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PERCOLATION TIMES IN TWO–DIMENSIONAL MODELS FOR EXCITABLE MEDIA

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Abstract. The three-color *Greenberg–Hastings model (GHM)* is a simple cellular automaton model for an excitable medium. Each site on the lattice $\mathbf{Z}^2$ is initially assigned one of the states 0, 1 or 2. At each tick of a discrete–time clock, the configuration changes according to the following synchronous rule: changes $1 \to 2$ and $2 \to 0$ are automatic, while an x in state 0 may either stay in the same state or change to 1, the latter possibility occurring iff there is at least one representative of state 1 in the local neighborhood of x. Starting from a product measure with just 1's and 0's such dynamics quickly die out (turn into 0's), but not before 1's manage to form infinite connected sets. A very precise description of this "transient percolation" phenomenon can be obtained when the neighborhood of x consists of 8 nearest points, the case first investigated by S. Fraser and R. Kapral. In addition, first percolation times for related monotone models are addressed.

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1. Introduction.

The *Greenberg-Hastings model (GHM)* ([WR], [GH]) on $\mathbf{Z}^2$ tries to capture two essential properties of a two–dimensional excitable medium: a refractory phase and excitation by contact. In one of its simplest incarnations, GHM dynamics have two parameters: an integer number $\kappa \ge 3$, which we refer to as the *number of colors* (or *states*), and a finite set $\mathcal{N} \subset \mathbf{Z}^2$, containing the origin, which is called the *neighborhood*. The neighborhood of an arbitrary site $x \in \mathbf{Z}^2$ is then obtained by translation: $\mathcal{N}_x = x + \mathcal{N}$. The state space of the GHM is $\{0, 1, \dots, \kappa-1\}^{\mathbf{Z}^2}$ and the update rule is given as follows:

$$\gamma_{t+1}(x) = \begin{cases} (\gamma_t(x)+1) \bmod \kappa, & \text{if } \gamma_t(x) > 0 \text{ or there is a } y \in \mathcal{N}_x \text{ with } \gamma_t(y) = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{1.1}$$

In words, all colors $1, \dots, \kappa-1$ update automatically to $2, \dots, 0$, respectively, while 0 updates by contact with a least one color 1 in its neighborhood. Readers interested in mathematical analysis of this GHM rule and various generalizations may consult [DS], [FGG1], [FGG2], [DN], [Dur], [DG], and [Ste]. Researchers closer to applications have considered many classes of multi–parameter rules related to the GHM (see, for example, [WTW] and the references contained therein).

Assume for a moment that $\mathcal{N}$ is the five point set $\{(0,0), (\pm 1, 0), (0, \pm 1)\}$, the *nearest neighbor* case. Assume moreover that γ_0 is a product measure with $P(\gamma_0(x) = 0) > 0$, $P(\gamma_0(x) = 1) > 0$ and $P(\gamma_0(x) = i) > 0$ for some $i \in \{2, \dots, \kappa-1\}$. Then it turns out that the dynamics is eventually run by "clocks," i.e. objects like

$$\begin{array}{ccccccccccc} 0 & - & 1 & - & 2 & - & 3 & - & \dots & - & \kappa-1 \\ | & & | & & | & & | & & & & | \\ \kappa-1 & - & \kappa-2 & - & \kappa-3 & - & \kappa-4 & - & \dots & - & 0 \end{array} ,$$

which repeat their pattern every κ units of time and eventually cause other sites to do the same, giving rise to *local periodicity* (see [FGG2]). The main aim of this paper is to study the case when γ_0 contains only 0's and 1's, which results in a markedly different dynamics. To see why, start a nearest neighbor GHM with just a single 1 in a sea of 0's. This initial state generates a diamond–shaped ring which keeps expanding forever. If the initial state contains just two 1's, the two rings proceed to annihilate on the boundary and combine into one big ring, which again expands without bound. We will show that such annihilating rings are the dominant feature of the dynamics started from a product measure consisting only of 0's and 1's. Indeed, one inevitably reaches such a conclusion by doing just a few computer simulations.

As in the simple example above, the number of colors turns out to be completely immaterial for the ring dynamics. Thus we assume that $\kappa = 3$ from now on. We also assume that the initial state γ_0 is the product measure given by $P(\gamma_0(x) = 1) = p = 1 - P(\gamma_0(x) = 0)$ for every x. Since our main results involve various percolation properties of subsets of $\mathbf{Z}^2$, we must start with a few definitions.

Fix a finite set $\mathcal{D} \subset \mathbf{Z}^2$. We say that a set $G \subset \mathbf{Z}^2$ $\mathcal{D}$*–percolates* if there exists an infinite sequence $x_1, x_2, \dots$ of distinct sites in G such that $x_{k+1} - x_k \in \mathcal{D}$ for $k = 1, 2, \dots$. The usual site percolation, which we call ℓ^1*–percolation*, is obtained by taking $\mathcal{D} = B_1(0,1)$, while its dual, which we refer to as ℓ^∞*–percolation*, is given by $\mathcal{D} = B_\infty(0,1)$. (We adopt the usual notation for balls: $B_r(x, R) = \{y : ||x - y||_r \le R\}$.)

The study of ring growth and interaction in this simple model of an excitable medium fits into the general framework of investigating domain growth in physical systems: namely, how spatial structures arise far from equilibrium. Real–life phenomena that may be related to the ones discussed here include aspects of the Belousov–Zhabotinsky reaction and processes in cardiac tissue ([FK], [Kap], [KW], [SLC]). Indeed, ring dynamics may be the dominant feature of conduction of electrical impulses in a healthy heart, while spatial defects (e.g. scar tissue) can break the rings and cause, through the formation of clock–like excitation centers, atrial fibrillation. These natural phenomena were the main motivation of S. Fraser and R. Kapral, who studied ring dynamics in excitable media in [FK]. They chose the Moore neighborhood (i.e. $\mathcal{N} = B_\infty(0,1)$) and combined computer simulations with various intuitively compelling but non-rigorous arguments, to arrive at the following three conclusions:

(1) Each point is in state 1 exactly once.

(2) The density of 1's goes to 0 very quickly as time t increases, and is of order $\sqrt{p}$ at its maximum.

(3) There exists a narrow time interval, lasting at most two time units, during which 1's percolate. ([FK] is vague as to what type of percolation this is.)

Assuming (1), (2) follows readily, as shown in [Bra]. But (1) is not very hard to show either (see Corollary 2.2). The most interesting statement is of course (3); our main aim in the present paper is to give a rigorous and more detailed treatment of this issue.

Before we proceed, let us introduce some standard regularity assumptions on the neighborhood $\mathcal{N}$ which arose in our previous work ([GG]). We call $\mathcal{N}$ *symmetric* if $-\mathcal{N} = \mathcal{N}$ and *supercritical* if the origin is an interior point of the convex hull $\mathrm{co}(\mathcal{N})$. Moreover, denote the sum $\mathcal{N}+\mathcal{N}+\cdots+\mathcal{N}$ of n copies of $\mathcal{N}$ by $\mathcal{N}^n$ (with the conventions $\mathcal{N}^0=\{0\}$, $\mathcal{N}^n=\emptyset$ when $n<0$), and call $\mathcal{N}$ *irreducible* if $\cup_{n\geq 1}\mathcal{N}^n=\mathbf{Z}^2$. Note that irreducibility implies supercriticality, but not vice versa. We assume throughout this paper that $\mathcal{N}$ is both symmetric and irreducible.

Theorem 1. *There exist sufficiently large integers a and b (depending only on $\mathcal{N}$) so that:*

(1) For every $p>0$ there exists a time $T(p)$ at which $\{\gamma_{T(p)}=1\}$ $B_\infty(0,a)$–percolates.

(2) The set $\{\gamma_t=1\}$ may $B_\infty(0,a)$–percolate only for $t\in[T(p),T(p)+b]$.

(3) $T(p)\sqrt{p}$ converges to a finite non-zero constant as $p\to 0$.

To prove a more precise theorem, we restrict to the special case of Moore neighborhood considered in [FK].

Theorem 2. *Assume that $\mathcal{N}=B_\infty(0,1)$.*

(1) For every $p\in(0,1]$ there is a time $T(p)$ such that the set $\{\gamma_t=1\}$ l^∞-percolates for $t=T(p)$ but does not l^∞-percolate for $t\notin[T(p),T(p)+1]$.

(2) There is a countable set $S\subset(0,1)$ with 0 as its only limit point, such that the following is true. Let $I_1,I_2,I_3,\ldots$ be the successive intervals in $(0,1]\setminus S$, where I_1 is the one that includes 1. Let G_1 be the union $I_1\cup I_3\cup I_5\cup\ldots$ and $G_2=(0,1]\setminus S\setminus G_1$. If $p\in G_1$, then 1's in γ_t ℓ^1-percolate at $t=T(p)$ but do not ℓ^∞-percolate for $t\neq T(p)$. If $p\in G_2$, then 1's in γ_t never ℓ^1-percolate, but they ℓ^∞-percolate for both $t=T(p)$ and $t=T(p)+1$. Finally, if $p\in S$, then 1's in γ_t ℓ^∞-percolate only for $t=T(p)$ and at this time they do not ℓ^1-percolate.

Figure 1 illustrates Theorem 2 for the density $p=0.01$. The frames represent the state of the system on a 200×200 square at times 3 and 4 (top), and 5 and 6 (bottom). At times 4 and 6 (resp. at time 5) the sites with color 1 that are connected via an l^∞-path (resp. an l^1-path) to the left edge of the rectangle are painted black. The remaining 1's are painted dark gray, while 2's are painted light gray. It is clear that at time 3 the rings of 1's are still too small to be able to ℓ^∞-percolate. This remains true at time 4, despite the existence of relatively long ℓ^∞-paths. At time 4, 1's form a left–right ℓ^1-connection of the square, suggesting ℓ^1-percolation. Finally, by time 5 there is enough annihilation between the rings so that 1's cannot even ℓ^∞–percolate. These figures leave little doubt that $T(0.01)=5$ and $0.01\in G_1$.

It is natural to ask what happens to Theorems 1 and 2 in more general circumstances. One possible generalization is to consider an arbitrary excitation threshold θ; this means that x will change color from 0 to 1 iff the number of sites with color 1 in $\mathcal{N}_x$ exceeds θ ([FGG1], [FGG2],

[DG], [GG]). In this case "failed centers" can break up the rings and cause local periodicity; computer simulations show that this is indeed a likely scenario in many regimes, though a rigorous general statement is hard to prove. Another interesting class of questions concerns the expanding "shells" in dimensions 3 or higher. These higher dimensional models are considerably harder to deal with not only because of technical difficulties, but also because of qualitatively different "intermediate" regimes. On a different note, present methods can be extended to handle various related two–dimensional models with *recurrent* ring dynamics ([Gra3]).

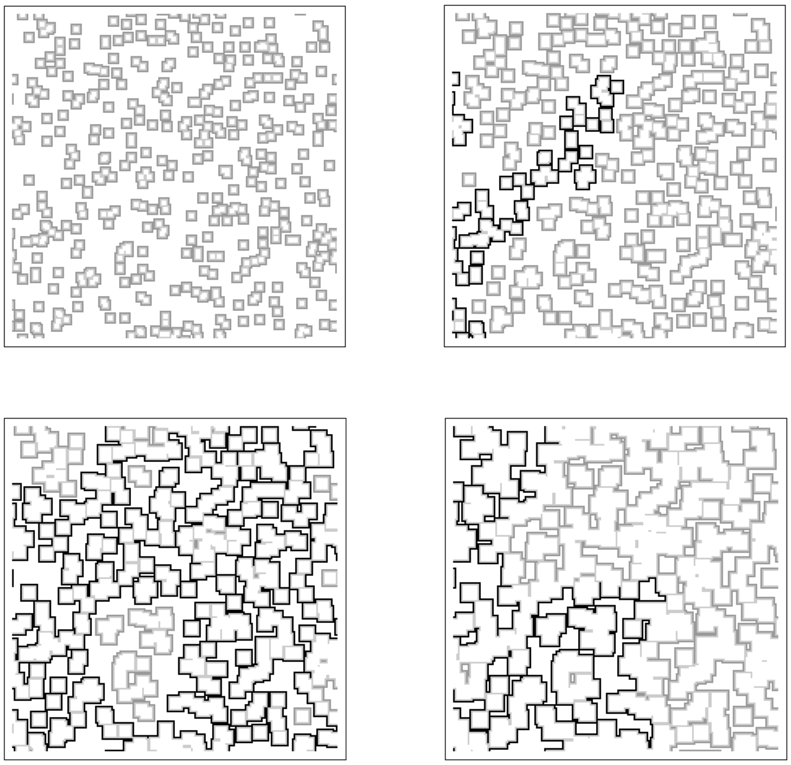

Figure 1. Ring dynamics in the GHM.

The first step in the proof of the above theorems is a comparison between the GHM and a monotone 2–color model which is in fact the threshold 1 (often called *additive*) version of threshold growth dynamics ([GG]). This auxiliary monotone automaton with state space $\{0,1\}^{\mathbf{Z}^2}$ is denoted by ξ_t and given by the update rule:

$$\xi_{t+1}(x) = \max\{\xi_t(y) : y \in \mathcal{N}_x\}.$$

Thus, $\{\xi_t = 1\} = \{\xi_0 = 1\} + \mathcal{N}^t$. Assume that ξ_0. like γ_0, is a product measure of 0's and 1's, with a density of 1's given by $p > 0$. Define $T_a(p)$ to be the first time for which the set $\{\xi_t = 1\}$ $B_\infty(0, a)$-percolates. By the ergodic theorem, T_a is a deterministic function of p. It is also not hard to prove the following result.

Theorem 3. *As $p \to 0$, $T_a(p)\sqrt{p}$ converges to a finite non-zero constant λ_c, which depends on $\mathcal{N}$ but is independent of a.*

Many models for crystal formation and growth, such as the famous Johnson–Mehl model ([JM]), include continuous seeding of nuclei. This inspired R. Kapral and M. Weinberg ([KW]) to propose a variant of ξ_t. This model, denoted by $\tilde{\xi}_t$, is a Markov chain with state space $\{0,1\}^{\mathbf{Z}^d}$ and the following transition rule at every site x:

$$
\begin{array}{ll}
0 \to 1 & \text{if there is a 1 in } \mathcal{N}_x, \\
0 \to 1 & \text{with probability } p, \text{ if there is no 1 in } \mathcal{N}_x, \\
0 \to 0 & \text{with probability } 1-p, \text{ if there is no 1 in } \mathcal{N}_x, \\
1 \to 1 & \text{automatically.}
\end{array}
$$

Hence this rule is a two–stage procedure: at the first stage the dynamics of ξ_t are applied to a configuration of 0's and 1's, and at the second stage a proportion p of the remaining 0's are turned to 1's.

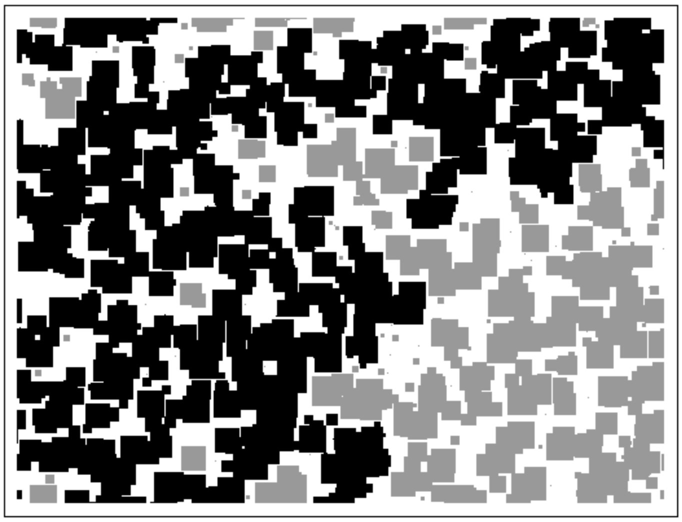

Figure 2. Threshold 1 growth rule with continuous nucleation.

Start $\tilde{\xi}_t$ from the quiescent state: $\tilde{\xi}_0(x) = 0$ for all x. Now define $\tilde{T}_a(p)$ to be the first time at which $\{\tilde{\xi}_t = 1\}$ $B_\infty(0,a)$-percolates. Our next theorem identifies the behavior of $\tilde{T}_a$ near $p = 0$. (We will see later that $\tilde{T}_a$ is a deterministic function of p as well.)

Theorem 4. *As $p \to 0$, $\tilde{T}_a(p)p^{1/3}$ converges to a finite non-zero constant.*

This theorem is illustrated by Figure 2, which shows the state of the 500×375 system with $\mathcal{N} = B_\infty(0,1)$ and $p = 0.0005$ at time $t = 12$, the first time when there is an ℓ^∞–path connecting left and right edges of the rectangle. The picture suggests that $\tilde{T}_1(0.0005) = 12$.

The rest of the paper is organized as follows. Section 2 contains a few results on additive growth, and a proof of Theorem 3. Theorem 4 is proved in Section 3. Section 4 states a few well known results about long–range percolation models, which we then use in Section 5 to prove Theorem 2. Finally, Section 6 completes the proof of Theorem 1.

2. Growth dynamics with initial nucleation.

We start with a simple lemma which establishes a connection between the growth automaton ξ_t and the GHM γ_t.

Lemma 2.1. *Assume that $\gamma_0 = \xi_0$. Then*

$$\{x : \gamma_{t+1}(x) = 1\} = \{x : \xi_t(x) = 0, \xi_{t+1}(x) = 1\}. \tag{2.1}$$

Proof. Denote the set on the right of (2.1) by A_t. To prove (2.1) by induction, observe first that it is true at $t = 0$, and then assume that $\{\gamma_{s+1} = 1\} = A_s$ for all $s < t$.

Firstly, we prove that $A_t \subset \{\gamma_{t+1} = 1\}$. Assume that $x \in A_t$. Then $x \notin A_s$ for any $s < t$. In particular, $x \notin A_{t-1}$, hence by hypothesis $\gamma_t(x) \neq 1$. Since $\gamma_t(x)$ cannot be 2 ($\gamma_t(x) = 2$ would imply $\gamma_{t-1}(x) = 1$ and thus $x \in A_{t-2}$, a contradiction), $\gamma_t(x) = 0$. Moreover, there is a $y \in \mathcal{N}_x$ such that $\xi_t(y) = 1$. If $\xi_{t-1}(y) = 1$, then $\xi_t(x) = 1$, a contradiction. It follows that $y \in A_{t-1}$, $\gamma_t(y) = 1$ and, consequently, $\gamma_{t+1}(x) = 1$.

Secondly, we prove that $\{\gamma_{t+1} = 1\} \subset A_t$. Assume now that $\gamma_{t+1}(x) = 1$, so $\gamma_t(x) = 0$ and $\gamma_t(y) = 1$ for some $y \in \mathcal{N}_x$. Hence $y \in A_{t-1}$ and, consequently $\xi_{t+1}(x) = 1$. Since $x \notin A_{t-1}$, we either have $\xi_t(x) = 0$ or $\xi_{t-1}(x) = 1$. But, since $x \notin A_{t-2}$, $\xi_{t-1}(x) = 1$ would imply that $\xi_{t-2}(x) = 1$ and hence $\xi_{t-1}(y) = 1$, a contradiction with $y \in A_{t-1}$. This leaves $\xi_t(x) = 0$ as the only possibility, proving that $x \in A_t$. □

Corollary 2.2. *For each site x there is a unique time t at which $\gamma_t(x) = 1$.*

Lemma 2.1 also immediately implies that $P(\gamma_t(x) = 1) = \left(1 - (1-p)^{|\mathcal{N}^t \setminus \mathcal{N}^{t-1}|}\right)(1-p)^{|\mathcal{N}^{t-1}|}$. To estimate the maximal value this expression can have, we first use the shape theorem; this result was first proved by Willson ([Wil]), but we give a short proof in Lemma 2.3 below. This theorem implies that $|\mathcal{N}^n|/n^2 \to \text{Area}(co(\mathcal{N}))$ as $n \to \infty$. (Here, and in the sequel, we denote by $|\cdot|$ the cardinality and by Area the Lebesgue measure.)

Assume, for a moment, that the growth at the boundary is also regular enough so that

$$\frac{1}{n} \cdot |\mathcal{N}^n \setminus \mathcal{N}^{n-1}| \to c \qquad \text{as } n \to \infty, \tag{2.2}$$

for some constant $c > 0$. Then $c = 2\mathrm{Area}(\mathrm{co}(\mathcal{N}))$, and a short computation gives

$$\max_t P(\gamma_t(x) = 1) \sim \sqrt{2e^{-1}\mathrm{Area}(\mathrm{co}(\mathcal{N}))} \cdot \sqrt{p} \qquad \text{as } p \to 0.$$

In fact, we show below (Lemma 2.4) that (2.2) is always true. Essentially, we show that the boundary of $\mathcal{N}^n$ is quickly trapped into a stable pattern. Figure 3 provides an illustration in which points in $\mathcal{N}^n \setminus \mathcal{N}^{n-1}$ are periodically shaded.

$$\mathcal{N} = \begin{array}{ccccccc} \bullet & \cdot & \cdot & \cdot & \cdot & \bullet & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \bullet \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \bullet & \cdot & \cdot & \cdot & \cdot & \bullet \end{array}$$

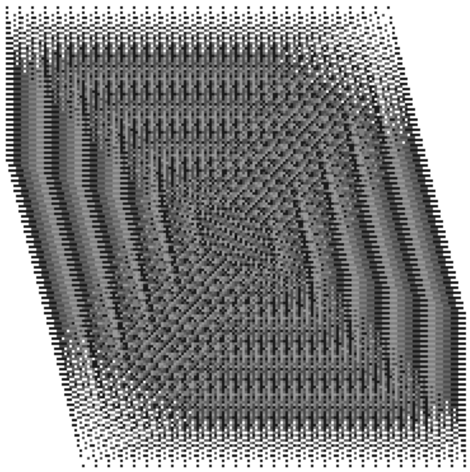

Figure 3. $\mathcal{N}^{30}$ for the specified $\mathcal{N}$.

Lemma 2.3. *There exists a constant M (depending only on $\mathcal{N}$) such that the set $\mathcal{N}^n$ is included in $n \cdot co(\mathcal{N})$ and includes all integer points in $(n - M) \cdot co(\mathcal{N})$.*

Proof. The inclusion $\mathcal{N}^n \subset n \cdot \mathrm{co}(\mathcal{N})$ is elementary. Next, it follows from the results in [Gra2] that there exists an $\epsilon > 0$ so that if A is a subset of $\mathbf{R}^2$ with $\mathcal{C}^2$ boundary whose curvature does not exceed ϵ, then $A + \mathcal{N} = A + \mathrm{co}(\mathcal{N})$. Moreover, the curvature of the boundary of $A + co(\mathcal{N})$ remains at most ϵ. Let A_0 now be any set with boundary curvature at most ϵ, say $A_0 = B_2(0, 1/\epsilon)$.

By irreducibility, there exists an n_0 so that $A_0 \cap \mathbf{Z}^2 \subset \mathcal{N}^{n_0}$. A simple conjugacy argument ([GG]) yields $(A_0 + (n - n_0) \cdot \mathrm{co}(\mathcal{N})) \cap \mathbf{Z}^2 \subset \mathcal{N}^n$. □

Lemma 2.4. *(2.2) holds.*

Proof. Take $m \geq 2$ and integers $x_1 < x_2 < \cdots < x_m$ and $\mathcal{M} = \{x_1, \ldots, x_m\}$. If $a = \text{g.c.d.}(\mathcal{M} - x_1)$, then elementary number theory implies that there exists a constant M_0 so that

$$[nx_1 + M_0, nx_m - M_0] \cap a\mathbf{Z} = [nx_1 + M_0, nx_m - M_0] \cap \mathcal{M}^n. \tag{2.3}$$

Let e_1, e_2 be the usual basis vectors and assume that β_M is such that $\mathcal{N} \subset \mathbf{Z}e_1 + (-\infty, \beta_M]e_2$. Moreover, assume that $N_l = \min\{\alpha : \alpha e_1 + \beta_M e_2 \in \mathcal{N}\}$ and $N_r = \max\{\alpha : \alpha e_1 + \beta_M e_2 \in \mathcal{N}\}$ are different, $N_l < N_r$. Then, we claim that, for any fixed $k \in \{0, 1, 2, \ldots\}$, either $(\mathbf{Z}e_1 + n(\beta_M - k)e_2) \cap \mathcal{N}^n = \emptyset$ for all n, or else there exists positive integer constants m_k, M_k and $a_1^{(k)}, \ldots, a_{m_k}^{(k)}$ so that

$$\begin{aligned} &([-nN_l + M_k, nN_r - M_k] \cap \left(\cup_{i=1}^{m_k} a_i^{(k)}\mathbf{Z}\right))e_1 + n(\beta_M - k)e_2 \\ =&([-nN_l + M_k, nN_r - M_k]e_1 + n(\beta_M - k)e_2) \cap \mathcal{N}^n. \end{aligned} \tag{2.4}$$

Indeed, for $k = 0$, (2.4) follows immediately from (2.3). In words, we know that the last horizontal line of $\mathcal{N}^n$ in the vertical direction is periodic, away from the edges. But then this argument shows that the same must be true for the next furthest line, and so on. Hence an induction argument with a repeated application of (2.3) establishes (2.4). By Lemma 2.3, for large enough k, $m_k = 1$, $a_1^{(k)} = 1$, and we can take $M_k = M$. This means that at least the boundary in the e_2 direction stabilizes away from the edges.

However, a similar argument (with a few more notational complications) will work in any direction u such that a line with normal u contains at least two extreme points of $\text{co}(\mathcal{N})$. Only finitely many such directions are determined by the boundary of $\text{co}(\mathcal{N})$ and it is easy to combine them to end the proof. $\square$

Next up is a well–known result about a continuous percolation model (for a proof, see [ZS] and [Roy]), which identifies the constant λ_c in the statement of Theorem 3.

Lemma 2.5. *Assume that $\mathcal{P} \subset \mathbf{R}^2$ is a Poisson random field with intensity 1. Attach to each $x \in \mathcal{P}$ a random variable $U_x > 0$; assume that U_x are i.i.d., and bounded. Define, for any fixed $\lambda > 0$, the random set $W(\lambda) = \mathcal{P} + \lambda \cdot U_x \cdot co(\mathcal{N}) \subset \mathbf{R}^2$. Let $W(\lambda, 0)$ be the connected component of $W(\lambda)$ which includes 0. Then there exists a $\lambda_c \in (0, \infty)$ such that*

$$\begin{aligned} &E(\text{Area}(W(\lambda, 0))) < \infty &&\text{for } \lambda < \lambda_c \\ &P(\text{Area}(W(\lambda, 0)) = \infty) > 0 &&\text{for } \lambda > \lambda_c. \end{aligned}$$

The above statement remains unchanged (and so does λ_c) if Area$(W(\lambda, 0))$ is replaced by $|\mathcal{P} \cap W(\lambda, 0))|$, or the diameter of $W(\lambda, 0)$. Moreover, $W(\lambda)$ percolates, *i.e. contains an unbounded* *connected set, if $\lambda > \lambda_c$, while all connected components are bounded if $\lambda < \lambda_c$. Finally, $W(\lambda)^c$ percolates if $\lambda < \lambda_c$ and it does not if $\lambda > \lambda_c$.*

In this section, we only need the case when all the U_x are point masses at 1. To prove Theorem 3, we use a basic coupling between $W(\lambda)$ and ξ_t which is similar to the one used by Zuev and Sidorenko in their pioneering papers ([ZS]).

Proof of Theorem 3. Fix the density p and define $Q(x,p) = \{y \in \mathbf{R}^2 : ||\sqrt{p}x - y||_\infty \le \sqrt{p}/2\}$. Let $\mathcal{P}$ be a Poisson random field with density 1. Declare a site $x \in \mathbf{Z}^2$ to be *occupied* if $Q(x,p) \cap \mathcal{P} \ne \emptyset$. Since $\text{Area}(Q(x,p)) = p$, and $\text{Area}(Q(x,p) \cap Q(y,p)) = 0$ for $x \ne y$, the sites in $\mathbf{Z}^2$ are independently occupied, each with probability $1 - e^{-p}$. Define ξ_0 by making $\{\xi_0 = 1\}$ equal to the set of occupied sites.

We claim that there exists a constant $N > 0$ (again dependent only on $\mathcal{N}$) so that,

$$W((t-N)\sqrt{p}) \subset \sqrt{p}(\{\xi_t = 1\} + B_\infty(0,1/2)) \subset W((t+N)\sqrt{p}), \tag{2.5}$$

for every t and p.

To prove (2.5), assume first that a point $x \in \mathbf{R}^2$ is in the leftmost set of (2.5). Then there is a $y \in \mathcal{P}$ such that $x \in y + (t-N)\sqrt{p} \cdot \text{co}(\mathcal{N})$. Find a $z \in \mathbf{Z}^2$ so that $||y - \sqrt{p}z||_\infty \le \sqrt{p}/2$. This means that $\xi_0(z) = 1$ and hence the middle set in (2.5) covers $\sqrt{p}z + \sqrt{p}(t-M) \cdot \text{co}(\mathcal{N})$ by Lemma 3.3. Since $\mathcal{N}$ is supercritical, $B_\infty(0,1/2) \subset \text{co}(\mathcal{N})$, and so x is in the middle set of (2.3) if we choose $N = M + 1$.

Assume now that x is in the middle set of (2.5). Hence there is a $y \in \mathbf{Z}^2$ so that $\xi_t(y) = 1$ and $||x - \sqrt{p}y||_\infty \le \sqrt{p}/2$. Then there exists a $z \in \mathbf{Z}^2$ so that $\xi_0(z) = 1$ and $y \in z + t \cdot \text{co}(\mathcal{N})$, and, in turn, some $w \in \mathcal{P} \cap Q(z,p)$. Therefore $x \in w + (t+2)\sqrt{p} \cdot \text{co}(\mathcal{N})$.

It is easy to translate (2.5) into a statement about T_a. Namely, $T_a(1 - e^{-p}) \in [\lambda_c/\sqrt{p} - 2a - N - 1, \lambda_c/\sqrt{p} + N + 1]$, finishing the proof. □

3. Growth dynamics with continuous nucleation.

We start the analysis of $\tilde{\xi}_t$ by associating to every site x, independently of other sites, a geometric random variable S_x with $P(S_x = k) = p(1-p)^{k-1}$, $k = 1, 2, \ldots$. Then,

$$\{\xi_t = 1\} = \bigcup \{x + \mathcal{N}^{t - S_x} : S_x \le t\}. \tag{3.1}$$

Note that $\tilde{T}_a(p)$ is a translation invariant function of the independent family $\{S_x : x \in \mathbf{Z}^2\}$, hence a determinstic quantity. Moreover, (3.1) and (2.2) yield, after a straightforward computation,

$$P(\tilde{\xi}_t(0) = 0) = \prod_{k=0}^{t} P(S_0 > t-k)^{|\mathcal{N}^k \setminus \mathcal{N}^{k-1}|} = (1-p)^{(\text{Area}(\text{co}(\mathcal{N}))/3)t^3 + o(t^3)}. \tag{3.2}$$

If we denote the first passage time to the origin by τ, i.e. $\tau = \min\{t : \xi_t(0) = 1\}$, then (3.2) immediately implies that $p^{1/3}\tau$ converges weakly as $p \to 0$ to a non–degenerate random variable. Therefore, most sites make the 0 to 1 transition at times of the order $p^{1/3}$, which certainly makes Theorem 4 plausible.

In this section, we use Lemma 2.5 with U_x uniformly distributed on $[0,1]$. Before proceeding with the proof of Theorem 3 we need a few preliminary results. Denote $W^{\epsilon}(\lambda) = W(\lambda)+\epsilon\cdot\mathrm{co}(\mathcal{N})$, and $W^{-\epsilon}(\lambda) = \mathcal{P} + \max\{\lambda U_x - \epsilon, 0\}\cdot\mathrm{co}(\mathcal{N})$.

Lemma 3.1. *If $\lambda < \lambda_c$ then there exists an $\epsilon > 0$ so that the ϵ-fattening $W^{\epsilon}(\lambda)$ does not percolate.*

Proof. For $0 \le a < b$, let $W'([a,b],\mu) = \bigcup\{x + U'_x \cdot \mathrm{co}(\mathcal{N}) : x \in \mathcal{P}(\mu)\}$, where U'_x are now independent and uniform on $[a,b]$ and $\mathcal{P}(\mu)$ is a Poisson point location with intensity μ. Note that $W^{\epsilon}(\lambda) = W'([\epsilon, \lambda+\epsilon], 1)$.

There exists a $\delta > 0$ so that $W((\lambda+\delta)(1+\delta))$ does not percolate, hence (as is easily seen by shrinking the space by a factor of $1/(1+\delta)$) neither does $W'([0,\lambda+\delta], 1+\delta)$. However, by discarding points $x \in \mathcal{P}(1+\delta)$ for which $U'_x < \epsilon$, $W'([0,\lambda+\delta],1+\delta)$ can be seen to dominate $W'([\epsilon,\lambda+\delta],(1+\delta)(1-\epsilon/(\lambda+\delta)))$. If ϵ is chosen small enough so that $\epsilon < \delta$ and $(1+\delta)(1-\epsilon/(\lambda+\delta)) \ge 1$, then $W'([\epsilon,\lambda+\epsilon],1)$ cannot percolate. □

Lemma 3.2. *If $\lambda > \lambda_c$ then there exists an $\epsilon > 0$ so that $W^{-\epsilon}(\lambda)$ percolates.*

Proof. The random variable $\max\{\lambda U_x - \epsilon, 0\}$ dominates a uniform random variable on $[0, \lambda-\delta]$ with probability $1 - \epsilon/\delta$. If one chooses $\epsilon = \delta^2$, then $W^{-\epsilon}(\lambda)$ dominates $W'([0,\lambda-\delta], 1-\delta)$, which does percolate for small enough δ. □

Lemma 3.3. *Given $p \in [0,1]$ and any integer $t > 0$ let*

$$u_r = \frac{p(1-p)^{t-r}}{1-(1-p)^t}, \qquad v_r = \sum_{i=1}^{r} u_i = \frac{(1-p)^{-r}-1}{(1-p)^{-t}-1},$$

for $r = 1,\ldots,t$. (Note that $v_t = 1$.) Define a real–valued function f on $[0,1]$ by $f(0) = 0$ and $f(v) = rp^{1/3}$ for $v \in (v_{r-1}, v_r]$, $r = 1,\ldots,t$. If there exists a constant $\lambda > 0$ so that $tp^{1/3} \to \lambda$ as $p \to 0$, then $\sup_{v\in[0,1]} |f(v) - \lambda v| \to 0$ as $p \to 0$.

Proof. Since $tp \to 0$, $(1-p)^{-r} = 1 + rp + a_r$, where $|a_r| \le C(rp)^2$, for some constant C. Therefore,

$$|f(v_r) - \lambda v_r| = \left| rp^{1/3} - \lambda \frac{r}{t} \frac{1 + a_r/(rp)}{1 + a_t/(tp)} \right| \le |tp^{1/3} - \lambda| + 2C\lambda tp.$$

Moreover, a simple calculation shows that $\sup_r(v_r - v_{r-1}) = \sup_r u_r \sim p^{1/3}/\lambda$ as $p \to 0$. Finally, if $v \in (v_{r-1}, v_r]$, then $|f(v) - \lambda v| \le |f(v_r) - \lambda v_r| + \lambda(v_{r+1} - v_r) \le |tp^{1/3} - \lambda| + Cp^{1/3}$, for a (different) constant C. □

Proof of Theorem 3. We prove that $T_2(p)p^{1/3} \to \lambda_c^{2/3}$ as $p \to 0$.

Let $R_t(x)$ equal 0 if $S_x \ge t$ and $t - S_x$ otherwise. This is the radius of the growing set which was generated by a seed at x during the time interval $[0, t]$. Observe that $P(R_{t+1}(x) = r \mid R_{t+1}(x) > 0) = u_r$, $r = 1, \dots, t$, where u_r are defined in the statement of Lemma 3.3. Now, assume that $\mathcal{P}$ is an intensity 1 Poisson random field and declare a site $x \in \mathbf{Z}^2$ *open at time* $t + 1$ if

$$B_\infty(x\sqrt{1-(1-p)^t}, \frac{1}{2}\sqrt{1-(1-p)^t}) \cap \mathcal{P} \ne \emptyset.$$

This makes sites open independently with probability $P(R_{t+1}(x) > 0)$. Adjoin to each site x that is open at time t a random variable $V_{t+1}(x)$ with $P(V_{t+1}(x) = r) = u_r$, $r = 1, \dots, t$, independently of other points. We get, by (3.1),

$$\{\tilde{\xi}_{t+1} = 1\} = \bigcup_{x \text{ open at time } t+1} \left(x + \mathcal{N}^{V_{t+1}(x)}\right) \tag{3.3}$$

Assume that the claim of the theorem is false. Then, we can find some sequence p_n converging to 0, and $\lambda \in [0, \infty] \setminus \{\lambda_c\}$, such that $\tilde{T}_a(p_n)p_n^{1/3} \to \lambda^{2/3}$ as $n \to \infty$. We will assume that $\lambda \in (0, \infty)$; the other two cases are then treated easily.

The first possibility is that $\lambda < \lambda_c$. Take $t = \tilde{T}_a(p_n)$. We claim that, if p_n is small enough, then the subset of $\mathbf{R}^2$: $\sqrt{1-(1-p_n)^t} \cdot (\{\tilde{\xi}_{t+1} = 1\} + B_\infty(0, (a+1)/2))$ does not percolate (i.e. contains no unbounded connected set). This will certainly show that $\{\tilde{\xi}_{t+1} = 1\}$ does not $B_\infty(0, a)$–percolate, implying $t < \tilde{T}_a(p_n)$, a contradiction. To prove the claim, note that $\sqrt{1-(1-p_n)^t}/p_n^{1/3} \to \lambda^{1/3}$, so (3.3) and Lemma 3.3 imply that for every $\epsilon > 0$ there exists a large enough n so that the set $\sqrt{1-(1-p_n)^t} \cdot (\{\tilde{\xi}_{t+1} = 1\} + B_\infty(0, (a+1)/2))$ is included in the fattening $W(\lambda)^\epsilon$. Now, use the ϵ from Lemma 3.1.

On the other hand, if $\lambda > \lambda_c$, then take $t = \tilde{T}_a(p_n) - M - 2$, with M as in Lemma 2.3. If p_n is small enough, then it follows from Lemma 2.3, (3.3), Lemma 3.3, and Lemma 3.2 that $\sqrt{1-(1-p_n)^t} \cdot (\{\tilde{\xi}_{t+M+1} = 1\} + B_\infty(0, (a+1)/2)) \subset \mathbf{R}^2$ percolates. Therefore, $\{\tilde{\xi}_{t+M+1} = 1\}$ $B_\infty(0, a)$-percolates, and so $t \ge T_2(p_n) - M - 1$, a contradiction. □

4. Some percolation preliminaries.

In this section, we state the results needed to prove Theorem 2. These are either applications of general percolation theory, or else follow by minor modifications of well–known arguments, so detailed proofs will be omitted.

Fix a positive integer k. Recall that γ_0 is an i.i.d. collection of Bernoulli random variables with density p. Call the sites of the set $\{\gamma_0 = 1\} + B_\infty(0,k)$ *k–occupied*, and other sites in $\mathbf{Z}^2$ *k–vacant*. Let $\mathcal{C}_k^1(x)$ (resp. $\mathcal{C}_k^\infty(x)$) be the ℓ^1–cluster (resp. ℓ^∞–cluster) of occupied sites which includes x, i.e. the set of all occupied points which can be connected to x by an ℓ^1–path (resp. ℓ^∞–path) of occupied sites. A self–avoiding ℓ^1–path (resp. ℓ^∞–path) which circles once around the origin is called an *ℓ^1–circuit* (resp. an *ℓ^∞–circuit*). Define:

$$p_c^k = \inf\{p : P(|\mathcal{C}_k^1(0)| = \infty) > 0\},$$
$$\tilde{p}_c^k = \inf\{p : P(|\mathcal{C}_k^\infty(0)| = \infty) > 0\}.$$

Note that $p_c^0 = p_c \approx 0.593$ is the critical density for the usual site percolation on $\mathbf{Z}^2$, while $\tilde{p}_c^0 = 1 - p_c$ is the critical density for its dual.

Lemma 4.1.

(1) If $p \le p_c^k$, then there exists an infinite sequence of disjoint k–vacant ℓ^∞–circuits.

(2) If $p \ge p_c^k$, then there exists an infinite sequence of disjoint k–occupied ℓ^1–circuits.

(3) The k–occupied sites ℓ^1–percolate iff $p > p_c^k$, while k–vacant sites ℓ^∞–percolate iff $p < p_c^k$.

Proof. The statements for $p \ne p_c$ follow by straightforward generalizations of the arguments in [Roy], together with standard geometric facts. (For example, an infinite k–occupied ℓ^∞–path does not exist iff there is an infinite sequence of disjoint k–vacant ℓ^1–circuits.) In fact, away from the critical values the necessary version of the Russo–Seymour–Welsh (RSW) lemma ([Gri]) can be proved in a more direct fashion ([Gra1]).

If $p = p_c$, then the k–vacant sites again cannot ℓ^∞–percolate by the RSW lemma for k–vacant sites ([Roy]) and the standard form of Russo's argument ([Gri]). To see that the k–occupied sites cannot ℓ^1–percolate, assume they do and fix a small $\epsilon > 0$. Then, for a large enough n, the probability that there is an ℓ^1–path consisting of k–occupied sites which stays inside the rectangle $[0, 1.2n] \times [0, n]$ and connects the left edge with the right edge is at least $1-\epsilon$. (Otherwise infinitely many k–vacant ℓ^∞–circuits could be constructed using the argument of [Roy] and Chapter 6 of [Kes].) This enables us to decrease p a little while keeping the probability of a left–right crossing of $[0, 1.2n] \times [0, n]$ larger than $1 - 2\epsilon$. A standard oriented percolation scheme (together with a Peierls argument) can now be utilized to show that, for a small enough ϵ, an infinite ℓ^1–path of k–occupied sites emanating from the origin exists with positive probability. This contradicts the fact that $p = p_c$. $\square$

Of course, almost exactly the same arguments handle the ℓ^∞–percolation case:

Lemma 4.2.

(1) If $p \le \tilde{p}_c^k$, then there exists an infinite sequence of disjoint k–vacant ℓ^1–circuits.

(2) If $p \ge \tilde{p}_c^k$, then there exists an infinite sequence of disjoint k–occupied ℓ^∞–circuits.

(3) The k–occupied sites ℓ^∞–percolate iff $p > \tilde{p}_c^k$, while k–vacant sites ℓ^1–percolate iff $p < \tilde{p}_c^k$.

It is obvious that $p_{k+1} \le \tilde{p}_k \le p_k$ for all $k \ge 0$. For many of our arguments it is however crucial that strict inequalities hold.

Lemma 4.3. $\cdots < p_c^3 < \tilde{p}_c^2 < p_c^2 < \tilde{p}_c^1 < p_c^1 < \tilde{p}_c^0 = 1 - p_c < p_c^0 = p_c$.

Proof. To apply the arguments from Section 3 of [AG], note that ℓ^1–percolation (resp. ℓ^∞–percolation) of k–occupied sites is equivalent to the site percolation on the graph with vertices $\mathbf{Z}^2$ and edges between any two sites x and y for which $y - x \in B_\infty(0, 2k) + B_1(0,1)$ (resp. $y - x \in B_\infty(0, 2k+1)$). □

5. Ring dynamics in the GHM: the Moore neighborhood case.

We begin with some notation. By dist_r we mean distance taken in the norm $||\cdot||_r$. Choose any set $A \subset \mathbf{Z}^d$. Define its *ℓ^∞–boundary* $\partial^\infty A$ to consist of all sites $x \in A$ for which $\text{dist}_\infty(x, A^c) = 1$, and the *outside ℓ^∞–boundary* by $\partial_o^\infty A = \partial^\infty(A^c)$. Note that Lemma 2.1 then says that $\{\gamma_{t+1} = 1\} = \partial_o^\infty(\{\xi_t = 1\})$.

The proof of Theorem 2 starts with a geometrical lemma which is probably well–known.

Lemma 5.1. *Let $A \subset \mathbf{Z}^2$. Let $x, y \in \partial_o^\infty A$. Let x and y be connected by an ℓ^1–path (resp. ℓ^∞–path) contained in A^c and connected by an ℓ^1–path (resp. ℓ^∞–path) contained in $A \cup \partial_o^\infty A$. Then x and y can be connected by an ℓ^1–path (resp. ℓ^∞–path) contained in $\partial_o^\infty A$.*

Proof. A path (an ℓ^1–path or an ℓ^∞–path) will be called *self–avoiding* if the induced path in $\mathbf{R}^2$ (obtained by adding line segments between successive points) does not cross itself. Moreover, for two self–avoiding paths with the same endpoints π_1 and π_2, we define $K(\pi_1, \pi_2)$ to be the subset of $\mathbf{R}^2$ consisting of the union of the two induced paths together with all bounded components of the complement of this union.

We start by proving the ℓ^1 version. Let π_1 be a self–avoiding ℓ^1–path connecting x to y through $A \cup \partial_o^\infty A$ and let π_2 be a self–avoiding ℓ^1–path connecting x to y through A^c.

Assume first that the only common sites of π_1 and π_2 are x and y. Then we can, without loss of generality, assume that $K(\pi_1, \pi_2)$ is seen on the left while moving from x to y on π_2. Let Π be the set of all self–avoiding ℓ^1–paths which connect x to y through A^c and have $K(\pi_1, \pi)$ on their left (but allow π to possibly intersect π_1). We let $\pi \leq \pi'$ iff $K(\pi_1, \pi) \subset K(\pi_1, \pi')$. Note that two paths $\pi, \pi' \in \Pi$ always have a path π'' such that $\pi'' \leq \pi$ and $\pi'' \leq \pi'$. (Proof: Start at x and find the first point z where π and π' split. At this point start to follow the left path until it crosses the other path again. If the crossing point is y then the procedure is completed, otherwise continue the same way.) This property, together with the fact that Π is non-empty and finite, imply that there exists the smallest path ρ in Π. We claim that ρ consists only of sites in $\partial_o^\infty A$.

If not, then there is a z on ρ which is not on $\partial_o^\infty A$. Figure 4 identifies (up to rotations) all possible local configurations around z; we assume that the previous site on ρ was below z. The symbol $\square$ then indicates sites in ρ, and $\bullet$ indicates sites which are not in A (since $z \notin \partial_o^\infty A$). Eliminate z from ρ, transforming the path in the indicated way. If this transformation creates a loop in the path, simply discard it. We thereby obtain a path $\rho' \in \Pi$ such that $\rho' \leq \rho$, but $\rho' \neq \rho$, a contradiction.

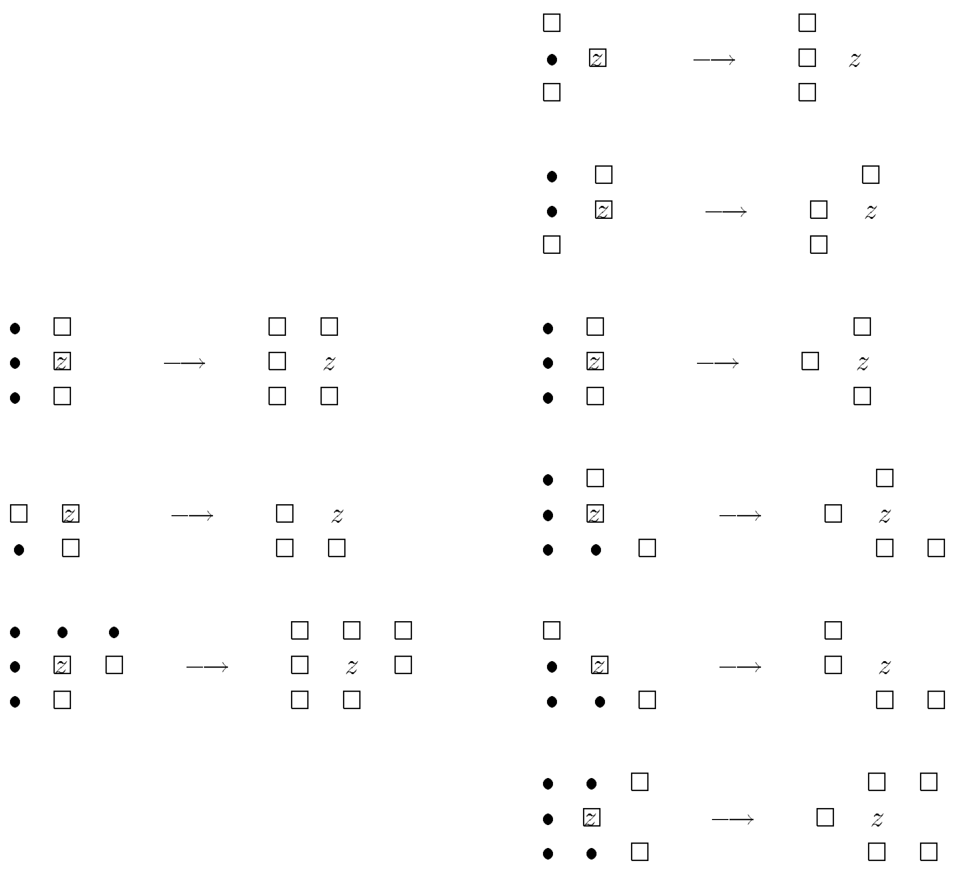

Figure 4. Transformation of ρ in the ℓ^1 case (left) and ℓ^∞ case (right).

If π_1 and π_2 do meet at sites other than x and y, let $x = v_0, v_1, \ldots, v_n = y$ be all points of intersection. Then do the above for x replaced by v_k and y replaced by v_{k+1}; $k = 0, \ldots, n-1$.

Finally, we prove the ℓ^∞ version. Let π_1 be a self–avoiding l^∞-path connecting x and y through $A \cup \partial_o^\infty A$, and let π_2 be a self–avoiding l^∞-path which connects x and y through A^c. Then ρ is defined in a way analogous to the ℓ^1 case. Again, to prove that ρ consists of sites on $\partial_o^\infty A$, one needs to define a transformation that will eliminate $z \notin \partial_o^\infty A$ from ρ. The transformation is given by Figure 4 (all other cases are rotations or reflections of those drawn). Note that every new point on ρ is within ℓ^1–distance 1 of z, and, because z is not on π_1, this implies that new points cannot be on the right of π_1. Apart from this issue, the argument is the same as before. □

Proof of Theorem 2. Start by defining $S = \cup_{k \ge 0}\{p_c^k, \tilde{p}_c^k\}$.

Assume first that $\tilde{p}_c^k < p < p_c^k$ for some $k \ge 0$. Then $\{\xi_t = 1\}$ ℓ^∞–percolates at $t = k$, but neither ℓ^1–percolates for $t \le k$ nor ℓ^∞–percolates for $t < k$. We first prove that $\{\gamma_t = 1\}$ ℓ^∞-percolates for $t = k$. This is obvious for $k = 0$, so we can assume that $k \ge 1$.

Assume that $v_0, v_1, v_2, v_3, \ldots$ is a self–avoiding infinite ℓ^1–path in $\{\xi_{k-1} = 0\}$ (which exists by Lemma 4.2(3)). Let $i_n, n = 1, 2, \ldots$ be such that all v_{i_n} belong to the infinite ℓ^∞–cluster of 1's in ξ_k. (Such indices must exist by Lemma 4.2(2), because 1's in ξ_k form an infinite sequence of disjoint ℓ^∞–circuits.) Then v_{i_n} and $v_{i_{n+1}}$ can be connected by an ℓ^∞–path in $\{\xi_k = 1\} = \{\xi_{k-1} = 1\} \cup \partial_o^\infty\{\xi_{k-1} = 1\}$. By Lemma 5.1, we can connect v_{i_n} and $v_{i_{n+1}}$ via an ℓ^∞–path π_n in $\partial_o^\infty\{\xi_{k-1} = 1\}$. The union of paths π_n is an infinite ℓ^∞–connected set, and consists of sites $v \in \partial_o^\infty\{\xi_{k-1} = 1\}$, hence such that $\xi_{k-1}(v) = 0$ and $\xi_k(v) = 1$, hence $\gamma_k(v) = 1$. Therefore $\{\gamma_k = 1\}$ ℓ^∞–percolates.

Moreover, by Lemma 4.1(3), there is an ℓ^∞–path of 0's in ξ_k. Since 1's in ξ_{k+1} ℓ^1–percolate and form ℓ^1–circuits (by Lemma 4.1(2)), this path must have infinitely many sites in the infinite ℓ^1–cluster of $\{\xi_{k+1} = 1\}$. As in the preceding paragraph, Lemma 5.1 assures that 1's in γ_{k+1} ℓ^∞–percolate.

Finally, 0's in ξ_{k+1} do not ℓ^∞–percolate (Lemma 4.1(3)). Hence 1's in γ_{k+2} cannot ℓ^∞–percolate and neither can they at any succeeding time. Of course $\gamma_t \subset \xi_t$, so $\{\gamma_t = 1\}$ cannot ℓ^∞–percolate for $t < k$. As neither $\{\xi_k = 1\}$ nor $\{\xi_{k+1} = 0\}$ ℓ^1–percolate, $\{\gamma_t = 1\}$ cannot ℓ^1–percolate for $t \in \{k, k+1\}$. Therefore we have proved the claim of the theorem with $T(p) = k$ in this case.

Assume now that $p_c^{k+1} < p < \tilde{p}_c^k$. Then $\{\xi_t = 1\}$ ℓ^1–percolates at time $t = k+1$, but it does not ℓ^∞–percolate for $t \le k$. There is an infinite ℓ^1–path of 0's in ξ_k (Lemma 4.2(3)), which intersects the infinite ℓ^1–cluster of 1's in ξ_{k+1} infinitely many times (Lemma 4.1(2)(3)). A procedure similar to the one above, ℓ^1 version of Lemma 5.1 shows that 1's in γ_{k+1} ℓ^1–percolate. They cannot ℓ^∞–percolate in γ_t for $t \geq k+2$ since 0's in ξ_{k+1} do not ℓ^∞–percolate (Lemma 4.1(3)). This proves the claim of the theorem, with $T(p) = k+1$ in this case.

Finally, we consider the case where p equals some critical value, assuming first that $p = p_c^k$ for some k. We then know that 1's in ξ_k ℓ^∞–percolate and form infinitely many disjoint ℓ^∞–circuits (Lemma 4.2(2)(3)), while 0's in ξ_{k-1} ℓ^1–percolate. As we have seen, this implies that 1's in γ_k ℓ^∞-percolate. An infinite collection of disjoint ℓ^1–circuits of 1's in ξ_k (Lemma 4.1(2)) prevents ℓ^∞–percolation of $\{\gamma_{k+1} = 1\}$. As $\{\xi_k = 1\}$ does not ℓ^1–percolate (Lemma 4.1(3)), neither can $\{\gamma_k = 1\}$. Hence $T(p) = k$ in this case.

If $p = \tilde{p}_c^k$, then there is an ℓ^∞–path of 0's in ξ_k (Lemma 4.1(3)), which is intersected infinitely many times by the infinite ℓ^1–cluster of 1's in ξ_{k+1} (Lemma 4.1(2)). Again, this implies that $\{\gamma_{k+1} = 1\}$ l^∞–percolates for $t = k+1$. Moreover, $\{\gamma_k = 1\}$ does not ℓ^∞–percolate (as $\{\xi_k = 1\}$ does not, by Lemma 4.2(3)), and neither can $\{\gamma_{k+2} = 1\}$ (as $\{\xi_{k+1} = 0\}$ does not). Since $\{\xi_k = 0\}$ does not ℓ^1–percolate (Lemma 4.2(3)), neither does $\{\gamma_{k+1} = 1\}$. Therefore, $T(p) = k+1$ in this final case. $\Box$

6. Ring dynamics in the GHM: the general neighborhood case.

Proof of Theorem 1. Throughout this proof C is a generic positive integer constant, which may change value from one appearance to another, and depends only on $\mathcal{N}$. Recall the coupling of ξ_t and W from the proof of Theorem 3 which establishes (2.5). That proof and Lemma 2.5 then imply that there exists a C so that if $t_2 > \lambda_c p^{-1/2} + C$, then $\{\xi_{t_2} = 1\}$ ℓ^1–percolates and the infinite ℓ^1–cluster contains an infinite sequence of disjoint ℓ^1–circuits. On the other hand, if $t_1 < \lambda_c p^{-1/2} - C$, then $\{\xi_{t_1} = 0\}$ ℓ^∞–percolates.

For every $x \in \mathbf{Z}^2$, let $e(x)$ be the "excitation time" of x, when $\gamma_{e(x)}(x) = 1$. Assume that t_1, t_2 are as above, with $t_2 - t_1 \leq C$. Let $v_0, v_1, \dots$ be a self–avoiding ℓ^∞–path inside $\{\xi_{t_1} = 0\}$. Let $i_k, k = 1, 2, \dots$ be such that v_{i_k} are in in the infinite ℓ^1–cluster of $\{\xi_{t_2} = 1\}$. By Lemma 5.1 we can connect v_{i_k} and $v_{i_{k+1}}$ by an ℓ^∞–path in $\partial_o^\infty\{\xi_{t_2} = 1\}$. However, Lemma 2.3 implies that there exists a large enough C so that $\xi_{t_2+C}(x) = 1$ for every $x \in \partial_o^\infty\{\xi_{t_2} = 1\}$. Thus we produce a self–avoiding infinite ℓ^∞–path $v_0', v_1', \dots$ inside $\{\xi_{t_1} = 0\} \cap \{\xi_{t_1+C} = 1\}$. By Lemma 2.1, $e(v_k') \in [t_1, t_1 + C]$ for every k, and thus there exists an $x_k \in B_\infty(v_k', C)$ so that $\gamma_{t_1}(x_k) = 1$. However, if a is large enough, then $\{x_k\}$ forms an infinite $B_\infty(0, a)$–percolating set, proving part (1).

Parts (2) and (3) follow easily from Theorem 3 and its proof. $\Box$

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REFERENCES

[AG] M. Aizenman, G. Grimmett, *Strict monotonicity for critical points in percolation and ferromagnetic models,* J. Stat. Phys. 63 (1991), 817–835.

[Bra] R. M. Bradley, *A Comment on "Ring Dynamics and percolation in an excitable medium,"* J. Chem. Phys. 86 (1987), 7245–7246.

[DG] R. Durrett, D. Griffeath, *Asymptotic behavior of excitable cellular automata,* Experimental Math. 2 (1993), 184–208.

[DN] R. Durrett, C. Neuhauser, *Epidemic with recovery in $d = 2$,* Ann. Appl. Prob. 1 (1991), 189–206.quit

[DS] R. Durrett, J. E. Steif, *Some rigorous results for the Greenberg-Hastings model,* J. Theor. Prob. 4 (1991), 669–690.

[Dur] R. Durrett, *Multicolor particle systems with large threshold and range,* J. Theor. Prob. 5 (1992), 127–152.

[FGG1] R. Fisch, J. Gravner, D. Griffeath, *Threshold-range scaling of excitable cellular automata,* Statistic and Computing 1 (1991), 23–39.

[FGG2] R. Fisch, J. Gravner, D. Griffeath, *Metastability in the Greenberg–Hastings model,* Ann. Appl. Prob. 3 (1993), 935–967.

[FK] S. Fraser, R. Kapral, *Ring dynamics and percolation in an excitable medium,* J. Chem. Phys. 85 (1986), 5682–5688.

[GH] J. M. Greenberg, S. P. Hastings, *Spatial patterns for discrete models of diffusion in exitable media,* SIAM J. Appl. Math. 34 (1978), 515–523.

[Gra1] J. Gravner, *Mathematical aspects of excitable media*, Ph. D. Thesis, University of Wisconsin, 1991.

[Gra2] J. Gravner, *The boundary of iterates in Euclidean growth models,* Trans. Amer. Math. Soc., to appear.

[Gra3] J. Gravner, *Recurrent ring dynamics in two–dimensional excitable cellular automata,* submitted.

[GG] J. Gravner, D. Griffeath, *First passage times for threshold growth dynamics on* $\mathbf{Z}^2$, Ann. Prob., to appear.

[Gri] G. Grimmett, "Percolation," Springer-Verlag, 1989.

[JM] W.A. Johnson and R.F. Mehl, *Reaction kinetics in processes of nucleation and growth*, Trans.

A.I.M.M.E. 135 (1939), 416–458.

[KW] R. Kapral, M. Weinberg, *Phase transformation kinetics in finite inhomogenuosly nucleated systems,* J. Chem. Phys. 11 (1989), 7146–7152.

[Kap] R. Kapral, *Discrete models for chemically reacting systems,* J. Math. Chem. 6 (1991), 113–163.

[Kes] H. Kesten, "Percolation Theory for Mathematicians," Birkhäuser, 1982.

[Pen] M. D. Penrose, *Single linkage clustering and continuum percolation,* J. of Multivariate Analysis 53(1995), 90–104.

[Roy] R. Roy, *The Russo–Seymour–Welsh theorem and the equality of critical densities and the "dual" critical densities for continuum percolation on* $\mathbf{R}^2$, Ann. Prob. 18 (1990), 1563–1575.

[SRC] J. M. Smith, A. L. Ritzenberg, R. J. Cohen, *Percolation theory and cardiac conduction,* Computers in Cardiology (1984), 175–178.

[Ste] J. Steif, *Two applications of percolation to cellular automata*, J. Stat. Phys. 78 (1995), 1325–1335.

[Wil] S. J. Willson, *On convergence of configurations*, Discrete Mathematics 23 (1978), 279–300.

[WR] N. Wiener, A. Rosenbluth, *The mathematical formulation of the problem of conduction of impulses in a network of connected excitable elements, specifically in cardiac muscle,* Arch. Inst. Cardiol. Mexico 16 (1946), 205–265.

[WTW] J. R. Weimar, J. J. Tyson, L. T. Watson, *Third generation cellular automaton for modeling excitable media,* Physica D 55 (1992), 328–339.

[ZS] S. A. Zuev, A. T. Sidorenko, *Continuous models of percolation theory I, II,* Theoretical and Mathematical Physics 62 (1985), 51–58, 171–177.