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# On the Small Deviation Problem for Some Iterated Processes

Frank Aurzada* and Mikhail Lifshits†

**Abstract**

We derive general results on the small deviation behavior for some classes of iterated processes. This allows us, in particular, to calculate the rate of the small deviations for $n$-iterated Brownian motions and, more generally, for the iteration of $n$ fractional Brownian motions. We also give a new and correct proof of some results in [24].

**Key words:** Small deviations; small ball problem; iterated Brownian motion; iterated fractional Brownian motion; iterated process; local time.

**AMS 2000 Subject Classification:** Primary 60G18; 60F99; 60G12; 60G52.

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*Technische Universität Berlin, Institut für Mathematik, Sekr. MA 7-5, Straße des 17. Juni 136, 10623 Berlin, Germany, aurzada@math.tu-berlin.de

†St. Petersburg State University, 198504 Stary Peterhof, Dept of Mathematics and Mechanics, Bibliotechnaya pl., 2, Russia, lifts@mail.rcom.ru

# 1 Introduction

This article is concerned with the small deviation problem for iterated processes. We consider two independent, real-valued stochastic processes $X$ and $Y$ (precise assumptions are given below), define the iterated process by $(X \circ Y)(t) := X(Y(t))$, $t \in [0,1]$, and investigate the small deviation function

$$-\log \mathbb{P}\left(\sup_{t\in[0,1]} |X(Y(t))| \leq \varepsilon\right), \tag{1}$$

when $\varepsilon \to 0$.

The goal of this article is

- to provide general results concerning the order of (1) – given that we know the small deviation probabilities for the processes $X$ and $Y$, respectively, and that $Y$ has a *continuous modification*;
- to study some illuminating examples of processes to which this technique can be applied, among them the iteration of $n$ (fractional) Brownian motions; and
- to show how the technique can be modified if $Y$ has jumps. This is illustrated by several examples, among them the $\alpha$-time Brownian motion, previously studied in [24]. Here, we give a correct proof of (a weaker version of) the results from [24].

Small deviation problems, also called small ball problems, were studied intensively during recent years, which is due to many connections to other subjects such as the law of the iterated logarithm of Chung type, strong limit laws in statistics, metric entropy properties of linear operators, quantization, and several other approximation quantities for stochastic processes. For a detailed account, we refer to the surveys [18] and [16] and to the literature compilation [19].

The interest in iterated processes, in particular iterated Brownian motion, started with the works of Burdzy (cf. [8] and [9], see also [31]). Iterated processes have interesting connections to higher order PDEs, cf. [1] and [25] for some recent results. Small deviations of iterated processes or the corresponding result for the law of the iterated logarithm are treated in [11] ($X$ and $Y$ Brownian motions, see also [29]), [12] ($X$ Brownian motion, $Y = |Y'|$ with $Y'$ being Brownian motion), [24] (see Section 5 below), [21] ($X$ fractional Brownian motion, $Y$ a subordinator), and, most recently, [22] ($X$ fractional Brownian motion, $Y$ a subordinator, and the sup-norm is taken over a possibly fractal index set).

In Section 2, we give general results under the assumption that the small deviation probabilities of $X$ and $Y$, respectively, are known to some extent and that $Y$ has a continuous modification. The proofs for these results are given in Section 3 and the results are illustrated with several examples in Section 4. In Section 5, we treat examples where $Y$ has jumps, in particular, the so-called $\alpha$-time Brownian motion, studied earlier in [24].

# 2 General results

Before we formulate our main results, let us define some notation. We write $f \preceq g$ or $g \succeq f$ if $\limsup f/g < \infty$, while the equivalence $f \approx g$ means that we have both $f \preceq g$ and $g \preceq f$.

Moreover, $f \lesssim g$ or $g \gtrsim f$ say that $\limsup f/g \leq 1$. Finally, the strong equivalence $f \sim g$ means that $\lim f/g = 1$.

We say that a process $X$ is $H$-self-similar if $(X(ct)) \overset{d}{=} (c^H X(t))$ for all $c > 0$, where $\overset{d}{=}$ means that the finite-dimensional distributions coincide. Recall that, for example, fractional Brownian motion with Hurst parameter $H$ is $H$-self-similar. However, there are many interesting self-similar processes outside the Gaussian framework, e.g. a strictly $\alpha$-stable Lévy process is $1/\alpha$-self-similar ([28], [10], [27]).

Let us consider stochastic processes $(X(t))_{t\geq 0}$ and $(Y(t))_{t\geq 0}$ that are independent and such that $X(0) = 0$ and $Y(0) = 0$ almost surely. We extend $X$ for $t < 0$ in the usual manner using an independent copy: namely, let $X'$ be an independent copy of $X$, and set $X(t) := X'(-t)$ for all $t < 0$. We call this process *two-sided*.

Recall that if $X$ is a classical fractional Brownian motion, it has dependent “wings” $(X(t))_{t\geq 0}$ and $(X(t))_{t\leq 0}$; hence it does not fit in the scope of the present section. Nevertheless we will show below how the technique can be adjusted by using the stationarity of increments instead of the independence.

In this section, we assume that

- $X$ is an $H$-self-similar, two-sided process and
- $Y$ has a continuous modification.

If we know the weak asymptotic order of the small deviation probability of the processes $X$ and $Y$, respectively, we can determine that of the process $X \circ Y$.

**Theorem 1.** *Let $\theta, \tau > 0$. Then, under the above assumptions, the relations*

$$\begin{aligned} -\log \mathbb{P}\left(\sup_{t\in[0,1]} |X(t)| \leq \varepsilon\right) &\approx \varepsilon^{-\theta} \\ -\log \mathbb{P}\left(\sup_{t\in[0,1]} |Y(t)| \leq \varepsilon\right) &\approx \varepsilon^{-\tau} \end{aligned} \tag{2}$$

*imply*

$$-\log \mathbb{P}\left(\sup_{t\in[0,1]} |X(Y(t))| \leq \varepsilon\right) \approx \varepsilon^{-1/(1/\theta + H/\tau)}.$$

*The implication also holds if $\approx$ is replaced by $\preceq$ or $\succeq$, respectively. For translating lower bounds (i.e. $\preceq$ in the relations above), the assumption that $Y$ is continuous can be dropped.*

**Remark 2.** Note that the resulting exponent is always less than $\theta$. Therefore, the small deviation probability of $X \circ Y$ is always larger than the one of $X$.

**Remark 3.** In fact, for the proof it is sufficient to know that

$$-\log \mathbb{P}\left(\sup_{t\in[0,T]} |X(t)| \leq \varepsilon\right) \approx T^{\theta H} \varepsilon^{-\theta}, \qquad \text{when } \varepsilon \to 0,$$

for all $T > 0$, instead of the self-similarity property and the given small deviations of $X$.

Furthermore, provided we know the *strong* order of the small deviation functions, we can prove a result for the strong asymptotic order for that of the iterated process.

**Theorem 4.** *Let $\tau > 0$ and $\theta := 1/H > 0$. Then, under the above assumptions, the relations*

$$\begin{aligned} -\log \mathbb{P}\left(\sup_{t\in[0,1]} |X(t)| \le \varepsilon\right) &\sim k\varepsilon^{-\theta} \\ -\log \mathbb{P}\left(\sup_{s,t\in[0,1]} |Y(s)-Y(t)| \le \varepsilon\right) &\sim \kappa\varepsilon^{-\tau} \end{aligned} \tag{3}$$

*imply*

$$-\log \mathbb{P}\left(\sup_{t\in[0,1]} |X(Y(t))| \le \varepsilon\right) \sim \kappa^{1/(1+\tau)}\tau^{-\tau/(1+\tau)}(1+\tau)k^{\tau/(1+\tau)}\varepsilon^{-\theta\tau/(1+\tau)}.$$

*The implication also holds if $\sim$ is replaced by $\lesssim$ or $\gtrsim$, respectively. For translating lower bounds (i.e. $\lesssim$ in the relations above), the assumption that $Y$ is continuous can be dropped.*

It is easy to check that this theorem recovers the results from [11], where $X$ and $Y$ are Brownian motions, and [12], where $X$ is a Brownian motion and $Y = |Y'|$ with $Y'$ being a Brownian motion.

**Remark 5.** As in Remark 3, it is sufficient to know that

$$-\log \mathbb{P}\left(\sup_{t\in[0,T]} |X(t)| \le \varepsilon\right) \sim kT^{\theta H}\varepsilon^{-\theta}, \qquad \text{when } \varepsilon \to 0,$$

for all $T > 0$, instead of the self-similarity property and the given small deviations of $X$.

**Remark 6.** A careful reader would wonder why the self-similarity index of $X$ and the small deviation index of $X$ should be related by $\theta := 1/H$. In fact, this relation is rather typical *for the supremum norm*. We refer to [17] for the explanation of this fact in the context of small deviations of general norms. Also see [27; 3].

One may argue that, typically, not the probability in (3) is given but rather the small deviation probability. The following lemma translates from the small deviation probability into (3) (and backwards) if we know that a process satisfies the *Anderson property*.

Recall that the Anderson property for a random vector $Y$ taking values in a linear space $E$ means that

$$\mathbb{P}(Y \in A) \ge \mathbb{P}(Y \in A + e) \tag{4}$$

for any $e \in E$ and any measurable symmetric convex set $A \subseteq E$, cf. [2]. It is known that any centered Gaussian vector has this property. Another example is given by symmetric $\alpha$-stable vectors since their distributions can be represented as mixtures of Gaussian ones.

**Lemma 7.** *Let $(Y(t))_{t\in T}$ be a stochastic process with $Y(t_0) = 0$ a.s. for some $t_0 \in T$. Furthermore, assume that $Y$ satisfies the Anderson property. Let $\tau > 0$ and $\ell$ be a slowly varying function. Then we have*

$$-\log \mathbb{P}\left(\sup_{s,t\in T} |Y(s)-Y(t)| \le \varepsilon\right) \sim \kappa\varepsilon^{-\tau}\ell(\varepsilon)$$

*if and only if*

$$-\log \mathbb{P}\left(\sup_{t\in T}|Y(s)|\le\varepsilon\right)\sim\kappa 2^{-\tau}\varepsilon^{-\tau}\ell(\varepsilon).$$

Note that the applicability of Lemma 7 depends on the use of the Anderson property. We now state that if $X$ satisfies the Anderson property so does $X\circ Y$. This makes it possible to use Theorem 4 iteratively.

**Lemma 8.** *Let $T$ be some non-empty index set and let $(X(u))_{u\in\mathbb{R}}$ and $(Y(t))_{t\in T}$ be independent stochastic processes, where $X$ satisfies the Anderson property. Then the process $(X(Y(t)))_{t\in T}$ satisfies the Anderson property.*

This shows that, in particular, iterated Brownian motion, the iteration of two (or more general $n$) fractional Brownian motions, $\alpha$-time Brownian motion (defined below), and many other non-Gaussian processes satisfy the Anderson property.

## 3 Proofs of the general results

Before we prove Theorem 1, we recall a result that translates the small deviation probability into a corresponding result for the Laplace transform.

**Lemma 9.** *Let $Y(0)=0$ almost surely, $p>0$, and $\tau>0$. Then*

$$-\log \mathbb{P}\left(\sup_{t\in[0,1]}|Y(t)|\le\varepsilon\right)\approx\varepsilon^{-\tau},\qquad \varepsilon\to 0,$$

*implies*

$$-\log \mathbb{E}\exp\left(-\lambda\sup_{s,t\in[0,1]}|Y(t)-Y(s)|^p\right)\approx\lambda^{1/(1+p/\tau)},\qquad \lambda\to\infty.$$

*The relation also holds if $\approx$ is replaced by $\preceq$ ($\succeq$ in the assertion) or $\succeq$ ($\preceq$ in the assertion), respectively.*

**Proof:** This follows simply from the fact that

$$\frac{1}{2}\sup_{s,t\in[0,1]}|Y(t)-Y(s)|\le\sup_{t\in[0,1]}|Y(t)|=\sup_{t\in[0,1]}|Y(t)-Y(0)|\le\sup_{s,t\in[0,1]}|Y(t)-Y(s)|.$$

and the de Bruijn Tauberian Theorem (Theorem 4.12.9 in [5]). □

Now we can prove Theorem 1.

**Proof of Theorem 1:** By assumption, for some constants $C_1,C_1',C_2,C_2'>0$ and *all* $\varepsilon>0$,

$$C_1'e^{-C_1\varepsilon^{-\theta}}\le\mathbb{P}\left(\sup_{t\in[0,1]}|X(t)|\le\varepsilon\right)\le C_2'e^{-C_2\varepsilon^{-\theta}}.\qquad(5)$$

Let

$$N := \inf_{t\in[0,1]} Y(t) \qquad \text{and} \qquad M := \sup_{t\in[0,1]} Y(t).$$

Note that, since $Y$ is continuous,

$$Y([0,1]) = [N, M].$$

Therefore, by independence of $X$ and $Y$ and by independence of $X$ for positive and negative arguments, we have

$$\begin{aligned} \mathbb{P}\left(\sup_{t\in[0,1]} |X(Y(t))| \le \varepsilon\right) &= \mathbb{P}\left(\sup_{s\in[N,0]} |X(s)| \le \varepsilon, \sup_{s\in[0,M]} |X(s)| \le \varepsilon\right) \\ &= \mathbb{E}\left[\mathbb{P}\left(\sup_{s\in[N,0]} |X(s)| \le \varepsilon \,\middle|\, Y\right) \mathbb{P}\left(\sup_{s\in[0,M]} |X(s)| \le \varepsilon \,\middle|\, Y\right)\right]. \quad (6) \end{aligned}$$

Now we use the $H$-self-similarity of $X$ to see that the last expression equals

$$\mathbb{E}\left[\mathbb{P}\left(\sup_{s\in[0,1]} |(-N)^H X(s)| \le \varepsilon \,\middle|\, Y\right) \mathbb{P}\left(\sup_{s\in[0,1]} |M^H X(s)| \le \varepsilon \,\middle|\, Y\right)\right]. \quad (7)$$

By (5), we have

$$\mathbb{P}\left(\sup_{s\in[0,1]} |(-N)^H X(s)| \le \varepsilon \,\middle|\, Y\right) = \mathbb{P}\left(\sup_{s\in[0,1]} |X(s)| \le \frac{\varepsilon}{(-N)^H} \,\middle|\, Y\right) \le C_2' e^{-C_2 \varepsilon^{-\theta}(-N)^{H\theta}}.$$

Analogously one can argue for the second term in (7), which yields that the whole expression in (7) is less than

$$C_2'^2 \mathbb{E} e^{-C_2 \varepsilon^{-\theta}((-N)^{H\theta} + M^{H\theta})} \le C_2'^2 \mathbb{E} e^{-\tilde{C}_2 \varepsilon^{-\theta}(M-N)^{H\theta}} = C_2'^2 \mathbb{E} e^{-\tilde{C}_2 \varepsilon^{-\theta}(\sup_{s,t\in[0,1]} |Y(t)-Y(s)|)^{H\theta}}.$$

By Lemma 9, the logarithmic order of this Laplace transform, when $\varepsilon \to 0$, is $\varepsilon^{-\theta/(1+H\theta/\tau)}$, which proves the upper bound in the assertion. The lower bound is established in exactly the same way using the lower bound in (5). Note that this argument fails when $Y$ is not continuous, because we only have $Y([0,1]) \subsetneq [N, M]$. □

Now let us prove the strong asymptotics result.

**Proof of Theorem 4:** Let $\delta > 0$. By assumption, for all $0 < \varepsilon < \varepsilon_0 = \varepsilon_0(\delta)$,

$$e^{-k(1+\delta)\varepsilon^{-\theta}} \le \mathbb{P}\left(\sup_{t\in[0,1]} |X(t)| \le \varepsilon\right) \le e^{-k(1-\delta)\varepsilon^{-\theta}}.$$

This implies that there are constants $C_1, C_2 > 0$ (depending on $\varepsilon_0$) such that for *all* $\varepsilon > 0$,

$$C_1 e^{-k(1+\delta)\varepsilon^{-\theta}} \le \mathbb{P}\left(\sup_{t\in[0,1]} |X(t)| \le \varepsilon\right) \le C_2 e^{-k(1-\delta)\varepsilon^{-\theta}}. \quad (8)$$

By repeating the previous proof with (5) replaced by (8), we arrive at

$$\mathbb{P}\left(\sup_{t\in[0,1]} |X(Y(t))| \le \varepsilon\right) \le C_2^2 \mathbb{E} e^{-k(1-\delta)\varepsilon^{-\theta}((-N)^{H\theta} + M^{H\theta})}.$$

By using the assumption $H\theta = 1$, we clearly have

$$C_2^2 \mathbb{E} e^{-k(1-\delta)\varepsilon^{-\theta}((-N)^{H\theta}+M^{H\theta})} = C_2^2 \mathbb{E} e^{-k(1-\delta)\varepsilon^{-\theta}(M-N)} = C_2^2 \mathbb{E} e^{-k(1-\delta)\varepsilon^{-\theta}\sup_{s,t\in[0,1]}|Y(t)-Y(s)|}.$$

Next, by the de Bruijn Tauberian Theorem (Theorem 4.12.9 in [5]), the strong asymptotic logarithmic order of this Laplace transform, when $\varepsilon \to 0$, is

$$\kappa^{1/(1+\tau)}\tau^{-1/(1+1/\tau)}(1+\tau)(k(1-\delta))^{1/(1+1/\tau)}\varepsilon^{-\theta/(1+1/\tau)}.$$

Letting $\delta \to 0$ proves the upper bound in the assertion. The lower bound follows in exactly the same way using the lower bound in (8). As in the previous theorem, the proof fails when $Y$ is not continuous, because we only have $Y([0,1]) \subsetneq [N,M]$. $\square$

**Proof of Lemma 7:** Clearly,

$$\sup_{s,t}|Y(s)-Y(t)| \le 2\sup_t |Y(t)|$$

and therefore

$$-\log \mathbb{P}\left(\sup_{s,t}|Y(s)-Y(t)| \le 2\varepsilon\right) \le -\log \mathbb{P}\left(\sup_t |Y(t)| \le \varepsilon\right),$$

which implies the inequality in one direction.

On the other hand, let $N := \inf_t Y(t)$ and $M := \sup_t Y(t)$. Fix $h > 1$ and $0 < \varepsilon < 1$.

Assume

$$\sup_{s,t}|Y(s)-Y(t)| \le 2\varepsilon. \tag{9}$$

Then since $Y(t_0) = 0$ we have $M \le 2\varepsilon$ and $N \ge -2\varepsilon$. Moreover, $Q := \frac{M+N}{2}$ must satisfy $|Q| \le \varepsilon$. Furthermore, we have

$$\sup_s |Y(s)-Q| \le \max\{M-Q, Q-N\} = \frac{M-N}{2} \le \varepsilon.$$

Let $m$ be the point in $\{k\varepsilon^h, k\in\mathbb{Z}\}$ closest to $Q$. There are at most $2(\varepsilon+\varepsilon^h)/\varepsilon^h$ possible values for $m$. Additionally, we have

$$\sup_s |Y(s)-m| = \sup_s \left|Y_s - Q + Q - m\right| \le \varepsilon + \varepsilon^h.$$

Therefore,

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t,s}|Y(s)-Y(t)| \le 2\varepsilon\right)\\
= &\sum_{\{k\in\mathbb{Z},\,|k\varepsilon^h|\le\varepsilon+\varepsilon^h\}} \mathbb{P}\left(\sup_{t,s}|Y(s)-Y(t)| \le 2\varepsilon, m = k\varepsilon^h\right)\\
\le &\sum_{\{k\in\mathbb{Z},\,|k\varepsilon^h|\le\varepsilon+\varepsilon^h\}} \mathbb{P}\left(\sup_s |Y(s)-m| \le \varepsilon+\varepsilon^h, m = k\varepsilon^h\right)\\
\le &\sum_{\{k\in\mathbb{Z},\,|k\varepsilon^h|\le\varepsilon+\varepsilon^h\}} \mathbb{P}\left(\sup_s |Y(s)-k\varepsilon^h| \le \varepsilon+\varepsilon^h\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{\{k\in\mathbb{Z}, |k\varepsilon^h|\leq \varepsilon+\varepsilon^h\}} \mathbb{P}\left(\sup_s |Y(s)| \leq \varepsilon + \varepsilon^h\right) \\
&\leq \frac{2(\varepsilon+\varepsilon^h)}{\varepsilon^h} \mathbb{P}\left(\sup_s |Y(s)| \leq \varepsilon + \varepsilon^h\right) \\
&\leq 4\varepsilon^{1-h} \mathbb{P}\left(\sup_s |Y(s)| \leq \varepsilon + \varepsilon^h\right),
\end{aligned}
$$

where we used the Anderson property in the fourth step.

Taking logarithms, multiplying with $-(2\varepsilon)^\tau \ell(2\varepsilon)^{-1}$, taking limits, and using that $\ell$ is a slowly varying function implies that

$$
\begin{aligned}
& \lim_{\varepsilon\to 0} \varepsilon^\tau \ell(\varepsilon)^{-1} \left(-\log \mathbb{P}\left(\sup_{s,t} |Y(s)-Y(t)| \leq \varepsilon\right)\right) \\
\geq\ & 2^\tau \lim_{\varepsilon\to 0} \varepsilon^\tau \ell(\varepsilon)^{-1} \left(-\log \mathbb{P}\left(\sup_s |Y(s)| \leq \varepsilon + \varepsilon^h\right)\right) \\
=\ & 2^\tau \lim_{\varepsilon\to 0} \left(\frac{\varepsilon}{\varepsilon+\varepsilon^h}\right)^\tau (\varepsilon+\varepsilon^h)^\tau \ell(\varepsilon+\varepsilon^h)^{-1} \left(-\log \mathbb{P}\left(\sup_s |Y(s)| \leq \varepsilon + \varepsilon^h\right)\right) \\
=\ & 2^\tau \lim_{\varepsilon\to 0} \varepsilon^\tau \ell(\varepsilon)^{-1} \left(-\log \mathbb{P}\left(\sup_s |Y(s)| \leq \varepsilon\right)\right),
\end{aligned}
$$

which finishes the proof.

□

**Proof of Lemma 8:** It is sufficient to check (4) for cylinder sets. Fix $d \geq 1$ and let $B$ be a symmetric convex set in $\mathbb{R}^d$. Let $t_1, \ldots, t_d \in T$ and fix any function $e : T \to \mathbb{R}^1$. Define a cylinder

$$
A := \{a : T \to \mathbb{R} : (a(t_1), \ldots, a(t_d)) \in B\}
$$

and the corresponding random cylinders

$$
A_{Y,e} := \{f : \mathbb{R} \to \mathbb{R} : (f(Y(t_1)) - e(t_1), \ldots, f(Y(t_d)) - e(t_d) \in B\}.
$$

Then we have

$$
\begin{aligned}
\mathbb{P}(X \circ Y \in A + e) &= \mathbb{P}\left((X(Y(t_1)) - e(t_1), \ldots, X(Y(t_d)) - e(t_d)) \in B\right) \\
&= \mathbb{E}\,\mathbb{P}\left(X \in A_{Y,e} | Y\right) = \mathbb{E}\,\mathbb{P}\left(X \in A_{Y,0} + (e(t_1), \ldots, e(t_d)) | Y\right) \\
&\leq \mathbb{E}\,\mathbb{P}\left(X \in A_{Y,0} | Y\right) = \mathbb{P}(X \circ Y \in A).
\end{aligned}
$$

□

# 4 Examples

## 4.1 Iterated Brownian motions

As a first example let us consider $n$-iterated Brownian motions:

$$
X^{(n)}(t) := X_n(X^{(n-1)}(t)), \qquad X^{(1)}(t) = X_1(t), \tag{10}
$$

where the $X_i$ are independent (two-sided) Brownian motions. This process is $2^{-n}$-self-similar. The small deviation problem can be solved by using $(n-1)$-times Theorem 4 and Lemmas 7 and 8.

We summarize the result in the following corollary. A simulation for $X^{(1)}$, $X^{(2)}$, and $X^{(4)}$ can be seen in Figure 4.1.

**Corollary 10.** *Let $X^{(n)}$ be the process given by (10), where the $X_i$ are independent two-sided Brownian motions. Then*

$$-\log \mathbb{P}\left(\sup_{t\in[0,1]} |X^{(n)}(t)| \leq \varepsilon\right) \sim \pi^2 \frac{1-2^{-n}}{2^{(n+1)/(2^n-1)}}\, \varepsilon^{-1/(1-2^{-n})}.$$

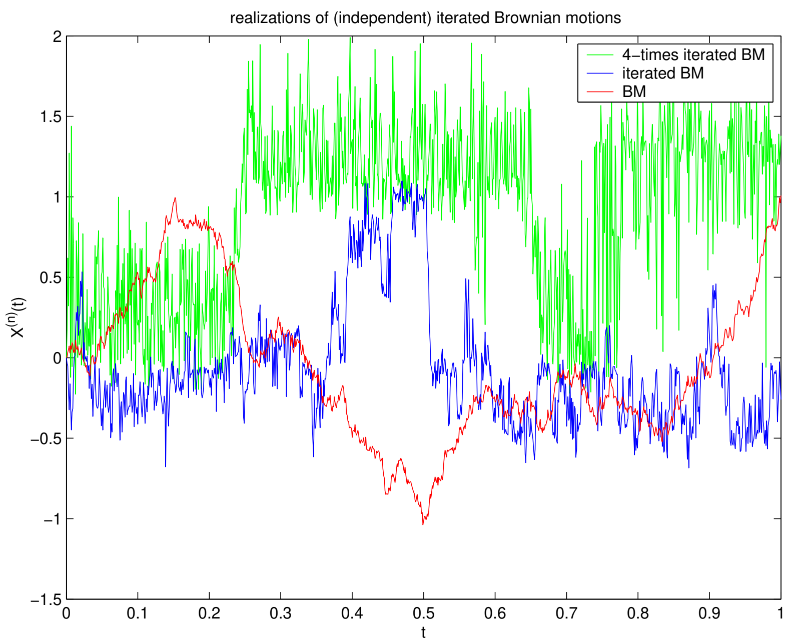


Figure 4.1: typical sample paths of iterated Brownian motions

### 4.2 Iterated two-sided fractional Brownian motions

More generally, one can consider $n$-iterated fractional Brownian motions, given by (10), where this time $X_1,\ldots,X_n$ are independent (two-sided) fractional Brownian motions with Hurst parameters $H_1,\ldots,H_n$, respectively. The process $X^{(n)}$ is $H_1\cdot\ldots\cdot H_n$-self-similar. Its small deviation order is given by

$$-\log \mathbb{P}\left(\sup_{t\in[0,1]} |X^{(n)}(t)| \leq \varepsilon\right) \sim c_n \varepsilon^{-\tau_n}, \qquad \text{where } \frac{1}{\tau_n} = \sum_{j=1}^{n} H_j\cdot\ldots\cdot H_n \tag{11}$$

and $c_n$ is defined iteratively by

$$c_n := (1+\tau_{n-1}) \left[ c_{n-1}^{1/\tau_{n-1}} \frac{2c(H_n)}{\tau_{n-1}} \right]^{\tau_{n-1}/(1+\tau_{n-1})}, \qquad c_1 = c(H_1),$$

and $c(H)$ is the small deviation constant of a fractional Brownian motion with Hurst parameter $H$.

Even for $n = 2$, i.e. fractional Brownian motions $X_1$ and $X_2$ with Hurst parameters $H_1$, $H_2$, respectively, this leads to the new result that the small deviation order is

$$-\log \mathbb{P}\left(\sup_{t\in[0,1]} |X_2(X_1(t))| \le \varepsilon\right) \sim \left(1+\frac{1}{H_1}\right)\left[c(H_1)^{H_1}\,2H_1c(H_2)\right]^{1/(1+H_1)} \varepsilon^{-\frac{1}{H_2(1+H_1)}}.$$

### 4.3 The 'true' iterated fractional Brownian motion

Note that in the last subsection we obtained the small deviation order for 'iterated fractional Brownian motion' $X \circ Y$, where $X$ was a *two-sided* fractional Brownian motion, i.e. a process consisting of two independent branches for positive and negative arguments, and $Y$ was another fractional Brownian motion (independent of the two branches of $X$). We shall now calculate the small deviation order for the 'true' iterated fractional Brownian motion, namely, using $Y$ as above but $X$ being a centered Gaussian process on $\mathbb{R}$ with covariance

$$\mathbb{E}X(t)X(s) = \frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |t-s|^{2H}\right), \qquad t,s \in \mathbb{R}. \tag{12}$$

The general result is as follows.

**Theorem 11.** *Let $X$ be a fractional Brownian motion with Hurst index $H$ as given in (12) and $Y$ be a continuous process independent of $X$ satisfying*

$$-\log \mathbb{P}\left(\sup_{t\in[0,1]} |Y(t)-Y(s)| \le \varepsilon\right) \sim \kappa\varepsilon^{-\tau}. \tag{13}$$

*Then*

$$-\log \mathbb{P}\left(\sup_{t\in[0,1]} |X(Y(t))| \le \varepsilon\right) \sim \kappa^{1/(1+\tau)}\tau^{-\tau/(1+\tau)}(1+\tau)c(H)^{\tau/(1+\tau)}\varepsilon^{-\tau/(H(1+\tau))},$$

*where $c(H)$ is the small deviation constant of a fractional Brownian motion.*

This theorem can be applied to many processes $Y$. We recall that (13) can be obtained e.g. via Lemma 7 from the small deviation order. In particular, if $Y$ is also a fractional Brownian motion we get the following result for the 'true' iterated Brownian motion.

**Corollary 12.** *Let $X$ be a fractional Brownian motion with Hurst index $H_2$ as given in (12) and $Y$ be a (continuous modification of a) fractional Brownian motion with Hurst index $H_1$ (independent of $X$). Then*

$$-\log \mathbb{P}\left(\sup_{t\in[0,1]} |X(Y(t))| \le \varepsilon\right) \sim \left(1+\frac{1}{H_1}\right)\left[c(H_1)^{H_1}\,2H_1c(H_2)\right]^{1/(1+H_1)} \varepsilon^{-\frac{1}{H_2(1+H_1)}}.$$

Recall that we obtain the same logarithmic small deviation order as for a two-sided fBM. Moreover, the iteration of $n$ 'true' fractional Brownian motions provides the same asymptotics as obtained in (11) for the two-sided fBM.

Note that, in spite of the identity of the assertions, Theorem 11 does not follow from Theorem 4, since $X$ is not two-sided. We will show now how the stationarity of increments of the 'true' fBM replaces the independence property of the two-sided process. For the proof of Theorem 11, we need the following lemma.

**Lemma 13.** *For any $\delta \in (0,1)$ there exists $K_\delta > 0$ such that for all $N \le 0 \le M$, for all $\varepsilon > 0$, and for any centered Gaussian process $X(t), t \in \mathbb{R}$, with stationary increments it is true that*

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\le t\le M+|N|} |X(t)| \le (1-\delta)\varepsilon\right)\mathbb{P}\left(|X(N)| \le \varepsilon/K_\delta\right) \\
\le\ &\mathbb{P}\left(\sup_{N\le t\le M} |X(t)| \le \varepsilon\right) \\
\le\ &\mathbb{P}\left(\sup_{0\le t\le M+|N|} |X(t)| \le (1+\delta)\varepsilon\right)\mathbb{P}\left(|X(N)| \le \varepsilon/K_\delta\right)^{-1}.
\end{aligned}$$

**Proof:** To see the upper bound observe that the stationarity of increments and weak correlation inequality (Theorem 1.1 in [15]) yield

$$\begin{aligned}
\mathbb{P}\left(\sup_{0\le t\le M+|N|} |X(t)| \le (1+\delta)\varepsilon\right) &= \mathbb{P}\left(\sup_{N\le t\le M} |X(t)-X(N)| \le (1+\delta)\varepsilon\right) \\
&\ge \mathbb{P}\left(\sup_{N\le t\le M} |X(t)| \le (1+\delta/2)\varepsilon, |X(N)| \le \delta\varepsilon/2\right) \\
&\ge \mathbb{P}\left(\sup_{N\le t\le M} |X(t)| \le \varepsilon\right)\mathbb{P}\left(|X(N)| \le \varepsilon/K_\delta\right).
\end{aligned}$$

For the lower bound, using the same arguments in inverse order we get

$$\begin{aligned}
\mathbb{P}\left(\sup_{N\le t\le M} |X(t)| \le \varepsilon\right) &= \mathbb{P}\left(\sup_{N\le t\le M} |X(t)-X(N)+X(N)| \le \varepsilon\right) \\
&\ge \mathbb{P}\left(\sup_{N\le t\le M} |X(t)-X(N)| \le (1-\delta/2)\varepsilon; |X(N)| \le \delta\varepsilon/2\right) \\
&\ge \mathbb{P}\left(\sup_{N\le t\le M} |X(t)-X(N)| \le (1-\delta)\varepsilon\right)\mathbb{P}\left(|X(N)| \le \varepsilon/K_\delta\right) \\
&= \mathbb{P}\left(\sup_{0\le t\le |N|+M} |X(t)| \le (1-\delta)\varepsilon\right)\mathbb{P}\left(|X(N)| \le \varepsilon/K_\delta\right).
\end{aligned}$$

□

**Proof of Theorem 11:** Let $\delta > 0$ and define as before $N := \inf_{t\in[0,1]} Y(t)$ and $M := \sup_{t\in[0,1]} Y(t)$. Then Lemma 13 yields that, for some constant $K_\delta > 0$,

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\in[0,1]} |X(Y(t))| \le \varepsilon\right) \\
=\ &\mathbb{E}\left[\mathbb{P}\left(\sup_{N\le t\le M} |X(t)| \le \varepsilon \,\middle|\, Y\right)\right] \\
\ge\ &\mathbb{E}\left[\mathbb{P}\left(\sup_{0\le t\le M+|N|} |X(t)| \le (1-\delta)\varepsilon \,\middle|\, Y\right)\mathbb{P}\left(|X(N)| \le \frac{\varepsilon}{K_\delta} \,\middle|\, Y\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\left[\mathbb{P}\left(\sup_{0\le t\le 1}|X(t)| \le \frac{(1-\delta)\varepsilon}{(M+|N|)^H}\middle| Y\right)\mathbb{P}\left(|X(1)| \le \frac{\varepsilon}{K_\delta |N|^H}\middle| Y\right)\right] \\
&\ge \mathbb{E}\left[\mathbb{P}\left(\sup_{0\le t\le 1}|X(t)| \le \frac{(1-\delta)\varepsilon}{(M+|N|)^H}\middle| Y\right)\mathbb{P}\left(|X(1)| \le \frac{\varepsilon}{K_\delta (M+|N|)^H}\middle| Y\right)\right] \\
&=: \mathbb{E}\left[f(M+|N|)g(M+|N|)\right].
\end{aligned}
$$

Note that $f, g \ge 0$ are non-increasing functions. Thus, by the FKG inequality (cf. e.g. [20], p. 65), the last term is bounded from below by

$$
\begin{aligned}
&\mathbb{E}f(M+|N|)\cdot\mathbb{E}g(M+|N|) \\
&\qquad = \mathbb{E}\left[\mathbb{P}\left(\sup_{0\le t\le 1}|X(t)| \le \frac{(1-\delta)\varepsilon}{(M+|N|)^H}\middle| Y\right)\right]\cdot\mathbb{E}\,\mathbb{P}\left(|X(1)| \le \frac{\varepsilon}{K_\delta (M+|N|)^H}\right).
\end{aligned}
$$

The first term can be handled as in the proof of Theorem 4, the resulting order is

$$
\begin{aligned}
&-\log\mathbb{E}\,\mathbb{P}\left(\sup_{0\le t\le 1}|X(t)| \le \frac{(1-\delta)\varepsilon}{(M+|N|)^H}\middle| Y\right) \\
&\qquad \sim \kappa^{1/(1+\tau)}\tau^{-\tau/(1+\tau)}(1+\tau)c(H)^{\tau/(1+\tau)}((1-\delta)\varepsilon)^{-\tau/(H(1+\tau))}.
\end{aligned}
$$

On the other hand, one easily proves that

$$
\mathbb{E}\,\mathbb{P}\left(|X(1)| \le \frac{\varepsilon}{K_\delta (M+|N|)^H}\right)
$$

admits a lower bound of order $\varepsilon$, as $\varepsilon \to 0$. Therefore,

$$
\begin{aligned}
&-\log\mathbb{P}\left(\sup_{t\in[0,1]}|X(Y(t))| \le \varepsilon\right) \\
&\qquad \lesssim \kappa^{1/(1+\tau)}\tau^{-\tau/(1+\tau)}(1+\tau)c(H)^{\tau/(1+\tau)}((1-\delta)\varepsilon)^{-\tau/(H(1+\tau))}.
\end{aligned}
$$

Letting $\delta \to 0$ finishes the proof. The upper bound can be proved along the same lines or by using the Hölder inequality instead of the FKG inequality.

□

# 5 The example of $\alpha$-time Brownian motion

## 5.1 Motivation

Let $X$ be a Brownian motion and $Y$ be a symmetric $\alpha$-stable Lévy process.

In [24], the small deviation problem for $X \circ Y$, called $\alpha$-time Brownian motion there, is studied and further applied to the LIL of Chung type and results for the local time for that processes. However, the method of proof in [24] is essentially the same as for our Theorem 4. Note that this proof is wrong in the case of $\alpha$-time Brownian motion, since the inner process $Y$ is *not continuous*, which is a main ingredient of the proof. In fact, it is used that $Y([0,1]) = [N,M]$, with as above $N := \inf_t Y(t)$ and $M := \sup_t Y(t)$, which is not true for this $Y$.

However, trivially $Y([0,1]) \subsetneq [N,M]$, and thus the proof in [24] does give a *lower bound* for the small deviation probability. The purpose of this section is, among other things, to give a correct proof of the upper bound.

More precisely, the case $\beta = 2$ in the following theorem provides a weaker version of Theorem 2.3 in [24].

**Theorem 14.** *Let $X$ be a two-sided strictly $\beta$-stable Lévy process ($0 < \beta \leq 2$) and $Y$ be a strictly $\alpha$-stable Lévy process ($0 < \alpha \leq 2$, independent of $X$) that is not a subordinator. Then*

$$-\log \mathbb{P}\left(\sup_{t\in[0,1]} |X(Y(t))| \leq \varepsilon\right) \approx \varepsilon^{-\beta\alpha/(1+\alpha)}.$$

This result implies weaker versions of the results in [24]. In particular, the existence of the small deviation constant and its value are not assured. The same concerns the constants in the law of the iterated logarithm. This should be subject to further investigation.

Note furthermore that we prove the result for general strictly $\alpha$-stable Lévy processes $Y$, symmetry (assumed in [24]) is not a feature that would be required here. The only property that is used is self-similarity.

For the sake of completeness let us mention that in the case that $Y$ is an $\alpha$-stable subordinator ($0 < \alpha < 1$) and, say $\beta = 2$, the result becomes wrong. Namely, in that case $X \circ Y$ is in fact a *symmetric* $(2\alpha)$-stable Lévy process itself, so that we then get that

$$-\log \mathbb{P}\left(\sup_{t\in[0,1]} |X(Y(t))| \leq \varepsilon\right) \approx \varepsilon^{-2\alpha}.$$

**Remark 15.** The assertion in Theorem 14 is also true if we take an $H$-fractional Brownian motion or $H$-Riemann-Liouville process as $X$. Then of course, $\beta = 1/H$.

The proof of Theorem 14 is given in several steps. First note that the lower bound follows from our Theorem 1 and Prop. 3, Section VIII, in [4] (the result actually dates back to [30], [23], and [7]). The upper bound follows from Proposition 17 below, as explained there.

## 5.2 Handling the outer Brownian motion

In order to prove Theorem 14, we shall proceed as follows. In a first step, we show that the small deviation problem of processes that are subordinated to Brownian motion (or more generally, to a strictly $\beta$-stable Lévy process) are closely connected to the (random) entropy numbers of the range of the inner process (i.e. $K = Y([0,1])$). This technique was previously used in [21] and [22] for fractional Brownian motion. Then we estimate the entropy numbers of the range of the inner process, in our case a strictly $\alpha$-stable Lévy process (the subordinator case was studied in [21]). This requires completely new arguments.

To formulate the first step, let us define the following notation. For given $\varepsilon > 0$ and a compact set $K \subseteq \mathbb{R}$, let

$$N(K,\varepsilon) := \min\left\{n \geq 1, \exists x_1,\ldots,x_n \in \mathbb{R} : \forall x \in K \exists i \leq n : |x - x_i| \leq \varepsilon\right\}.$$

These quantities are usually called covering numbers of the set $K$ and characterize its metric entropy. We can get rid of the randomness of the outer process $X$ in the following way. We remark that the following result is inspired by Proposition 3.1 in [22], where it was shown for $X$ being fractional Brownian motion.

**Proposition 16.** *Let $X$ be a (two-sided) strictly $\beta$-stable Lévy process, $0 < \beta \leq 2$. Then there is a constant $c_0 > 0$ such that, for all compact sets $K \subseteq \mathbb{R}$ and for all $\varepsilon > 0$,*

$$\mathbb{P}\left(\sup_{t\in K}|X(t)| \leq \varepsilon\right) \leq e^{1-N(K,c_0\varepsilon^\beta)}.$$

**Proof:** Let $c_0$ be chosen large enough such that $\sup_{t\geq c_0}\mathbb{P}(|X(t)| \leq 2) \leq e^{-1}$. For $N = N(K,c_0\varepsilon^\beta)$ find an increasing sequence $t_1,...,t_N$ in $K$ such that $t_{i+1} - t_i \geq c_0\varepsilon^\beta$ for all $i = 1,...,N-1$. Then by independence of increments and strict stability of $X$ we have

$$\begin{aligned}
\mathbb{P}\left(\sup_{t\in K}|X(t)| \leq \varepsilon\right) &\leq \mathbb{P}\left(\sup_{1\leq i\leq N}|X(t_i)| \leq \varepsilon\right) \leq \mathbb{P}\left(\sup_{1\leq i\leq N-1}|X(t_{i+1}) - X(t_i)| \leq 2\varepsilon\right) \\
&= \prod_{i=1}^{N-1}\mathbb{P}\left(|X(t_{i+1}) - X(t_i)| \leq 2\varepsilon\right) = \prod_{i=1}^{N-1}\mathbb{P}\left(|X(t_{i+1} - t_i)| \leq 2\varepsilon\right) \\
&= \prod_{i=1}^{N-1}\mathbb{P}\left(|X\left(\frac{t_{i+1} - t_i}{\varepsilon^\beta}\right)| \leq \varepsilon\right) \leq e^{-(N-1)}.
\end{aligned}$$

□

Recall that in order to prove Theorem 14 we want to get an upper bound for

$$\mathbb{P}\left(\sup_{t\in[0,1]}|X(Y(t))| \leq \varepsilon\right) = \mathbb{E}\left[\mathbb{P}\left(\sup_{t\in K}|X(t)| \leq \varepsilon \,\middle|\, Y\right)\right] \leq \mathbb{E}\left[e^{1-N(K,c_0\varepsilon^\beta)}\right],$$

where we let $K = Y([0,1])$. Since for any $R > 0$ we have

$$\mathbb{E}\left[e^{-N(K,c_0\varepsilon^\beta)}\right] \leq e^{-R} + \mathbb{P}\left(N(K,c_0\varepsilon^\beta) \leq R\right),$$

the upper bound in Theorem 14 follows immediately from the next result.

**Proposition 17.** *Let $Y$ be a strictly $\alpha$-stable Lévy process and set $K = Y([0,1])$. Then there exist small $c$ and $\delta$ depending on the law of $Y$ such that*

$$\mathbb{P}(N(K,\varepsilon) < \delta k) \leq e^{-ck}, \tag{14}$$

*for all $\varepsilon > 0$ and $k = \left\lceil \varepsilon^{-\alpha/(1+\alpha)}\right\rceil$.*

The proof of Proposition 17 is given in the next subsection. In Section 5.4, we provide an alternative proof, which is much shorter but involves local times and thus only works for $\alpha > 1$.

**Remark 18.** Actually, the investigation of the small deviation probabilities for covering numbers such as $\mathbb{P}(N(Y([0,1]),\varepsilon)<k)$ is an interesting problem in its own right; and we hope to handle it elsewhere extensively. Here, we just notice that the order of the estimate (14) is sharp and that it is a particular case of a more general fact that can be obtained similarly:

$$-\log \mathbb{P}(N(Y([0,1]),\varepsilon)<k) \approx (k\varepsilon)^{-\alpha},$$

which is valid for $1 \le k \le \varepsilon^{-1}$, $1<\alpha<2$, and for $1 \le k \le \delta\varepsilon^{-\frac{\alpha}{1+\alpha}}$, $0<\alpha<1$. More efforts are needed to understand the remaining cases, e.g. $\varepsilon^{-\frac{\alpha}{1+\alpha}} \ll k \ll \varepsilon^{-\alpha}$, $0<\alpha<1$.

We remark here that much is known about the Hausdorff dimension of the range of e.g. Lévy processes, cf. e.g. [13] for a recent survey or [26]. However, in our case, we require an estimate for the covering numbers $N(Y([0,1]),\varepsilon)$ on a set of large measure, which is a related question but requires different methods.

### 5.3 Proof of Proposition 17

We will now prove inequality (14). For this purpose, let us introduce the notation

$$N_{[0,t]}(\varepsilon) := N(Y([0,t]),\varepsilon), \qquad t \ge 0.$$

For a given $t \ge 0$, $N_{[0,t]}(\varepsilon)$ counts how many intervals are needed in order to cover the range of the process when only looking at the path until time $t$. A similar quantity was studied in [26], but the results do not seem directly applicable.

Let $T=\varepsilon^{-\alpha}$. By scaling we have

$$\mathbb{P}\left(N_{[0,1]}(\varepsilon)\le \delta k\right)=\mathbb{P}\left(N_{[0,T]}(T^{1/\alpha}\varepsilon)\le \delta k\right)=\mathbb{P}\left(N_{[0,T]}(1)\le \delta k\right).$$

By splitting the time interval $[0,T]$ in $k$ equal pieces, we get intervals of length $L=T/k \ge k^{\alpha}$, since $T \ge k^{1+\alpha}$.

Observe that if $N_{[0,T]}(1)\le \delta k$, then there are at most $\lceil \delta k\rceil$ points where the function $t\mapsto N_{[0,t]}(\varepsilon)$ increases. Therefore, there are at least $\lfloor (1-\delta)k\rfloor$ of the intervals, where this function does not increase. Thus, there exists a set of integers $J\subseteq\{0,\dots,k-1\}$ such that $|J|\ge\lfloor(1-\delta)k\rfloor$ and there is no increase of covering numbers

$$N_{[0,(j+1)L]}(1)=N_{[0,jL]}(1), \qquad \forall j\in J. \tag{15}$$

Let $\delta<1/2$. Notice that the number of choices for $J$ satisfying $|J|\ge\lfloor(1-\delta)k\rfloor$ can be expressed as $2^k\mathbb{P}\left(B_k\ge\lfloor(1-\delta)k\rfloor\right)$ where $B_k$ is a sum of $k$ Bernoulli random variables attaining the values 0 and 1 with equal probabilities. By the classical Chernoff bound for the large deviations of $B_k$ we see that this number is smaller than:

$$\left(\delta^{-\delta}(1-\delta)^{-(1-\delta)}\right)^k =: \exp(\delta_1 k),$$

where $\delta_1$ satisfies $\delta_1\to 0$, as $\delta\to 0$.

For a while, we fix an index set $J$. We enlarge the events from (15) as follows:

$$\Omega_j \quad := \quad \left\{N_{[0,(j+1)L]}(1)=N_{[0,jL]}(1)\le k\right\}$$

$$
\begin{aligned}
&\subseteq \quad \left\{Y((j+1)L) \in Y[0,jL] + [-1,1], N_{[0,jL]}(1) \le k\right\} \\
&= \quad \left\{Y((j+1)L) - Y(jL) \in Y[0,jL] + [-1,1] - Y(jL), N_{[0,jL]}(1) \le k\right\} \\
&=: \quad \Omega'_j.
\end{aligned}
$$

Let, as usual, $\mathscr{F}_t$ denote the filtration generated by the process $Y$ up to time $t$. We have, by stationarity and independence of increments,

$$
\begin{aligned}
\operatorname{esssup} \mathbb{P}(\Omega'_j|\mathscr{F}_{jL}) &\le \sup_A \{\mathbb{P}(Y(L) \in A + [-1,1]), N(A,1) \le k\} \\
&\le \sup_{A'} \{\mathbb{P}(Y(L) \in A'), N(A',2) \le k\} \\
&\le \sup_{A'} \{\mathbb{P}(Y(L) \in A'), |A'| \le 2k\} \\
&\le \sup_{A''} \{\mathbb{P}(Y(1) \in A''), |A''| \le 2\} =: c_1 < 1.
\end{aligned}
$$

By a standard conditioning argument, we find

$$
\mathbb{P}\left(\bigcap_{j\in J} \Omega_j\right) \le \mathbb{P}\left(\bigcap_{j\in J} \Omega'_j\right) \le \prod_{j\in J} \operatorname{esssup} \mathbb{P}\left(\Omega'_j|\mathscr{F}_{jL}\right) \le c_1^{|J|} \le c_1^{(1-\delta)k}.
$$

By summing up over all sets $J$, we have

$$
\begin{aligned}
\mathbb{P}(N_{[0,1]}(\varepsilon) < \delta k) &\le \sum_{J:|J|\ge\lceil(1-\delta)k\rceil} \mathbb{P}\left(\bigcap_{j\in J} \Omega_j\right) \\
&\le |\{J : |J| \ge \lceil(1-\delta)k\rceil\}|\, c_1^{(1-\delta)k} \\
&\le \exp(\delta_1 k) c_1^{(1-\delta)k};
\end{aligned}
$$

and we are done with (14) if $\delta$ is chosen so small that $\exp(\delta_1)c_1^{1-\delta} < 1$.

### 5.4 Alternative proof via local times

Here, we give an alternative proof for Proposition 17 when $\alpha > 1$. In this case, the strictly $\alpha$-stable process $Y$ possesses a continuous local time $L$:

$$
\int_B L(x)\,\mathrm{d}x = \int_0^1 \mathbb{1}_B(Y(t))\,\mathrm{d}t, \qquad \text{for all Borel sets } B.
$$

We define $L^* = \sup_x L(x)$, the maximum of local time of the $\alpha$-stable Lévy process considered on the time interval $[0,1]$. It was shown by Lacey ([14]) that

$$
\log \mathbb{P}(L^* > u) \sim -cu^\alpha, \qquad \text{as } u \to \infty,
$$

for some (explicitly known) constant $c > 0$.

Note that, if $N(K,\varepsilon) < k$ then

$$1 = \int_0^1 \mathbb{1}_K(Y(t))\,\mathrm{d}t = \int_K L(x)\,\mathrm{d}x \le L^* \int_K 1\,\mathrm{d}x \le L^* \cdot (\varepsilon k).$$

Therefore, for $\varepsilon$ small enough,

$$\mathbb{P}(N(K,\varepsilon) < k) \le \mathbb{P}(L^* > (\varepsilon k)^{-1}) \le \exp(-c'(\varepsilon k)^{-\alpha}).$$

In particular, by letting $k = \lceil \varepsilon^{-\alpha/(1+\alpha)} \rceil$ we get

$$\mathbb{P}(N(K,\varepsilon) < k) \le \exp(-c''\varepsilon^{-\alpha/(1+\alpha)}),$$

as required in (14).

A similar proof also works for other symmetric Lévy processes, cf. [6], where Lacey's result is generalized.

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