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# On the number of collisions in $\Lambda$-coalescents*

Alexander Gnedin[†] Yuri Yakubovich[‡]

**Abstract**

We examine the total number of collisions $C_n$ in the $\Lambda$-coalescent process which starts with $n$ particles. A linear growth and a stable limit law for $C_n$ are shown under the assumption of a power-like behaviour of the measure $\Lambda$ near 0 with exponent $0 < \alpha < 1$.

**Key words:** $\Lambda$-coalescent, stable laws.

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[†]Postal address: Department of Mathematics, Utrecht University, Postbus 80010, 3508 TA Utrecht, The Netherlands. E-mail address: gnedin@math.uu.nl

[‡]Postal address: St.Petersburg Department of Steklov Mathematical Institute, Fontanka 27, St. Petersburg 191023, Russia. E-mail address: yuyakub@gmail.com.

# 1 Introduction

A system of particles undergoes a random Markovian evolution according to the rules of the $\Lambda-$coalescent (introduced by Pitman [16] and Sagitov [17]) if the only possible type of interaction is a *collision* affecting two or more particles that merge together to form a single particle. When the total number of particles is $b \geq 2$, a collision affecting some $2 \leq j \leq b$ particles occurs at rate

$$\lambda_{b,j} = \binom{b}{j} \int_0^1 x^{j-2}(1-x)^{b-j}\Lambda(dx), \tag{1}$$

where $\Lambda$ is a given finite measure on $[0,1]$. Linear time change allows to rescale $\Lambda$ by its total mass, making it a probability measure, which is always supposed below. Two important special cases are Kingman's coalescent [13] with $\Lambda$ a unit mass at 0 (when only binary collisions are possible), and the Bolthausen–Sznitman coalescent [6] with $\Lambda$ the Lebesgue measure on $[0,1]$. See [1; 2; 10; 15] for recent work on the $\Lambda$-coalescents and further references.

A quantity of considerable interest is the number of collisions $C_n$ which occur as the system progresses from the initial state with $n$ particles to the terminal state with a single particle. Representing the coalescent process by a genealogical tree, $C_n$ can be also understood as the number of non-leave nodes. Asymptotic properties of $C_n$ are sensitive functions of the behaviour of $\Lambda$ near 0. In this paper we explore the class of measures which satisfy

$$\Lambda([0,x]) = Ax^\alpha + O(x^{\alpha+\varsigma}) \quad \text{as} \quad x \downarrow 0, \quad \text{with} \quad 0 < \alpha < 1 \quad \text{and} \quad \varsigma > 0. \tag{2}$$

Under this assumption we show that $C_n \sim (1-\alpha)\, n$ in probability as $n \to \infty$ (Lemma 4) and that the law of $C_n$ approaches a completely asymmetric stable distribution of index $2-\alpha$ (Theorem 7).

The same question for the Bolthausen–Sznitman coalescent has been addressed recently in [8; 9]. This can be viewed as a limiting case of (2) with $\alpha = 1$. However, the technique of [8; 9] is based on the particular form of $\Lambda$ in that case and hence cannot be applied to the general $\Lambda$ satisfying (2) with $\alpha = 1$.

If $\Lambda$ is the beta$(\alpha, 2-\alpha)$ distribution with parameter $0 < \alpha < 2$, a time-reversal of the coalescent describes the genealogy of a continuous-state branching process [5]. This connection was exploited recently to study a small-time behaviour of $\Lambda$-coalescents [1; 2] in the beta case.

The same stable law (as in our Theorem 7) has been derived also in [7] under different assumptions[1]. The technique used in [7] was based on the martingale theory and certain advanced estimates.

We develop here a more robust and straightforward approach based on the analysis of the decreasing Markov chain $\mathcal{M}_n$ counting the number of particles. The number of collisions $C_n$ is the number of steps needed for $\mathcal{M}_n$ to reach the absorbing state 1 from state $n$. In Kingman's case $\mathcal{M}_n$ has unit decrements, but in general the decrements of $\mathcal{M}_n$ are not stationary, which is a major source of difficulties preventing direct application of the classical renewal theorems for step distributions with infinite variance [11]. To override this obstacle we show that when (2) holds, in a certain range $\mathcal{M}_n$ can be bounded from above and below by processes with stationary decrements. This allows us to approximate $C_n$, and it happens that these bounds can be made

[1] Parameter $\alpha$ in the present paper corresponds to $2-\alpha$ in [1; 2; 7].

tight enough to derive the limit theorem. Our method may be of interest in the wider context of pure death processes.

By Schweinsberg's result [19] a coalescent satisfying (2) comes down from the infinity, hence the number of particles coexisting at a fixed time is uniformly bounded in the initial number of particles $n$. Therefore the asymptotics of the number of collisions that occur prior to some fixed time is the same as the asymptotics of $C_n$.

## 2 Markov chain $\mathcal{M}_n$

Let $\mathcal{M}_n$ be the embedded discrete-time Markov chain whose state coincides with the number of remaining particles. Since no two collisions occur simultaneously, the number of collisions $C_n$ in the $\Lambda$-coalescent starting with $n$ particles is the number of steps the Markov chain $\mathcal{M}_n$ needs to proceed from the initial state $n$ to the terminal state 1. Note that the number of particles decreases by $j-1$ when a collision affects $j$ particles, hence the probability of transition from $b$ particles to $b-j+1$ is

$$q_b(j) := \frac{\lambda_{b,j}}{\lambda_b}, \qquad 2 \le j \le b, \tag{3}$$

where $\lambda_b$ is the total collision rate of $b$ particles

$$\lambda_b = \sum_{j=2}^{b} \lambda_{b,j} = \int_0^1 \frac{1-(1-x)^b - bx(1-x)^{b-1}}{x^2} \Lambda(dx). \tag{4}$$

It is convenient to introduce the sequence of moments

$$\nu_b := \int_0^1 (1-x)^b \Lambda(dx), \qquad b = 0, 1, \ldots \tag{5}$$

In view of $\lambda_{b,2} = \binom{b}{2}\nu_{b-2}$ the rates $\lambda_{b,2}$ $(b = 2, 3, \ldots)$ uniquely determine the whole array $\lambda_{b,j}$, as one can also conclude from the consistency relation

$$(b+1)\lambda_{b,j} = (b+1-j)\lambda_{b+1,j} + (j+1)\lambda_{b+1,j+1},$$

which is equivalent to the integral representation of rates (1), see [16].

Simple computation shows that the rates can be derived from $\nu_b$'s as

$$\lambda_{b,j} = \binom{b}{j} \sum_{s=0}^{j-2} (-1)^{j-s} \binom{j-2}{s} \nu_{b-2-s}, \tag{6}$$

and, from $\lambda_{b+1} - \lambda_b = b\nu_{b-1}$, we have

$$\lambda_b = \sum_{i=1}^{b-1} i\nu_{i-1}. \tag{7}$$

The second order difference (in the sense of the finite difference calculus) of $\int_0^1 \frac{bx+(1-x)^b-1}{x^2}\Lambda(dx)$ is $\nu_b$, consequently

$$\sum_{j=2}^{b} (j-1)\lambda_{b,j} = \int_0^1 \frac{bx+(1-x)^b-1}{x^2}\Lambda(dx) = \sum_{i=1}^{b-1} (b-i)\nu_{i-1}. \tag{8}$$

We shall denote $J_b$ a random variable with distribution

$$\mathbb{P}(J_b = j) = q_b(j),$$

so the first decrement of $\mathcal{M}_n$ is distributed as $J_n - 1$. From (7) and (8) its mean value is

$$\mathbb{E}\left[J_n - 1\right] = \sum_{j=2}^{n}(j-1)q_n(j) = \frac{\sum_{i=1}^{n-1}(n-i)\nu_{i-1}}{\sum_{i=1}^{n-1} i\nu_{i-1}} = n\frac{\sum_{i=1}^{n-1}\nu_{i-1}}{\sum_{i=1}^{n-1} i\nu_{i-1}} - 1. \tag{9}$$

# 3 Asymptotics of the moments

From now on we only consider measures $\Lambda$ satisfying (2). Standard Tauberian arguments (see [12, Ch. XIII.5] or [4, Section 1.7.2]) show that in this case

$$\nu_n = A\Gamma(\alpha+1)n^{-\alpha} + O(n^{-\alpha-\varsigma'}), \qquad n \to \infty\,. \tag{10}$$

Here and henceforth

$$\varsigma' = \min\{1, \varsigma\}\,.$$

This behaviour will imply that the transition probabilities $q_n(j)$ stabilise as $n \to \infty$ for each fixed $j$. The relevant asymptotics of $\lambda_n$ and $\lambda_{n,j}$ appeared in [3, Lemma 4] under a less restrictive assumption of regular variation, but we need to explicitly control the error term.

**Lemma 1.** *Suppose $\Lambda$ satisfies (2). Then for $n$ sufficiently large*

$$\left|\sum_{j=m}^{n}\lambda_{n,j} - \frac{A\alpha}{2-\alpha}\,\frac{\Gamma(m+\alpha-2)}{\Gamma(m)}\,n^{2-\alpha}\right| < c\,\frac{\Gamma(m+\alpha+\varsigma'-2)}{\Gamma(m)}\,n^{2-\alpha-\varsigma'}\,,$$

*uniformly in $m = 2, \ldots, n$.*

*Proof.* Introduce the truncated moment

$$G_{-2}(x) = \int_x^1 \frac{\Lambda(dy)}{y^2}\,.$$

Integrating by parts we derive from (2) that for $x \to 0$

$$G_{-2}(x) = \frac{A\alpha}{2-\alpha}\,x^{\alpha-2} + O(\max\{x^{\alpha+\varsigma-2}, 1\})$$

Rewriting (1) in terms of $G_{-2}$ and integrating by parts we obtain

$$\sum_{j=m}^{n}\lambda_{n,j} = -\int_0^1 \sum_{j=m}^{n}\binom{n}{j}x^j(1-x)^{n-j}dG_{-2}(x) = m\binom{n}{m}\int_0^1 x^{m-1}(1-x)^{n-m}G_{-2}(x)\,dx\,,$$

because the sum telescopes and the integrated terms vanish. Plugging the expansion of $G_{-2}$ gives

$$\left|\sum_{j=m}^{n}\lambda_{n,j} - \frac{A\alpha}{2-\alpha}\,\frac{\Gamma(m+\alpha-2)\Gamma(n+1)}{\Gamma(m)\Gamma(n+\alpha-1)}\right| < c\,\frac{\Gamma(m+\alpha+\varsigma-2)\Gamma(n+1)}{\Gamma(m)\Gamma(n+\alpha+\varsigma-1)}$$

for $\varsigma < 2-\alpha$, in which case the result follows from the familiar asymptotics of the gamma function $\Gamma(n+\beta)/\Gamma(n) = n^\beta + O(n^{\beta-1})$ $(n \to \infty)$. If $\varsigma > 1$ the error term in this expansion constitutes the main part of the error, yielding the appearance of $\varsigma'$ instead of $\varsigma$. The case $\varsigma \geq 2-\alpha$ is treated in the same way. $\square$

**Corollary 2.** *If measure* $\Lambda$ *satisfies* (2) *then as* $n \to \infty$

$$\begin{aligned} \lambda_n &= \frac{A\Gamma(\alpha+1)}{2-\alpha}\, n^{2-\alpha} + O\left(n^{2-\alpha-\varsigma'}\right), \\ \lambda_{n,j} &= \frac{A\alpha\Gamma(j+\alpha-2)}{j!}\, n^{2-\alpha} + O\left(n^{2-\alpha-\varsigma'}\right), \\ q_n(j) &= (2-\alpha)\frac{(\alpha)_{j-2}}{j!} + O\left(n^{-\varsigma'}\right) \end{aligned} \tag{11}$$

*for every fixed* $j$*.*

*Proof.* The formula for $\lambda_n$ follows by the direct application of Lemma 1 with $m=2$. Expression for $\lambda_{n,j}$ is a difference between two subsequent tail sums. The ratio of these quantities gives $q_n(j)$. $\square$

Thus $J_n$ converge in distribution. The convergence in mean is also true. Note that the mean of the limiting distribution of jumps $J_b - 1$ is

$$\sum_{j=2}^{\infty}(j-1)(2-\alpha)\frac{(\alpha)_{j-2}}{j!} = \frac{1}{1-\alpha}\,. \tag{12}$$

**Lemma 3.** *If* (2) *holds then the mean decrease of the number of particles after collision satisfies*

$$\mathbb{E}\left[J_n - 1\right] = \frac{1}{1-\alpha} + O\left(n^{-\min\{1-\alpha,\,\varsigma\}}\right).$$

*Proof.* By assumption (2) relation (10) implies the existence of constants $n_0, c$ such that

$$\left|\nu_{n-1} - A\Gamma(\alpha+1)n^{-\alpha}\right| < cn^{-\alpha-\varsigma'}, \qquad n \geq n_0\,.$$

Approximating sums by integrals yields, as $n \to \infty$,

$$\begin{aligned} \sum_{i=1}^{n-1}\nu_{i-1} &= \sum_{i=n_0}^{n-1} A\Gamma(\alpha+1)i^{-\alpha} + O\left(\sum_{i=n_0}^{n-1} i^{-\alpha-\varsigma'}\right) + \sum_{i=1}^{n_0-1}\nu_{i-1} \\ &= A\Gamma(\alpha+1)n^{1-\alpha}(1+O(1/n))\int_{n_0/n}^{1} x^{-\alpha}dx + O(n^{1-\alpha-\varsigma'}) + O(1) \\ &= \frac{A\Gamma(\alpha+1)}{1-\alpha}n^{1-\alpha} + O(\max\{1, n^{1-\alpha-\varsigma'}\}) = \frac{A\Gamma(\alpha+1)}{1-\alpha}n^{1-\alpha} + O(\max\{1, n^{1-\alpha-\varsigma}\}) \end{aligned}$$

by definition of $\varsigma'$. Substitution of this expression into (9) and applying Corollary 2 finishes the proof. $\square$

**Example.** It is possible to choose a measure $\Lambda$ so that the decrement probabilities for $j < n$ are exactly the same as for the limiting distribution truncated at $n$, in which case the envisaged limit theorem for $C_n$ follows readily from [11]. To achieve

$$q_n(j) = (2-\alpha)\frac{(\alpha)_{j-2}}{j!} \quad (j = 2, \ldots, n-1), \quad q_n(n) = \sum_{j=n}^{\infty}(2-\alpha)\frac{(\alpha)_{j-2}}{j!} = \frac{\Gamma(n+\alpha-1)}{n!\,\Gamma(\alpha)}$$

one should take the measure

$$\Lambda(dx) = \alpha\left(1 - \frac{\alpha}{2}\right)x^{\alpha-1}dx + \frac{\alpha}{2}\boldsymbol{\delta}_1(dx),$$

which is a mixture of beta$(\alpha, 1)$ and a Dirac mass at 1. Adding $\boldsymbol{\delta}_1$ does not affect $\lambda_{n,j}$ for $j < n$, so the integration in (1) yields

$$\lambda_{n,j} = \binom{n}{j}\left(1 - \frac{\alpha}{2}\right)\frac{\Gamma(j+\alpha-2)(n-j)!}{\Gamma(n+\alpha-1)} \quad (j = 2, \ldots, n-1), \quad \lambda_{n,n} = \left(1 - \frac{\alpha}{2}\right)\frac{\alpha}{n+\alpha-2} + \frac{\alpha}{2}\,.$$

Summation (or direct integration of (4)) implies the desired expression for $q_n(j)$.

That a positive mass at 1 does not affect the asymptotics of $C_n$ is seen e.g. by observing that the probability of total collision implied by this mass is of the order smaller than $n^{-1}$, namely $q_n(n) = O(n^{\alpha-2})$. On the continuous time scale of the coalescent, the mass at 1 is responsible for the total coalescence time (independent of $n$), hence the insensibility of the asymptotics to $\Lambda(\{1\})$ may be explained by the effect of coming down from the infinity, as mentioned in Introduction.

The example also demonstrates that taking minimum in the error term of Lemma 3 is necessary. Indeed, $\varsigma' = 1$, however direct calculation using (12) shows that

$$\mathbb{E}\left[J_n - 1\right] = \frac{1}{1-\alpha} - \sum_{j=n}^{\infty}(j-1)(2-\alpha)\frac{(\alpha)_{j-2}}{j!} + n\,q_n(n) = \frac{1}{1-\alpha} - \frac{n^{\alpha-1}}{(1-\alpha)\Gamma(\alpha)}(1 + O(1/n))\,.$$

So the error term is $O(n^{\alpha-1})$, and not $O(n^{-1})$.

The fact that the jumps $J_b - 1$ become almost identically distributed for large $b$, with the mean close to $1/(1-\alpha)$, suggests that $C_n$ satisfies the same law of large numbers as it were the case for the $J_b$'s identically distributed. We state this now in the following lemma but postpone a rigorous proof to Section 5.

**Lemma 4.** *If* (2) *holds then*

$$C_n \sim (1-\alpha)n \quad (n \to \infty)$$

*in probability.*

# 4 Stochastic bounds on the jumps

In this section we construct stochastic bounds on the decrements $J_b - 1$ of Markov chain $\mathcal{M}_n$ in a range $b = k, \ldots, n$, to control the asymptotic behaviour of $C_n$. Specifically, we find random variables $J^+_{n\downarrow k}$ and $J^-_{n\downarrow k}$ to secure the distributional bounds

$$J^+_{n\downarrow k} \leq_d J_b \leq_d J^-_{n\downarrow k}\,. \tag{13}$$

which hold for all $b$ in some range $b = k, \dots, n$. Here $\le_d$ denotes the stochastic order, meaning that two random variables $X$ and $Y$ satisfy $X \le_d Y$ if and only if $\mathbb{P}[X \le t] \ge \mathbb{P}[Y \le t]$ for all $t$.

Our approach to establishing the limit theorem for the number of collisions is based on constructing random variables $J^+_{n\downarrow k}$ and $J^-_{n\downarrow k}$ which on the one hand comply with (13) and on the other hand yield the same limit distribution of the sum of their independent copies. These two requirements point in opposite directions, forcing an adequate choice of these random variables to be a compromise. We define the distributions which depend on parameters $\gamma, \beta \in ]0,1[$ and $\theta \in ]\beta, 1[$. The calibration of these constants will be done later. For $n \ge k > 0$ define

$$q^-_{n\downarrow k}(j) := \begin{cases} \frac{\lambda_{n,2} - n^{-\gamma}\lambda_n\left(3:\lfloor n^\beta\rfloor\right) + \lambda_n\left(\lfloor n^\beta\rfloor + 1:n\right)}{\lambda_n} - 2\max\limits_{\ell\in\{k,\dots,n\}} \frac{\lambda_\ell\left(\lfloor n^\beta\rfloor+1:\ell\right)}{\lambda_\ell}, & j = 2, \\ \frac{\lambda_{n,j}(1+n^{-\gamma})}{\lambda_n}, & j = 3, \dots, \lfloor n^\beta \rfloor, \\ 2\max\limits_{\ell\in\{k,\dots,n\}} \frac{\lambda_\ell\left(\lfloor n^\beta\rfloor+1:\ell\right)}{\lambda_\ell} - 2\max\limits_{\ell\in\{k,\dots,n\}} \frac{\lambda_\ell\left(\lfloor n^\theta\rfloor+1:\ell\right)}{\lambda_\ell}, & j = \lfloor n^\theta \rfloor, \\ 2\max\limits_{\ell\in\{k,\dots,n\}} \frac{\lambda_\ell\left(\lfloor n^\theta\rfloor+1:\ell\right)}{\lambda_\ell}, & j = n, \\ 0, & \text{otherwise}, \end{cases} \tag{14}$$

$$q^+_{n\downarrow k}(j) := \begin{cases} \frac{\lambda_{k,2} + n^{-\gamma}\lambda_k\left(3:\lfloor n^\beta\rfloor\right) + \lambda_k\left(\lfloor n^\beta\rfloor + 1:k\right)}{\lambda_k}, & j = 2, \\ \frac{\lambda_{k,j}(1-n^{-\gamma})}{\lambda_k}, & j = 3, \dots, \lfloor n^\beta \rfloor, \\ 0, & \text{otherwise}, \end{cases}$$

where

$$\lambda_n(m:k) = \sum_{j=m}^{k} \lambda_{n,j}\,.$$

Note that $\sum_j q^+_{n\downarrow k}(j) = \sum_j q^-_{n\downarrow k}(j) = 1$. Moreover, $q^\pm_{n\downarrow k}(j)$ are nonnegative for large enough $n$ and $k$. Indeed, the inequality $q^+_{n\downarrow k}(j) \ge 0$ is obvious. Lemma 1 implies that if $n$ and $k$ are large enough and $k > n^\beta$ then

$$\frac{c_1}{n^{\beta(2-\alpha)}} \le \frac{\lambda_\ell\left(\lfloor n^\beta\rfloor + 1 : \ell\right)}{\lambda_\ell} \le \frac{c_2}{n^{\beta(2-\alpha)}} \tag{15}$$

for some $c_2 > c_1 > 0$ uniformly in $\ell \in \{k, \dots, n\}$. Hence (15) holds for the maximum over these $\ell$, and it follows that $q^-_{n\downarrow k}(j) \ge 0$.

Hence, quantities $q^\pm_{n\downarrow k}(j)$ define some probability distributions on $\mathbb{N}$, at least for large enough $n$ and $k$. Let $J^+_{n\downarrow k}$ and $J^-_{n\downarrow k}$ be random variables with these distributions, so

$$\mathbb{P}[J^+_{n\downarrow k} = j] = q^+_{n\downarrow k}(j) \qquad \text{and} \qquad \mathbb{P}[J^-_{n\downarrow k} = j] = q^-_{n\downarrow k}(j). \tag{16}$$

*Remark.* Formulas for $q^-_{n\downarrow k}$ look more cumbersome than that for $q^+_{n\downarrow k}$. The reason for it is our desire to control the mean of $J^-_{n\downarrow k}$. A simpler choice is just to move all extra mass to its maximal value $J^-_{n\downarrow k} = n$ but it would lead to a relatively big mass at that point and affect the mean too strong for our purposes. Introduction of an intermediate mass at $\lfloor n^\theta \rfloor$ helps to avoid this obstacle.

**Lemma 5.** *Suppose that $\beta$, $\gamma$, $\theta$ and $\upsilon$ satisfy the inequalities*

$$1 > \upsilon > \theta > \beta > \gamma/(2-\alpha) > 0 \qquad \textit{and} \qquad \gamma < \frac{(\upsilon-\beta)(2-\alpha)\varsigma'}{2-\alpha-\varsigma'}\,. \tag{17}$$

*Then the stochastic bounds* (13) *hold for $n$ large enough and $b, k$ in the range $\lfloor n^{\upsilon} \rfloor \le k \le b \le n$.*

*Proof.* By definition of the stochastic order, we need to show that for all $m$

$$\mathbb{P}\left[J^{+}_{n\downarrow k} \ge m\right] \le \mathbb{P}\left[J_b \ge m\right] \le \mathbb{P}\left[J^{-}_{n\downarrow k} \ge m\right]\,. \tag{18}$$

The first inequality (18) is clearly true for $m \ge \lfloor n^{\beta} \rfloor + 1$ because the left-hand side is zero and for $m \le 2$ because both sides are 1. Suppose $3 \le m \le n^{\beta}$, then the first inequality reads as

$$\lambda_b \lambda_k \left(m : \lfloor n^{\beta} \rfloor\right)\left(1 - n^{-\gamma}\right) \le \lambda_k \lambda_b (m : b)\,. \tag{19}$$

Since $b \ge k \ge \lfloor n^{\upsilon} \rfloor$, taking $n$ sufficiently large enables us to apply Lemma 1 and Corollary 2 to get asymptotic estimates valid for all $b$ in the range $k \le b \le n$. From the definition of $\lambda_k(m : \lfloor n^{\beta} \rfloor)$, Lemma 1 and the inequality

$$\frac{\Gamma(m+\alpha+\varsigma'-2)}{\Gamma(m)} \ge \frac{\Gamma(\lfloor n^{\beta} \rfloor + \alpha + \varsigma' - 2)}{\Gamma(\lfloor n^{\beta} \rfloor)} \quad \text{for } \; m \le \left\lfloor n^{\beta} \right\rfloor$$

(which follows from the log-convexity of the Gamma function) we obtain

$$\begin{aligned}
\lambda_k\left(m : \left\lfloor n^{\beta} \right\rfloor\right) &= \lambda_k\left(m : k\right) - \lambda_k\left(\left\lfloor n^{\beta} \right\rfloor + 1 : k\right) \\
&= \frac{A\alpha}{2-\alpha}\left(\frac{\Gamma(m+\alpha-2)}{\Gamma(m)} - \frac{\Gamma(\lfloor n^{\beta} \rfloor + \alpha - 1)}{\Gamma(\lfloor n^{\beta} \rfloor + 1)}\right) k^{2-\alpha} + O\left(\frac{\Gamma(m+\alpha+\varsigma'-2)}{\Gamma(m)} k^{2-\alpha-\varsigma'}\right).
\end{aligned}$$

Hence we rewrite the inequality as

$$\begin{aligned}
&\left(\frac{A\alpha}{2-\alpha}\left(\frac{\Gamma(m+\alpha-2)}{\Gamma(m)} - \frac{\Gamma(\lfloor n^{\beta} \rfloor + \alpha - 1)}{\Gamma(\lfloor n^{\beta} \rfloor + 1)}\right) k^{2-\alpha} + O\left(\frac{\Gamma(m+\alpha+\varsigma'-2)}{\Gamma(m)} k^{2-\alpha-\varsigma'}\right)\right) \\
&\qquad\qquad \times \left(\frac{A\Gamma(\alpha+1)}{2-\alpha} b^{2-\alpha} + O\left(b^{2-\alpha-\varsigma'}\right)\right)\left(1 - n^{-\gamma}\right) \\
&\qquad \le \left(\frac{A\Gamma(\alpha+1)}{2-\alpha} k^{2-\alpha} + O\left(k^{2-\alpha-\varsigma'}\right)\right) \\
&\qquad\qquad \times \left(\frac{A\alpha}{2-\alpha}\,\frac{\Gamma(m+\alpha-2)}{\Gamma(m)} b^{2-\alpha} + O\left(\frac{\Gamma(m+\alpha+\varsigma'-2)}{\Gamma(m)} b^{2-\alpha-\varsigma'}\right)\right).
\end{aligned}$$

The leading terms on both sides cancel, and simplifying this inequality we are reduced to checking

$$\begin{aligned}
&O\left(\frac{\Gamma(m+\alpha+\varsigma'-2)}{\Gamma(m)} k^{2-\alpha-\varsigma'} b^{2-\alpha}\right) + O\left(\frac{\Gamma(m+\alpha-2)}{\Gamma(m)} b^{2-\alpha-\varsigma'} k^{2-\alpha}\right) \\
&\qquad \le \frac{A^2\alpha\Gamma(\alpha+1)}{(2-\alpha)^2}\left(\frac{\Gamma(\lfloor n^{\beta} \rfloor + \alpha - 1)}{\Gamma(\lfloor n^{\beta} \rfloor + 1)} b^{2-\alpha} k^{2-\alpha} + \frac{\Gamma(m+\alpha-2)}{\Gamma(m)} b^{2-\alpha} k^{2-\alpha} n^{-\gamma}\right).
\end{aligned}$$

(In the above lines we neglected certain lower order terms using inequalities like $b^{2-\alpha}k^{2-\alpha-\varsigma'} \geq b^{2-\alpha-\varsigma'}k^{2-\alpha}$.) The desired inequality follows from the following two inequalities

$$c_1 \leq \frac{c_3\Gamma(m)}{n^{\beta(2-\alpha)}\Gamma(m+\alpha+\varsigma'-2)}k^{\varsigma'} + \frac{\Gamma(m+\alpha-2)}{\Gamma(m+\alpha+\varsigma'-2)}k^{\varsigma'}n^{-\gamma}\,, \tag{20}$$

$$c_2 \leq \frac{c_3\Gamma(m)}{n^{\beta(2-\alpha)}\Gamma(m+\alpha-2)}k^{\varsigma'} + k^{\varsigma'}n^{-\gamma}\,, \tag{21}$$

with sufficiently large constants $c_1, c_2 > 0$ (twice the ratio of a constant implied by the corresponding $O(\cdot)$ and the constant in the right-hand side is enough) and $c_3 \in\, ]0,1[$.

The right-hand side of (20) considered as a function of $m \in [3, \lfloor n^\beta \rfloor]$ has a unique minimum which is attained at $m' \sim \left(\frac{\varsigma'}{c_3(2-\alpha-\varsigma')}\right)^{1/(2-\alpha)} n^{\beta-\gamma/(2-\alpha)}$ and has the value asymptotic to

$$\left(\frac{c_3\Gamma(m')}{n^{\beta(2-\alpha)}\Gamma(m'+\alpha+\varsigma'-2)} + \frac{\Gamma(m'+\alpha-2)}{\Gamma(m'+\alpha+\varsigma'-2)}n^{-\gamma}\right)k^{\varsigma'} \sim c_4 n^{-\beta\varsigma'-\gamma(2-\alpha-\varsigma')/(2-\alpha)}k^{\varsigma'}.$$

Since $k \geq \lfloor n^\upsilon \rfloor$ the right-hand side grows to infinity once (17) holds.

In inequality (21) we neglect the first summand in the right-hand side and still have the function $b^{1-\alpha}n^{-\gamma} \geq n^{\upsilon\varsigma'-\gamma}$ which grows to infinity with $n$ once (17) holds. Thus the first inequality in (18) holds for all sufficiently large $n$.

The second inequality in (18) is obvious for $m \geq \lfloor n^\beta \rfloor + 1$ and for $m \leq 2$. Suppose $3 \leq m \leq n^\beta$. The inequality can be rewritten as

$$\begin{aligned}\frac{\lambda_b(m:b)}{\lambda_b} \leq \frac{\lambda_n(m:n)}{\lambda_n}\left(1+n^{-\gamma}\right) + \left(1-n^{-\gamma}\right)\max_{\ell\in\{k,\dots,n\}}\frac{\lambda_\ell(\lfloor n^\beta\rfloor+1:\ell)}{\lambda_\ell} \\ + \left(1+n^{-\gamma}\right)\left(\max_{\ell\in\{k,\dots,n\}}\frac{\lambda_\ell(\lfloor n^\beta\rfloor+1:\ell)}{\lambda_\ell} - \frac{\lambda_n(\lfloor n^\beta\rfloor+1:n)}{\lambda_n}\right).\end{aligned}$$

The latter follows from a simpler inequality

$$\lambda_n\lambda_b(m:b) \leq \lambda_b\lambda_n(m:n)\left(1+n^{-\gamma}\right) + \lambda_b\lambda_n\left(1-n^{-\gamma}\right)\max_{\ell\in\{k,\dots,n\}}\frac{\lambda_\ell(\lfloor n^\beta\rfloor+1:\ell)}{\lambda_\ell}\,. \tag{22}$$

Since $k > \lfloor n^\beta \rfloor$, application of (15) implies

$$\max_{\ell\in\{k,\dots,n\}}\frac{\lambda_\ell(\lfloor n^\beta\rfloor+1:\ell)}{\lambda_\ell} \geq \frac{c_4}{n^{\beta(2-\alpha)}}$$

for some $c_4 > 0$. We suppose that $n$ is large enough to satisfy $1-n^{-\gamma} \geq 1/2$. These observations, Lemma 1 and Corollary 2 allow us to rewrite inequality (22) as

$$\begin{aligned}&\left(\frac{A\Gamma(\alpha+1)}{2-\alpha}n^{2-\alpha} + O\left(n^{2-\sigma-\varsigma'}\right)\right)\left(\frac{A\alpha}{2-\alpha}\frac{\Gamma(m+\alpha-2)}{\Gamma(m)}b^{2-\alpha} + O\left(\frac{\Gamma(m+\alpha+\varsigma'-2)}{\Gamma(m)}b^{2-\alpha-\varsigma'}\right)\right) \\ &\quad\leq \left(\frac{A\alpha}{2-\alpha}\frac{\Gamma(m+\alpha-2)}{\Gamma(m)}n^{2-\alpha} + O\left(\frac{\Gamma(m+\alpha+\varsigma'-2)}{\Gamma(m)}n^{2-\alpha-\varsigma'}\right)\right) \\ &\qquad\times\left(\frac{A\Gamma(\alpha+1)}{2-\alpha}b^{2-\alpha} + O\left(b^{2-\alpha-\varsigma'}\right)\right)\left(1+n^{-\gamma}\right) + c_5 b^{2-\alpha}n^{(1-\beta)(2-\alpha)}\end{aligned}$$

for some $c_5 > 0$. Simplification shows that this inequality holds provided

$$c_6 \frac{\Gamma(m+\alpha+\varsigma'-2)}{\Gamma(m)} b^{2-\alpha-\varsigma'} n^{2-\alpha} \le \frac{\Gamma(m+\alpha-2)}{\Gamma(m)} b^{2-\alpha} n^{2-\alpha-\gamma} + c_7 b^{2-\alpha} n^{(1-\beta)(2-\alpha)}$$

for suitable constants $c_6, c_7 > 0$. Further simplification gives

$$c_6 \le b^{\varsigma'} \left( \frac{\Gamma(m+\alpha-2)}{\Gamma(m+\alpha+\varsigma'-2)} n^{-\gamma} + c_7 \frac{\Gamma(m)}{\Gamma(m+\alpha+\varsigma'-2)} n^{-\beta(2-\alpha)} \right) . \tag{23}$$

Proceeding as above, the expression in brackets attains its minimum in $m \in [3, n^{\beta}]$ at $m'' \sim c_8 n^{\beta-\gamma/(2-\alpha)}$, $c_8 > 0$, with the minimum value asymptotic to

$$\frac{\Gamma(m''+\alpha-2)}{\Gamma(m''+\alpha+\varsigma'-2)} n^{-\gamma} + c_7 \frac{\Gamma(m'')}{\Gamma(m''+\alpha+\varsigma'-2)} n^{-\beta(2-\alpha)} \sim c_9 n^{-\beta\varsigma'-\gamma(2-\alpha-\varsigma')/(2-\alpha)}$$

where $c_9 > 0$. Since $b \ge \lfloor n^{\upsilon} \rfloor$ the right-hand side of (23) grows to infinity as $n \to \infty$ as long as (17) holds. This observation finishes the proof. □

We want to keep control over the difference between distributions of $J^{+}_{n\downarrow k}$, $J^{-}_{n\downarrow k}$ and $J_b$, $n \ge b \ge k$. In particular, the following statement provides bounds for divergence of means.

**Lemma 6.** *Suppose $\beta < \upsilon < 1$. Then there exists $c > 0$ such that for $n$ large enough and for $k$ in range $n \ge k \ge \lfloor n^{\upsilon} \rfloor$ the following inequalities hold:*

$$\begin{aligned} \left| \mathbb{E}\left[J^{-}_{n\downarrow k} - 1\right] - \mathbb{E}\left[J_n - 1\right] \right| &\le c \max\left\{ n^{-\gamma}, n^{-\beta(1-\alpha)}, n^{\theta-\beta(2-\alpha)}, n^{1-\theta(2-\alpha)} \right\} , \\ \left| \mathbb{E}\left[J^{+}_{n\downarrow k} - 1\right] - \mathbb{E}\left[J_k - 1\right] \right| &\le c \max\left\{ n^{-\beta(1-\alpha)}, n^{-\gamma} \right\} . \end{aligned}$$

*Proof.* We start with the following observation. For $2 \le m < b$, as $b \to \infty$ but $m/b \to 0$,

$$\begin{aligned} \sum_{j=m}^{b} j\lambda_{b,j} &= \int_0^1 \sum_{j=m}^{b} j \binom{b}{j} x^{j-2}(1-x)^{b-j} \Lambda(dx) = -\int_0^1 \sum_{j=m}^{b} j \binom{b}{j} x^{j-1}(1-x)^{b-j} dG_{-1}(x) \\ &= m(m-1)\binom{b}{m} \int_0^1 x^{m-2}(1-x)^{b-m} G_{-1}(x)\, dx \sim \frac{A\alpha\Gamma(b+1)\Gamma(m+\alpha-2)}{(1-\alpha)\Gamma(b+\alpha-1)\Gamma(m-1)} , \end{aligned}$$

where

$$G_{-1}(x) = \int_x^1 \frac{\Lambda(dy)}{y} \sim \frac{A\alpha}{1-\alpha} x^{\alpha-1}, \quad \text{as} \quad x \to 0 .$$

Taking $m = \lfloor n^{\beta} \rfloor + 1$, for some $\beta \in ]0, \upsilon[$ we see using Corollary 2 that

$$\sum_{j=\lfloor n^{\beta} \rfloor+1}^{k} \frac{j\lambda_{k,j}}{\lambda_k} \sim \frac{2-\alpha}{(1-\alpha)\Gamma(\alpha)} n^{-\beta(1-\alpha)} \tag{24}$$

as $n, k \to \infty$ with $n \ge k \ge \lfloor n^{\upsilon} \rfloor$.

Now the proof follows by a simple calculation. The mean of $J^-_{n\downarrow k}$ can be estimated using Lemma 3 and (24):

$$
\begin{aligned}
\mathbb{E}\left[J^-_{n\downarrow k}-1\right] &= \frac{\lambda_{n,2}-n^{-\gamma}\lambda_n(3:\lfloor n^\beta\rfloor)+\lambda_n(\lfloor n^\beta\rfloor:n)}{\lambda_n}+\sum_{j=3}^{\lfloor n^\beta\rfloor}\frac{(j-1)\lambda_{n,j}(1+n^{-\gamma})}{\lambda_n} \\
&\quad +2\left(\left\lfloor n^\theta\right\rfloor-2\right)\max_{\ell\in\{k,\dots,n\}}\frac{\lambda_\ell(\lfloor n^\beta\rfloor+1:\ell)}{\lambda_\ell}+2\left(n-\left\lfloor n^\theta\right\rfloor\right)\max_{\ell\in\{k,\dots,n\}}\frac{\lambda_\ell(\lfloor n^\theta\rfloor+1:\ell)}{\lambda_\ell} \\
&= \mathbb{E}\left[J_n-1\right]\left(1+n^{-\gamma}\right)-\frac{n^{-\gamma}\lambda_n(2:\lfloor n^\beta\rfloor)}{\lambda_n}+\frac{\lambda_n(\lfloor n^\beta\rfloor+1:n)}{\lambda_n} \\
&\quad -\sum_{j=\lfloor n^\beta\rfloor+1}^{n}\frac{(j-1)\lambda_{n,j}}{\lambda_n}+O\left(\max\left\{n^{\theta-\beta(2-\alpha)},n^{1-\theta(2-\alpha)}\right\}\right) \\
&= \mathbb{E}\left[J_n-1\right]+O\left(\max\left\{n^{-\gamma},n^{-\beta(1-\alpha)},n^{\theta-\beta(2-\alpha)},n^{1-\theta(2-\alpha)}\right\}\right).
\end{aligned}
$$

Similarly, since $\upsilon>\beta$ formula (24) is applicable and implies together with Lemma 1 that

$$
\begin{aligned}
\mathbb{E}\left[J^+_{n\downarrow k}-1\right] &= \frac{\lambda_{k,2}+n^{-\gamma}\lambda_k(3:\lfloor n^\beta\rfloor)+\lambda_k(\lfloor n^\beta\rfloor+1:k)}{\lambda_k}+\sum_{j=3}^{\lfloor n^\beta\rfloor}\frac{(j-1)\lambda_{k,j}}{\lambda_k}\left(1-n^{-\gamma}\right) \\
&= \mathbb{E}\left[J_k-1\right]\left(1-n^{-\gamma}\right)-\sum_{j=\lfloor n^\beta\rfloor+1}^{k}\frac{(j-1)\lambda_{k,j}}{\lambda_k}\left(1-n^{-\gamma}\right)+n^{-\gamma}\frac{\lambda_k(2:\lfloor n^\beta\rfloor)}{\lambda_k}+\frac{\lambda_k(\lfloor n^\beta\rfloor+1:k)}{\lambda_k} \\
&= \mathbb{E}\left[J_k-1\right]+O\left(\max\left\{n^{-\beta(1-\alpha)},n^{-\gamma}\right\}\right),
\end{aligned}
$$

so the claim follows. ☐

Using a standard coupling technique, Lemma 5 enables us to couple random variables $J^+_{n\downarrow k}$, $J_b$ and $J^-_{n\downarrow k}$ in such a way that

$$J^+_{n\downarrow k}\le J_b\le J^-_{n\downarrow k} \tag{25}$$

holds almost surely.

# 5 The total number of collisions

We are in position now to present our main result on the convergence of the number of collisions $C_n$ in the $\Lambda$-coalescent on $n$ particles.

**Theorem 7.** *Suppose that the measure $\Lambda$ satisfies (2) with $\varsigma>\max\left\{\frac{(2-\alpha)^2}{5-5\alpha+\alpha^2},1-\alpha\right\}$. Then, as $n\to\infty$, we have the convergence in distribution*

$$\frac{C_n-(1-\alpha)n}{(1-\alpha)n^{1/(2-\alpha)}}\to_d\mathcal{S}_{2-\alpha}$$

*to a stable random variable $\mathcal{S}_{2-\alpha}$ with the characteristic function*

$$\mathbb{E}\left[e^{\mathtt{i}u\mathcal{S}_{2-\alpha}}\right]=\exp\left(-e^{-\mathtt{i}\pi\alpha\,\mathrm{sign}(u)/2}|u|^{2-\alpha}\right). \tag{26}$$

We emphasize that $\varsigma = 1$ satisfies assumptions of the above Theorem for all $\alpha \in ]0,1[$. This is important because $\varsigma = 1$ appears, say, if $\Lambda$ is a beta-measure.

*Remark.* The characteristic function (26) is not a canonic form for the characteristic function of the stable distribution. There are several commonly used parametrisations for stable variables, see [18; 21]; the difference between them being a frequent source of confusion. Apparently the most common parametrisation involves the *index of stability* $\alpha \in ]0,2]$, the *skewness* $\beta \in [-1,1]$, the *scale* $\sigma > 0$ and the *location* $\mu \in \mathbb{R}$, so that a random variable $\mathcal{S}$ has the stable distribution with parameters $(\alpha, \beta, \sigma, \mu)$ if and only if

$$\mathbb{E}\left[e^{\mathtt{i}u\mathcal{S}}\right] = \begin{cases} \exp\left(-\sigma^\alpha |u|^\alpha \left(1 - \mathtt{i}\beta \tan \frac{\pi\alpha}{2} \operatorname{sign} u\right) + \mathtt{i}\mu u\right), & \alpha \neq 1, \\ \exp\left(-\sigma|u| \left(1 + \frac{2\mathtt{i}\beta}{\pi}(\operatorname{sign} u)\log|u|\right) + \mathtt{i}\mu u\right), & \alpha = 1. \end{cases}$$

In this parametrisation our random variable $\mathcal{S}_{2-\alpha}$ has $\left(2-\alpha, -1, (\cos\frac{\pi\alpha}{2})^{1/(2-\alpha)}, 0\right)$-stable distribution since its charactersitic function can be rewritten as

$$\begin{aligned} \exp\left(-e^{-\mathtt{i}\pi\alpha \operatorname{sign}(u)/2}|u|^{2-\alpha}\right) &= \exp\left(-\cos\tfrac{\pi\alpha}{2}|u|^{2-\alpha}(1 - \mathtt{i}\tan\tfrac{\pi\alpha}{2}\operatorname{sign} u)\right) \\ &= \exp\left(-\cos\tfrac{\pi\alpha}{2}|u|^{2-\alpha}(1 + \mathtt{i}\tan\tfrac{\pi(2-\alpha)}{2}\operatorname{sign} u)\right). \end{aligned}$$

Thus $\mathcal{S}_{2-\alpha}$ has $(2-\alpha)$-stable distribution totally skewed to the left.

The main idea of the proof is that the decrements $J_b$ are almost identically distributed for large $b$, as Corollary 2 suggests. However, the nonstationarity prevents us from any direct analysis. To override this, we use the technique of stochastic bounds described in the previous section. First we introduce some auxiliary notations.

For $1 \leq k \leq n$ the coalescent started with $n$ particles after some series of collisions will reach a state with less than $k+1$ particles; let $C_{n\downarrow k}$ denote the number of collisions until this time and let $B_{n,k} \leq k$ denote the number of particles as the coalescent enters such a state. In particular, $C_n = C_{n\downarrow 1}$. For $J^{\pm}_{n\downarrow k,m}$ independent copies of $J^{\pm}_{n\downarrow k}$, introduce

$$C^+_{n\downarrow k,\ell} := \min\left\{c : \sum_{m=1}^{c}\left(J^+_{n\downarrow k,m} - 1\right) \geq \ell\right\}, \qquad C^-_{n\downarrow k,\ell} := \min\left\{c : \sum_{m=1}^{c}\left(J^-_{n\downarrow k,m} - 1\right) \geq \ell\right\}, \tag{27}$$

the minimal number of decrements distributed as $J^+_{n\downarrow k} - 1$ (respectively, $J^-_{n\downarrow k} - 1$) needed to drop by at least $\ell$. We skip the index $\ell$ when it is equal to $n-k$, so that $C^{\pm}_{n\downarrow k} \equiv C^{\pm}_{n\downarrow k,n-k}$.

Under assumptions of Lemma 5 we can couple the corresponding Markov chains so that (25) holds almost surely for all large enough $n$ once $n \geq b \geq k \geq \lfloor n^v \rfloor$. Consequently, for such $n$ the coupled Markov chains satisfy

$$C^+_{n\downarrow\lfloor n^v\rfloor} \geq C_{n\downarrow\lfloor n^v\rfloor} \geq C^-_{n\downarrow\lfloor n^v\rfloor}.$$

In other words,

$$C^+_{n\downarrow\lfloor n^v\rfloor} \geq_d C_{n\downarrow\lfloor n^v\rfloor} \geq_d C^-_{n\downarrow\lfloor n^v\rfloor}. \tag{28}$$

Before we proceed with establishing limit theorems for $C^{\pm}_{n\downarrow\lfloor n^v\rfloor}$ let us finish the proof of the law of large numbers for $C_n$.

*Proof of Lemma 4.* Take parameters $\gamma$, $\beta$, $\theta$ and $\upsilon$ so that inequalities (28) hold. The random variable $C^+_{n\downarrow\lfloor n^\upsilon\rfloor}$ (respectively, $C^-_{n\downarrow\lfloor n^\upsilon\rfloor}$) is just the number of renewals in a renewal process with the step distributed as $J^+_{n\downarrow\lfloor n^\upsilon\rfloor}-1$ (respectively, $J^-_{n\downarrow\lfloor n^\upsilon\rfloor}-1$), and with the total time $n-\lfloor n^\upsilon\rfloor$. Lemmas 3 and 6 imply that the mean value of the step converges to $1/(1-\alpha)$ in both cases. Using a classical result of the renewal theory [11], we conclude that $C^\pm_{n\downarrow\lfloor n^\upsilon\rfloor}\sim(n-n^\upsilon)(1-\alpha)\sim n(1-\alpha)$. The lemma now follows from (28) by noting that, for $\upsilon<1$, the number of collisions among $n^\upsilon$ particles is $o(n)$. $\square$

In order to find the limit distributions for $C^\pm_{n\downarrow\lfloor n^\upsilon\rfloor}$ we need the following statement about the characteristic function

$$\phi_n(u):=\mathbb{E}\left[e^{\mathrm{i}u(J_n-1)}\right]$$

of the first decrement of $\mathcal{M}_n$.

**Lemma 8.** *Let $\Lambda$ satisfy* (2) *with $\varsigma>1-\alpha$. Then there exists $\delta>0$ such that*

$$\phi_n(s/m)=1+\frac{\mathrm{i}s}{(1-\alpha)m}-\frac{\omega(s)|s|^{2-\alpha}}{(1-\alpha)m^{2-\alpha}}+O\left(m^{\alpha-2-\delta}\right),$$

*as $n,m\to\infty$ with $m\le n^\upsilon$ for some $\upsilon<1$, where $\omega(s)=e^{\mathrm{i}\pi\alpha\,\mathrm{sign}(s)/2}$.*

*Proof.* We write for shorthand $u=s/m$. For $u=0$ the claim is obvious, so we suppose that $u\neq0$. The characteristic function of $J_n-1$ can be written in terms of $\Lambda$ as follows:

$$\phi_n(u)=e^{-\mathrm{i}u}\sum_{j=2}^{n}\frac{\lambda_{n,j}}{\lambda_n}e^{\mathrm{i}ju}=\frac{e^{-\mathrm{i}u}}{\lambda_n}\int_0^1\frac{\left(1-(1-e^{\mathrm{i}u})x\right)^n-(1-x)^n-nxe^{\mathrm{i}u}(1-x)^{n-1}}{x^2}\Lambda(dx)$$

using the integral representation of $\lambda_{n,j}$. Denote the numerator of the fraction under the integral above by $h_n(u,x)$; then

$$h_{n+1}(u,x)-h_n(u,x)=x(1-e^{\mathrm{i}u})\left((1-x)^n-(1-(1-e^{\mathrm{i}u})x)^n\right)+x^2n(1-x)^{n-1}e^{\mathrm{i}u}$$

so using (7) we obtain

$$\phi_n(u)=1-\frac{1-e^{-\mathrm{i}u}}{\lambda_n}\sum_{j=1}^{n-1}\int_0^1\frac{(1-x)^j-(1-(1-e^{\mathrm{i}u})x)^j}{x}\Lambda(dx)\tag{29}$$

because $h_1(u,x)=0$. Taking again differences of $(1-x)^j-(1-(1-e^{\mathrm{i}u})x)^j$ with respect to $j$ and calculating it directly for $j=0$ we represent the integral in (29) as

$$(1-e^{\mathrm{i}u})\sum_{k=0}^{j-1}\int_0^1(1-(1-e^{\mathrm{i}u})x)^k\Lambda(dx)-\sum_{k=0}^{j-1}\int_0^1(1-x)^k\Lambda(dx).$$

Exchanging the sums and utilising notation (5) for moments $\nu_k$ of $\Lambda$ we get

$$\phi_n(u)=1+\frac{(1-e^{-\mathrm{i}u})}{\lambda_n}\sum_{k=0}^{n-2}(n-k-1)\nu_k+\frac{(1-e^{\mathrm{i}u})^2e^{-\mathrm{i}u}}{\lambda_n}\sum_{k=0}^{n-2}(n-k-1)\int_0^1\left(1-(1-e^{\mathrm{i}u})x\right)^k\Lambda(dx).\tag{30}$$

By (8) and Lemma 3 the second term above is

$$\frac{(1-e^{-\mathtt{i}u})}{\lambda_n}\sum_{k=0}^{n-2}(n-k-1)\nu_k=(1-e^{-\mathtt{i}u})\mathbb{E}\left[J_n-1\right]=\frac{1-e^{-\mathtt{i}u}}{1-\alpha}\left(1+O\left(n^{\alpha-1}\right)\right)$$

since $\varsigma>1-\alpha$ by hypothesis. Recalling notation $u=s/m$ and inequality $n\geq m^{1/\upsilon}$ with $\upsilon<1$ we see that

$$\frac{(1-e^{-\mathtt{i}s/m})}{\lambda_n}\sum_{k=0}^{n-2}(n-k-1)\nu_k=\frac{\mathtt{i}s}{(1-\alpha)m}+O\left(m^{\alpha-2-\delta_1}\right)$$

for some $\delta_1>0$. Thus it remains to estimate the last summand in (30).

Integration by parts gives

$$\int_0^1\left(1-(1-e^{\mathtt{i}u})x\right)^k\Lambda(dx)=e^{\mathtt{i}ku}+k(1-e^{\mathtt{i}u})\int_0^1(1-(1-e^{\mathtt{i}u})x)^{k-1}\Lambda[0,x]dx.$$

Substitution of this relation into (30) leads to

$$\begin{aligned}\phi_n\left(s/m\right)&=1+\frac{\mathtt{i}s}{(1-\alpha)m}+\frac{(1-e^{\mathtt{i}s/m})^2e^{-\mathtt{i}s/m}}{\lambda_n}\sum_{k=0}^{n-2}(n-k-1)e^{\mathtt{i}ks/m}\\&+\frac{(1-e^{\mathtt{i}s/m})^3e^{-\mathtt{i}s/m}}{\lambda_n}\int_0^1\sum_{k=0}^{n-2}k(n-k-1)(1-(1-e^{\mathtt{i}s/m})x)^{k-1}\Lambda[0,x]dx+O(m^{-1-\varsigma/\upsilon}).\end{aligned}\quad(31)$$

Summation yields

$$\frac{(1-e^{\mathtt{i}s/m})^2e^{-\mathtt{i}s/m}}{\lambda_n}\sum_{k=0}^{n-2}(n-k-1)e^{\mathtt{i}ks/m}=\frac{e^{-\mathtt{i}s/m}(n(1-e^{\mathtt{i}s/m})+e^{\mathtt{i}sn/m}-1)}{\lambda_n}.$$

For $m$ big enough and $n\geq m^{1/\upsilon}$ with $\upsilon<1$

$$\left|\frac{n(1-e^{\mathtt{i}s/m})+e^{\mathtt{i}ns/m}-1}{\lambda_n}\right|\leq\frac{n|1-e^{\mathtt{i}s/m}|+|e^{\mathtt{i}ns/m}-1|}{\lambda_n}\leq\frac{\text{const}}{n^{1-\alpha}m}=O\left(m^{\alpha-2-\delta_2}\right)$$

for some $\delta_2>0$.

Let $\theta\in[-\pi/2,\pi/2]$ be such that $e^{\mathtt{i}\theta}=\frac{1-e^{\mathtt{i}s/m}}{|1-e^{\mathtt{i}s/m}|}$. Note that $\theta=-\pi\,\mathrm{sign}(s)/2+O(1/m)$ as $m\to\infty$. For any $\beta>0$ we have

$$\begin{aligned}&\int_0^1k(1-e^{\mathtt{i}s/m})(1-(1-e^{\mathtt{i}s/m})x)^{k-1}x^\beta dx\\&=e^{\mathtt{i}\theta}\int_0^{k|1-e^{\mathtt{i}s/m}|}\left(1-e^{\mathtt{i}\theta}t/k\right)^{k-1}\frac{t^\beta}{(k|1-e^{\mathtt{i}s/m}|)^\beta}dt=\frac{e^{-\mathtt{i}\beta\theta}}{\left(k|s|/m\right)^\beta}\Gamma(\beta+1)\left(1+O(1/m)\right)\end{aligned}$$

as $m,k\to\infty$ with $k\geq m^{1+\delta_3}$ for any $\delta_3>0$. By assumption (2) we can write $\Lambda[0,x]=Ax^\alpha+f(x)$ where $|f(x)|\leq cx^{\alpha+\varsigma}$ for some $c>0$ and all $x\in[0,1]$. Thus, as $m,k\to\infty$ with $k\geq m^{1+\delta_3}$,

$$\int_0^1k(1-e^{\mathtt{i}s/m})(1-(1-e^{\mathtt{i}s/m})x)^{k-1}\Lambda[0,x]\,dx=\frac{Ae^{\mathtt{i}\pi\alpha\,\mathrm{sign}(s)/2}}{\left(k|s|/m\right)^\alpha}\Gamma(\alpha+1)+O\left(\frac{m^{\alpha+\varsigma}}{k^{\alpha+\varsigma}}+\frac{1}{m^{1-\alpha}k^\alpha}\right).$$

Take $\delta_3 = (1/\upsilon - 1)/2$ and denote $n_0 = \lfloor m^{1+\delta_3} \rfloor$. Divide the last sum in (31) into two sums over $k \geq n_0$ and $k < n_0$. The first sum is estimated taking (11) into account as

$$\begin{aligned}
&\frac{(1-e^{\mathtt{i}s/m})^2}{\lambda_n} \sum_{k=n_0}^{n-2} (n-k-1) \int_0^1 k(1-e^{\mathtt{i}s/m})(1-(1-e^{\mathtt{i}s/m})x)^{k-1} \Lambda[0,x]\, dx \\
&\qquad = -\frac{|s|^{2-\alpha} e^{\mathtt{i}\pi\alpha\,\mathrm{sign}(s)/2}(2-\alpha)}{m^{2-\alpha} n^{2-\alpha}} \sum_{k=n_0}^{n-2} (n-k) k^{-\alpha} \\
&\qquad\quad + O\left( \frac{1}{n^{2-\alpha} m^{2-\alpha-\varsigma}} \sum_{k=n_0}^{n-2} \frac{n-k}{k^{\alpha+\varsigma}} + \frac{1}{n^{2-\alpha} m^{3-\alpha}} \sum_{k=n_0}^{n-2} \frac{n-k}{k^{\alpha}} \right) \\
&\qquad = -\frac{|s|^{2-\alpha} e^{\mathtt{i}\pi\alpha\,\mathrm{sign}(s)/2}(2-\alpha)}{m^{2-\alpha}} \int_{m^{1+\delta_3} n^{-1}}^1 x^{-\alpha}(1-x)dx + O\left( \frac{1}{n^{1-\alpha} m^{1+\varsigma\delta_3}} + \frac{1}{m^{3-\alpha}} \right) \\
&\qquad = -\frac{|s|^{2-\alpha} e^{\mathtt{i}\pi\alpha\,\mathrm{sign}(s)/2}}{(1-\alpha) m^{2-\alpha}} + O(m^{\alpha-2-\delta_4})
\end{aligned}$$

for some $\delta_4 > 0$. The same argument applied to the sum over $k = 0, \dots, n_0 - 1$ shows that it constitutes a lower order term to the whole sum. Thus it remains to combine the results above to get the statement of Lemma. □

Next we show that under certain assumptions the same asymptotic expansion is also valid for the characteristic functions of $J^{\pm}_{n\downarrow k}$

$$\phi^{+}_{n\downarrow k}(u) := \mathbb{E}\left[e^{\mathtt{i}uJ^{+}_{n\downarrow k}}\right] \qquad \text{and} \qquad \phi^{-}_{n\downarrow k}(u) := \mathbb{E}\left[e^{\mathtt{i}uJ^{-}_{n\downarrow k}}\right].$$

**Lemma 9.** *Suppose* (2) *holds with* $\varsigma > 1-\alpha$ *and that the parameters in* (14) *satisfy inequalities*

$$\gamma > \frac{1-\alpha}{2-\alpha}, \qquad \text{and} \qquad \upsilon > \theta > \beta > \frac{1}{2-\alpha}. \tag{32}$$

*Then there exists* $\delta > 0$ *such that*

$$\begin{aligned}
\phi^{+}_{n\downarrow k}(s/m) &= 1 + \frac{\mathtt{i}s}{(1-\alpha)m} - \frac{\omega(s)|s|^{2-\alpha}}{(1-\alpha)m^{2-\alpha}} + O\left(m^{\alpha-2-\delta}\right), \\
\phi^{-}_{n\downarrow k}(s/m) &= 1 + \frac{\mathtt{i}s}{(1-\alpha)m} - \frac{\omega(s)|s|^{2-\alpha}}{(1-\alpha)m^{2-\alpha}} + O\left(m^{\alpha-2-\delta}\right),
\end{aligned}$$

*as* $n, k, m \to \infty$ *in such a way that* $n \geq k \geq \lfloor n^{\upsilon} \rfloor$ *and* $m \leq cn^{1/(2-\alpha)}$ *for some* $c > 0$.

*Proof.* The characteristic function of $J^{-}_{n\downarrow k} - 1$ is by definition

$$\begin{aligned}
\phi^{-}_{n\downarrow k}(u) = e^{\mathtt{i}u} &\left( \frac{\lambda_{n,2} - n^{-\gamma}\lambda_n(3 : \lfloor n^{\beta} \rfloor) + \lambda_n(\lfloor n^{\beta} \rfloor + 1 : n)}{\lambda_n} - 2 \max_{\ell \in \{k,\dots,n\}} \frac{\lambda_\ell(\lfloor n^{\beta} \rfloor + 1 : \ell)}{\lambda_\ell} \right) \\
&+ \sum_{j=3}^{\lfloor n^{\beta} \rfloor} \frac{\lambda_{n,j}(1+n^{-\gamma})}{\lambda_n} e^{\mathtt{i}(j-1)u} + 2e^{\mathtt{i}(\lfloor n^{\theta} \rfloor - 1)u} \max_{\ell \in \{k,\dots,n\}} \frac{\lambda_\ell(\lfloor n^{\beta} \rfloor + 1 : \ell)}{\lambda_\ell} \\
&\qquad + 2\left(e^{\mathtt{i}(n-1)u} - e^{\mathtt{i}(\lfloor n^{\theta} \rfloor - 1)u}\right) \max_{\ell \in \{k,\dots,n\}} \frac{\lambda_\ell(\lfloor n^{\theta} \rfloor + 1 : \ell)}{\lambda_\ell}.
\end{aligned}$$

We rewrite it as follows:

$$\phi^-_{n\downarrow k}(u) = \phi_n(u)\left(1+n^{-\gamma}\right) - n^{-\gamma}e^{\mathtt{i}u} + \frac{\lambda_n(\lfloor n^\beta\rfloor+1:n)(1+n^{-\gamma})}{\lambda_n}e^{\mathtt{i}u} - \sum_{j=\lfloor n^\beta\rfloor+1}^{n}\frac{(1+n^{-\gamma})\lambda_{n,j}}{\lambda_n}e^{\mathtt{i}(j-1)u} + 2\left(e^{\mathtt{i}(\lfloor n^\theta\rfloor-1)u} - e^{\mathtt{i}u}\right)\max_{\ell\in\{k,\dots,n\}}\frac{\lambda_\ell(\lfloor n^\beta\rfloor+1:\ell)}{\lambda_\ell} + 2\left(e^{\mathtt{i}(n-1)u} - e^{\mathtt{i}(\lfloor n^\theta\rfloor-1)u}\right)\max_{\ell\in\{k,\dots,n\}}\frac{\lambda_\ell(\lfloor n^\theta\rfloor+1:\ell)}{\lambda_\ell}.$$

Four last summands are of the order of $n^{-\beta(2-\alpha)}$ by Lemma 1. From (32), $\beta > 1/(2-\alpha)$ and so the bound $m \le cn^{1/(2-\alpha)}$ guarantees that these four summands constitute $O(m^{\alpha-2-\delta_1})$ terms to the whole sum, for $\delta_1, \delta_2, \dots$ some positive constants. The same bound on $m$ allows application of Lemma 8 for $\phi_n(s/m)$ which leads to

$$\begin{aligned}\phi^-_{n\downarrow k}(s/m) &= \left(1+\frac{\mathtt{i}s}{(1-\alpha)m} - \frac{\omega(s)|s|^{2-\alpha}}{(1-\alpha)m^{2-\alpha}} + O(m^{\alpha-2-\delta_2})\right)\left(1+n^{-\gamma}\right)\\ &\quad - n^{-\gamma}\left(1+\frac{\mathtt{i}s}{m}+O(m^{-2})\right) + O(m^{\alpha-2-\delta_1})\\ &= 1+\frac{\mathtt{i}s}{(1-\alpha)m} - \frac{\omega(s)|s|^{2-\alpha}}{(1-\alpha)m^{2-\alpha}} + \left(\frac{1}{1-\alpha}-1\right)\frac{\mathtt{i}s\,n^{-\gamma}}{m} + O(m^{\alpha-2-\delta_3})\end{aligned}$$

and the claim about $\phi^-_{n\downarrow k}$ follows from inequality $\gamma > (1-\alpha)/(2-\alpha)$.
Analogously,

$$\begin{aligned}\phi^+_{n\downarrow k}(u) &= \frac{\lambda_{k,2}+n^{-\gamma}\lambda_k(3:\lfloor n^\beta\rfloor)+\lambda_k(\lfloor n^\beta\rfloor+1:k)}{\lambda_k}e^{\mathtt{i}u} + \sum_{j=3}^{\lfloor n^\beta\rfloor}\frac{\lambda_{k,j}(1-n^{-\gamma})e^{\mathtt{i}(j-1)u}}{\lambda_k}\\ &= \frac{\lambda_{k,2}}{\lambda_k}\left(1-n^{-\gamma}\right)e^{\mathtt{i}u} + n^{-\gamma}e^{\mathtt{i}u} + \frac{\lambda_k(\lfloor n^\beta\rfloor+1:b)}{\lambda_k}\left(1-n^{-\gamma}\right)e^{\mathtt{i}u}\\ &\quad + \sum_{j=3}^{n}\frac{\lambda_{k,j}(1-n^{-\gamma})}{\lambda_k}e^{\mathtt{i}(j-1)u} - \sum_{j=\lfloor n^\beta\rfloor+1}^{n}\frac{\lambda_{k,j}(1-n^{-\gamma})}{\lambda_k}e^{\mathtt{i}(j-1)u}\\ &= \phi_k(u)\left(1-n^{-\gamma}\right) + n^{-\gamma}e^{\mathtt{i}u} + \left(1-n^{-\gamma}\right)\sum_{j=\lfloor n^\beta\rfloor+1}^{n}\frac{\lambda_{k,j}(1-e^{\mathtt{i}ju})e^{\mathtt{i}u}}{\lambda_k}.\end{aligned}$$

Since $k \ge \lfloor n^\upsilon\rfloor$ with $\upsilon > \beta > 1/(2-\alpha)$ and $m \le cn^{1/(2-\alpha)}$, Lemma 1 ensures that the last sum above is $O(n^{-\beta(2-\alpha)}) = O(m^{\alpha-2-\delta_4})$. Since $m$ grows slower than $k$ by hypothesis, Lemma 8 can be applied for $\phi_k(s/m)$ and the claim again follows from inequality $\gamma > (1-\alpha)/(2-\alpha)$. □

**Lemma 10.** *Suppose* (2) *holds with* $\varsigma > 1-\alpha$ *and that the parameters in* (14) *satisfy inequalities*

$$\gamma > \frac{1-\alpha}{2-\alpha}, \qquad 1 > \upsilon > \theta > \beta > \frac{5-5\alpha+\alpha^2}{(2-\alpha)^3}, \qquad \beta(2-\alpha) - \frac{1-\alpha}{2-\alpha} > \theta > \frac{3-2\alpha}{(2-\alpha)^2}. \quad (33)$$

*Let $S^{+}_{n\downarrow k,h}$, respectively $S^{-}_{n\downarrow k,h}$, be the sum of $h$ independent copies of $J^{+}_{n\downarrow k}-1$, respectively $J^{-}_{n\downarrow k}-1$. Then*

$$\frac{S^{+}_{n\downarrow k,h}-h/(1-\alpha)}{(h/(1-\alpha))^{1/(2-\alpha)}}\to_d \bar{\mathcal{S}}_{2-\alpha} \qquad \textit{and} \qquad \frac{S^{-}_{n\downarrow k,h}-h/(1-\alpha)}{(h/(1-\alpha))^{1/(2-\alpha)}}\to_d \bar{\mathcal{S}}_{2-\alpha}$$

*as $n,h\to\infty$ with $h=O(n)$ and $n\ge k\ge \lfloor n^{\upsilon}\rfloor$, $1>\upsilon>\beta$, where $\bar{\mathcal{S}}_{2-\alpha}$ is a stable random variable with the characteristic function*

$$\mathbb{E}\left[e^{\mathtt{i}u\bar{\mathcal{S}}_{2-\alpha}}\right]=\exp\left[-\omega(u)|u|^{2-\alpha}\right], \qquad \omega(u)=\exp\left(\frac{\mathtt{i}\pi\alpha\operatorname{sign}u}{2}\right). \tag{34}$$

*Proof.* First note that a solution of inequality (33) always exists. Since (32) follows from (33), the bound $h=O(n)$ guarantees that Lemma 9 is applicable with $m=(h/(1-\alpha))^{1/(2-\alpha)}$. Lemmas 3 and 6 provide tough bounds for $\mathbb{E}\left[J^{\pm}_{n\downarrow k}-1\right]$. Namely, inequality (33) implies that

$$\left|\mathbb{E}\left[J^{-}_{n\downarrow k}-1\right]-\frac{1}{1-\alpha}\right|=O(n^{-(1-\alpha)/(2-\alpha)-\delta_1}) \tag{35}$$

for some $\delta_1>0$. Hence

$$\phi^{-}_{n\downarrow k}\left(\frac{s}{(h/(1-\alpha))^{1/(2-\alpha)}}\right)\exp\left(-\frac{\mathbb{E}\left[J^{-}_{n\downarrow k}-1\right]}{(h/(1-\alpha))^{1/(2-\alpha)}}\mathtt{i}s\right)=1-\frac{\omega(s)|s|^{2-\alpha}}{h}+O(h^{-1-\delta_2})$$

for some $\delta_2>0$. Moreover, equation (35) and $h=O(n)$ imply

$$\frac{\mathbb{E}\left[S^{-}_{n\downarrow k,h}\right]-\frac{h}{1-\alpha}}{(h/(1-\alpha))^{1/(2-\alpha)}}=O\left(h^{1-1/(2-\alpha)}n^{-(1-\alpha)/(2-\alpha)-\delta_1}\right)=O\left(h^{-\delta_1}\right),$$

as $n,h\to\infty$. Hence, for some $\delta_3>0$

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\frac{S^{-}_{n\downarrow k,h}-\frac{h}{1-\alpha}}{(h/(1-\alpha))^{1/(2-\alpha)}}\mathtt{i}s\right)\right]\\
&\qquad=\exp\left(\frac{\mathbb{E}\left[S^{-}_{n\downarrow k,h}\right]-\frac{h}{1-\alpha}}{(h/(1-\alpha))^{1/(2-\alpha)}}\mathtt{i}s\right)\mathbb{E}\left[\exp\left(\frac{S^{-}_{n\downarrow k,h}-\mathbb{E}\left[S^{-}_{n\downarrow k,h}\right]}{(h/(1-\alpha))^{1/(2-\alpha)}}\mathtt{i}s\right)\right]\\
&\qquad=\left(1+O(h^{-\delta_3})\right)\left(1-\frac{\omega(s)|s|^{2-\alpha}}{h}+O(h^{-1-\delta_2})\right)^{h}\\
&\qquad\to\exp\left[-e^{\frac{\mathtt{i}\pi\alpha\operatorname{sign}s}{2}}|s|^{2-\alpha}\right]
\end{aligned}$$

and the claim about $S^{-}_{n\downarrow k,h}$ follows.

Treatment of the limit theorem for $S^{+}_{n\downarrow k,h}$ literally repeats the above steps and is omitted. □

Let $F_{2-\alpha}(\cdot)$ be the distribution function of the stable random variable $\mathcal{S}_{2-\alpha}$ defined by (26) and $\bar{F}_{2-\alpha}(\cdot)$ be that of $\bar{\mathcal{S}}_{2-\alpha}$ defined by (34). Note that the random variables $\mathcal{S}_{2-\alpha}$ and $-\bar{\mathcal{S}}_{2-\alpha}$ have the same distributions, i.e. $F_{2-\alpha}(t)=1-\bar{F}_{2-\alpha}(-t)$.

**Lemma 11.** *Let the measure $\Lambda$ satisfy* (2) *with $\varsigma > \max\left\{\frac{(2-\alpha)^2}{5-5\alpha+\alpha^2}, 1-\alpha\right\}$. Then there exists $\upsilon < 1$ such that*

$$\frac{C_{n\downarrow\lfloor n^\upsilon\rfloor} - (n - \lfloor n^\upsilon\rfloor)(1-\alpha)}{n^{1/(2-\alpha)}(1-\alpha)} \to_d \mathcal{S}_{2-\alpha} \quad \text{as} \quad n \to \infty.$$

*Proof.* Suppose that $\beta$, $\gamma$ and $\theta$ satisfy both inequalities (17) and (33). This is always possible if $\varsigma$ satisfy the condition stated in this Lemma. Indeed, the only constraints which can become inconsistent by joining inequalities (17) and (33) are the constraints on $\gamma$

$$\frac{(\upsilon-\beta)(2-\alpha)\varsigma'}{2-\alpha-\varsigma'} > \gamma > \frac{1-\alpha}{2-\alpha}\,. \tag{36}$$

By (33) we can choose $\beta$ and $\upsilon$ such that $\upsilon - \beta < 1 - \frac{5-5\alpha+\alpha^2}{(2-\alpha)^3}$. Hence (36) is solvable for

$$\left(1 - \frac{5-5\alpha+\alpha^2}{(2-\alpha)^3}\right)\frac{(2-\alpha)\varsigma'}{2-\alpha-\varsigma'} > \frac{1-\alpha}{2-\alpha}\,.$$

Resolving $\varsigma'$ from this inequality and recalling its definition leads to the lower bound on $\varsigma$ in the claim.

By definition $C^+_{n\downarrow\lfloor n^\upsilon\rfloor, d}$ is the random number of decrements $J^+_{n\downarrow\lfloor n^\upsilon\rfloor} - 1$ needed to make a total move larger than $d$. Hence for all $h > 0$

$$\mathbb{P}\left[C^+_{n\downarrow\lfloor n^\upsilon\rfloor,d} \le h\right] = \mathbb{P}\left[S^+_{n\downarrow\lfloor n^\upsilon\rfloor,h} \ge d\right]. \tag{37}$$

Take now $d = n - \lfloor n^\upsilon\rfloor$ and $h = \left\lfloor\left(n - \lfloor n^\upsilon\rfloor + t_n n^{1/(2-\alpha)}\right)(1-\alpha)\right\rfloor$ where $t_n \to t$ as $n \to \infty$. Then $h \to \infty$ but $h = O(n)$ as $n \to \infty$. Moreover,

$$d = n - \lfloor n^\upsilon\rfloor = \frac{h}{1-\alpha} - t\left(\frac{h}{1-\alpha}\right)^{1/(2-\alpha)}(1+o(1)), \qquad n \to \infty,$$

and application of Lemma 10 ensures that the right-hand side of (37) converges to $\bar{F}_{2-\alpha}(-t)$ as $n \to \infty$, since $\bar{F}_{2-\alpha}$ is continuous [21]. Thus

$$\begin{aligned}\mathbb{P}&\left[\frac{C^+_{n\downarrow\lfloor n^\upsilon\rfloor} - (n-\lfloor n^\upsilon\rfloor)(1-\alpha)}{n^{1/(2-\alpha)}(1-\alpha)} \le t\right] \sim \mathbb{P}\left[C^+_{n\downarrow\lfloor n^\upsilon\rfloor} \le h\right]\\ &= \mathbb{P}\left[S^+_{n\downarrow\lfloor n^\upsilon\rfloor,h} \ge \frac{h}{1-\alpha} - t\left(\frac{h}{1-\alpha}\right)^{1/(2-\alpha)}(1+o(1))\right] \to 1 - \bar{F}_{2-\alpha}(-t) = F_{2-\alpha}(t)\,.\end{aligned}$$

Replacing $S^+_{n\downarrow\lfloor n^\upsilon\rfloor,h}$ with $S^-_{n\downarrow\lfloor n^\upsilon\rfloor,h}$ and $C^+_{n\downarrow\lfloor n^\upsilon\rfloor}$ with $C^-_{n\downarrow\lfloor n^\upsilon\rfloor}$ in the above argument we obtain

$$\mathbb{P}\left[\frac{C^-_{n\downarrow\lfloor n^\upsilon\rfloor} - (n-\lfloor n^\upsilon\rfloor)(1-\alpha)}{n^{1/(2-\alpha)}(1-\alpha)} \le t\right] \to F_{2-\alpha}(t).$$

Hence the claim follows from inequalities (28) since $F_{2-\alpha}$ is continuous. $\square$

*Proof of Theorem 7.* Recall that $B_{n,k}$ is the number of particles in the $\Lambda$-coalescent started with $n$ particles right after the number of particles drops below to $k+1$. Then for any $k \leq n$ the total number of collisions is decomposable as

$$C_n =_d C_{n\downarrow k} + C^{(1)}_{B_{n,k}}$$

where in the right-hand side $(C^{(1)}_b)$ is an independent copy of $(C_b)$. This can be iterated as

$$C_n = C_{n\downarrow k_1} + C^{(1)}_{B_{n,k_1}\downarrow k_2} + C^{(2)}_{B_{n,k_2}\downarrow k_3} + \cdots + C^{(\ell)}_{B_{n,k_{\ell-1}}\downarrow k_\ell} + C^{(\ell+1)}_{B_{n,k_\ell}} \tag{38}$$

for any finite sequence $k_\ell \leq k_{\ell-1} \leq \cdots \leq k_1 \leq n$, with the convention that $C_{b\downarrow k} = 0$ for $b \leq k$. Suppose that Lemma 11 holds for some $\upsilon < 1$. Let $\ell = \left\lfloor \frac{-\log(2-\alpha)}{\log \upsilon} \right\rfloor + 1$, then $\upsilon^\ell < 1/(2-\alpha)$. For each $m = 1, \ldots, \ell$, by Lemma 11 applied with $\lfloor n^{\upsilon^m} \rfloor$ instead of $n$,

$$\mathbb{P}\left[ \frac{C_{\lfloor n^{\upsilon^m}\rfloor \downarrow \lfloor n^{\upsilon^{m+1}}\rfloor} - \left( \lfloor n^{\upsilon^m}\rfloor - \lfloor n^{\upsilon^{m+1}}\rfloor \right)(1-\alpha)}{n^{\upsilon^m/(2-\alpha)}(1-\alpha)} \leq t \right] \to F_{2-\alpha}(t).$$

Consequently, since $\upsilon < 1$, for all $m > 0$

$$\frac{C_{\lfloor n^{\upsilon^m}\rfloor \downarrow \lfloor n^{\upsilon^{m+1}}\rfloor} - \left( \lfloor n^{\upsilon^m}\rfloor - \lfloor n^{\upsilon^{m+1}}\rfloor \right)(1-\alpha)}{n^{1/(2-\alpha)}(1-\alpha)} \to_d \boldsymbol{\delta}_0, \qquad n \to \infty,$$

where $\boldsymbol{\delta}_0$ is the $\delta$-measure at zero. Moreover, for some fixed $\tau \in \,]0, 1/(2-\alpha)[$ starting the coalescent with $\lfloor n^{\upsilon^m} - n^\tau \rfloor$ particles instead of $\lfloor n^{\upsilon^m} \rfloor$ particles results in the same asymptotic behaviour:

$$\frac{C_{\lfloor n^{\upsilon^m} - n^\tau\rfloor \downarrow \lfloor n^{\upsilon^{m+1}}\rfloor} - \left( \lfloor n^{\upsilon^m}\rfloor - \lfloor n^{\upsilon^{m+1}}\rfloor \right)(1-\alpha)}{n^{1/(2-\alpha)}(1-\alpha)} \to_d \boldsymbol{\delta}_0, \qquad n \to \infty.$$

Denote by $E_m$ the event $B_{n,\lfloor n^{\upsilon^m}\rfloor} \geq n^{\upsilon^m} - n^\tau$, i.e. that the Markov process $\mathcal{M}_n$ undershoots $n^{\upsilon^m}$ not more than by $n^\tau$. Thus, given $E_m$,

$$\frac{C_{B_{n,n^{\upsilon^m}} \downarrow \lfloor n^{\upsilon^{m+1}}\rfloor} - \left( \lfloor n^{\upsilon^m}\rfloor - \lfloor n^{\upsilon^{m+1}}\rfloor \right)(1-\alpha)}{n^{1/(2-\alpha)}(1-\alpha)} \to_d \boldsymbol{\delta}_0, \qquad n \to \infty,$$

by the monotonicity of the number of collisions in $n$. Using (38) with $k_m = \lfloor n^{\upsilon^m} \rfloor$, Lemma 11 and Slutsky's theorem yields the desired convergence for $C_n$, given $E_m$ holds for all $m = 1, \ldots, \ell$, because the last summand in (38) satisfies

$$0 \leq C^{(\ell+1)}_{B_{n,k_\ell}} \leq k_\ell \leq n^{\upsilon^\ell} = o(n^{1/(2-\alpha)}) \,.$$

Lemma 1 ensures that for any $\tau > 0$ the probability of $E_m$ grows to 1, as $n \to \infty$, for all $m = 1, \ldots, \ell$. Hence $\mathbb{P}\left[\cap_{m=1}^{\ell} E_m\right] \to 1$ and the convergence in distribution conditioned on $\cap_{m=1}^{\ell} E_m$ is equivalent to the unconditional convergence in distribution. $\square$

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