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Fragmentation of ordered partitions and intervals

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Abstract

Fragmentation processes of exchangeable partitions have already been studied by several authors. This paper deals with fragmentations of exchangeable compositions, i.e. partitions of $\mathbb{N}$ in which the order of the blocks matters. We will prove that such a fragmentation is bijectively associated to an interval fragmentation. Using this correspondence, we then study two examples : Ruelle's interval fragmentation and the interval fragmentation derived from the standard additive coalescent.

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1 Introduction

Random fragmentations describe objects that split as time goes on. Two types of fragmentation have received a special attention: fragmentations of partitions of $\mathbb{N}$ and mass-fragmentations, i.e. fragmentations on the space $\mathcal{S} = \{s_1 \geq s_2 \geq \ldots \geq 0, \sum_i s_i \leq 1\}$. Berestycki [3] has proved that to each homogeneous fragmentation process of exchangeable partitions, we can canonically associate a mass fragmentation. More precisely, let $\pi = (\pi_1, \pi_2, \ldots)$ be an exchangeable random partition of $\mathbb{N}$ (i.e. the distribution of π is invariant under finite permutations of $\mathbb{N}$) whose blocks $(\pi_i)_{i\geq 1}$ are ordered by increasing of their least elements. According to the work of Kingman and Pitman [15, 16], the asymptotic frequency of the i-th block π_i, $f_i = \lim_{n\to\infty} \frac{Card\{\pi_i \cap \{1,\ldots,n\}\}}{n}$, exists for every i a.s. We denote by $(|\pi_i|^{\downarrow})_{i\in\mathbb{N}}$ the sequence $(f_i)_{i\in\mathbb{N}}$ after a decreasing rearrangement. If $(\Pi(t), t \geq 0)$ is a fragmentation of exchangeable partitions, then $(|\Pi_i(t)|^{\downarrow}_{i\in\mathbb{N}}, t \geq 0)$ is a mass fragmentation. Conversely, a fragmentation of exchangeable partitions can be built from a mass fragmentation via a "paintbox process".

One goal of this paper is to develop a similar theory for fragmentations of exchangeable compositions and interval fragmentations. Let us recall that a composition of a natural number n is an ordered collection of natural numbers $(n_1, \ldots, n_k)$ with sum n. Here we will also use the definition of Gnedin [11]: a composition of the set $\{1, \ldots, n\}$ is an ordered collection of disjoint nonempty subsets $\gamma = (A_1, \ldots, A_k)$ with $\cup A_i = \{1, \ldots, n\}$. The vector of class size of γ, $(\sharp A_1, \ldots, \sharp A_k)$ is a composition of n and is called the shape of γ. Hence, there is a one to one correspondence between measures on compositions of n and measures on exchangeable compositions of the set $\{1, \ldots, n\}$. Gnedin proved a theorem analogous to Kingman's Theorem in the case of exchangeable compositions: for each probability measure P that describes the law of a random exchangeable composition, we can find a probability measure on the space of open subsets of [0,1], such that P can be recovered via a "paintbox process". This is why it seems very natural to look for a correspondence between fragmentations of compositions and interval fragmentations.

The first part of this paper develops the relation between probability laws of exchangeable compositions and laws of random open subsets, and its extension to infinite measures. Then we prove that there exists indeed a one to one correspondence between fragmentations of compositions and interval fragmentations. The next part gives some properties and characteristics of these processes and briefly presents how this theory can be extended to time-inhomogeneous fragmentations and self-similar fragmentations. Finally, as an application of this theory, the last section describes two well known interval fragmentations: first, the interval fragmentation introduced by Ruelle [2, 7, 9, 19] and second, the fragmentation derived from the standard additive coalescent [1, 4].

2 Exchangeable compositions and open subsets of $]0,1[$

2.1 Probability measures

In this section, we define exchangeable compositions following Gnedin [11], and recall some useful properties. For $n \in \mathbb{N}$, let $[n]$ be the set of integers $\{1, \ldots, n\}$.

Definition 2.1. *For $n \in \mathbb{N}$, a composition of the set $[n]$ is an ordered sequence of disjoint, non empty subsets of $[n]$, $\gamma = (A_1, \ldots, A_k)$, with $\cup A_i = [n]$. We denote by $\mathcal{C}_n$ the set of compositions of $[n]$.*

Let $k_n : \mathcal{C}_n \to \mathcal{C}_{n-1}$ be the restriction mapping from compositions of the set $[n]$ to compositions of the set $[n-1]$ and let $\mathcal{C}$ be the projective limit of $(\mathcal{C}_n, k_n)$. We endow $\mathcal{C}$ with the product topology, it is then a compact set. The composition of $[n]$ (resp. $\mathbb{N}$) with a single nonempty block will be denoted by $\mathbf{1}_n$ (resp. $\mathbf{1}_{\mathbb{N}}$) and we will write $\mathcal{C}_n^*$ for $\mathcal{C}_n \backslash \{\mathbf{1}_n\}$. In the sequel, for $n \in \mathbb{N} \cup \{\infty\}$, $\gamma \in \mathcal{C}_n$ and $A \subset [n]$, γ_A will denote the restriction of γ to A. Hence, for $m \leq n$, $\gamma_{[m]}$ will denote the restriction of γ to $[m]$. We say that a sequence $(P_n)_{n \in \mathbb{N}}$ of measures on $(\mathcal{C}_n)_{n \in \mathbb{N}}$ is consistent if, for all $n \geq 2$, P_{n-1} is the image of P_n by the projection k_n, i.e., for all $\gamma \in \mathcal{C}_{n-1}$, we have

$$P_{n-1}(\Gamma_{[n-1]} = \gamma) = \sum_{\gamma' \in \mathcal{C}_n : k_n(\gamma') = \gamma} P_n(\Gamma_{[n]} = \gamma').$$

By Kolmogorov's Theorem, such a sequence $(P_n)_{n \in \mathbb{N}}$ determines the law of a random composition of $\mathbb{N}$.

A random composition Γ of $\mathbb{N}$ is called exchangeable if for all $n \in \mathbb{N}$, for every permutation σ of $[n]$ and for all $\gamma \in \mathcal{C}_n$, we have

$$\mathbb{P}(\Gamma_{[n]} = \gamma) = \mathbb{P}(\sigma(\Gamma_{[n]}) = \gamma),$$

where $\sigma(\Gamma_{[n]})$ is the image of the composition $\Gamma_{[n]}$ by σ. Hence, given an exchangeable random composition Γ, we can associate a function defined on finite sequences of integers by

$$\forall k \in \mathbb{N}, \forall n_1, \ldots, n_k \in \mathbb{N}^k, p(n_1, \ldots, n_k) = \mathbb{P}(\Gamma_{[n]} = (B_1, \ldots, B_k)),$$

where $(B_1, \ldots, B_k)$ is a composition of the set $[n]$ with shape $(n_1, \ldots, n_k)$ and $n = n_1 + \ldots + n_k$. This function determines the law of Γ and is called the exchangeable composition probability function (ECPF) of Γ.

Notation 2.2. *Let γ be a composition of $\mathbb{N}$. For $(i, j) \in \mathbb{N}^2$, we will use the following notation:*

- *$i \sim j$, if i and j are in the same block.*
- *$i \prec j$, if the block containing i precedes the block containing j.*
- *$i \succ j$, if the block containing i follows the block containing j.*

Let $\mathcal{U}$ be the set of open subsets of $]0, 1[$. For $u \in \mathcal{U}$, let

$$\chi_u(x) = \min\{|x - y|, y \in u^c\}, \ x \in [0, 1],$$

where $u^c = [0, 1] \backslash u$. We also define a distance on $\mathcal{U}$ by:

$$d(u, v) = ||\chi_u - \chi_v||_\infty.$$

This makes $\mathcal{U}$ a compact metric space.

Definition 2.3. *Let u be an open subset of $[0,1]$. We construct a random composition of $\mathbb{N}$ in the following way: we draw $(X_i)_{i\in\mathbb{N}}$ iid random variables with uniform law on $[0,1]$ and we use the following rules:*

- *$i \sim j$, if $i = j$ or if X_i and X_j belong to the same component interval of u.*
- *$i \prec j$, if X_i and X_j do not belong to the same component interval of u and $X_i < X_j$.*
- *$i \succ j$, if X_i and X_j do not belong to the same component interval of u and $X_i > X_j$.*

This defines an exchangeable probability measure on $\mathcal{C}$ that we shall denote P^u; the projection of P^u on $\mathcal{C}_n$ will be denoted by P_n^u. If ν is a probability measure on $\mathcal{U}$, we denote by P^ν the law on $\mathcal{C}$ whose projections on $\mathcal{C}_n$ are:

$$P_n^\nu(\cdot) = \int_{\mathcal{U}} P_n^u(\cdot)\nu(du).$$

Let us recall here a useful theorem from Gnedin [11]:

Theorem 2.4. *[11] Let Γ be an exchangeable random composition of $\mathbb{N}$, $\Gamma_{[n]}$ its restriction to $[n]$. Let $(N_1,\ldots,N_k)$ be the shape of $\Gamma_{[n]}$ and $N_0 = 0$. For $i \in \{0,\ldots,k\}$, we write $M_i = \sum_{j=0}^i N_j$. Define $U_n \in \mathcal{U}$ by:*

$$U_n = \bigcup_{i=1}^k \left] \frac{M_{i-1}}{n}, \frac{M_i}{n} \right[.$$

Then U_n converges almost surely to a random element $U \in \mathcal{U}$. The conditional law of Γ given U is P^U. As a consequence, if P is an exchangeable probability measure on $\mathcal{C}$, then there exists a unique probability measure ν on $\mathcal{U}$ such that $P = P^\nu$.

Hence, with each exchangeable composition Γ, we can associate a random open set that we will call asymptotic open set of Γ and denote U_Γ. We shall also write $|\Gamma|^\downarrow$ for the decreasing sequence of the lengths of the interval components of U_Γ. More generally, for $u \in \mathcal{U}$, $u^\downarrow$ will be the decreasing sequence of the interval component lengths of u.

Let us notice that this theorem is the analogue of Kingman's Theorem for the representation of exchangeable partitions. Actually, let $\Pi = (\Pi_1, \Pi_2, \ldots)$ be an exchangeable random partition of $\mathbb{N}$ (the blocks of Π are listed by increase of their least elements). Pitman [16] has proved that each block of Π has almost surely a frequency, i.e.

$$\forall i \in \mathbb{N} \qquad f_i = \lim_{n\to\infty} \frac{Card\{\Pi_i \cap [n]\}}{n} \qquad \text{exists almost surely.}$$

One calls f_i the frequency of the block Π_i. Therefore, for all exchangeable random partitions, we can associate a probability on $\mathcal{S} = \{s = (s_1, s_2, \ldots), s_1 \geq s_2 \geq \ldots \geq 0, \sum_i s_i \leq 1\}$ which will be the law of the decreasing rearrangement of the sequence of the partition frequencies.

Conversely, given a law $\tilde{\nu}$ on $\mathcal{S}$, we can construct an exchangeable random partition whose law of its frequency sequence is $\tilde{\nu}$ (cf. [15]): we pick $S \in \mathcal{S}$ with law $\tilde{\nu}$ and we draw a sequence of independent random variables V_i with uniform law on $[0,1]$. Conditionally on S, two integers i and j are in the same block of Π iff there exists an integer k such that $\sum_{l=1}^k S_l \leq V_i < \sum_{l=1}^{k+1} S_l$

and $\sum_{l=1}^{k} S_l \le V_j < \sum_{l=1}^{k+1} S_l$. We denote by $\rho_{\tilde{\nu}}$ the law of this partition (and by a slight abuse of notation, ρ_s denotes the law of the partition obtained with $\tilde{\nu} = \delta_s$). Kingman's representation Theorem states that any exchangeable random partition can be constructed in this way.

Let $\wp_1$ be the canonical projection from the set of compositions $\mathcal{C}$ to the set of partitions $\mathcal{P}$ and $\wp_2$ the canonical projection from the set $\mathcal{U}$ to the set $\mathcal{S}$ that associates to an open set u the decreasing sequence $u^{\downarrow}$ of the lengths of its interval components. To sum up, we have the following commutative diagram between probability measures on $\mathcal{P}$, $\mathcal{C}$, $\mathcal{S}$, $\mathcal{U}$:

$$\begin{array}{ccc}
(\mathcal{C}, P^{\nu}) & \xleftrightarrow{\text{Gnedin}} & (\mathcal{U}, \nu) \\
{\scriptstyle \wp_1}\big\downarrow & & {\scriptstyle \wp_2}\big\downarrow \\
(\mathcal{P}, \rho_{\tilde{\nu}}) & \xleftrightarrow{\text{Kingman}} & (\mathcal{S}, \tilde{\nu}).
\end{array}$$

2.2 Representation of infinite measures on $\mathcal{C}$

In this section, we show how Theorem 2.4 can be extended to a class of infinite measures on $\mathcal{C}$.

Definition 2.5. *Let μ be a measure on $\mathcal{C}$. We call μ a fragmentation measure if the following conditions hold:*

- *μ is exchangeable.*
- $\mu(\mathbf{1}_{\mathbb{N}}) = 0$.
- $\mu(\{\gamma \in \mathcal{C}, \gamma_{[2]} \neq \mathbf{1}_2\}) < \infty$.

Notice that by exchangeability, the last condition implies that, for all $n \ge 2$, we have $\mu(\{\gamma \in \mathcal{C}, \gamma_{[n]} \neq \mathbf{1}_n\}) < \infty$. We will see in the sequel that such a measure can always be associated to a fragmentation process and conversely.

Definition 2.6. *A measure ν on $\mathcal{U}$ is called a dislocation measure if:*

$$\nu(]0,1[) = 0, \qquad \int_{\mathcal{U}} (1 - s_1)\nu(du) < \infty,$$

where s_1 is the length of the largest interval component of u.

In the sequel, for any ν measure on $\mathcal{U}$, we define the measure P^{ν} on $\mathcal{C}$ by

$$P^{\nu} = \int_{\mathcal{U}} P^u \nu(du).$$

Notice that if ν is a dislocation measure, then P^{ν} is a fragmentation measure. In fact, the measure P^{ν} is exchangeable since P^u is an exchangeable measure. For $u \neq]0,1[$, we have $P^u(\mathbf{1}_{\mathbb{N}}) = 0$, and as $\nu(]0,1[) = 0$, we have also $P^{\nu}(\mathbf{1}_{\mathbb{N}}) = 0$. We now have to check that $P^{\nu}(\{\gamma \in \mathcal{C}, \gamma_{[n]} \neq \mathbf{1}_n\}) < \infty$ for all $n \in \mathbb{N}$. Let us fix $u \in \mathcal{U}$. Set $u^{\downarrow} = s = (s_1, s_2, \ldots)$.

$$P^u(\{\gamma \in \mathcal{C}, \gamma_{[n]} \neq \mathbf{1}_n\}) = 1 - \sum_{i=1}^{\infty} s_i^n \le 1 - s_1^n \le n(1 - s_1)$$

and so $P^\nu(\{\gamma \in \mathcal{C}, \gamma_{[n]} \neq \mathbf{1}_n\}) < \infty$.

Let ϵ_l^i be the composition of $\mathbb{N}$ given by $(\{i\}, \mathbb{N} \setminus \{i\})$ and $\epsilon_l = \sum_i \delta_{\epsilon_l^i}$. Let ϵ_r^i be the composition of $\mathbb{N}$ given by $(\mathbb{N} \setminus \{i\}, \{i\})$ and $\epsilon_r = \sum_i \delta_{\epsilon_r^i}$. It is easy to check that ϵ_l and ϵ_r are also two fragmentation measures.

Theorem 2.7. *If μ is a fragmentation measure, there exists two unique nonnegative numbers c_l and c_r, called coefficients of erosion, and a unique dislocation measure ν on $\mathcal{U}$ such that:*

$$\mu = c_l\epsilon_l + c_r\epsilon_r + P^\nu.$$

Besides, we have $\mathbf{1}_{\{\gamma\in\mathcal{C},\, U_\gamma =]0,1[\}}\mu = c_l\epsilon_l + c_r\epsilon_r$ and $\mathbf{1}_{\{\gamma\in\mathcal{C},\, U_\Gamma \neq]0,1[\}}\mu = P^\nu$.

Recall that in the case of fragmentation measures on partitions, Bertoin [6] proved the following result: let $\tilde{\epsilon}^i$ be the partition of $\mathbb{N}$, $\big\{\{i\}, \mathbb{N}\setminus\{i\}\big\}$ and define the measure $\tilde{\epsilon} = \sum_i \delta_{\tilde{\epsilon}^i}$. Let $\tilde{\mu}$ be an exchangeable measure on $\mathcal{P}$ such that $\mu(\{\mathbb{N}\}) = 0$ and $\tilde{\mu}(\pi \in \mathcal{P}, \pi_n \neq \{[n]\})$ is finite for all $n \in \mathbb{N}$. Then there exists a measure $\tilde{\nu}$ on $\mathcal{S}$ such that $\tilde{\nu}((1,0,0,\ldots)) = 0$ and $\int_{\mathcal{S}}(1-s_1)\nu(ds) < \infty$, and a nonnegative number c such that:

$$\tilde{\mu} = \rho_{\tilde{\nu}} + c\tilde{\epsilon}.$$

Fragmentation measures on partitions fit in a more general framework of exchangeable semifinite measures on partitions as developed by Kerov (see [14], Chapter 1, Section 3).

Hence, Theorem 2.7 is an analogous decomposition in the case of fragmentation measures on compositions, except that, in this case, there are two coefficients of erosion, one characterizing the left side erosion and the other the right side erosion.

Proof. We adapt a proof due to Bertoin [6] for the exchangeable partitions to our case. Set $n \in \mathbb{N}$. Set $\mu_n = \mathbf{1}_{\{\Gamma_{[n]} \neq \mathbf{1}_n\}}\mu$, therefore μ_n is a finite measure. Let $\overrightarrow{\mu_n}$ be the image of μ_n by the n-shift, i.e.

$$i \overset{\overset{\rightarrow n}{\Gamma}}{\prec} j \Leftrightarrow i+n \overset{\Gamma}{\prec} j+n, \qquad i \overset{\overset{\rightarrow n}{\Gamma}}{\sim} j \Leftrightarrow i+n \overset{\Gamma}{\sim} j+n, \qquad i \overset{\overset{\rightarrow n}{\Gamma}}{\succ} j \Leftrightarrow i+n \overset{\Gamma}{\succ} j+n.$$

Then $\overrightarrow{\mu_n}$ is exchangeable since μ is, and furthermore it is a finite measure. So, we can apply Theorem 2.4:

$$\exists\,!\ \nu_n \text{ finite measure on } \mathcal{U} \text{ such that } \overrightarrow{\mu_n}(d\gamma) = \int_{\mathcal{U}} P^u(d\gamma)\nu_n(du).$$

According to Theorem 2.4, since $\overrightarrow{\mu_n}$ is an exchangeable finite measure, $\overrightarrow{\mu_n}$-almost every composition has an asymptotic open set and so μ_n-almost every composition has also an asymptotic open set, and as $\mu_n \uparrow \mu$, μ-almost every composition has also an asymptotic open set. Besides, we have

$$\forall A \subset \mathcal{U},\ \mu_n(|\gamma|^\downarrow \in A) = \overrightarrow{\mu_n}(|\gamma|^\downarrow \in A) = \nu_n(A).$$

Hence, since $\mu_n \leq \mu_{n+1}$, we deduce that $\nu_n \leq \nu_{n+1}$. Set $\nu = \lim_{n\to\infty} \uparrow \nu_n$. Furthermore, we have

$$\mu_n(n+1 \nsim n+2 \mid U_\Gamma = u) = \overrightarrow{\mu_n}(1 \nsim 2 \mid U_\Gamma = u) = P^u(1 \nsim 2) = 1 - \sum s_i \geq 1 - s_1.$$

So

$$\mu_n(n+1 \not\sim n+2) \geq \int (1-s_1)\nu_n(du).$$

Since

$$\mu_n(n+1 \not\sim n+2) \leq \mu(n+1 \not\sim n+2) = \mu(1 \not\sim 2) < \infty, \text{we deduce } \int (1-s_1)\nu(du) < \infty.$$

Hence ν is a dislocation measure. Set $\gamma_k \in \mathcal{C}_k$.

$$\begin{aligned}
\mu(\Gamma_{[k]} = \gamma_k, U_\Gamma \neq]0,1[) &= \lim_{n\to\infty} \mu(\Gamma_{[k]} = \gamma_k, \Gamma_{\{k+1,\ldots,k+n\}} \neq \mathbf{1}_n, U_\Gamma \neq]0,1[) \\
&= \lim_{n\to\infty} \mu(\overset{\rightarrow n}{\Gamma}_{[k]} = \gamma_k, \Gamma_{[n]} \neq \mathbf{1}_n, U_\Gamma \neq]0,1[) \\
&= \lim_{n\to\infty} \overrightarrow{\mu_n}(\Gamma_{[k]} = \gamma_k, U_\Gamma \neq]0,1[) \\
&= \int_{\mathcal{U}^*} P^u(\Gamma_{[k]} = \gamma_k)\nu(du) \qquad \text{with } \mathcal{U}^* = \mathcal{U}\backslash\{]0,1[\}.
\end{aligned}$$

Thus we have

$$\mu(\,\cdot\,, U_\gamma \neq]0,1[) = \int P^u(\,\cdot\,)\nu(du).$$

We now have to study μ on the event $\{U_\gamma =]0,1[\}$. Let $\tilde{\mu} = \mathbf{1}_{\{1 \not\sim 2, U_\gamma =]0,1[\}}\mu$. Let $\overrightarrow{\tilde{\mu}}$ be the image of $\tilde{\mu}$ by the 2-shift. The measure $\overrightarrow{\tilde{\mu}}$ is finite and exchangeable and its asymptotic open set is almost surely $]0,1[$, so $\overrightarrow{\tilde{\mu}} = a\delta_{]0,1[}$ where a is a nonnegative number. So $\tilde{\mu} = c_1\delta_{\gamma_1} + \ldots + c_{10}\delta_{\gamma_{10}}$ where $\gamma_1, \ldots, \gamma_6$ are the six possible compositions build from the blocks $\{1\}$, $\{2\}$, $\mathbb{N}\backslash\{1,2\}$, $\gamma_7 = (\{1\}, \mathbb{N}\backslash\{1\})$, $\gamma_8 = (\{2\}, \mathbb{N}\backslash\{2\})$, $\gamma_9 = (\mathbb{N}\backslash\{1\}, \{1\})$, $\gamma_{10} = (\mathbb{N}\backslash\{2\}, \{2\})$. We must have $c_1 = \ldots = c_6 = 0$, otherwise, by exchangeability, we would have $\mu(\{1\}, \{n\}, \mathbb{N}\backslash\{1,n\}) = c > 0$ and this would yield $\mu(\mathcal{C}_2^*) = \infty$. By exchangeability, we also have $c_7 = c_8$ and $c_9 = c_{10}$ and so, by exchangeability,

$$\mu\mathbf{1}_{\{U_\gamma =]0,1[\}} = c_l \sum_i \delta_{\epsilon_l^i} + c_r \sum_i \delta_{\epsilon_r^i}.$$

□

As in Section 2.1, we can now establish connections among fragmentation measures on $\mathcal{C}$ and $\mathcal{P}$ and dislocation measures on $\mathcal{U}$ and $\mathcal{S}$. Let us recall that $\wp_1$ is the canonical projection from $\mathcal{C}$ to $\mathcal{P}$, and denote $q : (\mathcal{U}, \mathbb{R}_+, \mathbb{R}_+) \mapsto (\mathcal{S}, \mathbb{R}_+)$ the operation defined by $q(u,a,b) = q(u^\downarrow, a+b)$. Then we have the following commutative diagram:

$$\begin{array}{ccc}
(\mathcal{C}, \mu) & \xleftrightarrow{\text{Theorem 2.7}} & \big(\mathcal{U}, (\nu, c_l, c_r)\big) \\
\wp_1 \big\downarrow & & \big\downarrow q \\
(\mathcal{P}, \tilde{\mu}) & \xleftrightarrow{\text{Bertoin}} & \big(\mathcal{S}, (\tilde{\nu}, c_l + c_r)\big).
\end{array}$$

Proof. It remains to prove that $\tilde{\mu} = \rho_{\tilde{\nu}} + (c_l + c_r)\tilde{\epsilon}$. Set $\tilde{\mu} = \rho_{\overline{\nu}} + c\tilde{\epsilon}$. Since $\tilde{\mu}$ is the image by $\wp_1$ of μ, we have

$$\tilde{\mu}(\tilde{\epsilon}^1) = \mu(\epsilon_l^1) + \mu(\epsilon_r^1) \text{ and then } c = c_r + c_l.$$

Let us fix $n \in \mathbb{N}$ and $\pi \in \mathcal{P}_n \backslash \{\mathbf{1}_n\}$. Set $A = \{\gamma \in \mathcal{C}_n, \wp_1(\gamma) = \pi\}$. Remark now that for all $u, v \in \mathcal{U}$ such that $u^{\downarrow} = v^{\downarrow}$, we have $P^u(A) = P^v(A)$. Moreover we have $P^u(A) = \rho_s(\pi)$ if $s = u^{\downarrow}$. So

$$P^{\nu}(A) = \int_{\mathcal{S}} P^u(A) \nu(u, u^{\downarrow} = ds) = \int_{\mathcal{S}} \rho_s(\pi) \tilde{\nu}(ds) = \rho_{\tilde{\nu}}(\pi).$$

We get

$$\mu(A) = P^{\nu}(A) + c_l \epsilon_l(A) + c_r \epsilon_r(A) = \rho_{\tilde{\nu}}(\pi) + (c_l + c_r)\tilde{\epsilon}(A) = \rho_{\overline{\nu}}(\pi) + (c_l + c_r)\tilde{\epsilon}(A) = \tilde{\mu}(\pi).$$

So we deduce that $\overline{\nu} = \tilde{\nu}$. □

3 Fragmentation of compositions and interval fragmentation

3.1 Fragmentation of compositions

Definition 3.1. *Let us fix $n \in \mathbb{N}$ and $\gamma \in \mathcal{C}_n$ with $\gamma = (\gamma_1, \dots, \gamma_k)$. Let $\gamma^{(\cdot)} = (\gamma^{(i)}, i \in \{1, \dots, n\})$ with $\gamma^{(i)} \in \mathcal{C}_n$ for all i. Set $m_i = \min \gamma_i$. We denote $\tilde{\gamma}^{(i)}$ the restriction of $\gamma^{(m_i)}$ to the set γ_i. So $\tilde{\gamma}^{(i)}$ is a composition of γ_i. We consider now $\tilde{\gamma} = (\tilde{\gamma}^{(1)}, \dots, \tilde{\gamma}^{(k)}) \in \mathcal{C}_n$. We denote by $FRAG(\gamma, \gamma^{(\cdot)})$ the composition $\tilde{\gamma}$. If $\Gamma^{(\cdot)}$ is a sequence of i.i.d. random compositions with law p, p-$FRAG(\gamma, \cdot)$ will denote the law of $FRAG(\gamma, \Gamma^{(\cdot)})$.*

We remark that the operator $FRAG$ has some useful properties. First, if $\mathbf{1}^{(\cdot)}$ denotes the constant sequence equal to $\mathbf{1}_n$, we have $FRAG(\gamma, \mathbf{1}^{(\cdot)}) = \gamma$. Furthermore, the fragmentation operator is compatible with the restriction i.e., for every $n' \leq n$,

$$FRAG(\gamma, \gamma^{(\cdot)})_{[n']} = FRAG(\gamma_{[n']}, \gamma^{(\cdot)}).$$

This implies that $FRAG$ is a consistent operator and we can extend this definition to the compositions of $\mathbb{N}$. Notice that we have this equality since we take care of fragmenting the block γ_i by $\gamma^{(m_i)}$. Indeed, if we have fragmented the block γ_i by $\gamma^{(i)}$, the operator $FRAG$ would not be anymore compatible with the restriction. For example, take $n = 3$, $\gamma = (\{3\}, \{1, 2\})$, $\gamma^{(1)} = (\{1, 2, 3\})$ and $\gamma^{(2)} = (\{1\}, \{2\}, \{3\})$.

Besides, the operator $FRAG$ preserves the exchangeability. More precisely, let $(\Gamma^{(i)}, i \in \{1, \dots, n\})$ be a sequence of random compositions which is *doubly exchangeable*, i.e. for each i, $\Gamma^{(i)}$ is an exchangeable composition, and moreover, the sequence $(\Gamma^{(i)}, i \in \{1, \dots, n\})$ is also exchangeable. Let Γ be an exchangeable composition of $\mathcal{C}_n$ independent of $\Gamma^{(\cdot)}$. Then $FRAG(\Gamma, \Gamma^{(\cdot)})$ is an exchangeable composition. Let us prove this property. We fix a permutation σ of $[n]$ and we shall prove that

$$FRAG(\Gamma, \Gamma^{(\cdot)}) \stackrel{law}{=} \sigma(FRAG(\Gamma, \Gamma^{(\cdot)})).$$

Let k be the number of blocks of Γ and denote by $m_1, \dots, m_k$ the minima of $\Gamma_1, \dots, \Gamma_k$. Let us define now $m'_1, \dots, m'_k$ the minima of $\sigma(\Gamma_1), \dots, \sigma(\Gamma_k)$. and $\Gamma'^{(\cdot)} = (\Gamma'^{(i)}, i \in \{1, \dots, n\})$ by

$$\Gamma'^{(m'_i)} = \sigma(\Gamma^{(m_i)}) \text{ for } 1 \leq i \leq k,$$

$$\Gamma'^{(j)} = \sigma(\Gamma^{(f(j))}) \text{ for } j \in \{1,\dots,n\} \setminus \{m'_i, 1 \leq i \leq k\},$$

where f is the increasing bijection from $\{1,\dots,n\}\backslash\{m'_i, 1 \leq i \leq k\}$ to $\{1,\dots,n\}\backslash\{m_i, 1 \leq i \leq k\}$. We get

$$\sigma(FRAG(\Gamma, \Gamma^{(.)})) = FRAG(\sigma(\Gamma), \Gamma'^{(.)}).$$

Since $\sigma(\Gamma) \overset{law}{=} \Gamma$ and $\Gamma'^{(.)} \overset{law}{=} \Gamma^{(.)}$ and $\Gamma'^{(.)}$ remains independent of Γ, we get

$$FRAG(\sigma(\Gamma), \Gamma'^{(.)}) \overset{law}{=} FRAG(\Gamma, \Gamma^{(.)}).$$

We can now define the notion of exchangeable fragmentation process of compositions.

Definition 3.2. *Let us fix $n \in \mathbb{N}$ and let $(\Gamma_n(t), t \geq 0)$ be a (possibly time-inhomogeneous) Markov process on $\mathcal{C}_n$ which is continuous in probability. We call Γ_n an exchangeable fragmentation process of compositions if:*

- *$\Gamma_n(0) = \mathbf{1}_n$ a.s.*
- *Its semi-group is described in the following way: there exists a family of probability measures on exchangeable compositions $(P_{t,s}, t \geq 0, s > t)$ such that for all $t \geq 0, s > t$ the conditional law of $\Gamma_n(s)$ given $\Gamma_n(t) = \gamma$ is the law of $P_{t,s}$-$FRAG(\gamma, \cdot)$.*

The fragmentation is homogeneous in time if $P_{t,s}$ depends only on $s - t$. A Markov process $(\Gamma(t), t \geq 0)$ on $\mathcal{C}$ is called an exchangeable fragmentation process of compositions if, for all $n \in \mathbb{N}$, the process $(\Gamma_{[n]}(t), t \geq 0)$ is an exchangeable fragmentation process of compositions on $\mathcal{C}_n$.

Hence, in our definition we impose that the blocks split independently by the same rule (the "branching property"). This hypothesis is crucial for most of the following results (see however Section 4.5 where more general processes are considered).

In the sequel, a *c*-fragmentation will denote an exchangeable fragmentation process on compositions.

Proposition 3.3. *The semi-group of transition of a time-homogeneous c-fragmentation has the Feller property.*

Proof. Let $\phi : \mathcal{C} \to \mathbb{R}$ be a continuous function (recall that $\mathcal{C}$ is compact, so ϕ is bounded). Then the function $\gamma \to \mathbb{E}\Big(\phi(FRAG(\gamma, \Gamma^{(.)}(t)))\Big)$ is also continuous on $\mathcal{C}$ since $FRAG$ is compatible with the restriction. Furthermore, for all $n \in \mathbb{N}$, $\lim_{t\to 0} \mathbb{P}(\Gamma_{[n]}(t) = \mathbf{1}_n) = 1$, so we have also

$$\lim_{t\to 0} \mathbb{E}\Big(\phi(FRAG(\gamma, \Gamma^{(.)}(t)))\Big) = \phi(\gamma).$$

$\square$

3.2 Interval fragmentation

In this section we recall the definition of a homogeneous[1] interval fragmentation [5]. We consider a family of probability measures $(q_{t,s}, t \geq 0, s > t)$ on $\mathcal{U}$. For every interval $I =]a, b[\subset]0, 1[$, we define the affine transformation $g_I :]0, 1[\to I$ given by $g_I(x) = a + x(b - a)$. We still denote g_I the induced map on $\mathcal{U}$, so, for $V \in \mathcal{U}$, $g_I(V)$ is an open subset of I. We define then $q^I_{t,s}$ as the image of $q_{t,s}$ by g_I. Hence $q^I_{t,s}$ is a probability measure on the space of open subsets of I. Finally, for $W \in \mathcal{U}$ with interval decomposition $(I_i, i \in \mathbb{N})$, $q^W_{t,s}$ is the distribution of $\cup X_i$ where the X_i are independent random variables with respective laws $q^{I_i}_{t,s}$.

Definition 3.4. *A process $(U(t), t \geq 0)$ on $\mathcal{U}$ is called a homogeneous interval fragmentation if it is a Markov process that fulfills the following properties:*

- *U is continuous in probability and $U(0) =]0, 1[$ a.s.*
- *U is nested i.e. for all $s > t$ we have $U(s) \subset U(t)$.*
- *There exists a family $(q_{t,s}, t \geq 0, s > t)$ of probability measures on $\mathcal{U}$ such that:*

$$\forall t \geq 0,\ \forall s > t,\ \forall A \subset \mathcal{U}, \quad \mathbb{P}(U(s) \in A |\, U(t)) = q^{U(t)}_{t,s}(A).$$

In the sequel, we abbreviate an interval fragmentation process as an i-fragmentation.

We remark that if we take the decreasing sequence of the sizes of the interval components of an i-fragmentation, we obtain a mass-fragmentation, denoted here a m-fragmentation (see [6] for a definition of m-fragmentations).

3.3 Link between i-fragmentation and c-fragmentation

From this point of the paper and until Section 4.4, the fragmentation processes we consider will always be homogeneous in time, i.e. $q_{t,s}$ depends only on $s - t$, hence we will just write q_{s-t} to denote $q_{t,s}$.

Let $(U(t), t \geq 0)$ be a process on $\mathcal{S}$. Let $(V_i)_{i \geq 0}$ be a sequence of independent random variables uniformly distributed on]0,1[. Using the same process as in Definition 2.3 with $U(t)$ and $(V_i)_{i \geq 1}$, we define a process $(\Gamma_U(t), t \geq 0)$ on $\mathcal{C}$.

Theorem 3.5. *There is a one to one correspondence between laws of i-fragmentations and laws of c-fragmentations. More precisely:*

- *If a process $(U(t), t \geq 0)$ is an i-fragmentation, then $(\Gamma_U(t), t \geq 0)$ defined as above is a c-fragmentation and we have $U_{\Gamma_U(t)} = U(t)$ a.s. for each $t \geq 0$.*
- *Let $(\Gamma(t), t \geq 0)$ be a c-fragmentation. Then $(U_{\Gamma_U(t)}, t \geq 0)$ is an i-fragmentation.*

[1]In [5], Bertoin defines more generally self-similar interval fragmentations with index α. Here, the term homogeneous means that we only consider the case $\alpha = 0$.

Proof. We start by proving the first point. For the sake of clarity, we will write in the sequel $\Gamma(t)$ instead of $\Gamma_U(t)$. We have by Theorem 2.4, $U_{\Gamma(t)} = U(t)$ a.s. for each $t \geq 0$. Let us fix $n \in \mathbb{N}$ and $t \geq 0$. We are going to prove that, for $s > t$, the conditional law of $\Gamma_{[n]}(s)$ given $\Gamma_{[n]}(t) = (\Gamma_1, \ldots, \Gamma_k)$ is the law of $FRAG(\Gamma_{[n]}(t), \Gamma^{(\cdot)})$, where $\Gamma^{(\cdot)}$ is a sequence of i.i.d. exchangeable compositions with law $\Gamma_{[n]}(s-t)$. Since $(U(s), s \geq 0)$ is a fragmentation process, we have $U(t+s) \subset U(t)$. By construction of $\Gamma_{[n]}(t)$, it is then clear that $\Gamma_{[n]}(t+s)$ is a finer composition than $\Gamma_{[n]}(t)$. Hence, each singleton of $\Gamma_{[n]}(t)$ remains a singleton of $\Gamma_{[n]}(t+s)$. For $1 \leq i \leq k$, fix $l \in \Gamma_i$ and define

$$a_i = \sup\{a \leq V_l, a \notin U(t)\}, \qquad b_i = \inf\{b \geq V_l, b \notin U(t)\}.$$

Notice that a_i and b_i do not depend on the choice of $l \in \Gamma_i$. Furthermore, we have $a_i < b_i$ if Γ_i is not a singleton. We also define

$$Y_j^i = \left(\frac{V_j - a_i}{b_i - a_i}\right) \qquad j \in \Gamma_i,\ i \in J,$$

where $J = \{1 \leq i \leq k,\ \Gamma_i \text{ is not a singleton}\}$.

Conditionally on $\Gamma_{[n]}(t)$, the random variables $(Y_j^i)_{j \in \Gamma_i, i \in J}$ are independent and uniformly distributed on $]0,1[$. Besides, $(]a_i, b_i[)_{i \in J}$ are $\sharp J$ distinct interval components of $U(t)$. Since $U(t)$ is a fragmentation process, the processes

$$\left(U^i(s) = \frac{1}{b_i - a_i}(U_{]a_i,b_i[}(s) - a_i), s \geq t\right)_{i \in J}$$

are $\sharp J$ independent i-fragmentations with law $(U(s-t), s \geq t)$ and are also independent of the singletons of $\Gamma(t)$. For $i \in J$, let $\Gamma^{(i)}(s)$ be the composition of Γ_i obtained from $U^i(s)$ and $(Y_j^i)_{j \in \Gamma_i}$ using Definition 2.3; for $i \notin J$, we set $\Gamma^{(i)} = \mathbf{1}_{\Gamma_i}$. Hence, $\Gamma^{(i)}(s)$ has the law of $\Gamma_{\Gamma_i}(s-t)$ and the processes $(\Gamma^{(i)}(s), s \geq t)_{1 \leq i \leq k}$ are independent. Furthermore, by construction we have $\Gamma_{[n]}(t+s) = FRAG(\Gamma_{[n]}(t), \Gamma^{(\cdot)}(s))$. Hence, $(\Gamma_{[n]}(t), t \geq 0)$ has the expected transition probabilities.

Let us now prove the second point. In the sequel, we will write U_t to denote $U_{\Gamma(t)}$. First, we prove that for all $s > t$, $U_s \subset U_t$. Fix $x \notin U_t$, we shall prove $x \notin U_s$. We have $\chi_{U_t}(x) = \min\{|x-y|, y \in U_t^c\} = 0$. Let U_t^n be the open subset of $]0,1[$ corresponding to $\Gamma_{[n]}(t)$ as in Theorem 2.4. So we have $\lim_{n\to\infty} d(U_t^n, U_t) = 0$. Fix $\varepsilon > 0$. Hence, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, $\chi_{U_t^n}(x) \leq \varepsilon$. This implies that:

$$\forall n \geq N, \exists y_n \notin U_t^n \text{ such that } |y_n - x| \leq \varepsilon.$$

Besides, as $(\Gamma(t), t \geq 0)$ is a fragmentation, we have for all $n \in \mathbb{N}$, $U_s^n \subset U_t^n$. Hence, we have also

$$\forall n \geq N, y_n \notin U_s^n,$$

and so $\chi_{U_s^n}(x) \leq \varepsilon$ for all $n \geq N$. We deduce that $\chi_{U_s}(x) = 0$ i.e. $x \notin U_s$.

We now have to prove the branching property. Fix $t > 0$. We consider the decomposition of U_t in disjoint intervals:

$$U_t = \coprod_{k \in \mathbb{N}} I_k(t).$$

Set $F_k(s) = U_{t+s} \cap I_k(t)$. We want to prove that, given U_t:

- $\forall l \in \mathbb{N},\ F_1, \dots, F_l$ are independent processes.
- F_k has the following law:

  $\forall A$ open subset of $]a,b[$, $\mathbb{P}((F_k(s), s \geq 0) \in A\ |I_k(t) =]a,b[) = \mathbb{P}((U_s, s \geq 0) \in (b-a)A+a)$.

For all $k \in \mathbb{N}$, there exists $i_k \in \mathbb{N}$ such that, if $J^n_{i_k}(t)$ denotes the interval component of U^n_t containing the integer i_k, then $J^n_{i_k}(t) \overset{n\to\infty}{\longrightarrow} I_k(t)$. Let B_k be the block of $\Gamma(t)$ containing i_k. As B_k has a positive asymptotic frequency, it is isomorphic to $\mathbb{N}$. Let f be the increasing bijection from the set of elements of B_k to $\mathbb{N}$. Let us re-label the elements of B_k by their image by f. The process $(U_{\Gamma_{B_k}(t+s)}, s \geq 0)$ has then the same law as $(U_s, s \geq 0)$ and is independent of the rest of the fragmentation. Besides, given $I_k(t) =]a,b[$, $F_k(s) = a + (b-a)U_{\Gamma_{B_k}(t+s)}$, so the two points above are proved. □

Hence, this result complements an analogous result due to Berestycki [3] in the case of m-fragmentations and p-fragmentation (i.e. fragmentations of exchangeable partitions). We can again draw a commutative diagram to represent the link between the four kinds of fragmentation:

$$\begin{array}{ccc} \big(\mathcal{C}, (\Gamma(t), t \geq 0)\big) & \xleftrightarrow{\text{Theorem 3.5}} & \big(\mathcal{U}, (U_{\Gamma(t)}, t \geq 0)\big) \\ {\scriptstyle\wp_1}\big\downarrow & & {\scriptstyle\wp_2}\big\downarrow \\ \big(\mathcal{P}, (\Pi(t), t \geq 0)\big) & \xleftrightarrow{\text{Berestycki}} & \big(\mathcal{S}, (U^{\downarrow}_{\Gamma(t)}, t \geq 0)\big). \end{array}$$

4 Some general properties of fragmentations

In this section, we gather general properties of i and c-fragmentations. Since the proofs of these results are simple variations of those in the case of m and p-fragmentations [6], we will be a bit sketchy.

4.1 Rate of a fragmentation process

Let $(\Gamma(t), t \geq 0)$ be a c-fragmentation. As in the case of p-fragmentation [6], for $n \in \mathbb{N}$ and $\gamma \in \mathcal{C}^*_n$, we define a jump rate from $\mathbf{1}_n$ to γ:

$$q_\gamma = \lim_{s\to 0} \frac{1}{s} \mathbb{P}\left(\Gamma_{[n]}(s) = \gamma\right).$$

With the same arguments as in the case of p-fragmentation, we can also prove that the family $(q_\gamma, \gamma \in \mathcal{C}^*_n, n \in \mathbb{N})$ characterizes the law of the fragmentation (you just have to use that distinct blocks evolve independently and with the same law). Furthermore, observing that we have

$$\forall n < m, \quad \forall \gamma' \in \mathcal{C}^*_n, \quad q_{\gamma'} = \sum_{\gamma\in\mathcal{C}_m, \gamma_{[n]}=\gamma'} q_\gamma,$$

and that

$$\forall n \in \mathbb{N}, \quad \forall \sigma \text{ permutation of } [n], \quad \forall \gamma \in \mathcal{C}^*_n, \quad q_\gamma = q_{\sigma(\gamma)},$$

we deduce that there exists a unique exchangeable measure μ on $\mathcal{C}$ such that $\mu(\mathbf{1}_{\mathbb{N}}) = 0$ and $\mu(\mathcal{Q}_{\infty,\gamma}) = q_\gamma$ for all $\gamma \in \mathcal{C}_n^*$ and $n \in \mathbb{N}$, where $\mathcal{Q}_{\infty,\gamma} = \{\gamma' \in \mathcal{C}, \gamma'_{[n]} = \gamma\}$. Furthermore, the measure μ characterizes the law of the fragmentation. We call μ the rate of the fragmentation.

We remark also that if a measure μ is the rate of a fragmentation process, we have for all $n \geq 2$,

$$\mu(\{\gamma \in \mathcal{C}, \gamma_{[n]} \neq \mathbf{1}_n\}) = \sum_{\gamma \in \mathcal{C}_n^*} q_\gamma < \infty.$$

So we can apply Theorem 2.7 to μ and we deduce the following result:

If μ is the rate of a c-fragmentation, then there exist a dislocation measure ν and two nonnegative numbers c_l and c_r such that:

- $\mu \mathbf{1}_{\{U_\gamma \neq]0,1[\}} = P^\nu$.
- $\mu \mathbf{1}_{\{U_\gamma =]0,1[\}} = c_l \epsilon_l + c_r \epsilon_r$.

With a slight abuse of notation, we will write sometimes in the sequel that $\mu = (\nu, c_l, c_r)$ when $\mu = P^\nu + c_l \epsilon_l + c_r \epsilon_r$.

4.2 The Poissonian construction

We notice that if μ is the rate of a c-fragmentation, then μ is a fragmentation measure in the sense of Definition 2.5. Conversely, we now prove that, if we consider a fragmentation measure μ, we can construct a c-fragmentation with rate μ.

We consider a Poisson measure M on $\mathbb{R}_+ \times \mathcal{C} \times \mathbb{N}$ with intensity $dt \otimes \mu \otimes \sharp$, where $\sharp$ is the counting measure on $\mathbb{N}$. Let M^n be the restriction of M to $\mathbb{R}_+ \times \mathcal{C}_n^* \times \{1, \ldots, n\}$. The intensity measure is then finite on the interval $[0,t]$, so we can rank the atoms of M^n according to their first coordinate. For $n \in \mathbb{N}$, $(\gamma, k) \in \mathcal{C} \times \mathbb{N}$, let $\Delta_n^{(.)}(\gamma, k)$ be the composition sequence of $\mathcal{C}_n$ defined by:

$$\Delta_n^{(i)}(\gamma, k) = \mathbf{1}_n \quad \text{if } i \neq k \qquad \text{and} \qquad \Delta_n^{(k)}(\gamma, k) = \gamma_{[n]}.$$

We construct then a process $(\Gamma_{[n]}(t), t \geq 0)$ on $\mathcal{C}_n$ in the following way:
$\Gamma_{[n]}(0) = \mathbf{1}_n$.
$(\Gamma_{[n]}(t), t \geq 0)$ is a pure jump process that only jumps at times when an atom of M^n appears. More precisely, if (s, γ, k) is an atom of M^n, set $\Gamma_{[n]}(s) = FRAG(\Gamma_{[n]}(s^-), \Delta_n^{(.)}(\gamma, k))$. We can check that this construction is compatible with the restriction; hence, this defines a process $(\Gamma(t), t \geq 0)$ on $\mathcal{C}$.

Proposition 4.1. *Let μ be a fragmentation measure. The construction above of a process on compositions from a Poisson point process on $\mathbb{R}_+ \times \mathcal{C} \times \mathbb{N}$ with intensity $dt \otimes \mu \otimes \sharp$, where $\sharp$ is the counting measure on $\mathbb{N}$, yields a c-fragmentation with rate μ.*

Proof. The proof is an easy adaptation of the Poissonian construction of p-fragmentations (cf. [6]). As the sequence $\Delta_n^{(.)}(\gamma, k)$ is doubly exchangeable, we also have that $\Gamma_{[n]}(t)$ is an exchangeable composition for each $t \geq 0$. Looking at the jump rates of the process $\Gamma_{[n]}(t)$, it is then easy to check that the constructed process is a c-fragmentation with rate μ. $\square$

A Poissonian construction of an i-fragmentation with no erosion is also possible with a Poisson measure on $\mathbb{R}_+ \times \mathcal{U} \times \mathbb{N}$ with intensity $dt \otimes \nu \otimes \sharp$. The proof of this result is not as simple as for compositions because it cannot be reduced to a discrete case as above. In fact, to prove this proposition, we must take the image of the Poisson measure M above by an appropriate application. For more details, we refer to Berestycki [3] who has already proved this result for m-fragmentation and the same approach works in our case.

To conclude this section, we turn our interest on how the two erosion coefficients affect the fragmentation. Let $(U(t), t \geq 0)$ be an i-fragmentation with parameter $(0, c_l, c_r)$. Set $c = c_l + c_r$. We have:

$$U(t) = \left]\frac{c_l}{c}(1 - e^{-tc}), 1 - \frac{c_r}{c}(1 - e^{-tc})\right[\text{ a.s.}$$

Indeed, consider a c-fragmentation $(\Gamma(t), t \geq 0)$ such that $U_{\Gamma(t)} = U(t)$ a.s. We define $\mu_{c_l,c_r} = c_l\epsilon_l + c_r\epsilon_r$. Hence $(\Gamma(t), t \geq 0)$ is a fragmentation with rate μ_{c_l,c_r}. Recall that the process $(\Gamma(t), t \geq 0)$ can be constructed from a Poisson measure on $\mathbb{R}_+ \times \mathcal{C} \times \mathbb{N}$ with intensity $dt \otimes \mu_{c_l,c_r} \otimes \sharp$. By the form of μ_{c_l,c_r}, we remark that, for all $t \geq 0$, $\Gamma(t)$ has only one non-singleton block. Furthermore, for all $n \in \mathbb{N}$, the integer n is a singleton at time t with probability $1 - e^{-tc}$, and, given n is a singleton of $\Gamma(t)$, $\{n\}$ is before the infinite block of $\Gamma(t)$ with probability c_l/c and after with probability c_r/c. By the law of large numbers, we deduce that the proportion of singletons before the infinite block of $\Gamma(t)$ is almost surely $\frac{c_l}{c}(1 - e^{-tc})$ and the proportion of singletons after the infinite block of $\Gamma(t)$ is almost surely $\frac{c_r}{c}(1 - e^{-tc})$.

Remark 4.2. *Berestycki [3] has proved a similar result for the m-fragmentation. He also proved that if $(F(t), t \geq 0)$ is a m-fragmentation with parameter $(\nu, 0)$, then $\tilde{F}(t) = e^{-ct}F(t)$ is a m-fragmentation with parameter (ν, c). There is no simple way to extend Berestycki's result to the case of an i-fragmentation since the Lebesgue measure of ${}^cU(t)$ squeezed between two successive interval components of $U(t)$ depends on the time where the two component intervals split.*

4.3 Projection from $\mathcal{U}$ to $\mathcal{S}$

We know that if $(U(t), t \geq 0)$ is an i-fragmentation, then its projection on $\mathcal{S}$, $(U^{\downarrow}(t), t \geq 0)$ is an m-fragmentation. More precisely, we can express the characteristics of the m-fragmentation in terms of the characteristics of the i-fragmentation.

Proposition 4.3. *The ranked sequence of the lengths of an i-fragmentation with rate (ν, c_l, c_r) is a m-fragmentation with parameter $(\tilde{\nu}, c_l + c_r)$ where $\tilde{\nu}$ is the image of ν by the projection $U \to |U|^{\downarrow}$.*

Proof. Let $(\Gamma(t), t \geq 0)$ be a c-fragmentation with rate $\mu = (\nu, c_l, c_r)$. Let $(\Pi(t), t \geq 0)$ be its image by $\wp_1$. The process $(\Pi(t), t \geq 0)$ is then a p-fragmentation. Set $n \in \mathbb{N}$ and $\pi \in \mathcal{P}_n^*$. We have

$$\begin{aligned} q_\pi &= \lim_{s \to 0} \frac{1}{s}\mathbb{P}(\Pi_{[n]}(s)) = \pi) \\ &= \lim_{s \to 0} \frac{1}{s}\mathbb{P}\left(\Gamma_{[n]}(s)) \in \wp^{-1}(\pi)\right) \\ &= \tilde{\mu}(\pi), \end{aligned}$$

where $\tilde{\mu}$ is the image of μ by $\wp_1$. Besides we have already proved that $\tilde{\mu} = (\tilde{\nu}, c_l + c_r)$. We consider now the i-fragmentation $(U_{\Gamma(t)}, t \geq 0)$ with rate (ν, c_l, c_r). We get that the process $(U^{\downarrow}_{\Gamma(t)}, t \geq 0)$ is a.s. equal to the m-fragmentation $(|\Pi(t)|^{\downarrow}, t \geq 0)$ with rate $(\tilde{\nu}, c_l + c_r)$. □

According to Proposition 4.3 and using the theory of m-fragmentation (see [6]), we deduce then the following results:

- Let $(\Gamma(t), t \geq 0)$ be a c-fragmentation with parameter (ν, c_l, c_r). We denote by B_1 the block of $\Gamma(t)$ containing the integer 1. Set $\sigma(t) = -\ln|B_1(t)|$. Then $(\sigma(t), t \geq 0)$ is a subordinator. If we denote $\zeta = \sup\{t > 0, \sigma_t < \infty\}$, then there exists a non-negative function ϕ such that

  $$\forall q, t \geq 0, \;\; \mathbb{E}[\exp(-q\sigma_t), \zeta > t] = \exp(-t\phi(q)).$$

  We call ϕ the Laplace exponent of σ and we have:

  $$\phi(q) = (c_l + c_r)(q+1) + \int_{\mathcal{U}} (1 - \sum_{i=1}^{\infty} |U_i|^{q+1}) \nu(dU),$$

  where $(|U_i|)_{i \geq 0}$ is the sequence of the lengths of the component intervals of U.

- An (ν, c_r, c_l) i-fragmentation $(U(t), t \geq 0)$ is proper (i.e. for each t, $U(t)$ has almost surely a Lebesgue measure equal to 1) iff

  $$c_l = c_r = 0 \text{ and } \nu\left(\sum_i s_i < 1\right) = 0.$$

4.4 Extension to the time-inhomogeneous case

We now briefly expose how the results of the preceding sections can be transposed in the case of time-inhomogeneous fragmentation. We will not always provide the details of the proofs since they are very similar to the homogeneous case. In the sequel, we shall focus on c-fragmentation $(\Gamma(t), t \geq 0)$ fulfilling the following properties:

- for all $n \in \mathbb{N}$, let τ_n be the time of the first jump of $\Gamma_{[n]}$ and λ_n be its law. Then λ_n is absolutely continuous with respect to Lebesgue measure with continuous and strictly positive density.
- for all $\gamma \in \mathcal{C}^*_n$, $\;h^n_\gamma(t) = \mathbb{P}(\Gamma_{[n]}(t) = \gamma \mid \tau_n = t)$ is a continuous function of t.

Remark that a time homogeneous fragmentation always fulfills these two conditions. Indeed, in that case, λ_n is an exponential random variable and the function $h^n_\gamma(t)$ does not depend on t. As in the case of fragmentation of exchangeable partitions [2], for $n \in \mathbb{N}$ and $\gamma \in \mathcal{C}^*_n$, we can define an instantaneous rate of jump from $\mathbf{1}_n$ to γ:

$$q_{\gamma,t} = \lim_{s \to 0} \frac{1}{s} \mathbb{P}\left(\Gamma_{[n]}(\tau_n) = \gamma \;\&\; \tau_n \in [t, t+s] \mid \tau_n \geq t\right).$$

With the same arguments as in the case of fragmentations of exchangeable partitions [2], we can prove that, for each $t > 0$, there exists a unique exchangeable measure μ_t on $\mathcal{C}$ such that $\mu_t(\mathbf{1}_{\mathbb{N}}) = 0$ and $\mu_t(\mathcal{Q}_{\infty,\gamma}) = q_{\gamma,t}$ for all $\gamma \in \mathcal{C}_n^*$ and $n \in \mathbb{N}$, where $\mathcal{Q}_{\infty,\gamma} = \{\gamma' \in \mathcal{C}, \gamma'_n = \gamma\}$. Furthermore, the family of measures $(\mu_t, t \geq 0)$ characterizes the law of the fragmentation. We call μ_t the instantaneous rate at time t of the fragmentation. We remark also that if $(\mu_t, t \geq 0)$ is the family of rates of a fragmentation process, we have for all $n \geq 2$,

$$\mu_t(\{\gamma \in \mathcal{C}, \gamma_{[n]} \neq \mathbf{1}_n\}) = \sum_{\gamma \in \mathcal{C}_n^*} q_{\gamma,t} < \infty \text{ and } \int_0^t \mu_u(\{\gamma \in \mathcal{C}, \gamma_{[n]} \neq \mathbf{1}_n\} = -\ln(\lambda_n(]t,\infty[)) < \infty.$$

So we can apply Theorem 2.7 to μ_t and we deduce the following proposition:

Corollary 4.4. *Let $(\mu_t, t \geq 0)$ be the family of rates of a c-fragmentation. Then there exist a family of dislocation measures $(\nu_t, t \geq 0)$ and two families of nonnegative numbers $(c_{l,t}, t \geq 0)$, $(c_{r,t}, t \geq 0)$ such that:*

- $\mu_t \mathbf{1}_{\{U_\pi \neq]0,1[\}} = P^{\nu_t}$.
- $\mu_t \mathbf{1}_{\{U_\pi =]0,1[\}} = c_{l,t}\epsilon_l + c_{r,t}\epsilon_r$.

Besides we have for all $T \geq 0$,

$$\int_0^T \int_{\mathcal{U}} (1 - s_1)\, \nu_t\, (dU)\, dt < \infty \text{ and } \int_0^T (c_{l,t} + c_{r,t}) dt < \infty.$$

Proof. The first part of the proposition comes from Theorem 2.7. For the second part, use that

$$\int_{\mathcal{U}} (1 - s_1)\, \nu_t\, (dU) \leq \mu_t \left(\{\pi \in \mathcal{P}, \pi_{[2]} \neq \mathbf{1}_2\}\right).$$

For the upper bound concerning the erosion coefficients, we remark that:

$$c_t + c'_t = \mu_t\left(\{1\}, \mathbb{N} \setminus \{1\}\right) + \mu_t\left(\mathbb{N} \setminus \{1\}, \{1\}\right).$$

□

In the same way as for homogeneous fragmentation, we define a family of fragmentation measures as a family $(\mu_t, t \geq 0)$ of exchangeable measures on $\mathcal{C}$ such that, for each $t \in [0, \infty[$, we have:

- $\mu_t(\mathbf{1}_{\mathbb{N}}) = 0$.
- $\forall n \geq 2,\ \mu_t(\{\gamma \in \mathcal{C}, \gamma_{[n]} \neq \mathbf{1}_n\}) < \infty$ and $\int_0^t \mu_u(\{\gamma \in \mathcal{C}, \gamma_{[n]} \neq \mathbf{1}_n\}) du < \infty$.
- $\forall n \in \mathbb{N},\ \forall A \subset \mathcal{C}_n^*,\ \mu_t(A)$ is a continuous function of t.

Proposition 4.5. *Let $(\mu_t, t \geq 0)$ be a family of fragmentation measures. A c-fragmentation with rate $(\mu_t, t \geq 0)$ can be constructed from a Poisson point process on $\mathbb{R}_+ \times \mathcal{C} \times \mathbb{N}$ with intensity $dt \otimes \mu_t \otimes \sharp$, where $\sharp$ is the counting measure on $\mathbb{N}$ in the same way as for time-homogeneous fragmentation.*

It is very easy to check that the proof of the homogeneous case applies here too. Of course, a Poissonian construction of a time-inhomogeneous i-fragmentation with no erosion is also possible with a Poisson measure on $\mathbb{R}_+ \times \mathcal{U} \times \mathbb{N}$ with intensity $dt \otimes \nu_t \otimes \sharp$. Concerning the law of the tagged fragment, if one defines $\sigma(t) = -\ln|B_1(t)|$, with B_1 the block containing the integer 1, we have now that $\sigma(t)$ is a process with independent increments. And so, if we denote $\zeta = \sup\{t > 0, \sigma_t < \infty\}$, then there exists a family of non-negative functions $(\phi_t, t \geq 0)$ such that

$$\forall q, t \geq 0,\ \mathbb{E}[\exp(-q\sigma_t), \zeta > t] = \exp(-\int_0^t \phi_u(q)du).$$

We call ϕ_t the instantaneous Laplace exponent of σ at time t and we have

$$\phi_t(q) = (c_{l,t} + c_{r,t})(q+1) + \int_{\mathcal{U}} (1 - \sum_{i=1}^{\infty} |U_i|^{q+1})\nu_t(dU),$$

where $(|U_i|)_{i\geq 0}$ is the sequence of the lengths of the component intervals of U. Furthermore, an $(\nu_t, c_t, c'_t)_{t\geq 0}$ i-fragmentation $(U(t), t \geq 0)$ is proper iff

$$\forall t > 0, \quad c_{l,t} = c_{r,t} = 0 \quad \text{ and } \quad \nu_t(\sum_i s_i < 1) = 0).$$

4.5 Extension to the self-similar case

A notion of self-similar fragmentations has been also introduced [5]. We recall here the definition of a self-similar p-fragmentation, the reader can easily adapt this definition to the three other instances of fragmentations.

Definition 4.6. *Let $\Pi = (\Pi(t), t \geq 0)$ be an exchangeable process on $\mathcal{P}$. We order the blocks of Π by their least elements. We call Π a self-similar p-fragmentation with index $\alpha \in \mathbb{R}$ if*

- $\Pi(0) = 1_{\mathbb{N}}$ *a.s.*
- Π *is continuous in probability*
- *For every $t \geq 0$, let $\Pi(t) = (\Pi_1, \Pi_2, \ldots)$ and denote by $|\Pi_i|$ the asymptotic frequency of the block Π_i. Then for every $s > 0$, the conditional distribution of $\Pi(t+s)$ given $\Pi(t)$ is the law of the random partition whose blocks are those of the partitions $\Pi^{(i)}(s_i) \cap \Pi_i$ for $i \in \mathbb{N}$, where $\Pi^{(1)}, \ldots$ is a sequence of independent copies of Π and $s_i = s|\Pi_i|^{\alpha}$.*

Notice that an homogeneous p-fragmentation corresponds to the case $\alpha = 0$.

We have still the same correspondence between the four types of fragmentation. In fact, a self-similar fragmentation can be constructed from a homogeneous fragmentation with a time change:

Proposition 4.7. *[5] Let $(U(t), t \geq 0)$ be an homogeneous interval fragmentation with dislocation measure ν. For $x \in]0,1[$, we denote by $I_x(t)$ the interval component of $U(t)$ containing x. We define*

$$T_t^{\alpha}(x) = \inf\{u \geq 0, \int_0^u |I_x(r)|^{-\alpha} dr > t\} \text{ and } U^{\alpha}(t) = U(T_t^{\alpha}) = \bigcup I_x(T_t^{\alpha}(x)).$$

Then $(U^{\alpha}(t), t \geq 0)$ is a self-similar interval fragmentation with index α.

A self-similar i-fragmentation (or c-fragmentation) is then characterized by a quadruple (ν, c_l, c_r, α) where ν is a dislocation measure on $\mathcal{U}$, c_l and c_r are two nonnegative numbers and $\alpha \in \mathbb{R}$ is the index of self-similarity.

5 Examples

5.1 Interval components in exchangeable random order

In this section we introduce the notion of random open set with interval components in exchangeable random order. In the next section, we will give an example of an i-fragmentation whose dislocation measure has its interval components in exchangeable random order.

Definition 5.1. *[13] Let $s \in \mathcal{S}$ such that $\sum s_i = 1$. Let $(V_i)_{i\in\mathbb{N}}$ be iid random variables uniform on $[0,1]$. We denote then U the random open subset of $]0,1[$ such that, if the decomposition of U in disjoint open intervals ranked by their length is $\coprod_{i=1}^{\infty} U_i$, we have*

- *For all $i \in \mathbb{N}$, $|U_i| = s_i$.*
- *For all $i, j \in \mathbb{N}$, $U_i \prec U_j \ \Leftrightarrow\ V_i < V_j$.*

Since we have $\sum_i s_i = 1$, there exists almost surely a unique open subset of $]0,1[$ fulfilling these two conditions. We denote by $\mathcal{Q}_s$ the distribution of U.
Let $\tilde{\nu}$ be a measure on $\mathcal{S}$ such that $\tilde{\nu}(\sum_i s_i < 1) = 0$. We denote by $\widehat{\nu}$ the measure on $\mathcal{U}$ defined by:

$$\widehat{\nu} = \int_{\mathcal{S}} \mathcal{Q}_s \tilde{\nu}(ds).$$

A measure on $\mathcal{U}$ which can be written in that form is said to have interval components in exchangeable random order.

Proposition 5.2. *Let $(U(t), t \geq 0)$ be an i-fragmentation with rate $(\nu, 0, 0)$ and such that for all $t \geq 0$, $U(t)$ has interval components in exchangeable random order. Then ν has also interval components in exchangeable random order.*

Proof. Let $(F(t), t \geq 0)$ be the projection of $(U(t), t \geq 0)$ on $\mathcal{S}$. We know that F is then an m-fragmentation with rate $(\tilde{\nu}, 0)$ where $\tilde{\nu}$ is the image of ν by the canonical projection $\mathcal{U} \to \mathcal{S}$. Let $\gamma \in \mathcal{C}_n$. Let $\pi \in \mathcal{P}_n$ be the image of γ by the canonical projection $\wp_1$ from $\mathcal{C}$ to $\mathcal{P}$. Let us now remark that we have

$$q_\gamma = \frac{1}{s} \lim_{s\to 0} \mathbb{P}(\Gamma_{[n]}(s) = \gamma) = \frac{1}{k!} q_\pi,$$

where k is the number of blocks of γ and q_π the jump rate of the p-fragmentation. Let $\widehat{\nu}$ be the measure on $\mathcal{U}$ obtained in Definition 5.1 from ν. Let us recall that $\mathcal{Q}_{\infty,\gamma} = \{\gamma' \in \mathcal{C}, \gamma'_{[n]} = \gamma\}$ and define also $\mathcal{P}_{\infty,\pi} = \{\pi' \in \mathcal{P}, \pi'_{[n]} = \pi\}$. We then have

$$P^{\widehat{\nu}}(\mathcal{Q}_{\infty,\gamma}) = \frac{1}{k!} P^{\tilde{\nu}}(\mathcal{P}_{\infty,\pi}) = \frac{1}{k!} q_\pi = q_\gamma = P^{\nu}(\mathcal{Q}_{\infty,\gamma}).$$

So we get that $\nu = \widehat{\nu}$ and hence ν has interval components in exchangeable random order. $\square$

Let us notice that the proof uses the identity $q_\gamma = \frac{1}{k!}q_\pi$, so if we want to extend this proposition to the time-inhomogeneous case, then we must suppose not only that $U(t)$ has interval components in exchangeable random order, but more generally that for all $s > t \geq 0$, the probability measure $q_{t,s}^{]0,1[}$ governing the transition probabilities of U from time t to time s (see Definition 3.4), has interval components in exchangeable random order.

Conversely, we may wonder: if $(U(t), t \geq 0)$ is an i-fragmentation with rate $(\nu, 0, 0)$ and ν has interval components in exchangeable random order, does this imply that $U(t)$ has interval components in exchangeable random order? The answer is clearly negative. Indeed, let ν be the following measure:

$$\nu = \delta_{U_1} + \delta_{U_2} \text{ with } {}^cU_1 = \left\{\frac{1}{3}, \frac{2}{3}\right\} \text{ and } {}^cU_2 = \left\{\frac{1}{2}\right\}.$$

Then ν has interval components in exchangeable random order, but $U(t)$ does not have this property since we have

$$\mathbb{P}\left({}^cU(t) = \left\{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right\}\right) > 0$$

and

$$\mathbb{P}\left({}^cU(t) = \left\{\frac{1}{6}, \frac{1}{2}, \frac{5}{6}\right\}\right) = 0.$$

5.2 Ruelle's fragmentation

In this section, we give the semi-group of Ruelle's fragmentation seen as an interval fragmentation. Let us recall the construction of this i-fragmentation [8].

Let $(\sigma_t^*, 0 < t < 1)$ be a family of stable subordinators such for every $0 < t_n < \ldots < t_1 < 1$, $(\sigma_{t_1}^*, \ldots, \sigma_{t_n}^*) \stackrel{law}{=} (\sigma_{t_1}, \ldots, \sigma_{t_n})$ where $\sigma_{t_i} = \tau_{\alpha_1} \circ \ldots \circ \tau_{\alpha_i}$ and $(\tau_{\alpha_i}, 1 \leq i \leq n)$ are n independent stable subordinators with indices $\alpha_1, \ldots, \alpha_n$ such that $t_i = \alpha_1 \ldots \alpha_i$. Fix $t_0 \in]0, 1[$ and for $t \in]t_0, 1[$ define T_t by:

$$\sigma_t^*(T_t) = \sigma_{t_0}^*(1).$$

Then consider the open set:

$$U(t) = \Big]0, 1\Big[\setminus \left\{\frac{\sigma_t^*(u)}{\sigma_{t_0}^*(1)}, 0 \leq u \leq T_t\right\}^{cl}.$$

Bertoin and Pitman [8] proved that $(U(t), t \in [t_0, 1[)$ is an i-fragmentation (with initial state $U(t_0) \neq]0, 1[$ a.s.) and the transition probabilities of $U(t)$ from time t to time s of the m-fragmentation $(U^\downarrow(t), t \in [t_0, 1[)$ is $PD(s, -t)$-FRAG where $PD(s, -t)$ denotes the Poisson-Dirichlet law with parameter $(s, -t)$ (see [18] for more details about the Poisson-Dirichlet laws). Moreover, the instantaneous dislocation measure of this m-fragmentation at time t is $\frac{1}{t}PD(t, -t)$ (cf. [2]). We would like now to calculate the dislocation measure of the i-fragmentation $(U(t), t \in [t_0, 1[)$.

Lemma 5.3. *Let us define $\widehat{PD}(t,0)$ as the measure on $\mathcal{U}$ obtained from $PD(t,0)$ by Definition 5.1. The distribution at time t of $U(t)$ is $\widehat{PD}(t,0)$.*

Proof. For $t \in]t_0, 1[$, we have $\sigma^*_{t_0} = \sigma^*_t \circ \tau_\alpha$ where $\alpha t = t_0$ and τ_α is a stable subordinator with index α and independent of σ^*_t. Hence we get

$$U(t) = \Big]0,1\Big[\Big\backslash \left\{ \frac{\sigma^*_t(u)}{\sigma^*_t(\tau_{\alpha(1)})}, 0 \le u \le \tau_\alpha(1) \right\}^{cl}.$$

We can thus write

$$U(t) = \Big]0,1\Big[\Big\backslash \left\{ \frac{\sigma_t(x)}{\sigma_t(a)}, x \in [0,a[\right\}^{cl},$$

where σ_t is a stable subordinator with index t and a is a random variable independent of σ_t. If we denote by $(t_i, s_i)_{i\ge 1}$ the time and size of the jump of σ_t in the interval $[0,a[$ ranked by decreasing order of the size of the jumps, this family has the same law of $(t_{\tau(i)}, s_i)_{i\ge1}$ for any τ permutation of $\mathbb{N}$. □

Proposition 5.4. *The semi-group of transition of Ruelle's interval fragmentation from time t to time s is $\widehat{PD}(s,-t)$-FRAG and the instantaneous dislocation measure at time t is $\frac{1}{t}\widehat{PD}(t,-t)$.*

Proof. We would like now to apply Proposition 5.2 to determine the instantaneous measure of dislocation of Ruelle's fragmentation, but this proposition holds only for time-homogeneous fragmentation. If the fragmentation is inhomogeneous in time, we must also check that, for all $s > t \ge 0$, the probability measure $q^{]0,1[}_{t,s}$ on $\mathcal{U}$ governing the transition probabilities of U from time t to time s (see Definition 3.4), has interval components in exchangeable random order. Fix $t \ge 0$ and $s > t$. Fix $y \in]0,1[$ and denote by $I(t)$ the interval component of $U(t)$ containing y. We shall prove that $U(s) \cap I(t)$ has its interval components in exchangeable random order. By the construction of $U(t)$, there exists $x \in]0, T_t[$ such that

$$I(t) = \Big] \frac{\sigma^*_t(x^-)}{\sigma^*_{t_0}(1)}, \frac{\sigma^*_t(x)}{\sigma^*_{t_0}(1)} \Big[.$$

We have $\sigma^*_t = \sigma^*_s \circ \tau_{t/s}$ where $\tau_{t/s}$ is a stable subordinator with index t/s and is independent of σ^*_{t+s}. Hence, we get:

$$U(s) \cap I(t) = I(t) \Big\backslash \left\{ \frac{\sigma^*_s(y)}{\sigma^*_{t_0}(1)},\ \tau_{t/s}(x^-) \le y \le \tau_{t/s}(x) \right\}^{cl}.$$

Since $\tau_{t/s}$ is independent of σ^*_s, the jump of σ^*_s on the interval $]\tau_{t/s}(x^-), \tau_{t/s}(x)[$ are in exchangeable random order. Since, as m-fragmentation, the semi-group of transition is $PD(s,-t)$-FRAG, we deduce that, as i-fragmentation, the semi-group is $\widehat{PD}(s,-t)$-FRAG. To prove that the dislocation measure at time t is $\frac{1}{t}\widehat{PD}(t,-t)$, we just have to apply the Proposition 5.2. □

We can also give the semi-group from time t to time s of the corresponding c-fragmentation. Indeed, the EPPF of a partition whose frequency law is a Poisson-Dirichlet law is well known (see [16, 17]), and since the blocks are in exchangeable random order, the semi-group from time t to time s of the c-fragmentation is $q_{t,s}$-FRAG$(\Gamma(t),\cdot)$ with

$$\forall n \in \mathbb{N}, \forall (A_1,\dots,A_k) \in \mathcal{C}_n,\ q_{t,s}(\Gamma_{[n]} = (A_1,\dots,A_k)) = \frac{[-t/s]_k}{k![-t]_n} \prod_{i=1}^{k} -[-s]_{n_i},$$

where $\sharp A_i = n_i$ and $[x]_n = \prod_{i=1}^{n}(x+i-1)$.

5.3 Dislocation measure of the Brownian fragmentation

We consider the m-fragmentation introduced by Aldous and Pitman [1] to study the standard additive coalescent. Bertoin [4] gave a construction of an i-fragmentation $(U(t), t \geq 0)$ whose projection on $\mathcal{S}$ is this fragmentation. More precisely, let $\varepsilon = (\varepsilon_s, s \in [0,1])$ be a standard positive Brownian excursion. For every $t \geq 0$, we consider

$$\varepsilon_s^{(t)} = ts - \varepsilon_s, \qquad S_s^{(t)} = \sup_{0\leq u \leq s} \varepsilon_u^{(t)}.$$

We define $U(t)$ as the constancy intervals of $(S_s^{(t)}, 0 \leq s \leq 1)$. Bertoin [5] proved also that $(U^{\downarrow}(t), t \geq 0)$ is an m-fragmentation with index of self-similarity $1/2$, with no erosion and its dislocation measure is carried by the subset of sequences

$$\{s = (s_1, s_2, \dots) \in \mathcal{S}, s_1 = 1 - s_2 \text{ and } s_i = 0 \text{ for } i \geq 3\}$$

and is given by

$$\tilde{\nu}_{AP}(s_1 \in dx) = \left(2\pi x^3(1-x)^3\right)^{-1/2} dx \quad \text{ for } x \geq 1/2.$$

Proposition 5.5. *The i-fragmentation derived from a Brownian motion [4] has dislocation measure ν_{AP} such that:*

- *ν_{AP} is supported by the sets of the form $]0, X[\cup]X, 1[$, so we shall identify each such set with X and write $\nu_{AP}(dx)$ for its distribution.*
- *For all $x \in]0,1[$, $\quad \nu_{AP}(dx) = (2\pi x(1-x^3))^{-1/2} dx.$*

Notice that we have $\nu_{AP}(dx) = x\tilde{\nu}_{AP}(s_1 \in dx \text{ or } s_2 \in dx)$ for all $x \in]0,1[$. Hence, given that the m-fragmentation splits in two blocks of size x and $1-x$, the left block of the i-fragmentation will be a size-biased pick from $\{x, 1-x\}$.

Proof. The first part of the proposition is straight forward since we have $\tilde{\nu}_{AP}(s_1 = 1 - s_2) = 1$. For the second part, let us use Theorem 9 in [4] which gives the distribution ρ_t of the leftmost fragment of $U(t)$:

$$\rho_t(dx) = t\frac{1}{\sqrt{2\pi x(1-x)^3}} \exp\left(-\frac{xt^2}{2(1-x)}\right) dx \quad \text{ for all } x \in]0,1[.$$

We get

$$\nu_{AP}(dx) = \lim_{t\to 0} \frac{1}{t}\rho_t(dx) = \frac{dx}{\sqrt{2\pi x(1-x)^3}}.$$

□

We can also give a description of the distribution at time $t > 0$ of $U(t)$. Recall the result obtained by Chassaing and Janson [10]. For a random process X on $\mathbb{R}$ and $t \geq 0$, we define $\ell_t(X)$ as the local time of X at level 0 on the interval $[0,t]$, i.e.

$$\ell_t(X) = \lim_{\varepsilon \to 0+} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{\{|X_s| < \varepsilon\}} ds,$$

whenever the limit makes sense.

Let X^t be a reflected Brownian bridge conditioned on $\ell_1(X^t) = t$. We define $\theta \in]0,1[$ such that

$$\ell_\theta(X^t) - t\theta = \max_{0 \leq u \leq 1} \ell_u(X^t) - tu.$$

It is well known that this equation has almost surely a unique solution. Let us define the process $(Z^t(s), 0 \leq s \leq 1)$ by

$$Z^t(s) = X^t(s + \theta \,[\mathrm{mod}\ 1]).$$

Chassaing and Janson [10] have proved that for each $t \geq 0$

$$U(t) \overset{law}{=}]0,1[\setminus \{x \in [0,1], Z^t(x) = 0\}.$$

Besides, as the inverse of the local time of X^t defined by

$$T_x = \inf\{u \geq 0,\ \ell_u(X^t) > x\}$$

is a stable subordinator with Lévy measure $(2\pi x^3)^{-1/2} dx$ conditioned to $T_t = 1$, we deduce the following description of the distribution of $U(t)$:

Corollary 5.6. *Let $t > 0$. Let T be a stable subordinator with Lévy measure $(2\pi x^3)^{-1/2} dx$ conditioned to $T_t = 1$. Let us define m as the unique real number in $[0,t]$ such that*

$$tT_{m^-} - m \leq tT_u - u \quad \text{for all } u \in [0,t],$$

where $T_{m^-} = \lim_{x \to m^-} T_x$. We set:

$$\begin{aligned} \tilde{T}_x &= T_{m+x} - T_{m^-} && \text{for } 0 < x < t - m, \\ & \quad T_{m+x-t} - T_{m^-} + 1 && \text{for } t - m \leq x \leq t. \end{aligned}$$

Then

$$U(t) \overset{law}{=}]0,1[\setminus \{\tilde{T}_x, x \in [0,t]\}^{cl}.$$

Proof. It is clear that $\{u, X^t(u) = 0\}$ coincides with $\{T_x, x \in [0,t]\}^{cl}$ when T is the inverse of the local time of X^t. Hence, we just have to check that if we set $m = \ell_\theta(X^t)$, then m verifies the equation $tT_{m^-} - m \leq tT_u - u$ for all $u \in [0,t]$. Since $X^t(\theta) = 0$, we have $T_{m^-} = \theta$, thus we get:

$$tT_{m^-} - m = t\theta - \ell_\theta(X^t) \leq tv - \ell_v(X^t) \text{ for all } v \in [0,1].$$

Let us fix $u \in [0,t]$. Since $\ell_v(X^t)$ is a continuous function, there exists $v \in [0,1]$ such that $\ell_v(X^t) = u$. Besides we have $T_u^- \leq v \leq T_u$, so we get

$$tT_{m^-} - m \leq tT_u - u.$$

□

Hence, the distribution of $[0,1] \setminus U(t)$ can be obtained as the closure of the shifted range of a stable subordinator $(T_s, 0 \leq s \leq t)$ with index 1/2 and conditioned on $T_t = 1$ (recall also that Chassaing and Janson [10] have proved that the leftmost fragment of $U(t)$ is size-biased picked).

Remark 5.7. *There exists another way to construct an i-fragmentation from a Brownian excursion [5]. Let $\varepsilon = (\varepsilon(r), 0 \leq r \leq 1)$ be a Brownian excursion with unit duration. We consider $U(t) = \{r \in]0,1[, \varepsilon(r) > t\}$. Bertoin has proved that the process $(U(t), t \geq 0)$ is an i-fragmentation whose rate as m-fragmentation is $(0, 2\tilde{\nu}_{AP})$ and index of self-similarity $\alpha = -1/2$. Let us define the open set $V(t) = \{x \in]0,1[, (1-x) \in U(t)\}$. Since $(\varepsilon(1-r), 0 \leq r \leq 1)$ has also the law of a Brownian excursion with unit duration, we deduce that $(V^{\downarrow}(t), t \geq 0)$ is also an m-fragmentation with the same characteristics as $(U^{\downarrow}(t), t \geq 0)$. Besides, if ν_ε (resp. $\nu_{\varepsilon'}$)denotes the dislocation measure of the i-fragmentation U (resp. V), we must have $\nu_\varepsilon(dx) = \nu_{\varepsilon'}(1-dx)$ (recall that since $\tilde{\nu}_{AP}$ is binary, we write $\nu_\varepsilon(dx)$ to denote the distribution of $]0,x[\cup]x,1[$). Hence, we deduce that $\nu_\varepsilon(dx) = \nu_\varepsilon(1-dx)$ and using that $2\tilde{\nu}_{AP}(s_1 \in dx) = \nu_\varepsilon(dx) + \nu_\varepsilon(1-dx)$ for $x \in]1/2,1[$, we get*

$$\nu_\varepsilon(dx) = \frac{1}{\sqrt{2\pi x^3(1-x)^3}}dx \quad \textit{for } x \in]0,1[,$$

and ν_ε has interval components in exchangeable random order.

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