove $R_2 \to 0$ as $m \to \infty$, it is sufficient to show that

$$|b_{nA}(z) - b(z)| = o(1) \text{ uniformly on } \gamma_{mh}, \tag{4.6}$$

where $b_{nA} = n(\mathbb{E}s_n(z)\mathbb{I}_{A_n} - s(z))$.

We first consider the case $z \in \gamma_{mh}$. Since

$$n\mathbb{E}\delta_n(z) = -\mathbb{E}\sum_{k=1}^{n} \frac{\varepsilon_k(z)}{\beta_k(z)} = -\mathbb{E}\sum_{k=1}^{n} \frac{\varepsilon_k(z)}{z + \mathbb{E}s_n(z) - \varepsilon_k(z)},$$

by the identity

$$\frac{1}{u-\epsilon} = \frac{1}{u}\left(1 + \frac{\epsilon}{u} + \ldots + \frac{\epsilon^p}{u^p} + \frac{\epsilon^{p+1}}{u^p(u-\epsilon)}\right),$$

we have

$$\begin{aligned} n\mathbb{E}\delta_n(z) &= -\mathbb{E}\sum_{k=1}^{n} \frac{\varepsilon_k(z)}{z + \mathbb{E}s_n(z)} - \mathbb{E}\sum_{k=1}^{n} \frac{\varepsilon_k^2(z)}{[z + \mathbb{E}s_n(z)]^2} \\ &\quad -\mathbb{E}\sum_{k=1}^{n} \frac{\varepsilon_k^3(z)}{[z + \mathbb{E}s_n(z)]^3} - \mathbb{E}\sum_{k=1}^{n} \frac{\varepsilon_k^4(z)}{[z + \mathbb{E}s_n(z)]^3\beta_k(z)} \\ &\triangleq S_1 + S_2 + S_3 + S_4. \end{aligned}$$

Now suppose $z \in \gamma_{mh}$. In order to analyze S_1, S_2, S_3 and S_4, we present some facts.

Fact 1.

$$\begin{aligned} |\mathbb{E}s_n(z) - s(z)| &= \left|\frac{1}{n}\mathbb{E}trD(z) - s(z)\right| = \left|\int \frac{1}{x-z}(\mathbb{E}F_n - F)(dx)\right| \\ &\leqslant \frac{1}{v}\|\mathbb{E}F_n - F\| = O\left(\frac{1}{\sqrt{n}v}\right) = O(n^{-\frac{27}{80}}). \end{aligned}$$

where we have used Lemma 6.1. This implies

$$|\mathbb{E}s_n(z)| \leqslant |\mathbb{E}s_n(z) - s(z)| + |s(z)| \leqslant O(\frac{1}{\sqrt{n}v}) + M \leqslant M.$$

Fact 2.

$$\begin{aligned} \mathbb{E}|\varepsilon_k(z)|^4 &= \mathbb{E}\left|\frac{1}{\sqrt{n}}x_{kk} - \frac{1}{n}\alpha_k^* D_k(z)\alpha_k + \mathbb{E}s_n(z)\right|^4 \\ &\leqslant M[\frac{1}{n^2}\mathbb{E}|x_{kk}|^4 + \frac{1}{n^4}\mathbb{E}|\alpha_k^* D_k(z)\alpha_k - trD_k(z)|^4 \\ &\quad + \frac{1}{n^4}\mathbb{E}|trD_k(z) - trD(z)|^4 + \frac{1}{n^4}\mathbb{E}|trD(z) - \mathbb{E}trD(z)|^4] \\ &\leq \frac{M}{n^2v^4} = Mn^{-\frac{27}{20}}, \end{aligned}$$

where we have used $|trD(z)-trD_k(z)|\leqslant 1/v$, $\mathbb{E}|trD(z)-EtrD(z)|^{2l}\leqslant cn^{-2l}v^{-4l}(\Delta+v)^l$ $(l\geq 1)$ which follows from the proof of (4.3) in [7], and

$$\begin{aligned}
\mathbb{E}|\alpha_k^* D_k(z)\alpha_k - trD_k(z)|^4 &\leqslant M\left[(\mathbb{E}|x_{12}|^4 trD_k(z)D_k^*(z))^2\right. \\
&\quad \left.+n\eta_n\mathbb{E}|x_{12}|^6 tr(D_k(z)D_k^*(z))^2\right] \\
&\leqslant M\left[\left(\frac{n}{v^2}\right)^2+\frac{n^2\eta_n}{v^4}\right]=\frac{Mn^2}{v^4}
\end{aligned}$$

which is derived from Lemma 6.2.

Fact 3.

From (3.2) we have

$$\mathbb{E}s_n(z)=\frac{-z+\sqrt{z^2-4+4\mathbb{E}\delta_n(z)}}{2}, \tag{4.7}$$

where

$$\begin{aligned}
|\mathbb{E}\delta_n(z)| &= |\frac{1}{n}\mathbb{E}\sum_{k=1}^n\frac{\varepsilon_k(z)}{\beta_k(z)}|\leqslant\frac{1}{nv}\sum_{k=1}^n\mathbb{E}|\varepsilon_k(z)|\leqslant\frac{1}{nv}\sum_{k=1}^n(\mathbb{E}|\varepsilon_k(z)|^4)^{\frac14} \\
&\leqslant \frac{M}{\sqrt{n}v^2}=Mn^{-\frac{7}{40}}.
\end{aligned}$$

Fact 4.

$$\begin{aligned}
\left|\frac{1}{z+\mathbb{E}s_n(z)}\right| &= \left|\frac{\mathbb{E}s_n(z)}{1-\mathbb{E}\delta_n(z)}\right|\leqslant M. \\
\left|\frac{1}{z+\mathbb{E}s_n(z)}-\frac{1}{z+s(z)}\right| &\leqslant M|\mathbb{E}s_n(z)-s(z)| \\
&\leqslant \frac{M}{\sqrt{n}v}=Mn^{-\frac{27}{80}}.
\end{aligned}$$

Now we can estimate S_1,S_2,S_3 and S_4. First, we prove $S_4\to 0$. By $|\beta_k(z)|^{-1}\leqslant v^{-1}$, Facts 2 and 4,

$$|S_4|=\left|\mathbb{E}\sum_{k=1}^n\frac{\varepsilon_k^4(z)}{[z+\mathbb{E}s_n(z)]^3\beta_k(z)}\right|\leqslant\sum_{k=1}^n\frac{M}{v}\mathbb{E}|\varepsilon_k(z)|^4=\frac{M}{nv^5}=Mn^{-\frac{3}{16}}.$$

Similarly, for S_3, we have

$$|S_3|=\left|\mathbb{E}\sum_{k=1}^n\frac{\varepsilon_k^3(z)}{[z+\mathbb{E}s_n(z)]^3}\right|\leqslant M\sum_{k=1}^n|\mathbb{E}\varepsilon_k(z)|^3=Mn\cdot\frac{1}{n^{3/2}v^3}=Mn^{-\frac{1}{80}}.$$

For S_1, we will prove that

$$S_1=s^2(z)[1+s'(z)]|+o(1),\quad \text{uniformly on }\ \gamma_{mh}.$$

By (4.7) and Fact 3, we have

$$\frac{1}{z+\mathbb{E}s_n(z)}=\frac{z-\sqrt{z^2-4(1-\mathbb{E}\delta_n(z))}}{2(1-\mathbb{E}\delta_n(z))}=-s(z)(1+o(1)), \tag{4.8}$$

where the $o(1)$ is uniform for $z \in \gamma_{mh}$.

Thus, we have

$$\begin{aligned} S_1 &= -\sum_{k=1}^n \frac{\mathbb{E}\varepsilon_k(z)}{z+\mathbb{E}s_n(z)} = \frac{1}{n}\sum_{k=1}^n \mathbb{E}\frac{1+\frac{1}{n}\alpha_k^* D_k^2(z)\alpha_k}{\beta_k(z+\mathbb{E}s_n(z))} \\ &= \frac{1}{n}\sum_{k=1}^n \frac{1+\frac{1}{n}\mathbb{E}trD_k^2(z)}{(z+\mathbb{E}s_n(z))^2} + \frac{1}{n}\sum_{k=1}^n \mathbb{E}\frac{(1+\frac{1}{n}\alpha_k^* D_k^2(z)\alpha_k)\varepsilon_k(z)}{\beta_k(z+\mathbb{E}s_n(z))^2} \\ &= S_{11}+S_{12}. \end{aligned}$$

Using the fact

$$\left|\frac{1+\frac{1}{n}\alpha_k^* D_k^2(z)\alpha_k}{\beta_k(z)}\right| \le \frac{1}{v},$$

and Fact 2, for $z \in \gamma_{mh}$, we obtain

$$|S_{12}| \le \frac{M}{nv}\sum_{k=1}^n \mathbb{E}|\varepsilon_k(z)| \le \frac{M}{nv}\cdot n \cdot n^{-1/2}v^{-1} \le Mn^{-7/20}. \tag{4.9}$$

To estimate S_{11}, we use Lemma 6.1 and the fact that $\|(n-1)F_{n-1}^k - nF_n\| \le 1$. For all $z \in \gamma_{mh}$,

$$\left|\frac{1}{n}\mathbb{E}trD_k^2(z) - s'(z)\right| = \left|\int \frac{(\frac{n-1}{n}\mathbb{E}F_{n-1}^k - F)(dx)}{(x-z)^2}\right|$$

$$\leqslant Mv^{-2}\|\frac{n-1}{n}\mathbb{E}F_{n-1}^k - F\| \le Mv^{-2}n^{-1/2} \le Mn^{-7/40}. \tag{4.10}$$

By (4.8) and (4.10), for all $z \in \gamma_{mh}$,

$$|S_{11} - s^2(z)(1+s'(z))| = o(1).$$

Finally we deal with

$$S_2 = -\sum_{k=1}^n \frac{\mathbb{E}\varepsilon_k^2(z)}{[z+\mathbb{E}s_n(z)]^2} = -s^2(z)(1+o(1))\sum_{k=1}^n \mathbb{E}\varepsilon_k^2(z).$$

By the previous estimate for $\mathbb{E}\varepsilon_k(z)$, it follows that

$$\begin{aligned} |\mathbb{E}\varepsilon_k(z)|^2 &= \frac{1}{n^2}\left[\mathbb{E}(trD(z)-trD_k(z))\right]^2 \leqslant \frac{M}{n^2v}. \\ \mathbb{E}\varepsilon_k^2(z) &= \mathbb{E}[\varepsilon_k(z)-\mathbb{E}\varepsilon_k(z)]^2 + O(\frac{1}{n^2v}) \\ &= \mathbb{E}\Big[\frac{1}{\sqrt{n}}x_{kk} - \frac{1}{n}[\alpha_k^* D_k(z)\alpha_k - trD_k(z)] \\ &\qquad -\frac{1}{n}[trD_k(z)-\mathbb{E}trD_k(z)]\Big]^2 + O(\frac{1}{n^2v}) \\ &= \frac{\sigma^2}{n} + \frac{1}{n^2}\mathbb{E}[\alpha_k^* D_k(z)\alpha_k - trD_k(z)]^2 \\ &\qquad -\frac{1}{n^2}\mathbb{E}[trD_k(z)-\mathbb{E}trD_k(z)]^2 + O(\frac{1}{n^2v}). \end{aligned} \tag{4.11}$$

By the proof of (4.3) in [7], we have

$$\frac{1}{n^2}\mathbb{E}|trD_k(z)-\mathbb{E}trD_k(z)|^2 \leqslant \frac{M}{n^2v^4}\left(\frac{1}{\sqrt{n}}+v\right)=\frac{M}{n^2v^3}.$$

Hence we neglect this item. In order to estimate the expectation in (4.11), we introduce the notation $\alpha_k=(x_{1k},x_{2k},...,x_{k-1k},x_{k+1k},...,x_{nk})^* \triangleq (x_i)$ and $D_k(z)\triangleq(d_{ij})$. Note that α_k and $D_k(z)$ are independent.

$$\begin{aligned}
&\mathbb{E}[\alpha_k^*D_k(z)\alpha_k-trD_k(z)]^2\\
=\;&\mathbb{E}\left[\sum_{i\neq j}d_{ij}\bar{x}_ix_j+\sum_i d_{ii}(|x_i|^2-1)\right]^2\\
=\;&\mathbb{E}\left[\sum_{i\neq j}\sum_{s\neq t}d_{ij}d_{st}\bar{x}_ix_j\bar{x}_sx_t\right]+\mathbb{E}\left[\sum_i d_{ii}^2(|x_i|^2-1)^2\right]\\
=\;&\mathbb{E}(\bar{x}_1)^2\mathbb{E}(x_2)^2\mathbb{E}\left[\sum_{i\neq j}d_{ij}^2\right]+\mathbb{E}|x_1|^2\mathbb{E}|x_2|^2\mathbb{E}\left[\sum_{i\neq j}d_{ij}d_{ji}\right]\\
&\quad+\mathbb{E}(|x_1|^2-1)^2\mathbb{E}\left[\sum_i d_{ii}^2\right].
\end{aligned}$$

Here we need to consider the difference between the CWE and RWE.

For the RWE, all the original and truncated variables have the properties: $\mathbb{E}x_{ij}=0$ and $\mathbb{E}|x_{ij}|^2=\mathbb{E}x_{ij}^2=1$. So

$$\begin{aligned}
\mathbb{E}[\alpha_k^*D_k(z)\alpha_k-trD_k(z)]^2 &= 2\mathbb{E}\left[\sum_{i\neq j}d_{ij}^2\right]+\mathbb{E}(|x_1|^2-1)^2\mathbb{E}\left[\sum_i d_{ii}^2\right]\\
&= 2\mathbb{E}\left[\sum_{i,j}d_{ij}^2\right]+[\mathbb{E}(|x_1|^2-1)^2-2]\mathbb{E}\left[\sum_i d_{ii}^2\right].
\end{aligned}$$

For the CWE, the truncated variables have the properties: $\mathbb{E}x_{ij}=0,\mathbb{E}|x_{ij}|^2=1$ and $\mathbb{E}x_{ij}^2=o\left(n^{-1}\eta_n^{-2}\right)$. Note that $|\sum\limits_{i,j}d_{ij}^2|\leqslant tr(D_k(z)D_k^*(z))\leqslant n/v^2$ and $\sum_i|d_{ii}^2|\leqslant n/v^2$. So

$$\begin{aligned}
&\mathbb{E}[\alpha_k^*D_k(z)\alpha_k-trD_k(z)]^2\\
=\;&\mathbb{E}\left[\sum_{ij}d_{ij}d_{ji}\right]+\left[\mathbb{E}(|x_1|^2-1)^2-1\right]\mathbb{E}\left[\sum_i d_{ii}^2\right]+o\left(\frac{1}{nv^2\eta_n^4}\right).
\end{aligned}$$

We introduce the parameters κ with values 1 and 2 for the CWE and RWE, and $\beta=E(|x_1|^2-1)^2-\kappa$, which allow us to have the following unified expression,

$$\mathbb{E}[\alpha_k^*D_k(z)\alpha_k-trD_k(z)]^2 = \kappa\mathbb{E}\left[\sum_{ij}d_{ij}d_{ji}\right]+\beta\mathbb{E}\left[\sum_i d_{ii}^2\right]+o\left(\frac{1}{nv^2\eta_n^4}\right).$$

Hence

$$\mathbb{E}\varepsilon_k^2(z) = \frac{\sigma^2}{n} + \frac{1}{n^2}\left[\kappa\mathbb{E}[trD_k^2(z)] + \beta\mathbb{E}\left[\sum_i d_{ii}^2\right] + o\left(\frac{1}{nv^2\eta_n^4}\right)\right] + \frac{M}{n^2v^3},$$

which combined with

$$\begin{aligned}\mathbb{E}\left|\frac{1}{n}trD_k^2(z) - s'(z)\right|^2 &\leqslant \frac{1}{v^4}(\|F_{n-1}^k - F_n\| + \|F_n - F\|)^2 \\ &\leqslant \frac{M}{v^4}\left(\frac{1}{n} + \frac{1}{n^{\frac{2}{5}-\eta}}\right)^2 \leqslant Mn^{-\frac{3}{20}+2\eta}\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}|d_{ii}^2 - s^2(z)| &\leqslant \mathbb{E}|[D_k(z)]_{ii} + s(z)||[D_k(z)]_{ii} - s(z)| \\ &\leqslant M\left(\frac{1}{v}+1\right)\mathbb{E}|[D_k(z)]_{ii} - s(z)| \leqslant \frac{M}{v}\left[\mathbb{E}|[D_k(z)]_{ii} - s(z)|^2\right]^{1/2} \\ &\leqslant \frac{M}{v}\left(\frac{1}{nv^4}\right)^{\frac{1}{2}} = \frac{M}{\sqrt{n}v^3},\end{aligned}$$

gives

$$|S_2 + s^2(z)[\sigma^2 + \kappa s'(z) + \beta s^2(z)]| \leqslant \frac{M}{\sqrt{n}v^3}.$$

Summarizing the estimates of $S_i, i = 1, ..., 4$, we have obtained that

$$|n\mathbb{E}\delta_n(z) + s^2(z)[\sigma^2 - 1 + (\kappa - 1)s'(z) + \beta s^2(z)]| \leqslant \frac{M}{\sqrt{n}v^3}$$

Note that

$$\begin{aligned}b_n(z) &= n[\mathbb{E}s_n(z) - s(z)] = \frac{2\text{sgn}(\Im\mathfrak{m}(z))n\mathbb{E}\delta_n(z)}{\sqrt{z^2 - 4(1 - \mathbb{E}\delta_n(z))} + \sqrt{z^2-4}} \\ &= \frac{\text{sgn}(\Im\mathfrak{m}(z))n\mathbb{E}\delta_n(z)}{\sqrt{z^2-4}}(1 + o(1)).\end{aligned}$$

and that

$$s'(z) = -\frac{s(z)}{z + 2s(z)} = -\frac{s(z)}{\text{sgn}(\Im\mathfrak{m}(z))\sqrt{z^2-4}}.$$

The second equation is equivalent to

$$\frac{\text{sgn}(\Im(z))}{\sqrt{z^2-4}} = s(z)(1 + s'(z)).$$

From the above two equations, we conclude

$$|b_n(z) - b(z)| = o(1), \text{ uniformly on } \gamma_{mh}.$$

Then, (4.6) is proved by noticing that $|b_n(z) - b_{nA}(z)| \le n v^{-1} P(A_n^c) \le o(n^{-t})$ for any fixed t.
Now, we proceed the proof of (4.5). We shall find a set Q_n such that $P(Q_n^c) = o(n^{-1})$ and

$$\oint_{\gamma_{mv}} f_m(z) n(s_n(z)\mathbb{I}_{Q_n} - s(z))dz = o_p(1). \tag{4.12}$$

By continuity of $s(z)$, there are positive constants M_l and M_u such that for all $z \in \gamma_{mv}$, $M_l \le |z + s(z)| \le M_u$. Define $B_n = \{\min_k |\beta_k(z)|\mathbb{I}_{A_n} > \epsilon\}$, where $\epsilon = M_l/4$. So

$$\begin{aligned}
P(B_n^c) &= P(\min_k |\beta_k(z)|\mathbb{I}_{A_n} \leqslant \epsilon) \leqslant \sum_{k=1}^n P(|\beta_k(z)|\mathbb{I}_{A_n} \leqslant \epsilon) \\
&\leqslant \sum_{k=1}^n P(|\beta_k(z) - z - s(z)|\mathbb{I}_{A_n} \geqslant \epsilon) + nP(A_n^c) \\
&\leqslant \frac{1}{\epsilon^4}\sum_{k=1}^n \mathbb{E}\Big(|\beta_k(z) - z - s(z)|^4 \mathbb{I}_{A_n}\Big) + nP(A_n^c) \\
&\leqslant M\Big[\frac{1}{n^2}\mathbb{E}|x_{kk}|^4 + \frac{1}{n^4}\mathbb{E}\Big(|\alpha_k^* D_k(z)\alpha_k - trD_k(z)|^4\mathbb{I}_{A_n}\Big) \\
&\qquad + \mathbb{E}\Big(|\frac{1}{n}trD_k(z) - s(z)|^4\mathbb{I}_{A_n}\Big)\Big] + nP(A_n^c).
\end{aligned}$$

Let $A_{nk} = \{|\lambda_{ext}(n^{-1/2}W_{nk})| \le 2 + \eta_0\}$. It is trivially obtained that $A_n \subseteq A_{nk}$ (see [11]). Noting the independence of $\mathbb{I}_{A_{nk}}$ and α_k, we have

$$\begin{aligned}
& n^{-4}\mathbb{E}\Big(|\alpha_k^* D_k(z)\alpha_k - trD_k(z)|^4\mathbb{I}_{A_n}\Big) \\
\le\ & n^{-4}\mathbb{E}\Big([\mathbb{E}^k|\alpha_k^* D_k(z)\alpha_k - trD_k(z)|^4]\mathbb{I}_{A_{nk}}\Big) \\
\le\ & Mn^{-4}\mathbb{E}\Big([\big(\mathbb{E}|x_{12}|^4 trD_k(z)D_k^*(z)\big)^2 + \mathbb{E}|x_{12}|^8 tr\big(D_k(z)D_k^*(z)\big)^2]\mathbb{I}_{A_{nk}}\Big) \\
\le\ & Mn^{-2},
\end{aligned}$$

where $\mathbb{E}^k$ denotes the expectation taken only about α_k. Therefore

$$P(B_n^c) \le M\big(n^{-2} + n^{-2} + (n^{-2/5+\eta})^4\big) + nP(A_n^c) \le Mn^{-8/5+4\eta}.$$

This gives $P(B_n^c) = o(n^{-1})$ as $n \to \infty$. Define $Q_n \triangleq A_n \cap B_n$, we have $P(Q_n) \to 1$, as $n \to \infty$.
Similar to (4.7), we have for $z \in \gamma_{mv}$,

$$\mathbb{E}s_n(z)\mathbb{I}_{Q_n} = -\frac{P(Q_n)}{z + \mathbb{E}s_n(z)\mathbb{I}_{Q_n}} + \frac{\delta_{nQ}(z)}{z + \mathbb{E}s_n(z)\mathbb{I}_{Q_n}}, \tag{4.13}$$

where

$$\begin{aligned}
\delta_{nQ}(z) &= \frac{1}{n}\sum_{k=1}^n \mathbb{E}\frac{\varepsilon_{kQ}(z)\mathbb{I}_{Q_n}}{\beta_k(z)} \\
\varepsilon_{kQ}(z) &= \frac{1}{\sqrt{n}}x_{kk} - \frac{1}{n}\alpha_k^* D_k(z)\alpha_k + \mathbb{E}s_n(z)\mathbb{I}_{Q_n}.
\end{aligned}$$

Similar to (4.7), we have

$$\mathbb{E}s_n(z)\mathbb{I}_{Q_n} = -\frac{1}{2}\Big(z - \mathrm{sgn}(\Im\mathfrak{m}(z))\sqrt{z^2 - 4(P(Q_n) - \delta_{nQ}(z))}\Big).$$

Therefore,

$$\begin{aligned} b_{nQ}(z) &= n(\mathbb{E}s_n(z)\mathbb{I}_{Q_n} - s(z)) \\ &= \frac{2n[P(Q_n^c) + \delta_{nQ}(z)]}{\mathrm{sgn}(\Im\mathfrak{m}(z))[\sqrt{z^2 - 4(P(Q_n) - \delta_{nQ}(z))} + \sqrt{z^2-4}]}. \end{aligned}$$

We shall prove that $b_{nQ}(z)$ is uniformly bounded for all $z \in \gamma_{mv}$. Noticing that $|z^2 - 4| > (a-2)^2$ and the fact $P(Q_n^c) = o(1)$, we only need to show that

$$n\delta_{nQ}(z) \text{ is uniformly bounded for } z \in \gamma_{mv}. \tag{4.14}$$

Rewrite

$$n\delta_{nQ}(z) = \sum_{k=1}^{n} \frac{\mathbb{E}\varepsilon_{kQ}\mathbb{I}_{Q_n}}{z + \mathbb{E}s_n(z)\mathbb{I}_{Q_n}} + \sum_{k=1}^{n} \frac{\mathbb{E}\varepsilon_{kQ}^2\mathbb{I}_{Q_n}}{\beta_k(z)[z + \mathbb{E}s_n(z)\mathbb{I}_{Q_n}]}.$$

At first we have

$$\begin{aligned} \mathbb{E}\varepsilon_{kQ}\mathbb{I}_{A_{nk}} &= \frac{1}{n}\mathbb{E}\Big[trD(z)\mathbb{I}_{Q_n}P(A_{nk}) - trD_k(z)\mathbb{I}_{A_{nk}}\Big] \\ &= \frac{1}{n}\mathbb{E}\left(\frac{\mathbb{I}_{Q_n}}{\beta_k}P(A_{nk}) - trD_k(z)(\mathbb{I}_{Q_n^cA_{nk}} + \mathbb{I}_{Q_n}P(A_{nk}^c))\right) = O(1/n). \end{aligned}$$

Here, the result follows from facts that $1/\beta_k$ is uniformly bounded when Q_n happens, $\frac{1}{n}D_k(z)$ is bounded when A_{nk} happens and $P(Q_n^c) = o(1/n)$. From this and the facts that $\mathbb{E}s_n(z)\mathbb{I}_{Q_n} \to s(z)$ uniformly on γ_{mv} and

$$\begin{aligned} \mathbb{E}|\varepsilon_{kQ}|\mathbb{I}_{Q_n^cA_{nk}} &\le \frac{1}{n}\mathbb{E}\Big[trD(z)\mathbb{I}_{Q_n}P(Q_n^cA_{nk}) - trD_k(z)\mathbb{I}_{Q_n^cA_{nk}}\Big] \\ &= O(1/n), \end{aligned}$$

we conclude that the first term in the expansion of $n\delta_{nQ}(z)$ is bounded.

By similar argument, one can prove that the second term of the expansion of $n\delta_{nQ}(z)$ is uniformly bounded on γ_{mv}. Therefore, we have

Proposition 4.2. $\oint_{\gamma_{mv}} f_m(z)n(\mathbb{E}s_n(z)\mathbb{I}_{Q_n} - s(z))dz \to 0$ *in probability as* $n \to \infty$.

□

Therefore, to complete the proof of (4.5), we only need to show that

Proposition 4.3. $\oint_{\gamma_{mv}} f_m(z)n(s_n(z)\mathbb{I}_{Q_n} - \mathbb{E}s_n(z)\mathbb{I}_{Q_n})dz \to 0$ *in probability as* $n \to \infty$.

We postpone the proof of Proposition 4.3 to the next section.

5 Convergence of $\Delta - \mathbb{E}\Delta$

Let $\mathscr{F}_k = (x_{ij}, k+1 \le i, j \le n)$ for $0 \le k \le n$ and $\mathbb{E}_k(\cdot) = \mathbb{E}(\cdot|\mathscr{F}_k)$. Based on the decreasing filtration $(\mathscr{F}_k)$, we have the following well known martingale decomposition

$$\begin{aligned} n[s_n(z) - \mathbb{E}s_n(z)] &= trD(z) - \mathbb{E}trD(z) \\ &= \sum_{k=1}^{n}(\mathbb{E}_{k-1} - \mathbb{E}_k)(trD(z) - trD_k(z)) \\ &= \sum_{k=1}^{n}(\mathbb{E}_{k-1} - \mathbb{E}_k)\frac{1 + n^{-1}\alpha_k^* D_k^2(z)\alpha_k}{n^{-1/2}x_{kk} - z - n^{-1}\alpha_k^* D_k(z)\alpha_k} \\ &= -\sum_{k=1}^{n}(\mathbb{E}_{k-1} - \mathbb{E}_k)\frac{d}{dz}\log\left[\frac{1}{\sqrt{n}}x_{kk} - z - \frac{1}{n}\alpha_k^* D_k(z)\alpha_k\right]. \end{aligned}$$

This decomposition gives

$$\begin{aligned} & \Delta - \mathbb{E}\Delta \\ = \ & \frac{1}{2\pi i}\sum_{k=1}^{n}(\mathbb{E}_{k-1} - \mathbb{E}_k)\oint_{\gamma_m} f_m(z)\frac{d}{dz}\log\left[\frac{1}{\sqrt{n}}x_{kk} - z - \frac{1}{n}\alpha_k^* D_k(z)\alpha_k\right]dz \\ = \ & -\frac{1}{2\pi i}\sum_{k=1}^{n}(\mathbb{E}_{k-1} - \mathbb{E}_k)\oint_{\gamma_m} f_m'(z)\log\left[\frac{1}{\sqrt{n}}x_{kk} - z - \frac{1}{n}\alpha_k^* D_k(z)\alpha_k\right]dz \\ = \ & -\frac{1}{2\pi i}\sum_{k=1}^{n}(\mathbb{E}_{k-1} - \mathbb{E}_k)\oint_{\gamma_m} f_m'(z)\log\left[1 + \frac{q_k(z)}{-z - n^{-1}trD_k(z)}\right]dz \\ = \ & R_{31} + R_{32} + R_{33} \end{aligned}$$

where $q_k(z) = \frac{1}{\sqrt{n}}x_{kk} - \frac{1}{n}\left[\alpha_k^* D_k(z)\alpha_k - trD_k(z)\right]$ and

$$\begin{aligned} R_{31} &= -\frac{1}{2\pi i}\sum_{k=1}^{n}(\mathbb{E}_{k-1} - \mathbb{E}_k)\oint_{\gamma_{mh}} f_m'(z)\log\left[1 + \frac{q_k(z)}{-z - n^{-1}trD_k(z)}\right]dz \\ R_{32} &= -\frac{1}{2\pi i}\sum_{k=1}^{n}\mathbb{I}_{A_{nk}}(\mathbb{E}_{k-1} - \mathbb{E}_k)\oint_{\gamma_{mv}} f_m'(z)\log\left[1 + \frac{q_k(z)}{-z - n^{-1}trD_k(z)}\right]dz \\ R_{33} &= -\frac{1}{2\pi i}\sum_{k=1}^{n}\mathbb{I}_{A_{nk}^c}(\mathbb{E}_{k-1} - \mathbb{E}_k)\oint_{\gamma_{mv}} f_m'(z)\log\left[1 + \frac{q_k(z)}{-z - n^{-1}trD_k(z)}\right]dz. \end{aligned}$$

At first, we note that

$$P(R_{33} \ne 0) \le P(\bigcup A_{nk}^c) \le P(A_n^c) \to 0.$$

Next, we note that R_{32} is a sum of martingale differences. Thus, we have

$$\begin{aligned} \mathbb{E}|R_{32}|^2 &\le \frac{\|f_m'\|^2 n^{-\frac{13}{40}}}{\pi^2}\sum_{k=1}^{n}\sup_{z\in\gamma_{mv}}\mathbb{E}\mathbb{I}_{A_{nk}}\left|\frac{q_k(z)}{z + n^{-1}trD_k(z)}\right|^2 \\ &= \frac{\|f_m'\|^2}{\pi^2 n^{\frac{53}{40}}}\sum_{k=1}^{n}\sup_{z\in\gamma_{mv}}\mathbb{E}\mathbb{I}_{A_{nk}}\left|\frac{\sigma^2 + \frac{2}{n}trD_k(z)D_k(\bar{z})}{z + n^{-1}trD_k(z)}\right|^2. \end{aligned} \tag{5.15}$$

When $z \in \gamma_{mv}$ and A_{nk} happens, we have $|n^{-1}trD_k(z)D_k(\bar{z})| \le \eta_0^{-2}$. Also, $z+n^{-1}trD_k(z) \to z+s(z)$ uniformly. Further, $|z+s(z)|$ has a positive lower bound on γ_{mv}. These facts, together with (5.15), imply that

$$R_{32} \to 0, \text{ in probability.}$$

The proof of Proposition 4.3 is the same as those for $R_{32} \to 0$ and $R_{33} \to 0$. We omit the details.

Note that when $z \in \gamma_{mh}$

$$\mathbb{E}q_k(z) = 0, \;\; \mathbb{E}\frac{q_k(z)}{z+n^{-1}trD_k(z)} = 0.$$

Taylor expansion of the log function implies

$$\begin{aligned} R_{31} &= \frac{1}{2\pi i}\sum_{k=1}^{n}\oint_{\gamma_{mh}} (\mathbb{E}_{k-1}-\mathbb{E}_k)\Big[\frac{q_k(z)}{z+n^{-1}trD_k(z)} \\ &\qquad\qquad +O\left(\frac{q_k(z)}{-z-n^{-1}trD_k(z)}\right)^2\Big]f_m'(z)dz \\ &= \frac{1}{2\pi i}\sum_{k=1}^{n}\oint_{\gamma_{mh}} \mathbb{E}_{k-1}\frac{q_k(z)}{z+n^{-1}trD_k(z)}f_m'(z)dz + o_p(1) \\ &\triangleq \frac{1}{2\pi i}\sum_{k=1}^{n} Y_{nk} + o_p(1), \end{aligned}$$

where $o_p(1)$ follows from the following Condition 5.1. Clearly $Y_{nk} \in \mathscr{F}_{k-1}$ and $\mathbb{E}_k Y_{nk} = 0$. Hence $\{Y_{nk}, k = 1,2,...,n\}$ is a martingale difference sequence and $\sum_{k=1}^{n} Y_{nk}$ is a sum of a martingale difference sequence. To save notation, we still use $G_n(f_m)$ to denote $\frac{1}{2\pi i}\sum_{k=1}^{n} Y_{nk}$ from now on. In order to apply CLT to $G_n(f_m)$, we need to check the following two conditions:

Condition 5.1 **Conditional Lyapunov condition.** For some $p > 2$,

$$\sum_{k=1}^{n}\mathbb{E}_k|Y_{nk}|^p \longrightarrow 0 \text{ in Probability.}$$

Condition 5.2. **Conditional covariance.** Note that Y_{nk} are complex, we need to show that $U^2 = \sum_{k=1}^{n}\mathbb{E}_k Y_{nk}^2$ converges to a constant limit to guarantee the convergence to complex normal. For simplicity, we may consider two functions $f, g \in C^4(\mathscr{U})$ and show that $Cov_k[G_n(f_m), G_n(g_m)]$ converges in probability, where f_m and g_m denote their Bernstein polynomial approximations. That is,

$$Cov_k[G_n(f_m), G_n(g_m)] \to C(f,g) \quad \text{in probability.}$$

The proof of condition 5.1 with $p = 3$.

$$\sum_{k=1}^{n}\mathbb{E}_k|Y_{nk}|^3 = \sum_{k=1}^{n}\mathbb{E}_k\left|\oint_{\gamma_{mh}} \mathbb{E}_{k-1}\frac{q_k(z)}{z+n^{-1}trD_k(z)}f_m'(z)dz\right|^3.$$

Since $f_m'(z)$ is bounded on γ_{mh}, it suffices to prove

$$\sum_{k=1}^{n}\mathbb{E}\left|\frac{q_k(z)}{z+n^{-1}trD_k(z)}\right|^3 \le \epsilon_n \quad \text{uniformly on } \ \gamma_{mh}.$$

For all $z\in\gamma_{mh}$, by Lemma 6.2, we have

$$\begin{aligned}
&\mathbb{E}\left|\frac{q_k(z)}{z+n^{-1}trD_k(z)}\right|^3 \\
\le\ & M\mathbb{E}\left[\frac{n^{-3/2}\mathbb{E}|x_{kk}|^3+\frac{1}{n^3}\{\left(trD_k(z)D_k(\bar z)\right)^{3/2}+tr([D_k(z)D_k(\bar z)]^{3/2})\}}{|z+n^{-1}trD_k(z)|^3}\right] \\
\le\ & \frac{M}{n^{3/2}v^3},
\end{aligned} \tag{5.16}$$

where we have used the facts that

$$\begin{aligned}
\left|\frac{\frac{1}{n}tr(D_k(z)D_k(\bar z))}{z+\frac{1}{n}trD_k(z)}\right| &< \frac{1}{v}, \\
\left(trD_k(z)D_k(\bar z)\right)^{3/2} &\le \frac{1}{v}\left(trD_k(z)D_k(\bar z)\right).
\end{aligned}$$

Then the above gives

$$\sum_{k=1}^{n}\mathbb{E}\left|\frac{q_k(z)}{z+\frac{1}{n}trD_k(z)}\right|^3 \le \frac{M}{\sqrt{n}v^3}=Mn^{-1/80} \ \text{ uniformly on } \ \gamma_{mh}.$$

<u>The proof of Condition 5.2.</u>

The conditional covariance is

$$\begin{aligned}
Cov_k[G_n(f_m),G_n(g_m)] &= -\frac{1}{4\pi^2}\oint_{\gamma_{mh}\times\gamma_{mh}}\oint\sum_{k=1}^{n}\mathbb{E}_k\left[\mathbb{E}_{k-1}\frac{q_k(z)}{z_1+n^{-1}trD_k(z_1)}\right] \\
&\quad\times\left[\mathbb{E}_{k-1}\frac{q_k(z)}{z_2+n^{-1}trD_k(z_2)}\right]f_m'(z_1)g_m'(z_2)dz_1dz_2.
\end{aligned}$$

For $z\in\gamma_{mh}$, since

$$\begin{aligned}
\mathbb{E}|[D(z)]_{kk}-s(z)|^2 &= \mathbb{E}\left|\frac{1}{\frac{1}{\sqrt{n}}x_{kk}-z-\frac{1}{n}\alpha_k^*D_k(z)\alpha_k}-\frac{1}{-z-s(z)}\right|^2 \\
=\ & \mathbb{E}\left|\frac{\frac{1}{\sqrt{n}}x_{kk}-\frac{1}{n}\alpha_k^*D_k(z)\alpha_k+s(z)}{[z+s(z)][\frac{1}{\sqrt{n}}x_{kk}-z-\frac{1}{n}\alpha_k^*D_k(z)\alpha_k]}\right|^2 \\
\le\ & \frac{M}{v^2}\left[\mathbb{E}|\varepsilon_k(z)|^2+|\mathbb{E}s_n(z)-s(z)|^2\right] \\
\le\ & \frac{M}{v^2}\left[\left(\frac{1}{n^2v^4}\right)^{\frac{1}{2}}+\left(\frac{1}{\sqrt{n}v}\right)^2\right]=\frac{M}{nv^4}=Mn^{-\frac{7}{20}}.
\end{aligned}$$

we obtain

$$\begin{aligned}\mathbb{E}\left|\frac{1}{z+\frac{1}{n}trD_k(z)}-\frac{1}{z+s(z)}\right|^2 &\leq \frac{1}{v^2}\mathbb{E}\left|\frac{1}{n}trD_k(z)-s(z)\right|^2\\ &= \frac{1}{v^2}\mathbb{E}\left|\frac{1}{n}\sum_i[D_k(z)]_{ii}-s(z)\right|^2\\ &\leq \frac{M}{nv^6}=Mn^{-1/40}.\end{aligned}$$

Therefore

$$\begin{aligned}&\mathrm{Cov}_k[G_n(f_m),G_n(g_m)]\\ =\ & -\frac{1}{4\pi^2}\oint\oint_{\gamma_{mh}\times\gamma_{mh}} s(z_1)s(z_2)\sum_{k=1}^{n}\mathbb{E}_k[\mathbb{E}_{k-1}q_k(z_1)\mathbb{E}_{k-1}q_k(z_2)]f_m'(z_1)g_m'(z_2)dz_1dz_2\\ &\qquad +o_p(1)\\ =\ & -\frac{1}{4\pi^2}\oint\oint_{\gamma_{mh}\times\gamma_{mh}} s(z_1)s(z_2)\Gamma(z_1,z_2)f_m'(z_1)g_m'(z_2)dz_1dz_2\\ & -\frac{1}{4\pi^2}\oint\oint_{\gamma_{mh}\times\gamma_{mh}} s(z_1)s(z_2)[\Gamma_n(z_1,z_2)-\Gamma(z_1,z_2)]f_m'(z_1)g_m'(z_2)dz_1dz_2+o_p(1)\\ =\ & Q_1+Q_2+o_p(1).\end{aligned}$$

where

$$\begin{aligned}&\Gamma(z_1,z_2)=\sigma^2-\kappa+\frac{1}{2}\beta s(z_1)s(z_2)-\frac{\kappa}{s(z_1)s(z_2)}\log[1-s(z_1)s(z_2)],\\ &\Gamma_n(z_1,z_2)=\sum_{k=1}^{n}\mathbb{E}_k[\mathbb{E}_{k-1}q_k(z_1)\mathbb{E}_{k-1}q_k(z_2)],\\ &A(z_1,z_2)=s(z_1)s(z_2)\Gamma(z_1,z_2)f_m'(z_1)g_m'(z_2)\end{aligned}$$

For Q_1, we have

$$\begin{aligned}Q_1 &= -\frac{1}{4\pi^2}\oint\oint_{\gamma_{mh}\times\gamma_{mh}} A(z_1,z_2)dz_1dz_2\\ &= -\frac{1}{4\pi^2}\int_{-2}^{2}\int_{-2}^{2}[A(t_1^-,t_2^-)-A(t_1^-,t_2^+)-A(t_1^+,t_2^-)+A(t_1^+,t_2^+)]dt_1dt_2\\ &= \frac{1}{4\pi^2}\int_{-2}^{2}\int_{-2}^{2}f_m'(t)g_m'(s)V(t,s)dtds\\ &\to \frac{1}{4\pi^2}\int_{-2}^{2}\int_{-2}^{2}f'(t)g'(s)V(t,s)dtds\end{aligned}$$

since $f_m'(t) \to f'(t)$, $g_m'(t) \to g'(t)$ uniformly on $[-2,2]$.

For Q_2, since $s(z_1)s(z_2)f_m'(z_1)g_m'(z_2)$ is bounded on $\gamma_{mh} \times \gamma_{mh}$, in order to prove $Q_2 \to 0$ in probability, it is sufficient to prove that $\Gamma_n(z_1,z_2)$ converges in probability to $\Gamma(z_1,z_2)$ uniformly on $\gamma_{mh}\times\gamma_{mh}$. Decompose $\Gamma_n(z_1,z_2)$ as

$$
\begin{aligned}
\Gamma_n(z_1,z_2) &= \sum_{k=1}^{n} \mathbb{E}_k[\mathbb{E}_{k-1}q_k(z_1)\mathbb{E}_{k-1}q_k(z_2)] \\
&= \sum_{k=1}^{n} \mathbb{E}_k\Big[\frac{x_{kk}^2}{n} + \frac{1}{n^2}\mathbb{E}_{k-1}[\alpha_k^* D_k(z_1)\alpha_k - trD_k(z_1)] \\
&\qquad \times\mathbb{E}_{k-1}[\alpha_k^* D_k(z_2)\alpha_k - trD_k(z_2)]\Big] \\
&= \sigma^2 + \frac{\kappa}{n^2}\sum_{k=1}^{n}\sum_{i,j>k}[\mathbb{E}_{k-1}D_k(z_1)]_{ij}[\mathbb{E}_{k-1}D_k(z_2)]_{ji} \\
&\quad + \frac{\beta}{n^2}\sum_{k=1}^{n}\mathbb{E}_k\sum_{i>k}[\mathbb{E}_{k-1}D_k(z_1)]_{ii}[\mathbb{E}_{k-1}D_k(z_2)]_{ii} \\
&\triangleq \sigma^2 + \mathscr{S}_1 + \mathscr{S}_2.
\end{aligned}
$$

Since $\mathbb{E}|[D_k(z)]_{ii} - s(z)|^2 \le M/(nv^4)$ uniformly on γ_{mh},

$$
\mathbb{E}|\mathbb{E}_k[\mathbb{E}_{k-1}D_k(z_1)]_{ii}[\mathbb{E}_{k-1}D_k(z_2)]_{ii} - s(z_1)s(z_2)| \le \frac{M}{nv^4}
$$

uniformly on $\gamma_{mh} \times \gamma_{mh}$. Hence

$$
\mathbb{E}|\mathscr{S}_2 - \frac{\beta}{2}s(z_1)s(z_2)| \le \frac{M}{nv^4}
$$

In the following, we consider the limit of $\mathscr{S}_1$. As proposed in [6], we use the following decomposition. Let

$$
e_j = (0,\dots,1,\dots,0)_{n-1}', \quad j = 1,2,\dots,k-1,k+1,\dots,n
$$

whose jth (or $(j-1)$th) element is 1, the rest being 0, if $j<k$ (or $j>k$). Then

$$
\begin{aligned}
D_k^{-1}(z) &= \frac{1}{\sqrt{n}}W_n(k) - zI_{n-1} = \sum_{i,j\neq k}\frac{1}{\sqrt{n}}x_{ij}e_ie_j' - zi_{n-1} \\
zD_k(z) + I &= \sum_{i,j\neq k}\frac{1}{\sqrt{n}}x_{ij}e_ie_j'D_k(z)
\end{aligned}
$$

We introduce the matrix

$$
D_{kij} = \left(\frac{1}{\sqrt{n}}[W_n(k) - \delta_{ij}(x_{ij}e_ie_j' + x_{ji}e_je_i')] - zI_{n-1}\right)^{-1}
$$

where $\delta_{ij} = 1$ for $i \neq j$ and $\delta_{ii} = 1/2$, such that D_{kij} is a perturbation of D_k and independent of x_{ij}.

From the formula $A^{-1}-B^{-1}=-B^{-1}(A-B)A^{-1}$, we have

$$D_k - D_{kij} = -D_{kij}\frac{1}{\sqrt{n}}\delta_{ij}(x_{ij}e_ie_j' + x_{ji}e_je_i')D_k.$$

From the above, we get

$$\begin{aligned}
zD_k(z) &= -I + \sum_{i,j\neq k} n^{-1/2}x_{ij}e_ie_j'D_{kij}(z) \\
&\quad - \sum_{i,j\neq k} n^{-1/2}x_{ij}e_ie_j'D_{kij}(z)n^{-1/2}\delta_{ij}(x_{ij}e_ie_j' + x_{ji}e_je_i')D_k(z) \\
&= -I + \sum_{i,j\neq k} n^{-1/2}x_{ij}e_ie_j'D_{kij}(z) - s(z)\frac{n-3/2}{n}\sum_{i\neq k} e_ie_i'D_k(z) \\
&\quad - \sum_{i,j\neq k}\delta_{ij}\left[n^{-1}|x_{ij}|^2([D_{kij}(z)]_{jj} - s(z)) + n^{-1}(|x_{ij}|^2-1)s(z)\right] \\
&\qquad \times e_ie_j'D_k(z) \\
&\quad - \sum_{i,j\neq k} n^{-1}\delta_{ij}x_{ij}^2[D_{kij}]_{ji}e_ie_j'D_k(z).
\end{aligned}$$

Therefore

$$\begin{aligned}
& z_1\sum_{i,j>k}[\mathbb{E}_{k-1}D_k(z_1)]_{ij}[\mathbb{E}_{k-1}D_k(z_2)]_{ji} \\
= & -\sum_{i>k}[\mathbb{E}_{k-1}D_k(z_2)]_{ii} \\
& + n^{-1/2}\sum_{i,j,l>k} x_{il}[\mathbb{E}_{k-1}D_{kil}(z_1)]_{lj}[\mathbb{E}_{k-1}D_k(z_2)]_{ji} \\
& - s(z_1)\frac{n-3/2}{n}\sum_{i,j>k}[\mathbb{E}_{k-1}D_k(z_1)]_{ij}[\mathbb{E}_{k-1}D_k(z_2)]_{ji} \\
& - \sum_{\substack{i,j>k\\ l\neq k}}\delta_{il}\mathbb{E}_{k-1}\left[\left(\frac{|x_{il}|^2-1}{n}s(z_1) + \frac{|x_{il}|^2}{n}[D_{kil}(z_1)_{ll} - s(z_1)]\right)[D_k(z_1)]_{ij}\right] \\
& \qquad \times[\mathbb{E}_{k-1}D_k(z_2)]_{ji} \\
& - \frac{1}{n}\sum_{\substack{i,j>k\\ l\neq k}}\delta_{il}\mathbb{E}_{k-1}x_{il}^2[D_{kil}(z_1)]_{li}[D_k(z_1)]_{lj}[\mathbb{E}_{k-1}D_k(z_2)]_{ji} \\
= & \; T_1 + T_2 + T_* + T_3 + T_4.
\end{aligned}$$

First note that the term T_* is proportional to the term of the left hand side. We now evaluate the contributions of the remaining four terms to the sum $\mathscr{S}_1$.

The term $T_1 \to s(z_2)$ in L_2 since

$$\mathbb{E}\left|n^{-2}\sum_k\sum_{i>k}([E_{k-1}D_k(z_2)]_{ii} - s(z_2))\right|^2 \le \frac{M}{nv^4}$$

by

$$\mathbb{E}|[D_k(z)]_{ii} - s(z)|^2 \leq \frac{M}{nv^4}.$$

The terms T_3 and T_4 are negligible. The calculations are lengthy but simple. We omit all the details. T_2 can not be ignored. We shall simplify it step by step. Split T_2 as

$$\begin{aligned}
T_2 &= \sum_{i,j,l>k} \mathbb{E}_{k-1} \frac{x_{lj}}{\sqrt{n}} [D_{kil}(z_1)]_{lj} [\mathbb{E}_{k-1} D_k(z_2)]_{ji} \\
&= \sum_{i,j,l>k} \mathbb{E}_{k-1} \frac{x_{il}}{\sqrt{n}} [D_{kil}(z_1)]_{lj} [\mathbb{E}_{k-1}(D_k - D_{kil})(z_2)]_{ji} \\
&\quad + \sum_{i,j,l>k} \mathbb{E}_{k-1} \frac{x_{il}}{\sqrt{n}} [D_{kil}(z_1)]_{lj} [\mathbb{E}_{k-1} D_{kil}(z_2)]_{ji} \\
&= T_{2a} + T_{2b}.
\end{aligned}$$

Again, T_{2b} can be shown to be ignored. As for the remaining term T_{2a}, we have

$$\begin{aligned}
& \mathbb{E} T_{2a} \\
= \; & n^{-2} \sum_{i,j,l>k} x_{il} [\mathbb{E}_{k-1} D_{kil}(z_1)]_{lj} [\mathbb{E}_{k-1}(D_k - D_{kil})(z_2)]_{ji} \\
= \; & -n^{-1} \sum_{i,j,l>k} [\mathbb{E}_{k-1} D_{kil}(z_1)]_{lj} [\mathbb{E}_{k-1} D_{kil}(z_2) \delta_{il} (x_{il}^2 e_i e_l' + |x_{il}|^2 e_l e_i') D_k(z_2)]_{ji} \\
= \; & J_1 + J_2,
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{E}|J_1| &= n^{-1} \sum_{i,j,l>k} \mathbb{E}|x_{il}^2| [\mathbb{E}_{k-1} D_{kil}(z_1)]_{lj} [D_{kil}(z_2)]_{ji} [D_k(z_2)]_{li}| \\
&\leq n^{-1} \Big(\sum_{i,j_1,,j_2,l>k} \mathbb{E}|x_{il}|^4 |[\mathbb{E}_{k-1} D_{kil}(z_1)]_{lj_1}|^2 |[D_{kil}(z_2)]_{j_2 i}|^2 \\
&\qquad \times \sum_{i,l>k} \mathbb{E}|(D_k(z_2))_{li}|^2 \Big)^{1/2} \\
&\leq M n^{1/2} v^{-3}.
\end{aligned}$$

Hence, the contribution of this term is negligible.
Finally, we have

$$\begin{aligned}
J_2 &= -\sum_{i,j,l>k} \mathbb{E}_{k-1} \frac{|x_{il}|^2}{n} [D_{kij}(z_1)]_{lj} [\mathbb{E}_{k-1} D_{kij}(z_2)]_{jl} [D_k(z_2)]_{ii} \\
&\simeq -s(z_2) \sum_{i,j,l>k} \mathbb{E}_{k-1} \frac{1}{n} [D_{kil}(z_1)]_{lj} [\mathbb{E}_{k-1} D_{kil} D_{kil}(z_2)]_{jl} \\
&= -\frac{n-k}{n} s(z_2) \sum_{j,l>k} [\mathbb{E}_{k-1} D_k(z_1)]_{lj} [\mathbb{E}_{k-1} D_k(z_2)]_{jl} + O(n^{1/2} v^{-3}),
\end{aligned}$$

where the last approximation follows from

$$\left|\sum_{i,j,l>k}[D_{kil}(z_1)]_{lj}[E_{k-1}D_{kil}(z_2)]_{jl}-(n-k)\sum_{j,l>k}[E_{k-1}D_k(z_1)]_{lj}[E_{k-1}D_k(z_2)]_{jl}\right|$$
$$\leq Mn^{3/2}v^{-3}.$$

Let us define

$$X_k=\sum_{i,j>k}[\mathbb{E}_{k-1}D_k(z_1)]_{ij}[\mathbb{E}_{k-1}D_k(z_2)]_{ji}$$

Summarizing the estimations of all the terms T_i, we have proved that

$$z_1X_k=-s(z_1)X_k-(n-k)s(z_2)X_k\frac{n-k}{n}+r_k,$$

where the residual term r_k is of order $O(\sqrt{n}v^{-3})$ uniformly in $k=1,\dots,n$ and $z_1,z_2\in\gamma_m$. By $z_1+s(z_1)=-1/s(z_1)$, the above identity is equivalent to

$$X_k=(n-k)s(z_1)s(z_2)+\frac{n-k}{n}s(z_1)s(z_2)X_k-s(z_1)r_k.$$

Consequently,

$$\frac{1}{n^2}\sum_{k=1}^{n}X_k \quad = \quad \frac{1}{n}\sum_{k=1}^{n}\frac{\frac{n-k}{n}s(z_1)s(z_2)}{1-\frac{n-k}{n}s(z_1)s(z_2)}+\frac{1}{n^2}\sum_{k=1}^{n}\frac{s(z_1)r_k}{1-\frac{n-k}{n}s(z_1)s(z_2)},$$

which converges in probability to

$$s(z_1)s(z_2)\int_0^1\frac{t}{1-ts(z_1)s(z_2)}dt=-1-\big(s(z_1)s(z_2)\big)^{-1}\log\big(1-s(z_1)s(z_2)\big).$$

As a conclusion, $\Gamma_n(z_1,z_2)$ converges in probability to

$$\Gamma(z_1,z_2)=\sigma^2-\kappa-\frac{\kappa}{s(z_1)s(z_2)}\log\big(1-s(z_1)s(z_2)\big)+\frac{1}{2}\beta s(z_1)s(z_2).$$

The proof of Condition 5.2 is then complete.

Although we have completed the proof of Theorem 1.1, we summarize the main steps of the proof for reader's convenience.

Proof of Theorem 1.1. In Section 2, the weak convergence of $G_n(f)$ is reduced to that of $G_n(f_m)$. Then in Section 4, we show that this is equivalent to the weak convergence of

$$\Delta=-\frac{1}{2\pi i}\oint_{\gamma_m}f_m(z)n[s_n(z)-s(z)]dz.$$

The weak convergence of Δ is proved in Section 5, where we also calculate the covariance function of the limiting process. The mean function of the limiting process is obtained in Section 4. ☐

6 Appendix

Lemma 6.1. *Under condition* (1.2)*, we have*

$$\begin{aligned} \|\mathbb{E}F_n - F\| &= O(n^{-1/2}), \\ \|F_n - F\| &= O_p(n^{-2/5}), \\ \|F_n - F\| &= O(n^{-2/5+\eta}) \ \ a.s., \ \ for\ any \ \ \eta > 0. \end{aligned}$$

This follows from Theorems 1.1, 1.2 and 1.3 in [5].

Lemma 6.2. *For $X = (x_1, ..., x_n)^T$ i.i.d. standardized (complex) entries with $\mathbb{E}x_i = 0$ and $\mathbb{E}|x_i|^2 = 1$, and C is an $n \times n$ matrix (complex), we have, for any $p \geqslant 2$*

$$\mathbb{E}|X^* CX - trC|^p \leqslant K_p \left[(\mathbb{E}|x_1|^4 trCC^*)^{p/2} + \mathbb{E}|x_1|^{2p} tr(CC^*)^{p/2} \right].$$

This is Lemma 2.7 in [7].

References

[1] Arnold, L. (1967). On the asymptotic distribution of the eigenvalues of random matrices. *J. Math. Anal. Appl.* **20** 262–268. MR0217833

[2] Arnold, L. (1971). On Wigner's semicircle law for the eigenvalues of random matrices. *Z. Wahrsch. Verw. Gebiete.* **19** 191–198. MR0348820

[3] Bai, Z. D. (1993). Convergence rate of expected spectral distributions of large random matrices. Part I. Wigner matrices. *Ann. Probab.* **21** 625–648. MR1217559

[4] Bai, Z. D. (1999). Methodologies in spectral analysis of large dimensional random matrices, a review. *Stat. Sin.* **9** 611–677. MR1711663

[5] Bai, Z. D., Miao, B. and Tsay, J. (2002). Convergence rates of the spectral distributions of large Wigner matrices. *Int. Math. J.* **1** 65–90. MR1825933

[6] Bai, Z. D. and Yao, J. (2005). On the convergence of the spectral empirical process of wigner matrices. *Bernoulli* **11** 1059-1092. MR2189081

[7] Bai, Z. D. and Silverstein, J. W. (1998). No eigenvalues outside the support of the limiting spectral distribution of large dimensional random matrices. *Ann. Probab.* **26** 316-345.6 MR1617051

[8] Bai, Z. D. and Silverstein, J. W. (2006). *Spectral analysis of large dimensional random matrices*. Mathematics Monograph Series 2, Science Press, Beijing.

[9] Bai, Z. D. and Yin, Y. Q. (1988). Necessary and sufficient conditions for the almost sure convergence of the largest eigenvalue of Wigner matrices. *Ann. Probab.* **16** 1729–1741. MR0958213

[10] Costin, O. and Lebowitz, J. (1995). Gaussian fluctuations in random matrices. *Physical Review Letters* **75** 69–72.

[11] Horn, R. A. and Johnson, C. R. (1990). *Matrix Analysis*. Cambridge University Press. MR1084815

[12] Johansson, K. (1998) On fluctuations of eigenvalues of random Hermitian matrices. *Duke. Math. J.* **91** 151–204. MR1487983

[13] Khorunzhy, A. M., Khoruzhenko, B. A. and Pastur, L. A. (1996) Asymptotic properties of large random matrices with independent entries. *J. Math. Phys.* **37** 5033¨C-5060.

[14] Collins, B., Mingo, J. A., Sniady, P., and Speicher, R. (2007) Second order freeness and fluctuations of random matrices. III. Higher order freeness and free cumulants. *Doc. Math.* **12**, 1–70 (electronic). MR2302524

[15] Mingo, J. A., Sniady, P., and Speicher, R. (2007) Second order freeness and fluctuations of random matrices. II. Unitary random matrices. *Adv. Math.* **209, no. 1**, 212–240. MR2294222

[16] Mingo, J. A. and Speicher, R. (2006) Second order freeness and fluctuations of random matrices. I. Gaussian and Wishart matrices and cyclic Fock spaces. *J. Funct. Anal.* **235, no. 1**, 226–270. MR2216446

[17] Pastur, L. A. and Lytova A. (2009) Central Limit Theorem for linear eigenvalue statistics of random matrices with independent entries. *Ann. Prob.* **37**, 1778-1840.

[18] Sinai, Y. and Soshnikov, A. (1998) Central limit theorem for traces of large random symmetric matrices with independent elements. *Bol. Soc. Brasil. Mat. (N.S.)* **29** 1–24. MR1620151

[19] Wigner, E. P. (1955). Characteristic vectors of bordered matrices with infinite dimensions. *Ann. Math.* **62** 548–564. MR0077805