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POISSON SNAKE AND FRAGMENTATION

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Abstract: Our main object that we call the Poisson snake is a Brownian snake as introduced by Le Gall. This process has values which are trajectories of standard Poisson process stopped at some random finite lifetime with Brownian evolution. We use this Poisson snake to construct a self-similar fragmentation as introduced by Bertoin. A similar representation was given by Aldous and Pitman using the Continuum Random Tree. Whereas their proofs used approximation by discrete models, our representation allows continuous time arguments.

Keywords and phrases: Path-valued process, Brownian snake, Poisson process, fragmentation, coalescence, self-similarity critical probability, percolation

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1 Introduction

The Brownian snake introduced by Le Gall [Lg1] has shown its usefulness in the study of superprocesses and semi-linear partial differential equations. The literature on the subject is now abundant. The process has values which are Brownian path in $\mathbb{R}^d$ stopped at a certain time that we call the lifetime. As time grows the lifetime evolves as a reflecting linear Brownian motion. Roughly speaking, when the lifetime decreases the corresponding Brownian trajectories in $\mathbb{R}^d$ are just erased and when the lifetime increases the corresponding Brownian trajectories in $\mathbb{R}^d$ are prolonged by an independent piece of Brownian trajectory. We refer to [Lg1] for a rigorous definition. Some authors (see [BLL] for instance) have replaced Brownian motion in $\mathbb{R}^d$ by a different "spatial motion". In the Poisson snake that we consider, the spatial motion is an ordinary Poisson process of parameter $\alpha > 0$. This object has already been considered by Warren [Wa] and Watanabe [Wt]. Moreover, in view of application to dislocation, we let α vary and more precisely we construct a family of Poisson snakes indexed by $\alpha > 0$, in a coherent way. This object is described by the following existence theorem.

Theorem 1. *There exists a family $(\zeta_s, N_s^\alpha(u);\ u, s \geq 0,\ \alpha > 0)$ of random variables with values in $\mathbb{R}_+ \times \mathbb{N}$ such that*

1. *$(\zeta_s;\ s \geq 0)$ has the law of the normalised Brownian excursion.*
2. *For every $\alpha > 0$, $(\zeta_s, N_s^\alpha(\cdot);\ s \geq 0)$ is a Markov process.*
3. *For every $\alpha > 0$ and conditionally on $(\zeta_s;\ s \geq 0)$, the process $(N_s^\alpha(\cdot);\ s \geq 0)$ is a Markov process such that, for $0 \leq s \leq s'$,*
   - (a) *$N_s^\alpha(\cdot)$ is a Poisson process with rate 2α stopped at time ζ_s.*
   - (b) *$N_s^\alpha(u) = N_{s'}^\alpha(u)$ for $u \leq \inf_{[s,s']} \zeta = m(s, s')$.*
   - (c) *$N_{s'}^\alpha(m(s, s') + \cdot) - N_{s'}^\alpha(m(s, s'))$ is independent of $N_s^\alpha(\cdot)$.*
4. *For $0 < \alpha \leq \beta$ and $s \geq 0$, $\mathrm{Supp}(d_u N_s^\alpha(u)) \subset \mathrm{Supp}(d_u N_s^\beta(u))$ and $\left((N_s^\beta(u) - N_s^\alpha(u)\right)_{u \geq 0}$ is independent of $\left(N_s^\alpha(u)\right)_{u \geq 0}$.*

In the following we will use the notation $\hat{N}_s^\alpha = N_s^\alpha(\zeta_s -)$. We must notice that this limit may be $+\infty$. In fact, as the lifetime process is of infinite variation, it is easy to see that the set of times s such that $\hat{N}_s^\alpha = +\infty$ is dense in $[0, 1]$.

We use this family of Poisson snakes to construct a fragmentation process. An exhaustive study of fragmentation processes has been made by Bertoin in the last few years, see [Be] for instance. Informally, a fragmentation process is a Markov process which describes the evolution of a mass that breaks down into small pieces. It must also fulfill two additional conditions:

- A fragmentation property (a sort of branching property): the fragments present at some time then disaggregate independently from each other.
- A self-similar property: each fragment disaggregate in the same way as the initial process, up to a different rate.

These statements will be made precise after Theorem 2.

For us the fragmentation process will appear as a process $(F(\alpha);\ \alpha \geq 0)$ taking its values in the set $\mathcal{S}$ of decreasing sequences of non-negative real numbers with total sum at most 1. We first set $F(0) = (1, 0, \dots)$. For every $\alpha > 0$ we introduce the equivalence relation $\mathcal{R}_\alpha$ defined on $[0,1]$ by

$$u\mathcal{R}_\alpha v \iff \left(\hat{N}_u^\alpha = \hat{N}_v^\alpha \leq \hat{N}_s^\alpha \text{ for all } s \in [u,v]\right).$$

Then $F(\alpha)$ is simply defined as the decreasing sequence of the lengths (Lebesgue measure) of the $\mathcal{R}_\alpha$-equivalence classes. The figures below describe first the lifetime process with the values of N_t^α then the different $\mathcal{R}_\alpha$-equivalence classes. Figure 1 represents how $F(\alpha)$ is constructed from the Poisson snake for a fixed α. The first part of the picture is supposed to be a Poisson snake for some fixed $\alpha > 0$. We pictured the lifetime process (which should be a Brownian excursion of length 1) and the jumps of the Poisson paths which are, due to the snake property, horizontal lines. This breaks the Brownian excursion into several parts (see the second picture) which are, after some suitable time-change, sub-excursions of the initial process (see the third part of the Figure). We then consider the lengths of these sub-excursions which form the state of $F(\alpha)$ after a decreasing reordering.

The construction of the process $F(\alpha)$ is based on the same ideas as in [AP] but the formalism is slightly different. The results below are therefore not new but the techniques to prove them are completely different from Aldous-Pitman's arguments. See section 4 for the connections between the two constructions.

Theorem 2. *The process $(F(\alpha);\ \alpha \geq 0)$ is a self-similar fragmentation process with index 1/2.*

This means that $(F(\alpha))$ is a Markov process with values in $\mathcal{S}$ such that, if p_α denotes the law of $F(\alpha)$, for $0 < \alpha < \beta$, the conditional law of $F(\beta)$ knowing $F(\alpha) = s = (s_1, s_2, \dots)$ is the law of the decreasing reordering of $\bigcup_{i \geq 1} s_i F_i$ where the F_i are independent $\mathcal{S}$-valued random variables of law $p_{(\beta-\alpha)\ s_i^{1/2}}$. When we write $s_i F_i$ we mean that we multiply all the terms of the sequence F_i by the real number s_i.

In our model the self similarity will only result from the independent increments properties of Brownian motion and Poisson process. The index 1/2 of the fragmentation derives from the Brownian scaling exponent. See Section 3.

In [Be] it is shown that a self-similar fragmentation is in fact determined by three parameters: the index of similarity, the so-called erosion coefficient (which characterises the loss of mass during the evolution) and the dislocation measure. In our case, we already know that the index is 1/2. The erosion coefficient is null; this corresponds to the lack of continuous loss of mass, which is clear in our representation. Roughly speaking, the dislocation measure describes how a mass is dislocated at every jump of the fragmentation process. In our case such dislocation arises when a new atom of the Poisson measure is created which leads to the breaking of the corresponding mass in two components. Figure 2 represents the effect of a new atom which appears in a sub-excursion. We see that this new atom leads to two fragments. For three fragments to appear, the new atom should arise at exactly the level of a local minimum of the Brownian excursion which is a zero probability event. In other words, the dislocation measure puts mass only on sequences $(s_1, s_2, \dots)$ with only s_1, s_2 being non-zero. Then, Bertoin remarked (cf [Be]) that such a self-similar fragmentation is completely determined when we can describe

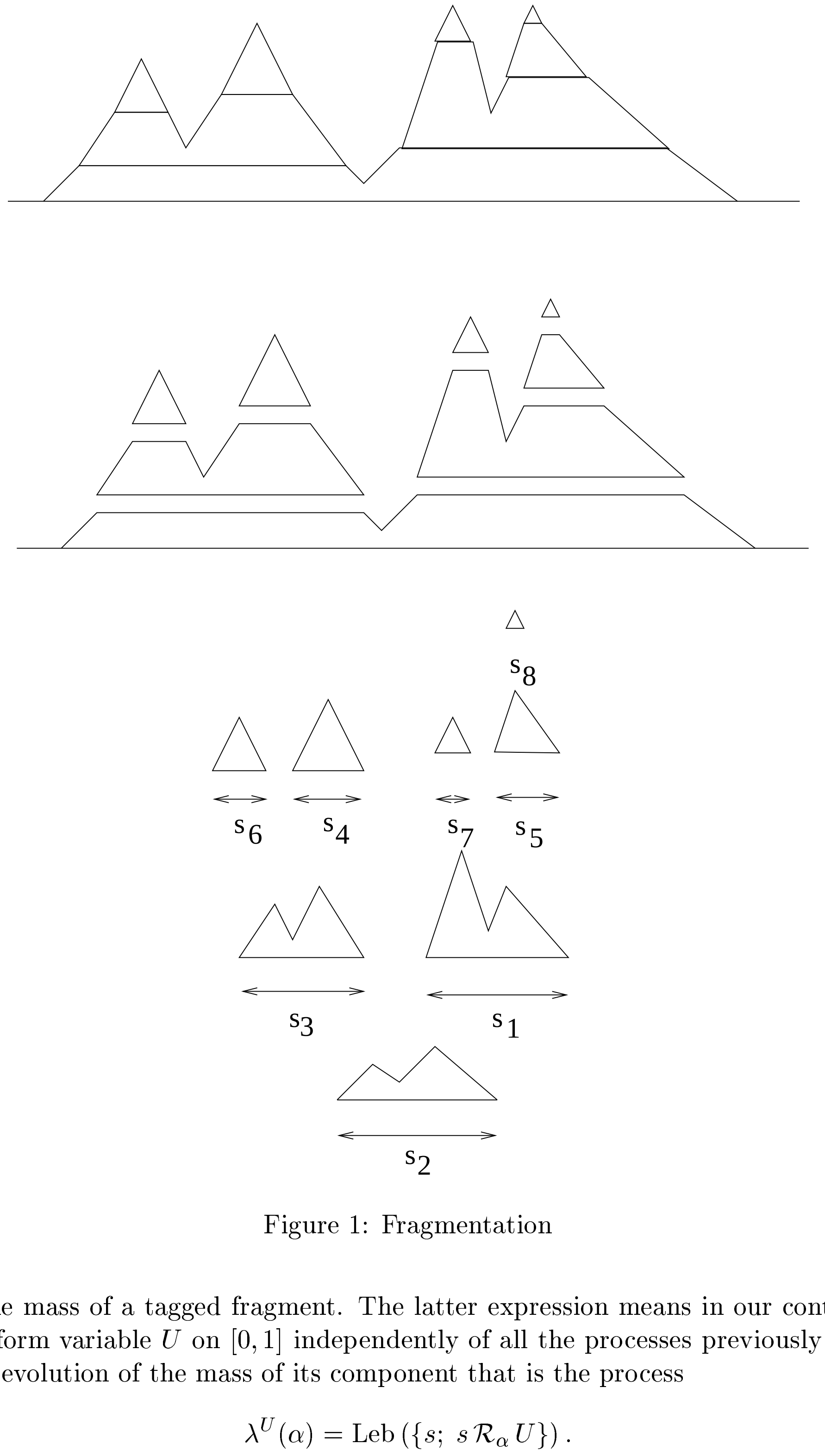

Figure 1: Fragmentation

the law of the mass of a tagged fragment. The latter expression means in our context that we choose a uniform variable U on $[0,1]$ independently of all the processes previously defined and we trace the evolution of the mass of its component that is the process

$$\lambda^U(\alpha) = \mathrm{Leb}\left(\{s;\ s\,\mathcal{R}_\alpha\,U\}\right).$$

Proposition 3. *The process* $(\lambda^U(\alpha))$ *has the same law as* $(1/(1+\sigma_\alpha))$ *where* (σ_α) *denotes a stable subordinator with index 1/2.*

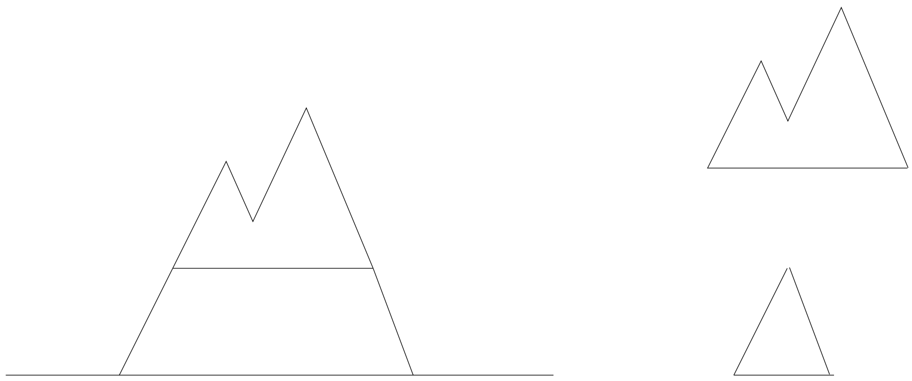

Figure 2: Dislocation of a sub-excursion

This proposition is essentially Theorem 6 of [AP]. See also the end of [Be]. As an application of our representation we will compute in Section 6 the dislocation measure independently of the previous results.

In reality our model could be considered as a replica of the Poisson fragmentation of the Aldous' Continuum Random Tree (CRT) explained in [AP]. The main difference is that we have built a "time"-parametrised process in the same way as Brownian excursion is a "time"-parametrised process" which can represent the CRT. As a consequence our proofs are completely different. Whereas Aldous and Pitman argue on discrete trees and pass to the limit, our main argument is obtained by continuous time stochastic calculus using [AS]. We first point out a symmetry property that would be called re-rooting symmetry in the tree terminology; see Subsection 3.2 and [A2] for a statement using the CRT. A consequence is that $(\lambda^U(\alpha))$ has the same law that $(\lambda^0(\alpha))$. We recall that $\lambda^0(\alpha) = \mathrm{Leb}(\{s;\ \hat{N}^\alpha_s = 0\})$. The law of this quantity, at least for fixed α, can be identified using a result of [AS] that we recall now.

Proposition 4. *Let us fix $\alpha > 0$ and set, for $s \geq 0$,*

$$A_s = \inf\left\{u;\ \int_0^u dv\ \mathbf{1}_{\{\hat{N}^\alpha_v = 0\}} > s\right\}$$

Then $(\zeta_{A_s};\ s \geq 0)$ has the same law as $(e(s) - \alpha\, s; s \geq 0)$ stopped at the first return at zero where e has the law of the normalised Brownian excursion.

How this result leads to Proposition 3 is detailed in Section 5.

2 Construction of Poisson snakes

Let M_u be the set of measures on $[0,u] \times \mathbb{R}_+$. We define the vague topology on M_u by

$$\mu_n \longrightarrow \mu \iff \forall f \in \mathcal{C}_c([0,u] \times \mathbb{R}_+),\ \int f d\mu_n \longrightarrow \int f d\mu$$

where $\mathcal{C}_c$ denotes the space of continuous functions with compact support. We recall that M_u endowed with this topology is a Polish space and we denote by d_u the corresponding distance.

We also denote by M the set of Radon measures on $\mathbb{R}_+ \times \mathbb{R}_+$ that we endow also with the vague topology.

Let $\mathcal{M}$ be the set of pairs (ζ, μ) where ζ is a nonnegative real number and μ is a Radon measure on $\mathbb{R}_+^2$ whose support is included in $[0,\zeta] \times \mathbb{R}_+$. We define, for $(\zeta,\mu),\ (\zeta',\mu') \in \mathcal{M}$

$$d\big((\zeta,\mu),(\zeta',\mu')\big) = |\zeta - \zeta'| + \int_0^{\zeta\wedge\zeta'} d_u(\mu_{(u)}, \mu'_{(u)}) \wedge 1\, du$$

where $\mu_{(u)}$ stands for the restriction of μ to $[0,u] \times \mathbb{R}_+$. It is easy to check that d is a distance on $\mathcal{M}$ and that $(\mathcal{M}, d)$ is a Polish space. Then, following the ideas of [BLL], we can construct a measure valued snake and so a family of probability measures $(\mathbb{P}^\ell)_{\ell>0}$ on $\mathcal{C}(\mathbb{R}_+, \mathcal{M})$ (the index ℓ corresponds to the duration of the lifetime of the snake):

Theorem 5. *There exists a continuous strong Markov process $(\zeta_s, \mathcal{N}_s;\ s \geq 0)$ with values in $\mathcal{M}$ such that*

1. *$(\zeta_s;\ s \geq 0)$ has the law of the Brownian excursion conditioned to have length ℓ.*
2. *Conditionally on $(\zeta_s;\ s \geq 0)$, the process $(\mathcal{N}_s;\ s \geq 0)$ is a Markov process such that, for $0 \leq s \leq s'$,*

$$\mathcal{N}_{s'}(dx\, dy) \overset{(d)}{=} \mathcal{N}_s(dx\, dy)\ \mathbf{1}_{x \leq m(s,s')} + \tilde{\mathcal{N}}(dx\, dy)\ \mathbf{1}_{m(s,s') \leq x \leq \zeta_{s'}}$$

where $m(s,s') = \inf_{[s,s']} \zeta$ and $\tilde{\mathcal{N}}$ denotes a Poisson point measure on $\mathbb{R}^2$ independent of $\mathcal{N}_s$ with intensity the Lebesgue measure on $(\mathbb{R}_+)^2$.

Sketch of proof. We follow the construction of Le Gall ([Lg1] Theorem 1.1). We first introduce a kernel $Q_{a,b}$ on M by

$$\int Q_{a,b}(\mathcal{N}, d\nu) F(\nu) = E\left[F\left(\mathcal{N}(dx\, dy)\ \mathbf{1}_{x \leq a} + \tilde{\mathcal{N}}(dx\, dy)\ \mathbf{1}_{a \leq x \leq b}\right)\right]$$

In this formula, $0 \leq a \leq b$, F is a bounded Borel function on M and $\tilde{\mathcal{N}}$ is a Poisson point measure with intensity the Lebesgue measure on $(\mathbb{R}_+)^2$. We denote by $\mathcal{F}(\mathbb{R}_+, M)$ the set of functions from $\mathbb{R}_+$ to M and by $\mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ the set of continuous function from $\mathbb{R}_+$ to $\mathbb{R}_+$. We fix $\eta \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ such that $\eta(0) = 0$. Using the Kolmogorov extension theorem we can define a probability R^η on $\mathcal{F}(\mathbb{R}_+, M)$ by

$$\begin{aligned} R^\eta\left(\nu(0) \in A_0, \nu(t_1) \in A_1, \dots, \nu(t_n) \in A_n\right) \\ = \mathbf{1}_{\{0 \in A_0\}} \int_{A_1 \times \cdots \times A_n} Q_{0,\eta(t_1)}(d\nu_1)\ Q_{m(t_1,t_2),\eta(t_2)}(d\nu_2) \\ \dots Q_{m(t_{n-1},t_n),\eta(t_n)}(d\nu_n) \end{aligned}$$

where $n \in \mathbb{N}^*$, $0 \leq t_1 \leq \cdots \leq t_n$, $m(r,s) = \inf_{[r,s]} \eta$, $A_0, A_1, \dots, A_n$ are measurable subsets of M. The compatibility conditions needed to apply the Kolmogorov extension theorem result from the properties of the kernels $Q_{a,b}$.

Let us denote by Γ^ℓ the law of the Brownian excursion of length ℓ. We define a probability $\mathbb{P}^\ell$ on $\mathcal{F}(\mathbb{R}_+, \mathcal{M})$ by $\mathbb{P}^\ell(d\eta\ d\nu) = \Gamma^\ell(d\eta)\ R^\eta(d\nu)$. The canonical process $(\zeta_s, \mathcal{N}_s;\ s \geq 0)$ then verifies, under $\mathbb{P}^\ell$, the properties 1. and 2.

It remains to check that there exists a continuous version of this process, which is rather easy since, for $r < s$,

$$d\big((\zeta_r, \mathcal{N}_r), (\zeta_s, \mathcal{N}_s)\big) = |\zeta_r - \zeta_s| + \int_0^{\zeta_r \wedge \zeta_s} \Big(d_u(\mathcal{N}_{r,(u)}, \mathcal{N}_{s,(u)}) \wedge 1\Big) du.$$

As $\mathcal{N}_{r,(u)} = \mathcal{N}_{s,(u)}$ for $u \leq \inf_{[r,s]} \zeta$, $\mathbb{P}^\ell$-a.s., we have

$$d\big((\zeta_r, \mathcal{N}_r), (\zeta_s, \mathcal{N}_s)\big) \leq |\zeta_r - \zeta_s| + (\zeta_r \wedge \zeta_s - \inf_{[r,s]} \zeta)$$

and the continuity of ζ under $\mathbb{P}^\ell$ gives the result.

The strong Markov property is eventually proved using the same arguments as in [Lg1]. ■

Now we can set

$$N_s^\alpha(u) = \mathcal{N}_s([0,u] \times [0, 2\alpha])$$

It is not difficult to prove that $(\zeta_s, N_s^\alpha(u);\ u, s \geq 0,\ \alpha > 0)$ is under $\mathbb{P}^1$ a Poisson snake in the sense of Theorem 1.

In what follows, we denote $\mathbb{P} = \mathbb{P}^1$. When the law is not mentioned, we implicitly use $\mathbb{P}$.

3 Self-similarity, scaling and re-rooting

3.1 Fragmentation property

We first prove Theorem 2. In what follows, we will work under the "excursion measure" $\mathbb{N}$ which is simply

$$\mathbb{N}(\cdot) = \int_0^{+\infty} \frac{d\ell}{\sqrt{8\pi}\ell^{3/2}} \mathbb{P}^\ell(\cdot).$$

We fix $\alpha > 0$. For a measure μ on $\mathbb{R}_+ \times \mathbb{R}_+$, we define

$$\tau^\alpha(\mu) = \inf\{u > 0, \mu([0,u] \times [0, 2\alpha]) > 0\}.$$

Then, we set

$$L_s^\alpha(\zeta, \mathcal{N}) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^s \mathbf{I}_{\{\tau^\alpha(\mathcal{N}_u) \leq \zeta_u < \tau^\alpha(\mathcal{N}_u) + \varepsilon\}} du$$

where the convergence holds $\mathbb{N}$-a.e. (see for instance [Lg1]). $L_\cdot^\alpha$ is called an exit local time. It is a continuous additive functional of the snake $(\zeta_s, \mathcal{N}_s)_{s \geq 0}$ which increases only when $\zeta_s = \tau^\alpha(\mathcal{N}_s)$.

Let us first recall the special Markov property for the snakes. We know that the set

$$\{s, \tau^\alpha(\mathcal{N}_s) < \zeta_s\}$$

is open $\mathbb{P}$-a.s. We denote by $\{(a_i, b_i), i \in I\}$ its connected components. For every $i \in I$, we define $(\zeta_s, \mathcal{N}_s^i) \in \mathcal{C}(\mathbb{R}_+, \mathcal{M})$ by

$$\zeta_s^i = \zeta_{(a_i + s) \wedge b_i} - \zeta_{a_i}$$

and

$$\mathcal{N}^i_s([0,t]\times A)=\mathcal{N}_{(a_i+s)\wedge b_i}([\zeta_{a_i},\zeta_{a_i}+t]\times A).$$

We denote by $\mathcal{F}^\alpha$ the σ-field generated by $\Big(\zeta_s\wedge\tau^\alpha(\mathcal{N}_s),\mathcal{N}_s\big(\cdot\cap([0,\tau^\alpha(\mathcal{N}_s)]\times\mathbb{R}_+)\big)\Big)_{s\geq 0}$. We also set

$$\kappa_s=\inf\left\{u>0,\int_0^u \mathbb{1}_{\zeta_r<\tau^\alpha(\mathcal{N}_r)}dr>s\right\}$$

and we denote by $\mathcal{G}^\alpha$ the σ-field generated by $\left(\zeta_{\kappa_s},\mathcal{N}_{\kappa_s}\right)_{s\geq 0}$. The σ-field $\mathcal{G}^\alpha$ corresponds to the knowledge of the paths of the Poisson snake N^α which stay at 0 whereas the σ-field $\mathcal{F}^\alpha$ corresponds to all the paths stopped when they reach 1. They only differ by the lengths of the excursions of the snake above level 1.

Then we have the following result (see for instance [BLL], Proposition 7).

Proposition 6. *The local time L^α_∞ is $\mathcal{G}^\alpha$-measurable. Moreover, if G is a nonnegative measurable function on $\mathcal{C}(\mathbb{R}_+,\mathcal{M})$,*

$$\mathbb{N}\left(\exp-\sum_{i\in I}G(\zeta^i,\mathcal{N}^i)\mid\mathcal{G}^\alpha\right)=\exp-L^\alpha_\infty\mathbb{N}(1-\exp-G).$$

This property is known as the "special Markov property". It describes how the excursions of the Poisson snake above level 1 behave knowing the number of excursions. ¿From this property, we immediately deduce from standard excursion theory the following lemma:

Lemma 7. *Conditionally on $\mathcal{F}^\alpha$, the excursions $(\zeta^i,\mathcal{N}^i)_{i\in I}$ form, under $\mathbb{N}$ and under $\mathbb{P}$, a sequence of independent $\mathcal{M}$-valued processes with respective distributions $\mathbb{P}^{b_i-a_i}$.*

We deduce that, conditionally on the lengths $(b_i-a_i)_{i\in I}$, the excursions $(\zeta^i,\mathcal{N}^i)_{i\in I}$ form under $\mathbb{P}$ a sequence of independent processes with respective distributions $\mathbb{P}^{b_i-a_i}$ and independent of $\mathcal{G}^\alpha$. Moreover, the law of the excursion (ζ_{κ_s}) is given by Proposition 4. So, conditionally on the lengths $(b_i-a_i)_{i\in I}$, the process $\left(\zeta_{\kappa_s}\right)$ has under $\mathbb{P}$ the law of the Brownian excursion of length $1-\ell$ where ℓ stands for the sum of these lengths: $\ell=\sum_{i\in I}(b_i-a_i)$.

Note that, for the fragmentation at time α, $1-\ell$ represents the mass of the component containing 0. Note however that the $(b_i-a_i)_{i\in I}$ are not the lengths of the other components. We now have to apply the results of lemma 7 for each sub-excursion $(\zeta^i,\mathcal{N}^i)$. For each of these excursions, we obtain a component of the fragmentation, associated to $\{\hat{N}^\alpha_s=1\}$, and sub-excursions, associated to $\{\hat{N}^\alpha_s\geq 2\}$. A proof by induction on the values of $\hat{N}^\alpha_s$ leads to the following result. The sub-excursions $(\zeta^j,\mathcal{N}^j)_{j\in J}$ associated to each equivalence class of the fragmentation at time α (as illustrated in figure 1) are, conditionally on their lengths $(\ell_j)_{j\in J}$, distributed as independent processes with respective distributions $\P^{\ell_j}$.

This result yields easily the fragmentation property: if $0<\alpha<\beta$, the law, under $\mathbb{P}$, of $F(\beta)$ knowing $F(\alpha)=(s_1,s_2,\dots)$ is the law of the decreasing reordering of $\bigcup_{i\geq 1}F_i$ where the F_i are independent random variables which have respectively the same law as $F(\beta-\alpha)$ under $\mathbb{P}_{s_i}$. The fact that this law is the same as the law of $s_iF\big((\beta-\alpha)s_i^{1/2}\big)$ under $\mathbb{P}$ follows directly from the scaling property of the Brownian motion and of the definition of the measure-valued snake. ■

3.2 Re-rooting

The goal of this section is to show that the sub-excursion created from the fragmentation that contains a uniform independent random variable is distributed as the sub-excursion that contains 0. To do so, we will use a trajectorial transformation that transforms the Poisson snake into another Poisson snake and that maps the excursion that contains the uniform random variable onto the excursion that contains 0. To describe this transformation, we need a Poissonian representation of the snake by sub-excursions (Proposition 8).

We define the measure $\tilde{\mathbb{N}}$ on $\mathbb{R}_+ \times \mathcal{C}(\mathbb{R}_+, \mathcal{M})$ by

$$\tilde{\mathbb{N}}(dt\, d\zeta\, d\mathcal{N}) = \mathbf{I}_{0\leq t\leq \sigma} dt\, \mathbb{N}(d\zeta\, d\mathcal{N})$$

where $\sigma = \inf\{s > 0, \zeta_s = 0\}$. Then, we combine Bismut's description of Itô measure (see for instance [RY], p. 502), the invariance of Itô measure under time reversal and Proposition 2.5 of [Lg2]. We obtain the following description of the measure $\tilde{\mathbb{N}}$:

Proposition 8. *Under $\tilde{\mathbb{N}}$, we have*

- *ζ_t is distributed as Lebesgue measure on $\mathbb{R}_+$.*
- *Conditionally on ζ_t, $\mathcal{N}_t(du\, dv)$ is a Poisson measure of intensity*

$$du\, dv\, \mathbf{I}_{u\leq \zeta_t}.$$

- *Conditionally on ζ_t and $\mathcal{N}_t$, if we set $(g_i, d_i)_{i\in I}$ the excursions intervals of $(\zeta_s - \inf_{[t,s]} \zeta_r, t \leq s \leq \sigma)$ above 0 and $(\alpha'_j, \beta'_j)_{j\in J}$ the excursions intervals of $(\zeta_{t-s} - \inf_{[t-s,t]} \zeta_r, 0 \leq s \leq t)$ above 0, and if we define $\mathcal{N}^i_s = \mathcal{N}_{(g_i+s)\wedge d_i}\big(\cdot \cap ([\zeta_{g_i}, +\infty) \times \mathbb{R}_+)\big)$, $\mathcal{N}'^j_s = \mathcal{N}_{(t-\beta'_j+s)\wedge(t-\alpha'_j)}\big(\cdot \cap ([\zeta_{t-\alpha'_j}, +\infty) \times \mathbb{R}_+)\big)$, then, the measures*

$$\sum_{i\in I} \delta_{(\zeta_{g_i}, \mathcal{N}^i)} \quad \textit{and} \quad \sum_{j\in J} \delta_{(\zeta_{t-\alpha'_j}, \mathcal{N}'^j)}$$

are independent Poisson measures with intensity

$$2\mathbf{I}_{[0,\zeta_t)}(u)du\, \mathbb{N}.$$

Moreover, these three properties characterise the measure $\tilde{\mathbb{N}}$.

Now, we construct from this process another process in the following way. We first take the measure $\mathcal{N}_t$ and reverse it. Then we consider the collection of excursions above the minimum of the lifetime after t and keep them as they are but attach them to $\mathcal{N}_t$ at the level $\zeta_t - s$ (if s is the level at which they were), so that the first excursions are now the last ones and vice-versa. Do the same kind of transformation to the excursions before t. This transformation is described in Figure 3 below.

More precisely, we consider, under $\tilde{\mathbb{N}}$, the snake $\check{\mathcal{N}}$ such that

- $\check{\zeta}_t = \zeta_t$.

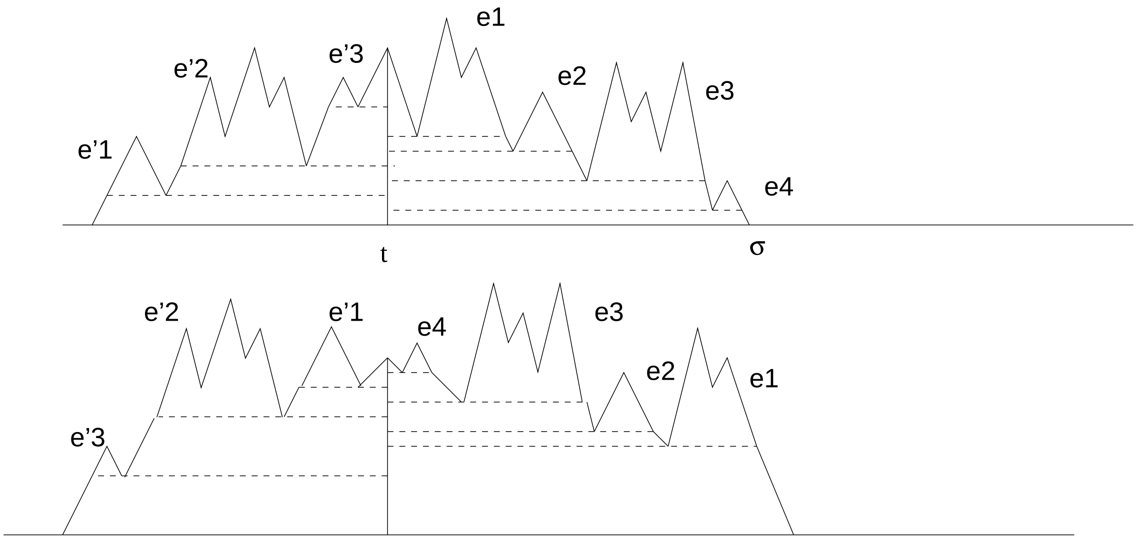

Figure 3: re-rooting

- $\forall s \geq 0\ \check{\mathcal{N}}_t([0,s] \cap \cdot) = \mathcal{N}_t([\zeta_t - s, \zeta_t] \cap \cdot)$.
- $\sum_{i \in \check{I}} \delta_{(\check{\zeta}_{\check{g}_i}, \check{\mathcal{N}}^i)} = \sum_{i \in I} \delta_{(\zeta_t - \zeta_{g_i}, \mathcal{N}^i)}$.
- $\sum_{j \in \check{J}} \delta_{(\check{\zeta}_{\check{g}'_j}, \check{\mathcal{N}}'^j)} = \sum_{j \in J} \delta_{(\zeta_t - \zeta_{\alpha'_j}, \mathcal{N}'^j)}$.

Then, it is clear from proposition 8 that $(t, \check{\zeta}, \check{\mathcal{N}})$ has the same law as $(t, \zeta, \mathcal{N})$ (under $\tilde{\mathbb{N}}$). Moreover, for every $\alpha > 0$, the equivalence class relative to $\mathcal{R}_\alpha$ which contains t is

$$\bigcup_{T_\alpha < \zeta_{a_i} < \zeta_t} \{s, \mathcal{N}^i_{s-g_i}(\mathbb{R}_+ \times [0,2\alpha]) = 0\} \cup \bigcup_{T_\alpha < \zeta_{a'_i} < \zeta_t} \{s, \mathcal{N}'^i_{s-t+\beta'_i}(\mathbb{R}_+ \times [0,2\alpha]) = 0\}$$

where $T_\alpha = \sup\{s, \mathcal{N}_t([0,s] \times [0,2\alpha]) = \mathcal{N}_t(\mathbb{R}_+ \times [0,2\alpha])\}$. In other words, we consider the excursions that are above the last jump of $N^\alpha_t(\cdot)$ (a time u straddled by such an excursion verifies $\hat{N}^\alpha_t \leq \hat{N}^\alpha_s$ for all $s \in [t,u]$) and we only keep the times u such that $\hat{N}^\alpha_t = \hat{N}^\alpha_u$, i.e. the times such that $\hat{N}^i = 0$. As these excursions appear as they are in $\check{\mathcal{N}}$ but are below the first jump of $N^\alpha_t(\cdot)$ by the reversal of $\mathcal{N}_t$, it is easy to see that the former equivalence class has the same Lebesgue measure as

$$\{s, \check{\mathcal{N}}_s(\mathbb{R}_+ \times [0,2\alpha]) = 0\}.$$

4 Connection with Aldous'CRT and coalescence

The Continuum Random Tree introduced by Aldous (see for instance Section 2.1 in [AP] or the relevant references) was constructed as a limit of discrete trees. However a mainstream idea in the literature is that the Brownian normalised excursion is a natural model for this object, because of the binary branching structure of the sub-excursions (see [A1]).

In [AP], fragmentation of the CRT is made by a random cut-set whose law is a Poisson measure with uniform intensity with respect to the "natural measure on the skeleton of the CRT". We can recover this model, or more precisely a translation of this model in our setting. We introduce a few notations. For $(e_s, s \in [0,1])$ being a normalised Brownian excursion, we define a family $(\mathcal{N}_\alpha(ds\,dt), \alpha > 0)$ of point measures on the epigraph $\mathcal{EG}(e) = \{(s,t);\ 0 \le s \le 1,\ 0 \le t \le e_s\} \subset \mathbb{R}^2$ of e such that their laws are (conditionally on e) Poisson point measures with intensity

$$\frac{2\alpha \ \ ds \ \ dt}{d(e,s,t) - g(e,s,t)} \mathbf{1}_{\{0 \le s \le 1,\, 0 \le t \le e(s)\}}.$$

For $(s,t) \in \mathcal{EG}(e)$, we have denoted $[g(e,s,t), d(e,s,t)]$ the excursion above level t containing time s that is

$$g(e,s,t) = \sup\{s' \le s;\ e_{s'} = t\} \text{ and } d(e,s,t) = \inf\{s' \ge s;\ e_{s'} = t\}.$$

Moreover we can impose that Supp $\mathcal{N}_\alpha \subset$ Supp $\mathcal{N}_{\alpha'}$ for $0 \le \alpha \le \alpha'$. (See Figure 4 below)

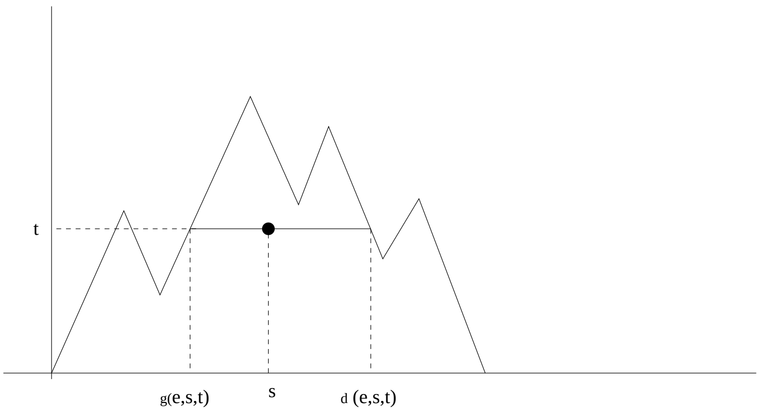

Figure 4: Atom of the point measure on the epigraph.

Proposition 9. *Let, for* $s \in [0,1]$,

$$N_s^\alpha(u) = \mathit{Card}((s_1,t_1); \mathcal{N}_\alpha((s_1,t_1) \ne 0),\ t_1 \le u, s \in [g(e,s_1,t_1), d(e,s_1,t_1)]).$$

Then, the family $(e_s, N_s^\alpha(u);\ u, s \ge 0,\ \alpha > 0)$ *is a Poisson snake in the sense of Theorem 1.*

5 Mass of a tagged fragment

This Section is devoted to the proof of Proposition 3. Recall that, as mentioned in the introduction Theorem 2 and Proposition 3 characterize (in that special case) the law of the fragmentation process $(F(\alpha), \alpha \ge 0)$.

First, we recall that, by the re-rooting property, the law of the mass of a tagged fragment is the same as the law of the mass of the fragment that contains 0.

From Proposition 4 we deduce that, for $\alpha > 0$, the law of $\lambda^0(\alpha)$ is equal to the law of $\inf\{s > 0;\ e(s) - \alpha s = 0\}$ under $\Gamma^1(de)$. Then, by the well known representation of the normalised

Brownian excursion by a 3-dimensional Bessel process R, we have $e(s) = s\,R_{(1/s)-1}$ (see [Bl] p. 42). So, the previous infimum is equal to $\inf\{s > 0;\ R_{(1/s)-1} - \alpha = 0\} = 1/(1+\tau_\alpha)$ where τ_α is the last passage time at α of the Bessel process R. By Pitman's theorem (see [RY] p. 253) this time has the law of the hitting time of α for a linear Brownian motion starting at 0. Thus, for a fixed $\alpha > 0$, the law of $\lambda^0(\alpha)$ is equal to the law $1/(1+\sigma_\alpha)$ where (σ_α) denotes a stable subordinator with index $1/2$. To show moreover that the process $(\lambda^0(\alpha))$ has the same law as the process $(1/(1+\sigma_\alpha))$ we have to show the equality of every finite dimensional marginal distribution, or equivalently that, for $0 < \alpha < \beta$ the conditional law of $\lambda^0(\beta)$ knowing $\sigma(\lambda^0(\alpha'),\ \alpha' \leq \alpha)$ is equal to the the conditional law of $X_\beta := 1/(1+\sigma_\beta)$ knowing $\sigma(X_{\alpha'} = 1/(1+\sigma_{\alpha'}),\ \alpha' \leq \alpha)$. Let us show first that the latter conditional law is the law of $X_\alpha/(1 + X_\alpha\,\tilde{\tau}_{\beta-\alpha})$ where $\tilde{\tau}$ is a stable subordinator with index $1/2$ independent of $X.$ (or $\sigma.$). By taking the inverse this reduces to showing that the conditional law of $1/X_\beta := 1 + \sigma_\beta$ knowing $\sigma(\sigma_{\alpha'},\ \alpha' \leq \alpha)$ is the law of $(1/X_\alpha) + \tilde{\tau}_{\beta-\alpha} = 1 + \sigma_\alpha + \tilde{\tau}_{\beta-\alpha}$. But the latter assertion is clear since $\sigma.$ is a process with stationary independent increments.

Let us now concentrate on the conditional law of $\lambda^0(\beta)$ knowing $\sigma(\lambda^0(\alpha'),\ \alpha' \leq \alpha)$. By definition of the Poisson snake, conditionally on $\zeta.$, we can write $N^\beta = N^\alpha + \tilde{N}^{\beta-\alpha}$ where $\tilde{N}^{\beta-\alpha}$ is a Poisson snake independent of N conditioned to have lifetime $\zeta.$. We recall that $(\zeta_{A_s^\alpha})$ has the same law as $e(s) - \alpha\, s$ where $e(\cdot)$ is the normalised excursion, up to the return time at zero which is precisely $\lambda^0(\alpha)$ (denoted by L in the following). This law is conditionally on L the law Γ^L of the Brownian excursion of length L. As a consequence, conditionally on $\sigma(\lambda^0(\alpha'), \alpha' \leq \alpha)$, the law of $N^\beta_{A_s}$ is the law of $\tilde{N}^{\beta-\alpha}$ where $\tilde{N}^{\beta-\alpha}$ is a Poisson snake independent of N with lifetime a Brownian excursion of length $L = \lambda^0(\alpha)$. Although we have only defined Poisson snake with lifetime a Brownian excursion of length 1, the same definition can be given replacing 1 by L. If we apply anew Proposition 4 with the slight modification that we replace length 1 by length L, we obtain that conditionally on $\sigma(\lambda^0(\alpha'), \alpha' \leq \alpha)$, the law of $\lambda^0(\beta)$ is the law of $\inf\{s > 0;\ e(s) - (\beta - \alpha)\, s = 0\}$ under $\Gamma^L(de)$. This law is, by a scaling argument, the law of $L\ \ \inf\{s > 0;\ e(s) - (\beta - \alpha)\sqrt{L}\, s = 0\}$ under $\Gamma^1(de)$. Thus it is the law of $L/(1 + \tau_{(\beta-\alpha)\sqrt{L}})$ or equivalently the law of $L/(1 + L\ \tau_{(\beta-\alpha)})$ where as before $\tau.$ denotes a stable subordinator with index $1/2$. We conclude as desired that the conditional law of $\lambda^0(\beta)$ knowing $\sigma(\lambda^0(\alpha'),\ \alpha' \leq \alpha)$ is the law of $\lambda^0(\alpha)/(1 +\ \lambda^0(\alpha)\tau_{(\beta-\alpha)})$.

6 Application: direct determination of the dislocation measure

In a fragmentation, if we consider the evolution of the mass of a tagged block, for instance if we consider the process $\lambda^0(\cdot)$, we can informally describe the evolution as a succession of jumps. At a jump–let us say at time α–the tagged block is dislocated into smaller blocks whose decreasing sequence of masses we denote $\lambda^0(\alpha-)\ s$ and we set $\Sigma_\alpha = s$. Then it is known (see [Be]) that the process Σ is a point process with intensity $\sqrt{\lambda^0(\alpha-)}\ \nu(ds)\ d\alpha$ where ν is the dislocation measure.

To see how this dislocation measure can be understood in our model, the better way is perhaps to think of Poisson atoms spread under the epigraph of the normalised Brownian excursion, as explained in Section 4. We see that the dislocation arises when a new atoms is put, creating two new blocks. The dislocation is binary that is ν puts mass on sequences s with only the two first term s_1, s_2 being non-zero. As the measure ν is defined, it amounts in our model to put an atom under the epigraph of a normalised excursion e with the intensity used in our Poisson measure, and see the resulting masses. Taking up the notations of Section 4, if the atom is placed in (s,t), we see that the excursion is cut into $[g(e,s,t), d(e,s,t)]$ and $[0, g(e,s,t)] \cup [d(e,s,t), 1]$. Thus ν is given by

$$\begin{aligned}
&\int F(s_1)\ \nu(ds) \\
&= \int \Gamma^1(de) \quad F[\ \max\{d(e,s,t) - g(e,s,t), 1 - (\ d(e,s,t) - g(e,s,t)\ )\}\ ] \\
&\qquad\qquad \times \frac{ds\ \ dt}{d(e,s,t) - g(e,s,t)}\ \mathbf{1}_{\{0 \le s \le 1,\, 0 \le t \le e(s)\}} \\
&= \int \Gamma^1(de) \int_0^1 ds \int_0^{e(s)} dt\ \ G(d(e,s,t) - g(e,s,t))
\end{aligned}$$

with (using abbreviated notations)

$$G(d-g) = \frac{1}{d-g}\left[F(d-g)\ \mathbf{1}_{\{d-g>1/2\}} + F(1-(d-g))\ \mathbf{1}_{\{d-g \le 1/2\}}\right].$$

It is convenient to come back for a moment to the Ito measure Γ of positive Brownian excursion. Under $\Gamma(de)$, the "law" of the length $\sigma(e)$ of e has the density $(1/\sqrt{8\pi})\ \sigma^{-3/2}$. Conditioning by the length then using the scaling properties of the conditional laws and finally making an obvious change of variables we get with an arbitrary function ψ,

$$\begin{aligned}
&\int \Gamma(de) \int_0^{\sigma(e)} ds \int_0^{e(s)} dt\ \ \psi(d(e,s,t) - g(e,s,t), \sigma(e)) \\
&\qquad = \frac{1}{\sqrt{8\pi}} \int \Gamma^1(de) \int_0^1 ds \int_0^{e(s)} dt \int d\sigma\ \ \psi(d(e,s,t) - g(e,s,t), \sigma). \quad (1)
\end{aligned}$$

The interest of coming back to Ito measure is that we can take advantage of Bismut decomposition of the excursion, see for instance [RY] p. 502. Denoting by $J_1(\cdot)$ and $J_2(\cdot)$ the last passage times of two independent 3-dimensional Bessel processes we obtain

$$\begin{aligned}
&\int \Gamma(de) \int_0^{\sigma(e)} ds \int_0^{e(s)} dt\ \ \psi(d(e,s,t) - g(e,s,t), \sigma(e)) \\
&\qquad = \int_0^{+\infty} dq \int_0^q dt\ E[\psi(J_1(q) - J_1(t) + J_2(q) - J_2(t), J_1(q) + J_2(q))].
\end{aligned}$$

By Pitman's theorem ([RY] p. 253), the processes $J_1(\cdot)$ and $J_2(\cdot)$ have the law of the process of hitting times of Brownian motion, thus are stable subordinators with index $1/2$. So $(J_1(q) - J_1(t) + J_2(q) - J_2(t), J_1(q) + J_2(q))$ is easily shown to have the law of $(4(q-t)^2)Z_1, 4(q-t)^2 Z_1 + t^2 Z_2])$ where Z_1, Z_2 are independent variables having the law of $J_1(1)$ that is having the density $(1/\sqrt{2\pi})\ x^{-3/2}\ \exp(-1/2x)$ over $[0, +\infty)$. We find, with the change of variables $t \to t/q$ and

$4q^2 \to q$,

$$
\begin{aligned}
&\int \Gamma(de) \int_0^{\sigma(e)} ds \int_0^{e(s)} dt \;\; \psi(d(e,s,t) - g(e,s,t), \sigma(e)) \\
&= \frac{1}{16\pi} \int_0^{+\infty} dq \int_0^1 dt \int_{[0,+\infty)^2} dx\; dy (xy)^{-3/2} \; \exp - \left(\frac{1}{2x} + \frac{1}{2y} \right) \\
&\qquad \times \psi(q(1-t)^2\, x, q(1-t)^2 x + qt^2 y) \\
&= \frac{1}{16\pi} \int_0^{+\infty} dq \int_0^1 dt \int_0^{+\infty} d\sigma \int_0^1 dz \;\; \psi(\sigma\, z, \sigma) \\
&\qquad \times \frac{qt(1-t)}{\sigma^2} z^{-3/2} (1-z)^{-3/2} \; \exp - \frac{q}{2\sigma} \left(\frac{t^2}{1-z} + \frac{(1-t)^2}{z} \right).
\end{aligned}
$$

The last equality is obtained by the change of variables

$$
\sigma = q((1-t)^2 x + t^2 y)\ , \qquad z = \frac{q(1-t)^2 x}{\sigma}.
$$

Then we evaluate easily the integral in q. For the resulting integral in t a glimpse at Mathematica spares some pain. We get finally

$$
\int \Gamma(de) \int_0^{\sigma(e)} ds \int_0^{e(s)} dt \;\; \psi(d(e,s,t) - g(e,s,t), \sigma(e)) = \frac{1}{8\pi} \int_0^1 \frac{dz}{z^{1/2}(1-z)^{1/2}} \int_0^{+\infty} d\sigma \; \psi(\sigma\, z, \sigma).
$$

If we compare with equation 1, we obtain

$$
\int \Gamma^1(de) \int_0^1 ds \int_0^{e(s)} dt \; H(d(e,s,t) - g(e,s,t)) = \frac{1}{\sqrt{8\pi}} \int_0^1 \frac{dz}{z^{1/2}(1-z)^{1/2}} \; H(z)
$$

for functions H of the form $H(z) = \int_0^{+\infty} d\sigma \psi(\sigma\, z, \sigma)$. Since this class is rich enough the formula holds in fact for all functions H and in particular for the function G defined previously. We find, summarizing our results, and making an obvious change of variable $z \to 1-z$,

$$
\begin{aligned}
&\int F(s_1)\; \nu(ds) \\
&= \int \Gamma^1(de) \int_0^1 ds \int_0^{e(s)} dt \;\; G(d(e,s,t) - g(e,s,t)) \\
&= \frac{1}{\sqrt{8\pi}} \int_0^1 \frac{dz}{z^{1/2}\, (1-z)^{1/2}} \;\; G(z) \\
&= \frac{1}{\sqrt{8\pi}} \int_0^1 \frac{dz}{z^{1/2}\, (1-z)^{1/2}} \;\; \frac{1}{z} \left(F(z)\, \mathbf{1}_{z>1/2} + F(1-z)\, \mathbf{1}_{z \le 1/2} \right) \\
&= \int_{1/2}^1 \frac{dz}{\sqrt{2\pi}\, z^{1/2}\, (1-z)^{1/2}} \;\; F(z) \;\; \left(\frac{1}{z} + \frac{1}{1-z} \right) \\
&= \int_{1/2}^1 \frac{dz}{\sqrt{2\pi}\; z^{3/2}\; (1-z)^{3/2}} \;\;\; F(z).
\end{aligned}
$$

This is the result that Bertoin had derived from the law of the tagged fragment using his general representation theorem on self-similar fragmentation processes.

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