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Insensitivity to Negative Dependence of the Asymptotic Behavior of Precise Large Deviations

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Abstract

Since the pioneering works of C.C. Heyde, A.V. Nagaev, and S.V. Nagaev in 1960's and 1970's, the precise asymptotic behavior of large-deviation probabilities of sums of heavy-tailed random variables has been extensively investigated by many people, but mostly it is assumed that the random variables under discussion are independent. In this paper, we extend the study to the case of negatively dependent random variables and we find out that the asymptotic behavior of precise large deviations is insensitive to the negative dependence.

Key words: Consistent variation; (lower/upper) negative dependence; partial sum; precise large deviations; uniform asymptotics; (upper) Matuszewska index. .

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1 Introduction

Let $\{X_k, k = 1, 2, \ldots\}$ be a sequence of random variables with common distribution F and mean 0 satisfying $\overline{F}(x) = 1 - F(x) > 0$ for all x, and let S_n be its nth partial sum, $n = 1, 2, \ldots$. In the present paper we are interested in precise large deviations of these partial sums in the situation that the random variables $\{X_k, k = 1, 2, \ldots\}$ are heavy tailed and negatively dependent. Following many researchers in this field, we aim to prove that for each fixed $\gamma > 0$, the relation

$$\Pr(S_n > x) \sim n\overline{F}(x), \qquad n \to \infty, \tag{1.1}$$

holds uniformly for all $x \geq \gamma n$. That is,

$$\lim_{n\to\infty} \sup_{x\geq\gamma n} \left| \frac{\Pr(S_n > x)}{n\overline{F}(x)} - 1 \right| = 0.$$

An important class of heavy-tailed distributions is $\mathcal{D}$, which consists of all distributions with dominated variation in the sense that the relation

$$\limsup_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)} < \infty$$

holds for some (hence for all) $0 < v < 1$. A slightly smaller class is $\mathcal{C}$, which consists of all distributions with consistent variation in the sense that

$$\lim_{v\searrow 1} \liminf_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)} = 1, \quad \text{or, equivalently,} \quad \lim_{v\nearrow 1} \limsup_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)} = 1. \tag{1.2}$$

The regularity property in (1.2) has been investigated in the literature; see Stadtmüller and Trautner (1979), Bingham et al. (1987), and Cline (1994), among others. Besides precise large deviations, the class $\mathcal{C}$ has recently been used in different studies of applied probability such as queueing systems and ruin theory. Closely related is the famous class $\mathcal{R}$ of all distributions with regular variation in the sense that for some $\alpha \geq 0$, the relation

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha} \tag{1.3}$$

holds for every $y > 0$. Clearly, the class $\mathcal{C}$ covers the class $\mathcal{R}$. Examples that illustrate the inclusion $\mathcal{R} \subset \mathcal{C}$ is proper were given in Cline and Samorodnitsky (1994) and Cai and Tang (2004).

For classical works of precise large deviations with heavy tails, we refer the reader to Heyde (1967a, 1967b, 1968), A.V. Nagaev (1969a, 1969b, 1969c), S.V. Nagaev (1973, 1979), Pinelis (1985), while for recent works, we refer to Cline and Hsing (1991), Rozovskiĭ (1993), Vinogradov (1994), Mikosch and A.V. Nagaev (1998), Tang et al. (2001), Ng et al. (2004), and Baltrūnas and Klüppelberg (2004), among many others.

Strolling in this literature, we find that most works were conducted only for independent random variables, though several dealing with non-identically distributed random variables.

The goal of this paper is to extend the study to certain dependent cases. More precisely, we shall consider negative dependence structures for the random variables $\{X_k, k = 1, 2, \ldots\}$. As defined below, these structures describe that the tails of finite-dimensional distributions of the random variables $\{X_k, k = 1, 2, \ldots\}$ in the lower left or/and upper right corners are dominated by the corresponding tails of the finite-dimensional distributions of an i.i.d. sequence with the same marginal distributions. As is pointed out by the referee, the intuition of the negative dependence structures is that they assist cancellation, and limit theorems in probability theory are basically cancellation phenomena. These dependence structures have been systematically investigated in the literature since they were introduced by Ebrahimi and Ghosh (1981) and Block et al. (1982).

Definition 1.1. *We call random variables* $\{X_k, k = 1, 2, \ldots\}$

(1) Lower Negatively Dependent (LND) if for each $n = 1, 2, \ldots$ *and all* $x_1, \ldots, x_n$*,*

$$\Pr(X_1 \leq x_1, \ldots, X_n \leq x_n) \leq \prod_{k=1}^{n} \Pr(X_k \leq x_k); \tag{1.4}$$

(2) Upper Negatively Dependent (UND) if for each $n = 1, 2, \ldots$ *and all* $x_1, \ldots, x_n$*,*

$$\Pr(X_1 > x_1, \ldots, X_n > x_n) \leq \prod_{k=1}^{n} \Pr(X_k > x_k); \tag{1.5}$$

(3) Negatively Dependent (ND) if both (1.4) and (1.5) hold for each $n = 1, 2, \ldots$ *and all* $x_1, \ldots, x_n$*.*

It is worth mentioning that for $n = 2$, the LND, UND, and ND structures are equivalent; see, for example, Lehmann (1966). In terms of the well-known Farlie-Gumbel-Morgenstern distributions (see Kotz et al. (2000)), it is not difficult to construct practically interesting examples that are (lower/upper) negatively dependent but not independent. We also remark that these notions of negative dependence are much more verifiable than the commonly used notion of negative association, the latter of which was introduced by Alam and Saxena (1981) and Joag-Dev and Proschan (1983). See also Bingham and Nili Sani (2004) for a recent account and for a list of relevant references.

The basic assumption of this paper is that the random variables $\{X_k, k = 1, 2, \ldots\}$ are ND with common distribution $F \in \mathcal{C}$. Below is our main result, which indicates that the asymptotic relation (1.1) is insensitive to the assumed ND structure.

Theorem 1.1. *Let $\{X_k, k = 1, 2, \ldots\}$ be ND with common distribution $F \in \mathcal{C}$ and mean 0 satisfying*

$$xF(-x) = o\left(\overline{F}(x)\right), \qquad x \to \infty. \tag{1.6}$$

Then for each fixed $\gamma > 0$, relation (1.1) holds uniformly for all $x \geq \gamma n$. Condition (1.6) is unnecessary when $\{X_k, k = 1, 2, \ldots\}$ are mutually independent.

Condition (1.6) indicates that the left tail of F should be lighter than the right tail. The result is new even when the random variables $\{X_k, k = 1, 2, \ldots\}$ are mutually independent. The main difficulties that we encounter in the proof are due to the negative dependence structure of $\{X_k, k = 1, 2, \ldots\}$ and the two-sided support of F.

The remaining part of this paper consists of two sections. After preparing several lemmas in Section 2, we formulate in Sections 3 the proof of Theorem 1.1 in two parts, which provide the probability $\Pr(S_n > x)$ with an asymptotic upper estimate and an asymptotic lower estimate, respectively.

2 Preliminaries

Throughout, every limit relation without explicit limit procedure is with respect to $n \to \infty$, letting the relation speak for itself. For positive functions $f(\cdot)$ and $g(\cdot)$, we write

$f = O(g)$ if $\limsup f/g < \infty$,

$f = o(g)$ if $\lim f/g = 0$,

$f \asymp g$ if both $f = O(g)$ and $g = O(f)$,

$f \lesssim g$ if $\limsup f/g \leq 1$,

$f \gtrsim g$ if $\liminf f/g \geq 1$, and

$f \sim g$ if both $f \lesssim g$ and $f \gtrsim g$.

For a distribution F, we define

$$J_F^* = -\lim_{v \to \infty} \frac{\log \overline{F}_*(v)}{\log v} \qquad \text{with} \quad \overline{F}_*(v) = \liminf_{x \to \infty} \frac{\overline{F}(vx)}{\overline{F}(x)} \qquad \text{for } v > 0,$$

and we call the quantity J_F^* the (upper) Matuszewska index of the distribution F. For details of the Matuszewska indices see Bingham et al. (1987, Chapter 2.1), while for further discussions and applications see Cline and Samorodnitsky (1994) and Tang and Tsitsiashvili (2003).

Clearly, if relation (1.3) holds with some $\alpha \geq 0$ then $J_F^* = \alpha$. Moreover, $F \in \mathcal{D}$ if and only if $\overline{F}(vx) \asymp \overline{F}(x)$ as $x \to \infty$ for each $v > 0$ if and only if $J_F^* < \infty$. We shall be using these equivalents without further comment.

The following lemma is a combination of Proposition 2.2.1 of Bingham et al. (1987) and Lemma 3.5 of Tang and Tsitsiashvili (2003):

Lemma 2.1. *If $F \in \mathcal{D}$, then,*

(1) for each $p > J_F^*$, there exist positive constants C and D such that the inequality

$$\frac{\overline{F}(y)}{\overline{F}(x)} \leq C\left(\frac{x}{y}\right)^p$$

holds for all $x \geq y \geq D$;

(2) it holds for each $p > J_F^*$ that $x^{-p} = o\left(\overline{F}(x)\right)$.

By the second item of this lemma it is easy to see that if $F(x)1_{(0\leq x<\infty)}$ has a finite mean then $J_F^* \geq 1$.

The following properties of LND or UND random variables are direct consequences of Definition 1.1 and were mentioned by Block et al. (1982, p. 769):

Lemma 2.2. *For random variables $\{X_k, k = 1, 2, \ldots\}$ and real functions $\{f_k(\cdot), k = 1, 2, \ldots\}$,*

(1) if $\{X_k, k = 1, 2, \ldots\}$ are LND (UND) and $\{f_k(\cdot), k = 1, 2, \ldots\}$ are all monotone increasing, then $\{f_k(X_k), k = 1, 2, \ldots\}$ are still LND (UND);

(2) if $\{X_k, k = 1, 2, \ldots\}$ are LND (UND) and $\{f_k(\cdot), k = 1, 2, \ldots\}$ are all monotone decreasing, then $\{f_k(X_k), k = 1, 2, \ldots\}$ are UND (LND);

(3) if $\{X_k, k = 1, 2, \ldots\}$ are ND and $\{f_k(\cdot), k = 1, 2, \ldots\}$ are either all monotone increasing or all monotone decreasing, then $\{f_k(X_k), k = 1, 2, \ldots\}$ are still ND;

(4) if $\{X_k, k = 1, 2, \ldots\}$ are nonnegative and UND, then for each $n = 1, 2, \ldots$,

$$\mathrm{E}\left(\prod_{k=1}^n X_k\right) \leq \prod_{k=1}^n \mathrm{E}X_k.$$

In the next lemma we establish general inequalities for the tail probabilities of sums of UND random variables. Relevant references in this direction but for independent random variables are Fuk and S.V. Nagaev (1971) and S.V. Nagaev (1979), whose ideas will be applied throughout the paper. See also Tang and Yan (2002). This lemma will be used in deriving a lower asymptotic estimate of the large-deviation probabilities.

We shall be using the symbols $x^+ = \max\{x, 0\}$, $m_- = \mathrm{E}X_1 1_{(X_1\leq 0)}$, and $m_+ = \mathrm{E}X_1 1_{(X_1>0)}$.

Lemma 2.3. *Let $\{X_k, k = 1, 2, \ldots\}$ be UND with common distribution F and mean 0 satisfying $\mathrm{E}\left(X_1^+\right)^r < \infty$ for some $r > 1$. Then for each fixed $\gamma > 0$ and $p > 0$, there exist positive numbers v and $C = C(v, \gamma)$ irrespective to x and n such that for all $x \geq \gamma n$ and $n = 1, 2, \ldots$,*

$$\Pr\left(\sum_{k=1}^n X_k \geq x\right) \leq n\overline{F}\left(vx\right) + Cx^{-p}. \tag{2.1}$$

Proof. If we have proven the result for all $x \geq \gamma n$ and all large n, say $n \geq n_0 + 1$, then using the inequality

$$\Pr\left(\sum_{k=1}^{n} X_k \geq x\right) \leq n \Pr\left(X_1 \geq \frac{x}{n_0}\right), \qquad \text{for } n = 1, 2, \ldots, n_0, \tag{2.2}$$

the result extends to all $n = 1, 2, \ldots$.

With arbitrarily fixed $v > 0$ we write $\widetilde{X}_k = \min\{X_k, vx\}$, $k = 1, 2, \ldots$, which, by Lemma 2.2(1), are still UND. A standard truncation argument gives that

$$\begin{aligned}\Pr\left(\sum_{k=1}^{n} X_k \geq x\right) &= \Pr\left(\sum_{k=1}^{n} X_k \geq x, \max_{1\leq k\leq n} X_k > vx\right) + \Pr\left(\sum_{k=1}^{n} X_k \geq x, \max_{1\leq k\leq n} X_k \leq vx\right) \\ &\leq n\overline{F}(vx) + \Pr\left(\sum_{k=1}^{n} \widetilde{X}_k \geq x\right). \end{aligned} \tag{2.3}$$

We estimate the second term above as follows. For a positive number $h = h(n, x)$, which we shall specify later, by Chebyshev's inequality and Lemma 2.2(1)(4),

$$\Pr\left(\sum_{k=1}^{n} \widetilde{X}_k \geq x\right) \leq \mathrm{e}^{-hx}\left(\mathrm{E}\mathrm{e}^{h\widetilde{X}_1}\right)^n. \tag{2.4}$$

Arbitrarily choose some $1 < q < \min\{r, 2\}$. We see that $\mathrm{E}\mathrm{e}^{h\widetilde{X}_1}$ is bounded from above by

$$\int_{-\infty}^{0} \left(\mathrm{e}^{hu} - 1\right) F(\mathrm{d}u) + \int_{0}^{vx} \frac{\mathrm{e}^{hu} - 1 - hu}{u^q} u^q F(\mathrm{d}u) + \left(\mathrm{e}^{hvx} - 1\right) \overline{F}(vx) + hm_+ + 1. \tag{2.5}$$

For the first term in (2.5), since

$$0 \leq \frac{\mathrm{e}^{hu} - 1 - hu}{h} \leq u(\mathrm{e}^{hu} - 1) \leq -u \qquad \text{for all } u \leq 0,$$

by the dominated convergence theorem we have

$$\lim_{h\searrow 0} \frac{\int_{-\infty}^{0} \left(\mathrm{e}^{hu} - 1\right) F(\mathrm{d}u)}{h} = \lim_{h\searrow 0} \int_{-\infty}^{0} \frac{\mathrm{e}^{hu} - 1 - hu}{h} F(\mathrm{d}u) + m_- = m_-.$$

Thus, there exists some real function $\alpha(\cdot)$ with $\alpha(h) \to 0$ as $h \searrow 0$ such that

$$\int_{-\infty}^{0} \left(\mathrm{e}^{hu} - 1\right) F(\mathrm{d}u) = (1 + \alpha(h))\, hm_-. \tag{2.6}$$

By virtue of the monotonicity in $u \in (0, \infty)$ of $\left(\mathrm{e}^{hu} - 1 - hu\right)/u^q$, we deal with the second term in (2.5) as

$$\int_{0}^{vx} \frac{\mathrm{e}^{hu} - 1 - hu}{u^q} u^q F(\mathrm{d}u) \leq \frac{\mathrm{e}^{hvx} - 1 - hvx}{(vx)^q} \mathrm{E}\left(X_1^+\right)^q. \tag{2.7}$$

Substituting (2.6) and (2.7) into (2.5), from (2.4) we obtain that

$$\begin{aligned}
&\Pr\left(\sum_{k=1}^{n}\widetilde{X}_k \geq x\right) \\
&\leq \mathrm{e}^{-hx}\left((1+\alpha(h))\,hm_- + \frac{\mathrm{e}^{hvx}-1-hvx}{(vx)^q}\mathrm{E}\left(X_1^+\right)^q + \left(\mathrm{e}^{hvx}-1\right)\overline{F}\left(vx\right) + hm_+ + 1\right)^n \\
&\leq \exp\left\{(1+\alpha(h))\,nhm_- + \frac{\mathrm{e}^{hvx}-1}{(vx)^q}n\mathrm{E}\left(X_1^+\right)^q + \left(\mathrm{e}^{hvx}-1\right)n\overline{F}\left(vx\right) + nhm_+ - hx\right\} \\
&= \exp\left\{\alpha(h)nhm_- + \frac{\mathrm{e}^{hvx}-1}{(vx)^q}n\mathrm{E}\left(X_1^+\right)^q + \left(\mathrm{e}^{hvx}-1\right)n\overline{F}\left(vx\right) - hx\right\}, \qquad (2.8)
\end{aligned}$$

where at the second step we used an elementary inequality $s+1 \leq \mathrm{e}^s$ for all s. In (2.8), take

$$h = \frac{1}{vx}\log\left(\frac{v^{q-1}x^q}{n\mathrm{E}\left(X_1^+\right)^q}+1\right),$$

which is positive and tends to 0 uniformly for all $x \geq \gamma n$. After some simple calculation we see that for all large n such that $\left|\alpha(h)m_-\right|/\gamma \leq 1/2$ holds for all $x \geq \gamma n$, the right-hand side of (2.8) is bounded from above by

$$\begin{aligned}
&\exp\left\{\frac{1}{2v}\log\left(\frac{v^{q-1}x^q}{n\mathrm{E}\left(X_1^+\right)^q}+1\right) + \frac{1}{v} + \frac{v^{q-1}x^q\overline{F}\left(vx\right)}{\mathrm{E}\left(X_1^+\right)^q} - \frac{1}{v}\log\left(\frac{v^{q-1}x^q}{n\mathrm{E}\left(X_1^+\right)^q}+1\right)\right\} \\
&\leq \exp\left\{\frac{1}{v} + \frac{v^{q-1}x^q\overline{F}\left(vx\right)}{\mathrm{E}\left(X_1^+\right)^q}\right\}\left(\frac{v^{q-1}\gamma x^{q-1}}{\mathrm{E}\left(X_1^+\right)^q}\right)^{-1/(2v)} \\
&\leq Cx^{-(q-1)/(2v)},
\end{aligned}$$

where the coefficient C is given by

$$C = \sup_{x\geq 0}\exp\left\{\frac{1}{v} + \frac{v^{q-1}x^q\overline{F}\left(vx\right)}{\mathrm{E}\left(X_1^+\right)^q}\right\}\left(\frac{v^{q-1}\gamma}{\mathrm{E}\left(X_1^+\right)^q}\right)^{-1/(2v)} < \infty.$$

Hence, with some $v > 0$ such that $(q-1)/(2v) > p$, from (2.3) we prove that inequality (2.1) holds for all $x \geq \gamma n$ and all large $n = 1, 2, \ldots$. ☐

3 Proof of Theorem 1.1

Hereafter, all limit relations are uniform for all $x \geq \gamma n$. We shall not repeat this explanation, but it remains in place.

3.1 An asymptotic upper estimate

Theorem 3.1. *Let $\{X_k, k = 1, 2, \ldots\}$ be UND with common distribution $F \in \mathcal{C}$ and mean 0. Then for each fixed $\gamma > 0$, the relation*

$$\Pr(S_n > x) \lesssim n\overline{F}(x) \tag{3.1}$$

holds uniformly for all $x \geq \gamma n$. That is,

$$\limsup_{n\to\infty} \sup_{x\geq\gamma n} \frac{\Pr(S_n > x)}{n\overline{F}(x)} \leq 1.$$

Proof. With arbitrarily fixed $0 < v < 1$, we write $\widetilde{X}_k = \min\{X_k, vx\}$, $k = 1, 2, \ldots$, and $\widetilde{S}_n = \sum_{k=1}^{n} \widetilde{X}_k$, $n = 1, 2, \ldots$. Analogously to (2.3),

$$\Pr(S_n > x) \leq n\overline{F}(vx) + \Pr\left(\widetilde{S}_n > x\right). \tag{3.2}$$

To estimate the second term above, we write $a = \max\{-\log(n\overline{F}(vx)), 1\}$, which tends to ∞. Analogously to (2.4), with a temporarily fixed number $h = h(x, n) > 0$ we have

$$\frac{\Pr\left(\widetilde{S}_n > x\right)}{n\overline{F}(vx)} \leq \mathrm{e}^{-hx+a}\left(\mathrm{E}\mathrm{e}^{h\widetilde{X}_1}\right)^n. \tag{3.3}$$

We split the expectation $\mathrm{E}\mathrm{e}^{h\widetilde{X}_1}$ into several parts as

$$\begin{aligned}\mathrm{E}\mathrm{e}^{h\widetilde{X}_1} &= \left(\int_{-\infty}^{0} + \int_{0}^{vx/a^2} + \int_{vx/a^2}^{vx}\right)(\mathrm{e}^{hu} - 1)F(\mathrm{d}u) + \left(\mathrm{e}^{hvx} - 1\right)\overline{F}(vx) + 1 \\ &\hat{=} (I_1 + I_2 + I_3) + \left(\mathrm{e}^{hvx} - 1\right)\overline{F}(vx) + 1. \end{aligned} \tag{3.4}$$

The idea of the division in (3.4) is from Cline and Hsing (1991), but the method is different. As done in (2.6), there exists some real function $\alpha(\cdot)$ with $\alpha(h) \to 0$ as $h \searrow 0$ such that

$$I_1 = (1 + \alpha(h))\, hm_-. \tag{3.5}$$

For I_2, using an elementary inequality $\mathrm{e}^s - 1 \leq s\mathrm{e}^s$ for all s we have

$$I_2 \leq \mathrm{e}^{hvx/a^2} h \int_0^{vx/a^2} uF(\mathrm{d}u) \leq \mathrm{e}^{hvx/a^2} hm_+. \tag{3.6}$$

For I_3, by Lemma 2.1(1) with some $p > J_F^*$, there exist some positive constants C and D irrespective to x and n such that for all $x \geq D$,

$$I_3 \leq \mathrm{e}^{hvx}\overline{F}(vx/a^2) \leq C\mathrm{e}^{hvx}a^{2p}\overline{F}(vx). \tag{3.7}$$

Substituting (3.5), (3.6), and (3.7) into (3.4) yields that

$$\mathrm{E}\mathrm{e}^{h\widetilde{X}_1} \leq (1+\alpha(h))\, hm_- + \mathrm{e}^{hvx/a^2} hm_+ + O(1)\mathrm{e}^{hvx} a^{2p}\overline{F}(vx) + 1. \tag{3.8}$$

Further substituting (3.8) into (3.3) and setting in the resulting inequality

$$h = \frac{a - 2p\log a}{vx},$$

which tends to zero, we obtain that

$$\frac{\Pr\left(\widetilde{S}_n > x\right)}{n\overline{F}(vx)} \leq \mathrm{e}^{-hx+a}\left((1+\alpha(h))\, hm_- + \mathrm{e}^{1/a} hm_+ + O(1)\mathrm{e}^{a}\overline{F}(vx) + 1\right)^n.$$

Then, as done in (2.8), using the elementary inequality $s + 1 \leq \mathrm{e}^s$ for all s gives that

$$\begin{aligned}
\frac{\Pr\left(\widetilde{S}_n > x\right)}{n\overline{F}(vx)} &\leq \exp\left\{(1+\alpha(h))\, nhm_- + \mathrm{e}^{1/a} nhm_+ + O(1)\mathrm{e}^{a} n\overline{F}(vx) - hx + a\right\} \\
&= \exp\left\{\left((1+\alpha(h))\, m_- + \mathrm{e}^{1/a} m_+\right) nh + O(1) - hx + a\right\} \\
&= O(1)\exp\left\{o(1)nh - hx + a\right\} \\
&= O(1)\exp\left\{o(a) + (1 - 1/v)\, a\right\} \\
&= o(1),
\end{aligned} \tag{3.9}$$

where the fact $(1+\alpha(h))\, m_- + \mathrm{e}^{1/a} m_+ \to 0$ used at the third step results from $\alpha(h) \to 0$, $a \to \infty$, and $m_- + m_+ = 0$. Substituting (3.9) into (3.2) yields that

$$\Pr(S_n > x) \lesssim n\overline{F}(vx). \tag{3.10}$$

By the arbitrariness of $0 < v < 1$ and the definition of $F \in \mathcal{C}$, we obtain that

$$\limsup_{n\to\infty} \sup_{x \geq \gamma n} \frac{\Pr(S_n > x)}{n\overline{F}(x)} \leq \lim_{v \nearrow 1} \limsup_{n\to\infty} \sup_{x\geq \gamma n} \frac{n\overline{F}(vx)}{n\overline{F}(x)} = 1. \tag{3.11}$$

This proves (3.1). $\square$

Note that during the proof of Theorem 3.1, the condition $F \in \mathcal{D}$ suffices for (3.10). Moreover, for each fixed positive integer n_0, from (2.2) we see that the inequality

$$\Pr\left(\sum_{k=1}^{n} X_k \geq x\right) \leq C(n_0) n\overline{F}(x)$$

holds for some $C(n_0) > 0$, all $x \geq 0$, and all $n = 1, 2, \ldots, n_0$. Hence, we conclude the following:

Corollary 3.1. *Let $\{X_k, k = 1, 2, \ldots\}$ be UND with common distribution $F \in \mathcal{D}$ and mean 0. Then for each fixed $\gamma > 0$ and some $C = C(\gamma)$ irrespective to x and n, the inequality*

$$\Pr(S_n > x) \le Cn\overline{F}(x) \tag{3.12}$$

holds for all $x \ge \gamma n$ and $n = 1, 2, \ldots$.

Inequality (3.12) is sharp in view of the fact that its bound is linear in n. This result is useful, particularly when dealing with tail probabilities of random sums of heavy-tailed random variables.

3.2 An asymptotic lower estimate

Theorem 3.2. *Let $\{X_k, k = 1, 2, \ldots\}$ be LND with common distribution $F \in \mathcal{C}$ and mean 0 satisfying (1.6). Then for each fixed $\gamma > 0$, the relation*

$$\Pr(S_n > x) \gtrsim n\overline{F}(x) \tag{3.13}$$

holds uniformly for all $x \ge \gamma n$. That is,

$$\liminf_{n\to\infty} \inf_{x\ge\gamma n} \frac{\Pr(S_n > x)}{n\overline{F}(x)} \ge 1.$$

Condition (1.6) is unnecessary when $\{X_k, k = 1, 2, \ldots\}$ are mutually independent.

Proof. With arbitrarily fixed $v > 1$, we have

$$\begin{aligned}\Pr(S_n > x) &\ge \Pr\left(S_n > x, \max_{1\le k\le n} X_k > vx\right)\\ &\ge \sum_{k=1}^{n} \Pr(S_n > x, X_k > vx) - \sum_{1\le k<l\le n} \Pr(S_n > x, X_k > vx, X_l > vx)\\ &\hat{=}\ J_1 - J_2. \end{aligned}\tag{3.14}$$

From the remark after Definition 1.1, it is clear that the random variables $\{X_k, k = 1, 2, \ldots\}$ are pairwise UND. Therefore,

$$J_2 \le \left(n\overline{F}(vx)\right)^2. \tag{3.15}$$

To deal with J_1, recall an elementary inequality $\Pr(E_1E_2) \ge \Pr(E_1) + \Pr(E_2) - 1$ for all events E_1 and E_2. We have

$$J_1 \ge \sum_{k=1}^{n} \Pr(S_n - X_k > (1-v)x, X_k > vx) \tag{3.16}$$

$$\begin{aligned} &\ge \sum_{k=1}^{n} (\Pr(S_n - X_k > (1-v)x) + \Pr(X_k > vx) - 1)\\ &= \sum_{k=1}^{n} \left(\overline{F}(vx) - \Pr\left(\sum_{l:\ 1\le l\le n, l\ne k} (-X_l) \ge (v-1)x\right)\right). \end{aligned}\tag{3.17}$$

By Lemma 2.2(2), the random variables $\{-X_k, k = 1, 2, \ldots\}$ are UND. Then for arbitrarily fixed $p > J_F^*$, by Lemma 2.3 there exist positive constants v_0 and C irrespective to x and n such that the inequality

$$\Pr\left(\sum_{l:\ 1\le l\le n, l\ne k}(-X_l) \ge (v-1)x\right) \le n \Pr\left(-X_1 \ge \frac{(v-1)x}{v_0}\right) + Cx^{-p}$$

holds for all $x \ge \gamma n$ and $n = 1, 2, \ldots$. Hence by Lemma 2.1(2) and condition (1.6),

$$\Pr\left(\sum_{l:\ 1\le l\le n, l\ne k}(-X_l) \ge (v-1)x\right) \le \frac{x}{\gamma} F\left(-\frac{(v-1)x}{v_0}\right) + Cx^{-p} = o\left(\overline{F}(vx)\right). \quad (3.18)$$

Substituting (3.18) into (3.17) yields that

$$J_1 \gtrsim n\overline{F}(vx). \quad (3.19)$$

Then substituting (3.15) and (3.19) into (3.14) yields that

$$\Pr(S_n > x) \gtrsim n\overline{F}(vx). \quad (3.20)$$

Thus, analogously to (3.11) we prove relation (3.13).

Now we assume that the random variables $\{X_k, k = 1, 2, \ldots\}$ are mutually independent and we show that condition (1.6) is unnecessary. Actually, in this case we only need to rewrite the treatment on J_1 in the segment between (3.16) and (3.19) as follows (the other part of the proof remains valid):

$$J_1 \ge \sum_{k=1}^{n} \overline{F}(vx) \Pr(S_n - X_k > (1-v)x) \sim n\overline{F}(vx),$$

where we used the law of large numbers to obtain that $\Pr(S_n - X_k > (1-v)x) \to 1$. This ends the proof of Theorem 3.2. □

Note that during the proof of Theorem 3.2 the condition $F \in \mathcal{D}$ suffices for (3.20). Moreover, for each fixed positive integer n_0 the inequalities

$$\Pr(S_n > x) \ge \Pr(X_1 > x)\Pr(X_2 > 0)\cdots\Pr(X_n > 0) \ge C(n_0)n\overline{F}(x)$$

trivially hold for some $C(n_0) > 0$, all $x \ge 0$, and all $n = 1, 2, \ldots, n_0$. Analogously to Corollary 3.1, we conclude the following:

Corollary 3.2. *Let $\{X_k, k = 1, 2, \ldots\}$ be LND with common distribution $F \in \mathcal{D}$ and mean 0 satisfying (1.6). Then for each fixed $\gamma > 0$ and some $C = C(\gamma) > 0$, the inequality*

$$\Pr(S_n > x) \ge Cn\overline{F}(x)$$

holds for all $x \ge \gamma n$ and $n = 1, 2, \ldots$. Condition (1.6) is unnecessary when $\{X_k, k = 1, 2, \ldots\}$ are mutually independent.

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