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Sample Path Large Deviations Principles for Poisson Shot Noise Processes, and Applications

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Abstract. This paper concerns sample path large deviations for Poisson shot noise processes, and applications in queueing theory. We first show that, under an exponential tail condition, Poisson shot noise processes satisfy a sample path large deviations principle with respect to the topology of pointwise convergence. Under a stronger superexponential tail condition, we extend this result to the topology of uniform convergence. We also give applications of this result to determining the most likely path to overflow in a single server queue, and to finding tail asymptotics for the queue lengths at priority queues.

Keywords: Poisson shot noise, large deviations, sample paths, queues, risk.

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1 Introduction and preliminaries

Shot noise processes have been widely studied in physics and electrical engineering. Relevant works on this subject are due to Campbell (1909), Schottky (1918), Rice (1944) and Lowen and Teich (1990); see also the references in the work of Bondesson (1988) and of Gubner (1996). They are also of interest in insurance risk theory where they are used to model incurred but not reported claims; see the papers of Klüppelberg and Mikosch (1995), Brémaud (2000), Klüppelberg, Mikosch and Schärf (2003), Macci and Torrisi (2004), Torrisi (2004) and Macci, Stabile and Torrisi (2004). Shot noises also arise naturally in queueing and teletraffic theory. Indeed, the well known $M/GI/\infty$ pure delay system is a Poisson shot noise with random impulse function. For the use of the $M/GI/\infty$ model in teletraffic theory, we refer the reader to Parulekar and Makowski (1997), and Konstantopoulos and Lin (1998).

In this paper, we study the sample path large deviations for Poisson shot noise processes, with a view towards applications in queueing and teletraffic theory. We start by recalling some basic definitions in large deviations theory (see, e.g., pages 4-5 in the book of Dembo and Zeitouni, 1998).

A family of probability measures $(\mu_\alpha, \alpha \in \mathbb{R}_+)$ on a topological space $(M, \mathcal{T}_M)$ satisfies the large deviations principle (LDP for short) with rate function I if $I : M \to [0, \infty]$ is a lower semicontinuous function and the following inequalities hold for every measurable set B:

$$-\inf_{x \in B^\circ} I(x) \leq \liminf_{\alpha \to \infty} \frac{1}{\alpha} \log \mu_\alpha(B) \leq \limsup_{\alpha \to \infty} \frac{1}{\alpha} \log \mu_\alpha(B) \leq -\inf_{x \in \overline{B}} I(x),$$

where B° denotes the interior of B and $\overline{B}$ denotes the closure of B.

Alternatively, we may refer to M-valued families of random variables instead of probability measures on $(M, \mathcal{T}_M)$. More precisely, if for each α we have $\mu_\alpha = P(V_\alpha \in \cdot)$ for some M-valued random variable V_α, we say that (V_α) satisfies the LDP with rate function I.

We point out that the lower semicontinuity of the rate function I means that its level sets

$$\Psi_I(c) = \{x \in M : I(x) \leq c\} \quad (\forall c > 0)$$

are closed; when the level sets are compact we say that I is a good rate function.

The stochastic process studied in this paper is a Poisson shot noise process of the following form:

$$S(t) = \sum_{n=1}^{N(t)} H(t - T_n, Z_n), \tag{1}$$

where $(N(t))$ is a homogeneous Poisson process with intensity λ and points (T_n); (Z_n) are iid E-valued random variables (for some measurable space $(E, \mathcal{E})$) and independent of $(N(t))$; $H : \mathbb{R} \times E \to [0, \infty[$ is a measurable function such that $H(\cdot, z)$ is non-decreasing and cadlag (right continuous with left limits) for each fixed $z \in E$ and $H(t, z) = 0$ for $t \leq 0$ and all $z \in E$. We use the notation $H(\infty, z)$ for the limit of $H(t, z)$ as $t \uparrow \infty$; the limit exists since $H(\cdot, z)$ is non-decreasing for all $z \in E$.

We give a couple of examples of such processes as a guide to intuition. In the insurance context, the Poisson process (T_n) specifies the times at which claims arise and $H(t, z)$ the amount of money that has to be paid out for a claim of "type" z within t time units of the claim; in this case, $H(\infty, z)$ is the total claim size. In the teletraffic context, (T_n) specifies arrival times of calls or sessions; a session of type z generates an amount of traffic $H(t,z)$ within t time units of entering the system. Here, $H(\infty,z)$ is the total traffic it generates before departing.

The paper is structured as follows. In Section 2 we show that, under a suitable exponential tail condition, the Poisson shot noise process considered in this paper satisfies the sample path large deviations principle with respect to the topology of pointwise convergence. Under a stronger superexponential condition, we show in Section 3 that the large deviations principle holds in the topology of uniform convergence, with the same rate function. It is then natural to ask whether the large deviations principle in the uniform topology can be expected to hold under the exponential tail condition of Section 2. We show in Section 4 that, if this were to be true, then the rate function would not be good in general. Finally, in Section 5, we describe some applications to queueing theory.

2 Sample path LDP in the topology of pointwise convergence

The results proved in this work are sample path LDPs for $(S(t))$. More precisely we consider $M = D[0,1]$, the space of cadlag functions on the interval $[0,1]$, and we prove LDPs for $(\frac{S(\alpha\cdot)}{\alpha})$ considering two different choices for $\mathcal{T}_M$: the topology of pointwise convergence and the topology of uniform convergence. The LDP is a refinement of the following functional law of large numbers:

$$\lim_{\alpha\to\infty} \frac{S(\alpha\cdot)}{\alpha} = \psi(\cdot) \quad a.s.,$$

where $\psi(t) = \lambda E[H(\infty, Z_1)]t$. As this is a corollary of the LDP we establish, we do not include a separate proof of this result.

In this section, we establish the LDP for $(\frac{S(\alpha\cdot)}{\alpha}, \alpha \in \mathbb{R}_+)$ with respect to the topology of pointwise convergence. But first, we need to define the following functions:

$$\Lambda_{H(\infty,Z)}(\theta) = \log \mathbb{E}[e^{\theta H(\infty,Z_1)}], \quad \Lambda(\theta) = \lambda(e^{\Lambda_{H(\infty,Z)}(\theta)} - 1), \tag{2}$$

and the Legendre transform of Λ

$$\Lambda^*(x) = \sup_{\theta\in\mathbb{R}}[\theta x - \Lambda(\theta)]. \tag{3}$$

The notion of steepness referred to in Proposition 2.1 below is defined in Definition 2.3.5 (c) of Dembo and Zeitouni (1998).

Proposition 2.1 *Assume the following light tail condition holds:*

$$(\mathbf{C1}): \ \Lambda_{H(\infty,Z)}(\theta) < \infty \ \textit{in a neighbourhood of} \ \theta = 0.$$

Moreover assume $\Lambda_{H(\infty,Z)}$ is steep. Then $(\frac{S(\alpha\cdot)}{\alpha})$ satisfies the LDP with respect to the topology of pointwise convergence, with good rate function

$$I(f) = \begin{cases} \int_0^1 \Lambda^*(\dot{f}(t))dt & \text{if} f \in AC_0[0,1], \\ \infty & \textit{otherwise}, \end{cases} \tag{4}$$

where $AC_0[0,1]$ is the family of all absolutely continuous functions f with $f(0) = 0$.

It is readily verified under the above assumptions that $\Lambda'(0) = \lambda E[H(\infty, Z_1)]$ and, consequently, that $\Lambda^*(\lambda E[H(\infty, Z_1)]) = 0$. Hence, if we take $\psi(t) = \lambda E[H(\infty, Z_1)]t$, then $I(\psi) = 0$ by (4). Moreover, this is the unique zero of $I(\cdot)$. Therefore, the probability law of $S(\alpha\cdot)/\alpha$ concentrates in arbitrarily small neighbourhoods of ψ as $\alpha \to \infty$, as stated by the functional law of large numbers. The LDP is a refinement of the law of large numbers in that it also tells us about the probability of fluctuations away from the most likely path.

The idea in proving Proposition 2.1 is to apply the Dawson-Gärtner Theorem (Theorem 4.6.1 in Dembo and Zeitouni, 1998) to "lift" an LDP for the finite dimensional distributions of $(\frac{S(\alpha\cdot)}{\alpha})$ to an LDP for the process. Thus we start by proving the following proposition.

Proposition 2.2 *Consider the same assumptions in Proposition 2.1. Then, for all* $n \geq 1$ *and* $0 \leq t_1 < \ldots < t_n \leq 1$, $(\frac{S(\alpha t_1)}{\alpha}, \ldots, \frac{S(\alpha t_n)}{\alpha})$ *satisfies the LDP in* $\mathbb{R}^n$ *with good rate function*

$$I_{t_1,\ldots,t_n}(x_1, \ldots, x_n) = \sum_{k=1}^{n} (t_k - t_{k-1})\Lambda^*(\frac{x_k - x_{k-1}}{t_k - t_{k-1}}), \tag{5}$$

where $x_0 = 0$ *and* $t_0 = 0$.

In order to prove this proposition, we will need the following technical result.

Lemma 2.3 *Let* $(\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$ *and let* $w_1, \ldots, w_n \geq 0$ *be such that* $w_1 \leq \ldots \leq w_n$. *Then* $\sum_{i=k}^{n} \theta_i w_i \leq \theta^* w^*$ *for all* $k \in \{1, \ldots, n\}$, *for any* $\theta^* \geq \max\{\max\{\sum_{i=k}^{n} \theta_i : k \in \{1, \ldots, n\}\}, 0\}$ *and any* $w^* \geq w_n$.

Proof. The proof proceeds by induction on $k \leq n$. The claim can be easily checked for $k = n$. Suppose the claim is true for some $k > 1$. We have

$$\begin{aligned}
\sum_{i=k-1}^{n} \theta_i w_i &= \sum_{i=k}^{n} \theta_i w_i + \theta_{k-1} w_{k-1} \\
&\leq \sum_{i=k}^{n} \theta_i w_i + \Big(\theta^* - \sum_{i=k}^{n} \theta_i\Big) w_{k-1} \quad (\text{since } \theta^* - \textstyle\sum_{i=k}^{n} \theta_i \geq \theta_{k-1} \text{ and } w_{k-1} \geq 0) \\
&= \sum_{i=k}^{n} \theta_i (w_i - w_{k-1}) + \theta^* w_{k-1} \\
&\leq \theta^*(w^* - w_{k-1}) + \theta^* w_{k-1} \;=\; \theta^* w^*.
\end{aligned}$$

The last inequality holds by the inductive hypothesis and the fact that $(w_i - w_{k-1})_{i \in \{k-1,\ldots,n\}}$ are non-decreasing, non-negative and bounded above by $w^* - w_{k-1}$. Thus, the induction is established. $\diamondsuit$

Proof of Proposition 2.2. Let $n \geq 1$ and $0 \leq t_1 < \ldots < t_n \leq 1$ be arbitrarily fixed. We prove that $(\frac{S(\alpha t_1)}{\alpha}, \ldots, \frac{S(\alpha t_n)}{\alpha})$ satisfies the LDP with rate function $I_{t_1,\ldots,t_n}$ in (5) as a consequence of the Gärtner-Ellis theorem (Theorem 2.3.6 in Dembo and Zeitouni, 1998) in $\mathbb{R}^n$. Our proof consists of three steps.

(a) For each $(\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$, we show that

$$\Lambda_{t_1,\ldots,t_n}(\theta_1, \ldots, \theta_n) := \lim_{\alpha\to\infty} \frac{1}{\alpha} \log \mathbb{E}\Big[\exp\Big(\alpha \sum_{i=1}^n \theta_i \frac{S(\alpha t_i)}{\alpha}\Big)\Big] = \sum_{k=1}^n (t_k - t_{k-1})\Lambda\Big(\sum_{i=k}^n \theta_i\Big), \quad (6)$$

where the existence of the limit (as an extended real number, i.e., in $\mathbb{R} \cup \{+\infty\}$) is part of the claim.

(b) We show that the function $\Lambda_{t_1,\ldots,t_n}$ satisfies the hypotheses of the Gärtner-Ellis theorem.

(c) We show that the rate function

$$I_{t_1,\ldots,t_n}(x_1, \ldots, x_n) = \sup_{(\theta_1,\ldots,\theta_n)\in\mathbb{R}^n} \Big[\sum_{i=1}^n \theta_i x_i - \Lambda_{t_1,\ldots,t_n}(\theta_1, \ldots, \theta_n)\Big]$$

coincides with the rate function defined in (5).

Proof of (a). We can write

$$\begin{aligned}
\sum_{i=1}^n \theta_i S(\alpha t_i) &= \sum_{i=1}^n \sum_{j=1}^{N(\alpha t_i)} \theta_i H(\alpha t_i - T_j, Z_j) = \sum_{i=1}^n \sum_{k=1}^i \sum_{j=N(\alpha t_{k-1})+1}^{N(\alpha t_k)} \theta_i H(\alpha t_i - T_j, Z_j) \\
&= \sum_{k=1}^n \sum_{i=k}^n \theta_i \sum_{j=N(\alpha t_{k-1})+1}^{N(\alpha t_k)} H(\alpha(t_i - t_k) + \alpha t_k - T_j, Z_j).
\end{aligned}$$

Hence, by the regenerative property of the Poisson process, we get

$$\mathbb{E}\Big[\exp \sum_{i=1}^n \theta_i S(\alpha t_i)\Big] = \prod_{k=1}^n \mathbb{E}\Big[\exp \sum_{i=k}^n \theta_i \sum_{j=N(\alpha t_{k-1})+1}^{N(\alpha t_k)} H(\alpha(t_i - t_k) + \alpha t_k - T_j, Z_j)\Big]. \quad (7)$$

Now, given $N(\alpha t_k)$ and $N(\alpha t_{k-1})$, the (unordered) times T_j, $N(\alpha t_{k-1}) + 1 \le j \le N(\alpha t_k)$ are conditionally iid, uniformly distributed on $(\alpha t_{k-1}, \alpha t_k]$ and independent of the marks Z_j. Denoting by $U_j^{(k)}$, $j = 1, 2, \ldots$, a sequence of iid random variables distributed uniformly on $(0, \alpha(t_k - t_{k-1})]$ and independent of (Z_j), we can write

$$\begin{aligned}
&\mathbb{E}\Big[\exp \sum_{i=k}^n \theta_i \sum_{j=N(\alpha t_{k-1})+1}^{N(\alpha t_k)} H(\alpha(t_i - t_k) + \alpha t_k - T_j, Z_j) \;\Big|\; N(\alpha t_k) - N(\alpha t_{k-1}) = m\Big] \\
&= \mathbb{E}\Big[\exp \sum_{i=k}^n \sum_{j=1}^m \theta_i H(\alpha(t_i - t_k) + U_j^{(k)}, Z_j)\Big] = \prod_{j=1}^m \mathbb{E}\Big[\exp \sum_{i=k}^n \theta_i H(\alpha(t_i - t_k) + U_j^{(k)}, Z_j)\Big] \\
&= \Big(\frac{1}{\alpha(t_k - t_{k-1})} \int_0^{\alpha(t_k - t_{k-1})} \mathbb{E}\Big[\exp \sum_{i=k}^n \theta_i H(\alpha(t_i - t_k) + s, Z_1)\Big] ds\Big)^m.
\end{aligned}$$

But $N(\alpha t_k) - N(\alpha t_{k-1})$ is Poisson distributed with mean $\lambda\alpha(t_k - t_{k-1})$, thus

$$\frac{1}{\alpha} \log \mathbb{E}\left[\exp \sum_{i=k}^n \theta_i \sum_{j=N(\alpha t_{k-1})+1}^{N(\alpha t_k)} H(\alpha(t_i - t_k) + \alpha t_k - T_j, Z_j)\right] = \lambda(t_k - t_{k-1}) I_k(\alpha), \quad (8)$$

where

$$I_k(\alpha) = \frac{1}{\alpha(t_k - t_{k-1})} \int_0^{\alpha(t_k - t_{k-1})} \Big(\mathbb{E}\big[\exp \sum_{i=k}^{n} \ \theta_i H(\alpha(t_i - t_k) + s, Z_1)\big] - 1 \Big) ds. \tag{9}$$

Therefore, by (7) and (8), we obtain

$$\frac{1}{\alpha} \log \mathbb{E}\left[\exp \sum_{i=1}^{n} \theta_i S(\alpha t_i)\right] = \lambda \sum_{k=1}^{n} (t_k - t_{k-1}) I_k(\alpha). \tag{10}$$

Let $\mathrm{dom}(\Lambda) = \{\theta \in \mathbb{R} : \Lambda(\theta) < \infty\}$ and $\mathrm{dom}(\Lambda_{H(\infty,Z)}) = \{\theta \in \mathbb{R} : \Lambda_{H(\infty,Z)}(\theta) < \infty\}$ be the effective domains of $\Lambda(\cdot)$ and $\Lambda_{H(\infty,Z)}(\cdot)$ respectively; obviously we have $\mathrm{dom}(\Lambda) = \mathrm{dom}(\Lambda_{H(\infty,Z)})$ by (2). Suppose $(\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$ is such that $\sum_{i=k}^n \theta_i \in \mathrm{dom}(\Lambda)$ for each $k \in \{1, \ldots, n\}$. Then, there is a $\theta^* \in \mathrm{dom}(\Lambda)$ such that $\theta^* \geq 0$ and $\sum_{i=k}^n \theta_i \leq \theta^*$ for all $k \in \{1, \ldots, n\}$. Now, fix Z_1 and s, and observe that for $i \in \{k, \ldots, n\}$, $H(\alpha(t_i - t_k) + s, Z_1)$ are non-negative constants, non- decreasing in i and bounded above by $H(\infty, Z_1)$. Hence, by Lemma 2.3,

$$\sum_{i=k}^{n} \ \theta_i H(\alpha(t_i - t_k) + s, Z_1) \leq \theta^* H(\infty, Z_1),$$

and it follows by (9) and the dominated convergence theorem, that

$$\lim_{\alpha \to \infty} I_k(\alpha) = \mathbb{E}\big[e^{\sum_{i=k}^n \ \theta_i H(\infty, Z_1)}\big] - 1 = e^{\Lambda_{H(\infty,Z)}(\sum_{i=k}^n \theta_i)} - 1.$$

Hence, from (2) and (10) we obtain (6) whenever $(\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$ satisfies $\sum_{i=k}^n \theta_i \in \mathrm{dom}(\Lambda)$ for every $k \in \{1, \ldots, n\}$.

Now fix $(\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$ such that $\sum_{i=j}^n \theta_i \notin \mathrm{dom}(\Lambda)$ for some $j \in \{1, \ldots, n\}$. Note that

$$I_j(\alpha) \geq \frac{1}{\alpha(t_j - t_{j-1})} \int_0^{\alpha(t_j - t_{j-1})} \mathbb{E}[\exp(\sum_{i=j}^{n} 1_{\theta_i < 0} \theta_i H(\infty, Z_1) + \sum_{i=j}^{n} 1_{\theta_i > 0} \theta_i H(s, Z_1)) - 1] ds.$$

The expectation above is bounded below by -1 for each s and goes to $\mathbb{E}[\exp(\sum_{i=j}^n \theta_i H(\infty, Z_1)) - 1]$ as $s \to \infty$ by the monotone convergence theorem. Hence we have

$$\lim_{\alpha \to \infty} I_j(\alpha) \geq \mathbb{E}[\exp(\sum_{i=j}^{n} \theta_i H(\infty, Z_1)) - 1] = e^{\Lambda_{H(\infty,Z)}(\sum_{i=j}^n \theta_i)} - 1 = \infty.$$

Thus, since the quantities $I_1(\alpha), \ldots, I_n(\alpha)$ are bounded below by -1, we get (6) also in this case.

Proof of (b). The differentiability of $\Lambda_{t_1,\ldots,t_n}$ in the interior of its domain is immediate from (6) and the differentiability of Λ. Moreover, Λ is lower semicontinuous hence, by (6), so is $\Lambda_{t_1,\ldots,t_n}$. Now, to check the conditions of the Gärtner-Ellis theorem, it only remains to establish its steepness. If $\mathrm{dom}(\Lambda) = \mathbb{R}$ the conclusion is immediate. Otherwise observe that $\Lambda_{H(\infty,Z)}$ is the logarithmic moment generating function of a positive random variable, and so $\mathrm{dom}(\Lambda) = \mathrm{dom}(\Lambda_{H(\infty,Z)}) = (-\infty, \overline{\theta})$ or $(-\infty, \overline{\theta}]$ for some $\overline{\theta} \in \mathbb{R}$; in fact, $\overline{\theta} > 0$ by Assumption **(C1)**. Let

$(\theta^{(m)}, m = 1, 2, \ldots)$ be a sequence of points in the interior of $\mathrm{dom}(\Lambda_{t_1,\ldots,t_n})$, converging to a point $\theta \in \mathbb{R}^n$ on the boundary of this domain. Thus

$$\sum_{i=k}^{n} \theta_i^{(m)} < \overline{\theta} \quad \forall\, m \geq 1 \text{ and } 1 \leq k \leq n, \qquad \text{and } \exists\, j : \sum_{i=j}^{n} \theta_i = \overline{\theta},$$

where $\theta = (\theta_1, \ldots, \theta_n)$. Since $\theta^{(m)}$ is in the interior of $\mathrm{dom}(\Lambda_{t_1,\ldots,t_n})$ for all m, we have by (6) that

$$\frac{\partial}{\partial \theta_j^{(m)}} \Lambda_{t_1,\ldots,t_n}(\theta^{(m)}) = \sum_{k=1}^{j} (t_k - t_{k-1}) \Lambda'\big(\sum_{i=k}^{n} \theta_i^{(m)}\big).$$

But, by the steepness of Λ, $\Lambda'\big(\sum_{i=j}^{n} \theta_i^{(m)}\big) \to \infty$ as $m \to \infty$, since $\sum_{i=j}^{n} \theta_i^{(m)} \to \overline{\theta}$, which is on the boundary of $\mathrm{dom}(\Lambda)$. Hence, the j^{th} partial derivative of $\Lambda_{t_1,\ldots,t_n}(\theta^{(m)})$ goes to infinity, which implies that $\|\nabla \Lambda_{t_1,\ldots,t_n}(\theta^{(m)})\| \to \infty$. This establishes that $\Lambda_{t_1,\ldots,t_n}$ is steep.

Proof of (c). By (a) and (b), we can use the Gärtner-Ellis theorem to obtain that $(\frac{S(\alpha t_1)}{\alpha}, \ldots, \frac{S(\alpha t_n)}{\alpha})$ satisfies the LDP in $\mathbb{R}^n$ with the good rate function

$$I_{t_1,\ldots,t_n}(x_1, \ldots, x_n) = \sup_{(\theta_1,\ldots,\theta_n) \in \mathbb{R}^n} \Big[\sum_{i=1}^{n} \theta_i x_i - \Lambda_{t_1,\ldots,t_n}(\theta_1, \ldots, \theta_n)\Big]. \tag{11}$$

Since

$$\sum_{i=1}^{n} \theta_i x_i = \sum_{i=1}^{n} \theta_i \sum_{j=1}^{i} (x_j - x_{j-1}) = \sum_{j=1}^{n} \sum_{i=j}^{n} \theta_i (x_j - x_{j-1}),$$

we have by (11) and (6) that

$$I_{t_1,\ldots,t_n}(x_1, \ldots, x_n) = \sup_{(\theta_1,\ldots,\theta_n) \in \mathbb{R}^n} \Big[\sum_{j=1}^{n} \sum_{i=j}^{n} \theta_i (x_j - x_{j-1}) - \sum_{j=1}^{n} (t_j - t_{j-1}) \Lambda\big(\sum_{i=j}^{n} \theta_i\big)\Big].$$

Setting $\nu_j = \sum_{i=j}^{n} \theta_i$ for all $j \in \{1, \ldots, n\}$, we have

$$\begin{aligned} I_{t_1,\ldots,t_n}(x_1, \ldots, x_n) &= \sup_{(\nu_1,\ldots,\nu_n) \in \mathbb{R}^n} \Big[\sum_{j=1}^{n} \nu_j (x_j - x_{j-1}) - \sum_{j=1}^{n} (t_j - t_{j-1}) \Lambda(\nu_j)\Big] \\ &= \sup_{(\nu_1,\ldots,\nu_n) \in \mathbb{R}^n} \sum_{j=1}^{n} (t_j - t_{j-1}) \Big[\nu_j \frac{x_j - x_{j-1}}{t_j - t_{j-1}} - \Lambda(\nu_j)\Big]. \end{aligned}$$

Thus

$$I_{t_1,\ldots,t_n}(x_1, \ldots, x_n) \leq \sum_{j=1}^{n} (t_j - t_{j-1}) \sup_{\nu \in \mathbb{R}} \Big[\nu \frac{x_j - x_{j-1}}{t_j - t_{j-1}} - \Lambda(\nu)\Big] = \sum_{j=1}^{n} (t_j - t_{j-1}) \Lambda^*\Big(\frac{x_j - x_{j-1}}{t_j - t_{j-1}}\Big).$$

Finally, we show the opposite inequality. For each fixed $j \in \{1, \ldots, n\}$, consider a sequence $(\nu_{j,k})$ such that

$$\lim_{k \to \infty} \nu_{j,k} \frac{x_j - x_{j-1}}{t_j - t_{j-1}} - \Lambda(\nu_{j,k}) = \Lambda^*\Big(\frac{x_j - x_{j-1}}{t_j - t_{j-1}}\Big).$$

Since, for all $k \geq 1$,

$$I_{t_1,\dots,t_n}(x_1,\dots,x_n) \geq \sum_{j=1}^{n}(t_j - t_{j-1})\Big[\nu_{j,k}\frac{x_j - x_{j-1}}{t_j - t_{j-1}} - \Lambda(\nu_{j,k})\Big],$$

by taking the limit as $k \to \infty$, we have

$$I_{t_1,\dots,t_n}(x_1,\dots,x_n) \geq \sum_{j=1}^{n}(t_j - t_{j-1})\Lambda^*\Big(\frac{x_j - x_{j-1}}{t_j - t_{j-1}}\Big).$$

This completes the proof of Propostion 2.2. $\diamondsuit$

Proof of Proposition 2.1. As a consequence of Proposition 2.2 and the Dawson-Gärtner Theorem, $(\frac{S(\alpha\cdot)}{\alpha})$ satisfies the LDP (with respect to the topology of pointwise convergence) with good rate function

$$\tilde{I}(f) = \sup\Big\{\sum_{k=1}^{n}(t_k - t_{k-1})\Lambda^*\Big(\frac{f(t_k) - f(t_{k-1})}{t_k - t_{k-1}}\Big) : \ n \geq 1, 0 \leq t_1 < \ldots < t_n \leq 1\Big\}.$$

The conclusion follows by noticing that this rate function $\tilde{I}$ coincides with the rate function I in (4); this can be checked following the same lines as in the proof of Lemma 5.1.6 in Dembo and Zeitouni (1998).

3 Sample path LDP in the topology of uniform convergence

In this section, we strengthen the LDP for $(\frac{S(\alpha\cdot)}{\alpha})$ to the topology of uniform convergence, which is of interest from the point of view of applications. In order to obtain the stronger result, we will need stronger assumptions. Specifically, we will need to assume that the tails of $H(\infty, Z_i)$ decay superexponentially. In the next section, we comment on the need for this assumption.

Proposition 3.1 *Assume the following superexponential condition holds:*

$$(\mathbf{C2}) : \ \Lambda_{H(\infty,Z)}(\theta) < \infty \ \text{for all } \theta \in \mathbb{R}.$$

Then $(\frac{S(\alpha\cdot)}{\alpha})$ satisfies the LDP on $D[0,1]$ with respect to the topology of uniform convergence with the same rate function as in (4). Moreover, under the above assumption, this rate function is good with respect to the uniform topology.

To prove this Proposition, we need the following Lemma (see, e.g., Borovkov (1967); see also de Acosta (1994) and the references cited therein).

Lemma 3.2 *Assume* **(C2)** *holds and consider the compound Poisson process*

$$C(t) = \sum_{n=1}^{N(t)} H(\infty, Z_n).$$

Then $(\frac{C(\alpha\cdot)}{\alpha})$ satisfies the LDP on $D[0,1]$ with respect to the topology of uniform convergence, with the same good rate function I as in (4).

We also need the following tail estimate.

Lemma 3.3 *Assume* **(C2)** *holds. Define* $T_0 = 0$ *and*

$$A_n = \sum_{k=0}^{n-1} [H(\infty, Z_k) - H(T_k, Z_k)], \ (n \geq 1).$$

For all $\theta > 0$ *and* $\delta > 0$*, we have*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}\left[e^{\theta A_n}\right] = 0, \quad \lim_{n\to\infty} \frac{1}{n} \log P(A_n \geq n\delta) = -\infty.$$

Proof. By the dominated convergence theorem,

$$\mathbb{E}[\exp\{\theta(H(\infty, Z_1) - H(t, Z_1))\}] \to 1 \text{ as } t \to \infty.$$

Therefore,

$$\forall\, \varepsilon > 0, \ \exists\, t(\varepsilon, \theta) > 0 : \quad \mathbb{E}[\exp\{\theta(H(\infty, Z_1) - H(t, Z_1))\}] < e^{\varepsilon} \quad \forall\, t \geq t(\varepsilon, \theta). \tag{12}$$

Fix $\theta > 0$ and $\varepsilon > 0$ and choose $t(\varepsilon, \theta)$ as above. Now, for all $n \geq 1$,

$$P(T_{[n\varepsilon]} \leq t(\varepsilon, \theta)) = P(N(t(\varepsilon, \theta)) \geq [n\varepsilon]), \tag{13}$$

where $[\cdot]$ denotes the integer part. But, by the Chernoff bound, we have for all $\eta > 0$ that

$$\begin{aligned} \frac{1}{n} \log P(N(t(\varepsilon, \theta)) \geq [n\varepsilon]) &\leq \frac{1}{n} \log(e^{-\eta[n\varepsilon]} \mathbb{E}[e^{\eta N(t(\varepsilon,\theta))}]) \\ &= \frac{1}{n} \log(e^{-\eta[n\varepsilon] + \lambda t(\varepsilon,\theta)(e^{\eta}-1)}) \to -\eta\varepsilon \ \ (\text{as } n \to \infty). \end{aligned}$$

Since $\eta > 0$ is arbitrary, it follows that

$$\limsup_{n\to\infty} \frac{1}{n} \log P(N(t(\varepsilon, \theta)) \geq [n\varepsilon]) = -\infty,$$

and, by (13),

$$\lim_{n\to\infty} \frac{1}{n} \log P(T_{[n\varepsilon]} \leq t(\varepsilon, \theta)) = -\infty. \tag{14}$$

We can write $\mathbb{E}[e^{\theta A_n}] = \mathbb{E}[e^{\theta A_n} \mathbf{1}(T_{[n\varepsilon]} < t(\varepsilon, \theta))] + \mathbb{E}[e^{\theta A_n} \mathbf{1}(T_{[n\varepsilon]} \geq t(\varepsilon, \theta))]$. We shall derive bounds on each of these terms. Note that $A_n \leq \sum_{k=0}^{n-1} H(\infty, Z_k)$, and that (Z_k) are iid and independent of (T_k). Hence, for $\theta > 0$, we have

$$\begin{aligned} \mathbb{E}\left[e^{\theta A_n} \mathbf{1}(T_{[n\varepsilon]} < t(\varepsilon, \theta))\right] &\leq \mathbb{E}\left[e^{\theta \sum_{k=0}^{n-1} H(\infty, Z_k)} \mathbf{1}(T_{[n\varepsilon]} < t(\varepsilon, \theta))\right] \\ &= e^{n\Lambda_{H(\infty,Z)}(\theta)} \ P(T_{[n\varepsilon]} < t(\varepsilon, \theta)). \end{aligned}$$

Since $\Lambda_{H(\infty,Z)}(\theta)$ is finite for all θ by assumption, we now get from (14) that

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}[e^{\theta A_n} \mathbf{1}(T_{[n\varepsilon]} < t(\varepsilon, \theta))] = -\infty. \tag{15}$$

Next, observe that if $T_{[n\varepsilon]} \geq t(\varepsilon,\theta)$, then

$$\sum_{k=0}^{n-1}\{H(\infty,Z_k)-H(T_k,Z_k)\} \leq \sum_{k=0}^{[\varepsilon n]-1} H(\infty,Z_k)+\sum_{k=[\varepsilon n]}^{n-1}\{H(\infty,Z_k)-H(t(\varepsilon,\theta),Z_k)\},$$

since $H(\cdot,z)$ is non-decreasing. Consequently, by (12),

$$\mathbb{E}\Big[e^{\theta A_n}\mathbf{1}(T_{[n\varepsilon]} \geq t(\varepsilon,\theta))\Big] \leq e^{[n\varepsilon]\Lambda_{H(\infty,Z)}(\theta)}\ e^{\varepsilon(n-[n\varepsilon])}, \tag{16}$$

for all $\theta > 0$. It is immediate from (15) and (16) that

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{E}[e^{\theta A_n}] \leq \varepsilon\Lambda_{H(\infty,Z)}(\theta)+\varepsilon(1-\varepsilon).$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, the limit superior is bounded above by zero. Moreover, A_n is a non-negative random variable, so $\log\mathbb{E}[e^{\theta A_n}] \geq 0$ for all n if $\theta > 0$. This yields

$$\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{E}[e^{\theta A_n}] \geq 0,$$

and hence the first claim of the lemma. To obtain the second claim, we use the Chernoff bound:

$$P(A_n \geq n\delta) \leq e^{-\theta n\delta}\mathbb{E}[e^{\theta A_n}]$$

and so

$$\frac{1}{n}\log P(A_n \geq n\delta) \leq -\theta\delta+\frac{1}{n}\log\mathbb{E}[e^{\theta A_n}],$$

for all $\theta > 0$. The claim follows by letting $n \to \infty$ and then $\theta \to \infty$. $\diamondsuit$

Proof of Proposition 3.1. We prove that $(\frac{S(\alpha\cdot)}{\alpha})$ and $(\frac{C(\alpha\cdot)}{\alpha})$ are exponentially equivalent (see e.g. Definition 4.2.10 in the book of Dembo and Zeitouni, 1998) with respect to the topology of uniform convergence. Therefore the conclusion follows by Lemma 3.2 and Theorem 4.2.13 in Dembo and Zeitouni (1998).

In order to establish the exponential equivalence of $(\frac{S(\alpha\cdot)}{\alpha})$ and $(\frac{C(\alpha\cdot)}{\alpha})$ in the uniform topology on $D[0,1]$, we need to show that for any $\delta > 0$,

$$\lim_{\alpha\to\infty}\frac{1}{\alpha}\log P(M_\alpha > \delta) = -\infty, \qquad \text{where } M_\alpha = \frac{1}{\alpha}\sup_{t\in[0,1]}|C(\alpha t)-S(\alpha t)|. \tag{17}$$

Since the function $H(\cdot,z)$ is non-decreasing, it is clear that the supremum over t is attained at one of the points T_n of the underlying Poisson process (on the interval $[0,T_1]$, $S(t)$ and $C(t)$ are both zero). Thus,

$$M_\alpha = \frac{1}{\alpha}\max_{n:T_n\leq\alpha}\sum_{k=1}^{n}[H(\infty,Z_k)-H(T_n-T_k,Z_k)]. \tag{18}$$

Since T_n is the sum of n exponential random variables with mean $1/\lambda$, we obtain using Chernoff bound that, for all $\eta > 0$ and all integers $K > \lambda$,

$$P(T_{K[\alpha]} < \alpha)\ \leq\ e^{\eta\alpha}\mathbb{E}\Big[e^{-\eta T_{K[\alpha]}}\Big]\ =\ e^{\eta\alpha}\Big(\frac{\lambda}{\lambda+\eta}\Big)^{K[\alpha]}.$$

Consequently,

$$\frac{1}{\alpha}\log P(T_{K[\alpha]} < \alpha) \;\le\; \eta + \frac{K[\alpha]}{\alpha}\log\Big(\frac{\lambda}{\lambda+\eta}\Big). \tag{19}$$

Next, observe from (18) that

$$P(M_\alpha > \delta, T_{K[\alpha]} \ge \alpha) \le P\Big(\max_{1\le n\le K[\alpha]} \sum_{k=1}^{n}[H(\infty, Z_k) - H(T_n - T_k, Z_k)] \ge \alpha\delta\Big)$$

$$\le K[\alpha] \max_{1\le n\le K[\alpha]} P\Big(\sum_{k=1}^{n}[H(\infty, Z_k) - H(T_n - T_k, Z_k)] \ge \alpha\delta\Big),$$

where we have used the union bound to obtain the last inequality. Now we remark that $(T_n - T_1, \ldots, T_n - T_{n-1})$ and $(T_{n-1}, \ldots, T_1)$ have the same joint distribution. Moreover, the (Z_k) are iid and independent of the (T_k). Hence, $\sum_{k=1}^{n}\{H(\infty, Z_k) - H(T_n - T_k, Z_k)\}$ has the same distribution as A_n, which was defined in the statement of Lemma 3.3. Thus, we can rewrite the above as

$$\begin{aligned} P(M_\alpha > \delta, T_{K[\alpha]} \ge \alpha) &\le K[\alpha] \max_{1\le n\le K[\alpha]} P\Big(\sum_{k=0}^{n-1}[H(\infty, Z_k) - H(T_k, Z_k)] \ge \alpha\delta\Big) \\ &= K[\alpha] \max_{1\le n\le K[\alpha]} P(A_n \ge \alpha\delta). \end{aligned}$$

The random variables A_n are clearly increasing. Hence, we have

$$P(M_\alpha > \delta, T_{K[\alpha]} \ge \alpha) \le K[\alpha] P(A_{K[\alpha]} \ge \alpha\delta),$$

and so, by Lemma 3.3,

$$\lim_{\alpha\to\infty} \frac{1}{\alpha}\log P(M_\alpha > \delta, T_{K[\alpha]} \ge \alpha) = -\infty. \tag{20}$$

Now, by the union bound,

$$\begin{aligned} P(M_\alpha > \delta) &= P(M_\alpha > \delta, T_{K[\alpha]} \ge \alpha) + P(M_\alpha > \delta, T_{K[\alpha]} < \alpha) \\ &\le P(M_\alpha > \delta, T_{K[\alpha]} \ge \alpha) + P(T_{K[\alpha]} < \alpha), \end{aligned}$$

for arbitrary $K > \lambda$. Hence by (19) and(20) we have

$$\limsup_{\alpha\to\infty} \frac{1}{\alpha}\log P(M_\alpha > \delta) \le \inf_{\eta>0} \eta + K\log\Big(\frac{\lambda}{\lambda+\eta}\Big) = K - \lambda - K\log\frac{K}{\lambda}.$$

Then we obtain (17) by taking $K \to \infty$ and this completes the proof. $\diamondsuit$

4 Some remarks on the exponential case

From the point of view of applications, one usually derives LDPs for continuous functions of sample paths of the original process by using the contraction principle (see e.g. Theorem 4.2.1 in Dembo and Zeitouni, 1998). Since the uniform topology is finer than the topology of pointwise convergence, it admits a richer class of continuous functions. This makes it interesting to ask if

$(\frac{S(\alpha\cdot)}{\alpha})$ satisfies an LDP in the uniform topology on $D[0,1]$, under the light tail condition **(C1)**. We do not know the answer to this question. However, we show that when the random variables $(H(\infty, Z_n))$ are exponentially distributed, the rate function defined in (4) is not good. Thus, even if Proposition 3.1 holds under the light tail condition **(C1)**, the contraction principle is not applicable, as it requires goodness of the rate function.

When the random variables $(H(\infty, Z_n))$ are exponentially distributed with parameter β, it can be easily checked that

$$\Lambda(\theta) = \begin{cases} \frac{\lambda\theta}{\beta-\theta} & \text{if } \theta < \beta \\ \infty & \text{if } \theta \geq \beta \end{cases} .$$

The equation $\Lambda'(\theta) = x$ is solved by $\theta = \beta - \sqrt{\lambda\beta/x}$. Therefore,

$$\Lambda^*(x) = \beta x - 2\sqrt{\lambda\beta x} + \lambda, \quad \text{and} \quad \lim_{x\to\infty} \frac{\Lambda^*(x)}{x} = \frac{\beta x - 2\sqrt{\lambda\beta x} + \lambda}{x} = \beta. \tag{21}$$

Now consider the sequence $(f_n) \subset C[0,1]$, defined as follows:

$$f_n(t) = \begin{cases} nt & \text{if } t \in [0, \frac{1}{n}] \\ \Lambda'(0)(t - \frac{1}{n}) + 1 & \text{if } t \in (\frac{1}{n}, 1] \end{cases} .$$

We have

$$I(f_n) = \int_0^{\frac{1}{n}} \Lambda^*(\dot{f}_n(t))dt + \int_{\frac{1}{n}}^1 \Lambda^*(\dot{f}_n(t))dt = \int_0^{\frac{1}{n}} \Lambda^*(n)dt + \int_{\frac{1}{n}}^1 \underbrace{\Lambda^*(\Lambda'(0))}_{=0} dt = \frac{\Lambda^*(n)}{n},$$

and therefore, $\lim_{n\to\infty} I(f_n) = \beta$, by (21). Thus, for some $\eta > 0$ and all n large enough, the functions f_n belong to the level set of I,

$$\Psi_I(\beta + \eta) = \{f \in C[0,1] : I(f) \leq \beta + \eta\}.$$

We claim that $\Psi_I(\beta+\eta)$ is not compact in the uniform topology on $D[0,1]$. We demonstrate this by showing that the sequence (f_n) does not have convergent subsequences. Indeed suppose by contradiction that there exists a subsequence which converges to a limit f. Then f has to be continuous and we must have $f(0) = 0$ since $f_n(0) = 0$ for all n. Since f is continuous, for any $\epsilon > 0$, there is a $\delta > 0$ such that $f(\delta) < \epsilon$. Now, for all $n > 1/\delta$, $f_n(\delta) \geq f_n(1/n)$ since $\Lambda'(0) \geq 0$. Hence, $f_n(\delta) \geq 1$, i.e., $\|f_n - f\| > 1 - \epsilon$ for all but finitely many n. This contradicts the assumption that f is a subsequential limit of the sequence (f_n).

We have thus shown that $\Psi_I(\beta + \eta)$ is not sequentially compact in the uniform topology on $D[0,1]$. But the uniform topology is metrizable (indeed, it is generated by the uniform norm), so this level set is not compact. Thus, the rate function I specified by (4) is not good.

Finally, we note that the arguments above can be adapted to the case when there exists $\theta_0 \in (0,\infty)$ such that $\Lambda(\theta) < \infty$ for $\theta < \theta_0$ and $\lim_{\theta\to\theta_0^-} \Lambda'(\theta) = \infty$. Indeed, in such a case, it can be shown that

$$\limsup_{x\to\infty} \frac{\Lambda^*(x)}{x} \leq \theta_0 < \infty.$$

5 Single server queues with Poisson shot noise traffic intensities

In this section we provide applications of the sample path LDP proved in Proposition 3.1 to single server queues with Poisson shot noise traffic intensities. In Subsection 5.1, we use the sample path LDP to determine the most likely path to exceed a large buffer level in a single server queue with constant server capacity and infinite buffer. Subsection 5.2 deals with a queue with two traffic classes, where one class receives priority over the other. The generalization to more than two classes is straightforward.

The sample path LDP described above is on the space $D[0,1]$, but it is clear that it can be extended to $D[0,t]$ for any fixed $t>0$, with the obvious modification that the rate function is

$$I(f) = \begin{cases} \int_0^t \Lambda^*(\dot{f}(t))dt & \text{if} f \in AC_0[0,t], \\ \infty & otherwise. \end{cases} \tag{22}$$

It is worthwhile to note that, following the ideas of Ganesh and O'Connell (2002), the sample path LDP on $D[0,1]$ can also be extended to a suitable subset of $D[0,\infty)$.

5.1 The most likely path to large exceedances

Consider a single server queue with deterministic service rate $c>0$ and infinite buffer capacity, fed by a Poisson shot noise $S(\cdot)$, as described in (1). We assume henceforth that $\Lambda'(0) = \lambda E[H(\infty, Z_1)] < c$, i.e., work enters the queue slower than the rate at which it can be served. If this were not the case, then the work backlogged at the queue would grow without bound. Let $Q(t)$ be the amount of work in the queue at time t. It was shown by Loynes (1962) that

$$Q(t) = \sup_{0 \le s \le t} S(t) - S(s) - c(t-s). \tag{23}$$

We are interested in the "most likely" sample path of the traffic process $(S(s), 0 \le s \le t)$, conditional on $Q(t) \ge q$, for fixed, large t and q. More precisely, we seek a sample path $\phi(\cdot)$ such that

$$\limsup_{\alpha\to\infty} P\Big(\frac{S(\alpha\cdot)}{\alpha} \notin B_\epsilon(\phi) \,\Big|\, Q(\alpha t) \ge \alpha q\Big) = 0,$$

for all $\epsilon > 0$, where $B_\epsilon(\phi)$ denotes the ϵ-ball around ϕ in $D[0,t]$. In fact, we shall show that this probability decays to zero at an exponential rate as α tends to infinity.

Intuitively, for the queue to build up to a large size q, work must arrive into the queue, over a period of length s, at rate $c + \frac{q}{s}$, strictly bigger than the service rate. Using a large deviations estimate for the logarithm of this probability leads us to consider the following optimization problem:

$$\inf_{0<s\le t} s\Lambda^*\Big(c + \frac{q}{s}\Big) = \inf_{0<s\le t} \sup_\theta \; \theta(cs+q) - s\Lambda(\theta) \tag{24}$$

Now, for a fixed $s \in (0,t]$, the supremum over θ is attained at the unique θ_s which solves $\Lambda'(\theta) = c + \frac{q}{s}$. The existence of θ_s follows from the fact that $\Lambda'(\cdot)$ is continuous, $\Lambda'(0) < c$ by assumption, and $\Lambda'(\theta) \to \infty$ as $\theta \to \infty$ (as can be readily seen from (2)), unless $H(\infty, Z_1)$ is identically zero, in which case the queue can never build up. The uniqueness of θ_s follows by the strict convexity of Λ. Finally, we note that θ_s is a decreasing function of s since $\Lambda'(\theta)$ is an increasing function of θ by the convexity of $\Lambda(\cdot)$.

We can now rewrite the optimization problem in (24) as

$$\inf_{0<s\le t} f(s) \quad \text{where } f(s) = \theta_s(cs+q) - s\Lambda(\theta_s). \tag{25}$$

We have

$$f'(s) = c\theta_s - \Lambda(\theta_s) + (cs + q - s\Lambda'(\theta_s))\frac{d\theta_s}{ds} = c\theta_s - \Lambda(\theta_s). \tag{26}$$

Now $\Lambda(\cdot)$ is convex and, as noted above, $\Lambda'(0) < c$ while $\Lambda(\theta) - c\theta \to \infty$ as $\theta \to \infty$. Therefore,

$$\exists\, w > 0 : \Lambda(w) - cw = 0, \quad \Lambda(\theta) - c\theta < 0 \;\forall\, \theta \in (0,w), \quad \Lambda(\theta) - c\theta > 0 \;\forall\, \theta > w. \tag{27}$$

Moreover, $f'(\cdot)$ is continuous and, as s decreases to zero, θ_s increases to infinity and consequently $f'(s) \to -\infty$. Hence, we see from (26) and (27) that either there is a $\tau \in (0,t]$ such that $f'(\tau) = 0$ or $f'(s) < 0$ on this interval. In the former case, we must necessarily have $\theta_\tau = w$ by (27); $\theta_\tau = 0$ is impossible since $\Lambda'(0) < c$. Since $s \mapsto \theta_s$ is strictly decreasing, the equation $f'(s) = 0$ can thus have at most one solution τ in $(0,t]$; the minimum of $f(s)$ on this interval is attained at τ, if it exists, and at t otherwise.

To summarise, we have the following two cases:

1. $\Lambda'(w) \ge c + \frac{q}{t}$: In this case, there is a $\tau \in (0,t]$ such that $\Lambda'(w) = c + \frac{q}{\tau}$, i.e., $w = \theta_\tau$. Thus, by (26) and (27), $f'(\tau) = 0$ and the infimum of f on $(0,t]$ is attained at

$$\tau = \frac{q}{\Lambda'(w) - c}.$$

   Therefore by (25)

$$\inf_{s\in(0,t]} s\Lambda^*\Big(c + \frac{q}{s}\Big) = f(\tau) = w(c\tau + q) - \tau cw = wq. \tag{28}$$

   We also notice that

$$\Lambda^*(\Lambda'(w)) = w\Lambda'(w) - \Lambda(w) = w(\Lambda'(w) - c). \tag{29}$$

2. $\Lambda'(w) < c + \frac{q}{t}$: In this case, clearly θ_t is bigger than w and, hence, so is θ_s for all $s \in (0,t]$. Thus, by (26) and (27), $f'(s) = 0$ has no solution in $(0,t]$. Therefore, the infimum of f on this interval is attained at t and by (25)

$$\inf_{s\in(0,t]} s\Lambda^*\Big(c + \frac{q}{s}\Big) = f(t) = (q + ct)\theta_t - t\Lambda(\theta_t).$$

The discussion above suggests that the most likely path to a large queue is as follows:

Proposition 5.1 *Assume the same hypotheses of Proposition 3.1 and the stability condition* $\Lambda'(0) < c$. *Let* w *be as in (27), and let* $q > 0$ *and* $t > 0$ *be fixed.*
(i) Suppose first that $\Lambda'(w) \ge c + \frac{q}{t}$. *Let* $\tau = q/(\Lambda'(w) - c)$, *and define*

$$\phi_1(s) = \begin{cases} \Lambda'(0)s, & 0 \le s \le t-\tau, \\ \Lambda'(0)(t-\tau) + \Lambda'(w)(s-t+\tau), & t-\tau < s \le t. \end{cases}$$

Then,

$$\limsup_{\alpha\to\infty} \frac{1}{\alpha} \log P\Big(\frac{S(\alpha\cdot)}{\alpha} \notin B_\epsilon(\phi_1) \;\Big|\; Q(\alpha t) \geq \alpha q\Big) < 0,$$

where $B_\epsilon(\phi_1)$ denotes the ϵ-ball around ϕ_1 in $D[0,t]$ equipped with the uniform topology, namely, the topology induced by the supremum norm.
(ii) Suppose next that $\Lambda'(w) < c+\frac{q}{t}$. As above, let $\theta_t > w$ be the unique solution of $\Lambda'(\theta_t) = c+\frac{q}{t}$. Define

$$\phi_2(s) = \Lambda'(\theta_t)s, \quad 0 \leq s \leq t.$$

Then,

$$\limsup_{\alpha\to\infty} \frac{1}{\alpha} \log P\Big(\frac{S(\alpha\cdot)}{\alpha} \notin B_\epsilon(\phi_2) \;\Big|\; Q(\alpha t) \geq \alpha q\Big) < 0,$$

where again $B_\epsilon(\phi_2)$ denotes the ϵ-ball around ϕ_2 in $D[0,t]$.

Proof. First consider case (i), corresponding to $\Lambda'(w) \geq c + \frac{q}{t}$. For a sample path $f \in D[0,t]$, define $\mathcal{Q}_t(f) = \sup_{0\leq s\leq t} f(t) - f(s) - c(t-s)$. Note that

$$\frac{Q(\alpha t)}{\alpha} = \frac{1}{\alpha} \sup_{0\leq s\leq t} [S(\alpha t) - S(\alpha s) - c(\alpha t - \alpha s)] = \mathcal{Q}_t\Big(\frac{S(\alpha\cdot)}{\alpha}\Big).$$

Moreover, it is readily verified that the function $\mathcal{Q}_t : D[0,t] \to \mathbb{R}_+$ is continuous with respect to the uniform topology on $D[0,t]$. Hence, by Proposition 3.1 and the contraction principle, the sequence $Q(\alpha t)/\alpha$ satisfies the LDP in $\mathbb{R}_+$ with rate function given by

$$J(q) = \inf\{I(f) : f \in D[0,t], \mathcal{Q}_t(f) = q\}, \tag{30}$$

where $I(f)$ is specified by (22). Since $\mathcal{Q}_t$ is continuous, the set $\{f \in D[0,t], \mathcal{Q}_t(f) = q\}$ is closed. Now, for ϕ_1 defined in the statement of the proposition, we have

$$\frac{d}{ds}[\phi_1(t) - \phi_1(s) - c(t-s)] = c - \dot{\phi}_1(s) = \begin{cases} c - \Lambda'(0), & 0 < s < t-\tau \\ c - \Lambda'(w), & t-\tau < s < t. \end{cases}$$

Since $\Lambda'(0) < c < c+\frac{q}{t} \leq \Lambda'(w)$, the supremum of $\phi_1(t)-\phi_1(s)-c(t-s)$ is attained at $s = t-\tau$. Therefore,

$$\mathcal{Q}_t(\phi_1) = \phi_1(t) - \phi_1(t-\tau) - c\tau = (\Lambda'(w) - c)\tau = q,$$

by the definition of τ. We have thus shown that ϕ_1 belongs to the set $\{f \in D[0,t], \mathcal{Q}_t(f) = q\}$. Hence, by (30),

$$J(q) \leq I(\phi_1) = \frac{q}{\Lambda'(w) - c}\Lambda^*(\Lambda'(w)) = wq, \tag{31}$$

where the equalities follow from the fact that $\Lambda^*(\Lambda'(0)) = 0$ and from (29).

Let $f \in D[0,t]$ be such that $\mathcal{Q}_t(f) \geq q$ and $I(f) < \infty$. Then f is continuous and there is a $\sigma \in (0,t]$ such that $\mathcal{Q}_t(f) = f(t) - f(t-\sigma) - c\sigma \geq q$. Now, by the non-negativity and convexity of Λ^*, and Jensen's inequality, we obtain that

$$I(f) \geq \int_{t-\sigma}^{t} \Lambda^*(\dot{f}(s))ds \geq \sigma\Lambda^*\Big(\frac{1}{\sigma}\int_{t-\sigma}^{t} \dot{f}(s)ds\Big) = \sigma\Lambda^*\Big(\frac{f(t) - f(t-\sigma)}{\sigma}\Big). \tag{32}$$

But $f(t)-f(t-\sigma)\geq q+c\sigma$, $\Lambda^*(\cdot)$ is increasing on $(\Lambda'(0),\infty)$ and $\Lambda'(0)<c$ by assumption. Hence, $I(f)\geq\sigma\Lambda^*\big(c+\frac{q}{\sigma}\big)$. Since such an equality holds for every f satisfying $\mathcal{Q}_t(f)\geq q$, we have

$$\inf_{f\in D[0,t]:\mathcal{Q}_t(f)\geq q} I(f)\geq \inf_{\sigma\in(0,t]}\sigma\Lambda^*\Big(c+\frac{q}{\sigma}\Big)=wq, \tag{33}$$

where the last equality follows from (28). Since this holds for all f such that $I(f)<\infty$ and $\mathcal{Q}_t(f)\geq q$, it follows that $J(q)\geq wq$. Combining this with (31), we see that $J(q)=I(\phi_1)=wq$, i.e., the infimum in (30) is attained at ϕ_1. In fact, the inequality in (32) is strict unless $\Lambda^*(\dot f(\cdot))=0$, i.e., $\dot f(\cdot)\equiv\Lambda'(0)$, on $(0,t-\sigma)$ and $\dot f(\cdot)$ is a constant on $(t-\sigma,t)$, say m (because of the strict convexity of Λ^*). Moreover the inequalities

$$\sigma\Lambda^*\Big(\frac{f(t)-f(t-\sigma)}{\sigma}\Big)\geq\sigma\Lambda^*\Big(c+\frac{q}{\sigma}\Big)\geq\inf_{\sigma\in(0,t]}\sigma\Lambda^*\Big(c+\frac{q}{\sigma}\Big)$$

are strict unless $m=\frac{f(t)-f(t-\sigma)}{\sigma}=c+\frac{q}{\sigma}$ with σ equal to τ; thus the inequality in (33) is strict unless $\sigma=\tau$ and $m=\Lambda'(w)$. This proves that $\inf\{I(f):f\in D[0,t],\mathcal{Q}_t(f)\geq q\}$ is uniquely attained at ϕ_1, and therefore ϕ_1 is the unique minimizer in (30). In this sense, ϕ_1 is the most likely path.

Recall that $Q(\alpha t)/\alpha$ satisfies the LDP in $\mathbb{R}_+$ with rate function $J(\cdot)$. Hence, by the large deviations lower bound for open sets,

$$\liminf_{\alpha\to\infty}\frac{1}{\alpha}\log P(Q(\alpha t)>\alpha q)\geq-\inf_{r>q}J(r)=-\inf_{r>q}rw=-qw. \tag{34}$$

Also, since $\Big\{\mathcal{Q}_t\Big(\frac{S(\alpha\cdot)}{\alpha}\Big)\geq q\Big\}=\{Q(\alpha t)\geq\alpha q\}$, we have by the large deviations upper bound for closed sets that

$$\limsup_{\alpha\to\infty}\frac{1}{\alpha}\log P\Big(\Big\{\frac{S(\alpha\cdot)}{\alpha}\notin B_\epsilon(\phi_1)\Big\}\bigcap\Big\{Q(\alpha t)\geq\alpha q\Big\}\Big)\leq-\inf\{I(f):\mathcal{Q}_t(f)\geq q, f\notin B_\epsilon(\phi_1)\}. \tag{35}$$

Since $I(\cdot)$ is a good rate function, the infimum over the closed set above can be restricted to a compact subset. Since $I(\cdot)$ is lower semicontinuous, there is a g at which the infimum is attained. Now $I(g)>I(\phi_1)=qw$ because ϕ_1 is the unique minimizer of $I(\cdot)$ over $\{f\in D[0,t]:\mathcal{Q}_t(f)\geq q\}$, as shown above. Hence, by (34) and (35),

$$\limsup_{\alpha\to\infty}\frac{1}{\alpha}\log P\Big(\frac{S(\alpha\cdot)}{\alpha}\notin B_\epsilon(\phi)\ \Big|\ Q(\alpha t)\geq\alpha q\Big)\leq -I(g)+qw<0.$$

This completes the proof of case (i).

The proof of case (ii) is very similar. It is readily verified that $\mathcal{Q}_t(\phi_2)=\phi_2(t)-\phi_2(0)-ct=q$. We need only to show that ϕ_2 is the unique minimizer of $I(f)$ over $D[0,t]$ subject to $\mathcal{Q}_t(f)\geq q$. As in the proof of case (i), this follows from the strict convexity of Λ^* and Jensen's inequality (to show that f must be piecewise linear), and the fact that the infimum of $\sigma\Lambda^*(c+\frac{q}{\sigma})$ over $(0,t]$ is attained uniquely at t in this case. The details are omitted. $\Diamond$

5.2 Priority queues

Consider a single server queue fed by two independent Poisson shot noise processes $(S_1(t))$, $(S_2(t))$, where, for $k \in \{1,2\}$,

$$S_k(t) = \sum_{n=1}^{N^{(k)}(t)} H^{(k)}(t - T_n^{(k)}, Z_n^{(k)}).$$

Here $(N^{(k)}(t))$ is a homogeneous Poisson process with intensity λ_k and points $(T_n^{(k)})$; $(Z_n^{(k)})$ are iid E_k-valued random variables (for some measurable space $(E_k, \mathcal{E}_k)$) and independent of $(N^{(k)}(t))$; $H^{(k)} : \mathbb{R} \times E_k \to [0, \infty)$ is a measurable function such that $H^{(k)}(t, z) = 0$ for $t \leq 0$ and $H^{(k)}(\cdot, z)$ is non-decreasing. The server has constant service capacity c and gives priority to class 1 traffic; class 2 traffic is served only when there is no class 1 traffic in the system.

Let $\mu_i = \lambda_i \mathbb{E}[H(\infty, Z_n^{(i)})]$, $i \in \{1,2\}$. We assume that $\mu_1 + \mu_2 < c$, so that the queues are stable. Let $(Q_1(t), Q_2(t))$ denote the amount of work stored in the two queues at time t. Analogous to (23), the total work in the system is described by

$$Q_1(t) + Q_2(t) = \sup_{0 \leq s \leq t} S_1(t) + S_2(t) - S_1(s) - S_2(s) - c(t - s). \tag{36}$$

Moreover, since the first queue has priority, it does not see the traffic entering the low priority queue. Thus,

$$Q_1(t) = \sup_{0 \leq s \leq t} S_1(t) - S_1(s) - c(t - s). \tag{37}$$

Define $\mathcal{Q}_t : D[0,t] \times D[0,t] \to \mathbb{R}_+^2$ by setting $(q_1, q_2) = \mathcal{Q}_t(\phi_1, \phi_2)$ if

$$\begin{aligned} q_1 &= \sup_{0 \leq s \leq t} [\phi_1(t) - \phi_1(s) - c(t-s)], \\ q_1 + q_2 &= \sup_{0 \leq s \leq t} [\phi_1(t) + \phi_2(t) - \phi_1(s) - \phi_2(s) - c(t-s)]. \end{aligned}$$

We saw in the proof of Proposition 5.1 that $\phi_1 \mapsto q_1$ is continuous. Likewise, $(\phi_1 + \phi_2) \mapsto (q_1 + q_2)$ is continuous. Hence, it is easy to see that $\mathcal{Q}_t$ is continuous. Note that

$$\Big(\frac{Q_1(\alpha t)}{\alpha}, \frac{Q_2(\alpha t)}{\alpha}\Big) = \mathcal{Q}_t\Big(\frac{S_1(\alpha \cdot)}{\alpha}, \frac{S_2(\alpha \cdot)}{\alpha}\Big).$$

Therefore, by the contraction principle, the random variables $(\frac{Q_1(\alpha t)}{\alpha}, \frac{Q_2(\alpha t)}{\alpha})$ satisfy the LDP in $\mathbb{R}_+^2$ with good rate function

$$J(q_1, q_2) = \inf\{I_1(\phi_1) + I_2(\phi_2) : \mathcal{Q}_t(\phi_1, \phi_2) = (q_1, q_2)\},$$

where $I_k(\cdot)$ is specified analagously to $I(\cdot)$ in (22).

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