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Level Sets of Multiparameter Brownian Motions

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Abstract

We use Girsanov's theorem to establish a conjecture of Khoshnevisan, Xiao and Zhong that $\phi(r) = r^{N-d/2}(\log\log(\frac{1}{r}))^{d/2}$ is the exact Hausdorff measure function for the zero level set of an N-parameter d-dimensional additive Brownian motion. We extend this result to a natural multiparameter version of Taylor and Wendel's theorem on the relationship between Brownian local time and the Hausdorff ϕ-measure of the zero set.

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1 Introduction and results

Let $\underline{X} : I\!R^N \to I\!R^d$ be a multiparameter additive Brownian motion, that is, $\underline{X}$ has the following decomposition

$$\underline{X}(\underline{t}) = \sum_{i=1}^{N} \underline{X}^i(t_i), \quad \underline{t} = (t_1, ..., t_N) \in I\!R^N,$$

where the $\underline{X}^i$ are independent, two sided d-dimensional Brownian motions.

The aim of this paper is to establish a conjecture of Khoshnevisan, Xiao and Zhong, c.f. [5, Problem 6.3], that if $2N > d$, for any bounded interval $I \subset I\!R^N$,

$$m_\phi(\{\underline{t} : \underline{X}(\underline{t}) = \underline{0}\} \cap I) < \infty \ \text{ a.s.},$$

where $\phi(r) = r^{N-d/2}(\log\log(\frac{1}{r}))^{d/2}$ and m_ϕ denotes the Hausdorff ϕ-measure.

Note that the question does not arise for $2N \leq d$ since a.s. for all $\underline{t} \neq \underline{0}$, $\underline{X}(\underline{t}) \neq \underline{0}$ (see e.g. [7, Proof of Theorem 1 (b), p.15] or [8]).

The conjecture followed the result of Xiao [14], as observed in [5], that $m_\phi(\{\underline{t} : \underline{X}(\underline{t}) = \underline{0}\} \cap I) > 0$ a.s. if $L(I) > 0$ for L the local time as defined below in (1.1), and so implies that ϕ is the exact measure Hausdorff function for the zero level set of $\underline{X}$.

In one dimension the first result of this kind was due to Taylor and Wendel [13], who showed that if X is a one-dimensional Brownian motion, there exists a positive finite constant c such that

$$m_\phi(\{s : X(s) = 0, \ s \leq t\}) = c\, L(t) \ \text{ a.s.},$$

for all $t > 0$, where L is the local time at zero of X, that is,

$$L(t) \ = \ \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t 1_{\{|X(s)| \leq \epsilon\}} ds,$$

where 1_A denotes indicator function of the event A.

Due to Perkins [9] much more is known, Perkins identified the constant c as $1/\sqrt{2}$ and in fact showed that a.s., simultaneously over all x,

$$m_\phi(\{s : X(s) = x, \quad s \leq t\}) = L_x(t)/\sqrt{2},$$

where

$$L_x(t) \ = \ \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t 1_{\{|X(s) - x| \leq \epsilon\}} ds.$$

We adopt arguments found in Perkins' and Taylor and Wendel's articles, and ultimately prove the following result.

Theorem 1.1 *Let $\underline{X} : I\!R^N \to I\!R^d$ be an additive Brownian motion. Then, there exists a strictly positive finite constant c such that for any multi-time interval $I \subset I\!R^N$,*

$$m_\phi(\{\underline{t} : \ \underline{X}(\underline{t}) = \underline{0}\} \cap I) = cL(I) \ \text{ a.s.},$$

where $L(I)$ denotes the local time at $\underline{0}$ on multi-time interval I, that is,

$$L(I) \ = \ \lim_{\epsilon \downarrow 0} \frac{1}{c_d \epsilon^d} \int_I 1_{\{\|\underline{X}(\underline{s})\| \leq \epsilon\}} d\underline{s}, \tag{1.1}$$

where c_d is the d-dimensional volume of the unit sphere and $\|\cdot\|$ denotes Euclidean norm in $I\!R^d$.

Remark 1.2 *It follows from [14], that the constant c is necessarily strictly positive.*

For proofs on existence of local times and their continuity properties in the multiparameter context see [3] and [4].

Recall that an exact Hausdorff measure function for a set E is meant a function $\psi(r)$ defined for small $r \geq 0$, vanishing at the origin, increasing and continuous, such that the Hausdorff ψ-measure of the set E defined by

$$m_\psi(E) = \lim_{\delta \to 0} \inf_{\{I_i\}} \{\sum_{i=1}^{\infty} \psi(|I_i|) : E \subset \cup_{i=1}^{\infty} I_i, |I_i| < \delta\},$$

is almost sure positive and finite, where $|I_i|$ is the diameter of the set I_i. See [2] or [11].

Essentially, there is at most one correct function ψ for a given set E, in the sense that if $m_{\psi_1}(E) \in (0, \infty)$ and if

$$\lim_{r \downarrow 0} \frac{\psi_1(r)}{\psi_2(r)} = 0, \quad \text{then} \quad m_{\psi_2}(E) = \infty,$$

while if

$$\lim_{r \downarrow 0} \frac{\psi_1(r)}{\psi_2(r)} = \infty, \quad \text{then} \quad m_{\psi_2}(E) = 0.$$

A natural covering of, say, the zero set of $\underline{X}$ in the interval $[1,2]^N$, is to divide up the cube into 2^{Nn} subcubes of side length 2^{-n} and to take as a cover the collection of subcubes which intersect the zero set. Now, since the variation of $\underline{X}$ on such a cube is of order $2^{-n/2}$, we can see that the probability that a given subcube intersects the zero set is of the order $2^{-nd/2}$ and so, for the resulting cover,

$$\mathbb{E}[\sum_{i=1}^{\infty} |I_i|^{N-d/2}] \leq K,$$

for some finite constant K not depending on n. By Fatou's lemma, a.s. there exists a sequence of covers of the zero set intersected with $[1,2]^N$, $\{I_i^n\}_{n \geq 1}$, for which the maximal diameter tends to zero and for which

$$\liminf_{n \to \infty} \sum_{i=1}^{\infty} |I_i^n|^{N-d/2} < \infty.$$

This implies that

$$m_\psi(\{\underline{t} : \underline{X}(\underline{t}) = \underline{0}\} \cap [1,2]^N) < \infty \text{ a.s.},$$

for $\psi(r) = r^{N-d/2}$. But it does not, conversely, show that $m_\psi > 0 : m_\psi$ is defined via optimal coverings rather than individual, given coverings. In fact there exist better coverings for which the lengths of the diameters vary, and ultimately it can be shown that $m_\psi(\{\underline{t} : \underline{X}(\underline{t}) = \underline{0}\}) = 0$.

For our purposes we note that, restricting to planar axes $P_i = \{\underline{t} : t_i = 0\}$, we have, for all R, $m_\psi(\{\underline{t} : \underline{X}(\underline{t}) = \underline{0}\} \cap P_i \cap B_R) < \infty$ a.s., for $\psi(r) = r^{N-1-d/2}$ and $B_R = \{\underline{t} : \|\underline{t}\| \leq R\}$. Finally, one can show the following result.

Lemma 1.3 *For $\underline{X}$ a multiparameter additive Brownian motion from $\mathbb{R}^N$ to $\mathbb{R}^d$ and ϕ as previously defined,*

$$m_\phi(\{\underline{t} : \underline{X}(\underline{t}) = \underline{0}\} \cap (\cup_{i=1}^{N} P_i)) = 0 \quad a.s.$$

A natural way to get good coverings of an awkward set such as the zero set of $\underline{X}$ is to exploit a kernel $L : \Omega \times \mathcal{B}(\mathbb{R}^N) \to \mathbb{R}_+$ which is a.s. supported by the zero set. If we could choose a disjoint random cover $\{I_i\}$ of, say, $\{\underline{t} : \underline{X}(\underline{t}) = \underline{0}\} \cap [1,2]^N$ for which a.s. $L(w, I_i(w)) \geq \phi(|I_i|)$, then, necessarily

$$\begin{aligned} \sum_{i=1}^{\infty} \phi(|I_i|) &\leq \sum_{i=1}^{\infty} L(w, I_i(w)) \\ &= \text{(by disjointness) } L(\omega, \cup_{i=1}^{\infty} I_i(w)) \\ &\leq L(\omega, [1,2]^N) < \infty. \end{aligned}$$

We propose to follow this heuristic with the kernel L given by multidimensional local time defined by (1.1).

A problem is that it may be unavoidable that a covering includes some intervals for which the inequality $\phi(|I_i|) \leq L(w, I_i)$ does not hold. We need a way of picking the covering so that $\sum \phi(|I_i|)$ is small where the sum is over I_i for which the relation $\phi(|I_i|) \leq L(w, I_i)$ fails.

For us this amounts to finding good probability bounds on the law of the iterated logarithm failing for local times in multidimentional setting and represents the bulk of our work. Essentially we want to get a reasonable lower bound on the probability of local time being large: we begin with product Wiener measure, $\mathbb{P}$, over the space of $\mathbb{R}^d$ valued continuous functions defined over $[0,t]$, $\underline{V}^i$, $i = 1, ..., N$. We then consider the (equivalent) measure $\mathbb{Q}$ with respect to which the $\underline{V}^i$ are independent Ornstein-Uhlenbeck processes $d\underline{V}^i(r) = d\underline{B}^i(r) - c\underline{V}^i(r)dr$, where the $\underline{B}^i$ are independent Wiener processes under $\mathbb{Q}$, and use the equality

$$\mathbb{P}\{A\} = \int_A \frac{d\mathbb{P}}{d\mathbb{Q}} d\mathbb{Q}$$

to estimate Wiener measure of an event A.

The paper is planned as follows. In Section Two we analyze certain additive Ornstein-Uhlenbeck processes and establish various "laws of large numbers". In terms of the above equation, this means to find good bounds on $\mathbb{Q}\{A\}$ for relevant A. In Section Three we use Girsanov's theorem to transform information from Section Two concerning Ornstein-Uhlenbeck processes to information on multiparameter Brownian motions. This results in good bounds for $\frac{d\mathbb{P}}{d\mathbb{Q}}$. This is then used to obtain a large deviations result for multi-parameter Brownian motions and local times. The fourth section proves (see Corollary 4.2) that for $\underline{X}$ an additive Brownian motion, there exists a positive finite constant K such that on any interval $I \subset \mathbb{R}^N$

$$m_\phi(\{\underline{t} : \underline{X}(\underline{t}) = \underline{0}\} \cap I) \leq K\, L(I) \;\text{ a.s.}$$

Finally, in Section 5 we arrive at the proof of Theorem 1.1.

Notation: in this paper we use the standard analogies for one dimensional terms, that is, an interval $I \subset \mathbb{R}^N$ is of the form $I_1 \times I_2 \times \cdots \times I_N$ where the I_i are one dimensional intervals, be they open, closed or half open, half closed. We say multi-time interval when we wish to emphasize the higher dimensional aspect. Two points $\underline{u} = (u_1, u_2, ..., u_N)$, $\underline{v} = (v_1, v_2, ..., v_N) \in \mathbb{R}^N$ satisfy the relation $\underline{u} < \underline{v}$ (resp. $\underline{u} \leq \underline{v}$) if and only if for every $i = 1, 2, ..., N$, $u_i < v_i$ (resp. $u_i \leq v_i$). For two such vectors, $[\underline{u}, \underline{v}]$ will denote the interval $[u_1, v_1] \times [u_2, v_2] \times \cdots \times [u_N, v_N]$.

Given a real number s and a given dimension, the vector $\underline{s}$ will denote the vector of the given dimension, all of whose components are equal to s in value. Given a vector $\underline{u}$ and a scaler s, $\underline{u}+s$ will denote the vector $\underline{u}+\underline{s}$.

As is common, throughout c, C, k, K will denote constants and their specific value may change from line to line or even from one side of an inequality to another. For $\underline{v}$ a vector in $\mathbb{R}^d$ and Σ a positive definite $d \times d$ matrix, $\mathcal{N}(\underline{v}, \Sigma)$ will denote the corresponding Gaussian distribution.

2 Local times for Ornstein-Uhlenbeck processes

Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a complete probability space and let $\underline{X}^c : \mathbb{R}_+^N \to \mathbb{R}^d$ be an additive Ornstein-Uhlenbeck process, that is,

$$\underline{X}^c(\underline{t}) = \sum_{i=1}^N \underline{X}^{c,i}(t_i), \qquad \underline{t} = (t_1, ..., t_N) \in \mathbb{R}_+^N,$$

where the $(\underline{X}^{c,i}(r),\ r \geq 0)$, $i = 1, ..., N$, are d-dimensional Ornstein-Uhlenbeck processes defined on $(\Omega, \mathcal{F}, \mathbb{Q})$, each independent of the others, and so that

$$d\underline{X}^{c,i}(r) = d\underline{W}^i(r) - c\underline{X}^{c,i}(r)dr, \ \ r > 0,$$

where the $\underline{W}^i$ are d-dimensional independent Brownian motions on $(\Omega, \mathcal{F}, \mathbb{Q})$.

As is well known, we can write

$$\underline{X}^{c,i}(r) \stackrel{D}{=} e^{-cr}\left(\underline{X}^{c,i}(0) + \underline{\tilde{W}}^i\left(\frac{e^{2cr}-1}{2c}\right)\right), \tag{2.1}$$

where $\underline{\tilde{W}}^i$ is a standard d-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{Q})$ independent of $\underline{X}^{c,i}(0)$.

We consider the local time at $\underline{0}$ of $\underline{X}^c$ in the time interval $[0,t]^N$, defined as in (1.1), that is,

$$L_t^c = \lim_{\epsilon \downarrow 0} \frac{1}{c_d \epsilon^d} \int_{[0,t]^N} 1_{\{\|\underline{X}^c(\underline{r})\| < \epsilon\}}\ d\underline{r}.$$

We start by proving the following result.

Lemma 2.1 *Let t be small but positive and let $c = \frac{h \log\log(\frac{1}{t})}{t}$, where h is a fixed positive finite constant. Then, on the event $\{\|\underline{X}^{c,i}(0)\| \leq \frac{1}{\sqrt{c}}\}$, as t tends down to zero,*

$$\mathbb{E}[L_t^c | \underline{X}^c(\underline{0})] = t^N (2c)^{d/2} (2\pi N)^{-d/2} + O\left(\frac{t^N c^{d/2}}{\sqrt{\log\log(\frac{1}{t})}}\right) \tag{2.2}$$

and

$$\mathbb{E}[(L_t^c)^2 | \underline{X}^c(\underline{0})] = (\mathbb{E}[L_t^c | \underline{X}^c(\underline{0})])^2 \left(1 + O\left(\frac{1}{\sqrt{\log\log(\frac{1}{t})}}\right)\right), \tag{2.3}$$

where for f, g real functions, $f = O(g)$ means that there exists a finite positive constant K such that $|f| \leq K|g|$.

Proof. We start by proving (2.2). One can easily check that

$$\mathbb{E}[L_t^c|\underline{X}^c(\underline{0})] = \int_{[0,t]^N} p^c(\underline{0},\underline{r})\ d\underline{r},$$

where $p^c(\underline{0},\underline{r})$ is the conditional density at $\underline{0}$ of $\underline{X}^c(\underline{r})$ given $\underline{X}^c(\underline{0})$. Note that by (2.1),

$$(\underline{X}^c(\underline{r})|\underline{X}^c(\underline{0})) \stackrel{D}{=} \mathcal{N}\left(\sum_{i=1}^N e^{-cr_i}\underline{X}^{c,i}(0), (\sum_{i=1}^N \frac{1-e^{-2cr_i}}{2c})I_d)\right),$$

where I_d denotes the $d \times d$ identity matrix.

Now, let $H = \{\underline{r} \in [0,t]^N : r_i \geq \frac{t}{\sqrt{\log\log(\frac{1}{t})}},\ \forall\ i = 1, ..., N\}$. Then, if $\underline{r} \in H$, using the condition on $\underline{X}^c(\underline{0})$, it is easily checked that, for $t << 1$,

$$p^c(\underline{0},\underline{r}) \in [(2c)^{d/2}(2\pi N)^{-d/2}e^{-N}, (2c)^{d/2}(2\pi N)^{-d/2}].$$

Therefore,

$$\left|\int_H p^c(\underline{0},\underline{r})\ d\underline{r} - t^N(2c)^{d/2}(2\pi N)^{-d/2}\right| \leq K\frac{t^N c^{d/2}}{\sqrt{\log\log(\frac{1}{t})}},$$

where K is a positive finite constant.

On the other hand, if $\underline{r} \in H^c \cap [0,t]^N$, then $r \in \bigcup_{i=1}^N B_i$, where,

$$B_i = [0,t]^{i-1} \times [0, \frac{t}{\sqrt{\log\log(\frac{1}{t})}}] \times [0,t]^{N-i}.$$

Now, for $\underline{r} \in B_i$, if $\max_{1\leq j\leq N} r_j \geq \frac{1}{c}$, the density $p^c(\underline{0},\underline{r})$ is bounded by

$$(2\pi)^{-d/2}(2c)^{d/2}(1-e^{-2})^{-d/2},$$

and, if $r_j \leq \frac{1}{c}$, for all $j = 1, ..., N$ the density $p^c(\underline{0},\underline{r})$ is bounded by

$$\frac{K}{(\sum_{j=1}^N r_j)^{d/2}},$$

where K is a positive finite constant not depending on t. Hence,

$$\begin{aligned}
\int_{H^c\cap[0,t]^N} p^c(\underline{0},\underline{r})\ d\underline{r} &\leq \sum_{i=1}^N \int_{B_i} p^c(\underline{0},\underline{r})\ d\underline{r} \\
&\leq Kc^{d/2}\sum_{i=1}^N |B_i| + K' \int_{[0,\frac{t}{h\log\log(\frac{1}{t})}]^N} \frac{1}{(\sum_{i=1}^N r_i)^{d/2}}\ d\underline{r} \\
&\leq K_h'' \frac{t^N c^{d/2}}{\sqrt{\log\log(\frac{1}{t})}},
\end{aligned}$$

and this concludes the proof of (2.2).

In order to prove (2.3) we write

$$\mathbb{E}[(L_t^c)^2|\underline{X}^c(\underline{0})] = \int_{[0,t]^N} d\underline{r} \int_{[0,t]^N} d\underline{r}' p^c(\underline{0},\underline{0},\underline{r},\underline{r}'),$$

where $p^c(\underline{0},\underline{0},\underline{r},\underline{r}')$ denotes the conditional joint density at $(\underline{0},\underline{0})$ of the random vector $(\underline{X}^c(\underline{r}),\underline{X}^c(\underline{r}'))$ given $\underline{X}^c(\underline{0})$.

Note that, by (2.1), $p^c(\underline{0},\underline{0},\underline{r},\underline{r}')$ can be written as the product of $p^c(\underline{0},\underline{r}')$ and the conditional density at the random point $-\sum_{i=1}^N e^{-cr_i}\underline{X}^{c,i}(0)$ of the (Gaussian) random variable $\sum_{i=1}^N e^{-cr_i}\underline{\tilde{W}}^i(\frac{e^{2cr_i}-1}{2c})$ given that

$$\sum_{i=1}^N e^{-cr'_i}\underline{\tilde{W}}^i\left(\frac{e^{2cr'_i}-1}{2c}\right) = -\sum_{i=1}^N e^{-cr'_i}\underline{X}^{c,i}(0).$$

The latter is a Gaussian density with mean

$$(\sum_{i=1}^N(1-e^{-2cr'_i}))^{-1}(\sum_{i=1}^N(1-e^{-2c(r_i\wedge r'_i)}))(-\sum_{i=1}^N e^{-cr'_i}\underline{X}^{c,i}(0))$$

and covariance matrix given by

$$\left(\frac{\sum_{i=1}^N(1-e^{-2cr_i})}{2c}\right)(1-\rho^2(\underline{r},\underline{r}'))I_d,$$

where

$$\rho(\underline{r},\underline{r}') = \frac{\sum_{i=1}^N(1-e^{-2c(r_i\wedge r'_i)})}{\sum_{i=1}^N(1-e^{-2cr'_i})\sum_{i=1}^N(1-e^{-2cr_i})}.$$

Then, the proof of (2.3) follows along the same lines as the proof of (2.2) by considering the different cases of $\underline{r},\underline{r}'$ of being in H and $H^c\cap[0,t]^N$, and using the estimates of $\rho(\underline{r},\underline{r}')$ obtained in [1, Section 3] and the condition on $\underline{X}^c(\underline{0})$.

♣

Using Lemma 2.1 one easily deduces the following result.

Proposition 2.2 *Let $\underline{X}^c$ be an additive Ornstein-Uhlenbeck process defined as above and such that $\|\underline{X}^{c,i}(0)\| \le \frac{1}{\sqrt{c}}$ a.s. with $c = \frac{h\log\log(\frac{1}{t})}{t}$, where h is a positive finite constant. Then, as t tends down to 0, $\frac{L_t^c}{\mathbb{E}[L_t^c|\underline{X}^c(\underline{0})]}$ tends to 1 in probability conditioning on $\underline{X}^c(\underline{0})$, uniformly over $\{\|\underline{X}^{c,i}(0)\|\le\frac{1}{\sqrt{c}}\}$.*

Finally, we shall need the following additional result.

Proposition 2.3 *Let $\underline{X}^{c,i}$, $i=1,...,N$ be independent Ornstein-Uhlenbeck processes defined as above and such that $\|\underline{X}^{c,i}(0)\|\le\frac{1}{\sqrt{c}}$ a.s. with $c=\frac{h\log\log(\frac{1}{t})}{t}$, where h is a positive finite constant. Then, as t tends down to zero,*

$$\frac{\displaystyle\int_0^t c^2\sum_{i=1}^N\|\underline{X}^{c,i}(s)\|^2ds}{ct/2}\xrightarrow{pr} Nd.$$

Proof. Let $Z(t) := \int_0^t c^2\sum_{i=1}^N\|\underline{X}^{c,i}(s)\|^2ds$. By (2.1),

$$\mathbb{E}[\|\underline{X}^{c,i}(s)\|^2] = e^{-2cs}\left(\mathbb{E}[\|\underline{X}^{c,i}(0)\|^2]+\frac{d(e^{2cs}-1)}{2c}\right).$$

Therefore,

$$\begin{aligned}
\mathbb{E}[Z(t)] &\geq \int_0^t c^2 N d\Big(\frac{1-e^{-2cs}}{2c}\Big) ds \\
&= \frac{dNct}{2} + \frac{dN}{4}(e^{-2ct} - 1) \\
&= \frac{dNct}{2} + O(1).
\end{aligned}$$

A similar calculation shows that

$$\begin{aligned}
\mathbb{E}[Z(t)^2] &= c^4 \int_0^t ds \left(\int_0^t \mathbb{E}[\sum_{i=1}^N \|\underline{X}^{c,i}(s)\|^2 \sum_{i=1}^N \|\underline{X}^{c,i}(s')\|^2] ds' \right) \\
&= 2c^4 \int_0^t ds \left(\int_0^s \mathbb{E}[\sum_{i=1}^N \|\underline{X}^{c,i}(s)\|^2 \sum_{i=1}^N \|\underline{X}^{c,i}(s')\|^2] ds' \right) \\
&\leq N^2 d^2 (\frac{c^2 t^2}{4} + O(h \log\log(\frac{1}{t})) + O(1)).
\end{aligned}$$

Therefore, as t tends down to 0,

$$E\left[\left(\frac{Z(t)}{\mathbb{E}[Z(t)]} - 1\right)^2\right] \to 0,$$

and the desired result follows. ♣

3 Girsanov's theorem

For the following, we refer the reader to [10, Chapter VIII].

Let $\underline{X} : I\!R^N \to I\!R^d$ be an additive Brownian motion on the standard d-dimensional Wiener space $(\Omega, \mathcal{F}, \mathbb{P})$, that is,

$$\underline{X}(\underline{t}) = \sum_{i=1}^N \underline{X}^i(t_i), \quad \underline{t} = (t_1, ..., t_N) \in I\!R^N,$$

where the $\underline{X}^i$ are independent d-dimensional Brownian motions. We define a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ such that

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_t} = \exp\Big(- \sum_{i=1}^N \int_0^t c\underline{X}^i(s) \cdot d\underline{X}^i(s) - \sum_{i=1}^N \frac{1}{2} c^2 \int_0^t \|\underline{X}^i(s)\|^2 ds \Big), \tag{3.1}$$

where $\{\mathcal{F}_t\}_{t \geq 0}$ denotes the natural filtration of the Brownian motion.

By Girsanov's theorem, under $\mathbb{Q}$, the processes $(\underline{X}^i(s),\ 0 \leq s \leq t)$ are independent d-dimensional Ornstein-Uhlenbeck processes, for $i = 1, ..., N$, and

$$\underline{M}^i(s) = \underline{X}^i(s) + \int_0^s c\underline{X}^i(u) du, \quad 0 \leq s \leq t,$$

are martingales, and in fact Brownian motions. Moreover,

$$\int_0^t c\underline{X}^i(s) \cdot d\underline{X}^i(s) = \int_0^t c\underline{X}^i(s) . d\underline{M}^i(s) - \int_0^t c^2 \|\underline{X}^i(s)\|^2 ds.$$

Therefore,

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_t} = \exp\Big(-\sum_{i=1}^{N}\int_0^t c\underline{X}^i(s)\cdot d\underline{M}^i(s) + \sum_{i=1}^{N}\frac{1}{2}c^2\int_0^t \|\underline{X}^i(s)\|^2 ds\Big),$$

and so

$$\frac{d\mathbb{P}}{d\mathbb{Q}}\bigg|_{\mathcal{F}_t} = \exp\Big(\sum_{i=1}^{N}\int_0^t c\underline{X}^i(s)\cdot d\underline{M}^i(s) - \sum_{i=1}^{N}\frac{1}{2}c^2\int_0^t \|\underline{X}^i(s)\|^2 ds\Big).$$

Then, we have the following result.

Lemma 3.1 *Let $\underline{X}$ be an additive Brownian motion on $(\Omega,\mathcal{F},\mathbb{P})$ defined as above and such that $\|\underline{X}^{c,i}(0)\| \leq \frac{1}{\sqrt{c}}$ a.s. with $c = \frac{h\log\log(1/t)}{t}$, where h a positive finite constant not depending on t. Consider the event*

$$A_t = \Big\{\frac{d\mathbb{P}}{d\mathbb{Q}}\bigg|_{\mathcal{F}_t} \geq e^{-hNd\log\log(1/t)}\Big\}.$$

Then, as t tends down to 0, $\mathbb{Q}\{A_t\}$ tends to 1.

Proof. Since the $\underline{M}^i(s) = \underline{X}^i(s) + \int_0^s c\underline{X}^i(u)du$, $i = 1,...,N$, are d-dimensional Brownian motions under $\mathbb{Q}$, $v \to \int_0^v c\underline{X}^i(s)\cdot d\underline{M}^i(s)$ are time-changed Brownian motions with clock $\int_0^v c^2\|\underline{X}^i(s)\|^2 ds$, that is, there exist independent Brownian motions W^i for $i = 1,2,...,N$ such that

$$\int_0^v c\underline{X}^i(s)\cdot d\underline{M}^i(s) = W^i(\int_0^v c^2\|\underline{X}^i(s)\|^2 ds).$$

Fix $\epsilon > 0$ and consider the event

$$\Big\{\inf_{v\leq\tau_{\frac{dh}{2}\log\log(\frac{1}{t})(1+\epsilon)}}\int_0^v c\underline{X}^i(s)\cdot d\underline{M}^i(s) \geq -\frac{dh}{4}\log\log(\frac{1}{t})\Big\},$$

where $\tau_g = \inf\{s : \int_0^s c^2\|\underline{X}^i(u)\|^2 du = g\}$. This event is equal to

$$\Big\{\inf_{s\ \leq\frac{dh}{2}\log\log(\frac{1}{t})(1+\epsilon)} W^i(s) \geq -\frac{dh}{4}\log\log(\frac{1}{t})\Big\},$$

which, by the reflection principle, has $\mathbb{Q}$-probability equal to

$$\frac{2}{\sqrt{2\pi}}\int_0^{\frac{\frac{dh}{4}\log\log(\frac{1}{t})}{\sqrt{\frac{dh}{2}\log\log(\frac{1}{t})(1+\epsilon)}}} e^{-x^2/2}dx,$$

which tends to 1 as t tends down to 0.

Now, consider the events

$$B_t = \Big\{\sum_{i=1}^{N}\int_0^t c^2\|\underline{X}^i(s)\|^2 ds \in \Big((1-\epsilon)\frac{Ndh}{2}\log\log(\frac{1}{t}), (1+\epsilon)\frac{Ndh}{2}\log\log(\frac{1}{t})\Big)\Big\}$$

and

$$C_t = \Big\{\sum_{i=1}^{N}\int_0^t c\underline{X}^i(s)\cdot d\underline{M}^i(s) \geq -\frac{Ndh}{4}\log\log(\frac{1}{t})\Big\}.$$

Under $\mathbb{Q}$, the processes $\underline{X}^i$ are Ornstein-Uhlenbeck processes with drift indexed by c and so, by the argument above and Proposition 2.3, we have

$$\mathbb{Q}\{B_t \cap C_t\} \to 1, \qquad \text{as } t \downarrow 0.$$

Thus, given ϵ fixed to be less than $1/2$, we have

$$\mathbb{Q}\{A_t\} \geq \mathbb{Q}\{B_t \cap C_t\} \to 1, \qquad \text{as } t \downarrow 0.$$

♣

We finally arrive at a lower bound for a local time large deviations of the process $\underline{X}$.

Proposition 3.2 *Let $\underline{X}$ be an additive Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ defined as above. Then, for h fixed and sufficiently small and for all t sufficiently small, we have, on the event $\{\|\underline{X}(\underline{0})\| \leq \sqrt{\frac{t}{h \log\log(1/t)}}\}$,*

$$\mathbb{P}\left\{L([0,t]^N) \geq (\pi N)^{-d/2} t^{N-d/2} (h \log\log(\frac{1}{t}))^{d/2} \,|\, \underline{X}(\underline{0})\right\} \geq \frac{1}{\log(\frac{1}{t})^{1/4}}.$$

Proof. Note that by linearity it makes no difference if we assume that

$$\|\underline{X}^i(0)\| \leq \sqrt{\frac{t}{h \log\log(1/t)}}, \qquad \text{for } i = 1, ..., N.$$

Let $\mathbb{Q}$ be the probability measure on $(\Omega, \mathcal{F})$ defined in (3.1) with $c = \frac{h \log\log(\frac{1}{t})}{t}$, and consider the event

$$D_t = \left\{L([0,t]^N) \geq (\pi N)^{-d/2} t^{N-d/2} (h \log\log(\frac{1}{t}))^{d/2} \,|\, \underline{X}(\underline{0})\right\}.$$

By Lemma 2.1 and Proposition 2.2, as t tends down to 0, $\mathbb{Q}\{D_t\}$ tends to 1, uniformly over $\{\|\underline{X}(\underline{0})\| \leq \sqrt{\frac{t}{h \log\log(1/t)}}\}$.

On the other hand, by Lemma 3.1, as t tends down to 0,

$$\mathbb{Q}\left\{\frac{d\mathbb{P}}{d\mathbb{Q}}\bigg|_{\mathcal{F}_t} \geq e^{-h \log\log(\frac{1}{t}) Nd} = \frac{1}{(\log(\frac{1}{t}))^{Ndh}}\right\} \to 1,$$

on the event $\{\|\underline{X}(\underline{0})\| \leq \sqrt{\frac{t}{h \log\log(1/t)}}\}$. Then, taking $h = \frac{1}{8Nd}$, we have, for t sufficiently small,

$$\begin{aligned}
\mathbb{P}\{D_t\} &= \int_{D_t} \frac{d\mathbb{P}}{d\mathbb{Q}} d\mathbb{Q} \\
&\geq \int_{D_t \cap \{\frac{d\mathbb{P}}{d\mathbb{Q}} \geq \frac{1}{(\log(\frac{1}{t}))^{Ndh}}\}} \frac{d\mathbb{P}}{d\mathbb{Q}} d\mathbb{Q} \\
&\geq \frac{1}{(\log(\frac{1}{t}))^{Ndh}} \mathbb{Q}\left\{D_t \cap \left\{\frac{d\mathbb{P}}{d\mathbb{Q}} > \frac{1}{(\log(\frac{1}{t}))^{Ndh}}\right\}\right\} \\
&\geq \frac{1}{(\log(\frac{1}{t}))^{1/4}}.
\end{aligned}$$

♣

We will use Proposition 3.2 in the following way. Fix $\underline{i} \in \mathbb{Z}^N$ and consider the 2^{-2n} side cube

$$[\frac{i_1}{2^{2n}}, \frac{(i_1+1)}{2^{2n}}] \times [\frac{i_2}{2^{2n}}, \frac{(i_2+1)}{2^{2n}}] \times \cdots \times [\frac{i_N}{2^{2n}}, \frac{(i_N+1)}{2^{2n}}].$$

We consider intervals

$$\Pi_{j=1}^N \left[T_j^r, T_j^r + \frac{n^{2r+1/4}}{2^{2n}} \right],$$

where

$$T_j^r = \frac{i_j}{2^{2n}} + \frac{n^{2r}}{2^{2n}}, \quad r = 1, ..., \sqrt{n}.$$

We note that, as for large n, $2^{-2n} \leq \frac{n^{2r}}{2^{2n}} + \frac{n^{2r+1/4}}{2^{2n}} << 2^{-n}$, for all $r = 1, ..., \sqrt{n}$, we have that

$$\sup_{r \leq \sqrt{n}} \left| \frac{\log\log((T_j^r + \frac{n^{2r+1/4}}{2^{2n}} - \frac{i_j}{2^{2n}})^{-1})}{\log(n)} - 1 \right| \to 0, \quad \text{as } n \to \infty. \tag{3.2}$$

Fix $h \in (0,1)$ and consider the event $A(r,h) := A(\underline{i}, r, h)$ defined by

$$\left\{ L\left(\prod_{j=1}^N [T_j^r, T_j^r + \frac{n^{2r+1/4}}{2^{2n}}] \right) \geq \left(\frac{n^{2r+1/4}}{2^{2n}} \right)^{N-d/2} (h \log n)^{d/2} \right\}.$$

Lemma 3.3 *Let $\mathcal{F}_r = \sigma(\underline{X}^j(s),\ j = 1, ..., N,\ s \leq T_j^r)$. Then, if h is chosen sufficiently small, for all n sufficiently large and uniformly over $1 \leq r \leq \sqrt{n}$, we have*

$$\mathbb{P}\{A(r,h)|\mathcal{F}_r\} \geq \frac{1}{n^{1/4}} \text{ on the event } \{\|\underline{X}(\underline{T}^r)\| \leq \frac{n^{r+1/16}}{2^n}\}.$$

Proof. Note that by the strong Markov property, the process $(\sum_{j=1}^N \underline{X}^j(T_j^r + s_j) : \underline{s} \geq \underline{0})$ conditioned on $\mathcal{F}_r$, is equal in law to the process $(\sum_{j=1}^N \underline{X}^j(s_j) : \underline{s} \geq \underline{0})$ started at $\sum_{j=1}^N \underline{X}^j(T_j^r)$.

Then, for n sufficiently large and h sufficiently small, and uniformly over $1 \leq r \leq \sqrt{n}$,

$$\|\sum_{j=1}^N \underline{X}^j(T_j^r)\| \leq \frac{n^{r+1/16}}{2^n} << \sqrt{\frac{t}{h \log\log(\frac{1}{t})}} \text{ for } t = \frac{n^{2r+1/4}}{2^{2n}}.$$

Therefore, using Proposition 3.2 we obtain the desired result for n large.

♣

We now fix $h > 0$ such that Lemma 3.3 holds for all n sufficiently small and let $A(r)$ denote the event $A(r,h)$ for this h fixed.

Corollary 3.4 *Uniformly for $\underline{i} > c > 0$,*

$$\mathbb{P}\left\{ \{\|\underline{X}(\frac{\underline{i}}{2^{2n}})\| \leq n2^{-n}\} \cap \{\cap_{r=1}^{\sqrt{n}} A(r)^c\} \right\} \leq kn^{d+1/2} 2^{-nd} e^{-n^{1/8}/4N},$$

for k depending on c but not on n.

Proof. Consider the event

$$B = \left\{ \|\underline{X}(\frac{\underline{i}}{2^{2n}})\| \leq n2^{-n},\ \exists 1 \leq r \leq \sqrt{n} \text{ so that } \|\underline{X}(\underline{T}^r)\| \geq \frac{n^{r+1/16}}{2^n} \right\}.$$

By the independent increments property scaling and standard Brownian bounds, we have

$$\begin{aligned}\mathbb{P}\{B\} &\leq \sum_{r=1}^{\sqrt{n}} \mathbb{P}\left\{ \|\underline{X}(\frac{\underline{i}}{2^{2n}})\| \leq n2^{-n}, \|\underline{X}(\underline{T}^r)\| \geq \frac{n^{r+1/16}}{2^n} \right\} \\ &\leq n^{1/2} C (n2^{-n})^d k e^{-n^{1/8}/4N}.\end{aligned}$$

Finally, by Lemma 3.3,

$$\begin{aligned}\mathbb{P}\left\{ \{\|\underline{X}(\frac{\underline{i}}{2^{2n}})\| \leq n2^{-n}\} \cap \{\cap_{r=1}^{\sqrt{n}} A(r)^c\} \cap B^c \right\} &\leq C(n2^{-n})^d \left(1 - \frac{1}{n^{1/4}}\right)^{\sqrt{n}} \\ &\leq C(n2^{-n})^d e^{-n^{1/4}}.\end{aligned}$$

♣

Corollary 3.5 *For h fixed sufficiently small, $\underline{i} > c > 0$, and n sufficiently large, the probability that $\underline{X}(\underline{t}) = \underline{0}$ for some $\underline{t}$ in $[\frac{\underline{i}}{2^{2n}}, \frac{\underline{i+1}}{2^{2n}}]$, but there is no $0 \leq s \leq 2^{-n}$ so that*

$$L\left(\left[\frac{\underline{i}}{2^{2n}}, \frac{\underline{i}}{2^{2n}} + s \right] \right) \geq s^{N-d/2} \left(h \log\log(\frac{1}{s}) \right)^{d/2},$$

is bounded by $kn^{d+1/2}2^{-nd}e^{-n^{1/8}/4N}$, for k not depending on n.

Proof. Let $s = \frac{n^{2r}}{2^{2n}} + \frac{n^{2r+1/4}}{2^{2n}}$, $r = 1, ..., \sqrt{n}$. Note that, by (3.2), uniformly over $1 \leq r \leq \sqrt{n}$,

$$\left| \frac{\log\log(\frac{1}{s})}{\log n} - 1 \right| \to 0, \quad \text{as } n \to \infty.$$

On the other hand, by definition, on the event $\bigcup_{r=1}^{\sqrt{n}} A(r)$,

$$L\left(\prod_{j=1}^{N} \left[T_j^r, T_j^r + \frac{n^{2r+1/4}}{2^{2n}} \right] \right) \geq \left(\frac{n^{2r+1/4}}{2^{2n}} \right)^{N-d/2} (h \log n)^{d/2},$$

for some r. As for n sufficiently large, the cube $[\frac{\underline{i}}{2^{2n}}, \frac{\underline{i}}{2^{2n}} + s]$ contains $\prod_{j=1}^{N}[T_j^r, T_j^r + \frac{n^{2r+1/4}}{2^{2n}}]$, we obtain, for n sufficiently large, that on this event

$$L\left(\left[\frac{\underline{i}}{2^{2n}}, \frac{\underline{i}}{2^{2n}} + s \right] \right) \geq s^{N-d/2} \left(h \log\log(\frac{1}{s}) \right)^{d/2}.$$

Thus, by Corollary 3.4, the probability that $\|\underline{X}(\frac{\underline{i}}{2^{2n}})\| \leq n2^{-n}$ and there does not exist $0 \leq s \leq 2^{-n}$ such that

$$L\left(\left[\frac{\underline{i}}{2^{2n}}, \frac{\underline{i}}{2^{2n}} + s \right] \right) \geq s^{N-d/2} \left(h \log\log(\frac{1}{s}) \right)^{d/2}$$

is bounded by $kn^{d+1/2}2^{-nd}e^{-n^{1/8}/4N}$.

The proof is completed by noting that by standard Brownian large deviations estimates

$$P\left\{ \|\underline{X}(\frac{\underline{i}}{2^{2n}})\| \geq n2^{-n}, \underline{X}(\underline{t}) = \underline{0} \text{ for some } \underline{t} \in \left[\frac{\underline{i}}{2^{2n}}, \frac{\underline{i+1}}{2^{2n}} \right] \right\} \leq Ke^{-n^2/4N^2}.$$

♣

4 The Hausdorff measure function

In this section, we suppose that h has been fixed at a strictly positive value small enough so that Corollary 3.5 holds. We prove the following result.

Proposition 4.1 *Let $\underline{X} : \mathbb{R}^N \to \mathbb{R}^d$ be an additive Brownian motion. Then, there exists a positive finite constant k such that for any closed interval I which does not intersect the planar axes,*

$$m_\phi(\{\underline{t} : \underline{X}(\underline{t}) = \underline{0}\} \cap I) \leq kL(I) \quad a.s.,$$

where $\phi(r) = r^{N-d/2}\left(\log\log(\frac{1}{r})\right)^{d/2}$.

Proof. Suppose without loss of generality that $I \subset (0,\infty)^N$. We first divide up I into the order of 2^{2nN} cubes of side length 2^{-2n}. Given a cube $[\frac{\underline{i}}{2^{2n}}, \frac{\underline{i}+1}{2^{2n}}]$, $\underline{i} \in \mathbb{Z}^N$, intersecting I, either

(i) it does not contain a point in $\{\underline{t} : \underline{X}(\underline{t}) = \underline{0}\}$;

(ii) it contains a point in $\{\underline{t} : \underline{X}(\underline{t}) = \underline{0}\}$ but the variation of $\underline{X}$ in the cube is greater than $n2^{-n}$;

(iii) (i) & (ii) above do not apply, but there is no $2^{-2n} \leq s \leq 2^{-n}$ so that

$$L\left(\left[\frac{\underline{i}}{2^{2n}}, \frac{\underline{i}}{2^{2n}} + s\right]\right) \geq s^{N-d/2}\left(h\log\log(\frac{1}{s})\right)^{d/2};$$

(iv) (i), (ii) & (iii) do not apply.

We now proceed to the construction of the covering of $\{\underline{t} : \underline{X}(\underline{t}) = \underline{0}\} \cap I$. We denote by $\underline{i}$ the cube $[\frac{\underline{i}}{2^{2n}}, \frac{\underline{i}+1}{2^{2n}}]$. Given a cube $\underline{i}$ satisfying (i), let $C_{\underline{i}}$ be $\emptyset$. For cubes satisfying (ii) or (iii), let $C_{\underline{i}} = \left[\frac{\underline{i}}{2^{2n}}, \frac{\underline{i}+1}{2^{2n}}\right]$. Finally, for cubes satisfying (iv), let $C_{\underline{i}} = \left[\frac{\underline{i}}{2^{2n}}, \frac{\underline{i}}{2^{2n}} + s\right]$, where s is the largest $s \leq 2^{-n}$ such that

$$L\left(\left[\frac{\underline{i}}{2^{2n}}, \frac{\underline{i}}{2^{2n}} + s\right]\right) \geq s^{N-d/2}\left(h\log\log(\frac{1}{s})\right)^{d/2}.$$

Now, let

$$D = \cup_{\underline{i} \text{ satisfies (iv)}} C_{\underline{i}}.$$

By Vitali covering theorem (see e.g. [2]), we can find a subcollection E of the set of $\underline{i}$ satisfying (iv), such that

(A) $C_{\underline{i}} \cap C_{\underline{j}} = \emptyset$, for all $\underline{i}, \underline{j} \in E$;

(B) $D \subset \cup_{\underline{i} \in E} D_{\underline{i}}$, where $D_{\underline{i}}$ is the cube with the same centre as $C_{\underline{i}}$ but 5 times the side length.

Consequently, we consider as a covering of $\{\underline{t} : \underline{X}(\underline{t}) = \underline{0}\} \cap I$,

$$(\cup_{\underline{i} \text{ satisfies (ii) or (iii)}} C_{\underline{i}}) \,\cup\, (\cup_{\underline{i} \in E} D_{\underline{i}}).$$

Now, by definition of the $D_{\underline{i}}$ and condition (iv), for $\underline{i} \in E$,

$$\phi(|D_{\underline{i}}|) \leq 5^{N-d/2}\phi(|C_{\underline{i}}|) \leq 5^{N-d/2}\left(\frac{1}{h}\right)^{d/2} L(C_{\underline{i}}),$$

where, as before, $\phi(r) = r^{N-d/2}\left(\log\log(\frac{1}{r})\right)^{d/2}$. Therefore, since the $C_{\underline{i}}$ are disjoint for $\underline{i} \in E$,

$$\begin{aligned}
&\sum_{\underline{i} \text{ satisfies (ii) or (iii)}} \phi(|C_{\underline{i}}|) + 5^{N-d/2}\left(\frac{1}{h}\right)^{d/2} \sum_{\underline{i}\in E} L(C_{\underline{i}}) \\
&\quad \leq \phi(2^{-2n}) \times \sharp \text{ of } \underline{i} \text{ satisfying (ii) or (iii)} + 5^{N-d/2}\left(\frac{1}{h}\right)^{d/2} L(\cup_{\underline{i}\in E} C_{\underline{i}}).
\end{aligned}$$

Now, by Corollary 3.5,

$$E\left[\, \sharp \text{ of } \underline{i} \text{ satisfying (ii) or (iii)}\right] \leq 2^{2Nn} k n^{d+1/2} 2^{-nd} e^{-cn^{1/8}},$$

and so by Fatou's lemma and the a.s. continuity of $L(\prod_{i=1}^{N}[x_i, y_i])$ with respect to x_i, y_i (see e.g. [3]), we have, a.s.,

$$\liminf_{n\to\infty} \sum_{\underline{i} \text{ satisfies (ii) or (iii)}} \phi(|C_{\underline{i}}|) + 5^{N-d/2}\left(\frac{1}{h}\right)^{d/2} L(\cup_{\underline{i}\in E} C_{\underline{i}}) \leq 5^{N-d/2}\left(\frac{1}{h}\right)^{d/2} L(I).$$

♣

Corollary 4.2 *Let $\underline{X} : \mathbb{R}^N \to \mathbb{R}^d$ be an additive Brownian motion. Then, for any interval I and h as fixed at the start of the section,*

$$m_\phi(\{\underline{t} : \underline{X}(\underline{t}) = \underline{0}\} \cap I) \leq 5^{N-d/2}\left(\frac{1}{h}\right)^{d/2} L(I) \quad a.s.,$$

where $\phi(r) = r^{N-d/2}\left(\log\log(\frac{1}{r})\right)^{d/2}$.

Proof. We write I as the disjoint sum of intersections of I with the planar axes P_i, and open rectangles which are disjoint from planar axes but whose closures are not. For the first case, by Lemma 1.3,

$$m_\phi(\{\underline{t} : \underline{X}(\underline{t}) = \underline{0}\} \cap (I \cap (\cup_{i=1}^{N} P_i))) = 0 \quad \text{a.s.}$$

For the second case, without loss of generality we consider a rectangle R contained in the "quadrant" $(0,\infty)^N$. Then, by Proposition 4.1 and the a.s. continuity of $L(\prod_{i=1}^{N}[x_i, y_i])$ with respect to x_i, y_i, we have, a.s.,

$$\begin{aligned}
m_\phi(\{\underline{t} : \underline{X}(\underline{t}) = \underline{0}\} \cap R) &= \lim_{\epsilon\downarrow 0} m_\phi(\{\underline{t} : \underline{X}(\underline{t}) = \underline{0}\} \cap R \cap [\epsilon,\infty)^N) \\
&\leq \lim_{\epsilon\downarrow 0} 5^{N-d/2}\left(\frac{1}{h}\right)^{d/2} L(R \cap [\epsilon,\infty)^N) \\
&= 5^{N-d/2}\left(\frac{1}{h}\right)^{d/2} L(R).
\end{aligned}$$

The final result follows from the additivity of the local time and the Hausdorff measure. ♣

5 Proof of Theorem 1.1

As a prelude to the proof of Theorem 1.1, we show the result for a fixed interval I bounded away from the axes. The extension to the ultimate result will then be standard.

Theorem 5.1 *Let $\underline{X} : I\!R^N \to I\!R^d$ be an additive Brownian motion. Let I be a closed interval bounded away from the planar axes. Then, there exists a positive finite constant c, not depending on I, such that*

$$m_\phi(\{\underline{t} : \ \underline{X}(\underline{t}) = \underline{0}\} \cap I) = cL(I), \quad a.s.,$$

where $\phi(r) = r^{N-d/2}(\log\log(\frac{1}{r}))^{d/2}$.

Proof. In order to simplify the notation we assume $I = [1,2]^N$. It will be clear that the proof covers all the cases claimed.

We will construct a set of random variables $\{V_M^n, n, M \geq 0\}$ such that

(i) for all $\delta > 0$ there exists M_0 such that for all n and $M \geq M_0$,

$$\mathbb{E}[\big| m_\phi(\{\underline{t} : \underline{X}(\underline{t}) = \underline{0}\} \cap [1,2]^N) - V_M^n\big|] < \delta; \tag{5.1}$$

(ii) for M fixed,

$$\mathbb{E}[(V_M^n - c_M L([1,2]^N))^2] \to 0, \ \text{ as } n \to \infty, \tag{5.2}$$

for some constants $c_M \to c \in (0,\infty)$, as $M \to \infty$. This will imply the desired result.

Let $I\!D^n$ denote the set of time points in $I\!R_+^N$ of the form $(\frac{i_1}{2^{2n}}, \frac{i_2}{2^{2n}}, \cdots, \frac{i_N}{2^{2n}})$, where the i_j are integers. For every n and $\underline{x} \in I\!D^n \cap [1,2]^N$, we write

$$\begin{aligned} L_{\underline{x}}^n &= L([\underline{x}, \underline{x} + 2^{-2n}]), \\ Y_{\underline{x}}^n &= m_\phi(\{\underline{t} : \underline{X}(\underline{t}) = \underline{0}\} \cap [\underline{x}, \underline{x} + 2^{-2n}]). \end{aligned}$$

Note that by Proposition 4.1, there exists a finite, non random constant k such that

$$Y_{\underline{x}}^n \leq k L_{\underline{x}}^n. \tag{5.3}$$

We need some preliminary lemmas.

Lemma 5.2 *There exist two positive finite constants c_1, c_2 not depending on n, such that uniformly over $\underline{x}, \underline{y} \in I\!D^n \cap [1,2]^N$,*

(a) $\mathbb{E}[L_{\underline{x}}^n] \leq c_1 2^{-2nN}$*;*

(b) $\mathbb{E}[L_{\underline{x}}^n L_{\underline{y}}^n] \leq c_2 2^{-4nN}(\|\underline{x} - \underline{y}\| + 2^{-2n})^{-d/2}$*.*

Proof. Let $g_{\underline{t}}(\cdot)$ denote the density of $\underline{X}(\underline{t})$. Then, for $\underline{x} \geq \underline{1}$,

$$\mathbb{E}[L_{\underline{x}}^n] = \int_{[\underline{x}, \underline{x}+2^{-2n}]} g_{\underline{t}}(\underline{0})\, d\underline{t} \leq (2\pi N)^{-d/2} \int_{[\underline{x}, \underline{x}+2^{-2n}]} d\underline{t} = (2\pi N)^{-d/2} 2^{-2Nn}.$$

This proves (a). In order to prove (b), let $g_{\underline{s},\underline{t}}(\cdot,\cdot)$ denote the joint density of the random vector $(\underline{X}(\underline{s}), \underline{X}(\underline{t}))$. Then,

$$\mathbb{E}[L_{\underline{x}}^n L_{\underline{y}}^n] = \iint_{[\underline{x}, \underline{x}+2^{-2n}] \times [\underline{y}, \underline{y}+2^{-2n}]} g_{\underline{s},\underline{t}}(\underline{0}, \underline{0})\, d\underline{s} d\underline{t}.$$

As is easily seen, for $\underline{s}, \underline{t} \geq \underline{1}$, $g_{\underline{s},\underline{t}}(\underline{0}, \underline{0}) \leq c\|\underline{t} - \underline{s}\|^{-d/2}$ for c not depending on $\underline{s}$ or $\underline{t}$, from which the desired result follows. ♣

Lemma 5.3 *For any $\epsilon > 0$ and $M \geq (dN)^{1/2}$, uniformly on n and $\underline{x} \geq \underline{1}$,*

(a) $\mathbb{E}[L_{\underline{x}}^n 1_{\{\|\underline{X}(\underline{x})\| \geq M2^{-n}\}}] \leq \epsilon 2^{-2nN}$;

(b) $\mathbb{E}[Y_{\underline{x}}^n 1_{\{\|\underline{X}(\underline{x})\| \geq M2^{-n}\}}] \leq \epsilon 2^{-2nN}$.

Proof. We start by proving (a). As before $g_{\underline{t}}(\underline{z})$ denotes the density of $\underline{X}(\underline{t})$ at $\underline{z}$. Given $\underline{z} \in I\!R^d$, with $\|\underline{z}\| \geq M2^{-n}$ and $M \geq (dN)^{1/2}$, we have

$$\begin{aligned}\mathbb{E}[L_{\underline{x}}^n | \underline{X}(\underline{x}) = \underline{z}] &= \int_{[0,2^{-2n}]^N} g_{\underline{t}}(\underline{z}) d\underline{t} \leq 2^{-2Nn} \sup_{\underline{t} \in [0,2^{-2n}]^N} g_{\underline{t}}(\underline{z}) \\ &= C 2^{-(2Nn-nd)} \exp(-\frac{\|\underline{z}\|^2}{c2^{-2n}}),\end{aligned}$$

where C, c do not depend on M, n or z. Consequently,

$$\begin{aligned}\mathbb{E}[L_{\underline{x}}^n 1_{\{\|\underline{X}(\underline{x})\| \geq M2^{-n}\}}] &\leq C2^{-(2Nn-nd)} \int_{\|\underline{z}\| \geq M2^{-n}} \exp(-\frac{\|\underline{z}\|^2}{c2^{-2n}}) d\underline{z} \\ &\leq C2^{-2Nn} \int_{\|z\| \geq M} \exp(-\frac{\|\underline{z}\|^2}{c}) d\underline{z} \\ &= C2^{-2Nn} \int_M^\infty \exp(-\frac{r^2}{c}) r^{d-1} dr \\ &\leq \epsilon 2^{-2Nn},\end{aligned}$$

if M is chosen sufficiently large. This proves (a).

The proof of (b) follows from (a) and inequality (5.3). ♣

We define

$$\begin{aligned}Y_{\underline{x}}^{n,M} &= Y_{\underline{x}}^n 1_{\{\|\underline{X}(\underline{x})\| \leq M2^{-n}\}} \\ V_M^n &= \sum_{\underline{x} \in D^n \cap [1,2]^N} Y_{\underline{x}}^{n,M}.\end{aligned}$$

Then, by Lemma 5.3 (b), we have that for all $\delta > 0$, there exists M_0 such that for all n and $M \geq M_0$,

$$\mathbb{E}[|m_\phi(\{\underline{t} : \underline{X}(\underline{t}) = \underline{0}\} \cap [1,2]^N) - V_M^n|] = \mathbb{E}[m_\phi(\{\underline{t} : \underline{X}(\underline{t}) = \underline{0}\} \cap [1,2]^N) - V_M^n] < \delta.$$

This proves (5.1).

We now address (5.2). Define $f_M : I\!R^d \to I\!R_+$ by

$$f_M(\underline{z}) = \mathbb{E}[m_\phi(\{\underline{t} : \underline{X}(\underline{t}) = \underline{0}\} \cap [0,1]^N) | \underline{X}(\underline{0}) = \underline{z}] 1_{\{\|\underline{z}\| \leq M\}}.$$

The function f_M is obviously bounded, given Proposition 4.1. By scaling, for $\underline{x} \geq \underline{0}$ and in particular for $\underline{x} \in [1,2]^N$,

$$\mathbb{E}[Y_{\underline{x}}^{n,M} | \underline{X}(\underline{x})] = \frac{f_M(2^n \underline{X}(\underline{x}))}{2^{2nN-nd}}.$$

In order to prove (5.2) (and therefore complete the proof of Theorem 5.1) it suffices to prove

Proposition 5.4 *For M fixed,*

$$\mathbb{E}\Big[\big(\sum_{\underline{x}\in \boldsymbol{D}^n\cap[1,2]^N}\frac{f_M(2^n\underline{X}(\underline{x}))}{2^{2nN-nd}}-\int_{\mathbb{R}^d} f_M(\underline{z})d\underline{z}\,L([1,2]^N)\big)^2\Big]\to 0,\quad \text{as } n\to\infty, \tag{5.4}$$

and

$$\mathbb{E}\Big[\big(\sum_{\underline{x}\in \boldsymbol{D}^n\cap[1,2]^N}\big(\frac{f_M(2^n\underline{X}(\underline{x}))}{2^{2nN-nd}}-Y^{n,M}_{\underline{x}}\big)\big)^2\Big]\to 0,\quad \text{as } n\to\infty. \tag{5.5}$$

This will ensure that equation (5.2) holds with $c_M=\int_{\mathbb{R}^d} f_M(\underline{z})d\underline{z}$. Since $\int_{\mathbb{R}^d} f_M(\underline{z})d\underline{z}$ is increasing as M increases, $\int_{\mathbb{R}^d} f_M(\underline{z})d\underline{z}\to c$ as M tends to infinity. This limit must be finite by inequality (5.3), while, as noted in Remark 1.2, it is necessarily strictly positive.

Proof of Proposition 5.4
We will detail the proof of (5.5). The proof of (5.4) follows along the same lines and is left to the reader. In order to simplify the notation we only treat the case $N=2$ but the approach extends to all time dimensions. The approach has some similarities with that of [6].

In order to prove (5.5), we write

$$\begin{aligned}
&E\Big[\big(\sum_{\underline{x}\in \boldsymbol{D}^n\cap[1,2]^N}\big(\frac{f_M(2^n\underline{X}(\underline{x}))}{2^{2nN-nd}}-Y^{n,M}_{\underline{x}}\big)\big)^2\Big]\\
&\quad=\sum_{\underline{x},\underline{y}\in \boldsymbol{D}^n\cap[1,2]^N} E\Big[\big(\frac{f_M(2^n\underline{X}(\underline{x}))}{2^{2nN-nd}}-Y^{n,M}_{\underline{x}}\big)\big(\frac{f_M(2^n\underline{X}(\underline{y}))}{2^{2nN-nd}}-Y^{n,M}_{\underline{y}}\big)\Big].
\end{aligned}$$

We consider two different cases:

(A) $|x_1-y_1|\wedge|x_2-y_2|\le 2^{-n/2}$.

$$\begin{aligned}
&\mathbb{E}\big[\big(\frac{f_M(2^n\underline{X}(\underline{x}))}{2^{2nN-nd}}-Y^{n,M}_{\underline{x}}\big)\big(\frac{f_M(2^n\underline{X}(\underline{y}))}{2^{2nN-nd}}-Y^{n,M}_{\underline{y}}\big)\big]\\
&\qquad\le \mathbb{E}\big[\frac{f_M(2^n\underline{X}(\underline{x}))f_M(2^n\underline{X}(\underline{y}))}{2^{4nN-2nd}}\big]+\mathbb{E}[Y^{n,M}_{\underline{x}}Y^{n,M}_{\underline{y}}]\\
&\qquad\le \frac{K}{2^{4Nn}}g_{\underline{x},\underline{y}}(\underline{0},\underline{0})+k^2\mathbb{E}[L^n_{\underline{x}}L^n_{\underline{y}}]\\
&\qquad\le \frac{K}{2^{4Nn}}\frac{1}{(\|\underline{x}-\underline{y}\|+2^{-2n})^{d/2}},
\end{aligned}$$

by bounds for $g_{\underline{x},\underline{y}}(\underline{0},\underline{0})$ and Lemma 5.2.

Thus, the sum of $\mathbb{E}[(\frac{f_M(2^n\underline{X}(\underline{x}))}{2^{2nN-nd}}-Y^{n,M}_{\underline{x}})(\frac{f_M(2^n\underline{X}(\underline{y}))}{2^{2nN-nd}}-Y^{n,M}_{\underline{y}})]$ over $\underline{x},\underline{y}$ satisfying (A) tends to zero as n tends to infinity.

(B) $|x_1-y_1|\wedge|x_2-y_2|> 2^{-n/2}$.

We need to consider different cases. We first assume that $y_1>x_1$ (and so $y_1>x_1+2^{-n/2}$) and $y_2>x_2$ (and so $y_2>x_2+2^{-n/2}$). Then, by the definition of f_M and Markov properties of $\underline{X}$, we have

$$\mathbb{E}\big[\big(\frac{f_M(2^n\underline{X}(\underline{y}))}{2^{2nN-nd}}-Y^{n,M}_{\underline{y}}\big)|\underline{X}(\underline{x}+2^{-2n})\big]=0.$$

Therefore,

$$\mathbb{E}[(\frac{f_M(2^n \underline{X}(\underline{y}))}{2^{2nN-nd}} - Y_{\underline{y}}^{n,M})(\frac{f_M(2^n \underline{X}(\underline{x}))}{2^{2nN-nd}} - Y_{\underline{x}}^{n,M})] = 0.$$

The case $x_1 > y_1$ and $x_2 > y_2$ follows similarly.

In order to treat the other cases we shall need some basic lemmas.

Lemma 5.5 *For $\underline{y} \in [1,2]^2$,*

$$\mathbb{E}[Y_{\underline{y}}^n | \underline{X}(\underline{y}), \underline{X}(\underline{y} + (0, 2^{-2n}))] \leq K 2^{-2nN+nd},$$

uniformly over $\underline{X}(\underline{y}), \underline{X}(\underline{y} + (0, 2^{-2n}))$.

Proof. It is sufficient to show the corresponding inequality for $L_{\underline{y}}^n$. We have

$$\mathbb{E}[L_{\underline{y}}^n | \underline{X}(\underline{y}), \underline{X}(\underline{y} + (0, 2^{-2n}))] = \int\int_{[0,2^{-2n}]^2} dsdt \; g_{s,t}^{2^{-2n}}(\underline{0}|\underline{u}, \underline{v}),$$

where $\underline{u} = \underline{X}(\underline{y})$, $\underline{v} = \underline{X}(\underline{y} + (0, 2^{-2n}))$ and $g_{s,t}^{2^{-2n}}(\underline{0}|\underline{u}, \underline{v})$ is the density at $\underline{0}$ of an independent $\mathcal{N}(\underline{0}, sI_d)$ in $I\!R^d$ convoluted with a d-dimensional Brownian bridge at time t going from $\underline{u}$ at time 0 to $\underline{v}$ at time 2^{-2n} (i.e. $\mathcal{N}(\underline{u} + (\underline{v} - \underline{u})\frac{t}{2^{-2n}}, \frac{t(2^{-2n}-t)}{2^{-2n}} I_d)$). Since

$$g_{s,t}^{2^{-2n}}(\underline{0}|\underline{u}, \underline{v}) \leq K\left(\frac{1}{(s+t)^{d/2}} + \frac{1}{(2^{-2n} - t + s)^{d/2}}\right)$$

the desired result follows. ♣

Lemma 5.6 *Let $\underline{x}, \underline{y} \in I\!D^n \cap [1,2]^2$ with $y_1 - x_1 > 2^{-n/2}$ and $x_2 - y_2 > 2^{-n/2}$. Then*

$$\mathbb{E}[W_{\underline{x}} W_{\underline{y}} 1_{\{\|\underline{X}^2(x_2)\| \geq n^2\}}] \leq K e^{-hn^2},$$

for some constants K and h independent of $\underline{x}$, $\underline{y}$ and n, where

$$W_{\underline{x}} = \frac{f_M(\underline{X}(\underline{x})2^n)}{2^{2nN-nd}} \;\; \textit{or}\; Y_{\underline{x}}^{n,M}$$
$$W_{\underline{y}} = \frac{f_M(\underline{X}(\underline{y})2^n)}{2^{2nN-nd}} \;\; \textit{or}\; Y_{\underline{y}}^{n,M}.$$

Remark 5.7 *The inequalities hold with $1_{\{\|\underline{X}^2(x_2)\| \geq n^2\}}$ replaced by $1_{\{\|\underline{X}^1(x_1)\| \geq n^2\}}$ and the corresponding inequalities hold for $x_1 > y_1$ and $x_2 < y_2$.*

Proof. By basic inequalities for the joint densities, we have

$$\mathbb{E}[W_{\underline{y}} | \underline{X}^2(x_2), W_{\underline{x}}] \leq K \frac{2^{-2nN}}{\|\underline{x} - \underline{y}\|^{d/2}},$$

so, it suffices to treat $\mathbb{E}[W_{\underline{x}} 1_{\{\|\underline{X}^2(x_2)\| \geq n^2\}}]$. On the other hand, it is easy to see that

$$\mathbb{E}[W_{\underline{x}} | \underline{X}^2(x_2)] \leq K 2^{-2nN}.$$

Therefore,

$$\begin{aligned} \mathbb{E}[W_{\underline{x}} 1_{\{\|\underline{X}^2(x_2)\| \geq n^2\}}] &\leq K 2^{-2nN} \mathbb{P}\{\|\underline{X}^2(x_2)\| \geq n^2\} \\ &\leq K 2^{-2nN} e^{-hn^2}, \end{aligned}$$

and the desired result follows. ♣

Lemma 5.8 *Let $(\underline{B}(t), t \geq 0)$ be a Brownian motion in $I\!R^d$ with $\underline{B}(0) = \underline{0}$. Let $g_{2^{-2n}}(\underline{y}|v, \underline{x})$ denote the conditional density of $\underline{B}(2^{-2n})$ given $\underline{B}(v) = \underline{x}$. Then, for $\|\underline{x}\| \leq 3n^2$ and $v \in [2^{-n/2}, 1]$, we have*

(i) $|g_{2^{-2n}}(\underline{y}|v, \underline{x}) - g_{2^{-2n}}(\underline{y})| \leq K g_{2^{-2n}}(\underline{y}) 2^{-n/3}$, *for* $\|\underline{y}\| \leq 2^{-n}n$,

(ii) $|g_{2^{-2n}}(\underline{y}|v, \underline{x})| \leq K 2^{nd} \exp(-\frac{\|\underline{y}\|^2}{4 2^{-2n}})$, *for* $\|\underline{y}\| \geq 2^{-n}n$,

where K does not depend on n, $\underline{y}$, v or $\underline{x}$.

Proof. By the independence of the increments of the Brownian motion, we have

$$|g_{2^{-2n}}(\underline{y}|v, \underline{x}) - g_{2^{-2n}}(\underline{y})| = g_{2^{-2n}}(\underline{y}) \left| \frac{g_{v-2^{-2n}}(\underline{x} - \underline{y})}{g_v(\underline{x})} - 1 \right|$$

$$= g_{2^{-2n}}(\underline{y}) \left| \frac{v^{d/2}}{(v - 2^{-2n})^{d/2}} \frac{\exp(-\frac{\|\underline{x}\|^2}{2(v-2^{-2n})}) \exp(\frac{\underline{x} \cdot \underline{y}}{v-2^{-2n}}) \exp(-\frac{\|\underline{y}\|^2}{2(v-2^{-2n})})}{\exp(-\frac{\|\underline{x}\|^2}{2v})} - 1 \right|$$

$$= g_{2^{-2n}}(\underline{y}) \left| \frac{v^{d/2}}{(v - 2^{-2n})^{d/2}} \exp(-\frac{\|\underline{x}\|^2 2^{-2n}}{2v(v - 2^{-2n})}) \exp(\frac{\underline{x} \cdot \underline{y}}{v - 2^{-2n}}) \exp(-\frac{\|\underline{y}\|^2}{2(v - 2^{-2n})}) - 1 \right|.$$

As $\|\underline{x}\| \leq 3n^2$, $v \in [2^{-n/2}, 1]$ and $\|\underline{y}\| \leq n2^{-n}$, we have

$$\begin{aligned}
\frac{\|\underline{x}\|^2 2^{-2n}}{v(v - 2^{-2n})} &\leq \frac{9n^4 2^{-2n}}{2^{-n}} \leq 9n^4 2^{-n}, \\
\frac{\underline{x} \cdot \underline{y}}{v - 2^{-2n}} &\leq \frac{3n^2 n 2^{-n}}{2^{-n/2} - 2^{-2n}} \leq \frac{3n^3 2^{-n/2}}{1 - 2^{-3n/2}}, \\
\frac{\|\underline{y}\|^2}{2(v - 2^{-2n})} &\leq \frac{4n^2 2^{-2n}}{2(2^{-n/2} - 2^{-2n})} \leq \frac{2n^2 2^{-3n/2}}{1 - 2^{-3n/2}}.
\end{aligned}$$

Therefore for $\|\underline{x}\| \leq 3n^2$, $v \in [2^{-n/2}, 1]$ and $\|\underline{y}\| \leq n2^{-n}$, and n sufficiently large,

$$|g_{2^{-2n}}(\underline{y}|v, \underline{x}) - g_{2^{-2n}}(\underline{y})| \leq g_{2^{-2n}}(\underline{y}) 2^{-n/3}.$$

It now suffices to choose K sufficiently large in order to cover the remaining finite number of n's less. This proves (i).

Equally, with the same hypotheses on v and $\underline{x}$ but with $\|\underline{y}\| \geq n2^{-n}$, it holds that

$$\begin{aligned}
g_{2^{-2n}}(\underline{y}|v, \underline{x}) &\leq K \exp(\frac{\underline{x} \cdot \underline{y}}{v - 2^{-2n}}) \exp(-\frac{\|\underline{y}\|^2}{2 2^{-2n}}) 2^{nd} \\
&\leq K \exp(-\frac{\|\underline{y}\|^2}{4 2^{-2n}}) 2^{nd}
\end{aligned}$$

for K not depending on n or y. ♣

Proposition 5.9 *Let $\underline{x}, \underline{y} \in I\!D^n \cap [1,2]^2$ with $y_1 - x_1 > 2^{-n/2}$ and $x_2 - y_2 > 2^{-n/2}$. Then*

$$E\left[(\frac{f_M(2^n \underline{X}(\underline{x}))}{2^{2nN - nd}} - Y_{\underline{x}}^{n,M})(\frac{f_M(2^n \underline{X}(\underline{y}))}{2^{2nN - nd}} - Y_{\underline{y}}^{n,M}) \right] \leq K \frac{2^{-4Nn}}{\|\underline{x} - \underline{y}\|^{d/2}} 2^{-n/3},$$

where K does not depend on n, $\underline{y}$ or $\underline{x}$.

Proof. By Lemma 5.6 it is enough to prove this upper bound for

$$E\Big[(\frac{f_M(2^n\underline{X}(\underline{x}))}{2^{2nN-nd}}-Y_{\underline{x}}^{n,M})(\frac{f_M(2^n\underline{X}(\underline{y}))}{2^{2nN-nd}}-Y_{\underline{y}}^{n,M})1_{\{\|\underline{X}^2(x_2)\|\le n^2\}}1_{\{\|\underline{X}^2(y_2)\|\le n^2\}}\Big].$$

Let $\mathcal{G}=\sigma(\underline{X}^2(y_2),\underline{X}^2(x_2),\underline{X}(\underline{x}),\underline{X}(\underline{y}),Y_{\underline{x}}^{n,M})$. It suffices to treat

$$\mathbb{E}[\frac{f_M(2^n\underline{X}(\underline{y}))}{2^{2nN-nd}}-Y_{\underline{y}}^{n,M}|\mathcal{G}]=\frac{f_M(2^n\underline{X}(\underline{y}))}{2^{2nN-nd}}-E[Y_{\underline{y}}^{n,M}|\mathcal{G}]$$

for $\|\underline{X}^2(x_2)\|\le n^2$ and $\|\underline{X}^2(y_2)\|\le n^2$.

In the following we need only consider $\underline{X}(\underline{x})$ such that $f_M(2^n\underline{X}(\underline{x}))>0$. Now let us define $f(\underline{x},\underline{z})$ by

$$\mathbb{E}[m_\phi(\{\underline{t}:\underline{X}(\underline{t})=\underline{0}\}\cap[0,1]^2)|\underline{X}(\underline{0})=\underline{x},\underline{X}(0,1)=\underline{x}+\underline{z}]1_{\{\|\underline{x}\|\le M\}}.$$

Note that this function is bounded, given Proposition 4.1 and Lemma 5.5. By definition,

$$f_M(\underline{x})=\int_{\mathbb{R}^d}\frac{e^{-\|\underline{z}\|^2/2}}{(\sqrt{2\pi})^d}f(\underline{x},\underline{z})d\underline{z},$$

while by scaling on $\|\underline{X}(\underline{y})\|\le M2^{-n}$, $\|\underline{X}^2(y_2)\|\le n^2$,

$$\mathbb{E}[Y_{\underline{y}}^{n,M}|\mathcal{G}]=2^{-2Nn+nd}\int_{\mathbb{R}^d}g_{2^{-2n}}(\underline{z}|x_2-y_2,\underline{X}^2(x_2)-\underline{X}^2(y_2))f(2^n\underline{X}(\underline{y}),2^n\underline{z})d\underline{z},$$

and so

$$\begin{aligned}&|\mathbb{E}[Y_{\underline{y}}^{n,M}-\frac{f_M(2^n\underline{X}(\underline{y}))}{2^{2Nn-nd}}|\mathcal{G}]|\\&=|2^{-2Nn+nd}\int_{\mathbb{R}^d}\left(g_{2^{-2n}}(\underline{z}|x_2-y_2,\underline{X}^2(x_2)-\underline{X}^2(y_2))-g_{2^{-2n}}(\underline{z})\right)f(2^n\underline{X}(\underline{y}),2^n\underline{z})d\underline{z}|,\end{aligned}$$

on $\|\underline{X}(\underline{y})\|\le M2^{-n}$, and is equal to zero on $\|\underline{X}(\underline{y})\|>M2^{-n}$.

By Lemmas 5.5 and 5.8 this is bounded by $K2^{-2Nn+nd}2^{-n/3}$ for K not depending on n. Thus, since the conditional probability that $\|\underline{X}(\underline{y})\|\le M2^{-n}$ given $Y_{\underline{x}}^{n,M}$ and $f_M(2^n\underline{X}(\underline{x}))$ is bounded by $\frac{K2^{-nd}}{\|\underline{x}-\underline{y}\|^{d/2}}$, we obtain the claimed bound. ♣

Proposition 5.9 concludes the proof of Proposition 5.4 and therefore the proof of Theorem 5.1.

♣

Proof of Theorem 1.1

Theorem 5.1 implies that for any interval I bounded away from the planar axes

$$m_\phi(\{\underline{t}\in I:\underline{X}(\underline{t})=\underline{0}\})\ =cL(I)\quad\text{a.s.}$$

for $c=\lim_{M\to\infty}\int_{\mathbb{R}^d}f_M(\underline{x})d\underline{x}$.

Thus, we have that outside a null set this relation holds for all such intervals I with rational endpoints. By the a.s. continuity of $L(\prod_{i=1}^N[x_i,y_i])$ with respect to x_i,y_i we deduce the relation simultaneously for all intervals not intersecting the planar axes. Using continuity again and Lemma 1.3, we extend the relation to all I contained in a closed orthant, as in the proof of Corollary 4.2. The final result follows from the additivity of the local time and the Hausdorff measure and the fact that any interval I can be split up into disjoint subintervals contained in orthants. ♣

References

[1] Dalang, R.C. and Nualart, E. (2004), Potential theory for hyperbolic SPDE's, *Ann. Probab.*, **32**, 2099-2148.

[2] Falconer, K.J. (1985), *The geometry of fractal sets.* Cambridge University Press.

[3] Geman, D. and Horowitz, J. (1980), Occupation densities, *Ann. Probab.*, **8**, 1-67.

[4] Geman, D., Horowitz, J. and Rosen, J. (1984), A local time analysis of intersections of Brownian paths in the plane, *Ann. Probab.*, **12**, 86-107.

[5] Khoshnevisan, D., Xiao, Y. and Zhong, Y. (2001), Local times of additive Lévy processes, I: Regularity. *Preprint.*

[6] Mountford, T.S. (2003), Brownian sheet bubbles and local time. *Séminaire de Probabilités. To appear.*

[7] Mueller, C. and Tribe, R. (2002), Hitting properties of a random string. *Electron. J. Probab.*, **7**, 1-29.

[8] Orey, S. and Pruitt, W.E. (1973) Sample functions of the N-parameter Wiener process. *Ann. Probab.*, **1**, 138-163.

[9] Perkins, E. (1981), The exact Hausdorff measure of the level sets of Brownian motion, *Z. Wahrsch. verw. Gebiete*, **58**, 373-388.

[10] Revuz, D. and Yor, M. (1998), *Continuous martingales and Brownian motion.* Springer-Verlag, New York.

[11] Rogers, C.A. (1970), *Hausdorff measures.* Cambridge University Press.

[12] Taylor, S.J. (1967), Sample path properties of a transient stable process, *J. Math. Mech.*, **16**, 1229-1246.

[13] Taylor, S.J. and Wendel, J.G. (1966), The exact Hausdorff measure for the zero set of a stable process. *Z. Wahrsch. verw. Gebiete*, **6**, 170-180.

[14] Xiao, Y. (1997), Hölder conditions for the local times and the Hausdorff measure of the level sets of Gaussian random fields, *Probab. Th. Rel. Fields*, **109**, 129-157.