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REFLECTED BACKWARD STOCHASTIC DIFFERENTIAL EQUATION WITH JUMPS AND RANDOM OBSTACLE

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Abstract: In this paper we give a solution for the one-dimensional reflected backward stochastic differential equation when the noise is driven by a Brownian motion and an independent Poisson point process. We prove existence and uniqueness of the solution in using penalization and the Snell envelope theory. However both methods use a contraction in order to establish the result in the general case. Finally, we highlight the connection of such reflected BSDEs with integro-differential mixed stochastic optimal control.

Keywords and phrases: Backward stochastic differential equation, Penalization, Poisson point process, Martingale representation theorem, Integral-differential mixed control.

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1 Introduction

The notion of non-linear Backward Stochastic Differential Equations (BSDE in short) was introduced by Pardoux & Peng ([11], 1990). Their aim was to give a probabilistic interpretation of a solution of second order quasi-linear partial differential equations. Since then, these equations have gradually become an important mathematical tool which is encountered in many fields such as financial mathematics, stochastic optimal control, partial differential equations, ...

In ([6], 1997), El-Karoui et al. introduced the notion of a reflected BSDE, which is actually a backward equation but one of the components of the solution is forced to stay above a given barrier, which is an adapted continuous process.

Recently in ([1], 1995), Barles et al. considered standard BSDEs when the noise is driven by a Brownian motion and an independent Poisson random measure. They have shown the existence and uniqueness of the solution, in addition, the link with integral-partial differential equations is studied.

In this paper our aim is to study the one-dimensional reflected BSDE (RBSDE in short) when the noise is driven by a Brownian motion and an independent Poisson random measure. This is the natural generalization of the work of El-Karoui et al.. The component $(Y_t)_{t\leq 1}$ of the solution which is forced to stay above a given barrier is, in our frame, no longer continuous but just right continuous and left limited (*rcll* in short) (see equation (1) below). It has jumps which arise naturally since the noise contains a Poisson random measure part.

The problem we consider here can be studied in a more general setting, namely, multivalued backward stochastic differential equations (see [10], for the continuous case). But for the sake of simplicity, we limit ourselves to the reflected framework. Finally, two other interesting papers on BSDEs with jumps but without reflection can be mentioned, namely those of R.Situ [12], and S. Tang & X. Li [13]. This latter is motivated by control problems.

In this work we mainly show the existence and uniqueness of the solution for the reflected BSDE with jumps (i.e. whose noise includes a Poisson random measure part) for a given,

- terminal value ξ which is square integrable random variable

- coefficient $f(t, \omega, y, z, v)$ which is a function, uniformly Lipschitz with respect to (y, z, v)

- barrier $(S_t)_{t\leq 1}$, which is the moving obstacle and which is a *rcll* process whose jumps are inaccessible.

In the proof of our result, we use two methods, the penalization and the Snell envelope theory. However in order to prove the result in the general case, both methods use a contraction (fixed point argument) since we do not have an efficient comparison theorem for solutions of standard BSDEs whose noise contains a Poisson measure part (see e.g. [1] for a counterexample). On the other hand, the fact that the jumping times of the moving barrier S are inaccessible is of crucial role. Finally we highlight the connection of our reflected BSDEs with integral-differential mixed stochastic optimal control. Nevertheless our results can be applied in mathematical finance, especially for the evaluation of American contingent claims when the dynamic of the prices contains a Poisson point process part.

This paper is divided into three sections.

In Section 1, we begin to show the uniqueness of the solution of the reflected BSDE when it exits. Then using the penalization method we show the existence of a solution when the function f does not depend on (y,z,v). Therefore we construct a contraction which has a fixed point which is the solution of our reflected BSDE with jumps. Finally we study the regularity of the non-decreasing process K which is absolutely continuous in the case when the barrier is regular.

In Section 2, using the Snell envelope theory, once again, we prove the existence of the solution if the function f does not depend on (y,z,v). Furthermore, as in Section 1, we obtain the solution in the general setting.

Section 3 is devoted to the link of our reflected BSDEs with integral-differential mixed stochastic optimal control. We show that the value function of the problem is solution of an appropriate reflected BSDE with jumps. In addition an optimal control exits and its expression is given. Our result generalizes that of J.P.Lepeltier & B.Marchal ([9]) on the same subject.

2 Reflected BSDEs with respect to Brownian motion and an independent Poisson point process.

Let $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\leq 1}))$ be a stochastic basis such that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$, $\mathcal{F}_{t+}=\bigcap_{\epsilon>0}\mathcal{F}_{t+\epsilon}=\mathcal{F}_t$, $\forall t<1$, and suppose that the filtration is generated by the two following mutually independent processes :

- a d-dimensional Brownian motion $(B_t)_{t\leq 1}$,

- a Poisson random measure μ on $I\!R^+\times U$, where $U:=I\!R^l\setminus\{0\}$ is equipped with its Borel fields $\mathcal{U}$, with compensator $\nu(dt,de)=dt\lambda(de)$, such that $\{\tilde{\mu}([0,t]\times A)=(\mu-\nu)([0,t]\times A)\}_{t\leq 1}$ is a martingale for every $A\in\mathcal{U}$ satisfying $\lambda(A)<\infty$. λ is assumed to be a σ-finite measure on $(U,\mathcal{U})$ satisfying

$$\int_U (1\wedge |e|^2)\lambda(de)<\infty.$$

On the other hand, let:

- $\mathcal{S}^2$ be the set of $\mathcal{F}_t$-adapted right continuous with left limit ($rcll$ in short) processes $(Y_t)_{t\leq 1}$ with values in $I\!R$ and $I\!E[\sup_{t\leq 1}|Y_t|^2]<\infty$.
- $H^{2,k}$ be the set of $\mathcal{F}_t$-progressively measurable processes with values in $I\!R^k$ such that $I\!E[\int_0^1 |Z_s|^2ds]<\infty$.
- $\mathcal{L}^2$ be the set of mappings $V:\Omega\times[0,1]\times U\to I\!R$ which are $\mathcal{P}\otimes\mathcal{U}$-measurable and $I\!E[\int_0^1 ds\int_U (V_s(e))^2\lambda(de)]<\infty$; $\mathcal{P}$ is the predictable tribe on $\Omega\times[0,1]$
- for a given $rcll$ process $(w_t)_{t\leq 1}$, $w_{t-}=\lim_{s\nearrow t}w_s, t\leq 1$ $(w_{0-}=w_0)$; $w_-:=(w_{t-})_{t\leq 1}$ $\Box$

We are now given three objects:

-a terminal value $\xi\in L^2(\Omega,F_1,P)$

-a map $f:\Omega\times[0,1]\times I\!R^{1+d}\times L^2(U,\mathcal{U},\lambda;I\!R)\longrightarrow I\!R$ which with (t,ω,y,z,v) associates $f(t,\omega,y,z,v)$ and which is $\mathcal{P}\otimes\mathcal{B}(I\!R^{1+d})\otimes\mathcal{B}(L^2(U,\mathcal{U},\lambda;I\!R))$-measurable. In addition we assume :

(i) the process $(f(t,0,0,0))_{t\leq 1}$ belongs to $L^2(\Omega\times[0,1],dP\otimes dt)$

(ii) f is uniformly Lipschitz with respect to (y, z, v), i.e., there exists a constant $k \geq 0$ such that for any y, y', z, $z' \in I\!R$ and v, $v' \in L^2(U, \mathcal{U}, \lambda; I\!R)$,

$$P - a.s., \ |f(\omega, t, y, z, v) - f(\omega, t, y', z', v')| \leq k(|y - y'| + |z - z'| + \|v - v'\|).$$

- an "obstacle" process $\{S_t, 0 \leq t \leq 1\}$, which is an $\mathcal{F}_t-$progressively measurable $rcll$, real valued process satisfying

$$I\!E[\sup_{0 \leq t \leq 1} \left(S_t^+\right)^2] < +\infty \, ; S_t^+ := \max\{S_t, 0\}.$$

Moreover we assume that its jumping times are inaccessible stopping times (see e.g. [3], p.58 for the definition). This assumption on S is satisfied if, for example, $\forall t \leq T$, $S_t = \tilde{S}_t + \tilde{P}_t$ where $\tilde{S}$ is continuous and, for any $t \leq T$, $\tilde{P}_t = \mu(t, \omega, A)$ where A is a Borel set such that $\lambda(A) < \infty$.

Let us now introduce our reflected BSDE with jumps (in short, RDBSDE; "D" for discontinuous) associated with (f, ξ, S). A solution is a quadruple $(Y, Z, K, V) := (Y_t, Z_t, K_t, V_t)_{t \leq 1}$ of processes with values in $I\!R^{1+d} \times I\!R^+ \times L^2(U, \mathcal{U}, \lambda; I\!R)$ and which satisfies :

$$\begin{cases} (i) & Y \in \mathcal{S}^2, \ Z \in H^{2,d} \text{ and } V \in \mathcal{L}^2; \ K \in \mathcal{S}^2 \ (K_0 = 0), \text{ is continuous and non-decreasing} \\ (ii) & Y_t = \xi + \int_t^1 f(s, Y_s, Z_s, V_s) ds + K_1 - K_t - \int_t^1 Z_s dB_s - \int_t^1 \int_U V_s(e) \tilde{\mu}(ds, de) \, , t \leq 1 \\ (iii) & \forall t \leq 1, Y_t \geq S_t \text{ and } \int_0^1 (Y_t - S_t) dK_t = 0. \end{cases} \quad (1)$$

In our definition, the jumps of Y are those of its Poisson part since K is continuous □

To begin with, we are going to show the uniqueness of the solution of the RDBSDE (1) under the above assumptions on f, ξ and S.

2.1 Uniqueness.

1.1.a. Proposition: Under the above assumptions on f, ξ and $(S_t)_{t \leq 1}$, the DRBSDE (1) associated with (f, ξ, S) has at most one solution.

Proof: Assume (Y, Z, K, V) and (Y', Z', K', V') are two solutions of (1). First let us underline that

$$\left(Y_t - Y_t'\right) \left(dK_t - dK_t'\right) \leq 0.$$

On the other hand, using Itô's formula with the discontinuous semi-martingale $Y - Y'$ and set

$$\Delta_t = \left|Y_t - Y_t'\right|^2 + \int_t^1 |Z_s - Z_s'|^2 ds \ + \int_t^1 \int_U \left(V_s\left(e\right) - V_s'\left(e\right)\right)^2 \lambda(de) ds$$

yields,

$$\begin{aligned} \Delta_t \ &= \ 2\int_t^1 (Y_s - Y_s')(f(s, Y_s, Z_s, V_s) - f(s, Y_s', Z_s', V_s')) ds \\ &-2\int_t^1 (Y_s - Y_s')(Z_s - Z_s') dB_s + 2\int_t^1 (Y_s - Y_s')(dK_s - dK_s') \\ &-\int_t^1 \int_U [(Y_{s-} - Y_{s-}' + V_s(e) - V_s'(e))^2 - (Y_{s-} - Y_{s-}')^2] \tilde{\mu}(ds, de). \end{aligned}$$

Now since $-\int_0^{\cdot}\int_U\left[\left(Y_{s-}-Y'_{s-}+V_s(e)-V'_s(e)\right)^2-\left(Y_{s-}-Y'_{s-}\right)^2\right]\tilde{\mu}(ds,de)$ and $\int_0^{\cdot}\left(Y_s-Y'_s\right)\left(Z_s-Z'_s\right)dB_s$ are $(\mathcal{F}_t,P)$-martingales, then taking the expectation in both sides yields, for any $t\le 1$,

$$I\!E[\Delta_t] \;\le\; 2\int_t^1 I\!E[\left(Y_s-Y'_s\right)\left(f(s,Y_s,Z_s,V_s)-f(s,Y'_s,Z'_s,V'_s)\right)]ds.$$

If we choose $\alpha\ge 4k$ and $\beta\ge 4k$, and denote

$$\Phi_t=I\!E\left[\left|Y_t-Y'_t\right|^2+\frac{1}{2}\int_t^1|Z_s-Z'_s|^2ds+\frac{1}{2}\int_t^1\int_U\left(V_s(e)-V'_s(e)\right)^2\lambda(de)\,ds\right]$$

we obtain,

$$\Phi_t \;\le\; 2k(1+\alpha+\beta)I\!E[\int_t^1\left(Y_s-Y'_s\right)^2ds].$$

Henceforth from Gronwall's lemma and the right continuity of $(Y_t-Y'_t)_{t\le 1}$, we get $Y=Y'$. Consequently $(Y,Z,V,K)=(Y',Z',V',K')$ whence the uniqueness of the solution of (1) □

We are going now to show that equation (1) has a solution in using two methods. Roughly speaking, the first one is based on the penalization and the second on the well known Snell envelope theory of processes. However, in order to obtain the result in the general frame, both methods use a contraction. The penalization, as it has been used e.g. in [6], is not workable since we do not have an efficient comparison theorem for solutions of BSDEs whose noise is driven by a Lévy process (see [1] for a counter-example). That is the reason for which, in a first time, we suppose that the map $f(t,\omega,y,z,v)$ does not depend on (y,z,v).

2.2 The penalization method.

First let us assume the map f does not depend on (y,z,v), i.e., P-*a.s.*, $f(t,\omega,y,z,v)\equiv g(t,\omega)$, for any t,y,z and v. In the following result, we establish the existence of the solution of the RDBSDE associated with (g,ξ,S).

1.2.a. Theorem : The RDBSDE associated with (g,ξ,S) has a unique solution $(Y_t,Z_t,K_t,V_t)_{t\le 1}$.

Proof: For each $n\in I\!N^*$, let $(Y^n_t,\,Z^n_t,V^n_t)_{t\le 1}$ be the $\mathcal{F}_t$-progressively measurable process with values in $I\!R^{1+d}\times L^2(U,\mathcal{U},\lambda;I\!R)$, unique solution of the BSDE associated with $(g(t,\omega)+n(y-S_t)^-,\xi)$ ($(y-S_t)^-:=\max\{0,S_t-y\}$). It exists according to Barles et al.'s result [1]. So,

$$I\!E[\sup_{t\le 1}|Y^n_t|^2+\int_0^1|Z^n_t|^2\,dt+\int_0^1\int_U|V^n_s(e)|^2\lambda(de)\,ds]<+\infty \qquad (2)$$

and

$$Y^n_t=\xi+\int_t^1 g(s)ds-\int_t^1 Z^n_s dB_s+\int_t^1 n\left(Y^n_s-S_s\right)^-ds-\int_t^1\int_U V^n_s(e)\,\tilde{\mu}(ds,de)\,,\forall t\le 1.$$

From now on the proof will be divided into six steps.

Step 1: For any $n \geq 0$, $Y^n \leq Y^{n+1}$.

Indeed, using the generalized Itô's formula with the convex function $x \longmapsto (x^+)^2$ and $Y^n - Y^{n+1}$ implies that the following process $(A_t^n)_{t\leq 1}$, where

$$\begin{aligned}A_t^n &:= (Y_t^n - Y_t^{n+1})^{+2} - (Y_0^n - Y_0^{n+1})^{+2} - 2\int_{]0,t]} (Y_{s-}^n - Y_{s-}^{n+1})^+ d(Y_s^n - Y_s^{n+1}) \\ &- \sum_{0<s\leq t} \{(Y_s^n - Y_s^{n+1})^{+2} - (Y_{s-}^n - Y_{s-}^{n+1})^{+2} - 2(Y_{s-}^n - Y_{s-}^{n+1})^+ \Delta_s(Y^n - Y^{n+1})\},\end{aligned}$$

is continuous non-decreasing ([4], pp.349). Here $\Delta_s(Y^n - Y^{n+1}) = (Y_s^n - Y_s^{n+1}) - (Y_{s-}^n - Y_{s-}^{n+1})$, $\forall s \leq 1$. Henceforth for any $t \leq 1$ we have,

$$\begin{aligned}&(Y_t^n - Y_t^{n+1})^{+2} + \sum_{t<s\leq 1} \{(Y_s^n - Y_s^{n+1})^{+2} - (Y_{s-}^n - Y_{s-}^{n+1})^{+2} \\ &\quad -2(Y_{s-}^n - Y_{s-}^{n+1})^+ \Delta_s(Y^n - Y^{n+1})\} \leq 2\int_{]t,1]} (Y_{s-}^n - Y_{s-}^{n+1})^+ \{d(K_s^n - K_s^{n+1}) \\ &\qquad -(Z_s^n - Z_s^{n+1})dB_s - \int_U (V_s^n(e) - V_s^{n+1}(e))\tilde{\mu}(ds, de)\}.\end{aligned}$$

where for any $n \geq 0$ and $t \leq 1$, $K_t^n := n\int_0^t (Y_s^n - S_s)^- ds$. But
$\sum_{t<s\leq 1} \{(Y_s^n - Y_s^{n+1})^{+2} - (Y_{s-}^n - Y_{s-}^{n+1})^{+2} - 2(Y_{s-}^n - Y_{s-}^{n+1})^+ \Delta_s(Y^n - Y^{n+1})\} \geq 0$
since $(y^+)^2 - (x^+)^2 - 2x^+(y-x) \geq 0, \forall x, y \in I\!\!R$. Then

$$\begin{aligned}(Y_t^n - Y_t^{n+1})^{+2} \leq 2\int_{]t,1]} (Y_{s-}^n - Y_{s-}^{n+1})^+ \{d(K_s^n - K_s^{n+1}) - (Z_s^n - Z_s^{n+1})dB_s \\ - \int_U (V_s^n(e) - V_s^{n+1}(e))\tilde{\mu}(ds, de)\}.\end{aligned} \tag{3}$$

Through estimates (2) we deduce that $\int_0^{\cdot} (Y_{s-}^n - Y_{s-}^{n+1})^+ \{(Z_s^n - Z_s^{n+1})dB_s +$
$\int_U (V_s^n(e) - V_s^{n+1}(e))\tilde{\mu}(ds, de)\}$ is an $(\mathcal{F}_t, P)$-martingale. Now taking expectation in both sides of (3) yields

$$\begin{aligned}I\!\!E[(Y_t^n - Y_t^{n+1})^{+2}] &\leq 2I\!\!E[\int_{]t,1]} (Y_{s-}^n - Y_{s-}^{n+1})^+ d(K_s^n - K_s^{n+1})] \\ &\leq 2I\!\!E[\int_{]t,1]} (Y_s^n - Y_s^{n+1})^+ [n(Y_s^n - S_s)^- - n(Y_s^{n+1} - S_s)^-]ds \\ &\leq 2nI\!\!E[\int_{]t,1]} (Y_s^n - Y_s^{n+1})^{+2} ds]\end{aligned}$$

since the function $y \longmapsto n(y - S_t)^-$ is Lipschitz. Finally Gronwall's inequality implies, for any $t \leq 1$, $Y_t^n \leq Y_t^{n+1}$, P-*a.s.* and then $Y^n \leq Y^{n+1}$ since Y^n and Y^{n+1} are right continuous processes.

Step 2: There exists a constant $C \geq 0$ such that

$$\forall n \geq 0 \text{ and } t \leq 1,\ I\!\!E[|Y_t^n|^2 + \int_0^1 |Z_s^n|^2 ds + \int_0^1 ds \int_U (V_s^n(e))^2 \lambda(de) + (K_1^n)^2] \leq C. \tag{4}$$

Indeed by Itô's rule we obtain,

$$\begin{aligned}Y_t^{n2} &+\int_t^1 |Z_s^n|^2ds+\int_{]t,1]}ds\int_U (V_s^n(e))^2\lambda(de)+\sum_{t<s\leq 1}(\Delta_s Y^n)^2=\xi^2+2\int_{]t,1]}Y_s^n g(s)ds\\ &+2\int_{]t,1]}nY_s^n(Y_s^n-S_s)^-ds-2\int_{]t,1]}Y_{s-}^n Z_s^n dB_s-2\int_{]t,1]}Y_{s-}^n\int_U V_s^n(e)\tilde{\mu}(ds,de), t\leq 1.\end{aligned}$$

Taking the expectation in both sides yields,

$$\begin{aligned}&I\!\!E[|Y_t^n|^2+\int_t^1|Z_s^n|^2ds+\int_{]t,1]}ds\int_U(V_s^n(e))^2\lambda(de)]\\ &\leq I\!\!E[\xi^2]+2I\!\!E[\int_{]t,1]}Y_s^n g(s)ds]+2I\!\!E[\int_{]t,1]}nY_s^n(Y_s^n-S_s)^-ds]\\ &\leq I\!\!E[\xi^2]+I\!\!E[\int_{]t,1]}(Y_s^n)^2ds]+I\!\!E[\int_{]t,1]}(g(s))^2ds]+\epsilon^{-1}I\!\!E[\sup_{t\leq s\leq 1}(S_s^+)^2]+\epsilon I\!\!E[(K_1^n-K_t^n)^2];\end{aligned}$$

ϵ is a universal non-negative real constant. But for any $t\leq 1$ we have,

$$\begin{aligned}I\!\!E[(K_1^n-K_t^n)^2] &\leq C\{I\!\!E[\xi^2+|Y_t^n|^2+(\int_t^1|g(s)|ds)^2+(\int_t^1 Z_s^n dB_s)^2\\ &\quad+(\int_{]t,1]}\int_U V_s^n(e)\tilde{\mu}(ds,de))^2]\}\\ &\leq CI\!\!E[\xi^2+|Y_t^n|^2+(\int_t^1|g(s)|ds)^2+\int_t^1|Z_s^n|^2ds)\\ &\quad+\int_{]t,1]}ds\int_U|V_s^n(e)|^2\lambda(de)]\end{aligned}$$

where C is a constant. Now plugging this inequality in the previous one yields,

$$\begin{aligned}&I\!\!E[|Y_t^n|^2+\int_t^1|Z_s^n|^2ds+\int_{]t,1]}ds\int_U(V_s^n(e))^2\lambda(de)]\leq\\ &(1+\epsilon C)I\!\!E[\xi^2]+\epsilon CI\!\!E[|Y_t^n|^2]+I\!\!E[\int_{]t,1]}(Y_s^n)^2ds]+(1+\epsilon C)I\!\!E[\int_{]t,1]}(g(s))^2ds]\\ &+\epsilon^{-1}I\!\!E[\sup_{t\leq s\leq 1}(S_s^+)^2]+\epsilon CI\!\!E[\int_t^1|Z_s^n|^2ds)+\int_{]t,1]}ds\int_U(V_s^n(e))^2\lambda(de)], t\leq 1.\end{aligned}$$

Choosing $\epsilon C=1/2$ yields

$$I\!\!E[|Y_t^n|^2+\int_t^1|Z_s^n|^2ds+\int_{]t,1]}ds\int_U(V_s^n(e))^2\lambda(de)]\leq\bar{C}(1+I\!\!E[\int_t^1(Y_s^n)^2ds]), t\leq 1,$$

where $\bar{C}$ is an appropriate real constant. Now Gronwall's inequality leads to the desired result for $I\!\!E[|Y_t^n|^2]$ and then also for $I\!\!E[\int_0^1|Z_s^n|^2ds]$, $I\!\!E[\int_0^1 ds\int_U(V_s^n(e))^2\lambda(de)]$ and $I\!\!E[(K_1^n)^2]$.

Step 3: There exists a constant $C\geq 0$ such that for any $n\geq 0$ we have $I\!\!E[\sup_{0\leq t\leq 1}|Y_t^n|^2]\leq C$. In addition there exists an $\mathcal{F}_t$-adapted process $(Y_t)_{t\leq 1}$ such that $I\!\!E[\int_0^1|Y_s^n-Y_s|^2ds]\to 0$ as $n\to\infty$.

Indeed for $n\geq 0$, using once again Itô's formula we obtain,

$$\begin{aligned}Y_t^{n2} &+\int_t^1 |Z_s^n|^2ds+\int_{]t,1]}ds\int_U (V_s^n(e))^2\lambda(de)+\sum_{t<s\leq 1}(\Delta_s Y^n)^2=\xi^2+2\int_{]t,1]}Y_s^n g(s)ds\\ &+2\int_{]t,1]}nY_s^n(Y_s^n-S_s)^-ds-2\int_{]t,1]}Y_{s-}^n Z_s^n dB_s-2\int_{]t,1]}Y_{s-}^n\int_U V_s^n(e)\tilde{\mu}(ds,de), t\leq 1.\end{aligned}\tag{5}$$

But $|\int_t^1 Y_s^n g(s)ds| \leq \int_t^1 \{c_1|Y_s^n|^2 + c_1^{-1}|g(s)|^2\}ds$, $\int_t^1 Y_s^n dK_s^n \leq c_2 \sup_{t\leq s\leq 1} |S_s^+|^2 + c_2^{-1}(K_1^n - K_t^n)^2$. On the other hand using the Burkholder-Davis-Gundy inequality ([4],p.304) we get,

$$I\!\!E[\sup_{t\leq s\leq 1} |\int_{]s,1]} Y_{r-}^n Z_r^n dB_r|] \leq c_3 I\!\!E[\sup_{t\leq s\leq 1} |Y_s^n|^2] + c_3^{-1} I\!\!E[\int_t^1 |Z_r^n|^2 dr]$$

and

$$\begin{aligned} I\!\!E[\sup_{t\leq s\leq 1} |\int_{]s,1]} \int_U Y_{r-}^n V_r^n(e)\tilde{\mu}(dr,de)|] &\leq C I\!\!E[\{\int_{]t,1]} dr \int_U |Y_{r-}^n V_r^n(e)|^2 \lambda(de)\}^{1/2}] \\ &\leq c_4 I\!\!E[\sup_{t\leq s\leq 1} |Y_s^n|^2] + \tfrac{1}{c_4} I\!\!E[\int_t^1 ds \int_U (V_s^n(e))^2 \lambda(de)]. \end{aligned}$$

Here c_1, c_2, c_3 and c_4 are universal non-negative real constants. Now combining these inequalities with (5) yields,

$$\begin{aligned} I\!\!E[\quad & \sup_{t\leq s\leq 1} |Y_s^n|^2 + \int_t^1 |Z_s^n|^2 ds + \int_{]t,1]} ds \int_U (V_s^n(e))^2 \lambda(de)] \\ & \leq I\!\!E[\xi^2] + 2I\!\!E[\int_{]t,1]} \{c_1|Y_s^n|^2 + c_1^{-1}|g(s)|^2\}ds + 2c_2 I\!\!E[\sup_{t\leq s\leq 1} |S_s^+|^2] \\ & \qquad +2c_2^{-1} I\!\!E[(K_1^n - K_t^n)^2] + 2c_3 I\!\!E[\sup_{t\leq s\leq 1} |Y_r^n|^2] + 2c_3^{-1} I\!\!E[\int_t^1 |Z_r^n|^2 dr] \\ & \qquad +2c_4 I\!\!E[\sup_{t\leq s\leq 1} |Y_s^n|^2] + 2c_4^{-1} I\!\!E[\int_t^1 ds \int_U (V_s^n(e))^2 \lambda(de)], \forall t \leq 1. \end{aligned}$$

Finally choosing $2(c_3 + c_4) < 1$ we obtain $I\!\!E[\sup_{t\leq 1} |Y_s^n|^2] \leq C$.

Now let $Y_t = \liminf_{n\to\infty} Y_t^n$, $t \leq 1$. Since the sequence $(Y^n)_{n\geq 0}$ is non-decreasing then, using Fatou's lemma, $I\!\!E[Y_t^1] \leq I\!\!E[Y_t] \leq \liminf_{n\to\infty} I\!\!E[Y_t^n] \leq C$. It follows that for any $t \leq 1$, $Y_t < \infty$ and then P-*a.s.*, $Y_t^n \to Y_t$ as $n \to \infty$. In addition the Lebesgue's dominated convergence theorem implies that $I\!\!E[\int_0^1 |Y_s^n - Y_s|^2 ds] \to 0$ as $n \to \infty$.

Step 4: $\lim_{n\to\infty} I\!\!E[\sup_{t\leq 1} |(Y_t^n - S_t)^-|^2] = 0$. This property is the key point in the proof of our result.

Let $\left(\bar{Y}_t^n, \bar{Z}_t^n, \bar{V}_t^n\right)_{t\leq 1}$ be the solution of the following BSDE :

$$\bar{Y}_t^n = \xi + \int_t^1 \{g(s) - n(\bar{Y}_s^n - S_s)\}ds - \int_t^1 \bar{Z}_s^n dB_s - \int_t^1 \int_U \bar{V}_s^n(e)\,\tilde{\mu}(ds,de)$$

By comparison we have P-*a.s.*, $\forall t \leq 1$, $Y_t^n \geq \bar{Y}_t^n$, for any $n \geq 0$ (the proof is similar to the one we have done to prove $Y^n \leq Y^{n+1}$ in *Step* 1). Now let τ be an $\mathcal{F}_t$-stopping time such that $\tau \leq 1$. Then,

$$\bar{Y}_\tau^n = E[\xi \exp -n(1-\tau) + \int_\tau^1 (g(s) + nS_s) \exp -n(s-\tau)\,ds\,|\mathcal{F}_\tau].$$

Since S is a right continuous then

$$\xi \exp -n(1-\tau) + n\int_\tau^1 S_s \exp -n(s-\tau)\,ds \to \xi 1_{[\tau=1]} + S_\tau 1_{[\tau<1]} \text{ as } n \to \infty$$

P$-a.s.$ and in $L^2(\Omega, P)$. Henceforth we have also the convergence of the conditional expectation in $L^2(\Omega, P)$. In addition

$$\left|\int_\tau^1 g(s) exp\{-n\,(s-\tau)\} ds\right| \leq \frac{1}{\sqrt{n}} \left(\int_\tau^1 g^2(s) ds\right)^{\frac{1}{2}}$$

then

$$\int_\tau^1 g(s) exp - n\,(s-\tau)\, ds \longrightarrow 0 \text{ in } L^2(\Omega, P) \text{ as } n \to \infty.$$

Consequently

$$\bar{Y}_\tau^n \longrightarrow \xi 1_{[\tau=1]} + S_\tau 1_{[\tau<1]} \text{ in } L^2(\Omega, P) \text{ as } \quad n \to \infty.$$

Therefore $Y_\tau \geq S_\tau\ P-a.s.$ From that and the section theorem ([4], p.220), we deduce that $Y_t \geq S_t,\ \forall t \leq 1,\ P-a.s.$ and then $(Y_t^n - S_t)^- \searrow 0,\ \forall t \leq 1$, P-$a.s.$

Now since $Y^n \nearrow Y$ then, if we denote by ${}^pX$ the predictable projection of any process X, ${}^pY^n \nearrow {}^pY$ and ${}^pY \geq {}^pS$. But for any n the jumping times of the process $(\int_0^t \int_U \bar{V}_s^n(e)\, \tilde{\mu}\,(ds, de))_{0\leq t\leq 1}$ are inaccessible since μ is a Poisson random measure. It follows that the jumping times of Y^n are also inaccessible. Then for any predictable stopping time δ we have $Y_\delta^n = Y_{\delta-}^n$, henceforth the predictable projection of Y^n is Y_-^n, i.e., ${}^pY^n = Y_-^n$. In the same way we have ${}^pS = S_-$ since we have supposed the jumping times of S inaccessible.

So we have proved that ${}^pY^n \nearrow {}^pY \geq {}^pS$, i.e., $Y_-^n \nearrow {}^pY \geq S_-$, hence $Y_-^n - S_- \nearrow {}^pY - S_- \geq 0$. It follows that $\left(Y_{t-}^n - S_{t-}\right)^- \searrow 0, \forall t \leq 1$ P-$a.s.$ as $n \to \infty$. Consequently, from a weak version of the Dini's theorem ([4], p.202), we deduce that $\sup_{t\leq 1} (Y_t^n - S_t)^- \searrow 0$ P-$a.s.$ as $n \to \infty$. Therefore the dominated convergence theorem implies

$$E[\sup_{t\leq 1} |(Y_t^n - S_t)^-|^2] \longrightarrow 0\, a.s. \text{ as } n \to \infty$$

since for any $n \geq 0$, $Y_t^1 - S_t^+ \leq Y_t^n - S_t$ and then $(Y_t^n - S_t)^- \leq \left|Y_t^1\right| + S_t^+$.

Step 5: $\lim_{n\to\infty} I\!E[\sup_{t\leq 1} |Y_t^n - Y_t|^2] = 0$ and there exist $\mathcal{F}_t$-adapted processes $Z = (Z_t)_{t\leq 1}$, $K = (K_t)_{t\leq 1}$ (K non-decreasing and $K_0 = 0$) and $V = (V_t)_{t\leq 1}$ such that

$$I\!E[\int_0^1 |Z_s^n - Z_s|^2 ds + \sup_{t\leq 1} |K_t^n - K_t|^2 + \int_0^1 ds \int_U |V_s^n(e) - V_s^p(e)|^2 \lambda(de)] \to \infty \text{ as } n \to \infty.$$

Indeed using Itô's formula we have for any $p \geq n \geq 0$ and $t \leq 1$,

$$\begin{aligned}(Y_t^n - Y_t^p)^2 \quad &+ \int_t^1 |Z_s^n - Z_s^p|^2 ds + \int_t^1 ds \int_U |V_s^n(e) - V_s^p(e)|^2 \lambda(de) + \sum_{t<s\leq 1} \Delta_s (Y^n - Y^p)^2 \\ &= 2\int_t^1 (Y_s^n - Y_s^p)(dK_s^n - dK_s^p) - 2\int_t^1 (Y_{s-}^n - Y_{s-}^p)(Z_s^n - Z_s^p) dB_s \\ &-2\int_t^1 \int_U ds (Y_{s-}^n - Y_{s-}^p)(V_s^n(e) - V_s^p(e))\tilde{\mu}(ds, de).\end{aligned} \tag{6}$$

So since $p \geq n$, then $\displaystyle\int_t^1 (Y_s^n - Y_s^p)(dK_s^n - dK_s^p) \leq -\int_t^1 (Y_s^n - S_s) dK_s^p \leq \sup_{t\leq 1} (Y_s^n - S_s)^- K_1^p$. Therefore taking expectation in (6) and using the results of *Step* 2 and *Step* 4, yields

$$I\!E[\int_t^1 |Z_s^n - Z_s^p|^2 ds + \int_0^1 ds \int_U |V_s^n(e) - V_s^p(e)|^2 \lambda(de)] \leq 2I\!E[\sup_{t\leq 1} (Y_s^n - S_s)^- K_1^p] \to 0 \text{ as } n \to \infty.$$

It follows that $(Z^n)_{n\geq 0}$ and $(V^n)_{n\geq 0}$ are Cauchy sequences in complete spaces then there exist processes Z and V, respectively $\mathcal{F}_t$-progressively measurable and $\mathcal{P}\otimes\mathcal{U}$-measurable such that the sequences $(Z^n)_{n\geq 0}$ and $(V^n)_{n\geq 0}$ converge respectively toward Z and V in $L^2(dP\otimes dt)$ and $L^2(dP\otimes dt\lambda(de))$ respectively.

Now going back to (6), taking first the supremum then the expectation and using the Burkholder-Davis-Gundy inequality ([4], p.304) yields,

$$\begin{aligned}
I\!\!E[\ & \sup_{t\leq s\leq 1}(Y_s^n-Y_s^p)^2+\int_t^1|Z_s^n-Z_s^p|^2ds+\int_0^1 ds\int_U|V_s^n(e)-V_s^p(e)|^2\lambda(de)]\\
&\leq 2I\!\!E[\sup_{t\leq s\leq 1}(Y_s^n-S_s)^-.K_1^p]+2\alpha I\!\!E[\sup_{t\leq s\leq 1}|Y_s^n-Y_s^p|^2]+\\
&\quad \alpha^{-1}I\!\!E[\int_t^1|Z_s^n-Z_s^p|^2ds]+\alpha^{-1}I\!\!E[\int_t^1\int_U ds|V_s^n(e)-V_s^p(e)|^2\lambda(de)],t\leq 1,
\end{aligned}$$

where α is a universal real non-negative constant. Henceforth choosing $\alpha<1/2$ implies that $I\!\!E[\sup_{0\leq s\leq 1}(Y_s^n-Y_s^p)^2]\to 0$ as $p,n\to\infty$ and then $I\!\!E[\sup_{0\leq s\leq 1}(Y_s^n-Y_s)^2]\to 0$ as $n\to\infty$, moreover $Y=(Y_t)_{t\leq 1}$ is an $\mathcal{F}_t$-adapted *rcll* process.

Finally since for any $n\geq 0$ and $t\leq 1$,

$$K_t^n=Y_0^n-Y_t^n-\int_0^1 g(s)ds+\int_0^t Z_s^n dB_s+\int_0^t\int_U V_s^n(e)\tilde{\mu}(ds,de),$$

then we have also, $I\!\!E[\sup_{0\leq s\leq 1}|K_s^n-K_s^p|^2]\to 0$ as $n,p\to\infty$. Hence there exists an $\mathcal{F}_t$-adapted non-decreasing and continuous process $(K_t)_{t\leq 1}$ $(K_0=0)$ such that $I\!\!E[\sup_{0\leq s\leq 1}|K_s^n-K_s|^2]\to 0$ as $n\to\infty$.

Step 6: The limiting process $(Y,Z,K,V)=(Y_t,Z_t,K_t,V_t)_{t\leq 1}$ is the solution of the reflected DBSDE associated with (g,ξ,S).

Obviously the process (Y,Z,K,V) satisfies

$$Y_t=\xi+\int_t^1 g(s)ds+K_1-K_t-\int_t^1 Z_s dB_s-\int_t^1 ds\int_U V_s(e)\tilde{\mu}(ds,de),\forall t\leq 1.$$

On the other hand since $\lim_{n\to\infty}I\!\!E[\sup_{t\leq 1}((Y_t^n-S_t)^-)^2]=0$ then P-$a.s.$, $\forall t\leq 1$, $Y_t\geq S_t$. Finally let us prove $\int_0^1(Y_s-S_s)dK_s=0$.

First there exists a subsequence of $(K^n)_{n\geq 0}$ which we still denote $(K^n)_{n\geq 0}$ such that P$-a.s.$ $\lim_{n\to\infty}\sup_{t\leq 1}|K_t^n-K_t|=0$. Now let ω be fixed. Since the function $Y(\omega)-S(\omega):t\in[0,1]\longmapsto Y_t(\omega)-S_t(\omega)$ is *rcll* then there exists a sequence of step functions $(f^m(\omega))_{m\geq 0}$ which converges uniformly on $[0,1]$ to $Y(\omega)-S(\omega)$. Now

$$\int_0^1(Y_s-S_s)dK_s=\int_0^1(Y_s-S_s)d(K_s-K_s^n)+\int_0^1(Y_s-S_s)dK_s^n. \tag{7}$$

On the other hand the result stated in *Step* 5 implies, for any $\epsilon>0$, there exists $n_0(\omega)$ such that for any $n\geq n_0(\omega)$, $\forall t\leq 1$, $Y_t(\omega)-S_t(\omega)\leq Y_t^n(\omega)-S_t(\omega)+\epsilon$ and $K_1^n(\omega)\leq K_1(\omega)+\epsilon$. Therefore for $n\geq n_0(\omega)$ we have

$$\int_0^1(Y_s-S_s)dK_s^n\leq\epsilon K_1(\omega)+\epsilon^2 \tag{8}$$

since

$$\int_0^1 (Y^n_s - S_s)\, dK^n_s = -n \int_0^1 \left((Y^n_s - S_s)^-\right)^2 ds \leq 0.$$

Now there exists $m_0(\omega) \geq 0$ such that for $m \geq m_0(\omega)$ we have $\forall t \leq 1$, $|Y_t(\omega) - S_t(\omega) - f^m_t(\omega)| < \epsilon$. It follows that

$$\begin{aligned}\int_0^1 (Y_s - S_s)d(K_s - K^n_s) &= \int_0^1 (Y_s - S_s - f^m_s(\omega))d(K_s - K^n_s) + \int_0^1 f^m_s(\omega)d(K_s - K^n_s)\\ &\leq \int_0^1 f^m_s(\omega)d(K_s - K^n_s) + \epsilon(K_1(\omega) + K^n_1(\omega)).\end{aligned}$$

But the right-hand side converge to $2\epsilon K_1(\omega)$, as $n \to \infty$, since $f^m(\omega)$ is a step function and then $\int_0^1 f^m_s(\omega)d(K_s - K^n_s) \to 0$. Therefore we have

$$\limsup_{n\to\infty} \int_0^1 (Y_s - S_s)d(K_s - K^n_s) \leq 2\epsilon K_1(\omega). \tag{9}$$

Finally from (7), (8) and (9) we deduce that

$$\int_0^1 (Y_s - S_s)dK_s \leq 3\epsilon K_1(\omega) + \epsilon^2.$$

As ϵ is whatever and $Y \geq S$ then

$$\int_0^1 (Y_s - S_s)dK_s = 0.$$

The other properties are satisfied by construction of the quadruple of processes (Y, Z, K, V) and the proof is complete $\Box$.

We are now ready to give the main result of this section.

1.2.b. Theorem: The reflected BSDE with jumps (1) associated with (f, ξ, S) has a unique solution (Y, Z, K, V).

Proof: It remains to show the existence which will be obtained via a fixed point of the contraction of the function Φ defined as follows:

Let $\mathcal{D} := \mathcal{S}^2 \times H^{2,d} \times \mathcal{L}^2$ endowed with the norm,

$$\|(Y, Z, V)\|_\alpha = \{I\!E[\int_0^1 e^{\alpha s}(|Y_s|^2 + |Z_s|^2 + \int_U |V_s(e)|^2\lambda(de))ds]\}^{1/2};\ \alpha > 0.$$

Let Φ be the map from $\mathcal{D}$ into itself which with (Y, Z, V) associates $\Phi(Y, Z, V) = (\tilde{Y}, \tilde{Z}, \tilde{V})$ where $(\tilde{Y}, \tilde{Z}, \tilde{K}, \tilde{V})$ is the solution of the reflected DBSDE associated with $(f(t, Y_t, Z_t, V_t), \xi, S)$. Let (Y', Z', V') be another triple of $\mathcal{D}$ and $\Phi(Y', Z', V') = (\tilde{Y}', \tilde{Z}', \tilde{V}')$, then using Itô's formula we obtain, for any $t \leq 1$,

$$\begin{aligned}&e^{\alpha t}(\tilde{Y}_t - \tilde{Y}'_t)^2 + \alpha\int_t^1 e^{\alpha s}(\tilde{Y}_s - \tilde{Y}'_s)^2 ds + \int_t^1 e^{\alpha s}|\tilde{Z}_s - \tilde{Z}'_s|^2 ds +\\ &\int_t^1 e^{\alpha s}ds\int_U (\tilde{V}_s(e) - \tilde{V}'_s(e))^2\lambda(de) + \sum_{t<s\leq 1} e^{\alpha s}(\Delta_s\tilde{Y} - \Delta_s\tilde{Y}')^2 = (M_1 - M_t) +\\ &2\int_t^1 e^{\alpha s}(\tilde{Y}_s - \tilde{Y}'_s)(d\tilde{K}_s - d\tilde{K}'_s) + 2\int_t^1 e^{\alpha s}(\tilde{Y}_s - \tilde{Y}'_s)(f(s, Y_s, Z_s, V_s) - f(s, Y'_s, Z'_s, V'_s))ds\end{aligned}$$

where $(M_t)_{t\leq 1}$ is a martingale. But $\int_t^1 e^{\alpha s}(\tilde{Y}_s - \tilde{Y}'_s)(d\tilde{K}_s - d\tilde{K}'_s) \leq 0$ then

$$\begin{aligned}
&\alpha \mathbb{E}[\int_t^1 e^{\alpha s}(\tilde{Y}_s - \tilde{Y}'_s)^2 ds] + \mathbb{E}[\int_t^1 e^{\alpha s}|\tilde{Z}_s - \tilde{Z}'_s|^2 ds] + \mathbb{E}[\int_t^1 e^{\alpha s} ds \int_U (\tilde{V}_s(e) - \tilde{V}'_s(e))^2 \lambda(de)] \\
&\leq 2\mathbb{E}[\int_t^1 e^{\alpha s}(\tilde{Y}_s - \tilde{Y}'_s)(f(s, Y_s, Z_s, V_s) - f(s, Y'_s, Z'_s, V'_s))ds] \\
&\leq k\epsilon \mathbb{E}[\int_t^1 e^{\alpha s}(\tilde{Y}_s - \tilde{Y}'_s)^2 ds] + \frac{k}{\epsilon}\mathbb{E}[\int_t^1 e^{\alpha s}\{|Y_s - Y'_s|^2 + |Z_s - Z'_s|^2 + \\
&\qquad \int_U |V_s(e) - V'_s(e)|^2 \lambda(de)\} ds].
\end{aligned}$$

It implies that

$$\begin{aligned}
&(\alpha - k\epsilon)\mathbb{E}[\int_t^1 e^{\alpha s}(\tilde{Y}_s - \tilde{Y}'_s)^2 ds] + \mathbb{E}[\int_t^1 e^{\alpha s}(\tilde{Z}_s - \tilde{Z}'_s)^2 ds] + \\
&\qquad \mathbb{E}[\int_t^1 e^{\alpha s} ds \int_U (\tilde{V}_s(e) - \tilde{V}'_s(e))^2 \lambda(de)] \leq \\
&\quad \tfrac{k}{\epsilon}\mathbb{E}[\int_t^1 e^{\alpha s}\{|Y_s - Y'_s|^2 + |Z_s - Z'_s|^2 + \int_U |V_s(e) - V'_s(e)|^2 \lambda(de)\} ds].
\end{aligned}$$

Now let α great enough and ϵ such that $k < \epsilon < \frac{\alpha - 1}{k}$, then Φ is a contraction on $\mathcal{D}$, henceforth there exists a triple (Y, Z, V) such that $\Phi(Y, Z, V) = (Y, Z, V)$ which, with K, is the unique solution of the reflected DBSDE associated with (f, ξ, S) $\Box$

2.3 Regularity of the process K.

We now focus on the regularity of the process K. We are going to show that the process $K = (K_t)_{t\leq 1}$ is absolutely continuous if the barrier $S = (S_t)_{t\leq 1}$ is regular. Precisely we have :

1.3.a. Proposition: Assume the barrier $S = (S_t)_{t\leq 1}$ satisfies:

$$P - a.s. \ S_t = H^{2,1} - \lim_{m\to\infty} S_t^m$$

where for any $m \geq 0$, $(S_t^m)_{t\leq 1}$ is a semimartingale which satisfies

$$S_t^m = S_0^m + \int_0^t l_s^m dB_s + \int_0^t \int_U w_s^m(e)\tilde{\mu}(ds, de) + \int_0^t a_s^m ds, t \leq 1,$$

with

$$\mathbb{E}[\int_0^1 \{(l_s^m)^2 ds + \int_U (w_s^m(e))^2 \lambda(de)\} ds] < +\infty \ , \forall m \in \mathbb{N},$$

and

$$\sup_{m\geq 0} \mathbb{E}[\int_0^1 |(a_s^m)^-|^2 ds] < \infty.$$

In addition $(S_1^m)_{m\geq 0}$ converges to S_1 in $L^2(\Omega, dP)$. Then the process K of the solution of the reflected DBSDE associated with (f, ξ, S) is absolutely continuous with respect to the Lebesgue measure dt.

Proof: Let (Y, Z, K, V) be the solution of the reflected DBSDE associated with (f, ξ, S) and for $n \geq 0$, let (Y^n, Z^n, V^n) be the solution of the following standard BSDE :

$$Y_t^n = \xi + \int_t^1 \{f(s, Y_s, Z_s, V_s) + n(Y_s^n - S_s)^-\}ds - \int_t^1 Z_s^n dB_s - \int_t^1 \int_U V_s^n(e)\tilde{\mu}(ds, de), t \leq 1.$$

Since the solution of the reflected DBSDE associated with (f, ξ, S) is unique then, as it has been shown in Thm.1.2.a, the sequence $((Y^n, Z^n, \int_0^\cdot n(Y_s^n - S_s)^- ds), V^n))_{n\geq 0}$ converges toward (Y, Z, K, V) in $\mathcal{S}^2 \times H^{2,d} \times \mathcal{S}^2 \times \mathcal{L}^2$.

Now using the generalized Itô's formula with the convex function $x \longmapsto x^{-2}$ and the process $Y^n - S^m$ implies that $A_t^{n,m}$, defined below, is non-decreasing in t ;

$$\begin{aligned} A_t^{n,m} &= (Y_t^n - S_t^m)^{-2} - (Y_0^n - S_0^m)^{-2} + \int_{]0,t]} (Y_{s-}^n - S_{s-}^m)^- d(Y_s^n - S_s^m) \\ &- \textstyle\sum_{0<s\leq t}\{(Y_s^n - S_s^m)^{-2} - (Y_{s-}^n - S_{s-}^m)^{-2} + 2(Y_{s-}^n - S_{s-}^m)^- \Delta_s(Y^n - S^m)\}. \end{aligned}$$

Then for any $t \leq 1$ we have $A_1^{n,m} - A_t^{n,m} \geq 0$ which yields,

$$\begin{aligned} (Y_t^n - S_t^m)^{-2} &+ \textstyle\sum_{t<s\leq 1}\{(Y_s^n - S_s^m)^{-2} - (Y_{s-}^n - S_{s-}^m)^2 + 2(Y_{s-}^n - S_{s-}^m)^- \Delta_s(Y^n - S^m)\} \\ &\leq (\xi - S_1^m)^{-2} + 2\int_{]t,1]} (Y_{s-}^n - S_{s-}^m)^- d(Y_s^n - S_s^m) \\ &\leq (\xi - S_1^m)^{-2} + 2\int_{]t,1]} (Y_{s-}^n - S_{s-}^m)^- \{-f(s, Y_s, V_s) - n(Y_s^n - S_s)^- - a_s^m\}ds \\ &\quad +2\int_t^1 (Y_{s-}^n - S_{s-}^m)^- \{(Z_s^n - l_s^m)dB_s + \int_U (V_s^n(e) + w_s^m(e))\tilde{\mu}(ds, de)\}. \end{aligned}$$

Now since $(y^-)^2 - (x^-)^2 + 2x^-(y - x) \geq 0, \forall x, y \in I\!R$ then

$$\sum_{t<s\leq 1} \{(Y_s^n - S_s^m)^{-2} - (Y_{s-}^n - S_{s-}^m)^{-2} + 2(Y_{s-}^n - S_{s-}^m)^- \Delta_s(Y^n - S^m)\} \geq 0.$$

Taking the expectation in both sides above yields, for any $t \leq 1$,

$$I\!E[(Y_t^n - S_t^m)^{-2}] \leq I\!E[(\xi - S_1^m)^{-2}] - 2I\!E[\int_t^1 (Y_s^n - S_s^m)^- \{f(s, Y_s, Z_s, V_s) + n(Y_s^n - S_s)^- + a_s^m\}ds].$$

Then

$$\begin{aligned} I\!E[(Y_t^n - S_t^m)^{-2}] &\leq I\!E[(\xi - S_1^m)^{-2}] - 2nI\!E[\int_t^1 (Y_s^n - S_s^m)^-(Y_s^n - S_s)^- ds] \\ &\quad + \tfrac{1}{\epsilon^2} I\!E[\int_0^1 (Y_s^n - S_s^m)^{-2}]ds + \epsilon^2 C. \end{aligned}$$

It follows that

$$2nI\!E[\int_0^1 (Y_s^n - S_s^m)^-(Y_s^n - S_s)^- ds] \leq I\!E[(\xi - S_1^m)^{-2}] + \frac{1}{\epsilon^2} I\!E[\int_0^1 (Y_s^n - S_s^m)^{-2} ds] + \epsilon^2 C.$$

Now since the sequence of processes $(S^m)_{m\geq 0}$ converges to S in $H^{2,1}$ and $S_1 = L^2 - \lim_{m\to\infty} S_1^m$ then taking the limit in the previous inequality as $m \to \infty$ yields,

$$2nI\!E[\int_0^1 (Y_s^n - S_s)^{-2} ds] \leq \frac{1}{\epsilon^2} I\!E[\int_0^1 (Y_s^n - S_s)^{-2}]ds + \epsilon^2 C,$$

and choosing $\epsilon = n^{-\frac{1}{2}}$ implies

$$\mathbb{E}[\int_0^1 (Y_s^n - S_s)^{-2} ds] \leq \frac{C}{n^2}. \tag{10}$$

But the inequality (10) can be written as

$$\sup_{n\in\mathbb{N}^*} \mathbb{E}[\|K^n\|_{H^1(0,1;\mathbb{R}^d)}] < \infty$$

where $K_t^n = n\int_0^t (Y_s^n - S_s)^- ds, t \leq 1$, and $H^1(0,1;\mathbb{R}^d)$ is the usual Sobolev space consisting of all absolutely continuous functions with derivative in $L^2(0,1)$. Hence the sequence $(K^n)_n$ is bounded in the Hilbert space $L^2(\Omega; H^1(0,1;\mathbb{R}^d))$ and then there exists a subsequence of $(K^n)_n$ which converges weakly. The limiting process, which is actually K, belongs to $L^2(\Omega; H^1(0,1;\mathbb{R}^d))$ and then P-*a.s.*, $K_.(\omega) \in H^1(0,1;\mathbb{R}^d)$ i.e. K is absolutely continuous with respect to Lebesgue measure dt $\Box$

2.4 The Snell envelope method.

The aim of this part is to give another proof of the existence and uniqueness result using the so called Snell envelope of processes (see El Karoui et al. [6] for the continuous case). However as it is pointed out in the beginning of Section 1.2, first we assume the function f does not depend on (y,z,v), i.e. $f(t,y,z,v) = g(t)$, then we have the following result.

1.4.a. Proposition: There exists a process $(Y_t, Z_t, V_t, K_t)_{t\leq 1}$ solution of the reflected DBSDE associated with (g, ξ, S).

Proof: Let $\eta := (\eta_t)_{t\leq 1}$ be the process defined as follows:

$$\eta_t = \xi 1_{\{t=1\}} + S_t 1_{\{t<1\}} + \int_0^t g(s)ds,$$

then η is *rcll* and its jumping times τ before 1 are the same as the ones of S and then they are inaccessible since those of this latter process are so. Moreover

$$\sup_{0\leq t\leq 1} |\eta_t| \in L^2(\Omega). \tag{11}$$

The Snell envelope of η is the smallest *rcll* supermartingale which dominates the process η, it is given by :

$$\mathcal{S}_t(\eta) = ess \sup_{\nu\in\mathcal{T}_t} \mathbb{E}[\eta_\nu | F_t]$$

where $\mathcal{T}_t$ is the set of stopping time ν such that $t \leq \nu \leq 1$ *a.s.* Now due to (11), we have $\mathbb{E}[\sup_{t\leq 1} |\mathcal{S}_t|^2] < \infty$ and then $(\mathcal{S}_t(\eta))_{t\leq 1}$ is of class [D], i.e. the set of random variables $\{\mathcal{S}_\tau(\eta), \tau \in \mathcal{T}_0\}$ is uniformly integrable. Henceforth it has the following Doob-Meyer decomposition

$$\mathcal{S}_t(\eta) = \mathbb{E}\left[\xi + \int_0^1 g(s)ds + K(1) | F_t\right] - K(t)$$

where $(K(t))_{t\leq 1}$ is an $(\mathcal{F}_t)_{t\leq 1}$-adapted *rcll* non-decreasing process such that $K(0) = 0$. Furthermore we have $\mathbb{E}[K(1)^2] < \infty$ since $\mathbb{E}[\sup_{t\leq 1} |\mathcal{S}_t|^2] < \infty$ (see e.g. [4], p.221). It follows that

$I\!E[\sup_{t\leq 1}|I\!E[\xi+K(1)|F_t]|^2]<\infty$ and then, through the representation theorem of martingales with respect to $(\mathcal{F}_t)_{t\leq 1}$ (see [8]), there exist two processes $Z=(Z_t)_{t\leq 1}$ and $V=(V_t)_{t\leq 1}$ which belong respectively to $H^{2,d}$ and $\mathcal{L}^2$ such that,

$$M_t := I\!E[\xi+\int_0^1 g(s)ds+K(1)|F_t] = I\!E[\xi+K(1)]+\int_0^t Z_s dB_s+\int_0^t\int_U V_s(e)\tilde{\mu}(ds,de), \forall t\leq 1.$$

Now let us show that the process K is continuous.

First let us underline that the jumping times of K are included in the set $\{\mathcal{S}_-(\eta)=\underline{\eta}\}$ where $\underline{\eta}_t=\limsup_{s\nearrow t}\eta_s=\eta_{t-}$ since the process η is $rcll$ (see e.g. [EK], pp.131).

Now let τ be a predictable stopping time, then

$$\begin{aligned} I\!E[\mathcal{S}_{\tau-}(\eta)1_{\{\Delta K(\tau)>0\}}] &= I\!E[\eta_{\tau-}1_{\{\Delta K(\tau)>0\}}] \\ &\leq I\!E[\eta_\tau 1_{\{\Delta K(\tau)>0\}}] \leq I\!E[\mathcal{S}_\tau(\eta)1_{\{\Delta K(\tau)>0\}}]. \end{aligned} \tag{12}$$

The second inequality is obtained through the fact that the process η has inaccessible jumping times, and may have a positive jump at $t=1$. On the other hand,

$$\begin{aligned} I\!E[\mathcal{S}_{\tau-}(\eta)1_{\{\Delta K(\tau)=0\}}] &= I\!E[(M_{\tau-}+K(\tau))1_{\{\Delta K(\tau)=0\}}] \\ &= I\!E[(M_\tau+K(\tau))1_{\{\Delta K(\tau)=0\}}] = I\!E[\mathcal{S}_\tau(\eta)1_{\{\Delta K(\tau)=0\}}]. \end{aligned} \tag{13}$$

The second equality stems from the fact that τ is predictable, then $M_{\tau-}=M_\tau$ since the jumping times of $(M_t)_{t\leq 1}$ are those of its Poisson part and those latter are inaccessible. Now combining (12) and (13) yields $I\!E[\mathcal{S}_{\tau-}(\eta)]\leq I\!E[\mathcal{S}_\tau(\eta)]$ and then, since $\mathcal{S}(\eta)$ is a supermartingale, $I\!E[\mathcal{S}_{\tau-}(\eta)]=I\!E[\mathcal{S}_\tau(\eta)]$ for any predictable stopping time τ. Henceforth the supermartingale $(\mathcal{S}_t(\eta))_{t\leq 1}$ is regular, i.e. ${}^p\mathcal{S}(\eta)=\mathcal{S}_-(\eta)$, and then the process K is continuous (see [3], p.119).

Now let us set

$$Y_t = ess\sup_{v\in\mathcal{T}_t} I\!E\left[\xi 1_{\{v=1\}}+S_v 1_{\{v<1\}}+\int_t^v g(s)ds\,|F_t\right],$$

then $Y_t+\int_0^t g(s)ds=\mathcal{S}_t(\eta)=M_t-K(t), t\leq 1$, henceforth we have

$$Y_t+\int_0^t g(s)ds = I\!E[\xi+K(1)]+\int_0^t Z_s dB_s+\int_0^t\int_U V_s(e)\tilde{\mu}(ds,de)-K(t), \forall t\leq 1,$$

and then for any $t\leq 1$ we have,

$$Y_t=\xi+\int_t^1 g(s)ds-\int_t^1 Z_s dW_s-\int_t^1\int_U V_s(e)\tilde{\mu}(ds,de)+K(1)-K(t).$$

Now since $Y_t+\int_0^t g(s)ds=\mathcal{S}_t(\eta)$ then $Y_t\geq S_t$ for any $t\leq 1$.

Finally it remains to show that $\int_0^1(Y_t-S_t)dK(t)=0$. The Snell envelope process $(\mathcal{S}_t(\eta))_{t\leq 1}$ is regular i.e. $\mathcal{S}_-(\eta)={}^p\mathcal{S}(\eta)$. Now let $t\leq 1$ and $\delta_t:=\inf\{s\geq t, K(s)>K(t)\}\wedge 1$. As $\mathcal{S}(\eta)$ is regular then δ_t is the largest optimal stopping time after t (see e.g. [EK], p.140). It implies that $\mathcal{S}_{\delta_t}(\eta)=\eta_{\delta_t}$ ([EK], p.111) . Henceforth for any $s \in [t, \delta_t]$ we have $(\mathcal{S}_s(\eta) - \eta_s)dK(s) = 0$ and then $(Y_s - S_s)dK(s) = 0$ which implies $\int_0^1 (Y_t - S_t)dK(t) = 0$.

The process (Y, Z, K, V) is then solution of the reflected BSDE associated with (g, ξ, S) $\Box$

Now we argue as in Thm.1.2.b. to obtain the existence and uniqueness of the solution of the reflected discontinuous BSDE associated with coefficients f which depend on (y, z, v) and which are uniformly Lipschitz with respect to those variables. Therefore we have,

1.4.b. Theorem: There exists a unique solution $(Y, Z, K, V) = (Y_t, Z_t, K_t, V_t)_{t\leq 1}$ for the reflected backward stochastic differential equation (1) with jumps associated with (f, ξ, S) $\Box$

2.5 Application of Reflected DBSDEs in mixed stochastic control.

Now we are going to highlight the link between mixed stochastic optimal control, when the noise is of gaussian and Poisson types, and RDBSDEs.

Let us consider D_1 and D_2 two compact metric spaces whose Borel σ-algebras are respectively $B(D_1)$ and $B(D_2)$, and f, g two functions defined as :

(i) f maps $[0,1] \times \Omega \times D_1$ into $I\!R^d$, bounded and $\mathcal{P} \otimes B(D_1)/B(I\!R^d)$-measurable. Moreover for any $(t, \omega) \in [0,1] \times \Omega$, the function $f(t, \omega, .) : d_1 \in D_1 \longmapsto f(t, \omega, d_1)$ is continuous.

(ii) g maps $[0,1] \times \Omega \times D_2 \times U$ into $I\!R$, is $\mathcal{P} \times B(D_2 \times U)/B(I\!R)$-measurable and there exist two constants α_1 and α_2 such that $|g(t, \omega, d_2, e)| \leq \alpha_1 |e| 1_{[|e|\leq 1]} + \alpha_2 1_{[|e|>1]}$ for any $(t, \omega, d_2, e) \in [0,1] \times \Omega \times D_2 \times U$. Moreover for any (t, ω, e) the function $g(t, \omega, e, .) : d_2 \in D_2 \longmapsto g(t, \omega, d_2, e)$ is continuous.

Now let $\mathcal{D}_1$ (resp. $\mathcal{D}_2$) be the set of $\mathcal{P}$-measurable processes with values in D_1 (resp. D_2). The set $\mathcal{D} := \mathcal{D}_1 \times \mathcal{D}_2$ is called of admissible controls. For any $\delta = (\delta_1, \delta_2) \in \mathcal{D}$ we associate a process L^δ defined as follows:

$$
\begin{aligned}
L_t^\delta \;=\; & \exp[\int_0^t f(s, \delta_1(s))dB_s - \frac{1}{2}\int_0^t |f(s, \delta_1(s))|^2 ds + \int_0^t \int_U g(s, \delta_2(s), e)\tilde{\mu}(de, ds) \\
& - \int_0^t \int_U \{e^{g(s,\delta_2(s),e)} - 1 - g(s, \delta_2(s), e)\}\lambda(de)ds],\ t \leq 1.
\end{aligned}
$$

The above assumptions on f and g imply that L^δ is an $(\mathcal{F}_t, P)$-martingale and the measure P^δ on $(\Omega, \mathcal{F})$ defined by $dP^\delta = L^\delta.dP$ is a probability (see e.g. [9]). Moreover under P^δ, $\tilde{\mu}^\delta(dt, de) :=$ $\tilde{\mu}(dt, de) - (e^{g(t,\delta_2(t),e)} - 1)dt\lambda(de)$ is an $\mathcal{F}_t-$martingale measure and $(W_t^\delta = W_t - \int_0^t f(s, \delta_1(s))ds)_{t\leq 1}$ is an $\mathcal{F}_t-$Brownian motion.

Now let us consider $\delta = (\delta_1, \delta_2) \in \mathcal{D}$, τ an $\mathcal{F}_t$-stopping time such that $\tau \leq 1$, P-a.s. and $J(\delta, \tau)$ a functional whose expression is given by:

$$
J(\delta, \tau) = E^\delta[\int_0^\tau ds\{c(s, \delta_1(s)) + \int_U h(s, \delta_2(s), e)e^{g(s,\delta_2(s),e)}\lambda(de)\} + S_\tau 1_{[\tau<1]} + \xi 1_{[\tau=1]}]
$$

where c and h are two bounded measurable functions defined respectively on $[0,1]\times\Omega\times D_1$ and $[0,1]\times\Omega\times D_2\times U$. Furthermore we suppose that c is continuous with respect to d_1, h continuous with respect to d_2 and satisfies $\forall e\in U$, $|h(t,\omega,d_2,e)|\leq a(1\wedge|e|^2)$ for some $a>0$.

The problem on which we are interested in is to look for (δ^*,τ^*) which maximizes $J(\delta,\tau)$ i.e. $J(\delta^*,\tau^*)\geq J(\delta,\tau)$ for any other δ and τ.

We can think of $J(\delta,\tau)$ as the profit that makes an agent who intervenes on a system whose evolution is described by a stochastic process, say $(X_t)_{t\leq 1}$. An intervention strategy for the agent is a pair (δ,τ), δ is his control action and τ is the time he chooses to stop controlling. A strategy (δ^*,τ^*) which maximizes $J(\delta,\tau)$, if it exists, is called *optimal* for the agent.

In the expression of $J(\delta,\tau)$, the term which is absolutely continuous with respect to dt is the instantaneous reward and the other is the reward at stopping for the agent.

This problem is called of *mixed control* type because it combines optimal control and stopping.

Assume the state evolution $(X_t)_{t\leq 1}$ of the non-controlled system is described by a stochastic differential equation of the following type:

$$X_t = x + W_t + \sum_{s\leq t}\Delta X_s 1_{[|\Delta X_s|>1]} + \int_0^t\int_{|e|\leq 1} e\tilde{\mu}(ds,de), t\leq 1.$$

The control action of the agent consists in choosing a probability P^δ under which the system will evolve. So under P^δ the state evolution of the controlled system is described by :

$$\begin{aligned} X_t &= x + W_t^\delta + \int_0^t f(s,\delta_1(s))ds + \sum_{s\leq t}\Delta X_s 1_{[|\Delta X_s|>1]} + \int_0^t\int_{|e|\leq 1} e\tilde{\mu}^\delta(ds,de) \\ &\quad + \int_0^t\int_{|e|\leq 1} e(e^{g(s,\delta_2(s),e)}-1)\lambda(de)ds,\ t\leq 1 \end{aligned}$$

It means that the agent control action generates a drift for the dynamic of the system and a reward which is equal to $J(\delta,\tau)$, therefore he looks for optimal strategies $\Box$

We now go back to our general mixed control problem. Let H_1 and H_2 be the hamiltonian functions associated with this control problem, defined on $[0,1]\times\Omega\times I\!R^d\times D_1$ and $[0,1]\times\Omega\times L^2(U,\mathcal{U},\lambda;I\!R)\times D_2$ respectively, as follows:

$$H_1(t,\omega,p,d_1) = pf(t,\omega,d_1) + c(t,\omega,d_1)$$

and

$$H_2(t,\omega,v,d_2) = \int_U v(e)(e^{g(t,\omega,d_2,e)}-1)\lambda(de) + \int_U h(t,\omega,d_2,e)e^{g(t,\omega,d_2,e)}\lambda(de).$$

According to Benes'selection theorem [2], through the above assumptions on f and g, there exist two measurable functions $d_1^*(t,\omega,p)$ and $d_2^*(t,\omega,v)$ with values respectively in D_1 and D_2 such that $H_1(t,\omega,p,d_1^*) = \sup_{d_1\in D_1} H_1(t,\omega,p,d_1)$ and $H_2(t,\omega,v,d_2^*) = \sup_{d_2\in D_2} H_2(t,\omega,v,d_2)$. Moreover the function from $[0,1]\times\Omega\times R^d\times L^2_{I\!R}(U,\mathcal{U},\lambda)$ which with (t,ω,p,v) associates $H_1(t,\omega,p,d_1^*(t,\omega,p))+H_2(t,\omega,v,d_2^*(t,\omega,v))$ is Lipschitz in (p,v) uniformly on (t,ω). Indeed,

$$\begin{array}{l}|H_1(t,\omega,p,d_1^*(t,\omega,p))-H_1(t,\omega,p',d_1^*(t,\omega,p'))|\\ \qquad =|\sup_{d_1\in D_1}H_1(t,\omega,p,d_1)-\sup_{d_1\in D_1}H_1(t,\omega,p',d_1)|\\ \qquad \leq \sup_{d_1\in D_1}|H_1(t,\omega,p,d_1)-H_1(t,\omega,p',d_1)|=\sup_{d_1\in D_1}|f(t,\omega,d_1)||p-p'|\leq C|p-p'|,\end{array}$$

since f is a bounded function. On the other hand

$$\begin{array}{l}|H_2(t,\omega,v,d_2^*(t,\omega,v))-H_2(t,\omega,v',d_2^*(t,\omega,v'))|\leq \sup_{d_2\in D_2}|\int_U(v(e)-v'(e))(e^{g(t,\omega,d_2,e)}-1)\lambda(de)|\\ \qquad\qquad \leq \|v-v'\|\sup_{d_2\in D_2}\{\int_U|e^{g(t,\omega,d_2,e)}-1|^2\lambda(de)\}^{\frac{1}{2}}\leq C\|v-v'\|,\end{array}$$

since $|e^{g(t,d_2,e)}-1|^2\leq C(1\wedge|e|^2)$, for any (t,d_2,e) $\Box$

Now we are ready to give the main result of this part. Let (W^*,Z^*,K^*,V^*) be the solution of the reflected DBSDE associated with $[H_1(t,z,d_1^*(t,z))+H_2(t,v,d_2^*(t,v)),\xi,S]$ namely,

$$\left\{\begin{array}{l}W^*,\,K^*\in\mathcal{S}^2,\,Z^*\in H^{2,d},\,V^*\in\mathcal{L}^2;K^*\text{ is moreover continuous non-decreasing and }K_0^*=0\\ W_t^*=\xi+\int_t^1\{H_1(s,Z_s^*,d_1^*(s,Z_s^*))+H_2(s,V_s^*,d_2^*(s,V_s^*))\}ds+K_1^*-K_t^*-\int_t^1 Z_s^*dW_s\\ \qquad -\int_t^1 ds\int_U V_s^*(e)\tilde{\mu}(ds,de);\ W_t^*\geq S_t,\forall t\leq 1;\ \int_0^1(W_s^*-S_s^*)dK_s^*=0.\end{array}\right.$$

On the other hand for any $t\leq 1$, let $\tau_t^*=\inf\{s\geq t,W_s^*=S_s\}\wedge 1$ and $\delta^*=(d_1(t,Z_t^*),d_2(t,V_t^*))_{t\leq 1}$. Then we have :

1.5.a. Theorem: The process $(W_t^*)_{t\leq 1}$ is the value function of the *mixed optimal control* problem, i.e., for any $t\leq 1$,

$$\begin{array}{rl}W_t^*=&\operatorname{esssup}_{\delta\in\mathcal{D}}\operatorname{esssup}_{\tau\geq t}I\!E^\delta[\int_t^\tau ds\{c(s,d_1(s))+\int_U h(s,d_2(s),e)e^{g(s,d_2(s),e)}\lambda(de)\}\\ &\qquad +S_\tau 1_{[\tau<1]}+\xi 1_{[\tau=1]}|\mathcal{F}_t];\ \tau\text{ is a stopping time.}\end{array}$$

Moreover the strategy (δ^*,τ_0^*) is optimal and $W_0^*=J(\delta^*,\tau^*)$.

Proof: Through the Burkholder-Davis-Gundy inequality ([4], p.304), the processes $(\int_0^t\int_U V_s^*(e)\tilde{\mu}^{\delta^*}(ds,de))_{t\leq 1}$ and $(\int_0^t Z_s^*dW_s^{\delta^*})_{t\leq 1}$ are P^{δ^*}-martingales. In addition we have

$$W_{\tau_t^*}^*=W_{\tau_t^*}^*1_{[\tau_t^*<1]}+W_1^*1_{[\tau_t^*=1]}=S_{\tau_t^*}1_{[\tau_t^*<1]}+\xi 1_{[\tau_t^*=1]}.$$

It follows that, since W_t^* is $\mathcal{F}_t$-measurable and $K_t^*=K_{\tau_t^*}^*$,

$$\begin{array}{rl}W_t^*&=I\!E^{\delta^*}[S_{\tau_t^*}1_{[\tau_t^*<1]}+\xi 1_{[\tau_t^*=1]}+\\ &\int_t^{\tau_t^*}c(s,d_1^*(s,Z_s^*))ds+\int_t^{\tau_t^*}ds\int_U h(s,d_2^*(s,V_s^*(e)),e)e^{g(s,d_2^*(s,V_s^*(e),e))}\lambda(de)|\mathcal{F}_t].\end{array}$$

Now let $\delta = (d_1, d_2)$ be another control and τ a stopping time such that $\tau \geq t$, P-$a.s.$. Once again since W^*_t is $\mathcal{F}_t$-measurable, we have,

$$\begin{aligned} W^*_t &= \mathbb{E}^\delta[\int_t^\tau \{H_1(s, Z^*_s, d^*_1(s, Z^*_s)) + H_2(s, V^*_s, d^*_2(s, V^*_s))\}ds + K^*_\tau - K^*_t - \int_0^\tau Z^*_s dB_s \\ &\quad - \int_t^\tau \int_U V^*_s(e)\tilde{\mu}(ds, de) + W^*_\tau | \mathcal{F}_t] \end{aligned}$$

But

$$H_1(s, Z^*_s, d^*_1(s, Z^*_s)) \geq Z^*_s f(s, d_1(s)) + c(s, d_1(s)) \text{ and } H_2(s, V^*_s, d^*_2(s, V^*_s)) \geq H_2(s, V^*_s, d_2(s)),$$

then

$$\begin{aligned} W^*_t &\geq \mathbb{E}^\delta[\int_t^\tau \{Z^*_s f(s, d_1(s)) + c(s, d_1(s)) + H_2(s, V^*_s, d_2(s))\}ds + K^*_\tau - K^*_t \\ &\quad - \int_t^\tau Z^*_s dB_s - \int_t^\tau \int_U V^*_s(e)\tilde{\mu}(ds, de) + W^*_\tau | \mathcal{F}_t] \end{aligned}$$

which implies, since $K^*_\tau - K^*_t \geq 0$ and $W^*_\tau \geq S_\tau 1_{[\tau<1]} + \xi 1_{[\tau=1]}$,

$$W^*_t \geq \mathbb{E}^\delta[\int_t^\tau ds\{c(s, d_1(s)) + \int_U h(s, d_2(s), e)e^{g(s,d_2(s),e)}\lambda(de)\} + S_\tau 1_{[\tau<1]} + \xi 1_{[\tau=1]} | \mathcal{F}_t]$$

The last inequality is due to the fact that $\int_0^\cdot Z^*_s dB^\delta_s$ and $\int_0^\cdot \int_U V^*_s(e)\tilde{\mu}^\delta(ds, de)$ are $(\mathcal{F}_t, P^\delta)$-martingales. Henceforth we have, for any $t \leq 1$,

$$\begin{aligned} W^*_t = \ & \text{esssup}_{\delta\in\mathcal{D}} \text{esssup}_{\tau \geq t} \mathbb{E}^\delta[\int_t^\tau ds\{c(s, d_1(s)) + \int_U h(s, d_2(s), e)e^{g(s,d_2(s),e)}\lambda(de)\} \\ & + S_\tau 1_{[\tau<1]} + \xi 1_{[\tau=1]}] | \mathcal{F}_t]; \ \tau \text{ is a stopping time.} \end{aligned}$$

Now taking $t = 0$ we have $W^*_0 = J(\delta^*, \tau^*_0)$ and $W^*_0 \geq J(\delta, \tau)$ for any $\delta \in \mathcal{D}$ and τ a stopping time, since $\mathcal{F}_0$ is the trivial tribe. It follows that $J(\delta^*, \tau^*_0) \geq J(\delta, \tau)$ for any δ, τ, i.e., (δ^*, τ^*_0) is an optimal strategy for the agent. $\square$

This problem has been considered yet by N. El-Karoui [5] in a general case and J.P.Lepeltier & B.Marchal [9] in a particular case. Using martingale methods, which are a heavy tool, all of them show the existence of an optimal strategy. We show here that this problem can be solved in a simple way.

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