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TWO COALESCENTS DERIVED FROM THE RANGES OF STABLE SUBORDINATORS

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Abstract Let M_α be the closure of the range of a stable subordinator of index $\alpha \in]0,1[$. There are two natural constructions of the M_α's simultaneously for all $\alpha \in]0,1[$, so that $M_\alpha \subseteq M_\beta$ for $0 < \alpha < \beta < 1$: one based on the intersection of independent regenerative sets and one based on Bochner's subordination. We compare the corresponding two coalescent processes defined by the lengths of complementary intervals of $[0,1] \backslash M_{1-\rho}$ for $0 < \rho < 1$. In particular, we identify the coalescent based on the subordination scheme with the coalescent recently introduced by Bolthausen and Sznitman [6].

Keywords stable regenerative set, Bochner's subordination, Bolthausen-Sznitman coalescent, Poisson covering, zero sets of Bessel processes, two parameter Poisson-Dirichlet distribution, fragmentation

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Contents

1 Introduction

A *ranked coalescent process* describes the evolution of a system of masses which aggregate randomly as time passes. More formally, write $\mathcal{S}^{\downarrow}$ for the state space of decreasing positive sequences $v := (v(n), n \in \mathbb{N})$ with $\sum_n v(n) = 1$, so each term $v(n)$ can be viewed as the mass of some fragment of a unit mass. In a ranked coalescent process $(V_\rho, \rho \in I)$ parameterized by some interval $I \subseteq \mathbb{R}$, for each $\rho < \rho'$

$$V_{\rho'}(n) = \sum_{k \in \Pi(\rho, \rho', n)} V_\rho(k)$$

where $\Pi(\rho, \rho', n)$ indicates which of the masses present at time ρ have coalesced by time ρ' to form the nth largest mass present at time ρ'. The $\Pi(\rho, \rho', n)$ for $n \in \mathbb{N}$ are the blocks of some random partition of $\mathbb{N}$, and these partitions are subject to a consistency requirement as ρ and ρ' vary. We call the time reversal of a ranked coalescent process a *ranked fragmentation process.* See [8] for a general framework for the analysis of such processes, and [1, 3, 6, 8, 17] for some specific examples.

Our purpose here is to discuss some instances of the following general construction, where we assume with little loss of generality that $I =]0,1[$. Consider a family $(M_\alpha, 0 < \alpha < 1)$ of closed subsets of $[0, \infty[$ which is *nested*, meaning that $M_\alpha \subseteq M_\beta$ for $0 < \alpha < \beta < 1$, and suppose that each M_α has zero Lebesgue measure. Let $V(M_\alpha) \in \mathcal{S}^{\downarrow}$ be the sequence of ranked lengths of component intervals of $[0,1] \backslash M_\alpha$. Then $(V(M_{1-\rho}), 0 < \rho < 1)$ is a ranked coalescent process. Kingman [12] gave such a construction of his coalescent using discrete random sets M_α. Here we focus on two natural constructions of nested $(M_\alpha, 0 < \alpha < 1)$ such that

$$M_\alpha = \{\sigma_\alpha(t), t \geq 0\}^{\text{cl}} \tag{1}$$

is a random *stable regenerative subset* of $[0,\infty[$ defined by the closure of the range of $\sigma_\alpha := (\sigma_\alpha(t), t \geq 0)$, a stable subordinator of index α. Following [22], the distribution of $V(M_\alpha)$ on $\mathcal{S}^\downarrow$ derived from M_α in (1) will be called *Poisson-Dirichlet with parameters* $(\alpha, 0)$, denoted $PD(\alpha, 0)$. See also [20, 18] for further background. Our interest in constructions of nested M_α was spurred by the following result:

Theorem 1 (Bolthausen-Sznitman [6]) *There exists a (time-inhomogeneous) Markovian ranked fragmentation process $(V_\alpha^{\mathrm{BS}}, 0 < \alpha < 1)$ such that $(V_{\mathrm{e}^{-t}}^{\mathrm{BS}}; t > 0)$ is a time-homogeneous ranked Markovian coalescent with Feller semigroup, and for each $\alpha \in]0,1[$ the law of V_α^{BS} is $PD(\alpha, 0)$.*

We recall in Section 2.3 the precise definition of the semigroup of this *Bolthausen-Sznitman coalescent* $(V_{\mathrm{e}^{-t}}^{\mathrm{BS}}; t > 0)$. The equality in distribution $V_\alpha^{\mathrm{BS}} \stackrel{d}{=} V(M_\alpha)$, for each fixed $\alpha \in]0,1[$, suggests the possibility of constructing a family of nested stable regenerative sets $(M_\alpha, 0 < \alpha < 1)$ such that $(V(M_\alpha), 0 < \alpha < 1)$ is a realization of $(V_\alpha^{\mathrm{BS}}, 0 < \alpha < 1)$. To this end we consider Bochner's subordination [5]. Recall that if σ_α and $\sigma_{\alpha'}$ are independent stable subordinators, then the subordinate process $\sigma_\alpha \circ \sigma_{\alpha'}$ has the same law as $\sigma_{\alpha\alpha'}$. By application of Kolmogorov's extension theorem we can justify the following construction:

Construction 2 *Let*

$$M_\alpha^* := \{\sigma_\alpha^*(t), t \geq 0\}^{\mathrm{cl}}, \qquad 0 < \alpha < 1$$

where $(\sigma_\alpha^*, 0 < \alpha < 1)$ is a family of stable subordinators such that for every $0 < \alpha_n < \ldots < \alpha_1 < 1$, the joint distribution of $\sigma_{\alpha_1}^*, \ldots, \sigma_{\alpha_n}^*$ is the same as that of $\sigma_{\alpha_1}, \ldots, \sigma_{\alpha_n}$ defined as follows. Consider a family of n independent stable subordinators, $\tau_{\beta_1}, \ldots, \tau_{\beta_n}$ with indices $\beta_1, \ldots, \beta_n \in]0,1]$ such that $\alpha_i = \beta_1 \ldots \beta_i$ for $i = 1, \ldots, n$, and set $\sigma_{\alpha_i} = \tau_{\beta_1} \circ \ldots \circ \tau_{\beta_i}$.

Our main result, which we prove in Section 2.3 using excursion theory and special properties of Poisson-Dirichlet laws, is the following identity:

Theorem 3 *There is the equality of finite-dimensional distributions of ranked fragmentation processes*

$$(V(M_\alpha^*), 0 < \alpha < 1) \stackrel{d}{=} \left(V_\alpha^{\mathrm{BS}}, 0 < \alpha < 1\right). \tag{2}$$

This result is closely related to another recent construction of the Bolthausen-Sznitman coalescent in [4], based on the genealogy of Neveu's continuous-state branching process. We explain the connection between the two constructions in Section 2.4. See also [17] for quite a different construction, which yields generalizations of the Bolthausen-Sznitman coalescent.

There is another natural construction of nested stable regenerative sets M_α which is implicit in the literature. Recall from [15] that M_α can be constructed for each fixed $\alpha \in]0,1[$ as the zero set of a Bessel process of dimension $2 - 2\alpha$ started at 0. The additivity property of squares of Bessel processes then justifies the following construction [24, 19]:

Construction 4 *Construct on a common probability space the squares of Bessel processes with dimension δ, $X_{\delta,\bullet}$, in such a way that the path-valued process $(X_{\delta,\bullet}, \delta \geq 0)$ is a $C[0,\infty[$-valued process with independent increments and càdlàg paths, and set $M_\alpha^\cap = \{t \geq 0 : X_{2-2\alpha,t} = 0\}$.*

The Poisson structure of jumps of $(X_{\delta,\bullet}, \delta \geq 0)$, described in [19] (see also [16, Proposition 14]), shows that the process created by Construction 4 can be represented by a simpler random covering scheme [10]:

Construction 5 *Generate a Poisson point process of triples (a, x, z) in the space $]0,1[\times]0,\infty[^2$ with intensity $da\,dx\,z^{-2}dz$, and let $M_\alpha^\cap$ be the set left uncovered by intervals $]x, x+z[$ corresponding to points (a, x, z) of the Poisson process with $a \leq 1-\alpha$.*

That $M_\alpha^\cap$ is the closure of the range of a stable subordinator of index α can be read from [10, p. 180]. Implicit in Constructions 4 and 5 is the well known fact that if M_α and $M_{\alpha'}$ are independent, then the intersection $M_\alpha \cap M_{\alpha'}$ has the same law as $M_{\alpha+\alpha'-1}$, where $M_\alpha := \{0\}$ for $\alpha \leq 0$.

Plainly, the set-valued processes $(M_\alpha^\cap, 0 < \alpha < 1)$ and $(M_\alpha^*, 0 < \alpha < 1)$ have the same one-dimensional distributions, meaning that $M_\alpha^\cap$ and M_α^* have the same law as M_α in (1) for each fixed α. But these families do not have the same finite-dimensional distributions. Indeed, we will show in Section 3 that the ranked fragmentation processes $(V(M_\alpha^\cap), 0 < \alpha < 1)$ and $(V(M_\alpha^*), 0 < \alpha < 1)$ do not have the same laws. So Theorem 3 is false with $M_\alpha^\cap$ instead of M_α^*. Nonetheless, there are some striking resemblances between the two families, and it is interesting to investigate the similarities and differences.

One similarity involves the so-called *age* A_α of M_α at time 1. That is, A_α is the length of the component interval of $[0,1]\backslash M_\alpha$ that contains 1 (or equivalently, $1 - A_\alpha$ is the largest point in $[0,1] \cap M_\alpha$). Note that A_α is one of the components of $V(M_\alpha)$. To be more precise [20], A_α is a *size-biased pick* from the components of $V(M_\alpha)$, meaning that

$$P(A_\alpha = V(M_\alpha)(n) \mid V(M_\alpha)) = V(M_\alpha)(n) \qquad (n = 1, 2, \ldots)$$

where $V(M_\alpha)(n)$ is the length of the nth longest component interval of $[0,1] \cap M_\alpha$. Here and in the sequel, we develop notation for M_α, and modify by a superscript $\cap$ or $*$ to replace M_α by $M_\alpha^\cap$ or M_α^*. It is known from [7] that there is a gamma subordinator $(\Gamma_\rho, \rho \geq 0)$ such that for each fixed $\rho \in]0,1[$

$$A_{1-\rho}^\cap \overset{d}{=} A_{1-\rho}^* \overset{d}{=} \Gamma_\rho / \Gamma_1. \tag{3}$$

As an extension of this, we obtain the following proposition:

Proposition 6 *Let $(\Gamma_\rho; \rho \geq 0)$ be a gamma subordinator. There is the equality of finite-dimensional distributions*

$$(A_{1-\rho}^\cap; 0 < \rho < 1) \overset{d}{=} (A_{1-\rho}^*; 0 < \rho < 1) \overset{d}{=} (\Gamma_\rho/\Gamma_1; 0 < \rho < 1).$$

In other words, each of these process has a right-continuous version which is the cumulative distribution function of a Dirichlet random measure on $]0,1[$ governed by Lebesgue measure.

This Proposition is a consequence of [2, Proposition 8], as it is easily checked that for every $0 < \alpha_1 < \ldots < \alpha_n < 1$, the embeddings

$$M_{\alpha_1}^\cap \subseteq \ldots \subseteq M_{\alpha_n}^\cap \quad \text{and} \quad M_{\alpha_1}^* \subseteq \ldots \subseteq M_{\alpha_n}^*$$

are compatible with the regenerative property in the sense [2]. We also give another proof of Proposition 6 in Section 4. Finally, we mention some open problems about the coalescent $\big(V(M_{1-\rho}^\cap), 0 < \rho < 1\big)$ in section 5.

2 The Subordination Scheme

2.1 Preliminaries

It is known that if $V_\alpha = (V_\alpha(1), V_\alpha(2), \ldots)$ has the $PD(\alpha, 0)$ law, meaning $V_\alpha \stackrel{d}{=} V(M_\alpha)$, then the limit

$$L_{1,\alpha} := \lim_{n\to\infty} n\ (V_\alpha(n))^\alpha \tag{4}$$

exists with probability 1. Moreover, if $V_\alpha = V(M_\alpha)$ then $L_{1,\alpha}$ coincides with the local time of the random set $[0,1] \cap M_\alpha$. Alternatively, $L_{1,\alpha}$ can be constructed as the first passage time above level 1 for the stable subordinator σ_α, provided that the latter has been suitably normalized (which induces no loss of generality by the scaling property). See equations (2.c-h) in [20] for details.

Definition 7 [22] *For $\alpha \in]0,1[$ and $\theta > -\alpha$, the Poisson-Dirichlet distribution with parameters (α, θ), denoted $PD(\alpha, \theta)$, is the law on $\mathcal{S}^\downarrow$ that is absolutely continuous with respect to $PD(\alpha, 0)$, with density proportional to $L_{1,\alpha}^{\theta/\alpha}$.*

The rest of this section is organized as follows. In the next subsection, we develop material on the 'excursions' of the random set M_α^* away from M_γ^* for arbitrary $0 < \gamma < \alpha < 1$. This is used in the third subsection to prove Theorem 3. In the final subsection, we relate Theorem 3 to a different construction of the Bolthausen-Sznitman coalescent process based on Bochner's subordination that has been recently obtained in [4], using a multidimensional extension of an identity due to Pitman and Yor [20].

2.2 Normalized excursion and meander

In this subsection, we work in the canonical space Ω of closed subsets $\omega \subseteq [0, \infty[$ with $0 \in \omega$. We give Ω the topology of Matheron [14] and the corresponding Borel field $\mathcal{B}(\Omega)$. We write $\mathtt{n}$ for the operator of normalization of compact sets. That is, if $\max \omega < \infty$, then

$$\mathtt{n}(\omega) \ = \ \{x \in [0,1] : x \max \omega \in \omega\} \ ,$$

and $\mathtt{k}_r$ for the killing operator at $r \geq 0$

$$\mathtt{k}_r(\omega) \ = \ \omega \cap [0, r] \, .$$

For $\omega \in \Omega$ let $V(\omega)$ be the sequence of ranked lengths of component intervals of $[0,1]\backslash\omega$. Note that $V(\omega) = V(\mathtt{k}_1(\omega))$, and that $V(\omega) \in \mathcal{S}^\downarrow$ provided the Lebesgue measure of ω equals 0, as it is for almost all ω with respect to the distribution of M_α for each $0 < \alpha < 1$. Here as before, $M_\alpha := \{\sigma_\alpha(t), t \geq 0\}^{\mathrm{cl}}$ for σ_α a stable subordinator with index α, and we now regard M_α as a random variable with values in Ω. Our analysis relies on the following identity in distribution [20, Theorem 1.1]:

$$V(M_\alpha) \ \stackrel{d}{=} \ V\left(\mathtt{n} \circ \mathtt{k}_{\sigma_\alpha(1)}(M_\alpha)\right) \tag{5}$$

where the common distribution of both sides is the Poisson-Dirichlet law $PD(\alpha, 0)$.

We focus now on the joint law of (M_γ^*, M_α^*) for arbitrary $0 < \gamma < \alpha < 1$. Let $\beta := \gamma/\alpha$, so $0 < \beta < 1$. Suppose that σ_α and τ_β are independent stable subordinators of indices $\alpha, \beta \in]0,1[$, and that M^*_α and M^*_γ are the closed ranges of σ_α and $\sigma_\alpha \circ \tau_\beta$ respectively. Consider the random countable subset of $[0,\infty[\times\Omega$ defined by the points

$$(t, \{\sigma_\alpha(s) - \sigma_\alpha(\tau_\beta(t-)), \tau_\beta(t-) \le s \le \tau_\beta(t)\}^{\mathrm{cl}}) \quad \text{with} \quad \tau_\beta(t-) < \tau_\beta(t). \tag{6}$$

As both σ_α and τ_β have independent and stationary increments, and τ_β is independent of σ_α, it follows by the argument of Itô [11] that the points (6) are points of a Poisson point process $[0,\infty[\times\Omega$ with intensity measure $dt\,\nu_{\alpha,\gamma}(d\omega)$, for some measure $\nu_{\alpha,\gamma}$ on Ω, call it the *law of excursions of M^*_α away from M^*_γ*. By construction of the point process there is the formula

$$\nu_{\alpha,\gamma}(B) = c\int_0^\infty \mathtt{P}\left(\mathtt{k}_{\sigma_\alpha(t)}(M^*_\alpha) \in B\right) t^{-\beta-1}dt\,, \quad B \in \mathcal{B}(\Omega)\,.$$

Here $ct^{-\beta-1}dt$ is the Lévy measure of τ_β, with $c > 0$ a constant whose value is not relevant. Note that c may change from line to line in the sequel. We define the *meander* $M^{*\mathbf{me}}_{\alpha,\gamma}$ to be a random set distributed according to an excursion conditioned to have length at least 1 and restricted to the unit interval. That is,

$$\mathtt{P}(M^{*\mathbf{me}}_{\alpha,\gamma} \in B) = c\int_0^\infty \mathtt{P}\left(\mathtt{k}_1(M^*_\alpha) \in B, \sigma_\alpha(t) > 1\right) t^{-\beta-1}dt\,, \quad B \in \mathcal{B}(\Omega).$$

We also define a *normalized excursion* $M^{*\mathbf{ex}}_{\alpha,\gamma}$ to be a random set distributed as the excursion conditioned to have length at least 1 and then normalized in order to have unit length. That is

$$\mathtt{P}(M^{*\mathbf{ex}}_{\alpha,\gamma} \in B) = c\int_0^\infty \mathtt{P}\left(\mathtt{n}\circ\mathtt{k}_{\sigma_\alpha(t)}(M^*_\alpha) \in B, \sigma_\alpha(t) > 1\right) t^{-\beta-1}dt\,, \quad B \in \mathcal{B}(\Omega)\,.$$

Consider now $V(M^{*\mathbf{ex}}_{\alpha,\gamma})$ and $V(M^{*\mathbf{me}}_{\alpha,\gamma})$, the ranked lengths of intervals that result from the partition of $[0,1]$ induced by $M^{*\mathbf{me}}_{\alpha,\gamma}$ and $M^{*\mathbf{ex}}_{\alpha,\gamma}$ respectively.

Lemma 8 *For $0 < \gamma < \alpha < 1$, the two sequences $V(M^{*\mathbf{ex}}_{\alpha,\gamma})$ and $V(M^{*\mathbf{me}}_{\alpha,\gamma})$ have the same law, which is $PD(\alpha,-\gamma)$.*

Proof: Recall $\gamma = \alpha\beta$. First, observe that the scaling property combined with Fubini's theorem yields that the distribution of the normalized excursion is given for every $B \in \mathcal{B}(\Omega)$ by

$$\begin{aligned}
\mathtt{P}(M^{*\mathbf{ex}}_{\alpha,\gamma} \in B) &= c\int_0^\infty \mathtt{P}\left(\mathtt{n}\circ\mathtt{k}_{\sigma_\alpha(t)}(M^*_\alpha) \in B, \sigma_\alpha(t) > 1\right) t^{-\beta-1}dt \\
&= c\int_0^\infty \mathtt{P}\left(\mathtt{n}\circ\mathtt{k}_{\sigma_\alpha(t)}(M^*_\alpha) \in B, t^{-1/\alpha}\sigma_\alpha(t) > t^{-1/\alpha}\right) t^{-\beta-1}dt \\
&= c\int_0^\infty \mathtt{P}\left(\mathtt{n}\circ\mathtt{k}_{\sigma_\alpha(1)}(M^*_\alpha) \in B, \sigma_\alpha(1) > t^{-1/\alpha}\right) t^{-\beta-1}dt \\
&= c\,\mathtt{E}\left((\sigma_\alpha(1))^\gamma; \mathtt{n}\circ\mathtt{k}_{\sigma_\alpha(1)}(M^*_\alpha) \in B\right)\,.
\end{aligned}$$

In particular, the distribution of $V(M^{*\mathbf{ex}}_{\alpha,\gamma})$ is absolutely continuous with respect to that of $V(\mathtt{n}\circ\mathtt{k}_{\sigma_\alpha(1)}(M^*_\alpha))$ with density proportional to $(\sigma_\alpha(1))^\gamma$. As $(\sigma_\alpha(1))^{-\alpha}$ coincides with the local time of the normalized set $\mathtt{n}\circ\mathtt{k}_{\sigma_\alpha(1)}(M^*_\alpha)$, we deduce from (5) that the distribution of $V(M^{*\mathbf{ex}}_{\alpha,\gamma})$ is absolutely continuous with respect to that of $V(M^*_\alpha)$ with density proportional to $L_{1,\alpha}^{-\gamma/\alpha}$, where $L_{1,\alpha}$ stands for the local time of M^*_α at 1. So by Definition 7, the law of $V(M^{*\mathbf{ex}}_{\alpha,\gamma})$ is $PD(\alpha,-\gamma)$. On the other hand, $\{\sigma_\alpha(t)>1\}=\{L_{1,\alpha}<t\}$ up to a null set, and it then follows that the law of the meander is given for every $B\in\mathcal{B}(\Omega)$ by

$$\begin{aligned}
\mathtt{P}(M^{*\mathbf{me}}_{\alpha,\gamma}\in B) &= c\int_0^\infty \mathtt{P}\left(\mathtt{k}_1(M^*_\alpha)\in B,\sigma_\alpha(t)>1\right)t^{-\beta-1}dt\\
&= c\int_0^\infty \mathtt{P}\left(\mathtt{k}_1(M^*_\alpha)\in B,L_{1,\alpha}<t\right)t^{-\beta-1}dt\\
&= c\,\mathtt{E}\left(L_{1,\alpha}^{-\beta}\,,\,\mathtt{k}_1(M^*_\alpha)\in B\right).
\end{aligned}$$

In particular, the law of $V(M^{*\mathbf{me}}_{\alpha,\gamma})$ is absolutely continuous with respect to that of $V(M^*_\alpha)$ with density proportional to $L_{1,\alpha}^{-\beta}$. So the law of $V(M^{*\mathbf{ex}}_{\alpha,\gamma})$ is also $PD(\alpha,-\gamma)$. □

Next, following [21, 17], we can associate to any parameters $\alpha\in]0,1[$ and $\theta>-\alpha$ an (α,θ)-*fragmentation kernel* on $\mathcal{S}^\downarrow$ as follows. We introduce first a sequence $Y_1,Y_2,\ldots$ of i.i.d. variables with law $PD(\alpha,\theta)$. Then for an arbitrary $v=(v(n),n\in\mathtt{N})\in\mathcal{S}^\downarrow$, we write (α,θ)-FRAG$(v,\cdot)$ for the distribution of the decreasing rearrangement of the elements of the sequences $v(1)Y_1,v(2)Y_2,\ldots$.

Lemma 9 *For each choice of α and γ with $0<\gamma<\alpha<1$, the conditional law of $V(M^*_\alpha)$ given $\mathtt{k}_1\left(M^*_\gamma\right)$ is $(\alpha,-\gamma)$-FRAG$(V(M^*_\gamma),\cdot)$.*

Proof: Let I_0 be the right-most interval component of $[0,1]\backslash M^*_\gamma$, and write $(I_k,k\in\mathtt{N})$, for the sequence of the remaining open intervals of this decomposition, ranked according to the decreasing order of lengths. For every integer $k\geq 0$, let $\ell_k\in]0,1[$ be the left-end point of the interval I_k and denote by Z_k the Ω-valued random variable such that

$$Z_k=\{x\in[0,1]:\ell_k+|I_k|x\in M^*_\alpha\}\,.$$

Note that the sequence of the lengths $|I_0|,|I_1|,\ldots$ is measurable with respect to $\mathtt{k}_1(M^*_\gamma)$ and that its increasing rearrangement is $V(M^*_\gamma)$. Standard arguments of excursion theory using the scaling property imply that $Z_0,Z_1,\ldots$ is a sequence of independent variables which is also independent of $\mathtt{k}_1(M^*_\gamma)$. Moreover, Z_0 has the law of the meander $M^{*\mathbf{me}}_{\alpha,\gamma}$, and for $k\geq 1$, each Z_k has the law of the normalized excursion $M^{*\mathbf{ex}}_{\alpha,\gamma}$. By construction, $V(M^*_\alpha)$ is the decreasing rearrangement of the elements of the sequences $|I_0|V(Z_0),|I_1|V(Z_1),\ldots$ As we know from Lemma 8 that the variables $V(Z_0),V(Z_1),\ldots$ all have the $PD(\alpha,-\gamma)$ distribution, our claim is proven. □

We recall next a basic duality relation between Poisson-Dirichlet fragmentation and coagulation kernels. Associated with each probability distribution Q on $\mathcal{S}^\downarrow$ there is a Markov kernel on $\mathcal{S}^\downarrow$, the *Q-coagulation kernel*, denoted Q-COAG, which is defined as follows [6, 17]. Let V be a random element of $\mathcal{S}^\downarrow$ with distribution Q, and given V let $X_1,X_2,\ldots$ be a sequence of i.i.d. $\mathtt{N}$-valued random variables with law $\mathtt{P}(X_i=n|V)=V(n)$. For $v\in\mathcal{S}^\downarrow$, let Q-COAG$(v,\cdot)$ be the distribution on $\mathcal{S}^\downarrow$ of the ranked rearrangement of the sequence

$$\left(\sum_n v(n)1(X_n=1),\sum_n v(n)1(X_n=2),\ldots\right).$$

For $Q = PD(\alpha, \theta)$, this kernel will be denoted simply (α, θ)-COAG.

Lemma 10 [17, Corollary 13] *Fix $\alpha, \beta \in]0,1[$ and $\theta > -\alpha\beta$, and let V and V' be two $\mathcal{S}^{\downarrow}$-valued random variables. Then the following are equivalent:*

(i) *V has the $PD(\alpha, \theta)$-law and the conditional law of V' given V is* $(\beta, \theta/\alpha)$-COAG$(V, \cdot)$.

(ii) *V' has the $PD(\alpha\beta, \theta)$-law and the conditional law of V given V' is* $(\alpha, -\alpha\beta)$-FRAG$(V', \cdot)$.

By Lemma 9, condition (ii) of Lemma 10 holds for $V = V(M^*_\alpha)$ and $V' = V(M^*_\gamma)$, with $\beta = \gamma/\alpha$ and $\theta = 0$. So Lemmas 9 and 10 yield:

Lemma 11 *For each choice of α and γ with $0 < \gamma < \alpha < 1$, the conditional law of $V(M^*_\gamma)$ given $V(M^*_\alpha) = v$ is* $(\gamma/\alpha, 0)$-COAG$(v, \cdot)$.

2.3 Proof of Theorem 3

As shown in [6], the semigroup of the Bolthausen-Sznitman coalescent $(V^{\mathrm{BS}}_{\mathrm{e}^{-t}}; t > 0)$ introduced in Theorem 1 is provided by the family of coagulation kernels $\left((\mathrm{e}^{-t}, 0)\text{-COAG}, t > 0\right)$. It follows using (5) and Lemma 11 that there is equality of two-dimensional distributions in (2). To complete the proof, it remains only to establish the Markov property of ranked fragmentation process $(V^*_\alpha, 0 < \alpha < 1)$, where we abbreviate $V^*_\alpha := V(M^*_\alpha)$. Denote by $\mathcal{G}_\alpha$ the sigma field generated by the family of random closed sets

$$\mathtt{k}_1(M^*_{\alpha'}) \,, \qquad 0 < \alpha' \le \alpha \,.$$

Because of the way that $\sigma^*_{\alpha'}$ can be recovered from $M^*_{\alpha'}$, the sigma field $\mathcal{G}_\alpha$ coincides with the sigma field generated by the family of processes

$$(\sigma^*_{\alpha'}(t) \wedge 1, t \ge 0) \,, \qquad 0 < \alpha' \le \alpha \,.$$

Combined with Bochner's subordination, this shows that for every $0 < \alpha' < \alpha < 1$, the conditional distribution of $\mathtt{k}_1(M^*_\alpha)$ given $\mathcal{G}_{\alpha'}$ only depends on $\mathtt{k}_1(M^*_{\alpha'})$. In other words,

$$\text{the set-valued process } (\mathtt{k}_1(M^*_\alpha), 0 < \alpha < 1) \text{ is Markovian with respect to } (\mathcal{G}_\alpha). \tag{7}$$

We now fix $\alpha, \beta \in]0,1[$, set $\gamma = \alpha\beta$ and use the notation of the preceding subsection. Consider the conditional distribution of $V^*_\alpha := V(M^*_\alpha)$ given $\mathcal{G}_\gamma$. Because V^*_α is measurable with respect to $\mathtt{k}_1(M^*_\alpha)$, we deduce from (7) that this conditional law only depends on $\mathtt{k}_1(M^*_\gamma)$. Lemma 9 now shows that

$$\text{the conditional law of } V^*_\alpha \text{ given } \mathcal{G}_\gamma \text{ is } (\alpha, -\gamma)\text{-FRAG}(V^*_\gamma, \cdot) \tag{8}$$

so the process $(V^*_\alpha, 0 < \alpha < 1)$ has the Markov property. $\square$

2.4 Another construction of the Bolthausen-Sznitman coalescent

Recently, another simple connection linking the nested family $(M^*_\alpha, 0<\alpha<1)$ to the Bolthausen-Sznitman coalescent has been obtained in [4]. Specifically, fix $\alpha_1 \in]0,1[$, and for $\alpha \in [\alpha_1, 1[$ define T_α by the identity

$$\sigma^*_\alpha(T_\alpha) \;=\; \sigma^*_{\alpha_1}(1)\,.$$

Then consider the normalized sets

$$N^*_\alpha := \left\{\sigma^*_\alpha(t)/\sigma^*_{\alpha_1}(1), 0 \le t \le T_\alpha\right\}^{\mathrm{cl}}\,.$$

It is immediately checked that $(N^*_\alpha, \alpha_1 \le \alpha < 1)$ is a nested family of closed subsets of the unit interval with zero Lebesgue measure. Theorem 8 in [4] states the following identity of finite dimensional distributions of ranked fragmentation processes:

$$(V(N^*_\alpha), \alpha_1 \le \alpha < 1) \;\stackrel{d}{=}\; \left(V^{\mathrm{BS}}_\alpha, \alpha_1 \le \alpha < 1\right)\,. \tag{9}$$

It follows from (9) and Theorem 3 that

$$(V(N^*_\alpha), \alpha_1 \le \alpha < 1) \;\stackrel{d}{=}\; (V(M^*_\alpha), \alpha_1 \le \alpha < 1)\,. \tag{10}$$

Observe that identity of one-dimensional distributions in (10) just rephrases (5). The purpose of this subsection is to point out that our approach yields a refinement of (10), and hence enables us to recover (9) via Theorem 3.

We first need some definitions. Fix an integer $k \ge 1$ and k real numbers $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_k < 1$, and consider k arbitrary positive integers, $n_1, \ldots, n_k$. We write $V^*(n_1)$ for the n_1-th term of the decreasing sequence $V^*_{\alpha_1}$, that is the length of the n_1-th largest interval component of $[0,1]\backslash M^*_{\alpha_1}$. We then denote by $V^*(n_1,n_2)$ the length of the n_2-th largest interval component of $[0,1]\backslash M^*_{\alpha_2}$ which is contained into the n_1-th largest interval in $[0,1]\backslash M^*_{\alpha_1}$. More generally, we define $V^*(n_1,\ldots,n_k)$ by an obvious induction. Note that

$$\sum_{n_k=1}^{\infty} V^*(n_1,\ldots,n_k) \;=\; V^*(n_1,\ldots,n_{k-1})\,.$$

We define $W^*(n_1,\ldots,n_k)$ analogously by replacing $M^*_{\alpha_i}$ by $N^*_{\alpha_i}$ for $i = 1,\ldots,k$. We can now state the following result which obviously encompasses (10):

Proposition 12 *For each $k \ge 1$, and each choice of $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_k < 1$,*

$$(V^*(n_1,\ldots,n_k), n_1,\ldots,n_k \in \mathbb{N}) \;\stackrel{d}{=}\; (W^*(n_1,\ldots,n_k), n_1,\ldots,n_k \in \mathbb{N})\,.$$

Proof: For $k=1$, the statement reduces to (5). We will establish the claim for $k=2$, the general case follows by an easy iteration of the argument. Set

$$V'(n_1,n_2) \;=\; \frac{V^*(n_1,n_2)}{V^*(n_1)} \quad , \quad W'(n_1,n_2) \;=\; \frac{W^*(n_1,n_2)}{W^*(n_1)}\,, \quad n_1,n_2 \in \mathbb{N}\,.$$

On the one hand, it follows readily from Itô's excursion theory and the scaling property that the $\mathcal{S}^{\downarrow}$-valued random variables $W'(1,\cdot), W'(2,\cdot), \ldots$ are i.i.d. and are jointly independent of $V(N^*_{\alpha_1}) = (W^*(n), n \in \mathbb{N})$. Moreover, using the notation of Section 3.2, each has the same law as $V(M^{*\mathbf{ex}}_{\alpha_1,\alpha_2})$.

On the other hand, recall that $A^*_{\alpha_1}$ denotes the age for $M^*_{\alpha_1}$, and work conditionally on $A^*_{\alpha_1} = V^*(p)$ for an arbitrary $p \in \mathbb{N}$. In other words, we work conditionally on the event that the right-most interval component of $[0,1]\backslash M^*_{\alpha_1}$ has rank p when the interval components are ordered by decreasing lengths. It follows again from Itô's excursion theory and the scaling property that the $\mathcal{S}^{\downarrow}$-valued random variables $V'(1,\cdot), V'(2,\cdot), \ldots$ are independent and are jointly independent of $V^*_{\alpha_1} = (V^*(n), n \in \mathbb{N})$. Moreover, in the notation of Section 3.2, the law of $V'(p,\cdot)$ is that of $V(M^{*\mathbf{me}}_{\alpha_1,\alpha_2})$, and for $r \neq p$ the law of $V'(r,\cdot)$ is that of $V(M^{*\mathbf{ex}}_{\alpha_1,\alpha_2})$. But we know from Lemma 8 that $V(M^{*\mathbf{ex}}_{\alpha_1,\alpha_2})$ and $V(M^{*\mathbf{me}}_{\alpha_1,\alpha_2})$ are identical in distribution, and the conclusion follows. $\square$

3 The Intersection Scheme

3.1 The ranked coalescent

In this section, we show that the ranked coalescent based on the intersection scheme is different from the one based on the subordination scheme. First recall Constructions 2 and 5, the definition (4) of the local time, and that we are using superscripts in the obvious notation. For every $0 < \alpha < \beta < 1$ the joint laws of (V^*_α, V^*_β) and $(V^\cap_\alpha, V^\cap_\beta)$ are distinct because

$$\mathtt{E}\left(L^*_{1,\alpha} \mid V^*_\beta = v\right) \neq \mathtt{E}\left(L^\cap_{1,\alpha} \mid V^\cap_\beta = v\right) , \tag{11}$$

where the generic sequence v belongs to some subset of $\mathcal{S}^{\downarrow}$ with positive $PD(\beta,0)$-measure. The proof of (11) relies on the following lemma.

Lemma 13 *For every $0 < \alpha < \beta < 1$, there is a measurable function $f_{\alpha/\beta}$ such that*

$$\mathtt{E}\left(L^*_{1,\alpha} \mid M^*_\beta\right) = f_{\alpha/\beta}\left(L^*_{1,\beta}\right) \tag{12}$$

and a real number $c_{\alpha,\beta} > 0$ such that

$$\mathtt{E}\left(L^\cap_{1,\alpha} \mid M^\cap_\beta\right) = c_{\alpha,\beta} \int_0^1 s^{\alpha-\beta} dL^\cap_{s,\beta} . \tag{13}$$

Proof: The identity $\sigma^*_\alpha = \sigma^*_\beta \circ \tau_{\alpha/\beta}$ (where $\tau_{\alpha/\beta}$ is a stable subordinator with index α/β which is independent of σ^*_β) and the fact that local times are inverse of subordinators yield $L^*_{1,\alpha} = \lambda(L^*_{1,\beta})$, where $(\lambda(s), s \geq 0)$ is the local time process corresponding to $\tau_{\alpha/\beta}$. The assertion (12) follows immediately.

Next, recall Construction 5. For every $\varepsilon > 0$, write $N^{(\varepsilon)}_{1+\alpha-\beta}$ for the set left uncovered by intervals $]x, x+z[$ corresponding to points (a,x,z) of the Poisson process with $1-\beta < a \leq 1-\alpha$ and $z \geq \varepsilon$. It is easily checked that

$$\mathtt{P}\left(s \in N^{(\varepsilon)}_{1+\alpha-\beta}\right) = (s/\varepsilon)^{\alpha-\beta} , \qquad s \geq \varepsilon.$$

We then introduce

$$L^{(\varepsilon)}_{\alpha}(t) \;:=\; \varepsilon^{\alpha-\beta}\int_0^t \mathbf{1}_{\{s\in N^{(\varepsilon)}_{1+\alpha-\beta}\}}\, dL^{\cap}_{s,\beta}$$

and point out the straightforward estimate

$$0 \le \mathtt{E}\left(L^{(\varepsilon)}_{\alpha}(t)\mid M^{\cap}_{\beta}\right) - \int_{\varepsilon}^{t} s^{\alpha-\beta}dL^{\cap}_{s,\beta} \;\le\; \varepsilon^{\alpha-\beta}L^{\cap}_{\varepsilon,\beta}\,.$$

Combining this with the fact that $\varepsilon^{\alpha-\beta}\mathtt{E}\left(L^{\cap}_{\varepsilon,\beta}\right) = c\varepsilon^{\alpha}$ yields

$$\lim_{\varepsilon\to 0+}\mathtt{E}\left(L^{(\varepsilon)}_{\alpha}(t)\mid M^{\cap}_{\beta}\right) \;=\; \int_0^t s^{\alpha-\beta}dL^{\cap}_{s,\beta} \qquad \text{in } L^1(\mathtt{P})\,. \tag{14}$$

All that we need now is to check that there is some constant number $c>0$ such that

$$\lim_{\varepsilon\to 0+} L^{(\varepsilon)}_{\alpha}(t) \;=\; cL^{\cap}_{t,\alpha} \qquad \text{in } L^1(\mathtt{P})\,. \tag{15}$$

To see this, observe that for $\eta\in]0,\varepsilon[$, one has

$$L^{(\eta)}_{\alpha}(t) \;=\; (\eta/\varepsilon)^{\alpha-\beta}\int_0^t \mathbf{1}_{\{s\in N^{(\eta,\varepsilon)}_{1+\alpha-\beta}\}}\, dL^{(\varepsilon)}_{\alpha}(s)\,,$$

where $N^{(\eta,\varepsilon)}_{1+\alpha-\beta}$ denotes the set left uncovered by intervals $]x,x+z[$ corresponding to points (a,x,z) of the Poisson process with $1-\beta<a\le 1-\alpha$ and $\eta\le z<\varepsilon$. Using the elementary identity

$$\mathtt{P}(s\in N^{(\eta,\varepsilon)}_{1+\alpha-\beta}) = (\varepsilon/\eta)^{\alpha-\beta}\,, \qquad s\ge\varepsilon\,,$$

we deduce that for every fixed $0<s<t$, the process $\left(L^{(\varepsilon)}_{\alpha}(t)-L^{(\varepsilon)}_{\alpha}(s),\varepsilon\in]0,s[\right)$ is a reversed martingale.

Standard arguments involving the martingale convergence theorem and (14) show that $L^{(\varepsilon)}_{\alpha}(t)$ converges in $L^1(\mathtt{P})$ as $\varepsilon\to 0+$ to, say, $L_{\alpha}(t)$, and that the increasing process $(L_{\alpha}(t),t\ge 0)$ is continuous. It is plain from the construction that $L_{\alpha}(\cdot)$ is an additive functional that only increases on $M^{\cap}_{\alpha}$, and thus is must be proportional to the local time process $L^{\cap}_{\cdot,\alpha}$ on $M^{\cap}_{\alpha}$. Thus (15) is established and the proof of (13) is complete. □

Now if (11) failed, then the conditional expectation

$$\mathtt{E}\left(L^{\cap}_{1,\alpha}\mid V^{\cap}_{\beta}\right) \;=\; c_{\alpha,\beta}\mathtt{E}\left(\int_0^1 s^{\alpha-\beta}dL^{\cap}_{s,\beta}\mid V^{\cap}_{\beta}\right)$$

would be given by some functional $f_{\alpha/\beta}(L^{\cap}_{1,\beta})$ of the local time of $V^{\cap}_{\beta}$ (by Lemma 13, because $L^{\cap}_{1,\beta}$ is measurable with respect to $V^{\cap}_{\beta}$). To see that this is absurd, observe that for every $\eta>0$, we would have for a.e. $\ell>0$

$$\begin{aligned} c^{-1}_{\alpha,\beta}f_{\alpha/\beta}(\ell) &= \mathtt{E}\left(\int_0^1 s^{\alpha-\beta}dL^{\cap}_{s,\beta}\mid V^{\cap}_{\beta}, L^{\cap}_{1,\beta}=\ell\right)\\ &\ge \eta^{\alpha-\beta}\ell\,\mathtt{P}\left(1-A^{\cap}_{\beta}\le\eta\mid V^{\cap}_{\beta}, L^{\cap}_{1,\beta}=\ell\right)\,, \end{aligned}$$

where $1 - A^{\cap}_{\beta} = \max\left([0,1] \cap M^{\cap}_{\beta}\right)$. According to [20, Prop. 6.3], given $V^{\cap}_{\beta}$, $A^{\cap}_{\beta}$ is a size-biased choice from $V^{\cap}_{\beta}$. It follows readily that the random variable $\mathtt{P}\left(A^{\cap}_{\beta} \geq 1-\eta \mid V^{\cap}_{\beta}, L^{\cap}_{1,\beta} = \ell\right)$ is greater than $1-\eta$, with positive probability (because the conditional probability given $L^{\cap}_{1,\beta} = \ell$ that the first element of $V^{\cap}_{\beta}$ being larger than $1-\eta$ is strictly positive). We conclude that we would have

$$f_{\alpha/\beta}(\ell) \geq (1-\eta)c_{\alpha,\beta}\eta^{\alpha-\beta}\ell\,,$$

and since $\eta > 0$ can be chosen arbitrarily small and $\alpha - \beta < 0$, that $f_{\alpha/\beta} \equiv \infty$. Hence the conditional expectation $\mathtt{E}\left(L^{\cap}_{1,\alpha} \mid V^{\cap}_{\beta}\right)$ cannot be a functional of $L^{\cap}_{1,\beta}$, and by (12), this establishes (11).

3.2 The partition-valued process

In this section, we shall investigate the partition-valued process for the intersection scheme.

For $n \in \mathtt{N}$, let $\mathcal{P}_n$ be the finite set of all partitions of the set $[n] := \{1, \ldots, n\}$. Following [12, 13, 8, 6] a ranked coalescent $(V_{1-\rho}; 0 < \rho < 1)$, defined by $V_\alpha := V(M_\alpha)$ for a family of nested closed sets with zero Lebesgue measure, $(M_\alpha, 0 < \alpha < 1)$, is conveniently encoded as a family of $\mathcal{P}_n$-valued processes with step-function paths $(\Pi_{n,\alpha}; 0 < \alpha < 1)$, $n = 1, 2, \ldots$ as follows. Let $U_1, U_2, \ldots$ be i.i.d. uniform $[0,1]$ variables, independent of $(M_\alpha, 0 < \alpha < 1)$. Let $\Pi_{n,\alpha}$ be the partition of $[n]$ generated by the random equivalence relation $\sim_\alpha$ where $i \sim_\alpha j$ if and only if U_i and U_j fall in the same component interval of the complement of M_α. It is plain that $\Pi_{n,\alpha}$ a refinement of $\Pi_{n,\alpha'}$ for $\alpha > \alpha'$, the distribution of each $\Pi_{n,\alpha}$ is *exchangeable*, that is invariant under the natural action of permutations of $[n]$ on $\mathcal{P}_n$, and the $\Pi_{n,\alpha}$ are *consistent* as n varies, meaning that $\Pi_{m,\alpha}$ is the restriction to $[m]$ of $\Pi_{n,\alpha}$ for $n < m$. For each α the sequence $\Pi_{n,\alpha}, n = 1, 2, \ldots$ then induces a random partition of the set of all positive integers, each of whose classes has an almost sure limiting frequency; the ranked values of these frequencies define the random vector $V_\alpha \in \mathcal{S}^{\downarrow}$.

Each of the partition-valued processes $(\Pi^{\mathrm{BS}}_{n,\mathrm{e}^{-t}}; t > 0)$ corresponding to the Bolthausen-Sznitman coalescent is a Markov chain with stationary transition probabilities such that whenever that the state of the process is a partition with b blocks, each k-tuple of these blocks is merging to form a single block at rate

$$\lambda_{b,k} := \int_0^1 x^{k-2}(1-x)^{b-k}dx = \frac{(k-2)!(b-k)!}{(b-1)!}. \tag{16}$$

As observed in [17], collision rates $\lambda_{b,k}$ specified by the above integral, with $\Lambda(dx)$ instead of dx, serve to define a consistent family of coalescent Markov chains and hence an $\mathcal{S}^{\downarrow}$-valued coalescent process for an arbitrary positive and finite measure Λ on $[0,1]$. The case $\Lambda = \delta_0$, a unit mass at 0, is Kingman's coalescent in which every pair of blocks coalesces at rate 1.

In the remainder of this section, we shall discuss some features of the ranked coalescent $\left(V^{\cap}_{1-\rho}, 0 < \rho < 1\right)$. Let $(\Pi^{\cap}_{n,1-\rho}; 0 < \rho < 1)$, $n = 1, 2, \ldots$ be the partition-valued coalescents derived from $(M^{\cap}_{1-\rho}, 0 < \beta < 1)$ made by the random covering scheme of Construction 5, and let $(V^{\cap}_{1-\rho}, 0 < \rho < 1)$ denote the associated $\mathcal{S}^{\downarrow}$-valued coalescent process. As $V^{\cap}_{1-\rho} \stackrel{d}{=} V^{\mathrm{BS}}_{1-\rho}$ for each ρ, it follows that $\Pi^{\cap}_{n,1-\rho} \stackrel{d}{=} \Pi^{\mathrm{BS}}_{n,1-\rho}$ for each $0 < \rho < 1$ and $n = 1, 2, \ldots$. According to the

Bolthausen-Sznitman description of the homogeneous Markov chain $(\Pi^{\rm BS}_{2,e^{-t}}; t \geq 0)$, this process starts in state $\{\{1\},\{2\}\}$ at time 0, holds there for an exponential time T with rate 1, when it jumps to the absorbing state $\{\{1,2\}\}$. Since $1 - e^{-T}$ has uniform distribution on $[0,1]$, it follows from $\Pi^{\cap}_{2,1-\rho} \stackrel{d}{=} \Pi^{\rm BS}_{2,1-\rho}$ that the process $(\Pi^{\cap}_{2,1-\rho}; 0 < \rho < 1)$ starts in state $\{\{1\},\{2\}\}$ at time $\rho = 0+$, holds there for a time with uniform distribution on $[0,1]$, when it jumps to the absorbing state $\{\{1,2\}\}$. As a check, this can be verified as follows.

Suppose that $I :=]G_I, D_I[$ and $J :=]G_J, D_J[$ are two distinct component intervals of $M^{\cap}_{1-\rho}$ with I to the left of J, that is $D_I < G_J$. Then in the coalescent process derived from $(M_{1-\rho}; 0 < \rho < 1)$, a collision event involving the merger of I and J occurs at time ρ if and only if there is an interval in the Poisson covering process which contains $]D_I, G_J[$. Given $D_I = a$ and $G_J = b$ with $0 < a < b$ the rate per unit increment of ρ at which such a merger occurs is

$$\int_0^a \frac{dx}{b-x} = -\log\left(\frac{b-a}{b}\right).$$

For t in the complement of $M^{\cap}_\alpha$ let $]G_{\alpha,t}, D_{\alpha,t}[$ denote the interval component of the complement $M^{\cap}_\alpha$ containing t. Let $U_{(1)} < U_{(2)}$ be ordered values of independent uniform variables U_1, U_2 used to generate the process $(\Pi^{\cap}_{2,1-\rho}; 0 < \rho < 1)$. Set $\alpha = 1 - \rho$. According to [18, Prop. 16], conditionally given $\Pi^{\cap}_{2,\alpha} = \{\{1\},\{2\}\}$ the triple $(D_{\alpha,U_{(1)}}, G_{\alpha,U_{(2)}} - D_{\alpha,U_{(1)}}, 1 - G_{\alpha,U_{(1)}})$ has a Dirichlet$(1, \alpha, 2-\alpha)$ distribution. Therefore, given $\Pi^{\cap}_{2,\alpha} = \{\{1\},\{2\}\}$ the rate of transition of the process $(\Pi^{\cap}_{2,1-\rho}; 0 < \rho < 1)$ into state $\{\{1,2\}\}$ at time $\rho = 1 - \alpha$ is

$$-\mathtt{E}\left[\log\left(\frac{G_{\alpha,U_{(2)}} - D_{\alpha,U_{(1)}}}{G_{\alpha,U_{(2)}}}\right)\middle| \Pi_{2,\alpha} = \{\{1\},\{2\}\}\right] = -\mathtt{E}\left[\log Z_{\alpha,1}\right] = \frac{1}{\alpha}$$

where $Z_{a,b}$ denotes a beta(a,b) variable, and the last equality is checked as follows. Let $U = Z_{1,1}$ be uniform on $[0,1]$. Then $-\log U$ is exponential with rate 1, and $Z_{\alpha,1} \stackrel{d}{=} U^{1/\alpha}$, so

$$-\mathtt{E}\left[\log Z_{\alpha,1}\right] = -\mathtt{E}\left[\log U^{1/\alpha}\right] = \frac{1}{\alpha}\mathtt{E}\left[-\log U\right] = \frac{1}{\alpha}$$

It follows that the time of the transition of $(\Pi^{\cap}_{2,1-\rho}; 0 < \rho < 1)$ into state $\{\{1,2\}\}$ has uniform distribution on $[0,1]$, as claimed. A similar application of [18, Prop. 16] yields the following:

Proposition 14 *Let $U_{(1)} < U_{(2)} < U_{(3)}$ be the order statistics of three independent uniform variables U_1, U_2, U_3 used to generate the process $(\Pi^{\cap}_{3,1-\rho}; 0 < \rho < 1)$. Then*

(i) *conditionally given $\Pi^{\cap}_{3,1-\rho} = \{\{1\},\{2\},\{3\}\}$ as $\rho = 1-\alpha$ increases the rate of appearance of intervals that would cover $]D_{\alpha,U_{(1)}}, G_{\alpha,U_{(3)}}[$, corresponding to* a double collision, *meaning a jump of the partition of* [3] *into state* $\{[3]\}$, *is*

$$-\mathtt{E}\left(\log Z_{1+\alpha,1}\right) = \frac{1}{1+\alpha};$$

(ii) *given $\Pi^{\cap}_{3,1-\rho} = \{\{1\},\{2\},\{3\}\}$ the rate of appearance of intervals that would cover $]D_{\alpha,U_{(1)}}, G_{\alpha,U_{(2)}}[$, causing a jump in the $\mathcal{P}_3$-valued process is*

$$-\mathtt{E}\left(\log Z_{\alpha,1}\right) = \frac{1}{\alpha};$$

(iii) *given* $\Pi^{\cap}_{3,1-\rho} = \{\{1\},\{2\},\{3\}\}$ *the rate of appearance of intervals that would cover* $]D_{\alpha,U_{(2)}}, G_{\alpha,U_{(3)}}[$, *also causing a jump in the* $\mathcal{P}_3$*-valued process, is*

$$-\mathtt{E}\,(\log Z_{\alpha,2}) = \frac{2\alpha+1}{\alpha(\alpha+1)}.$$

As a check on these rates (i)-(ii)-(iii), the total rate of jumps out of all kinds out of state $\{\{1\},\{2\},\{3\}\}$ is the sum of the rates in (ii) and (iii) minus the rate in (i), that is

$$\frac{1}{\alpha} + \frac{2\alpha+1}{\alpha(\alpha+1)} - \frac{1}{1+\alpha} = \frac{2}{\alpha}.$$

This calculation is consistent with the case $n=3$ of the following consequence of the equality in distribution $\Pi^{\mathrm{BS}}_{n,1-\rho} \stackrel{d}{=} \Pi^{\cap}_{n,1-\rho}$ for all ρ: the exit time of $(\Pi^{\cap}_{n,1-\rho}; 0<\rho<1)$ from its initial state has the same distribution as the exit time of $(\Pi^{\mathrm{BS}}_{n,1-\rho}; 0<\rho<1)$ from its initial state. We know that the exit time of $(\Pi^{\mathrm{BS}}_{n,\mathrm{e}^{-t}}; t>0)$ from its initial state is exponential with rate $n-1$, so the exit rate of $(\Pi^{\mathrm{BS}}_{n,1-\rho}; 0<\rho<1)$ from its initial state when $1-\rho=\alpha$ is $(n-1)/\alpha$.

In the Bolthausen-Sznitman coalescent, when the $\mathcal{P}_3$-valued process exits from its initial state it does so by a double collision with probability $1/4$, independently of the holding time of the initial state. However, the previous calculations show that in the $\mathcal{P}_3$-valued coalescent $(\Pi^{\cap}_{3,1-\rho}; 0<\rho<1)$, if the exit from the initial state occurs at time $\rho = 1-\alpha$, then the chance that it occurs by a double collision is

$$\frac{1}{1+\alpha}\left(\frac{2}{\alpha}\right)^{-1} = \frac{\alpha}{2+2\alpha}.$$

It follows that $(\Pi^{\cap}_{3,1-\rho}; 0<\rho<1)$ does not have the same distribution as $(\Pi^{\mathrm{BS}}_{3,1-\rho}; 0<\rho<1)$, and hence that $(V^{\cap}_{1-\rho}; 0<\rho<1)$ does not have the same distribution as $(V^{\mathrm{BS}}_{1-\rho}; 0<\rho<1)$. The above calculations also show that $(\Pi^{\cap}_{3,\mathrm{e}^{-t}}; t>0)$ is not a time-homogeneous Markov process.

Finally, we stress that the partition-valued coalescents $(\Pi^{*}_{n,1-\rho}; 0<\rho<1)$, $n=1,2,\ldots$ derived from $(M^{*}_{1-\rho}, 0<\beta<1)$ have the same distributions as the Bolthausen-Sznitman partition-valued coalescents. The proof is a slight variation of that in Section 2 (replace the present Lemma 10 by [17, Theorem 12]).

4 The age

In this section, we shall give a direct proof of Proposition 6. Recall that the age A_α is defined by $A_\alpha = 1 - \max(M_\alpha \cap [0,1])$. Consider first the case of $M^{\cap}_{\alpha}$, and suppose that $M^{\cap}_{\alpha}$ is constructed as in Construction 4 as the zero set of a Bessel process $X_{2-2\alpha,\bullet}$. If we pick parameters $0 < \alpha_1 < \ldots < \alpha_n < 1$, set $2-2\alpha_i = \beta_i + \ldots + \beta_n$ for $i=1,\ldots,n$, and introduce independent squares of Bessel processes $X^{(1)},\ldots,X^{(n)}$ started at 0 with respective dimensions $\beta_1,\ldots,\beta_n$, then $X^{(i)}+\cdots+X^{(n)}$ is the square of a Bessel process of dimension $(2-2\alpha_i)$, and its zero set is a version of $M^{\cap}_{\alpha_i}$. Consider an exponential time T which is independent of the family of Bessel processes, and write

$$g^{(i)} = \sup\{t<T : X^{(i)}_t = \cdots = X^{(n)}_t = 0\} = \sup\{t<T : X^{(i)}_t + \cdots + X^{(n)}_t = 0\}$$

for the last passage time at the origin of $(X^{(i)},\dots,X^{(n)})$ before T.

On the one hand, the decomposition at a last passage time tells us that for each $i=1,\dots,n$, the processes $\left((X^{(i)}_t,\dots,X^{(n)}_t);0\le t<g^{(i)}\right)$ and $\left((X^{(i)}_{g^{(i)}+t},\dots,X^{(n)}_{g^{(i)}+t});0\le t<T-g^{(i)}\right)$ are independent. It follows that

$$g^{(1)},g^{(2)}-g^{(1)},\dots,g^{(n)}-g^{(n-1)},T-g^{(n)}$$

are independent random variables.

On the other hand, we know from [7] that there is some gamma process $(\Gamma_t,0\le t\le 1)$ such that for each i, $g^{(i)}$ has the same distribution as Γ_{α_i}. We deduce from above that in fact the $(n+1)$-tuples $\left(g^{(1)},\dots,g^{(n)},T\right)$ and $\left(\Gamma_{\alpha_1},\dots,\Gamma_{\alpha_{n+1}}\right)$ have the same law, where $\alpha_{n+1}=1$. The scaling property now implies that

$$\left(1-A^{\cap}_{\alpha_1},\dots,1-A^{\cap}_{\alpha_n}\right)\ \stackrel{d}{=}\ \left(\frac{g^{(1)}}{T},\dots,\frac{g^{(n)}}{T}\right)\ \stackrel{d}{=}\ \left(\frac{\Gamma_{\alpha_1}}{\Gamma_{\alpha_{n+1}}},\dots,\frac{\Gamma_{\alpha_n}}{\Gamma_{\alpha_{n+1}}}\right),$$

which is equivalent to our statement for $A^{\cap}$.

A similar argument applies for A^*. More precisely, if we write

$$A^*_{\alpha_i}(t)\ =\ \inf\left\{s\in]0,t[:\,t-s\in M^*_{\alpha_i}\right\}\,,\qquad t>0$$

for the age process related to $M^*_{\alpha_i}$, then it is easy to check that for each $i=1,\dots,n$, the $(n-i+1)$-tuple

$$\left(\left(A^*_{\alpha_i}(t),\dots,A^*_{\alpha_n}(t)\right),t\ge 0\right)$$

is a Markov process, and the set of its passage times at the origin coincides with $M^*_{\alpha_i}$. This enables us to follow the same argument as above. $\square$

5 Some Open Problems

For any $\mathcal{S}^{\downarrow}$-valued coalescent process, say $(V_{1-\rho},0<\rho<1)$, we can define an increasing process $(\widetilde{V}_{1-\rho},0<\rho<1)$ with values in $[0,1]$ be letting $\widetilde{V}_{1-\rho}$ be the component of $V_{1-\rho}$ that contains a point picked at random from the mass distribution at time $0+$. In the present setting, with $V_{1-\rho}$ constructed as $V_{1-\rho}=V(M_{1-\rho})$, this can be achieved by introducing an independent uniform random variable U on $[0,1]$, and letting $\widetilde{V}_\alpha$ be the length of the component interval of $[0,1]\backslash M_\alpha$ that contains U. It is known [7, 20] that for M_α derived from the stable subordinator of index α, for each fixed $\alpha\in]0,1[$ there is the equality in distribution

$$\widetilde{V}_\alpha \stackrel{d}{=} A_\alpha \stackrel{d}{=} \Gamma_{1-\alpha}/\Gamma_1\,. \tag{17}$$

where we use the notation of Proposition 6. According to [17, Corollary 16],

$$(\widetilde{V}^{\mathrm{BS}}_{1-\rho};0<\rho<1)\ \stackrel{d}{=}\ (\Gamma_\rho/\Gamma_1;0<\rho<1)\,. \tag{18}$$

Proposition 6, Theorem 3 and (18) imply the following extension of (17):

$$(\widetilde{V}^*_\alpha; 0 < \alpha < 1) \stackrel{d}{=} (A^*_\alpha; 0 < \alpha < 1)\,.$$

We do not believe the same identity should hold for $\left(\widetilde{V}^\cap_\alpha, 0 < \alpha < 1\right)$, but we do not have a rigorous argument. The problem of specifying the distribution of the process $\left(\widetilde{V}^\cap_\alpha, 0 < \alpha < 1\right)$ is open.

More generally, it is a natural problem to describe more explicitly such features of the coalescent process $(V^\cap_{1-\rho}; 0 < \rho < 1)$ as the laws of associated partition-valued processes, as considered in [6, 17] for the Bolthausen-Sznitman coalescent. The study of such problems is complicated by the fact in contrast to the Bolthausen-Sznitman coalescent, neither $(V^\cap_{1-\rho}; 0 < \rho < 1)$ nor its associated partition-valued processes appear to have the Markov property. While it is easily seen from the Poisson construction that the set-valued process $(M^\cap_\alpha; 0 < \alpha < 1)$ is Markov, this Markov property does not propagate to $(V^\cap_\alpha; 0 < \alpha < 1)$ by Dynkin's criterion for a function of a Markov process to be Markov, because the way that the restriction of $M^\cap_\alpha$ to $[0, 1]$ evolves depends on the ordering of the intervals, not just $V^\cap_\alpha$. It might be that $(V^\cap_\alpha)$ is Markov by the criterion of Rogers-Pitman [23]. According to [20, Prop. 6.3] there is a conditional law for $[0, 1] \cap M^\cap_\alpha$ given $V^\cap_\alpha$ which is the same for all α: to reconstruct $[0, 1] \cap M^\cap_\alpha$ from $V^\cap_\alpha$, place an interval whose length is a size-biased choice from $V^\cap_\alpha$ at the right end, then precede it by intervals with lengths from the rest of $V^\cap_\alpha$ put in exchangeable random order. But to apply the result of [23] there is an intertwining identity of kernels to be verified, and it does not seem easy to decide if this identity holds.

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