Electronic Journal of Probability

Vol. 11 (2006), Paper no. 9, pages 249–275.

Journal URL
http://www.math.washington.edu/~ejpecp/

On random walk simulation of one-dimensional diffusion processes with discontinuous coefficients

Pierre Étoré*
Projet OMEGA, IECN, BP 239, F-54506 Vandœuvre-lès-Nancy CEDEX
Pierre.Etore@iecn.u-nancy.fr
http://www.iecn.u-nancy.fr/ etore/

Résumé

In this paper, we provide a scheme for simulating one-dimensional processes generated by divergence or non-divergence form operators with discontinuous coefficients. We use a space bijection to transform such a process in another one that behaves locally like a Skew Brownian motion. Indeed the behavior of the Skew Brownian motion can easily be approached by an asymmetric random walk.

Key words: Monte Carlo methods, random walk, Skew Brownian motion, one-dimensional process, divergence form operator, local time **AMS 2000 Subject Classification:** Primary 660J60, 65C.

Submitted to EJP on May 11, 2005. Final version accepted on February 21, 2006.

*Support acknowledgement: This work has been supported by the GdR MOMAS.

1 Introduction

In this paper we provide a random walk based scheme for simulating one-dimensional processes generated by operators of type

$$L = \frac{\rho}{2}\nabla\Big(a\nabla\Big) + b\nabla. \tag{1.1}$$

These operators appear in the modelisation of a wide variety of diffusion phenomena, for instance in fluid mechanics, in ecology, in finance (see [DDG05]). If a, ρ, and b are measurable and bounded and if a and ρ are uniformly elliptic it can be shown that L is the infinitesimal generator of a Markov process X. Note that these operators contain the case of operators of type $L = \frac{a}{2}\Delta + b\nabla$ whose interpretation in terms of Stochastic Differential Equation is well known. In the general case there is still such an interpretation. Indeed if we assume for simplicity that $b = 0$ it can be shown (see Section 4) that X solves

$$X_t = X_0 + \int_0^t \sqrt{a(X_s)\rho(X_s)}dW_s + \int_0^t \frac{\rho(X_s)a'(X_s)}{2}ds + \sum_{x\in\mathcal{I}} \frac{a(x+) - a(x-)}{a(x+) + a(x-)} L_t^x(X), \tag{1.2}$$

where $\mathcal{I}$ is the set of the points of discontinuity of a and $L_t^x(X)$ is the symmetric local time of X in x.

For the coefficients a, ρ, and b may be discontinuous, providing a scheme to simulate trajectories of X is challenging: we cannot use the panel of Monte-Carlo methods available for smooth coefficients (see [KP92]). However, some authors have recently provided schemes to simulate X in the case of coefficients having some discontinuities.

In [Mar04] (see also [MT06]) M. Martinez treated the case of a coefficient a having one point of discontinuity. He applied an Euler scheme after a space transformation that allows to get rid of the local time in (1.2). To estimate the speed of convergence of his method he needs a to be $\mathcal{C}^6$ outside the point of discontinuity. The initial condition has to be $\mathcal{C}^4$ almost everywhere and to satisfy other restrictive conditions.

In [LM06] (see also [Mar04]) A. Lejay and M. Martinez proposed a different scheme. After a piecewise constant approximation of the coefficients and a different space transformation, they propose to use an exact simulation method of the Skew Brownian Motion (SBM). This one is based on the simulation of the exit times of a Brownian motion. In general the whole algorithm is slow and costly but allows to treat the case of coefficients a, ρ and b being right continuous with left limits, and of class $\mathcal{C}^1$ except on countable set of points, without cluster point. Besides the initial condition can be taken in H^1, and the algorithm is well adapted to the case of coefficients being flat on large intervals outside their points of discontinuity.

Here, under the same hypotheses on a, ρ and b, but with the initial condition in $\mathrm{W}^{1,\infty} \cap \mathrm{H}^1 \cap \mathcal{C}_0$ we propose a new scheme based mostly on random walks.

Roughly the idea is the following: assume for simplicity that $b = 0$. First, like in [LM06], we replace a and ρ by piecewise constant a^n and ρ^n in order to obtain X^n that is a good weak approximation of X and that solves

$$X_t^n = X_0^n + \int_0^t \sqrt{a^n(X_s^n)\rho^n(X_s^n)}dW_s + \sum_{x_i^n\in\mathcal{I}^n} \frac{a(x_i^n+) - a(x_i^n-)}{a(x_i^n+) + a(x_i^n-)} L_t^{x_i^n}(X^n),$$

where $\mathcal{I}^n$ is the set of the points of discontinuity of a^n. Second by a proper change of scale we transform X^n into Y^n that solves

$$Y_t^n = Y_0^n + W_t + \sum_{x_k^n\in\mathcal{I}^n} \beta_k^n L_t^{k/n}(Y^n),$$

where the β_k^n's are explicitely known. Thus Y^n behaves around each k/n like a SBM of parameter β_k^n (see Subsection 5.2 for a brief presentation of the SBM). That is, heuristically, Y^n when in k/n moves up with probability $(\beta_k^n+1)/2$ and down with probability $(\beta_k^n-1)/2$, and behaves like a standard Brownian motion elsewhere. Thus a random walk on the grid $\{k/n : k \in \mathbb{Z}\}$ can reflect the behaviour of Y^n as was shown in [Leg85]. We use this to finally construct an approximation $\widehat{Y}^n$ of Y^n.

We obtain a very easy to implement algorithm that only requires simulations of Bernoulli random variables. We estimate the speed of weak convergence of our algorithm by mixing an estimate of a weak error and an estimate of a strong error. Indeed computing the strong error of the algorithm presents difficulties we were not able to overcome. On the other hand computing directly the weak error without using a strong error estimate would lead to more complicated computations without any improvement of the speed of convergence: basically our approach relies on the Donsker theorem and we cannot get better than an error in $\mathcal{O}(n^{-1/2})$. Moreover to make such computations we should require additionnal smoothness on the data (see [Mar04]).

We finally make numerical experiments: the proposed scheme appears to be satisfying compared to the ones proposed in [Mar04] or [LM06].

Hypothesis. We make some assumptions from now till the end of the paper, for the sake of simplicity but without loss of generality.

(A1) $b = 0$.

Indeed, as explained in [LM06] Section 2, if we can treat the case

$$L = \frac{\rho}{2}\nabla\Big(a\nabla\Big), \tag{1.3}$$

we can treat the case (1.1) for any measurable bounded b by defining the coefficients ρ and a in (1.3) in the following manner:

$$\text{If } \ a_b := a\exp^{\Psi} \ \text{ and } \rho_b := \rho\exp^{-\Psi},$$

$$\text{with } \Psi(x) = \int_0^x h(y)dy \ \text{ and } h(x) = 2\frac{b(x)}{\rho(x)a(x)},$$

$$\text{then } \ \frac{\rho_b}{2}\nabla\Big(a_b\nabla\Big) = \frac{\rho}{2}\nabla\Big(a\nabla\Big) + b\nabla.$$

(A2) Let be $G = (l, r)$ an open bounded interval of $\mathbb{R}$. We will assume the process X starts from $x \in G$ and is killed when reaching $\{l, r\}$. >From a PDEs point of view this means the parabolic PDEs involving L we will study are submitted to uniform Dirichlet boundary conditions. We could treat Neumann boundary conditions (thanks to the results of [BC05] for instance) and, by localization arguments, the case of an unbounded domain G (see [LM06]). But this assumption will make the material of the paper simpler and clearer.

Outline of the paper. In Section 2 we define precisely Divergence Form Operators (DFO) and recall some of their properties. In Section 3 we speak of Stochastic Differential Equations involving Local Time (SDELT): we state a general change of scale formula and recall some convergence results established by J.F. Le Gall in [Leg85]. In Section 4 we link DFO and SDELT: a process generated by a DFO of coefficients a and ρ having countable discontinuities without cluster points is solution of a SDELT with coefficients determined by a and ρ. In Section 5 we present our scheme. In Section 6 we estimate the speed of convergence of this scheme. Section 7 is devoted to numerical experiments.

Some notations. For $1 \leq p < \infty$ we denote by $\mathrm{L}^p(G)$ the set of measurable functions f on G such that

$$\|f\|_p := \big(\int_G |f(x)|^p dx \big)^{1/p} < \infty.$$

Let be $0 < T < \infty$ fixed. For $1 \leq p, q < \infty$ we denote by $\mathrm{L}^q(0,T;\mathrm{L}^p(G))$ the set of measurable functions f on $(0,T) \times G$ such that

$$|||f|||_{p,q} := \big(\int_0^T \|f\|_p^q \, dt \big)^{1/q} < \infty.$$

For $u \in \mathrm{L}^p(G)$ we denote by $\frac{du}{dx}$ the first derivative of u in the distributional sense. It is standard to denote by $\mathrm{W}^{1,p}(G)$ the space of functions $u \in \mathrm{L}^p(G)$ such that $\frac{du}{dx} \in \mathrm{L}^p(G)$, and by $\mathrm{W}_0^{1,p}(G)$ the closure of $\mathcal{C}_c^\infty(G)$ in $\mathrm{W}^{1,p}(G)$ equipped with the norm $(\|u\|_p^p + \big\|\frac{du}{dx}\big\|_p^p)^{1/p}$. We denote by $\mathrm{H}^1(G)$ the space $\mathrm{W}^{1,2}(G)$, and by $\mathrm{H}_0^1(G)$ the space $\mathrm{W}_0^{1,2}(G)$.
For $u \in \mathrm{L}^2(0,T;\mathrm{L}^2(G))$ we denote by $\partial_t u$ the distribution such that for all $\varphi \in \mathcal{C}_c^\infty((0,T) \times G)$, we have, $\langle \partial_t u, \varphi \rangle = -\int_0^T \int_G u \partial_t \varphi$. We still denote by $\frac{du}{dx}$ the first derivative of u with respect to x in the distributional sense.
We will classicaly denote by $\|.\|_\infty$ and $|||.|||_{\infty,\infty}$ the supremum norms.

For the use of probability we will denote by $\mathcal{C}_0(G)$ the set of continuous bounded functions on G. The symbol $\cong$ will denote equality in law.

2 On divergence form operators

For $0 < \lambda < \Lambda < \infty$ let us denote by $\mathfrak{Ell}(\lambda, \Lambda)$ the set of functions f on G that are measurable and such that

$$\forall x \in G, \quad \lambda \leq f(x) \leq \Lambda.$$

For $\rho \in \mathfrak{Ell}(\lambda, \Lambda)$ let us define the measure $m_\rho(dx) := \rho^{-1}(x)dx$.
For any measure m with a bounded density with respect to the Lebesgue measure we then denote by $\mathrm{L}^2(G, m)$ the Hilbert space of functions in $\mathrm{L}^2(G)$ equipped with the scalar product

$$(f, g) \longmapsto \int_G f(x)g(x)m(dx).$$

This is done in order that the operator we define below is symmetric on $\mathrm{L}^2(G, m_\rho)$.

Definition 2.1 *Let a and ρ be in $\mathfrak{Ell}(\lambda, \Lambda)$ for some $0 < \lambda < \Lambda < \infty$. We call Divergence form operator of coefficients a and ρ, and we note $\mathfrak{L}(a, \rho)$, the operator $(L, D(L))$ on $\mathrm{L}^2(G, m_\rho)$ defined by*

$$L = \frac{\rho}{2} \frac{d}{dx} \Big(a \frac{d}{dx} \Big),$$

and

$$D(L) = \{ \, u \in \mathrm{H}_0^1(G) \; , \; Lu \in \mathrm{L}^2(G) \}.$$

Actually if $a, \rho \in \mathfrak{Ell}(\lambda, \Lambda)$ the operator $\mathfrak{L}(a, \rho)$ has sufficient properties to generate a continuous Markov process. We sum up these properties in the next theorem.

Theorem 2.1 *Let a and ρ be in $\mathfrak{Ell}(\lambda,\Lambda)$ for some $0<\lambda<\Lambda<\infty$. Then we have:*

i) The operator $\mathfrak{L}(a,\rho)$ on $\mathrm{L}^2(G,m_\rho)$ is closed and self-adjoint, with dense domain.

ii) This operator is the infinitesimal generator of a strongly continuous semigroup of contraction $(S_t)_{t\geq 0}$ on $\mathrm{L}^2(G,m_\rho)$.

iii) Moreover $(S_t)_{t\geq 0}$ is a Feller semigroup. Thus $\mathfrak{L}(a,\rho)$ is the infinitesimal generator of a Markov process $(X_t, t\geq 0)$.

iv) The process $(X_t, t\geq 0)$ has continuous trajectories.

Proof. We give the great lines of the proof and refer the reader to [Lej00] and [Str82] for details.

We set $(L,D(L))=\mathfrak{L}(a,\rho)$. First it is possible to build a symmetric bilinear form $\mathcal{E}$ on $\mathrm{L}^2(G,m_\rho)$ defined by

$$\mathcal{E}(u,v)=\int_G \frac{a}{2}\frac{du}{dx}\frac{dv}{dx}dx,\ \ \forall (u,v)\in\ D(\mathcal{E})\times D(\mathcal{E}),\ \ \text{and}\ \ D(\mathcal{E})=\mathrm{H}^1_0(G),$$

that verifies,

$$\mathcal{E}(u,v)=-\left\langle Lu,v\right\rangle_{\mathrm{L}^2(G,m_\rho)},\ \ \forall (u,v)\in D(L)\times D(\mathcal{E}).$$

Thus the resolvent of $(L,D(L))$ can be built and we get i). An application of the Hille-Yosida theorem then leads to ii).

Further it is a classical result of PDEs that the semigroup $(P_t)_{t\geq 0}$ has a density $p(t,x,y)$ with respect to the measure m_ρ such that

$$u(t,x)=\int_G p(t,x,y)f(y)\rho^{-1}(y)dy \tag{2.1}$$

is a continuous version of $P_tf(x)$. Then for f in $\mathcal{C}_0(G)$, P_tf belongs to $\mathcal{C}_0(G)$. By the use of the maximum principle it can be shown that $(P_t)_{t\geq 0}$ is semi-markovian and we get iii).

Finally Aronson estimates on the density $p(t,x,y)$ can be used to show for example that

$$\forall\varepsilon>0,\ \ \forall x\in G,\ \ \lim_{t\downarrow 0}\frac{1}{t}\int_{|y-x|>\varepsilon}p(t,x,y)\rho^{-1}(y)dy=0,$$

and thus we get iv) (see Proposition 2.9 in chapter 4 of [EK86]).

□

We have a consistency theorem.

Theorem 2.2 *Let be $0<\lambda<\Lambda<\infty$. Let a and ρ be in $\mathfrak{Ell}(\lambda,\Lambda)$ and (a^n,ρ^n) be a sequence of $\mathfrak{Ell}(\lambda,\Lambda)\times\mathfrak{Ell}(\lambda,\Lambda)$.*

Let us denote by S and X respectively the semigroup and the process generated by $\mathfrak{L}(a,\rho)$ and by (S^n) and (X^n) the sequences of semigroups and processes generated by the sequence of operators $\mathfrak{L}(a^n,\rho^n)$.

Assume that

$$\frac{1}{a^n}\xrightharpoonup[n\to\infty]{\mathrm{L}^2(G)}\frac{1}{a},\ \ \text{and}\ \ \frac{1}{\rho^n}\xrightharpoonup[n\to\infty]{\mathrm{L}^2(G)}\frac{1}{\rho}.$$

Then for any $T>0$ and any $f\in\mathrm{L}^2(G)$ we have :

i) The function $S_t^n f(x)$ *converges weakly in* $\mathrm{L}^2(0,T;\mathrm{H}_0^1(G))$ *to* $S_t f(x)$.

ii) The continuous version of $S_t^n f(x)$ *given by* (2.1) *with* p *replaced by* p^n *converges uniformly on each compact of* $(0,T)\times G$ *to the continuous version of* $S_t f(x)$ *given by* (2.1).

iii)

$$(X_t^n,\ t\geq 0)\xrightarrow[n\to\infty]{\mathcal{L}}(X_t,\ t\geq 0).$$

Proof. See in [LM06] the proofs of Propositions 3 and 4.

□

3 On SDE involving local time

First we introduce a new class of coefficients. For $0<\lambda<\Lambda<\infty$ we denote by $\mathfrak{Coeff}(\lambda,\Lambda)$ the set of the elements f of $\mathfrak{Ell}(\lambda,\Lambda)$ that verify:

i) f is right continuous with left limits (r.c.l.l.).
ii) f belongs to $\mathcal{C}^1(G\setminus\mathcal{I})$, where $\mathcal{I}$ is a countable set without cluster point.

Let us also denote by $\mathfrak{M}$ the space of all bounded measures ν on G such that $|\nu(\{x\})|<1$ for all x in G.

Definition 3.1 *Let* σ *be in* $\mathfrak{Coeff}(\lambda,\Lambda)$ *for some* $0<\lambda<\Lambda<\infty$*, and* ν *be in* $\mathfrak{M}$*. We call Stochastic Differential Equation with Local Time of coefficients* σ *and* ν*, and we note* $\mathfrak{Sde}(\sigma,\nu)$*, the following SDE*

$$X_t=X_0+\int_0^t\sigma(X_s)dW_s+\int_{\mathbb{R}}\nu(dx)L_t^x(X),$$

where $L_t^x(X)$ *is the symmetric local time of the unknown process* X.

In [Leg85] J.F. Le Gall studied some properties of SDEs of the type $\mathfrak{Sde}(\sigma,\nu)$. We will recall here some results of this work we will use in the sequel.

We will see below that $\sigma\in\mathfrak{Coeff}(\lambda,\Lambda)$ and $\nu\in\mathfrak{M}$ is a sufficient condition to have a unique strong solution to $\mathfrak{Sde}(\sigma,\nu)$. We first fix some additionnal notations.

For f in $\mathfrak{Coeff}(\lambda,\Lambda)$ we denote by $f'(dx)$ the bounded measure corresponding to the first derivative of f in the generalized sense. We denote by $f(x+)$ and $f(x-)$ respectively the right and left limits of f in x. We will also denote by $f'(x)$ the r.c.l.l. density of the absolutely continuous part of $f'(dx)$ (that it is to say we take for $f'(x)$ the right derivative of f in x).

For ν in $\mathfrak{M}$ we denote by ν^c the absolutely continuous part of ν.

3.1 A change of scale formula

Let us define the class of bijections we will use in our change of scale.

For $0<\lambda<\Lambda<\infty$ we denote by $\mathfrak{T}(\lambda,\Lambda)$ the set of all functions Φ on G that have a first derivative Φ' that belongs to $\mathfrak{Coeff}(\lambda,\Lambda)$. The assumption made on the bijection is minimal and we can then state a very general change of scale formula.

Proposition 3.1 *Let σ be in $\mathfrak{Coeff}(\lambda,\Lambda)$ for some $0<\lambda<\Lambda<\infty$. Let*

$$\nu(dx)=b(x)dx+\sum_{x_i\in\mathcal{I}}c_{x_i}\delta_{x_i}(dx),$$

be in $\mathfrak{M}$ (i.e., b is measurable and bounded, and each $|c_{x_i}|<1$).
Let Φ be in $\mathfrak{T}(\lambda',\Lambda')$ for some $0<\lambda'<\Lambda'<\infty$ and let $\mathcal{J}$ be the set of the points of discontinuity of Φ'.
Then the next statements are equivalent:

i) The process X solves $\mathfrak{Sde}(\sigma,\nu)$.
ii) The process $Y:=\Phi(X)$ solves $\mathfrak{Sde}(\gamma,\mu)$ with

$$\gamma(y)=(\sigma\Phi')\circ\Phi^{-1}(y),$$

and

$$\mu(dy)=\frac{\Phi' b+\frac{1}{2}(\Phi')'}{(\Phi')^2}\circ\Phi^{-1}(y)dy+\sum_{x_i\in\mathcal{I}\cup\mathcal{J}}\beta_{x_i}\delta_{\Phi(x_i)}(dy),$$

where,

$$\beta_x=\frac{\Phi'(x+)(1+c_x)-\Phi'(x-)(1-c_x)}{\Phi'(x+)(1+c_x)+\Phi'(x-)(1-c_x)},$$

with $c_x=0$ if $x\in\mathcal{J}\setminus\mathcal{I}$.

Remark 3.1 Note that γ obviously belongs to $\mathfrak{Coeff}(\lambda'',\Lambda'')$ for some $0<\lambda''<\Lambda''<\infty$, and that μ is in $\mathfrak{M}$, so it makes sense to speak of $\mathfrak{Sde}(\gamma,\mu)$.
We can say that the class of SDE of type $\mathfrak{Sde}(\sigma,\nu)$ is stable by transformation by a bijection belonging to $\mathfrak{T}(\lambda,\Lambda)$ for some $0<\lambda<\Lambda<\infty$.

Proof of Proposition 3.1. We prove $i)\Rightarrow ii)$. The converse can be proven in the same manner quite being technically more cumbersome.
By the symmetric Itô-Tanaka formula we first get:

$$\begin{aligned}
Y_t=\Phi(X_t) &= \Phi(X_0)+\textstyle\int_0^t(\sigma\Phi')(X_s)dW_s+\int_0^t(\sigma^2 b\Phi')(X_s)ds\\
&\quad+\textstyle\sum_{x_i\in\mathcal{I}}\frac{\Phi'(x_i+)+\Phi'(x_i-)}{2}c_{x_i}L_t^{x_i}(X)\\
&\quad+\textstyle\frac{1}{2}\int_0^t[\sigma^2(\Phi')'](X_s)ds+\sum_{x_i\in\mathcal{J}}\frac{\Phi'(x_i+)-\Phi'(x_i-)}{2}L_t^{x_i}(X)\\
&= \Phi(X_0)+\textstyle\int_0^t(\sigma\Phi')\circ\Phi^{-1}(Y_s)dW_s\\
&\quad+\textstyle\int_0^t[\sigma^2(\Phi' b+\frac{1}{2}(\Phi')')]\circ\Phi^{-1}(Y_s)ds+\sum_{x_i\in\mathcal{I}\cup\mathcal{J}}K_{x_i}L_t^{x_i}(X),
\end{aligned}$$

with $K_x=c_x(\Phi'(x+)+\Phi'(x-))/2+(\Phi'(x+)-\Phi'(x-))/2$.
We have then to express $L_t^x(X)$ in function of $L_t^{\Phi(x)}(Y)$ for $x\in\mathcal{I}\cup\mathcal{J}$.

Using Corollary VI.1.9 of [RY91] it can be shown that

$$L_t^{\Phi(x)^\pm}(Y) = \Phi'(x\pm) L_t^{x\pm}(X). \tag{3.1}$$

Besides theorem VI.1.7 of [RY91] leads to $(L_t^{x+}(X) - L_t^{x-}(X))/2 = c_x L_t^x(X)$ and combining with $(L_t^{x+}(X) + L_t^{x-}(X))/2 = L_t^x(X)$ we get

$$L_t^{x+}(X) = (1 + c_x) L_t^x(X). \tag{3.2}$$

In a similar manner we have $(L_t^{\Phi(x)+}(Y) - L_t^{\Phi(x)-}(Y))/2 = K_x L_t^x(X)$ and we can get

$$K_x L_t(X) + L_t^{\Phi(x)}(Y) = L_t^{\Phi(x)+}(Y).$$

Then using (3.1) and (3.2) we get

$$L_t^{\Phi(x)}(Y) = (\Phi'(x+)(1 + c_x) - K_x) L_t^x(X),$$

and the formula is proved.

□

To prove the proposition below, Le Gall used in [Leg85] a space bijection that enters in the general setting of Proposition 3.1.

Proposition 3.2 (Le Gall 1985) *Let σ be in $\mathfrak{Coeff}(\lambda, \Lambda)$ for some $0 < \lambda < \Lambda < \infty$ and ν be in $\mathfrak{M}$. There is a unique strong solution to $\mathfrak{Sde}(\sigma, \nu)$.*

We need two lemmas.

Lemma 3.1 (Le Gall 1985) *Let σ be in $\mathfrak{Coeff}(\lambda, \Lambda)$ for some $0 < \lambda < \Lambda < \infty$. There is a unique strong solution to $\mathfrak{Sde}(\sigma, 0)$.*

Proof. See [Leg85]. □

The next lemma will play a great role for calculations in the sequel.

Lemma 3.2 *Let ν be in $\mathfrak{M}$. There exists a function f_ν in $\mathfrak{Coeff}(\lambda, \Lambda)$ (for some $0 < \lambda < \Lambda < \infty$), unique up to a multiplicative constant, such that:*

$$f_\nu'(dx) + (f_\nu(x+) + f_\nu(x-))\nu(dx) = 0. \tag{3.3}$$

Proof of Proposition 3.2. It suffices to set

$$\Phi_\nu(x) = \int_0^x f_\nu(y) dy.$$

By Lemma 3.2 Φ_ν obviously belongs to $\mathfrak{T}(\lambda, \Lambda)$ for some $0 < \lambda < \Lambda < \infty$. By Proposition 3.1 and Lemma 3.2 we get that X solves $\mathfrak{Sde}(\sigma, \nu)$ if and only if $Y := \Phi_\nu(X)$ solves $\mathfrak{Sde}\big((\sigma f_\nu) \circ \Phi_\nu^{-1}, 0\big)$. By Lemma 3.1 the proof is completed.

□

3.2 Convergence results

Le Gall proved the next consistency result for equations of the type $\mathfrak{Sde}(\sigma,\nu)$.

Theorem 3.1 (Le Gall 1985) *Let be two sequences (σ^n) and (ν^n) for which there exist $0<\lambda<\Lambda<\infty$, $0<M<\infty$ and $\delta>0$ such that*

$$
\begin{aligned}
&(H1) \quad \sigma^n \in \mathfrak{Coeff}(\lambda,\Lambda), \quad \forall n\in\mathbb{N},\\
&(H2) \quad |\nu^n(\{x\})| \leq 1-\delta, \quad \forall n\in\mathbb{N}, \forall x\in G.\\
&(H3) \quad |\nu^n|(G)\leq M, \qquad \forall n\in\mathbb{N},
\end{aligned}
$$

so that each ν^n is in $\mathfrak{M}$. Assume that there exist two functions σ and f in $\mathfrak{Coeff}(\lambda',\Lambda')$ (for some $0<\lambda'<\Lambda'<\infty$) such that

$$\sigma_n \xrightarrow[n\to\infty]{L^1_{loc}(\mathbb{R})} \sigma \qquad \text{and} \qquad f_{\nu^n} \xrightarrow[n\to\infty]{L^1_{loc}(\mathbb{R})} f,$$

and set:

$$\nu(dx) = -\frac{f'(dx)}{f(x+)+f(x-)}. \tag{3.4}$$

Let $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P}^x)$ be a filtered probability space carrying an adapted Brownian motion W. On this space, for each $n\in\mathbb{N}$ let be X^n the strong solution of $\mathfrak{Sde}(\sigma^n,\nu^n)$, and let be X the strong solution of $\mathfrak{Sde}(\sigma,\nu)$. Then:

$$\mathbb{E}\big[\sup_{0\leq s\leq t}|X^n_s - X_s|\big] \xrightarrow[n\to\infty]{} 0 \qquad \text{and} \qquad (X^n_t,\, t\geq 0) \xrightarrow[n\to\infty]{\mathcal{L}} (X_t,\, t\geq 0).$$

Remark 3.2 In this theorem ν^n approaches ν in the sense that f_{ν^n} tends to f_ν for the L^1_{loc} convergence. Note we can have $\nu^n \rightharpoonup \nu_1$, but $f_{\nu^n}\to f_{\nu_2}$ for the L^1_{loc} convergence, with $\nu_1\neq\nu_2$ (See [Leg85] p 65 for an example). The theorem asserts that X^n tends to X that solves $\mathfrak{Sde}(\sigma,\nu_2)$ and not $\mathfrak{Sde}(\sigma,\nu_1)$!

Le Gall also proved a Donsker theorem for solution to SDEs of the type $\mathfrak{Sde}(\sigma,\nu)$ for $\sigma\equiv 1$. Let be μ in $\mathfrak{M}$ and Y be the solution to $\mathfrak{Sde}(1,\mu)$.

We define some coefficients β^n_k for all $k\in\mathbb{Z}$, and all $n\in\mathbb{N}^*$, by:

$$\frac{1-\beta^n_k}{1+\beta^n_k} = \exp\big(-2\mu^c(]\frac{k}{n},\frac{k+1}{n}])\big)\prod_{\frac{k}{n}<y\leq\frac{k+1}{n}}\Big(\frac{1-\mu(\{y\})}{1+\mu(\{y\})}\Big) = \frac{f_\mu(\frac{k+1}{n})}{f_\mu(\frac{k}{n})}. \tag{3.5}$$

We define a sequence (μ^n) of measures in $\mathfrak{M}$ by

$$\mu^n = \sum \beta^n_k \delta_{\frac{k}{n}}, \tag{3.6}$$

and a sequence (Y^n) of processes such that each Y^n solves $\mathfrak{Sde}(1,\mu^n)$.

Finally we define for all $n\in\mathbb{N}^*$ a sequence $(\tau^n_p)_{p\in\mathbb{N}}$ of stopping times by,

$$\begin{aligned} &\tau_0^n = 0 \\ &\tau_{p+1}^n = \inf\{t > \tau_p^n : |Y_t^n - Y_{\tau_p^n}^n| = \tfrac{1}{n}\}. \end{aligned} \tag{3.7}$$

We have the next theorem.

Theorem 3.2 (Le Gall 1985) *In the previous context $S_p^n := nY_{\tau_p^n}^n$ defines a sequence of random walks on the integers such that:*

i)

$$\begin{aligned} &S_0^n = 0, && \forall n \in \mathbb{N}^*, \\ &\mathbb{P}[S_{p+1}^n = k+1 | S_p^n = k] = \tfrac{1}{2}(1+\beta_k^n), && \forall n, p \in \mathbb{N}^*, \forall k \in \mathbb{Z}, \\ &\mathbb{P}[S_{p+1}^n = k-1 | S_p^n = k] = \tfrac{1}{2}(1-\beta_k^n), && \forall n, p \in \mathbb{N}^*, \forall k \in \mathbb{Z}. \end{aligned}$$

ii)

The sequence of processes defined by $\widetilde{Y}_t^n := (1/n) S_{[n^2 t]}^n$, where $\lfloor . \rfloor$ stands for the integer part of a non negative real number, verifies for all $0 < T < \infty$:

$$\mathbb{E}|\widetilde{Y}_t^n - Y_t| \xrightarrow[n\to\infty]{} 0, \ \forall t \in [0,T] \quad \text{and} \quad (\widetilde{Y}_t^n, \ t \geq 0) \xrightarrow[n\to\infty]{\mathcal{L}} (Y_t, \ t \geq 0).$$

4 Link between DFO and SDELT

This link is stated by the following proposition.

Proposition 4.1 *Let a and ρ be in $\mathfrak{Coeff}(\lambda, \Lambda)$ for some $0 < \lambda < \Lambda < \infty$. Let us denote by $\mathcal{I}$ the set of the points of discontinuity of a. Then $\mathfrak{L}(a, \rho)$ is the infinitesimal generator of the unique strong solution of $\mathfrak{Sde}(\sqrt{a\rho}, \nu)$ with,*

$$\nu(dx) = \Big(\frac{a'(x)}{2a(x)}\Big) dx + \sum_{x_i \in \mathcal{I}} \frac{a(x_i+) - a(x_i-)}{a(x_i+) + a(x_i-)} \delta_{x_i}(dx). \tag{4.1}$$

In [LM06] the authors proved the proposition above by the use of Dirichlet forms and Revuz measures. We give here a more simple proof, based on smoothing the coefficients and using the consistency theorems of the two preceeding sections.

Proof of Proposition 4.1. As a and ρ are in $\mathfrak{Coeff}(\lambda, \Lambda)$ the function $\sqrt{\rho a}$ is in $\mathfrak{Coeff}(\lambda, \Lambda)$. Besides, as $|a-b|/|a+b| < 1$ for any a,b in $\mathbb{R}_+^*$, the measure ν defined by (4.1) is in $\mathfrak{M}$. The existence of a unique strong solution X to $\mathfrak{Sde}(\sqrt{a\rho}, \nu)$ follows from Proposition 3.2.

We then identify the infinitesimal generator of X. We can build two sequences (a^n) and (ρ^n) of functions in $\mathfrak{Coeff}(\lambda, \Lambda) \cap \mathcal{C}^\infty(G)$, such that

$$a^n \xrightarrow[n\to\infty]{} a \ \text{ a.e.} \quad \text{and} \quad \rho^n \xrightarrow[n\to\infty]{} \rho \ \text{ a.e.}$$

For any n in $\mathbb{N}$ we denote by X^n the process generated by $\mathfrak{L}(a^n, \rho^n)$.

On one hand, by dominated convergence the hypotheses of Theorem 2.2 are fulfilled, and we have,

$$(X_t^n,\ t \geq 0) \xrightarrow[n\to\infty]{\mathcal{L}} (\widetilde{X}_t,\ t \geq 0), \tag{4.2}$$

where the process $\widetilde{X}$ is generated by $\mathfrak{L}(a, \rho)$.
On the other hand we will show by Theorem 3.1 that

$$(X_t^n,\ t \geq 0) \xrightarrow[n\to\infty]{\mathcal{L}} (X_t,\ t \geq 0). \tag{4.3}$$

Thus taking in care (4.2) and (4.3) we will conclude that the infinitesimal generator of X is $\mathfrak{L}(a, \rho)$.
As a^n and ρ^n are $\mathcal{C}^\infty$, $(L^n, D(L^n)) = \mathfrak{L}(a^n, \rho^n)$ can be written,

$$L^n = \frac{\rho^n}{2}\Big[a^{n'}\frac{d}{dx} + a^n\frac{d^2}{dx^2}\Big],$$

so it is standard to say that X^n solves

$$X_t^n = x + \int_0^t \sqrt{\rho^n(X_s^n)a^n(X_s^n)}dW_s + \int_0^t \frac{\rho^n(X_s^n)a^{n'}(X_s^n)}{2}ds. \tag{4.4}$$

As $d\langle X^n\rangle_s = \rho^n(X_s^n)a^n(X_s^n)ds$, by the occupation time density formula we can rewrite (4.4) and assert that X^n solves,

$$X_t^n = x + \int_0^t \sqrt{\rho^n(X_s^n)a^n(X_s^n)}dW_s + \int_{\mathbb{R}} \nu_n(dx)L_t^x(X^n),$$

where $\nu_n(dx) = (a^{n'}(x)/2a^n(x))\lambda(dx)$.

Then elementary calculations show that the function f_{ν_n} associated to ν_n by Lemma 3.2 is of the form $f_{\nu_n}(x) = K/a^n(x)$ with K a real number. This obviously tends to $K/a(x) =: f(x)$ for the $L^1_{loc}(\mathbb{R})$-convergence. We then determine the measure ν associated to f by (3.4). First we check that $\nu^c(dx) = (a'(x)/2(a(x))\lambda(dx)$. Second, the set $\{x \in G : \nu(\{x\}) \neq 0\}$ is equal to $\mathcal{I}$, and we have for all $x \in \mathcal{I}$,

$$\nu(\{x\}) = -\frac{f(x+) - f(x-)}{f(x+) + f(x-)} = \frac{a(x+) - a(x-)}{a(x+) + a(x-)}.$$

So the measure ν is equal to the one defined by (4.1). As it is obvious that

$$\sqrt{\rho^n a^n} \xrightarrow[n\to\infty]{L^1_{loc}(\mathbb{R})} \sqrt{\rho a},$$

and that the hypotheses $(H1) - (H3)$ of Theorem 3.1 are fulfilled, we can say that (4.3) holds. The proof is completed. □

5 Random walk approximation

5.1 Monte Carlo Approximation

From now the horizon $0 < T < \infty$ is fixed. For any $a, \rho \in \mathfrak{Coeff}(\lambda, \Lambda)$ and any initial condition f we denote by $(\mathcal{P})(a, \rho, f)$ the parabolic PDE

$$
(\mathcal{P})(a,\rho,f)\begin{cases}\frac{\partial u(t,x)}{\partial t}=Lu(t,x), & \text{for } (t,x)\in[0,T]\times G,\\ u(t,l)=u(t,r)=0 & \text{for } t\in[0,T],\\ u(0,x)=f(x) & \text{for } x\in G,\end{cases}
$$

with $(L,D(L))=\mathfrak{L}(a,\rho)$.

Let be $0<\lambda<\Lambda<\infty$. From now till the end of this paper we assume that a and ρ are in $\mathfrak{Coeff}(\lambda,\Lambda)$. We denote by $\mathcal{I}=\{x_i\}_{i\in I}$ the set of the points of discontinuity of a ($I=\{0\le i\le k_1\}\subset\mathbb{Z}$ is finite). We set X to be the process generated by $\mathfrak{L}(a,\rho)$.

We seek for a probabilistic numerical method to approximate the solution of $(\mathcal{P})(a,\rho,f)$. By Theorem 2.1 and some standard PDEs refinements we know that for all $f\in\mathrm{L}^2(G)$, $(\mathcal{P})(a,\rho,f)$ has a unique weak solution $u(t,x)$ in $\mathcal{C}([0,T],\mathrm{L}^2(G,m_\rho))\cap\mathrm{L}^2(0,T;\mathrm{H}^1_0(G))$. We know that $\mathbb{E}^x[f(X_t)]$ is a continuous version of $u(t,x)$.

Our goal is to build a process $\widehat{X}^n$ easy to simulate and such that

$$
(\widehat{X}^n_t,\ t\ge0)\xrightarrow[n\to\infty]{\mathcal{L}}(X_t,\ t\ge0). \tag{5.1}
$$

Thus $\mathbb{E}^x[f(\widehat{X}^n_t)]\to\mathbb{E}^x[f(X_t)]$ for any $t\in[0,T]$, and the strong law of large numbers asserts that,

$$
\frac{1}{N}\sum_{i=1}^{N}f(\widehat{X}^{n,(i)}_t)\xrightarrow[N\to\infty]{n\to\infty}u(t,x), \tag{5.2}
$$

where for each i, $\widehat{X}^{n,(i)}$ is a realisation of the random variable $\widehat{X}^n$.

5.2 Skew Brownian Motion

The Skew Brownian Motion (SBM) of parameter $\beta\in(-1,1)$ starting from y, which we denote by $Y^{\beta,y}$ is known to solve:

$$
Y^{\beta,y}_t=y+W_t+\beta L^y_t(Y^{\beta,y}), \tag{5.3}
$$

i.e. $Y^{\beta,y}$ solves $\mathfrak{SDe}(1,\beta\delta_0)$ (see [HS81]).

It was first constructed by Itô and McKean in [IM74] (Problem 1 p115) by flipping the excursions of a reflected Brownian motion with probability $\alpha=(\beta+1)/2$. On SBM see also [Wal78] . It behaves like a Brownian motion except in y where its behaviour is pertubated, so that

$$
\mathbb{P}(Y^{\beta,y}_t>y)=\alpha,\quad\forall t>0. \tag{5.4}
$$

We denote by $\mathcal{T}(\Delta)$ the law of the stopping time $\tau=\inf\{t\ge0,\ |W_t|=\Delta\}$ where W is a standard Brownian motion starting at zero. For the SBM $Y^{\beta,0}$ and Δ in $\mathbb{R}^*_+$ we define the stopping time $\tau_\Delta=\inf\{t\ge0,\ Y^{\beta,0}_t\in\{\Delta,-\Delta\}\}$. Our approach relies on the following lemma.

Lemma 5.1 *Let y and x be in $\mathbb{R}$, Δ in $\mathbb{R}^*_+$ and β in $(-1,1)$. Set $\alpha=(\beta+1)/2$. Then*
i) $Y^{\beta,y}+x\cong Y^{\beta,y+x}$, and ii) $(\tau_\Delta,Y^{\beta,0}_{\tau_\Delta})\cong\mathcal{T}(\Delta)\otimes Ber(\alpha)$.

Proof. This follows from the construction of the SBM by Itô and McKean in [IM74]. □

5.3 Some possible approaches

Recently two methods have been proposed to build $\widehat{X}^n$ satisfying (5.1).

In [Mar04] M. Martinez proposed to use an Euler scheme. We know by Proposition 4.1 that X solves $\mathfrak{Sde}(\sqrt{a\rho}, \nu)$ with ν defined by (4.1). We have seen in the proof of Proposition 3.2 that if we define $\Phi(x) = \int_0^x f_\nu(y)$ then $Y = \Phi(X)$ solves $\mathfrak{Sde}(\gamma, 0)$ with γ in some $\mathfrak{Coeff}(m, M)$. Thus an Euler scheme approximation $\widehat{Y}^n$ of Y can be built and by setting $\widehat{X}^n = \Phi^{-1}(\widehat{Y}^n)$ we get an approximation of X. Because the coefficient γ is not Lipschitz if a and ρ are not, evaluating the speed of convergence of such a scheme is not easy.

In [LM06], A. Lejay and M. Martinez proposed to use the SBM. They first build a piecewise constant approximation (a^n, ρ^n) of (a, ρ) in order that the process X^n generated by $\mathfrak{L}(a^n, \rho^n)$ solves $\mathfrak{Sde}(\sqrt{a^n\rho^n}, \nu^n)$ with ν^n satisfying $(\nu^n)^c = 0$. Second by a proper bijection $\Phi^n \in \mathfrak{T}(m, M)$, they get that $Y^n = \Phi^n(X^n)$ solves $\mathfrak{Sde}(1, \mu^n)$ with $\mu^n = \sum \beta_k \delta_{y_k}$, i.e. Y^n behaves locally like a SBM. Third they proposed a scheme $\widehat{Y}^n$ for Y^n based on Lemma 5.1 and simulations of exit times of the SBM and they finally set $\widehat{X}^n = (\Phi^n)^{-1}(\widehat{Y}^n)$.

Our method can be seen as a variation of this last approach because it also deeply relies on getting such a Y^n and using Lemma 5.1. But we then use random walks instead of the scheme proposed in [LM06].

5.4 The basic idea of our approach

We focus on weak convergence and propose a three-step approximation scheme differing slightly from the one proposed by Theorem 3.2.

We fix $n \in \mathbb{N}^*$, and $1/n$ will be the spatial discretization step size.

STEP 1. We build (a^n, ρ^n) in $\mathfrak{Coeff}(\lambda, \Lambda) \times \mathfrak{Coeff}(\lambda, \Lambda)$ such that:

i) The functions a^n and ρ^n are piecewise constant. The points of discontinuity of either a^n and ρ^n are included in some set $\mathcal{I}^n$. We assume $\mathcal{I}^n = \{x_k^n\}_{k \in I^n}$ for $I^n = \{0 \le k \le k_1^n\} \subset \mathbb{Z}$ finite and $x_k^n < x_{k+1}^n$, $\forall k \in I^n$.

ii) For each $x_k^n \in \mathcal{I}^n$ we have $a^n(x_k^n) = a(x_k^n)$ and $\rho^n(x_k^n) = \rho(x_k^n)$.

iii) Consider the function

$$\Phi^n(x) = \sum_{k=0}^{k_{n,x}-1} \frac{x_{k+1}^n - x_k^n}{\sqrt{a(x_k^n)\rho(x_k^n)}} + \frac{x - x_{k_{n,x}}^n}{\sqrt{a(x_{k_{n,x}}^n)\rho(x_{k_{n,x}}^n)}}, \tag{5.5}$$

where the integer $k_{n,x}$ verifies $x_{k_{n,x}}^n \le x \le x_{k_{n,x}+1}^n$.

The set $\mathcal{I}^n$ satisfies $\Phi^n(\mathcal{I}^n) = \{k/n,\, k \in \mathbb{Z}\} \cap \Phi^n(G)$. From now we assume x_k^n is the point of $\mathcal{I}^n$ such that $\Phi^n(x_k^n) = k/n$.

Remark 5.1 In fact the first thing to do is to construct the grid $\mathcal{I}^n$ satisfying iii). It is very easy and only requires to know the coefficients a and ρ (see point 1 of the algorithm in Subsection 5.5). Then a^n and ρ^n can be constructed.

Remark 5.2 The sets $\mathcal{I}$ and $\mathcal{I}^n$ may have no common points.

We take X^n to be the process generated by $\mathfrak{L}(a^n, \rho^n)$.

Remark 5.3 It can be shown by Theorem 2.2 that X^n converges in law to X.

STEP 2. By Proposition 4.1 the process X^n solves $\mathfrak{S}\mathfrak{D}\mathfrak{e}(\sqrt{a^n\rho^n},\nu^n)$ with

$$\nu^n=\sum_{x_k^n\in\mathcal{I}^n}\frac{a^n(x_k^n+)-a^n(x_k^n-)}{a^n(x_k^n+)+a^n(x_k^n-)}\delta_{x_k^n}.$$

The function Φ^n defined by (5.5) belongs to $\mathfrak{T}(1/\Lambda,1/\lambda)$. The points of discontinuity of $\Phi^{n'}$ are those in $\mathcal{I}^n$, and $(\Phi^{n'})'=0$, so by Proposition 3.1 the process $Y^n=\Phi^n(X^n)$ solves

$$Y_t^n=Y_0^n+W_t+\sum_{x_k^n\in\mathcal{I}^n}\beta_k^nL_t^{k/n}(Y^n),\tag{5.6}$$

where

$$\beta_k^n=\frac{\sqrt{a(x_k^n)/\rho(x_k^n)}-\sqrt{a(x_{k-1}^n)/\rho(x_{k-1}^n)}}{\sqrt{a(x_k^n)/\rho(x_k^n)}+\sqrt{a(x_{k-1}^n)/\rho(x_{k-1}^n)}}.\tag{5.7}$$

To write these coefficients we have used the fact that a^n and ρ^n are r.c.l.l. and that for instance $a^n(x_k^n+)=a(x_k^n)$ and $a^n(x_k^n-)=a(x_{k-1}^n)$.

Remark 5.4 We have got Y^n that solves $\mathfrak{S}\mathfrak{D}\mathfrak{e}(1,\sum\beta_k^n\delta_{k/n})$ in a different way than the one used by Le Gall in Theorem 3.2. We now use his method to get $\widetilde{Y}^n$ that verifies

$$\mathbb{E}|\widetilde{Y}_t^n-Y_t^n|\xrightarrow[n\to\infty]{}0,\ \forall t\in[0,T].\tag{5.8}$$

STEP 3. Like in (3.7) we define a sequence $(\tau_p^n)_{p\in\mathbb{N}}$ of stopping times by,

$$\tau_0^n=0,\qquad\text{and}\qquad\tau_{p+1}^n=\inf\{t>\tau_p^n:|Y_t^n-Y_{\tau_p^n}^n|=1/n\}.$$

Thanks to the uniformity of the grid $\{k/n,\ k\in I^n\}$ we have the following lemma.

Lemma 5.2 *i) For all $k\in\mathbb{Z}$ and all $p\in\mathbb{N}$, $(Y_{\tau_p^n+u}^n-Y_{\tau_p^n}^n,\ 0\le u\le\tau_{p+1}^n-\tau_p^n)$ knowing that $\{Y_{\tau_p^n}^n=k/n\}$ has the same law as $(Y_t^{\beta_k^n,0},0\le t\le\tau_{1/n})$.*
ii)

$$\forall p\in\mathbb{N},\ \ \sigma_p^n:=n^2\big(\tau_p^n-\tau_{p-1}^n\big)\cong\mathcal{T}(1),$$

and the σ_p^n's are independent.

Proof. The statement i) follows simply from point i) of Lemma 5.1 and the comparison between (5.3) and (5.6). From i), the strong Markov property, and point ii) of Lemma 5.1, we get that $(\tau_p^n-\tau_{p-1}^n)\cong\mathcal{T}(1/n^2)$, and the statement ii) follows by scaling. Using again the Markov property we get the independence of the σ_p^n's. □

Le Gall used this lemma in his proof of Theorem 3.2. Indeed it is obvious by the i) of Lemma 5.2 and the ii) of Lemma 5.1 that $S_p^n:=nY_{\tau_p^n}^n$ satisfies

$$
\begin{aligned}
&S_0^n = 0, \\
&\mathbb{P}[S_{p+1}^n = k+1 | S_p^n = k] = \tfrac{1}{2}(1+\beta_k^n) =: \alpha_k^n, \qquad \forall p \in \mathbb{N}^*, \forall k \in I^n, \\
&\mathbb{P}[S_{p+1}^n = k-1 | S_p^n = k] = \tfrac{1}{2}(1-\beta_k^n) = 1-\alpha_k^n, \quad \forall p \in \mathbb{N}^*, \forall k \in I^n.
\end{aligned}
\tag{5.9}
$$

Moreover the ii) of Lemma 5.2 allows to show that $\widetilde{Y}_t^n := (1/n) S_{[n^2 t]}^n = Y_{\tau_{[n^2 t]}^n}^n$ satisfies (5.8).

Thus the idea is to take

$$
\widehat{X}_t^n := (\Phi^n)^{-1}\big(\frac{1}{n}\widehat{S}_{[n^2 t]}^n\big), \tag{5.10}
$$

where $\widehat{S}^n$ is a random walk on the integers defined by (5.9). The process $\widehat{X}^n$ is a random walk on the grid $\mathcal{I}^n$. In fact this grid is made in order that $\widehat{X}^n$ spends the same average time in each of its cells. Combining remark 5.3 and theorem 3.2 we should have (5.1). To sum up this section we write our scheme in the algorithm form. In the next section we will estimate the approximation error of our scheme.

5.5 The algorithm

Note that by construction $(\Phi^n)^{-1}\big(k/n\big) = x_k^n$ for all $k \in I^n$.

We define a function `ALGO` in the next manner:

INPUT DATA: the coefficients a and ρ, the starting point x, the precision order n and the final time t.

OUTPUT DATA: an approximation in law $\widehat{X}^n$ of X at time t.

1. Set $x_0^n \leftarrow l$.

while $x_k^n \leq r$

set $x_k^n \leftarrow \sqrt{a(x_k^n)\rho(x_k^n)}(1/n) + x_k^n$ and $k \leftarrow k+1$.

endwhile

2. Compute the $\alpha_k^n = (1+\beta_k^n)/2$ with β_k^n defined by (5.7).

3. Set $y \leftarrow \Phi^n(x)$.

if $(n\,y - \lfloor n\,y \rfloor) < 0.5$

set $s_0 \leftarrow \lfloor n\,y \rfloor$.

else

set $s_0 \leftarrow \lfloor n\,y \rfloor + 1$.

endif

4. **for** $i = 0$ to $i = \lfloor n^2\,t \rfloor =: N$

if $x_{s_i}^n \in \mathbb{R} \setminus (l, r)$

Return $x_{s_i}^n$.

endif

We have $s_i = k$ for some $k \in I^k$. Simulate a realization B of $Ber(\alpha_k^n)$.

Then set $s_i \leftarrow s_i + B$.

endfor

5. Return $x^n_{s_N}$.

6 Speed of convergence

In this section we will prove the following theorem.

Theorem 6.1 *Assume that $a, \rho \in \mathfrak{Coeff}(\lambda, \Lambda)$ for some $0 < \lambda \leq \Lambda < \infty$. Let be $0 < T < \infty$ and X the process generated by $\mathfrak{L}(a, \rho)$. For $n \in \mathbb{N}$ consider the process $\widehat{X}^n$ starting from x defined by,*

$$\forall t \in [0, T], \quad \widehat{X}^n_t = \mathtt{ALGO}(a, \rho, x, n, t).$$

For all $f \in \mathrm{W}^{1,\infty}_0(G) \cap \mathcal{C}_0(G)$, all $\varepsilon > 0$, and all $\gamma \in (0, 1/2)$ there exists a constant C depending on ε, γ, T, λ, Λ, G, $\|a'\|_\infty$, $\|\rho'\|_\infty$ $\|f\|_\infty$, $\|df/dx\|_2$, $\|df/dx\|_\infty$, $\sup_{i \in I} 1/(x_{i+1} - x_i)$, and the two first moments of $\mathcal{T}(1)$ such that, for n large enough,

$$\sup_{(t,x) \in [\varepsilon, T] \times \bar{G}} \Big| \mathbb{E}^x f(X_t) - \mathbb{E}^x f(\widehat{X}^n_t) \Big| \leq C n^{-\gamma}.$$

We have,

$$\begin{aligned} |\mathbb{E}^x f(X_t) - \mathbb{E}^x f(\widehat{X}^n_t)| &\leq |\mathbb{E}^x f(X_t) - \mathbb{E}^x f(X^n_t)| + |\mathbb{E}^x f(X^n_t) - \mathbb{E}^x f(\widehat{X}^n_t)| \\ &=: e_1(t, x, n) + e_2(t, x, n). \end{aligned} \tag{6.1}$$

We will estimate $e_1(t, x, n)$ by PDEs techniques and $e_2(t, x, n)$ by very simple probabilistic techniques.

6.1 Estimate of a weak error

In this subsection we prove the following proposition.

Proposition 6.1 *Assume f belongs to $\mathrm{H}^1_0(G) \cap \mathcal{C}_0(G)$. Let be $u(t, x)$ and $u^n(t, x)$ respectively the solutions of $(\mathcal{P})(a, \rho, f)$ and $(\mathcal{P})(a^n, \rho^n, f)$, with a^n and ρ^n like in Subsection 5.4, Step 1. Then for all $\varepsilon > 0$ there is a constant C_1 depending on ε, T, λ, Λ, G, $\|f\|_\infty$, $\|df/dx\|_2$, $\|a'\|_\infty$, $\|\rho'\|_\infty$, and $\sup_{i \in I} 1/(x_{i+1} - x_i)$ such that for n large enough,*

$$\sup_{(t,x) \in [\varepsilon, T] \times \bar{G}} |u(t, x) - u^n(t, x)| \leq C_1 \frac{1}{\sqrt{n}}.$$

As we will see in Proposition 6.2, if we had $\mathcal{I} \subset \mathcal{I}^n$ we could obtain an upper bound for $|||u - u^n|||_{\infty,\infty}$ of the form $K(\|a - a^n\|^2_\infty + \|\rho - \rho^n\|_\infty)$. But this is not necessary the case (see Remark 5.2). However it is possible to modify a and ρ in order to refind us in a situation close to this one, and we will do that to prove Proposition 6.1.

Proposition 6.2 *Let be $f \in \mathrm{H}^1_0(G) \cap \mathcal{C}_0(G)$. Let be $a_1, \rho_1, a_2, \rho_2 \in \mathfrak{Coeff}(\lambda, \Lambda)$, and $\mathcal{I}_1$ and $\mathcal{I}_2$ respectively the set of points of discontinuity of a_1 and ρ_1 and a_2 and ρ_2. Assume $\mathcal{I}_1 \subset \mathcal{I}_2$. Let be $u_1(t, x)$ the weak solution of $(\mathcal{P})(a_1, \rho_1, f)$ and $u_2(t, x)$ the weak solution of $(\mathcal{P})(a_2, \rho_2, f)$. There exists a constant $\widetilde{C}_1$ depending on T, λ, Λ, G, $\|f\|_\infty$, and $\|df/dx\|_2$, such that,*

$$|||u_1 - u_2|||_{\infty,\infty} \leq \widetilde{C}_1 \Big(\|a_1 - a_2\|^2_\infty + \|\rho_1 - \rho_2\|_\infty \Big).$$

We need a lemma asserting some standard estimates.

Lemma 6.1 *i) Let be $f \in \mathrm{H}_0^1(G)$. Let be $u(t,x)$ the weak solution of $(\mathcal{P})(a,\rho,f)$. Then $\partial_t u$ is in $\mathrm{L}^2(0,T;\mathrm{L}^2(G))$ and more precisely,*

$$|||\partial_t u|||_{2,2} \leq \frac{\Lambda}{2}\left\|\frac{df}{dx}\right\|_2. \tag{6.2}$$

ii) Let be $f \in \mathrm{L}^2(G)$. Let be $u(t,x)$ the weak solution of $(\mathcal{P})(a,\rho,f)$. Then $\frac{du}{dx}$ is in $\mathrm{L}^2(0,T;\mathrm{L}^2(G))$ and more precisely,

$$\left|\left|\left|\frac{du}{dx}\right|\right|\right|_{2,2} \leq \frac{1}{\lambda}\|f\|_2. \tag{6.3}$$

Proof. i) *Step 1.* Assume first that a and ρ are $\mathcal{C}^\infty(G)$ and that f is $\mathcal{C}_c^\infty(G)$ so that $u(t,x)$ is itself $\mathcal{C}^\infty((0,T)\times G)$. As $u(t,x)$ is a weak solution of $(\mathcal{P})(a,\rho,f)$, and u, Lu and $\partial_t u$ are $\mathcal{C}^\infty$, using $\partial_t u$ as a test function and integrating by parts with respect to x, we get

$$\int_0^T\int_G |\partial_t u|^2 m_\rho(dx)dt = \int_0^T\int_G \partial_t u\, Lu\, m_\rho(dx)dt = -\frac{1}{2}\int_0^T\int_G a\frac{du}{dx}\frac{d(\partial_t u)}{dx}dx\,dt.$$

Then using Fubini's theorem, interverting the partial derivatives, integrating by parts with respect to t and then again with respect to x, we get

$$2\int_0^T\int_G |\partial_t u|^2\, m_\rho(dx)dt = \frac{1}{2}\int_G a\left|\frac{du(0,x)}{dx}\right|^2 dx - \frac{1}{2}\int_G a\left|\frac{du(T,x)}{dx}\right|^2 dx,$$

which leads to (6.2).

Step 2. In the general case, with a and ρ in $\mathfrak{Coeff}(\lambda,\Lambda)$, and f in $\mathrm{H}_0^1(G)$, we use a regularization argument, Theorem 2.2, Step 1, a compactness argument and an integration by parts with respect to t, to exhibit a function $w \in \mathrm{L}^2(0,T;\mathrm{L}^2(G))$ satifying:

$$\forall\varphi \in \mathcal{C}_c^\infty((0,T)\times G),\ \int_0^T\int_G w\varphi = -\int_0^T\int_G u\partial_t\varphi, \text{ and } |||w|||_{2,2} \leq \frac{\Lambda}{2}\left\|\frac{df}{dx}\right\|_2.$$

That is $\partial_t u$ is in $\mathrm{L}^2(0,T;\mathrm{L}^2(G))$ and verifies (6.2).

ii) Thanks to point i) and because $Lu(t,.) \in \mathrm{L}^2(G)$ we can write

$$\int_\delta^T \langle\partial_t u, u\rangle_{\mathrm{L}^2(G,m_\rho)}\,dt = \int_\delta^T \langle Lu, u\rangle_{\mathrm{L}^2(G,m_\rho)}\,dt, \tag{6.4}$$

for all $\delta > 0$. As u is in $\mathcal{C}^1([\delta,T],\mathrm{L}^2(G,m_\rho))dt$ and we have (see [Bre83]),

$$2\langle\partial_t u, u\rangle_{\mathrm{L}^2(G,m_\rho)} = \frac{d}{dt}\|u\|^2_{\mathrm{L}^2(G,m_\rho)}, \qquad \forall t \in [\delta,T], \tag{6.5}$$

using an integration by part with respect to x in the right hand side of (6.4), and making δ tend to 0 we get,

$$\frac{\lambda}{2}\left|\left|\left|\frac{du}{dx}\right|\right|\right|^2_{2,2} \leq \frac{1}{2}\Big(\|u(0,.)\|^2_{\mathrm{L}^2(G,m_\rho)} - \|u(T,.)\|^2_{\mathrm{L}^2(G,m_\rho)}\Big),$$

which leads to (6.3). □

Proof of Proposition 6.2. *Step 1.* We introduce the following norm on $\mathcal{C}(0,T;\mathrm{L}^2(G))\cap \mathrm{L}^2(0,T;\mathrm{H}^1_0(G))$:

$$|v|_{G,T} := \Big(\sup_{t\in[0,T]}\|v(t,.)\|_2^2 + \Big|\Big|\Big|\frac{dv}{dx}\Big|\Big|\Big|^2_{2,2}\Big)^{1/2}.$$

We have the following estimate:

$$|||v|||_{\infty,\infty} \leq K|v|_{G,T}, \tag{6.6}$$

where the constant K depends only on T and G (see [LSU68], II.§.3 inequality (3) p 74 with $r=\infty$ and $q=\infty$).

We set $v := u_2 - u_1$. Our goal is now to estimate $|v|_{G,T}$.

Step 2. Set $(L_2, D(L_2)) = \mathfrak{L}(a_2,\rho_2)$. Elementary computations show that, $v(t,x)$ is a weak solution to

$$\partial_t v(t,x) = L_2 v(t,x) + \frac{\rho_2(x)}{2}\frac{d}{dx}\big((a_2(x)-a_1(x))\frac{du_1(t,x)}{dx}\big) + (\frac{\rho_2(x)}{\rho_1(x)}-1)\partial_t u_1(t,x),$$

that is to say for all $\varphi\in\mathcal{C}_c^\infty((0,T)\times G)$ we have,

$$\begin{aligned} -\textstyle\int_0^T\int_G v\partial_t\varphi m_{\rho_2}(dx)dt = -\int_0^T\int_G a_2\frac{dv}{dx}\frac{d\varphi}{dx}dx\,dt \\ +\textstyle\int_0^T\int_G\big[\frac{\rho_2}{2}\frac{d}{dx}((a_2-a_1)\frac{du_1}{dx}) + (\frac{\rho_2}{\rho_1}-1)\partial_t u_1\big]\varphi m_{\rho_2}(dx)dt. \end{aligned}\tag{6.7}$$

We take v as a test function in (6.7). Using (6.5) with v, integration by parts, $ab\leq(\lambda/2)a^2+(2/\lambda)b^2$ and $a_1,\rho_2\in\mathfrak{Coeff}(\lambda,\Lambda)$ we finally get,

$$|v|_{G,T} \leq \frac{\kappa'}{\kappa}\Big(\int_0^T\int_G(a_1-a_2)^2\Big(\frac{du_1}{dx}\Big)^2dxdt + \int_0^T\int_G\Big(\frac{1}{\rho_1}-\frac{1}{\rho_2}\Big)\partial_t u_1 v dxdt\Big), \tag{6.8}$$

where $\kappa=\min(1/\Lambda,\lambda/4)$ and $\kappa'=\max(1/\lambda,1)$.

Step 3. As $f\in\mathcal{C}_0(G)$ and the semigroup $(S_t^1)_{t\geq0}$ and $(S_t^2)_{t\geq0}$ generated respectively by $(L_1,D(L_1)) = \mathfrak{L}(a_1,\rho_1)$ and $(L_2,D(L_2))$ are Feller, we can consider that

$$\forall t\in[0,T],\ \ \|v(t,.)\|_\infty \leq \|u_2(t,.)\|_\infty + \|u_1(t,.)\|_\infty \leq 2\,\|f\|_\infty.$$

Thus $|||v|||_{\infty,\infty}\leq 2\,\|f\|_\infty$; besides $f\in\mathrm{H}^1_0(G)$ thus using point i) of Lemma 6.1 and Hölder inequality we get,

$$\begin{aligned} \textstyle\int_0^T\int_G(\frac{1}{\rho_1}-\frac{1}{\rho_2})\partial_t u_1 v dxdt &\leq 2\Big\|\tfrac{1}{\rho_1}-\tfrac{1}{\rho_2}\Big\|_\infty \|f\|_\infty\,|||\partial_t u_1|||_{1,1} \\ &\leq \tfrac{2}{\lambda^2}\sqrt{T|G|}\,\|\rho_1-\rho_2\|_\infty\,\|f\|_\infty\,|||\partial_t u_1|||_{2,2}. \\ &\leq \tfrac{\Lambda}{\lambda^2}\sqrt{T|G|}\,\|f\|_\infty\Big\|\tfrac{df}{dx}\Big\|_2\|\rho_1-\rho_2\|_\infty. \end{aligned}\tag{6.9}$$

To finish we have,

$$\int_0^T\int_G(a_1-a_2)^2(\frac{du_1}{dx})^2dxdt \leq \|a_1-a_2\|^2_\infty\Big|\Big|\Big|\frac{du_1}{dx}\Big|\Big|\Big|^2_{2,2},$$

and point ii) of Lemma 6.1 completes the proof because $f \in \mathrm{L}^2(G)$.
Note that the fact that $\mathcal{I}_1 \subset \mathcal{I}_2$ allows to consider the quantities $\|\rho_1 - \rho_2\|_\infty$ and $\|a_1 - a_2\|_\infty$.

$\square$

We are then ready to prove Proposition 6.1.

Proof of Proposition 6.1. The idea is to build a bijection ϕ_n such that the points of discontinuity of $a \circ \phi_n$ and $\rho \circ \phi_n$ are included in $\mathcal{I}^n$.

Step 1. We build a piecewise linear bijection ϕ_n such that $\phi_n^{-1}(\mathcal{I}) \subset \mathcal{I}^n$ in the following manner:
We first define a projection $\pi_n : G \to \mathcal{I}^n$ by,

$$\pi_n(x) = \begin{cases} x^n_{k_{n,x}} & \text{if } (x - x^n_{k_{n,x}})/(x^n_{k_{n,x}+1} - x^n_{k_{n,x}}) < 1/2, \\ x^n_{k_{n,x}+1} & \text{if } (x - x^n_{k_{n,x}})/(x^n_{k_{n,x}+1} - x^n_{k_{n,x}}) \geq 1/2. \end{cases}$$

Then we set

$$\phi_n(x) = \begin{cases} \frac{x_0 - l}{\pi_n(x_0) - l}(x - l) + l & \text{if } x \in [l, \pi_n(x_0)[, \\ \frac{x_{i+1} - x_i}{\pi_n(x_{i+1}) - \pi_n(x_i)}(x - \pi_n(x_i)) + x_i & \text{if } x \in [\pi_n(x_i), \pi_n(x_{i+1})[, \\ & \text{for}\, 0 < i < k_1 - 1, \\ \frac{r - x_{k_1}}{r - \pi_n(x_{k_1})}(x - \pi_n(x_{k_1})) + x_{k_1} & \text{if } x \in [\pi_n(x_{k_1}), r]. \end{cases}$$

Note that we then have for all $i \in I$, $\phi_n(\pi_n(x_i)) = x_i$.

Step 2. We set $\tilde{u}^n(t,x) := u(t, \phi_n(x))$. If a and ρ are smooth simple computations show that $\tilde{u}^n(t,x)$ solves $(\mathcal{P})(\tilde{a}^n, \tilde{\rho}^n, f \circ \phi_n)$ with,

$$\tilde{a}^n(x) = \frac{a \circ \phi_n(x)}{\phi_n'(x)}, \qquad \text{and} \qquad \tilde{\rho}^n(x) = \rho \circ \phi_n(x).$$

It can be shown that this is still the case for a and ρ in $\mathfrak{Coeff}(\lambda, \Lambda)$, using a regularization argument and again Theorem 2.2.
We also define $\bar{u}^n(t,x)$ to be the solution of $(\mathcal{P})(\tilde{a}^n, \tilde{\rho}^n, f)$.

Step 3. For all $(t,x) \in [\varepsilon, T] \times \bar{G}$, we have,

$$|u(t,x) - u^n(t,x)| \leq |u(t,x) - u(t, \phi_n(x))| + |||\tilde{u}^n - \bar{u}^n|||_{\infty,\infty} + |||\bar{u}^n - u^n|||_{\infty,\infty}.$$

The points of discontinuity of $\tilde{a}^n$ and $\tilde{\rho}^n$ belong to $\mathcal{I}^n$. So by Proposition 6.2 there is a constant $\widetilde{C}_1$ not depending on n such that

$$|||\bar{u}^n - u^n|||_{\infty,\infty} \leq \widetilde{C}_1 \Big(\|\tilde{a}^n - a^n\|^2_\infty + \|\tilde{\rho}^n - \rho^n\|_\infty \Big). \tag{6.10}$$

Besides, as for each n the semigroup $(\widetilde{S}^n_t)_{t \geq 0}$ generated by $\mathfrak{L}(\tilde{a}^n, \tilde{\rho}^n)$ is Feller, we have

$$|||\tilde{u}^n - \bar{u}^n|||_{\infty,\infty} \leq \|f - f \circ \phi_n\|_\infty \leq \left\| \frac{df}{dx} \right\|_2 \sqrt{\|id - \phi_n\|_\infty}. \tag{6.11}$$

Finally, we know that $u(t,x)$ is continuous on $[0,T]\times\bar{G}$ and of class $\mathcal{C}^1$ on each $[\varepsilon,T]\times(x_i,x_{i+1})$ (see in [LRU68] Theorems 6 and 7). So if x and $\phi_n(x)$ belong to the same interval (x_i,x_{i+1}) we have,

$$|u(t,x)-u(t,\phi_n(x))|\leq \sup_{(t,x)\in[\varepsilon,T]\times(x_i,x_{i+1})}|\frac{du}{dx}(t,x)|\cdot|x-\phi_n(x)|.$$

Let us set

$$M=\sup_{i\in I}\sup_{(t,x)\in[\varepsilon,T]\times(x_i,x_{i+1})}|\frac{du}{dx}(t,x)|.$$

If for instance $x\in(x_{i-1},x_i)$ and $\phi_n(x)\in(x_i,x_{i+1})$ we have,

$$|u(t,x)-u(t,\phi_n(x))|\leq|u(t,x)-u(t,x_i)|+|u(t,x_i)-u(t,\phi_n(x))|\leq 2M\left\|id-\phi_n\right\|_\infty. \tag{6.12}$$

We will see below that $\|id-\phi_n\|_\infty\to 0$ as $n\to\infty$, so for n large enough we are always at least in the last situation.

Step 4. By construction (see point 1 of the algorithm) the grid $\mathcal{I}^n$ satisfies $|x^n_{k+1}-x^n_k|\leq\Lambda/n$, for all $k\in I^n$.
So elementary computations show that

$$\|id-\phi_n\|_\infty\leq\frac{3\Lambda}{n}\qquad\text{and}\qquad\left\|1-\frac{1}{\phi_n'}\right\|_\infty\leq 2\Lambda\sup_{i\in I}\frac{1}{x_{i+1}-x_i}.\frac{1}{n}.$$

As we have said above, for n large enough (6.12) is valid and we then have,

$$|u(t,x)-u(t,\phi_n(x))|\leq 6M\Lambda\frac{1}{n}. \tag{6.13}$$

It remains to evaluate $\|\tilde{a}^n-a^n\|_\infty$ and $\|\tilde{\rho}^n-\rho^n\|_\infty$. On each (x_i,x_{i+1}), a is of class $\mathcal{C}^1$ and a' is r.c.l.l., so it makes sense to speak of $\|a'\|_\infty$. Moreover each $\phi_n([x^n_k,x^n_{k+1}))$ is included in some $[x_i,x_{i+1}]$ that contains x^n_k, so we have,

$$\begin{aligned}\|\tilde{a}^n-a^n\|_\infty &\leq \sup_{k\in I^n}\sup_{x\in[x^n_k,x^n_{k+1})}|a(\phi_n(x))/\phi_n'(x)-a^n(x)|\\ &\leq \|a'\|_\infty\sup_{x\in[x^n_k,x^n_{k+1})}|\phi_n(x)-x^n_k|+\Lambda\sup_{x\in[x^n_k,x^n_{k+1})}|1-\tfrac{1}{\phi_n'(x)}|.\end{aligned}$$

In addition $|\phi_n(x)-x^n_k|\leq|\phi_n(x)-x|+|x-x^n_k|\leq 4\Lambda/n$, for all $x\in[x^n_k,x^n_{k+1})$, and we can get a similar bound for $|1-\frac{1}{\phi_n'(x)}|$ so finally there exists K_1 such that

$$\|\tilde{a}^n-a^n\|_\infty\leq K_1\frac{1}{n}. \tag{6.14}$$

In a similar manner we get K_2 such that,

$$\|\tilde{\rho}^n-\rho^n\|_\infty\leq K_2\frac{1}{n}. \tag{6.15}$$

Thus, combining (6.10), (6.11), (6.13), (6.14) and (6.15), we complete the proof.

□

6.2 Estimate of a strong error

Proposition 6.3 *In the context of Subsection 5.4, for all $\gamma \in (0,1/2)$ there exist a constant C_2 depending on T, γ and the two first moments of $\mathcal{T}(1)$ such that,*

$$\forall n \in \mathbb{N}^*, \quad \forall t \in [0,T], \quad \mathbb{E}|Y^n_{\tau^n_{[n^2t]}} - Y^n_t| \leq C_2 n^{-\gamma}.$$

Note that in all this subsection we drop any reference to the starting point x in the notation of the expectation. Indeed the calculations we make are uniform with respect to this variable.

The proof of Proposition 6.3 will follow from two simple lemmas.

For all $\gamma \in (0,1/2)$, all $T > 0$, and for any process Y let us introduce the notation

$$M^\gamma_T(Y) := \sup_{s \neq t,\ s,t \in [0,T]} \frac{|Y_t - Y_s|}{|t-s|^\gamma}.$$

Lemma 6.2 *Let (μ^n) be a sequence in $\mathfrak{M}$ and let be (Y^n) the sequence of processes such that each Y^n solves $\mathfrak{SDe}(1,\mu^n)$. Assume there exist two positive constants m and M such that,*

$$\forall n \in \mathbb{N}, \quad f_{\mu^n} \in \mathfrak{Coeff}(m, M). \tag{6.16}$$

Then for all $\gamma \in (0,1/2)$, all $k \in \mathbb{N}^*$ such that $0 < \gamma < 1/2 - 1/(2k)$, and all $T > 0$, there exists a positive constant $C^{\gamma,T}_k$, not depending on n, which verifies,

$$\forall n \in \mathbb{N}, \quad \Big(\mathbb{E}[(M^\gamma_T(Y^n))^{2k}]\Big)^{1/2k} \leq C^{\gamma,T}_k < \infty. \tag{6.17}$$

Proof. We have to use the Kolmogorov-Čentsov theorem. Let be $\gamma \in (0,1/2)$ and $k \in \mathbb{N}^*$ such that $0 < \gamma < 1/2 - 1/(2k)$. For all $n \in \mathbb{N}$ let us define

$$F_n(x) := \int_0^x f_{\mu^n}(y)dy,$$

and

$$h^n := f_{\mu^n} \circ F_n^{-1}$$

Let us evaluate $\mathbb{E}|Y^n_t - Y^n_s|^{2k}$ for $n \in \mathbb{N}$ and $t, s > 0$. We have

$$\mathbb{E}|Y^n_t - Y^n_s|^{2k} = \mathbb{E}|F_n^{-1}(Z^n_t) - F_n^{-1}(Z^n_s)|^{2k}, \tag{6.18}$$

where $Z^n := F_n(Y^n)$ is solution of $\mathfrak{SDe}(h_n, 0)$ by the virtue of Proposition 3.1.

By (6.16) we have that $||(F_n^{-1})'||_\infty \leq 1/m$, so a simple use of the mean value theorem leads from (6.18) to,

$$\mathbb{E}|Y^n_t - Y^n_s|^{2k} \leq \frac{1}{m^{2k}} \mathbb{E}|Z^n_t - Z^n_s|^{2k} = \frac{1}{m^{2k}} \mathbb{E}|\int_s^t h_n(Y^n_r)dW_r|^{2k}.$$

Using now the Burkholder-Davis-Gundy inequality and the majoration part of (6.16) we get

$$\mathbb{E}|Y^n_t - Y^n_s|^{2k} \leq \frac{1}{m^{2k}} C_k\, \mathbb{E}\Big(\int_s^t h_n^2(Y^n_r)ds\Big)^k = \Big(\frac{M}{m}\Big)^{2k} C_k (t-s)^{1+(k-1)},$$

where C_k is a constant not depending on n. The constant $\left(\frac{M}{m}\right)^{2k} C_k$ does not depend on n and we have $2k > 1$. Thus, having a look at the proof of the Kolmogorov-Čentsov theorem (see [RY91] for instance), and identifying each Y^n with its γ-Hölder modification we can say that there is a positive constant $C_k^{\gamma,T}$, such that (6.17) holds.

□

Lemma 6.3 *There exists a constant K depending on T and the two first moments of $\mathcal{T}(1)$ such that:*

$$\forall n \in \mathbb{N}^*,\ \forall t \in [0,T], \quad \mathbb{E}|\tau^n_{[n^2t]} - t|^2 \leq K\frac{1}{n^2}. \tag{6.19}$$

Proof. First we notice that if $t < 1/n^2$ then (6.19) holds with $K = 1$. We then assume now that $t \geq 1/n^2$. For all $t \in [0,T]$ and with the σ^n_p defined in Lemma 5.2 we set $\sigma^{n,t}_p := t\sigma^n_p$. Let $T_1 \cong \mathcal{T}(1)$. Using,

$$(v_1 + \ldots + v_k)^2 \leq k(v_1^2 + \ldots + v_k^2), \tag{6.20}$$

we get

$$\mathbb{E}|\tau^n_{[n^2t]} - t|^2 \leq 2\mathbb{E}\big|\tau^n_{[n^2t]} - \frac{1}{[n^2t]}\sum_{p=1}^{[n^2t]} \sigma^{n,t}_p\big|^2 + 2\mathbb{E}\big|\frac{1}{[n^2t]}\sum_{p=1}^{[n^2t]} \sigma^{n,t}_p - t\big|^2. \tag{6.21}$$

We have $(\tau^n_p - \tau^n_{p-1}) \cong \mathcal{T}(1/n^2)$ so we get,

$$\begin{aligned}
\mathbb{E}\big|\tau^n_{[n^2t]} - \tfrac{1}{[n^2t]}\textstyle\sum_{p=1}^{[n^2t]} \sigma^{n,t}_p\big|^2 \quad &= \mathbb{E}\Big|\textstyle\sum_{p=1}^{[n^2t]}(\tau^n_p - \tau^n_{p-1})\big(\tfrac{[n^2t]-n^2t}{[n^2t]}\big)\Big|^2 \\
&\leq \tfrac{1}{[n^2t]}\textstyle\sum_{p=1}^{[n^2t]} \mathbb{E}(\tau^n_p - \tau^n_{p-1})^2 \\
&\leq \mathbb{E}(T_1^2)\tfrac{1}{n^4}.
\end{aligned} \tag{6.22}$$

For the second term of (6.21), as $\mathbb{E}T_1 = 1$ (see [Bre68] for instance) and the σ^n_p's are independent, we have

$$\begin{aligned}
\mathbb{E}\big|\tfrac{1}{[n^2t]}\textstyle\sum_{p=1}^{[n^2t]} \sigma^{n,t}_p - t\big|^2 \quad &= \tfrac{t^2}{[n^2t]^2}\mathbb{E}\big|\textstyle\sum_{p=1}^{[n^2t]}(\sigma^n_p - 1)\big|^2 \\
&= \tfrac{t^2}{[n^2t]^2}\textstyle\sum_{p=1}^{[n^2t]} \mathbb{V}(\sigma^n_p) \\
&\leq 2T\mathbb{V}(T_1)\tfrac{1}{n^2}.
\end{aligned} \tag{6.23}$$

Taking in account (6.22), (6.23) and (6.21) we complete the proof.

□

Proof of Proposition 6.3. Let be $\gamma \in (0,1/2)$, $k \in \mathbb{N}^*$ such that $0 < \gamma < 1/2 - 1/(2k)$, and $n \in \mathbb{N}^*$. By the Hölder inequality we have for p and q conjugate

$$\forall t \in [0,T], \quad \mathbb{E}|Y^n_{\tau^n_{[n^2t]}} - Y^n_t| \leq [\mathbb{E}\big(M^\gamma_T(Y^n)\big)^p]^{1/p}.[\mathbb{E}|\tau^n_{[n^2t]} - t|^{\gamma q}]^{1/q}. \tag{6.24}$$

Let us fix $q = 2/\gamma$. We have $1 < p < 4/3$. Each Y^n solves $\mathfrak{S}\mathfrak{D}\mathfrak{e}(1, \mu^n)$ with $\mu^n = \sum_{k \in I^n} \beta_k^n \delta_{k/n}$ with the β_k^n defined by (5.7). The function f_{μ_n} is unique up to a multiplicative constant. If we impose $f_{\mu_n}(l) = 1$ then simple calculations show that

$$f_{\mu_n}(x) = \prod_{\frac{k}{n} \leq x, k \in I^n} \frac{1 - \beta_k^n}{1 + \beta_k^n} = \frac{\sqrt{\rho(x_{k_{n,x}}^n)}}{\sqrt{a(x_{k_{n,x}}^n)}}.$$

Thus each f_{μ_n} is in $\mathfrak{Coeff}(\sqrt{\Lambda/\lambda}, \sqrt{\lambda/\Lambda})$ and by Lemma 6.2, there exists $C_k^{\gamma,T}$ verifying (6.17).
As $p < 2k$ by Jensen inequality there exists K_γ uniform in n such that

$$\forall n \in \mathbb{N}^*, \quad [\mathbb{E}(M_T^\gamma(Y^n)^p)]^{1/p} < K_\gamma < \infty. \tag{6.25}$$

By Lemma 6.3 there exists K such that

$$\forall n \in \mathbb{N}^*, \forall t \in [0,T], \quad [\mathbb{E}|\tau_{[n^2 t]}^n - t|^{\gamma q}]^{1/q} = [\mathbb{E}|\tau_{[n^2 t]}^n - t|^2]^{1/q} \leq K\, n^{-2/q} = K\, n^{-\gamma}. \tag{6.26}$$

Combining (6.24), (6.25) and (6.26) we can complete the proof.

□

6.3 Proof of Theorem 6.1

Combining the two preceeding subsections we can finally prove Theorem 6.1.

Proof of Theorem 6.1. We have $e_1(t,x,n) = |u(t,x) - u^n(t,x)|$ where $u(t,x)$ and $u^n(t,x)$ are those of Proposition 6.1.
Moreover, by construction, $\widehat{X}^n$ of Theorem 6.1 is distributed as $\widetilde{X}^n := (\Phi^n)^{-1}(\widetilde{Y}^n)$ with the $\widetilde{Y}^n$ of Subsection 5.4 and Proposition 6.3, that lives in the same probality space as Y^n. So, as $\left\|\nabla(\Phi^n)^{-1}\right\|_\infty \leq 1/\Lambda$, we have,

$$e_2(t,x,n) = |\mathbb{E}^x[f(\widetilde{X}_t^n) - f(X_t^n)]| \leq \left\|\frac{df}{dx}\right\|_\infty \mathbb{E}^x\,|\widetilde{X}_t^n - X_t^n| \leq \frac{\left\|\frac{df}{dx}\right\|_\infty}{\Lambda} \mathbb{E}^x\,|\widetilde{Y}_t^n - Y_t^n|.$$

Thus using Propositions 6.1 and 6.3 we complete the proof.

□

7 Numerical experiments

Example 1. We take a and ρ to be

$$a(x) = \begin{cases} 1 & \text{if } x < 0, \\ 5 & \text{if } x \geq 0, \end{cases} \qquad \text{and} \qquad \rho(x) = \begin{cases} 1 & \text{if } x < 0, \\ 1/5 & \text{if } x \geq 0. \end{cases}$$

Then by Proposition 4.1, the process X generated by $\mathfrak{L}(a,\rho)$ solves
$\mathfrak{S}\mathfrak{D}\mathfrak{e}(1, 2/3\,\delta_0)$, *i.e.* X is distributed as the simple SBM $Y^{x,\beta}$ of parameter $\beta = 2/3$.
We know the exact density $p(t,x,y)$ of the transition probability of $Y^{x,\beta}$ (see [Wal78]). For $x = 0$ we have:

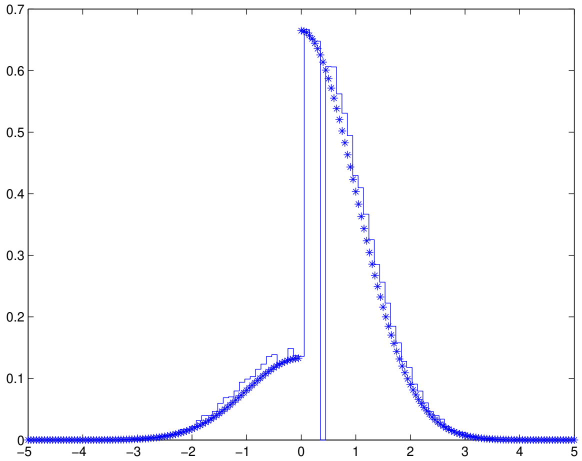

FIG. 1 – Approximated density $p(t,x,y)$ for X with $t=1$ and $x=0$ (with 50000 particles), together with the exact density (represented by the $*$-line).

$$\sqrt{2\pi t}\, p(t,0,y) = \begin{cases} (\beta+1)\exp\{-y^2/2t\} & \text{if } y \geq 0, \\ (1-\beta)\exp\{-y^2/2t\} & \text{if } y < 0. \end{cases}$$

We simulate $N = 50000$ random variables $\widehat{X}_t^n = \text{ALGO}(a, \rho, x, n, t)$ with $x = 0$, $t = 1$ and the precision order $n = 20$. We plot on the same graph (Figure 1) the histogram we get and the exact $p(1, 0, y)$.

Example 2. We take $\rho \equiv 1$ and the coefficient a represented by Figure 2. We plot a histogram approximating $p(t,x,y)$ for $x = 0$ at three successive times, $t = 1$, $t = 2$ and $t = 3.5$. We used $N = 10000$ particules and took $n = 20$ (Figure 3).

Example 3. We take the same n, a and ρ as in example 2. For $f \in \mathrm{W}_0^{1,\infty}(G) \cap \mathcal{C}_0(G)$, we know

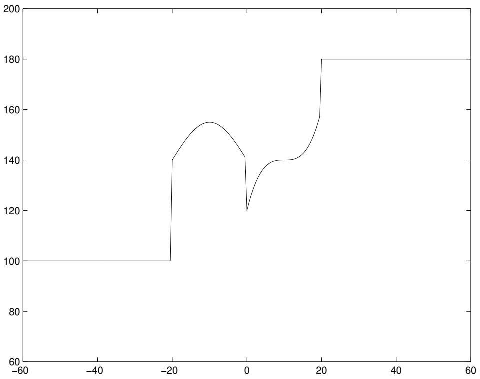

FIG. 2 – Graph of a.

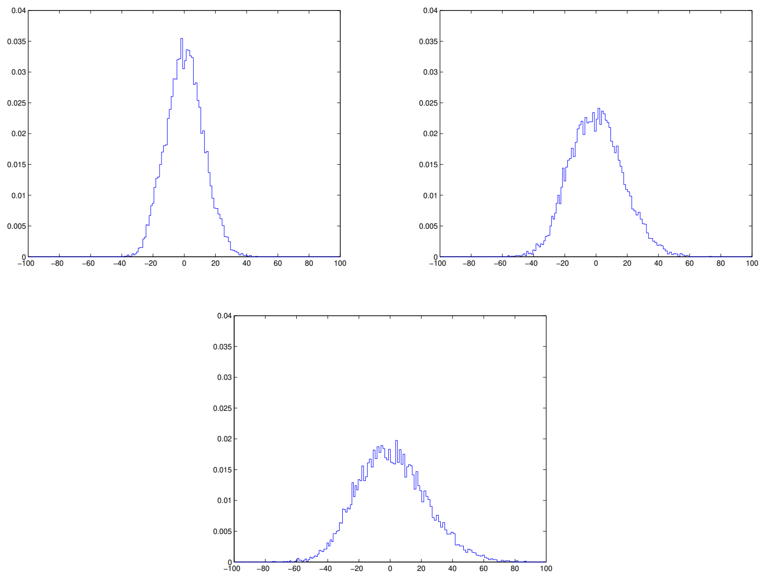

FIG. 3 – Approximation of $p(t,x,y)$ for $x=0$ at times $t=1$, $t=2$ and $t=3.5$ (with N=10000 particules).

(Subsection 5.1) that if each $\widehat{X}^{n,(i)}$ is a realisation of $\widehat{X}^n$ starting at x, the quantity $(1/N)\sum_{i=1}^N f[\widehat{X}^{n,(i)}]$ approaches $u(t,x)$, the solution of $(\mathcal{P})(a,\rho,f)$.

We then consider that $G=(-100,100)$ and take $f(x)=\sin(\pi\,(x+100)/200)$. We take $x=-10$. We compute $u_{sto}(t,x) := (1/N)\sum_{i=1}^N f[\widehat{X}_t^{n,(i)}]$, with $N=10000$, for t belonging to a time grid that is a discretisation of $[0,4]$. We use a deterministic algorithm to compute an approximation $u_{det}(t,x)$ of $u(t,x)$ for $t\in[0,4]$. We plot $u_{sto}(t,x)$ and $u_{det}(t,x)$ on the same graph (Figure 4).

Example 4. We wish to compare our scheme with the one proposed by Lejay and Martinez in [LM06]. In order to do that we take $\rho\equiv 1$ and a defined by

$$a(x)=\begin{cases} 2+\sin(x) & \text{if } x<0,\\ 5+\sin(x+\pi) & \text{if } x\geq 0.\end{cases}$$

We take $t=1$, $x=0.5$, $n=10$ and $N=10000$. We plot a histogram of the values of $\widehat{X}_1^n$ approximating $p(1,0.5,y)$ on Figure 5. We plot on the same figure the histogram obtained in [LM06] for the same parameters. Note that in [LM06] the authors also compared their histogram with the one obtained by the Euler Scheme of M. Martinez in [Mar04].

Acknowledgement: The author would like to thank Antoine Lejay for his encouragements and constructive remarks.

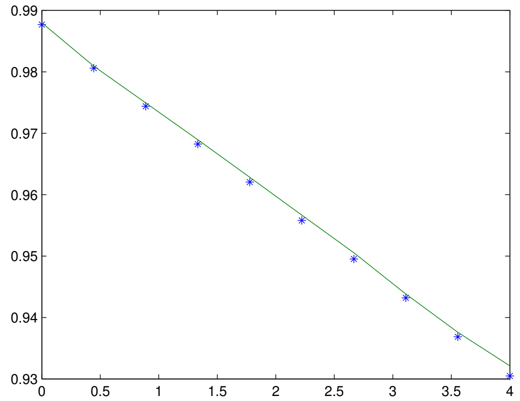

FIG. 4 – Graphs of $u_{sto}(t, x)$ and $u_{det}(t, x)$ (represented by the $*$-line) for $x = -10$ and $t \in [0, 4]$.

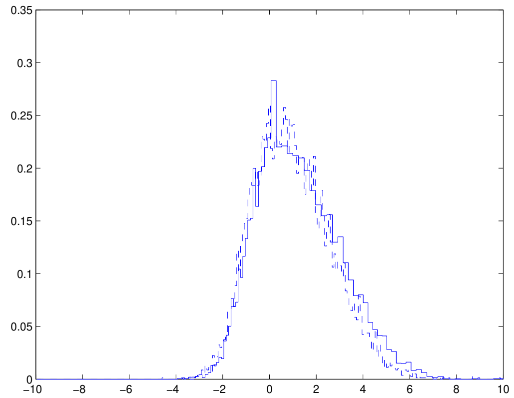

FIG. 5 – Approximation of $p(t, x, y)$ in example 4 by our algorithm together with the one by Lejay and Martinez (represented by the dashed line).

Références

[BC05] R.F. Bass Z.-Q. Chen *One-dimensional Stochastic Differential Equations with Singular and Degenerate Coefficients.* Sankhya 67 (2005), 19-45.

[Bre68] L. Breiman. *Probability.* Addison-Wesley Series in Statistics, 1968. MR229267

[Bre83] H. Brezis. *Analyse fonctionnelle.* Masson, 1983. MR697382

[DDG05] M. Decamps, A. De Schepper and M. Goovaerts. *Applications of δ-function pertubation to the pricing of derivative securities.* Physica A 342:3-4 (2004), 677-692. MR2105796

[EK86] S.N. Ethier and T.G. Kurz *Markov processes. Characterization and convergence.* Wiley Series in Probability and Mathematical Statistics, 1986. MR838085

[HS81] J.M. Harrison and L.A. Shepp. *On skew Brownian Motion*, The Annals of Probability 9 (1981), 309-313. MR606993

[IM74] K. Itô and H.P. McKean. *Diffusion Processes and Their Sample Paths.* Springer, Berlin, 1974.

[KP92] P.E. Kloeden and E. Platen. *Numerical Solution of Stochastic Differential Equations.* Springer, Berlin, 1992. MR1214374

[LSU68] O.A. Ladyzenskaja, V.A. Solonnikov and N.N. Ural'ceva. *Linear and Quasilinear Equations of Parabolic Type.* American Mathematical Society, 1968. MR241822

[LRU68] O.A. Ladyzenskaja, V.J. Rivkind and N.N. Ural'ceva. *The classical solvability of diffraction problems.* Trudy Mat. Inst. Steklov 92 (1966), 116-146. MR211050

[Lej00] A. Lejay. *Méthodes probabilistes pour l'homogénéisation des opérateurs sous forme divergence: cas linéaires et semi-linéaires.* PhD thesis, Université de Provence, Marseille, France, 2000.

[LM06] A. Lejay and M. Martinez. *A scheme for simulating one-dimensional diffusion processes with discontinuous coefficients.* To appear in The Annals of Applied Probability.

[Leg85] J.F. Le Gall. *One-dimensional stochastic differential equations involving the local time of the unknown process*, Stochastic Analysis and Applications, vol. 1095 of Lect. Notes in Math. 51-82, Springer 1985. MR777514

[Mar04] M. Martinez. *Interprétations probabilistes d'opérateurs sous forme divergence et analyse de méthodes numériques probabilistes associées.* PhD thesis, Université de Provence, Marseille, France, 2004.

[MT06] M. Martinez and D. Talay. *Discrétisation d'équations différentielles stochastiques unidimensionnelles à générateur sous forme divergence avec coefficient discontinu.* C.R. Acad. Sci. Paris 342:1 (2006), 51-56.

[RY91] D. Revuz and M. Yor. *Continuous Martingale and Brownian Motion.* Springer, Heidelberg, 1991. MR1083357

[Str82] D. W. Stroock. *Aronson's Estimate for Elliptic Operators in Divergence Form.* Lecture Notes in Mathematics, Seminaire de Probabilités XXII (1321), Springer 1988, 316-347.

[Wal78] J.B. Walsh. *A Diffusion with a Discontinuous Local Time.* Société Mathématique de France, Astérisque 52-53 (1978) 37-45.