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Cube root fluctuations for the corner growth model associated to the exclusion process

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Abstract

We study the last-passage growth model on the planar integer lattice with exponential weights. With boundary conditions that represent the equilibrium exclusion process as seen from a particle right after its jump we prove that the variance of the last-passage time in a characteristic direction is of order $t^{2/3}$. With more general boundary conditions that include the rarefaction fan case we show that the last-passage time fluctuations are still of order $t^{1/3}$, and also that the transversal fluctuations of the maximal path have order $t^{2/3}$. We adapt and then build on a recent study of Hammersley's process by Cator and Groeneboom, and also utilize the competition interface introduced by Ferrari, Martin and Pimentel. The arguments are entirely probabilistic, and no use is made of the combinatorics of Young tableaux or methods of asymptotic analysis

Key words: Last-passage, simple exclusion, cube root asymptotics, competition interface, Burke's theorem, rarefaction fan

AMS 2000 Subject Classification: Primary 60K35, 82C43.

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1 Introduction

We construct a version of the corner growth model that corresponds to an equilibrium exclusion process as seen by a typical particle right after its jump, and show that along a characteristic direction the variance of the last-passage time is of order $t^{2/3}$. This last-passage time is the maximal sum of exponential weights along up-right paths in the first quadrant of the integer plane. The interior weights have rate 1, while the boundary weights on the axes have rates $1-\varrho$ and ϱ where $0<\varrho<1$ is the particle density of the exclusion process. By comparison to this equilibrium setting, we also show fluctuation results with similar scaling in the case of the rarefaction fan.

The proof is based on a recent work of Cator and Groeneboom (3) where corresponding results are proved for the planar-increasing-path version of Hammersley's process. A key part of that proof is an identity that relates the variance of the last-passage time to the point where the maximal path exits the axes. This exit point itself is related to a second-class particle via a time reversal. The idea that the current and the second-class particle should be connected goes back to a paper of Ferrari and Fontes (4) on the diffusive fluctuations of the current away from the characteristic. However, despite this surprising congruence of ideas, article (3) and our work have no technical relation to the Ferrari-Fontes work.

The first task of the present paper is to find the connection between the variance of the last-passage time and the exit point, in the equilibrium corner growth model. The relation turns out not as straightforward as for Hammersley's process, for we also need to include the amount of weight collected on the axes. However, once this difference is understood, the arguments proceed quite similarly to those in (3).

The notion of competition interface recently introduced by Ferrari, Martin and Pimentel (6; 7) now appears as the representative of a second-class particle, and as the time reversal of the maximal path. As a by-product of the proof we establish that the transversal fluctuations of the competition interface are of the order $t^{2/3}$ in the equilibrium setting.

In the last section we take full advantage of our probabilistic approach, and show that for initial conditions obtained by decreasing the equilibrium weights on the axes in an *arbitrary* way, the fluctuations of the last-passage time are still of order $t^{1/3}$. This includes the situation known as the *rarefaction fan*. We are also able to show that in this case the transversal fluctuations of the longest path are of order $t^{2/3}$. In this more general setting there is no direct connection between a maximal path and a competition interface (or trajectory of a second class particle).

Our results for the competition interface, and our fluctuation results under the more general boundary conditions are new. The variance bound for the equilibrium last-passage time is also strictly speaking new. However, the corresponding distributional limit has been obtained by Ferrari and Spohn (8) with a proof based on the RSK machinery. But they lack a suitable tightness property that would give them also control of the variance. [Note that Ferrari and Spohn start by describing a different set of equilibrium boundary conditions than the ones we consider, but later in their paper they cover also the kind we define in (2.5) below.] The methods of our paper can also be applied to geometrically distributed weights, with the same outcomes.

In addition to the results themselves, our main motivation is to investigate new methods to attack the last-passage model, methods that do not rely on the RSK correspondence of Young tableaux. The reason for such a pursuit is that the precise counting techniques of Young tableaux appear to work only for geometrically distributed weights, from which one can then take a limit to obtain the case of exponential weights. New techniques are needed to go beyond the geometric and exponential cases, although we are not yet in a position to undertake such an advance.

For the class of totally asymmetric stochastic interacting systems for which the last-passage approach works, this point of view has been extremely valuable. In addition to the papers already mentioned above, we list Seppäläinen (14; 15), Johansson (9), and Prähofer and Spohn (13).

Organization of the paper. The main results are discussed in Section 2. Section 3 describes the relationship of the last-passage model to particle and deposition models, and can be skipped without loss of continuity. The remainder of the paper is for the proofs. Section 4 covers some preliminary matters. This includes a strong form of Burke's theorem for the last-passage times (Lemma 4.2). Upper and lower bounds for the equilibrium results are covered in Sections 5 and 6. Lastly, fluctuations under more general boundary conditions are studied in Section 7.

Notation. $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ denotes the set of nonnegative integers. The integer part of a real number is $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$. C denotes constants whose precise value is immaterial and that do not depend on the parameter (typically t) that grows. $X \sim \text{Exp}(\varrho)$ means that X has the exponential distribution with rate ϱ, in other words has density $f(x) = \varrho e^{-\varrho x}$ on $\mathbb{R}_+$. For clarity, subscripts can be replaced by arguments in parentheses, as for example in $G_{ij} = G(i, j)$.

2 Results

We start by describing the corner growth model with boundaries that correspond to a special view of the equilibrium. Section 3 and Lemma 4.2 justify the term equilibrium in this context. Our results for more general boundary conditions are in Section 2.2.

2.1 Equilibrium results

We are given an array $\{\omega_{ij}\}_{i,j \in \mathbb{Z}_+}$ of nonnegative real numbers. We will always have $\omega_{00} = 0$. The values ω_{ij} with either $i = 0$ or $j = 0$ are the boundary values, while $\{\omega_{ij}\}_{i,j \geq 1}$ are the interior values.

Figure 1 depicts this initial set-up on the first quadrant $\mathbb{Z}_+^2$ of the integer plane. A $\star$ marks $(0, 0)$, ∇'s mark positions $(i, 0)$, $i \geq 1$, $\triangle$'s positions $(0, j)$, $j \geq 1$, and interior points (i, j), $i, j \geq 1$ are marked with $\circ$'s. The coordinates of a few points around $(5, 2)$ have been labeled.

For a point $(i, j) \in \mathbb{Z}_+^2$, let Π_{ij} be the set of directed paths

$$\pi = \{(0, 0) = (p_0, q_0) \to (p_1, q_1) \to \cdots \to (p_{i+j}, q_{i+j}) = (i, j)\} \tag{2.1}$$

with up-right steps

$$(p_{l+1}, q_{l+1}) - (p_l, q_l) = (1, 0) \quad \text{or} \quad (0, 1) \tag{2.2}$$

along the coordinate directions. Define the *last passage time* of the point (i, j) as

$$G_{ij} = \max_{\pi \in \Pi_{ij}} \sum_{(p,q) \in \pi} \omega_{pq}.$$

G satisfies the recurrence

$$G_{ij} = (G_{\{i-1\}j} \vee G_{i\{j-1\}}) + \omega_{ij} \qquad (i,\, j \geq 0) \tag{2.3}$$

(with formally assuming $G_{\{-1\}j} = G_{i\{-1\}} = 0$). A common interpretation is that this models a growing cluster on the first quadrant that starts from a seed at the origin (bounded by the thickset line in Figure 1). The value ω_{ij} is the time it takes to occupy point (i,j) after its neighbors to the left and below have become occupied, with the interpretation that a boundary point needs only one occupied neighbor. Then G_{ij} is the time when (i,j) becomes occupied, or joins the growing cluster. The occupied region at time $t \geq 0$ is the set

$$\mathcal{A}(t) = \{(i,j) \in \mathbb{Z}_+^2 \,:\, G_{ij} \leq t\}. \tag{2.4}$$

Figure 2 shows a possible later situation. Occupied points are denoted by solidly colored symbols, the occupied cluster is bounded by the thickset line, and the arrows mark an admissible path π from $(0,0)$ to $(5,2)$. If $G_{5,2}$ is the smallest among $G_{0,5}$, $G_{1,4}$, $G_{5,2}$ and $G_{6,0}$, then $(5,2)$ is the next point added to the cluster, as suggested by the dashed lines around the $(5,2)$ square.

To create a model of random evolution, we pick a real number $0 < \varrho < 1$ and take the variables $\{\omega_{ij}\}$ mutually independent with the following marginal distributions:

$$\begin{aligned}
&\omega_{00} = 0, && \text{where the } \star \text{ is,}\\
&\omega_{i0} \sim \mathrm{Exp}(1-\varrho),\ i \geq 1, && \text{where the } \triangledown\text{'s are,}\\
&\omega_{0j} \sim \mathrm{Exp}(\varrho),\ j \geq 1, && \text{where the } \vartriangle\text{'s are,}\\
&\omega_{ij} \sim \mathrm{Exp}(1),\ i,j \geq 1, && \text{where the } \circ\text{'s are.}
\end{aligned} \tag{2.5}$$

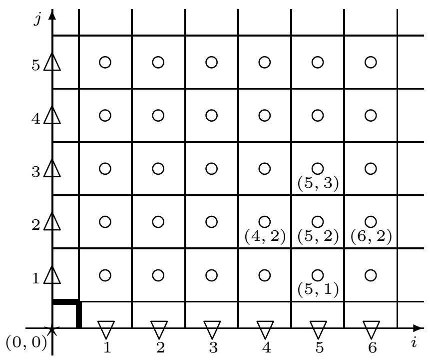

Figure 1: The initial situation

Ferrari, Prähofer and Spohn (8), (13) consider the Bernoulli-equilibrium of simple exclusion, which corresponds to a slightly more complicated boundary distribution than the one described above. However, Ferrari and Spohn (8) early on turn to the distribution described by (2.5), as it is more natural for last-passage. We will greatly exploit the simplicity of (2.5) in Section 4. In fact, (2.5) is also connected with the stationary exclusion process of particle density ϱ. To see this point, we need to look at a particle of simple exclusion in a specific manner that we explain below in Section 3.1.

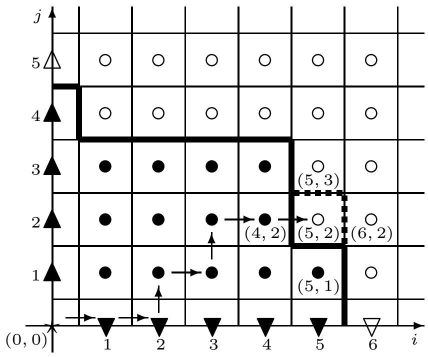

Figure 2: A possible later situation

Once the parameter ϱ has been picked we denote the last-passage time of point (m,n) by G^{ϱ}_{mn}. In order to see interesting behavior we follow the last-passage time along the ray defined by

$$(m(t), n(t)) = (\lfloor (1-\varrho)^2 t \rfloor, \lfloor \varrho^2 t \rfloor) \tag{2.6}$$

as $t \to \infty$. In Section 3.4 we give a heuristic justification for this choice. It represents the characteristic speed of the macroscopic equation of the system. Let us abbreviate

$$G^{\varrho}(t) = G^{\varrho}\big(\lfloor (1-\varrho)^2 t \rfloor, \lfloor \varrho^2 t \rfloor\big).$$

Once we have proved that all horizontal and vertical increments of G-values are distributed exponentially like the boundary increments, we see that

$$\mathbf{E}(G^{\varrho}(t)) = \frac{\lfloor (1-\varrho)^2 t \rfloor}{1-\varrho} + \frac{\lfloor \varrho^2 t \rfloor}{\varrho}.$$

The first result is the order of the variance.

Theorem 2.1. *With $0 < \varrho < 1$ and independent $\{\omega_{ij}\}$ distributed as in (2.5),*

$$0 < \liminf_{t\to\infty} \frac{\mathbf{Var}(G^{\varrho}(t))}{t^{2/3}} \le \limsup_{t\to\infty} \frac{\mathbf{Var}(G^{\varrho}(t))}{t^{2/3}} < \infty.$$

We emphasize here that our results do not imply a similar statement for the particle flux variance, nor for the position of the second class particle.

For given (m,n) there is almost surely a unique path $\widehat{\pi}$ that maximizes the passage time to (m,n), due to the continuity of the distribution of $\{\omega_{ij}\}$. The *exit point* of $\widehat{\pi}$ is the last boundary point on the path. If (p_l, q_l) is the exit point for the path in (2.1), then either $p_0 = p_1 = \cdots = p_l = 0$ or $q_0 = q_1 = \cdots = q_l = 0$, and $p_k, q_k \ge 1$ for all $k > l$. To distinguish between exits via the i- and j-axis, we introduce a non-zero integer-valued random variable Z such that if $Z > 0$ then the exit point is $(p_{|Z|}, q_{|Z|}) = (Z, 0)$, while if $Z < 0$ then the exit point is $(p_{|Z|}, q_{|Z|}) = (0, -Z)$. For the sake of convenience we abuse language and call the variable Z also the "exit point." $Z^{\varrho}(t)$ denotes the exit point of the maximal path to the point $(m(t), n(t))$ in (2.6) with boundary condition parameter ϱ. Transposition $\omega_{ij} \mapsto \omega_{ji}$ of the array shows that $Z^{\varrho}(t)$ and $-Z^{1-\varrho}(t)$ are equal in distribution. Along the way to Theorem 2.1 we establish that $Z^{\varrho}(t)$ fluctuates on the scale $t^{2/3}$.

Theorem 2.2. *Given $0 < \varrho < 1$ and independent $\{\omega_{ij}\}$ distributed as in* (2.5).
(a) *For $t_0 > 0$ there exists a finite constant $C = C(t_0, \varrho)$ such that, for all $a > 0$ and $t \geq t_0$,*

$$\mathbf{P}\{Z^{\varrho}(t) \geq at^{2/3}\} \leq Ca^{-3}.$$

(b) *Given $\varepsilon > 0$, we can choose a $\delta > 0$ small enough so that for all large enough t*

$$\mathbf{P}\{1 \leq Z^{\varrho}(t) \leq \delta t^{2/3}\} \leq \varepsilon.$$

Competition interface. In (6; 7) Ferrari, Martin and Pimentel introduced the *competition interface* in the last-passage picture. This is a path $k \mapsto \varphi_k \in \mathbb{Z}_+^2$ ($k \in \mathbb{Z}_+$), defined as a function of $\{G_{ij}\}$: first $\varphi_0 = (0,0)$, and then for $k \geq 0$

$$\varphi_{k+1} = \begin{cases} \varphi_k + (1,0) & \text{if } G(\varphi_k + (1,0)) < G(\varphi_k + (0,1)), \\ \varphi_k + (0,1) & \text{if } G(\varphi_k + (1,0)) > G(\varphi_k + (0,1)). \end{cases} \tag{2.7}$$

In other words, φ takes up-right steps, always choosing the smaller of the two possible G-values.

The term "competition interface" is justified by the following picture. Instead of having the unit squares centered at the integer points as in Figure 1, draw the squares so that their corners coincide with integer points. Label the squares by their northeast corners, so that the square $(i-1, i] \times (j-1, j]$ is labeled the (i, j)-square. Regard the last-passage time G_{ij} as the time when the (i, j)-square becomes occupied. Color the square $(0, 0)$ white. Every other square gets either a red or a blue color: squares to the left and above the path φ are colored red, and squares to the right and below φ blue. Then the red squares are those whose maximal path $\widehat{\pi}$ passes through $(0, 1)$, while the blue squares are those whose maximal path $\widehat{\pi}$ passes through $(1, 0)$. These can be regarded as two competing "infections" on the (i, j)-plane, and φ is the interface between them.

The competition interface represents the evolution of a second-class particle, and macroscopically it follows the characteristics. This was one of the main points for (7). In the present setting the competition interface is the time reversal of the maximal path $\widehat{\pi}$, as we explain more precisely in Section 4 below. This connection allows us to establish the order of the transversal fluctuations of the competition interface in the equilibrium setting. To put this in precise notation, we introduce

$$v(n) = \inf\{i \,:\, (i, n) = \varphi_k \text{ for some } k \geq 0\} \tag{2.8}$$

and

$$w(m) = \inf\{j \,:\, (m, j) = \varphi_k \text{ for some } k \geq 0\}$$

with the usual convention $\inf \emptyset = \infty$. In other words, $(v(n), n)$ is the leftmost point of the competition interface on the horizontal line $j = n$, while $(m, w(m))$ is the lowest such point on the vertical line $i = m$. They are connected by the implication

$$v(n) \geq m \implies w(m) < n \tag{2.9}$$

as can be seen from a picture. Transposition $\omega_{ij} \mapsto \omega_{ji}$ of the ω-array interchanges v and w.

Given m and n, let

$$Z^{*\varrho} = [m - v(n)]^+ - [n - w(m)]^+ \tag{2.10}$$

denote the signed distance from the point (m, n) to the point where φ_k first hits either of the lines $j = n$ ($Z^{*\varrho} > 0$) or $i = m$ ($Z^{*\varrho} < 0$). Precisely one of the two terms contributes to the difference. When we let $m = m(t)$ and $n = n(t)$ according to (2.6), we have the t-dependent version $Z^{*\varrho}(t)$. Time reversal will show that in distribution $Z^{*\varrho}(t)$ is equal to $Z^{\varrho}(t)$. (The notation Z^* is used in anticipation of this time reversal connection.) Consequently

Corollary 2.3. *Theorem 2.2 is true word for word when $Z^{\varrho}(t)$ is replaced by $Z^{*\varrho}(t)$.*

2.2 Results for the rarefaction fan

We now partially generalize the previous results to arbitrary boundary conditions that are bounded by the equilibrium boundary conditions of (2.5). Let $\{\omega_{ij}\}$ be distributed as in (2.5). Let $\{\hat\omega_{ij}\}$ be another array defined on the same probability space such that $\hat\omega_{00} = 0$, $\hat\omega_{ij} = \omega_{ij}$ for $i, j \geq 1$, and

$$\hat\omega_{i0} \leq \omega_{i0} \quad \text{and} \quad \hat\omega_{0j} \leq \omega_{0j} \quad \forall\, i, j \geq 1. \tag{2.11}$$

In particular, $\hat\omega_{i0} = \hat\omega_{0j} = 0$ is admissible here. Sections 3.2 and 3.4 below explain how these boundary conditions can represent the so-called rarefaction fan situation of simple exclusion and cover all the characteristic directions contained within the fan.

Let $\hat G(t)$ denote the weight of the maximal path to (m, n) of (2.6), using the $\{\hat\omega_{ij}\}$ array.

Theorem 2.4. *Fix $0 < \alpha < 1$. There exists a constant $C = C(\alpha, \varrho)$ such that for all $t \geq 1$ and $a > 0$,*

$$\mathbf{P}\{|\hat G(t) - t| > at^{1/3}\} \leq Ca^{-3\alpha/2}.$$

As a last result we show that even with these general boundary conditions, a maximizing path does not fluctuate more than $t^{2/3}$ around the diagonal of the rectangle. Define $\hat Z_l(t)$ as the i-coordinate of the right-most point on the horizontal line $j = l$ of the right-most maximal path to (m, n), and $\hat Y_l(t)$ as the i-coordinate of the left-most point on the horizontal line $j = l$ of the left-most maximal path to (m, n). (In this general setting we no longer necessarily have a unique maximizing path because we have not ruled out a dependence of $\{\hat\omega_{i0}, \hat\omega_{0j}\}$ on $\{\hat\omega_{ij}\}_{i,j\geq 1}$.)

Theorem 2.5. *For all $0 < \alpha < 1$ there exists $C = C(\alpha, \varrho)$, such that for all $a > 0$, $s \leq t$ with $t \geq 1$ and $(k, l) = (\lfloor (1-\varrho)^2 s \rfloor, \lfloor \varrho^2 s \rfloor)$,*

$$\mathbf{P}\{\hat Z_l(t) \geq k + at^{2/3}\} \leq Ca^{-3\alpha} \quad \textit{and} \quad \mathbf{P}\{\hat Y_l(t) \leq k - at^{2/3}\} \leq Ca^{-3\alpha}.$$

3 Particle systems and queues

The proofs in our paper will only use the last-passage description of the model. However, we would like to point out several other pictures one can attach to the last-passage model. An immediate one is the totally asymmetric simple exclusion process (TASEP). The boundary conditions (2.5) of the last-passage model correspond to TASEP in equilibrium, as seen by a "typical" particle right after its jump. We also briefly discuss queues, and an augmentation of the last-passage picture that describes a deposition model with column growth, as in (1).

3.1 The totally asymmetric simple exclusion process

This process describes particles that jump unit steps to the right on the integer lattice $\mathbb{Z}$, subject to the exclusion rule that permits at most one particle per site. The state of the process is a $\{0,1\}$-valued sequence $\underline{\widetilde{\eta}} = \{\widetilde{\eta}_x\}_{x\in\mathbb{Z}}$, with the interpretation that $\widetilde{\eta}_x = 1$ means that site x is occupied by a particle, and $\widetilde{\eta}_x = 0$ that x is vacant. The dynamics of the process are such that each $(1,0)$ pair in the state becomes a $(0,1)$ pair at rate 1, independently of the rest of the state. In other words, each particle jumps to a vacant site on its right at rate 1, independently of other particles. The extreme points of the set of spatially translation-invariant equilibrium distributions of this process are the Bernoulli(ϱ) distributions ν^ϱ indexed by particle density $0 \le \varrho \le 1$. Under ν^ϱ the occupation variables $\{\widetilde{\eta}_x\}$ are i.i.d. with mean $\mathbf{E}^\varrho(\widetilde{\eta}_x) = \varrho$.

The Palm distribution of a particle system describes the equilibrium distribution as seen from a "typical" particle. For a function f of $\underline{\widetilde{\eta}}$, the Palm-expectation is

$$\widehat{\mathbf{E}}^\varrho(f(\underline{\widetilde{\eta}})) = \frac{\mathbf{E}^\varrho(f(\underline{\widetilde{\eta}})\cdot\widetilde{\eta}_0)}{\mathbf{E}^\varrho(\widetilde{\eta}_0)}$$

in terms of the equilibrium expectation, see e.g. Port and Stone (12). Due to $\widetilde{\eta}_x \in \{0,\,1\}$, for TASEP the Palm distribution is the original Bernoulli(ϱ)-equilibrium conditioned on $\widetilde{\eta}_0 = 1$.

Theorem 3.1 (Burke)**.** *Let $\underline{\widetilde{\eta}}$ be a totally asymmetric simple exclusion process started from the Palm distribution (i.e. a particle at the origin, Bernoulli measure elsewhere). Then the position of the particle started at the origin is marginally a Poisson process with jump rate $1-\varrho$.*

The theorem follows from considering the inter-particle distances as M/M/1 queues. Each of these distances is geometrically distributed, which is the stationary distribution for the corresponding queue. Departure processes from these queues, which correspond to TASEP particle jumps, are marginally Poisson due to Burke's Theorem for queues, see e.g. Brémaud (2) for details. The Palm distribution is important in this argument, as selecting a "typical" TASEP-particle assures that the inter-particle distances (or the lengths of the queues) are geometrically distributed. For instance, the first particle to the left of the origin in an ordinary Bernoulli equilibrium will *not* see a geometric distance to the next particle on its right.

Shortly we will explain how the boundary conditions (2.5) correspond to TASEP started from Bernoulli(ϱ) measure, *conditioned on* $\widetilde{\eta}_0(0) = 0$ *and* $\widetilde{\eta}_1(0) = 1$, i.e. a hole at the origin and a particle at site one initially. It will be convenient to give all particles and holes labels that they retain as they jump (particles to the right, holes to the left). The particle initially at site one is labeled P_0, and the hole initially at the origin is labeled H_0. After this, all particles are labeled with integers *from right to left*, and all holes *from left to right*. The position of particle P_j at time t is $P_j(t)$, and the position of hole H_i at time t is $H_i(t)$. Thus initially

$$\begin{aligned}\cdots < P_3(0) < P_2(0) < P_1(0) < H_0(0) = 0&\\ < 1 = P_0(0) < H_1(0) < H_2(0) < H_3(0) < \cdots&\end{aligned}$$

Since particles never jump over each other, $P_{j+1}(t) < P_j(t)$ holds at all times $t \ge 0$, and by the same token also $H_i(t) < H_{i+1}(t)$.

It turns out that this perturbation of the Palm distribution does not entirely spoil Burke's Theorem.

Corollary 3.2. *Marginally, $P_0(t) - 1$ and $-H_0(t)$ are two independent Poisson processes with respective jump rates $1 - \varrho$ and ϱ.*

Proof. The evolution of $P_0(t)$ depends only on the initial configuration $\{\widetilde{\eta}_x(0)\}_{x>1}$ and the Poisson clocks governing the jumps over the edges $\{x \to x+1\}_{x\geq 1}$. The evolution of $H_0(t)$ depends only on the initial configuration $\{\widetilde{\eta}_x(0)\}_{x<0}$ and the Poisson clocks governing the jumps over the edges $\{x \to x+1\}_{x<0}$. Hence $P_0(t)$ and $H_0(t)$ are independent. Moreover, $\{\widetilde{\eta}_x(0)\}_{x>1,\ x<0}$ is Bernoulli(ϱ) distributed, just like in the Palm distribution. Hence Burke's Theorem applies to $P_0(t)$. As for $H_0(t)$, notice that $\underline{1} - \underline{\widetilde{\eta}}(t)$, with $\underline{1}_x \equiv 1$, is a TASEP with holes and particles interchanged and particles jumping to the left. Hence Burke's Theorem applies to $-H_0(t)$. □

Now we can state the precise connection with the last-passage model. For $i, j \geq 0$ let T_{ij} denote the time when particle P_j and hole H_i exchange places, with $T_{00} = 0$. Then

the processes $\{G_{ij}\}_{i,j\geq 0}$ and $\{T_{ij}\}_{i,j\geq 0}$ are equal in distribution.

For the marginal distributions on the i- and j-axes we see the truth of the statement from Corollary 3.2. More generally, we can compare the growing cluster

$$\mathcal{C}(t) = \{(i, j) \in \mathbb{Z}_+^2 : T_{ij} \leq t\}$$

with $\mathcal{A}(t)$ defined by (2.4), and observe that they are countable state Markov chains with the same initial state and identical bounded jump rates.

Since each particle jump corresponds to exchanging places with a particular hole, one can deduce that at time T_{ij},

$$P_j(T_{ij}) = i - j + 1 \quad \text{and} \quad H_i(T_{ij}) = i - j. \tag{3.1}$$

By the queuing interpretation of the TASEP, we represent particles as servers, and the holes between P_j and P_{j-1} as customers in the queue of server j. Then *the occupation of the last-passage point (i, j) is the same event as the completion of the service of customer i by server j*. This infinite system of queues is equivalent to a constant rate totally asymmetric zero range process.

3.2 The rarefaction fan

The classical rarefaction fan initial condition for TASEP is constructed with two densities $\lambda_\ell > \lambda_r$. Initially particles to the left of the origin obey Bernoulli λ_ℓ distributions, and particles to the right of the origin follow Bernoulli λ_r distributions. Of interest here is the behavior of a second-class particle or the competition interface, and we refer the reader to articles (5; 7; 6; 11; 16)

Following the development of the previous section, condition this initial measure on having a hole H_0 at 0, and a particle P_0 at 1. Then as observed earlier, H_0 jumps to the left according to a Poisson(λ_ℓ) process, while P_0 jumps to the right according to a Poisson($1 - \lambda_r$) process. To represent this situation in the last-passage picture, choose boundary weights $\{\hat{\omega}_{i0}\}$ i.i.d. Exp($1 - \lambda_r$), and $\{\hat{\omega}_{0j}\}$ i.i.d. Exp(λ_ℓ), corresponding to the waiting times of H_0 and P_0. Suppose $\lambda_\ell > \varrho > \lambda_r$ and ω is the ϱ-equilibrium boundary condition defined by (2.5). Then we have the stochastic domination $\omega_{i0} \geq \hat{\omega}_{i0}$ and $\omega_{0j} \geq \hat{\omega}_{0j}$, and we can realize these inequalities by coupling the boundary weights. The proofs of Section 7 show that in fact one need not insist on exponential boundary weights $\{\hat{\omega}_{i0}, \hat{\omega}_{0j}\}$, but instead only inequality (2.11) is required for the fluctuations.

3.3 A deposition model

In this section we describe a deposition model that gives a direct graphical connection between the TASEP and the last-passage percolation. This point of view is not needed for the later proofs, hence we only give a brief explanation.

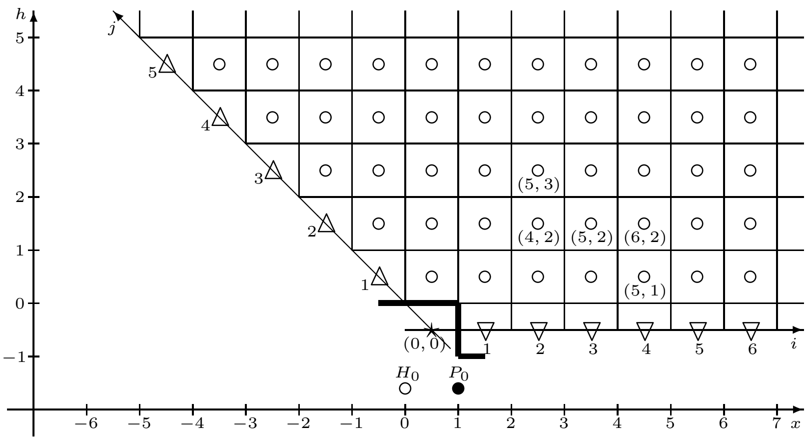

Figure 3: The initial configuration

We start by tilting the j-axis and all the vertical columns of Figure 1 by 45 degrees, resulting in Figure 3. This picture represents the same initial situation as Figure 1, but note that now the j-coordinates must be read in the direction $\nwarrow$. (As before, some squares are labeled with their (i, j)-coordinates.) The $i - j$ tilted coordinate system is embedded in an $x - h$ orthogonal system.

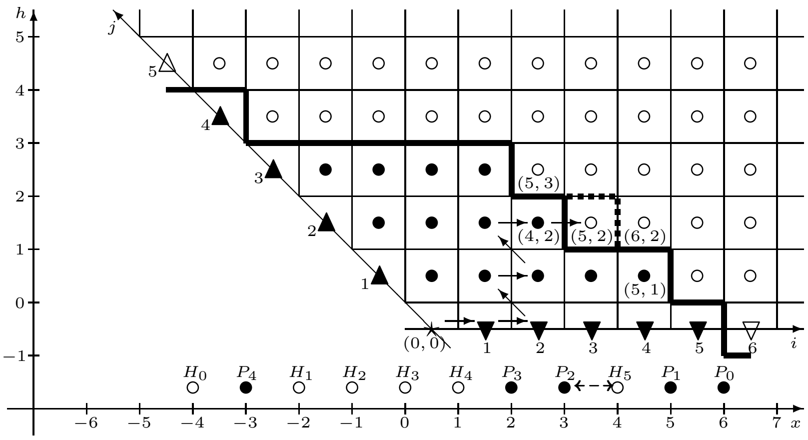

Figure 4: A possible move at a later time

Figure 4 shows the later situation that corresponds to Figure 2. As before, the thickset line is the boundary of the squares belonging to $\mathcal{A}(t)$ of (2.4). Whenever it makes sense, the height h_x of a column x is defined as the h-coordinate (i.e. the vertical height) of the thickset line above the edge $[x,\, x+1]$ on the x-axis. Define the increments $\eta_x = h_{x-1} - h_x$ and notice that, whenever defined, $\eta_x \in \{0,\, 1\}$ due to the tilting we made. The last passage rules, converted for this picture, tell us that occupation of a new square happens at rate one unless it would violate $\eta_x \in \{0,\, 1\}$ for some x. Moreover, one can read that the occupation of a square $(i,\, j)$ is the same event as the pair $(\eta_{i-j},\, \eta_{i-j+1})$ changing from $(1,\, 0)$ to $(0,\, 1)$. Comparing this to (3.1) leads us to the conclusion that η_x, whenever defined, is the occupation variable of the simple exclusion process that corresponds to the last passage model. This way one can also conveniently include the particles ($\eta_x = 1$) and holes ($\eta_x = 0$) on the x-axis, as seen on the figures. Notice also that the time-increment $h_x(t) - h_x(0)$ is the cumulative particle current across the bond $[x,\, x+1]$.

3.4 The characteristics

One-dimensional conservative particle systems have the conservation law

$$\partial_t \varrho(t,\, x) + \partial_x f(\varrho(t,\, x)) = 0 \tag{3.2}$$

under the Eulerian hydrodynamic scaling, where $\varrho(t,\, x)$ is the expected particle number per site and $f(\varrho(t,\, x))$ is the macroscopic particle flux around the rescaled position x at the rescaled time t, see e.g. (10) for details. Disturbances of the solution propagate with the characteristic speed $f'(\varrho)$. The macroscopic particle flux for TASEP is $f(\varrho) = \varrho(1-\varrho)$, and consequently the characteristic speed is $f'(\varrho) = 1 - 2\varrho$. Thus the characteristic curve started at the origin is $t \mapsto (1-2\varrho)t$. To identify the point (m, n) in the last-passage picture that corresponds to this curve, we reason approximately. Namely, we look for m and n such that hole H_m and particle P_n interchange positions at around time t and the characteristic position $(1-2\varrho)t$. By time t, that particle P_n has jumped over approximately $(1-\varrho)t$ sites due to Burke's Theorem. Hence at time zero, P_n is approximately at position $(1-2\varrho)t - (1-\varrho)t = -\varrho t$. Since the particle density is ϱ, the particle labels around this position are $n \approx \varrho^2 t$ at time zero. Similarly, holes travel at a speed $-\varrho$, so hole H_m starts from approximately $(1-2\varrho)t + \varrho t$. They have density $1-\varrho$, which indicates $m \approx (1-\varrho)^2 t$. Thus we are led to consider the point $(m, n) = (\lfloor (1-\varrho)^2 t \rfloor, \lfloor \varrho^2 t \rfloor)$ as done in (2.6).

In the rarefaction fan situation with initial density

$$\varrho(0, x) = \begin{cases} \lambda_\ell, & x < 0 \\ \lambda_r, & x > 0 \end{cases} \qquad \text{with } \lambda_\ell > \lambda_r$$

all curves $t \mapsto (1-2\varrho)t$ for $\varrho \in [\lambda_r, \lambda_\ell]$ are characteristics emanating from the origin. With this initial density the entropy solution of the conservation law (3.2) is

$$\varrho(t, x) = \begin{cases} \lambda_\ell, & x < (1-2\lambda_\ell)t \\ \frac{1}{2} - \frac{x}{2t}, & (1-2\lambda_\ell)t < x < (1-2\lambda_r)t \\ \lambda_r, & x > (1-2\lambda_r)t. \end{cases}$$

For given densities $\lambda_\ell > \lambda_r$ and Bernoulli initial occupations, we can take any $\varrho \in [\lambda_r, \lambda_\ell]$ and construct the coupled boundary values (2.11) with the correct exponential distributions.

A macroscopic calculation utilizing $\varrho(t,x)$ and the flux $f(\varrho(t,x))$ concludes that again, roughly speaking, particle $\varrho^2 t$ meets hole $(1-\varrho)^2 t$ at point $(1-2\varrho)t$ at time t, for each $\varrho \in [\lambda_r, \lambda_\ell]$. Thus the endpoints $(m,n) = (\lfloor (1-\varrho)^2 t \rfloor, \lfloor \varrho^2 t \rfloor)$ that are possible in Theorem 2.4 cover the entire range of characteristic directions for given $\lambda_\ell > \lambda_r$.

4 Preliminaries

We turn to establish some basic facts and tools. First an extension of Corollary 3.2 to show that Burke's Theorem holds for every hole and particle in the last-passage picture. Define

$$\begin{aligned} I_{ij} &:= G_{ij} - G_{\{i-1\}j} \quad \text{for } i \ge 1,\ j \ge 0, \qquad \text{and} \\ J_{ij} &:= G_{ij} - G_{i\{j-1\}} \quad \text{for } i \ge 0,\ j \ge 1. \end{aligned} \tag{4.1}$$

I_{ij} is the time it takes for particle P_j to jump again after its jump to site $i-j$. J_{ij} is the time it takes for hole H_i to jump again after its jump to site $i-j+1$. Applying the last passage rules (2.3) shows

$$\begin{aligned} I_{ij} &= G_{ij} - G_{\{i-1\}j} \\ &= (G_{\{i-1\}j} \vee G_{i\{j-1\}}) + \omega_{ij} - G_{\{i-1\}\{j-1\}} \\ &\qquad\qquad - (G_{\{i-1\}j} - G_{\{i-1\}\{j-1\}}) \\ &= (J_{\{i-1\}j} \vee I_{i\{j-1\}}) + \omega_{ij} - J_{\{i-1\}j} \\ &= (I_{i\{j-1\}} - J_{\{i-1\}j})^+ + \omega_{ij}. \end{aligned} \tag{4.2}$$

Similarly,

$$J_{ij} = (J_{\{i-1\}j} - I_{i\{j-1\}})^+ + \omega_{ij}. \tag{4.3}$$

For later use, we define

$$X_{\{i-1\}\{j-1\}} = I_{i\{j-1\}} \wedge J_{\{i-1\}j}. \tag{4.4}$$

Lemma 4.1. *Fix $i,\ j \ge 1$. If $I_{i\{j-1\}}$ and $J_{\{i-1\}j}$ are independent exponentials with respective parameters $1-\varrho$ and ϱ, then I_{ij}, J_{ij}, and $X_{\{i-1\}\{j-1\}}$ are jointly independent exponentials with respective parameters $1-\varrho$, ϱ, and 1.*

Proof. As the variables $I_{i\{j-1\}}$, $J_{\{i-1\}j}$ and ω_{ij} are independent, we use (4.2), (4.3) and (4.4) to write the joint moment generating function as

$$\begin{aligned} M_{I_{ij},\,J_{ij},\,X_{\{i-1\}\{j-1\}}}(s,\,t,\,u) &:= \mathbf{E}\mathrm{e}^{sI_{ij}+tJ_{ij}+uX_{\{i-1\}\{j-1\}}} \\ &= \mathbf{E}\mathrm{e}^{s(I_{i\{j-1\}}-J_{\{i-1\}j})^+ + t(J_{\{i-1\}j}-I_{i\{j-1\}})^+ + u(I_{i\{j-1\}}\wedge J_{\{i-1\}j})} \cdot \mathbf{E}\mathrm{e}^{(s+t)\omega_{ij}} \end{aligned}$$

where it is defined. Then, with the assumption of the lemma and the definition of ω_{ij}, elementary calculations show

$$M_{I_{ij},\,J_{ij},\,X_{\{i-1\}\{j-1\}}}(s,\,t,\,u) = \frac{\varrho \cdot (1-\varrho)}{(1-\varrho-s)\cdot(\varrho-t)\cdot(1-u)}.$$

□

Let Σ be the set of doubly-infinite down-right paths in the first quadrant of the (i, j)-coordinate system. In terms of the sequence of points visited a path $\sigma \in \Sigma$ is given by

$$\sigma = \{\cdots \to (p_{-1},\, q_{-1}) \to (p_0,\, q_0) \to (p_1,\, q_1) \to \cdots \to (p_l,\, q_l) \to \dots\}$$

with all $p_l, q_l \geq 0$ and steps

$$(p_{l+1},\, q_{l+1}) - (p_l,\, q_l) = \begin{cases} (1, 0) & \text{(direction} \to \text{in Figure 1), or} \\ (0, -1) & \text{(direction} \downarrow \text{in Figure 1).} \end{cases}$$

The interior of the set enclosed by σ is defined by

$$\mathcal{B}(\sigma) = \{(i,\, j) \,:\, 0 \leq i < p_l,\; 0 \leq j < q_l \text{ for some } (p_l,\, q_l) \in \sigma\}.$$

The last-passage time increments along σ are the variables

$$Z_l(\sigma) = G_{p_{l+1}q_{l+1}} - G_{p_l q_l} = \begin{cases} I_{p_{l+1}q_{l+1}}, & \text{if } (p_{l+1},\, q_{l+1}) - (p_l,\, q_l) = (1, 0), \\ J_{p_l q_l}, & \text{if } (p_{l+1},\, q_{l+1}) - (p_l,\, q_l) = (0, -1), \end{cases}$$

for $l \in \mathbb{Z}$. We admit the possibility that σ is the union of the i- and j-coordinate axes, in which case $\mathcal{B}(\sigma)$ is empty.

Lemma 4.2. *For any $\sigma \in \Sigma$, the random variables*

$$\big\{\{X_{ij} \,:\, (i,\, j) \in \mathcal{B}(\sigma)\},\; \{Z_l(\sigma) : l \in \mathbb{Z}\}\big\} \tag{4.5}$$

are mutually independent, I's with $\mathrm{Exp}(1-\varrho)$*, J's with* $\mathrm{Exp}(\varrho)$*, and X's with* $\mathrm{Exp}(1)$ *distribution.*

Proof. We first consider the countable set of paths that join the j-axis to the i-axis, in other words those for which there exist finite $n_0 < n_1$ such that $p_n = 0$ for $n \leq n_0$ and $q_n = 0$ for $n \geq n_1$. For these paths we argue by induction on $\mathcal{B}(\sigma)$. When $\mathcal{B}(\sigma)$ is the empty set, the statement reduces to the independence of ω-values on the i- and j-axes which is part of the set-up.

Now given an arbitrary $\sigma \in \Sigma$ that connects the j- and the i-axes, consider a *growth corner* (i, j) for $\mathcal{B}(\sigma)$, by which we mean that for some index $l \in \mathbb{Z}$,

$$(p_{l-1}, q_{l-1}), (p_l, q_l), (p_{l+1}, q_{l+1}) \;=\; (i, j+1), (i, j), (i+1, j).$$

A new valid $\widetilde{\sigma} \in \Sigma$ can be produced by replacing the above points with

$$(\widetilde{p}_{l-1}, \widetilde{q}_{l-1}), (\widetilde{p}_l, \widetilde{q}_l), (\widetilde{p}_{l+1}, \widetilde{q}_{l+1}) \;=\; (i, j+1), (i+1, j+1), (i+1, j)$$

and now $\mathcal{B}(\widetilde{\sigma}) = \mathcal{B}(\sigma) \cup \{(i, j)\}$.

The change inflicted on the set of random variables (4.5) is that

$$\{I_{\{i+1\}j},\; J_{i\{j+1\}}\} \tag{4.6}$$

has been replaced by

$$\{I_{\{i+1\}\{j+1\}},\; J_{\{i+1\}\{j+1\}},\; X_{ij}\}. \tag{4.7}$$

By (4.2)–(4.3) variables (4.7) are determined by (4.6) and $\omega_{\{i+1\}\{j+1\}}$. If we assume inductively that σ satisfies the conclusion we seek, then so does $\widetilde{\sigma}$ by Lemma 4.1 and because in the situation under consideration $\omega_{\{i+1\}\{j+1\}}$ is independent of the variables in (4.5).

For an arbitrary σ the statement follows because the independence of the random variables in (4.5) follows from independence of finite subcollections. Consider any square $R = \{0 \le i, j \le M\}$ large enough so that the corner (M, M) lies outside $\sigma \cup \mathcal{B}(\sigma)$. Then the X- and $Z(\sigma)$-variables associated to σ that lie in R are a subset of the variables of a certain path $\widetilde{\sigma}$ that goes through the points $(0, M)$ and $(M, 0)$. Thus the variables in (4.5) that lie inside an arbitrarily large square are independent. $\square$

By applying Lemma 4.2 to a path that contains the horizontal line $q_l \equiv j$ we get a version of Burke's theorem: particle P_j obeys a Poisson process after time G_{0j} when it "enters the last-passage picture." The vertical line $p_l \equiv i$ gives the corresponding statement for hole H_i.

Example 2.10.2 of Walrand (17) gives an intuitive understanding of this result. Our initial state corresponds to the situation when particle P_0 and hole H_0 have just exchanged places in an equilibrium system of queues. H_0 is therefore a customer who has just moved from queue 0 to queue 1. By that Example, this customer sees an equilibrium system of queues every time he jumps. Similarly, any new customer arriving to the queue of particle P_1 sees an equilibrium queue system in front, so Burke's theorem extends to the region between P_0 and H_0.

Up-right turns do not have independence: variables I_{ij} and $J_{i\{j+1\}}$, or J_{ij} and $I_{\{i+1\}j}$ are not independent.

The same inductive argument with a growing cluster $\mathcal{B}(\sigma)$ proves a result that corresponds to a coupling of two exclusion systems η and $\widetilde{\eta}$ where the latter has a higher density of particles. However, the lemma is a purely deterministic statement.

Lemma 4.3. *Consider two assignments of values $\{\omega_{ij}\}$ and $\{\widetilde{\omega}_{ij}\}$ that satisfy $\omega_{00} = \widetilde{\omega}_{00} = 0$, $\omega_{0j} \ge \widetilde{\omega}_{0j}$, $\omega_{i0} \le \widetilde{\omega}_{i0}$, and $\omega_{ij} = \widetilde{\omega}_{ij}$ for all $i, j \ge 1$. Then all increments satisfy $I_{ij} \le \widetilde{I}_{ij}$ and $J_{ij} \ge \widetilde{J}_{ij}$.*

Proof. One proves by induction that the statement holds for all increments between points in $\sigma \cup \mathcal{B}(\sigma)$ for those paths $\sigma \in \Sigma$ for which $\mathcal{B}(\sigma)$ is finite. If $\mathcal{B}(\sigma)$ is empty the statement is the assumption made on the ω- and $\widetilde{\omega}$-values on the i- and j-axes. The induction step that adds a growth corner to $\mathcal{B}(\sigma)$ follows from equations (4.2) and (4.3). $\square$

4.1 The reversed process.

Fix $m > 0$ and $n > 0$, and define

$$H_{ij} = G_{mn} - G_{ij}$$

for $0 \le i \le m$, $0 \le j \le n$. This is the time needed to "free" the point (i, j) in the reversed process, started from the moment when (m, n) becomes occupied. For $0 \le i < m$ and $0 \le j < n$,

$$\begin{aligned} H_{ij} &= -((G_{\{i+1\}j} - G_{mn}) \wedge (G_{i\{j+1\}} - G_{mn})) \\ &\quad + ((G_{\{i+1\}j} - G_{ij}) \wedge (G_{i\{j+1\}} - G_{ij})) \\ &= H_{\{i+1\}j} \vee H_{i\{j+1\}} + X_{ij} \end{aligned}$$

with definition (4.4) of the X-variables. Taking this and Lemma 4.2 into account, we see that the H-process is a copy of the original G-process, but with reversed coordinate directions. Precisely speaking, define $\omega^*_{00} = 0$, and then for $0 < i \le m$, $0 < j \le n$: $\omega^*_{i0} = I_{\{m-i+1\}n}$, $\omega^*_{0j} = J_{m\{n-j+1\}}$, and $\omega^*_{ij} = X_{\{m-i\}\{n-j\}}$. Then $\{\omega^*_{ij} : 0 \le i \le m, 0 \le j \le n\}$ is distributed like $\{\omega_{ij} : 0 \le i \le m, 0 \le j \le n\}$ in (2.5), and the process

$$G^*_{ij} = H_{\{m-i\}\{n-j\}} \tag{4.8}$$

for $0 \le i \le m$, $0 \le j \le n$ satisfies

$$G^*_{ij} = (G^*_{\{i-1\}j} \vee G^*_{i\{j-1\}}) + \omega^*_{ij}, \qquad 0 \le i \le m\,,\, 0 \le j \le n$$

(with the formal assumption $G^*_{\{-1\}j} = G^*_{i\{-1\}} = 0$), see (2.3). Thus the pair (G^*, ω^*) has the same distribution as (G, ω) in a fixed rectangle $\{0 \le i \le m\} \times \{0 \le j \le n\}$. Throughout the paper quantities defined in the reversed process will be denoted by a superscript *, and they will always be equal in distribution to their original forward versions.

4.2 Exit point and competition interface.

For integers x define

$$U^\varrho_x = G_{\{x^+\}\{x^-\}} = \begin{cases} \sum_{i=0}^{x} \omega_{i0}, & x \ge 0 \\ \sum_{j=0}^{-x} \omega_{0j}, & x \le 0. \end{cases} \tag{4.9}$$

Referring to the two coordinate systems in Figure 3, this is the last-passage time of the point on the (i, j)-axes above point x on the x-axis. This point is on the i-axis if $x \ge 0$ and on the j-axis if $x \le 0$.

Fix integers $m \ge x^+ \vee 1$, $n \ge x^- \vee 1$, and define $\Pi_x(m, n)$ as the set of directed paths π connecting $(x^+ \vee 1,\, x^- \vee 1)$ and $(m,\, n)$ using allowable steps (2.2). Then let

$$A_x = A_x(m, n) = \max_{\pi \in \Pi_x(m,n)} \sum_{(p,q) \in \pi} \omega_{pq} \tag{4.10}$$

be the maximal weight collected by a path from $(x^+,\, x^-)$ to $(m,\, n)$ that immediately exits the axes, and does not count $\omega_{x^+x^-}$. Notice that $A_{-1} = A_0 = A_1$ and this value is the last-passage time from $(1, 1)$ to (m, n) that completely ignores the boundaries, or in other words, sets the boundary values ω_{i0} and ω_{0j} equal to zero.

By the continuity of the exponential distribution there is an a.s. unique path $\widehat{\pi}$ from $(0,\, 0)$ to $(m,\, n)$ which collects the maximal weight G^ϱ_{mn}. Earlier we defined the *exit point* $Z^\varrho \in \mathbb{Z}$ to represent the last point of this path on either the i-axis or the j-axis. Equivalently we can now state that Z^ϱ is the a.s. unique integer for which

$$G^\varrho_{mn} = U^\varrho_{Z^\varrho} + A_{Z^\varrho}.$$

Simply because the maximal path $\widehat{\pi}$ necessarily goes through either $(0, 1)$ or $(1, 0)$, Z^ϱ is always nonzero.

Recall the definition of the competition interface in (2.7). Now we can observe that the competition interface is the time reversal of the maximal path $\widehat{\pi}$. Namely, the competition interface of the reversed process follows the maximal path $\widehat{\pi}$ backwards from the corner (m,n), until it hits either the i- or the j-axis. To make a precise statement, let us represent the a.s. unique maximal last-passage path, with exit point Z^{ϱ} as defined above, as

$$\widehat{\pi} = \{(0,0) = \widehat{\pi}_0 \to \widehat{\pi}_1 \to \cdots \to \widehat{\pi}_{|Z^{\varrho}|} \to \cdots \to \widehat{\pi}_{m+n} = (m,n)\},$$

where $\{\widehat{\pi}_{|Z^{\varrho}|+1} \to \cdots \to \widehat{\pi}_{m+n}\}$ is the portion of the path that resides in the interior $\{1,\dots,m\}\times\{1,\dots,n\}$.

Lemma 4.4. *Let φ^* be the competition interface constructed for the process G^* defined by (4.8). Then* $\varphi^*_k = (m,n) - \widehat{\pi}_{m+n-k}$ for $0 \le k \le m+n-|Z^{\varrho}|$.

Proof. Starting from $\widehat{\pi}_{m+n} = (m,n)$, the maximal path $\widehat{\pi}$ can be constructed backwards step by step by always moving to the maximizing point of the right-hand side of (2.3). This is the same as constructing the competition interface for the reversed process G^* by (2.7). Since G^* is not constructed outside the rectangle $\{0,\dots,m\}\times\{0,\dots,n\}$, we cannot assert what the competition interface does after the point

$$\varphi^*_{m+n-|Z^{\varrho}|} = \widehat{\pi}_{|Z^{\varrho}|} = \begin{cases} (Z^{\varrho},0) & \text{if } Z^{\varrho} > 0 \\ (0,-Z^{\varrho}) & \text{if } Z^{\varrho} < 0. \end{cases} \qquad \square$$

Notice that, due to this lemma, $Z^{*\varrho}$ defined in (2.10) is indeed Z^{ϱ} defined in the reversed process, which justifies the argument following (2.10).

The competition interface bounds the regions where the boundary conditions on the axes are felt. From this we can get useful bounds between last-passage times under different boundary conditions. This is the last-passage model equivalent of the common use of second-class particles to control discrepancies between coupled interacting particle systems. In the next lemma, the superscript $\mathcal{W}$ represents the west boundary (j-axis) of the (i,j)-plane. Remember that $(v(n),n)$ is the left-most point of the competition interface on the horizontal line $j=n$ computed in terms of the G-process (see (2.8)).

Lemma 4.5. *Let $G^{\mathcal{W}=0}$ be the last-passage times of a system where we set $\omega_{0j}=0$ for all $j\ge 1$. Then for $v(n) < m_1 < m_2$,*

$$\begin{aligned} A_0(m_2,n) - A_0(m_1,n) &\le G^{\mathcal{W}=0}(m_2,n) - G^{\mathcal{W}=0}(m_1,n) \\ &= G(m_2,n) - G(m_1,n). \end{aligned}$$

Proof. The first inequality is a consequence of Lemma 4.3, because computing A_0 is the same as computing G with all boundary values $\omega_{i0} = \omega_{0j} = 0$ (and in fact this inequality is valid for all $m_1 < m_2$.) The equality $G(m,n) = G^{\mathcal{W}=0}(m,n)$ for $m > v(n)$ follows because the maximal path $\widehat{\pi}$ for $G(m,n)$ goes through $(1,0)$ and hence does not see the boundary values ω_{0j}. Thus this same path $\widehat{\pi}$ is maximal for $G^{\mathcal{W}=0}(m,n)$ too. $\square$

If we set $\omega_{i0} = 0$ (the south boundary denoted by $\mathcal{S}$) instead, we get this statement: for $0 \le m_1 < m_2 \le v(n)$,

$$\begin{aligned} A_0(m_2,n) - A_0(m_1,n) &\ge G^{\mathcal{S}=0}(m_2,n) - G^{\mathcal{S}=0}(m_1,n) \\ &= G(m_2,n) - G(m_1,n). \end{aligned} \tag{4.11}$$

4.3 A coupling on the i-axis

Let $1 > \lambda > \varrho > 0$. As a common realization of the exponential weights ω_{i0}^{λ} of $\text{Exp}(1-\lambda)$ and ω_{i0}^{ϱ} of $\text{Exp}(1-\varrho)$ distribution, we write

$$\omega_{i0}^{\lambda} = \frac{1-\varrho}{1-\lambda} \cdot \omega_{i0}^{\varrho}. \tag{4.12}$$

We will use this coupling later for different purposes. We will also need

$$\mathbf{Var}(\omega_{i0}^{\lambda} - \omega_{i0}^{\varrho}) = \left(\frac{1-\varrho}{1-\lambda} - 1\right)^2 \cdot \frac{1}{(1-\varrho)^2} = \left(\frac{1}{1-\lambda} - \frac{1}{1-\varrho}\right)^2.$$

4.4 Exit point and the variance of the last-passage time

With these preliminaries we can prove the key lemma that links the variance of the last-passage time to the weight collected along the axes.

Lemma 4.6. *Fix m, n positive integers. Then*

$$\begin{aligned}\mathbf{Var}(G_{mn}^{\varrho}) &= \frac{n}{\varrho^2} - \frac{m}{(1-\varrho)^2} + \frac{2}{1-\varrho} \cdot \mathbf{E}(U_{Z^{\varrho+}}^{\varrho}) \\ &= \frac{m}{(1-\varrho)^2} - \frac{n}{\varrho^2} + \frac{2}{\varrho} \cdot \mathbf{E}(U_{-Z^{\varrho-}}^{\varrho}),\end{aligned} \tag{4.13}$$

where Z^{ϱ} is the a.s. unique exit point of the maximal path from $(0, 0)$ to (m, n).

Proof. We label the total increments along the sides of the rectangle by compass directions:

$$\mathcal{W} = G_{0n}^{\varrho} - G_{00}^{\varrho}, \quad \mathcal{N} = G_{mn}^{\varrho} - G_{0n}^{\varrho}, \quad \mathcal{E} = G_{mn}^{\varrho} - G_{m0}^{\varrho}, \quad \mathcal{S} = G_{m0}^{\varrho} - G_{00}^{\varrho}.$$

As $\mathcal{N}$ and $\mathcal{E}$ are independent by Lemma 4.2, we have

$$\begin{aligned}\mathbf{Var}(G_{mn}^{\varrho}) &= \mathbf{Var}(\mathcal{W} + \mathcal{N}) \\ &= \mathbf{Var}(\mathcal{W}) + \mathbf{Var}(\mathcal{N}) + 2\mathbf{Cov}(\mathcal{S} + \mathcal{E} - \mathcal{N}, \mathcal{N}) \\ &= \mathbf{Var}(\mathcal{W}) - \mathbf{Var}(\mathcal{N}) + 2\mathbf{Cov}(\mathcal{S}, \mathcal{N}).\end{aligned} \tag{4.14}$$

We now modify the ω-values in $\mathcal{S}$. Let $\lambda = \varrho + \varepsilon$ and apply (4.12), without changing the other values $\{\omega_{ij} : i \geq 0, j \geq 1\}$. Quantities of the altered last-passage model will be marked with a superscript ε. In this new process, $\mathcal{S}^{\varepsilon}$ has a Gamma$(m, 1-\varrho-\varepsilon)$ distribution with density

$$f_{\varepsilon}(s) = \frac{(1-\varrho-\varepsilon)^m \cdot \mathrm{e}^{-(1-\varrho-\varepsilon)s} \cdot s^{m-1}}{(m-1)!}$$

for $s > 0$, whose ε-derivative is

$$\partial_{\varepsilon} f_{\varepsilon}(s) = s f_{\varepsilon}(s) - \frac{m}{1-\varrho-\varepsilon} \cdot f_{\varepsilon}(s). \tag{4.15}$$

Given the sum $\mathcal{S}^\varepsilon$, the joint distribution of $\{\omega_{i0}\}_{1\le i\le m}$ is independent of the parameter ε, hence the quantity $\mathbf{E}(\mathcal{N}^\varepsilon \,|\, \mathcal{S}^\varepsilon = s) = \mathbf{E}(\mathcal{N} \,|\, \mathcal{S} = s)$ does not depend on ε. Therefore, using (4.15) we have

$$\begin{aligned}
\partial_\varepsilon \mathbf{E}(\mathcal{N}^\varepsilon)\Big|_{\varepsilon=0} &= \partial_\varepsilon \int_0^\infty \mathbf{E}(\mathcal{N} \,|\, \mathcal{S} = s) f_\varepsilon(s)\ \mathrm{d}s\Big|_{\varepsilon=0} \\
&= \int_0^\infty \mathbf{E}(\mathcal{N} \,|\, \mathcal{S} = s)\cdot s \cdot f_0(s)\ \mathrm{d}s - \frac{m}{1-\varrho}\int_0^\infty \mathbf{E}(\mathcal{N} \,|\, \mathcal{S} = s) f_0(s)\ \mathrm{d}s \\
&= \mathbf{E}(\mathcal{N}\mathcal{S}) - \frac{m}{1-\varrho}\cdot \mathbf{E}(\mathcal{N}) = \mathbf{Cov}(\mathcal{N},\, \mathcal{S}).
\end{aligned} \tag{4.16}$$

Next we compute the same quantity by a different approach. Let Z and Z^ε be the exit points of the maximal paths to $(m,\, n)$ in the original and the modified processes, respectively. Similarly, U_x and U_x^ε are the weights as defined by (4.9) for the two processes. Hence U_Z is the weight collected on the i or j axis by the maximal path of the original process. Then

$$\begin{aligned}
\mathcal{N}^\varepsilon - \mathcal{N} &= (\mathcal{N}^\varepsilon - \mathcal{N})\cdot \mathbf{1}\{Z^\varepsilon = Z\} + (\mathcal{N}^\varepsilon - \mathcal{N})\cdot \mathbf{1}\{Z^\varepsilon \neq Z\} \\
&= (U_Z^\varepsilon - U_Z)\cdot \mathbf{1}\{Z^\varepsilon = Z\} + (\mathcal{N}^\varepsilon - \mathcal{N})\cdot \mathbf{1}\{Z^\varepsilon \neq Z\} \\
&= (U_Z^\varepsilon - U_Z) + (\mathcal{N}^\varepsilon - \mathcal{N} - U_Z^\varepsilon + U_Z)\cdot \mathbf{1}\{Z^\varepsilon \neq Z\}.
\end{aligned}$$

As ω values are only changed on the i-axis, the first term is rewritten as

$$U_Z^\varepsilon - U_Z = U_{Z^+}^\varepsilon - U_{Z^+} = \left(\frac{1-\varrho}{1-\varrho-\varepsilon} - 1\right) U_{Z^+} = \frac{\varepsilon}{1-\varrho-\varepsilon}\cdot U_{Z^+}$$

by (4.12). We show that the expectation of the second term is $\mathfrak{o}(\varepsilon)$. Note that the increase $\mathcal{N}^\varepsilon - \mathcal{N}$ is bounded by $\mathcal{S}^\varepsilon - \mathcal{S}$. Hence

$$\begin{aligned}
&\mathbf{E}[(\mathcal{N}^\varepsilon - \mathcal{N} - U_Z^\varepsilon + U_Z)\cdot \mathbf{1}\{Z^\varepsilon \neq Z\}] \\
&\quad \le \mathbf{E}[(\mathcal{N}^\varepsilon - \mathcal{N})\cdot \mathbf{1}\{Z^\varepsilon \neq Z\}] \ \le\ \mathbf{E}[(\mathcal{S}^\varepsilon - \mathcal{S})\cdot \mathbf{1}\{Z^\varepsilon \neq Z\}] \\
&\quad \le \big(\mathbf{E}[(\mathcal{S}^\varepsilon - \mathcal{S})^2]\big)^{\frac{1}{2}}\cdot \big(\mathbf{P}\{Z^\varepsilon \neq Z\}\big)^{\frac{1}{2}}.
\end{aligned} \tag{4.17}$$

To show that the probability is of the order of ε, notice that the exit point of the maximal path can only differ in the modified process from the one of the original process, if for some $Z < k \le m$, $U_k^\varepsilon + A_k > U_Z^\varepsilon + A_Z$ with Z of the original process (see (4.10) for the definition of A_i). Therefore,

$$\begin{aligned}
\mathbf{P}\{Z^\varepsilon \neq Z\} &= \mathbf{P}\{U_k^\varepsilon + A_k > U_Z^\varepsilon + A_Z \text{ for some } Z < k \le m\} \\
&= \mathbf{P}\{U_k^\varepsilon - U_Z^\varepsilon > A_Z - A_k \text{ for some } Z < k \le m\} \\
&= \mathbf{P}\{U_k^\varepsilon - U_Z^\varepsilon > A_Z - A_k \ge U_k - U_Z \text{ for some } Z < k \le m\} \\
&\le \mathbf{P}\{U_k^\varepsilon - U_i^\varepsilon > A_i - A_k \ge U_k - U_i \text{ for some } 0 \le i < k \le m\} \\
&\le \sum_{0\le i<k\le m} \mathbf{P}\{U_k^\varepsilon - U_i^\varepsilon > A_i - A_k \ge U_k - U_i\}.
\end{aligned}$$

We also used the definition of Z in the third equality, via $A_Z + U_Z \ge A_k + U_k$. Notice that A's and U's are independent for fixed indices. Hence with μ denoting the distribution of $A_i - A_k$, we write

$$\begin{aligned}
\mathbf{P}\{U_k^\varepsilon - U_i^\varepsilon > A_i - A_k \geq U_k - U_i\} \\
&= \int \mathbf{P}\{U_k^\varepsilon - U_i^\varepsilon > x \geq U_k - U_i\}\,\mathrm{d}\mu(x) \\
&\leq \sup_x \mathbf{P}\{U_k^\varepsilon - U_i^\varepsilon > x \geq U_k - U_i\} \\
&= \sup_x \mathbf{P}\Big\{\frac{1-\varrho}{1-\varrho-\varepsilon}\cdot(U_k - U_i) > x \geq U_k - U_i\Big\} \\
&= \sup_x \mathbf{P}\Big\{x \geq U_k - U_i > x\Big(1 - \frac{\varepsilon}{1-\varrho}\Big)\Big\}.
\end{aligned}$$

Since $U_k - U_i$ has a Gamma distribution, the supremum above is $\mathcal{O}(\varepsilon)$, which shows the bound on $\mathbf{P}\{Z^\varepsilon \neq Z\}$. The first factor on the right-hand side of (4.17),

$$\big(\mathbf{E}[(\mathcal{S}^\varepsilon - \mathcal{S})^2]\big)^{1/2} = \frac{\varepsilon}{1-\varrho-\varepsilon}\cdot\big(\mathbf{E}[\mathcal{S}^2]\big)^{1/2},$$

is of order ε. Hence the error term (4.17) is $\mathfrak{o}(\varepsilon)$, and we conclude

$$\partial_\varepsilon \mathbf{E}(\mathcal{N}^\varepsilon)\Big|_{\varepsilon=0} = \frac{1}{1-\varrho}\cdot\mathbf{E}(U_{Z^+}).$$

The proof of the first statement is then completed by this display, (4.14) and (4.16), as $\mathcal{W}$ and $\mathcal{N}$ are Gamma-distributed by Lemma 4.2. The second statement follows in a similar way, using $\mathbf{Cov}(\mathcal{W},\,\mathcal{E})$. □

Lemma 4.7. *Let $0 < \varrho \leq \lambda < 1$. Then*

$$\mathbf{Var}(G_{mn}^\lambda) \leq \frac{\varrho^2}{\lambda^2}\cdot\mathbf{Var}(G_{mn}^\varrho) + m\cdot\bigg(\frac{1}{(1-\lambda)^2} - \frac{\varrho^2}{\lambda^2(1-\varrho)^2}\bigg).$$

Proof. The proof is based on the coupling described by (4.12), and a similar one $\omega_{0j}^\lambda = \frac{\varrho}{\lambda}\cdot\omega_{0j}^\varrho$ on the j axis. Note that in this coupling, when changing from ϱ to λ, we are increasing the weights on the i-axis and decreasing the weights on the j-axis, which clearly implies $Z^\varrho \leq Z^\lambda$. Also, we remain in the stationary situation, so (4.13) remains valid for λ. As $U_{-x^-}^\varrho$ is non-increasing in x, this implies

$$U_{-Z^{\lambda-}}^\lambda = \frac{\varrho}{\lambda}\cdot U_{-Z^{\lambda-}}^\varrho \leq \frac{\varrho}{\lambda}\cdot U_{-Z^{\varrho-}}^\varrho.$$

We substitute this into the second line of (4.13) to get

$$\begin{aligned}
\mathbf{Var}(G_{mn}^\lambda) &= \frac{m}{(1-\lambda)^2} - \frac{n}{\lambda^2} + \frac{2}{\lambda}\cdot\mathbf{E}(U_{-Z^{\lambda-}}^\lambda) \\
&\leq \frac{m}{(1-\lambda)^2} - \frac{n}{\lambda^2} + \frac{2\varrho}{\lambda^2}\cdot\mathbf{E}(U_{-Z^{\varrho-}}^\varrho) \\
&= \frac{\varrho^2}{\lambda^2}\cdot\bigg(\frac{m}{(1-\varrho)^2} - \frac{n}{\varrho^2} + \frac{2}{\varrho}\cdot\mathbf{E}(U_{-Z^{\varrho-}}^\varrho)\bigg) \\
&\qquad + m\cdot\bigg(\frac{1}{(1-\lambda)^2} - \frac{\varrho^2}{\lambda^2(1-\varrho)^2}\bigg) \\
&= \frac{\varrho^2}{\lambda^2}\cdot\mathbf{Var}(G_{mn}^\varrho) + m\cdot\bigg(\frac{1}{(1-\lambda)^2} - \frac{\varrho^2}{\lambda^2(1-\varrho)^2}\bigg).
\end{aligned}$$

$\square$

5 Upper bound

We turn to proving the upper bounds in Theorems 2.1 and 2.2. We have a fixed density $\varrho \in (0, 1)$, and to study the last-passage times G^ϱ along the characteristic, we define the dimensions of the last-passage rectangle as

$$m(t) = \lfloor (1-\varrho)^2 t \rfloor \qquad \text{and} \qquad n(t) = \lfloor \varrho^2 t \rfloor \tag{5.1}$$

with a parameter $t \to \infty$. The quantities A_x, Z and G_{mn} connected to these indices are denoted by $A_x(t)$, $Z(t)$, $G(t)$. In the proofs we need to consider different boundary conditions (2.5) with ϱ replaced by λ. This will be indicated by a superscript. However, the superscript λ only changes the boundary conditions and not the dimensions $m(t)$ and $n(t)$, always defined by (5.1) with a fixed ϱ. Moreover, we apply the coupling (4.12) on the i-axis and $\omega_{0j}^\lambda = \frac{\varrho}{\lambda} \cdot \omega_{0j}^\varrho$ on the j-axis. The weights $\{\omega_{ij}\}_{i,j \geq 1}$ in the interior will not be affected by changes in boundary conditions, so in particular $A_x(t)$ will not either. Since $G^\lambda(t)$ chooses the maximal path,

$$U_z^\lambda + A_z(t) \leq G^\lambda(t)$$

for all $1 \leq z \leq m(t)$ and all densities $0 < \lambda < 1$. Consequently, for integers $u \geq 0$ and densities $\lambda \geq \varrho$,

$$\begin{aligned} \mathbf{P}\{Z^\varrho(t) > u\} &= \mathbf{P}\{\exists z > u \,:\, U_z^\varrho + A_z(t) = G^\varrho(t)\} \\ &\leq \mathbf{P}\{\exists z > u \,:\, U_z^\varrho - U_z^\lambda + G^\lambda(t) \geq G^\varrho(t)\} \\ &= \mathbf{P}\{\exists z > u \,:\, U_z^\lambda - U_z^\varrho \leq G^\lambda(t) - G^\varrho(t)\} \\ &\leq \mathbf{P}\{U_u^\lambda - U_u^\varrho \leq G^\lambda(t) - G^\varrho(t)\}. \end{aligned} \tag{5.2}$$

The last step is justified by $\lambda \geq \varrho$ and the coupling (4.12).

To put the subsequent technicalities into context, we give a rough summary of the argument that follows. To get an upper bound for $\mathbf{Var}(G^\varrho(t))$, by Lemma 4.6 it suffices to find an upper bound for the mean of $U^\varrho_{Z^\varrho(t)^+}$. The goal will be to obtain an inequality of the type

$$\mathbf{P}\{U^\varrho_{Z^\varrho(t)^+} > y\} \leq C\frac{t^2}{y^4} \cdot \mathbf{E}(U^\varrho_{Z^\varrho(t)^+}) + (\text{error terms}). \tag{5.3}$$

This is sufficient to guarantee that $\mathbf{E}(U^\varrho_{Z^\varrho(t)^+})$ does not grow faster than $t^{2/3}$, provided the error terms can be handled. Informally speaking, here is the logic that produces inequality (5.3). If $U^\varrho_{Z^\varrho(t)^+}$ is too large, then except for a large deviation, it must be that $Z^\varrho(t)$ is too large. When $G^\varrho(t)$ collects too much weight on the i-axis, we compare it to a λ-system with $\lambda > \varrho$, as was done in (5.2) above. By choosing λ appropriately we can make $\mathbf{E}(U_u^\lambda - U_u^\varrho - G^\lambda(t) + G^\varrho(t))$ strictly positive for a range of u (Lemma 5.2), and thereby the last probability in (5.2) is already a deviation. In Lemma 5.5 we control this deviation simply with the variances of the terms. In preparation for that Lemmas 5.3 and 5.4 give bounds for the variances. In particular, by another application of Lemma 4.6, bounds on the variances of the last-passage times bring back the mean $\mathbf{E}(U^\varrho_{Z^\varrho(t)^+})$ on the right-hand side of (5.3), and thereby close the loop.

We turn to the details of the derivation of the upper bound, beginning with the optimal choice of λ. Set

$$\lambda_u = \frac{\varrho}{\sqrt{(1-\varrho)^2 - u/t} + \varrho}. \tag{5.4}$$

This density maximizes

$$\mathbf{E}(U_u^\lambda) - \mathbf{E}(G^\lambda(t)) = \frac{u}{1-\lambda} - \frac{\lfloor (1-\varrho)^2 t \rfloor}{1-\lambda} - \frac{\lfloor \varrho^2 t \rfloor}{\lambda}$$

if the integer parts are dropped. The expectation $\mathbf{E}(G_{mn}^\lambda)$ is computed as $\mathbf{E}(G_{0n}^\lambda) + \mathbf{E}(G_{mn}^\lambda - G_{0n}^\lambda)$ with the help of Lemma 4.2. Some useful identities for future computations:

$$\lambda_u \ge \varrho, \quad \frac{1}{\lambda_u} = 1 + \frac{\sqrt{(1-\varrho)^2 - u/t}}{\varrho}, \quad \frac{1}{1-\lambda_u} = 1 + \frac{\varrho}{\sqrt{(1-\varrho)^2 - u/t}}. \tag{5.5}$$

Lemma 5.1. *With* $0 \le u \le (1-\varrho)^2 t$ *and* λ_u *of* (5.4),

$$\mathbf{E}(U_u^{\lambda_u} - U_u^\varrho - G^{\lambda_u}(t) + G^\varrho(t)) \\ \ge \frac{t\varrho}{1-\varrho}\Big((1-\varrho) - \sqrt{(1-\varrho)^2 - u/t}\Big)^2 - \frac{u/t}{\varrho(1-\varrho)}.$$

Proof. By Lemma 4.2 and (5.1)

$$\begin{aligned} &\mathbf{E}(U_u^{\lambda_u} - U_u^\varrho - G^{\lambda_u}(t) + G^\varrho(t)) \\ &= \frac{u}{1-\lambda_u} - \frac{u}{1-\varrho} - \frac{\lfloor (1-\varrho)^2 t \rfloor}{1-\lambda_u} + \frac{\lfloor (1-\varrho)^2 t \rfloor}{1-\varrho} - \frac{\lfloor \varrho^2 t \rfloor}{\lambda_u} + \frac{\lfloor \varrho^2 t \rfloor}{\varrho}. \end{aligned}$$

First we remove the integer parts. Since $\lambda_u \ge \varrho$,

$$-\frac{\lfloor (1-\varrho)^2 t \rfloor}{1-\lambda_u} + \frac{\lfloor (1-\varrho)^2 t \rfloor}{1-\varrho} \ge -\frac{(1-\varrho)^2 t}{1-\lambda_u} + \frac{(1-\varrho)^2 t}{1-\varrho}.$$

For the other integer parts

$$\begin{aligned} -\frac{\lfloor \varrho^2 t \rfloor}{\lambda_u} + \frac{\lfloor \varrho^2 t \rfloor}{\varrho} &\ge -\frac{\varrho^2 t}{\lambda_u} + \frac{\varrho^2 t}{\varrho} - \frac{1}{\varrho} + \frac{1}{\lambda_u} \\ &= -\frac{\varrho^2 t}{\lambda_u} + \frac{\varrho^2 t}{\varrho} - \frac{1 - \varrho - \sqrt{(1-\varrho)^2 - u/t}}{\varrho} \\ &\ge -\frac{\varrho^2 t}{\lambda_u} + \frac{\varrho^2 t}{\varrho} - \frac{u/t}{\varrho(1-\varrho)} \end{aligned}$$

The last term above is the last term of the bound in the statement of the lemma. It remains to check that after the integer parts have been removed from the mean, the remaining quantity equals the main term of the bound.

$$\frac{u}{1-\lambda_u} - \frac{u}{1-\varrho} - \frac{(1-\varrho)^2 t}{1-\lambda_u} + \frac{(1-\varrho)^2 t}{1-\varrho} - \frac{\varrho^2 t}{\lambda_u} + \frac{\varrho^2 t}{\varrho}$$
$$= [u-(1-\varrho)^2 t] \cdot \left[1 + \frac{\varrho}{\sqrt{(1-\varrho)^2 - u/t}} - \frac{1}{1-\varrho}\right]$$
$$- \varrho^2 t \cdot \left[1 + \frac{\sqrt{(1-\varrho)^2 - u/t}}{\varrho} - \frac{1}{\varrho}\right]$$
$$= t \cdot \frac{\varrho}{1-\varrho}\left((1-\varrho) - \sqrt{(1-\varrho)^2 - u/t}\right)^2.$$

□

Lemma 5.2. *For any* $8\varrho^{-2}(1-\varrho)^2 \le u \le (1-\varrho)^2 t$,

$$\mathbf{E}(U_u^{\lambda_u} - U_u^{\varrho} - G^{\lambda_u}(t) + G^{\varrho}(t)) \ge \frac{\varrho}{8(1-\varrho)^3} \cdot \frac{u^2}{t}.$$

Proof. Assumption $u \ge 8\varrho^{-2}(1-\varrho)^2$ implies that the last term of the bound from the previous lemma satisfies

$$-\frac{u/t}{\varrho(1-\varrho)} \ge -\frac{\varrho}{8(1-\varrho)^3} \cdot \frac{u^2}{t}.$$

Thus it remains to prove

$$\left((1-\varrho) - \sqrt{(1-\varrho)^2 - u/t}\right)^2 \ge \frac{1}{4(1-\varrho)^2} \cdot \frac{u^2}{t^2}.$$

This is easy to check in the form

$$\left(C - \sqrt{C^2 - x}\right)^2 \ge \frac{1}{4C^2} \cdot x^2,$$

where $x = u/t$, $C = 1-\varrho$ and then $x \le C^2$. □

Lemma 5.3. *For any* $0 \le u \le \frac{3}{4}(1-\varrho)^2 t$,

$$\mathbf{Var}(G^{\lambda_u}(t) - G^{\varrho}(t)) \le \frac{8}{1-\varrho} \cdot \mathbf{E}(U^{\varrho}_{Z^{\varrho}(t)^+}) + \frac{8(u+1)}{(1-\varrho)^2}$$

Proof. We start with substituting (5.1) into Lemma 4.7 (integer parts can be dropped without violating the inequality):

$$\mathbf{Var}(G^{\lambda_u}(t)) \le \frac{\varrho^2}{\lambda_u^2} \cdot \mathbf{Var}(G^{\varrho}(t)) + t \cdot \left(\frac{(1-\varrho)^2}{(1-\lambda_u)^2} - \frac{\varrho^2}{\lambda_u^2}\right).$$

Utilizing (5.5),

$$\frac{(1-\varrho)^2}{(1-\lambda_u)^2} - \frac{\varrho^2}{\lambda_u^2} = \left(\sqrt{(1-\varrho)^2 - u/t} + \varrho\right)^2 \cdot \frac{u/t}{(1-\varrho)^2 - u/t}.$$

Since the expression in parentheses is not larger than 1, $u/t \le \frac{3}{4}(1-\varrho)^2$, and $\varrho \le \lambda_u$, it follows that

$$\mathbf{Var}(G^{\lambda_u}(t)) \le \mathbf{Var}(G^{\varrho}(t)) + \frac{4}{(1-\varrho)^2} \cdot u.$$

Then we proceed with Lemma 4.6 and (5.1):

$$\begin{aligned}
\mathbf{Var}(G^{\lambda_u}(t) - G^{\varrho}(t)) &\le 2\mathbf{Var}(G^{\lambda_u}(t)) + 2\mathbf{Var}(G^{\varrho}(t)) \\
&\le 4\mathbf{Var}(G^{\varrho}(t)) + \frac{8}{(1-\varrho)^2} \cdot u \\
&= \frac{8}{1-\varrho} \cdot \mathbf{E}(U^{\varrho}_{Z^{\varrho}(t)^+}) + 4\frac{\lfloor \varrho^2 t \rfloor}{\varrho^2} - 4\frac{\lfloor (1-\varrho)^2 t \rfloor}{(1-\varrho)^2} + \frac{8}{(1-\varrho)^2} \cdot u \\
&\le \frac{8}{1-\varrho} \cdot \mathbf{E}(U^{\varrho}_{Z^{\varrho}(t)^+}) + \frac{8(u+1)}{(1-\varrho)^2}
\end{aligned}$$

□

Lemma 5.4. *With the application of the coupling* (4.12)*, for any* $0 \le u \le \frac{3}{4}(1-\varrho)^2 t$ *we have*

$$\mathbf{Var}(U^{\lambda_u}_u - U^{\varrho}_u) \le u \cdot \frac{\varrho^2}{(1-\varrho)^2}.$$

Proof. By that coupling,

$$\mathbf{Var}[U^{\lambda_u}_u - U^{\varrho}_u] = \mathbf{Var}\left[\left(\frac{1-\varrho}{1-\lambda_u} - 1\right) U^{\varrho}_u\right] = u \cdot \left(\frac{1-\varrho}{1-\lambda_u} - 1\right)^2 \cdot \frac{1}{(1-\varrho)^2},$$

as U^{ϱ}_u is the sum of u many independent $\mathrm{Exp}(1-\varrho)$ weights. Write

$$\begin{aligned}
\left(\frac{1-\varrho}{1-\lambda_u} - 1\right) \cdot \frac{1}{(1-\varrho)} &= \frac{\sqrt{(1-\varrho)^2 - u/t} + \varrho}{\sqrt{(1-\varrho)^2 - u/t}} - \frac{1}{(1-\varrho)} \\
&\le \frac{\frac{1}{2}(1+\varrho)}{\frac{1}{2}(1-\varrho)} - \frac{1}{(1-\varrho)} = \frac{\varrho}{1-\varrho}.
\end{aligned}$$

□

After these preparations, we continue the main argument from (5.2).

Lemma 5.5. *There exists a constant* $C_1 = C_1(\varrho)$ *such that for any* $u \ge 8\varrho^{-2}(1-\varrho)^2$ *and* $t > 0$*,*

$$\mathbf{P}\{Z^{\varrho}(t) > u\} \le C_1\Big(\frac{t^2}{u^4} \cdot \mathbf{E}(U^{\varrho}_{Z^{\varrho}(t)^+}) + \frac{t^2}{u^3}\Big).$$

Proof. If $8\varrho^{-2}(1-\varrho)^2 \le u \le (1-\varrho)^2 t$, then continuing from (5.2) and taking Lemma 5.2 into account, we write

$$\begin{aligned}
\mathbf{P}\{Z^\varrho(t)>u\} &\le \mathbf{P}\Big\{U_u^{\lambda_u}-U_u^\varrho \le \mathbf{E}(U_u^{\lambda_u}-U_u^\varrho)-\frac{\varrho}{16(1-\varrho)^3}\cdot\frac{u^2}{t}\Big\}\\
&\quad+\mathbf{P}\Big\{G^{\lambda_u}(t)-G^\varrho(t)\ge \mathbf{E}(G^{\lambda_u}(t)-G^\varrho(t))+\frac{\varrho}{16(1-\varrho)^3}\cdot\frac{u^2}{t}\Big\}\\
&\le \mathbf{Var}(U_u^{\lambda_u}-U_u^\varrho)\cdot\frac{16^2(1-\varrho)^6}{\varrho^2}\cdot\frac{t^2}{u^4}\\
&\quad+\mathbf{Var}(G^{\lambda_u}(t)-G^\varrho(t))\cdot\frac{16^2(1-\varrho)^6}{\varrho^2}\cdot\frac{t^2}{u^4}
\end{aligned}$$

by Chebyshev's inequality. If $8\varrho^{-2}(1-\varrho)^2\le u\le\frac{3}{4}(1-\varrho)^2t$, use Lemmas 5.4 and 5.3 to conclude

$$\begin{aligned}
\mathbf{P}\{Z^\varrho(t)>u\} &\le 16^2(1-\varrho)^4\cdot\frac{t^2}{u^3}+8\cdot 16^2\cdot\frac{(1-\varrho)^5}{\varrho^2}\cdot\frac{t^2}{u^4}\cdot\mathbf{E}(U^\varrho_{Z^\varrho(t)^+})\\
&\quad+8\cdot16^2\cdot\frac{(1-\varrho)^4}{\varrho^2}\cdot\frac{t^2(u+1)}{u^4}.
\end{aligned}$$

When $\frac{3}{4}(1-\varrho)^2t<u\le(1-\varrho)^2t$, the previous display works for $\frac{3}{4}u$. Hence by

$$\mathbf{P}\{Z^\varrho(t)>u\}\le\mathbf{P}\{Z^\varrho(t)>3u/4\},$$

the statement still holds, modified by a factor of a power of 4/3.
Finally, the probability is trivially zero if $u>(1-\varrho)^2t$. □

Fix a number $0<\alpha<1$, and define

$$y=\frac{u}{\alpha(1-\varrho)}. \tag{5.6}$$

Lemma 5.6. *We have the following large deviations estimate:*

$$\mathbf{P}\{U_u^\varrho>y\}\le \mathrm{e}^{-(1-\varrho)(1-\sqrt{\alpha})^2y}.$$

Proof. We use the fact that $U_u^\varrho=\sum_{i=1}^u\omega_{i0}$, where the ω's are iid. Exp$(1-\varrho)$ variables. Fix s with $1-\varrho>s>0$. By the Markov inequality, we write

$$\begin{aligned}
\mathbf{P}\{U_u^\varrho>y\}&=\mathbf{P}\{\mathrm{e}^{sU_u^\varrho}>\mathrm{e}^{sy}\}\le\mathrm{e}^{-sy}\mathbf{E}(\mathrm{e}^{sU_u^\varrho})=\mathrm{e}^{-sy}\cdot\Big(\frac{1-\varrho}{1-\varrho-s}\Big)^u\\
&\le\exp\Big(-sy+u\cdot\frac{s}{1-\varrho-s}\Big).
\end{aligned}$$

Substituting $u=\alpha(1-\varrho)y$, the choice $s=(1-\varrho)(1-\sqrt{\alpha})$ minimizes the exponent, and yields the result. □

Lemma 5.7. *There exist finite positive constants $C_2 = C_2(\alpha, \varrho)$ and $C_3 = C_3(\alpha, \varrho)$ such that, for all*

$$r \geq \frac{8(1-\varrho)}{\alpha\varrho^2 \mathbf{E}(U^{\varrho}_{Z^{\varrho}(t)^+})},$$

we have the bound

$$\mathbf{P}\{U^{\varrho}_{Z^{\varrho}(t)^+} > r\mathbf{E}(U^{\varrho}_{Z^{\varrho}(t)^+})\}$$
$$\leq \frac{C_2 t^2}{[\mathbf{E}(U^{\varrho}_{Z^{\varrho}(t)^+})]^3} \cdot \left(\frac{1}{r^3} + \frac{1}{r^4}\right) + \exp\{-C_3 r\mathbf{E}(U^{\varrho}_{Z^{\varrho}(t)^+})\}.$$

Proof. By (5.6) and Lemmas 5.5 and 5.6, for any $y \geq 8\alpha^{-1}\varrho^{-2}(1-\varrho)$, and with an appropriately defined new constant,

$$\begin{aligned}\mathbf{P}\{U^{\varrho}_{Z^{\varrho}(t)^+} > y\} &\leq \mathbf{P}\{Z^{\varrho}(t)^+ > u\} + \mathbf{P}\{U^{\varrho}_u > y\} \\ &\leq C_2\Big(\frac{t^2}{y^4} \cdot \mathbf{E}(U^{\varrho}_{Z^{\varrho}(t)^+}) + \frac{t^2}{y^3}\Big) + \mathrm{e}^{-(1-\varrho)(1-\sqrt{\alpha})^2 y}.\end{aligned}$$

Choose $y = r\mathbf{E}(U^{\varrho}_{Z^{\varrho}(t)^+})$. □

Theorem 5.8.

$$\limsup_{t\to\infty} \frac{\mathbf{E}(U^{\varrho}_{Z^{\varrho}(t)^+})}{t^{2/3}} < \infty, \quad \textit{and} \quad \limsup_{t\to\infty} \frac{\mathbf{Var}(G^{\varrho}(t))}{t^{2/3}} < \infty.$$

Proof. The first inequality implies the second one by Lemma 4.6 and (5.1). To prove the first one, suppose that there exists a sequence $t_k \nearrow \infty$ such that

$$\lim_{k\to\infty} \frac{\mathbf{E}(U^{\varrho}_{Z^{\varrho}(t_k)^+})}{t_k^{2/3}} = \infty.$$

Then $\mathbf{E}(U^{\varrho}_{Z^{\varrho}(t_k)^+}) > t_k^{2/3}$ for all large k's, and consequently by the above lemma

$$\mathbf{P}\{U^{\varrho}_{Z^{\varrho}(t_k)^+} > r\mathbf{E}(U^{\varrho}_{Z^{\varrho}(t_k)^+})\} \leq C_2\left(\frac{1}{r^3} + \frac{1}{r^4}\right)\frac{t_k^2}{[\mathbf{E}(U^{\varrho}_{Z^{\varrho}(t_k)^+})]^3} + \exp(-C_3 r t_k^{2/3})$$

for all $r \geq C_4 t_k^{-2/3}$. This shows by dominated convergence that

$$\int_0^\infty \mathbf{P}\{U^{\varrho}_{Z^{\varrho}(t_k)^+} > r\mathbf{E}(U^{\varrho}_{Z^{\varrho}(t_k)^+})\}\,\mathrm{d}r \underset{k\to\infty}{\longrightarrow} 0,$$

which leads to the contradiction

$$1 = \mathbf{E}\left(\frac{U^{\varrho}_{Z^{\varrho}(t_k)^+}}{\mathbf{E}(U^{\varrho}_{Z^{\varrho}(t_k)^+})}\right) \underset{k\to\infty}{\longrightarrow} 0.$$

□

Combining Lemma 5.5 and Theorem 5.8 gives a tail bound on Z:

Corollary 5.9. *Given any $t_0 > 0$ there exists a finite constant $C_4 = C_4(t_0, \varrho)$ such that, for all $a > 0$ and $t \ge t_0$,*

$$\mathbf{P}\{Z^\varrho(t) \ge at^{2/3}\} \le C_4 a^{-3}.$$

6 Lower bound

The lower bound is a little more subtle than the upper bound for in a sense there is "less room in the estimates." The objective is to show that $U^\varrho_{Z^\varrho(t)^+}$ genuinely fluctuates on the scale $t^{2/3}$, and then again Lemma 4.6 translates this into a lower bound on $\mathbf{Var}(G^\varrho(t))$. So the main point is to prove this lemma:

Lemma 6.1. *We have the asymptotics*

$$\lim_{\varepsilon \searrow 0} \limsup_{t\to\infty} \mathbf{P}\{0 < U^\varrho_{Z^\varrho(t)^+} \le \varepsilon t^{2/3}\} = 0.$$

Note that part of the event is the requirement $Z^\varrho(t) > 0$. Since U^ϱ is a sum of i.i.d. exponentials, the work lies really in controlling the point $Z^\varrho(t)$. To make this explicit write

$$\mathbf{P}\{0 < U^\varrho_{Z^\varrho(t)^+} \le \varepsilon t^{2/3}\} \le \mathbf{P}\{0 < Z^\varrho(t) \le \delta t^{2/3}\} + \mathbf{P}\{U^\varrho_{\lfloor \delta t^{2/3} \rfloor} \le \varepsilon t^{2/3}\}.$$

Given $\delta > 0$, the last probability vanishes as $t \to \infty$ for any $\varepsilon < \delta(1-\varrho)^{-1}$. Thus it remains to show that the first probability on the right can be made arbitrarily small for large t, by choosing a small enough δ.

Let $0 < \delta, b < 1$. Utilizing the fact that $Z^\varrho(t)$ marks the exit point of the maximal path, we split this probability into two parts.

$$\begin{aligned}
&\mathbf{P}\{0 < Z^\varrho(t) \le \delta t^{2/3}\} \\
&\le \mathbf{P}\Big\{ \sup_{x > \delta t^{2/3}} \big(U^\varrho_x + A_x(t)\big) < \sup_{1 \le x \le \delta t^{2/3}} \big(U^\varrho_x + A_x(t)\big) \Big\} \\
&\le \mathbf{P}\Big\{ \sup_{x > \delta t^{2/3}} \big(U^\varrho_x + A_x(t) - A_1(t)\big) < bt^{1/3} \Big\} && (6.1) \\
&\quad + \mathbf{P}\Big\{ \sup_{1 \le x \le \delta t^{2/3}} \big(U^\varrho_x + A_x(t) - A_1(t)\big) > bt^{1/3} \Big\}. && (6.2)
\end{aligned}$$

Next we bound the probabilities (6.1) and (6.2) separately. We abbreviate $(m, n) = \big(\lfloor (1-\varrho)^2 t \rfloor, \lfloor \varrho^2 t \rfloor\big)$ throughout this section.

We do not have direct control over the quantity $A_x(t) - A_1(t)$ that appears in both probabilities (6.1) and (6.2). As defined in (4.10), these are last passage times that use only the internal weights $\{\omega_{ij} : i, j \ge 1\}$ and thereby are not constrained by the boundary conditions. But we can relate $A_x(t) - A_1(t)$ to equilibrium G-values by using the competition interface to restrict the influence of the boundary conditions, as shown in Lemma 4.5 and equation (4.11). The location of the competition interface is controlled by Corollary 5.9 which applies to $Z^{*\varrho}$ of (2.10) by time reversal (recall discussion before and after Lemma 4.4).

The only impediment appears to be that Lemma 4.5 and equation (4.11) concern a difference $A_0(m_2, n) - A_0(m_1, n)$ of two last-passage times emanating from the common point $(1, 1)$ but ending up at different points on the same horizontal line at level n, while we need to treat $A_x(t) - A_1(t)$ where the last-passage times emanate from different points $(x, 1)$ and $(1, 1)$ but end up at (m, n). However, these situations can be turned into each other by reversing the coordinate directions of the internal weights. The maximal path between two points on the lattice does not depend on which point we declare the starting point and which the ending point.

This argument is made precise in the next lemma which develops a bound for probability (6.2). After that we handle (6.1) with similar reasoning.

Lemma 6.2. *Let $a, b > 0$ be arbitrary positive numbers. There exist finite constants $t_0 = t_0(a, b, \varrho)$ and $C = C(\varrho)$ such that, for all $t \geq t_0$,*

$$\mathbf{P}\Big\{ \sup_{1\leq z\leq at^{2/3}} \big(U_z^\varrho + A_z(t) - A_1(t)\big) \geq bt^{1/3} \Big\} \leq Ca^3(b^{-3} + b^{-6}).$$

Proof. The process $\{U_z^\varrho\}$ depends on the boundary $\{\omega_{i0}\}$. Pick a version $\{\omega_{ij}\}_{1\leq i\leq m, 1\leq j\leq n}$ of the interior variables independent of $\{\omega_{i0}\}$. If we use the reversed system

$$\{\widetilde{\omega}_{ij} = \omega_{m-i+1, n-j+1}\}_{1\leq i\leq m, 1\leq j\leq n} \tag{6.3}$$

to compute $A_z(m, n)$, then this coincides with $A_1(m-z+1, n)$ computed with $\{\omega_{ij}\}$. Thus with this coupling (and some abuse of notation) we can replace $A_z(m, n) - A_1(m, n)$ with $A_1(m - z + 1, n) - A_1(m, n)$. [Note that $A_1(m, n)$ is the same for ω and $\widetilde{\omega}$.] Next pick a further independent version of boundary conditions (2.5) with density λ. Use these and $\{\omega_{ij}\}_{i,j\geq 1}$ to compute the last-passage times G^λ, together with a competition interface φ^λ defined by (2.7) and the projections v^λ defined by (2.8). Then by (4.11), on the event $v^\lambda(n) \geq m$,

$$A_1(m, n) - A_1(m - z + 1, n) \geq G^\lambda(m, n) - G^\lambda(m - z + 1, n).$$

Set

$$V_z^\lambda = G^\lambda(m, n) - G^\lambda(m - z, n),$$

a sum of z i.i.d. Exp$(1 - \lambda)$ variables. V^λ is independent of U^ϱ. Combining these steps we get the bound

$$\begin{aligned} &\mathbf{P}\Big\{ \sup_{1\leq z\leq at^{2/3}} \big(U_z^\varrho + A_z(t) - A_1(t)\big) \geq bt^{1/3} \Big\} \\ &\quad\leq \mathbf{P}\big\{v^\lambda(\lfloor \varrho^2 t\rfloor) < \lfloor (1-\varrho)^2 t\rfloor\big\} + \mathbf{P}\Big\{ \sup_{1\leq z\leq at^{2/3}} (U_z^\varrho - V_{z-1}^\lambda) \geq bt^{1/3} \Big\} \end{aligned} \tag{6.4}$$

Introduce a parameter $r > 0$ whose value will be specified later, and define

$$\lambda = \varrho - rt^{-1/3}. \tag{6.5}$$

For the second probability on the right-hand side of (6.4), define the martingale $M_z = U_z^\varrho - V_{z-1}^\lambda - \mathbf{E}(U_z^\varrho - V_{z-1}^\lambda)$, and note that for $z \le at^{2/3}$,

$$\begin{aligned}\mathbf{E}(U_z^\varrho - V_{z-1}^\lambda) &= \frac{z}{1-\varrho} - \frac{z-1}{1-\lambda} = \frac{zrt^{-1/3}}{(1-\varrho)(1-\lambda)} + \frac{1}{1-\lambda}\\ &\le \frac{rat^{1/3}}{(1-\varrho)^2} + \frac{1}{1-\varrho}.\end{aligned}$$

As long as

$$b > ra(1-\varrho)^{-2} + t^{-1/3}(1-\varrho)^{-1}, \tag{6.6}$$

we get by Doob's inequality, for any $p \ge 1$,

$$\begin{aligned}&\mathbf{P}\Big\{\sup_{1\le z\le at^{2/3}} (U_z^\varrho - V_{z-1}^\lambda) \ge bt^{1/3}\Big\}\\ &\le \mathbf{P}\Big\{\sup_{1\le z\le at^{2/3}} M_z \ge t^{1/3}\Big(b - \frac{ra}{(1-\varrho)^2} - \frac{t^{-1/3}}{1-\varrho}\Big)\Big\}\\ &\le \frac{C(p)t^{-p/3}}{\big(b - ra(1-\varrho)^{-2} - t^{-1/3}(1-\varrho)^{-1}\big)^p}\mathbf{E}\big[\,|M_{\lfloor at^{2/3}\rfloor}|^p\,\big]\\ &\le \frac{C(p,\varrho)a^{p/2}}{\big(b - ra(1-\varrho)^{-2} - t^{-1/3}(1-\varrho)^{-1}\big)^p}.\end{aligned} \tag{6.7}$$

Now choose $t_0 = 4^3 b^{-3}(1-\varrho)^{-3}$. Then for $t \ge t_0$ the above bound is dominated by

$$\frac{C(p,\varrho)a^{p/2}}{\Big(\frac{3b}{4} - \frac{ra}{(1-\varrho)^2}\Big)^p}$$

which becomes $C(p,\varrho)a^3b^{-6}$ once we choose

$$r = \frac{b(1-\varrho)^2}{4a} \tag{6.8}$$

and $p = 6$, and change the constant $C(p,\varrho)$.

For the first probability on the right-hand side of (6.4), introduce the time $s = (\varrho/\lambda)^2 t$. Then

$$\mathbf{P}\big\{v^\lambda(\lfloor\varrho^2 t\rfloor) < \lfloor(1-\varrho)^2 t\rfloor\big\} = \mathbf{P}\big\{v^\lambda(\lfloor\lambda^2 s\rfloor) < \lfloor\lambda^2(1-\varrho)^2\varrho^{-2}s\rfloor\big\}.$$

Notice that since $\lambda < \varrho$ here, $\lfloor\lambda^2(1-\varrho)^2\varrho^{-2}s\rfloor \le \lfloor(1-\lambda)^2 s\rfloor$ and so by redefining (2.6) and (2.10) with s and λ, we have that the event $v^\lambda(\lfloor\lambda^2 s\rfloor) < \lfloor\lambda^2(1-\varrho)^2\varrho^{-2}s\rfloor$ is equivalent to

$$\begin{aligned}Z^{*\lambda}(s) &= [\lfloor(1-\lambda)^2 s\rfloor - v^\lambda(\lfloor\lambda^2 s\rfloor)]^+ - [\lfloor\lambda^2 s\rfloor - w^\lambda(\lfloor(1-\lambda)^2 s\rfloor)]^+\\ &= \lfloor(1-\lambda)^2 s\rfloor - v^\lambda(\lfloor\lambda^2 s\rfloor)\\ &> \lfloor(1-\lambda)^2 s\rfloor - \lfloor\lambda^2(1-\varrho)^2\varrho^{-2}s\rfloor.\end{aligned}$$

By $Z^{*\lambda} \overset{\mathrm{d}}{=} Z^\lambda$, we conclude

$$\begin{aligned}&\mathbf{P}\big\{v^\lambda(\lfloor\varrho^2 t\rfloor) < \lfloor(1-\varrho)^2 t\rfloor\big\}\\ &\qquad = \mathbf{P}\big\{Z^\lambda(s) > \lfloor(1-\lambda)^2 s\rfloor - \lfloor\lambda^2(1-\varrho)^2\varrho^{-2}s\rfloor\big\}.\end{aligned} \tag{6.9}$$

Utilizing the definitions (6.5) and (6.8) of λ and r, one can check that by increasing $t_0 = t_0(a, b, \varrho)$ if necessary, one can guarantee that for $t \geq t_0$ there exists a constant $C = C(\varrho)$ such that

$$\lfloor (1-\lambda)^2 s \rfloor - \lfloor \lambda^2 (1-\varrho)^2 \varrho^{-2} s \rfloor \geq Crs^{2/3}.$$

Combining this with Corollary 5.9 and definition (6.8) of r we get the bound

$$\begin{aligned}\mathbf{P}\big\{v^\lambda(\lfloor \varrho^2 t \rfloor) < \lfloor (1-\varrho)^2 t \rfloor\big\} &\leq \mathbf{P}\big\{Z^\lambda(s) > Crs^{2/3}\big\} \\ &\leq Cr^{-3} \leq C(a/b)^3.\end{aligned}$$

Returning to (6.4) to combine all the bounds, we have

$$\mathbf{P}\Big\{\sup_{1 \leq z \leq at^{2/3}} \big(U_z^\varrho + A_z(t) - A_1(t)\big) \geq bt^{1/3}\Big\} \leq C\Big(\frac{a^3}{b^3} + \frac{a^3}{b^6}\Big). \qquad \square$$

Completion of the proof of Lemma 6.1. By Lemma 6.2 the probability (6.2) is bounded by $C\delta^3(b^{-3} + b^{-6})$. Bound the probability (6.1) by

$$\begin{aligned}&\mathbf{P}\Big\{\sup_{\delta t^{2/3} < x \leq t^{2/3}} \big(U_x^\varrho + A_x(t) - A_1(t)\big) < bt^{1/3}\Big\} \\ &\quad \leq \mathbf{P}\big\{v^\lambda(\lfloor \varrho^2 t \rfloor) > \lfloor (1-\varrho)^2 t \rfloor - t^{2/3}\big\} &(6.10)\\ &\quad\quad + \mathbf{P}\Big\{\sup_{\delta t^{2/3} < x \leq t^{2/3}} (U_x^\varrho - V_x^\lambda) < bt^{1/3}\Big\} &(6.11)\end{aligned}$$

where, following the example of the previous proof, we have introduced a new density, this time

$$\lambda = \varrho + rt^{-1/3},$$

and then used the reversal trick of equation (6.3) and Lemma 4.5 to deduce

$$A_x(m,n) - A_1(m,n) \geq G^\lambda(m - x + 1, n) - G^\lambda(m,n) \equiv -V_{x-1}^\lambda \geq -V_x^\lambda,$$

whenever $v^\lambda(\lfloor \varrho^2 t \rfloor) \leq \lfloor (1-\varrho)^2 t \rfloor - t^{2/3}$. We claim that, given $\eta > 0$ and parameter r from above, we can fix $\delta, b > 0$ small enough so that, for some $t_0 < \infty$, the probability in (6.11) satisfies

$$\mathbf{P}\Big\{\sup_{\delta t^{2/3} < x \leq t^{2/3}} (U_x^\varrho - V_x^\lambda) < bt^{1/3}\Big\} \leq \eta \qquad \text{for all } t \geq t_0. \tag{6.12}$$

As $t \to \infty$,

$$\begin{aligned}&t^{-1/3}\mathbf{E}(U^\varrho_{\lfloor yt^{2/3} \rfloor} - V^\lambda_{\lfloor yt^{2/3} \rfloor}) \longrightarrow \frac{-ry}{(1-\varrho)^2} \\ &\text{and} \quad t^{-2/3}\mathbf{Var}(U^\varrho_{\lfloor yt^{2/3} \rfloor} - V^\lambda_{\lfloor yt^{2/3} \rfloor}) \longrightarrow \frac{2y}{(1-\varrho)^2} \equiv \sigma^2(\varrho) y\end{aligned}$$

uniformly over $y \in [\delta, 1]$. Since we have a sum of i.i.d's, the probability in (6.12) converges, as $t \to \infty$, to

$$\mathbf{P}\Big\{\sup_{\delta \leq y \leq 1} \Big(\sigma(\varrho) B(y) - \frac{ry}{(1-\varrho)^2}\Big) \leq b\Big\}$$

where $B(\cdot)$ is standard Brownian motion. The random variable

$$\sup_{0\le y\le 1}\Bigl(\sigma(\varrho)B(y)-\frac{ry}{(1-\varrho)^2}\Bigr)$$

is positive almost surely, so the above probability is less than $\eta/2$ for small δ and b. This implies (6.12).

The probability in (6.10) is bounded by

$$\begin{aligned}
&\mathbf{P}\bigl\{v^{\lambda}(\lfloor\varrho^2t\rfloor)>\lfloor(1-\varrho)^2t-t^{2/3}-1\rfloor\bigr\}\\
&\le\mathbf{P}\bigl\{v^{1-\lambda}(\lfloor(1-\varrho)^2t-t^{2/3}-1\rfloor)<\lfloor\varrho^2t\rfloor\bigr\}\\
&\le\mathbf{P}\bigl\{v^{1-\lambda}(\lfloor(1-\lambda)^2s\rfloor)<\lfloor\lambda^2s\rfloor-qs^{2/3}\bigr\}\\
&=\mathbf{P}\bigl\{Z^{*1-\lambda}(s)>qs^{2/3}\bigr\}\\
&=\mathbf{P}\bigl\{Z^{1-\lambda}(s)>qs^{2/3}\bigr\}\\
&\le Cq^{-3}.
\end{aligned}$$

Above we first used (2.9) and transposition of the array $\{\omega_{ij}\}$. Because this exchanges the axes, density λ becomes $1-\lambda$. Then we defined s by

$$(1-\lambda)^2s=(1-\varrho)^2t-t^{2/3}-1$$

and observed that for large enough t, the second inequality holds for some $q=C(\varrho)((1-\varrho)r-\varrho)$. We used (2.10) and the distributional identity of Z and Z^* thereafter. The last inequality is from Corollary 5.9.

Now given $\eta>0$, choose r large enough so that $Cq^{-3}<\eta$. Given this r, choose δ, b small enough so that (6.12) holds. Finally, shrink δ further so that $C\delta^3(b^{-3}+b^{-6})<\eta$ (shrinking δ does not violate (6.12)). To summarize, we have shown that, given $\eta>0$, if δ is small enough, then for all large t

$$\mathbf{P}\{1\le Z^{\varrho}(t)\le\delta t^{2/3}\}\le 3\eta. \tag{6.13}$$

This concludes the proof of the lemma. □

Via transpositions we get Lemma 6.1 also for the j-axis:

Corollary 6.3. *We have the asymptotics*

$$\lim_{\varepsilon\searrow 0}\limsup_{t\to\infty}\mathbf{P}\{0<U^{\varrho}_{-Z^{\varrho}(t)^-}\le\varepsilon t^{2/3}\}=0.$$

Proof. Let $\{\omega_{ij}\}$ be an initial assignment with density ϱ. Let $\widetilde{\omega}_{ij}=\omega_{ji}$ be the transposed array, which is an initial assignment with density $1-\varrho$. Under transposition the $\lfloor(1-\varrho)^2t\rfloor\times\lfloor\varrho^2t\rfloor$ rectangle has become $\lfloor\varrho^2t\rfloor\times\lfloor(1-\varrho)^2t\rfloor$, the correct characteristic dimensions for density $1-\varrho$. Since transposition exchanges the coordinate axes, after transposition $U^{\varrho}_{Z^{\varrho}(t)^+}$ has become $U^{1-\varrho}_{-Z^{1-\varrho}(t)^-}$, and so these two random variables have the same distribution. The corollary is now a consequence of Lemma 6.1 because this lemma is valid for each density $0<\varrho<1$. □

(6.13) proves part (b) of Theorem 2.2. The theorem below gives the lower bound for Theorem 2.1 and thereby completes its proof.

Theorem 6.4.
$$\liminf_{t\to\infty} \frac{\mathbf{E}(U^{\varrho}_{Z^{\varrho}(t)^+})}{t^{2/3}} > 0, \quad \textit{and} \quad \liminf_{t\to\infty} \frac{\mathbf{Var}(G^{\varrho}(t))}{t^{2/3}} > 0.$$

Proof. Suppose there exists a density ϱ and a sequence $t_k \to \infty$ such that $t_k^{-2/3}\mathbf{Var}(G^{\varrho}(t_k)) \to 0$. Then by Lemma 4.6
$$\frac{\mathbf{E}(U^{\varrho}_{Z^{\varrho}(t_k)^+})}{t_k^{2/3}} \to 0 \quad \text{and} \quad \frac{\mathbf{E}(U^{\varrho}_{-Z^{\varrho}(t_k)^-})}{t_k^{2/3}} \to 0.$$
From this and Markov's inequality
$$\mathbf{P}\{U^{\varrho}_{Z^{\varrho}(t_k)^+} > \varepsilon t_k^{2/3}\} \to 0 \quad \text{and} \quad \mathbf{P}\{U^{\varrho}_{-Z^{\varrho}(t_k)^-} > \varepsilon t_k^{2/3}\} \to 0$$
for every $\varepsilon > 0$. This together with Lemma 6.1 and Corollary 6.3 implies
$$\mathbf{P}\{U^{\varrho}_{Z^{\varrho}(t_k)^+} > 0\} \to 0 \quad \text{and} \quad \mathbf{P}\{U^{\varrho}_{-Z^{\varrho}(t_k)^-} > 0\} \to 0.$$
But these statements imply that
$$\mathbf{P}\{Z^{\varrho}(t_k) > 0\} \to 0 \quad \text{and} \quad \mathbf{P}\{Z^{\varrho}(t_k) < 0\} \to 0,$$
which is a contradiction since these two probabilities add up to 1 for each fixed t_k. This proves the second claim of the theorem.

The first claim follows because it is equivalent to the second. □

7 Rarefaction boundary conditions

In this section we prove results on the longitudinal and transversal fluctuations of a maximal path under more general boundary conditions. Abbreviate as before
$$(m,n) = \big(\lfloor (1-\varrho)^2 t\rfloor, \lfloor \varrho^2 t\rfloor\big).$$
We start by studying $A_0(t) = A_0(m,n)$, the length of the maximal path to (m,n) when there are no weights on the axes, and we will show that it has fluctuations of the order $t^{1/3}$. We still use the boundary conditions (2.5), so that we have coupled $A_0(t)$ and $G^{\varrho}(t)$. We first need another version of Lemma 6.2 to make it applicable for all $t \geq 1$.

Lemma 7.1. *Fix $0 < \alpha < 1$. There exists a constant $C = C(\alpha,\varrho)$ such that, for each $t \geq 1$ and $b \geq C$,*
$$\mathbf{P}\{G^{\varrho}(t) - A_0(t) \geq bt^{1/3}\} \leq Cb^{-3\alpha/2}.$$

Proof. Note that
$$\begin{aligned} &\mathbf{P}\{G^{\varrho}(t) - A_0(t) \geq bt^{1/3}\} \\ &\quad\leq \mathbf{P}\{\sup_{|z|\leq at^{2/3}} \big(U^{\varrho}_z(t) + A_z(t) - A_0(t)\big) \geq bt^{1/3}\} \\ &\qquad + \mathbf{P}\{\sup_{|z|\leq at^{2/3}} \big(U^{\varrho}_z(t) + A_z(t)\big) \neq G^{\varrho}(t)\}. \end{aligned} \tag{7.1}$$

The last term of (7.1) can easily be dealt with using Corollary 5.9: there exists a $C = C(\varrho)$ such that

$$\begin{aligned}\mathbf{P}\{\sup_{|z|\le at^{2/3}} \big(U_z^\varrho(t) + A_z(t)\big) \neq G^\varrho(t)\} &\le \mathbf{P}\{Z^\varrho(t) \ge at^{2/3}\} \\ &+ \mathbf{P}\{Z^\varrho(t) \le -at^{2/3}\} \le Ca^{-3}.\end{aligned} \tag{7.2}$$

For the first term of (7.1) we will use the results from the proof of Lemma 6.2. We split the range of z into $[1, at^{2/3}]$ and $[-1, -at^{2/3}]$ and consider for now only the first part. Define

$$\lambda = \varrho - rt^{-1/3}.$$

We can use (6.4) and (6.7), where we choose $a = b^{\alpha/2}$, $p = 2$, and $r = b^{\alpha/2}$. Choose $C = C(\alpha, \varrho) > 0$ large enough so that for $b \ge C$ (6.6) is satisfied and the denominator of the last bound in (6.7) is at least $b/2$. Then we can claim that, for all $b \ge C$ and $t \ge 1$,

$$\begin{aligned}\mathbf{P}\{\sup_{1\le z\le at^{2/3}} &\big(U_z^\varrho(t) + A_z(t) - A_0(t)\big) \ge bt^{1/3}\} \\ &\le \mathbf{P}\{v^\lambda(\lfloor \varrho^2 t\rfloor) < \lfloor (1-\varrho)^2 t\rfloor\} + Cb^{\alpha/2-2}.\end{aligned} \tag{7.3}$$

From (6.9) we get with $s = (\varrho/\lambda)^2 t$

$$\mathbf{P}\{v^\lambda(\lfloor \varrho^2 t\rfloor) < \lfloor (1-\varrho)^2 t\rfloor\} = \mathbf{P}\{Z^\lambda(s) > \lfloor (1-\lambda)^2 s\rfloor - \lfloor \lambda^2(1-\varrho)^2\varrho^{-2} s\rfloor\}.$$

Now we continue differently than in Lemma 6.2 so that t is not forced to be large. An elementary calculation yields

$$\lfloor (1-\lambda)^2 s\rfloor - \lfloor \lambda^2(1-\varrho)^2\varrho^{-2} s\rfloor \ge 2\,\frac{1-\varrho}{\varrho}\, rs^{2/3} + \frac{2\varrho - 1}{\varrho^2}\, r^2 s^{1/3} - 1.$$

We want to write down conditions under which the right-hand side above is at least $\delta r s^{2/3}$ for some constant δ and all $s \ge 1$. First increase the above constant $C = C(\alpha, \varrho)$ so that if $b = r^{2/\alpha} \ge C$, then

$$\frac{1-\varrho}{\varrho}\, rs^{2/3} - 1 \ge \frac{1-\varrho}{2\varrho}\, rs^{2/3} \qquad \text{for all } s \ge 1.$$

Then choose $\eta = \eta(\alpha, \varrho) > 0$ small enough such that whenever $b \in [C, \eta t^{2/(3\alpha)}]$ (in this case r is small enough compared to $t^{1/3}$, but notice that the interval might as well be empty when t is small),

$$\frac{1-\varrho}{\varrho}\, rs^{2/3} \ge -\frac{2\varrho-1}{\varrho^2}\, r^2 s^{1/3}.$$

This last condition is vacuously true if $\varrho \ge 1/2$.

Now we have for $C \le b \le \eta t^{2/(3\alpha)}$ and with $\delta = (1-\varrho)/(2\varrho)$,

$$\lfloor (1-\lambda)^2 s\rfloor - \lfloor \lambda^2(1-\varrho)^2\varrho^{-2} s\rfloor \ge \delta r s^{2/3} \qquad \text{for all } s \ge 1.$$

If we combine this with (7.3) and Corollary 5.9, we can state that for all $C \le b \le \eta t^{2/(3\alpha)}$ and $t \ge 1$,

$$\mathbf{P}\{\sup_{1\le z\le at^{2/3}} U_z^\varrho(t) + A_z(t) - A_0(t) \ge bt^{1/3}\} \le Cb^{-3\alpha/2} + Cb^{\alpha/2-2}.$$

The same argument works (or just apply transposition) for the values $-at^{2/3} \le z \le 1$, so this same upper bound is valid for the first probability on the right-hand side of (7.1).

Taking (7.2) also into consideration, at this point we have shown that whenever $C \le b \le \eta t^{2/(3\alpha)}$ and $t \ge 1$,

$$\mathbf{P}\{G^\varrho(t) - A_0(t) \ge bt^{1/3}\} \le \frac{1}{2}C(b^{\alpha/2-2} + b^{-3\alpha/2}) \le Cb^{-3\alpha/2}.$$

What if $b \ge \eta t^{2/(3\alpha)}$? Note that

$$\mathbf{P}\{G^\varrho(t) - A_0(t) \ge bt^{1/3}\} \le \mathbf{P}\{G^\varrho(t) \ge bt^{1/3}\}.$$

Since $G^\varrho(t)$ is the sum of two (dependent) random variables, each of which in turn is the sum of i.i.d. exponentials, and since

$$\mathbf{E}(G^\varrho(t)b^{-1}t^{-1/3}) \le C(\varrho, \eta)b^{\alpha-1}$$

($\mathbf{E}(G^\varrho(t))$ is basically linear in t by (2.6) and Lemma 4.2), we conclude that $\mathbf{P}\{G^\varrho(t) - A_0(t) \ge bt^{1/3}\}$ goes to zero faster than any polynomial in b, if $b \ge \eta t^{2/(3\alpha)}$. This proves the lemma for all $b \ge C$. □

Now we can establish that the fluctuations of $A_0(t)$ are of order $t^{1/3}$.

Corollary 7.2. *Fix $0 < \alpha < 1$. There exists a constant $C = C(\alpha, \varrho)$ such that for all $a > 0$ and $t \ge 1$,*

$$\mathbf{P}\{|A_0(t) - t| > at^{1/3}\} \le Ca^{-3\alpha/2}.$$

In particular this means that

$$\mathbf{E}(|A_0(t) - t|) = O(t^{1/3}) \quad \textit{and} \quad \mathbf{E}(A_0(t)) = t - O(t^{1/3}).$$

Proof. Lemma 7.1 together with Theorem 5.8 implies for $a \ge C(\alpha, \varrho)$

$$\begin{aligned}\mathbf{P}\{|A_0(t) - t| > at^{1/3}\} &\le \mathbf{P}\{G^\varrho(t) - A_0(t) > at^{1/3}/2\} \\ &\quad + \mathbf{P}\{|G^\varrho(t) - t| > at^{1/3}/2\} \\ &\le C_1 a^{-3\alpha/2} + C_2 a^{-2} \\ &\le Ca^{-3\alpha/2}.\end{aligned}$$

Finally, we can always increase C in order to take all $0 < a \le C(\alpha, \varrho)$ values into account. □

We can also consider the fluctuations of the position of a maximal path. To this end we extend the definition of $Z(t)$, the exit point from the axes. We define $Z_l(t)$ as the i-coordinate of the right-most point on the horizontal line $j = l$ of the right-most maximal path to (m, n) (we say right-most path, because later in this section we will consider boundary conditions that no longer necessarily have a unique longest path). We will use the notation Z_l^ϱ to denote the stationary situation and Z_l^0 to denote the situation where all the weights on the axes are zero. Note that in all cases

$$Z(t)^+ = Z_0(t).$$

Lemma 7.3. *Define* $(k,l) = (\lfloor (1-\varrho)^2 s \rfloor, \lfloor \varrho^2 s \rfloor)$ *for* $s \le t$. *There exists a constant* $C = C(\rho)$ *such that for all* $0 \le s \le t-1$ *and all* $a > 0$

$$\mathbf{P}\{Z_l^{\varrho}(t) \ge k + a(t-s)^{2/3}\} \le Ca^{-3}.$$

Proof. There are several ways to see this, for example by time-reversal. One can also pick a new origin at (k,l), and define a last-passage model in the rectangle $[k,m] \times [l,n]$ with boundary conditions given by the I- and J-increments (4.1) of the original G-process. The maximizing path in this new model connects up with the original maximizing path. Hence in this new model it looks as though the maximal path to $(m-k, n-l)$ exits the i-axis beyond the point $a(t-s)^{2/3}$, and so

$$\mathbf{P}\{Z_l^{\varrho}(t) \ge k + a(t-s)^{2/3}\} = \mathbf{P}\{Z^{\varrho}(t-s) \ge a(t-s)^{2/3}\}.$$

We have ignored the integer parts here, but this can be dealt with uniformly in $a > 0$. Now we can use Corollary 5.9 to conclude that

$$\mathbf{P}\{Z_l^{\varrho}(t) \ge k + a(t-s)^{2/3}\} \le Ca^{-3}.$$

□

To get a similar result for $Z_l^0(t)$ we need a more convoluted argument and the conclusion is a little weaker.

Lemma 7.4. *Define* $(k,l) = (\lfloor (1-\varrho)^2 s \rfloor, \lfloor \varrho^2 s \rfloor)$ *for* $s \le t$. *There exists a constant* $C = C(\alpha, \varrho)$ *such that for all* $a > 0$ *and* $t \ge 1$

$$\mathbf{P}\{Z_l^0(t) > k + a\,t^{2/3}\} \le Ca^{-3\alpha}.$$

Proof. The event $\{Z_l^0(t) > k + u\}$ is equivalent to the event

$$E = \left\{ A_0(t) = \sup_{z \ge u}\{A_0(k+z+1, l) + \tilde{A}_0(z-u, 0)\} \right\}.$$

Here, $A_0(i,j)$ is the weight of the maximal path (not using the axes) from $(0,0)$ to (i,j), including the endpoint, whereas $\tilde{A}_0(i,j)$ is the weight of the maximal path from $(k+u+i, l+j)$ to (m,n), including the endpoint but excluding the starting point *and excluding all the weights directly to the right or directly above* $(k+u+i, l+j)$. This corresponds to choosing $(k+u+i, l+j)$ as a new origin, and making sure that the axes through this origin have no weights. Note that the processes $A_0(\cdot, l)$ and $\tilde{A}_0(\cdot, 0)$ are independent. The idea is to bound A_0 and $\tilde{A}_0$ by appropriate stationary processes G^{λ} and $G^{\tilde{\lambda}}$ to show that, with high probability, this supremum will be too small if u is too large. We can couple the processes G^{λ} and $G^{\tilde{\lambda}}$, where $\tilde{\lambda} > \lambda$, in the following way: G^{λ} induces weights on the horizontal line $j = l$ through the increments of G^{λ}, see Lemma 4.2. The process $G^{\tilde{\lambda}}$ takes the point $(k+u, l)$ as origin and uses as boundary weights on the horizontal line $j = l$, with a slight abuse of notation, for $i \ge 1$

$$\tilde{\omega}_{i0} = \frac{1-\lambda}{1-\tilde{\lambda}} I_{k+u+i+1,l} = \frac{1-\lambda}{1-\tilde{\lambda}}(G^{\lambda}(k+u+i+1, l) - G^{\lambda}(k+u+i, l)).$$

These weights are independent $\mathrm{Exp}(1-\tilde{\lambda})$ random variables. The weights $\tilde{\omega}_{0j} \sim \mathrm{Exp}(\tilde{\lambda})$ on the line $i = k+u$ can be chosen independently of everything else, whereas for $i, j \geq 1$

$$\tilde{\omega}_{ij} = \omega_{u+k+i,l+j}.$$

So $G^{\tilde{\lambda}}(i,j)$ equals the weight of the maximal path from $(k+u,l)$ to $(k+u+i,l+j)$, using as weights on the points $(k+u+i,l)$ the $\tilde{\omega}_{i0}$ (for $i \geq 1$), on the points $(k+u,l+j)$ the $\tilde{\omega}_{0j}$ (for $j \geq 1$) and on the points $(k+u+i,l+j)$ the original $\omega_{k+u+i,l+j}$ (for $i,j \geq 1$). This construction leads to

$$A_0(i,j) \leq G^{\lambda}(i,j) \quad \text{and} \quad \tilde{A}_0(i,j) \leq G^{\tilde{\lambda}}(m-k-u,n-l) - G^{\tilde{\lambda}}(i,j).$$

Also, for all $0 \leq i \leq m-k-u-1$,

$$G^{\lambda}(k+u+i+1,l) - G^{\lambda}(k+u+1,l) \leq G^{\tilde{\lambda}}(i,0).$$

Therefore, for all $0 \leq i \leq m-k-u-1$,

$$\begin{aligned} A_0(k+u+i+1,l) + \tilde{A}_0(i,0) &\leq G^{\lambda}(k+u+i+1,l) - G^{\tilde{\lambda}}(i,0) + \\ &\quad G^{\tilde{\lambda}}(m-k-u,n-l) \\ &\leq G^{\lambda}(k+u+1,l) + G^{\tilde{\lambda}}(m-k-u,n-l). \end{aligned}$$

So we get

$$\mathbf{P}(E) \leq \mathbf{P}\{A_0(t) - G^{\lambda}(k+u+1,l) - G^{\tilde{\lambda}}(m-k-u,n-l) \leq 0\}. \tag{7.4}$$

Here, we can still choose λ and $\tilde{\lambda}$ as long as $0 < \lambda < \tilde{\lambda}$, but it is not hard to see that for the optimal choices (in expectation) of λ and $\tilde{\lambda}$ are determined by

$$\frac{(1-\lambda)^2}{\lambda^2} = \frac{k+u+1}{l} \quad \text{and} \quad \frac{(1-\tilde{\lambda})^2}{\tilde{\lambda}^2} = \frac{m-k-u}{n-l}. \tag{7.5}$$

With these choices we get

$$\mathbf{E}(G^{\lambda}(k+u+1,l)) = (\sqrt{k+u+1} + \sqrt{l})^2$$

and

$$\mathbf{E}(G^{\tilde{\lambda}}(m-k-u,n-l)) = (\sqrt{m-k-u} + \sqrt{n-l})^2.$$

This particular choice of $(\lambda, \tilde{\lambda})$ is valid (i.e., $\tilde{\lambda} > \lambda$) as soon as $u \geq C(\varrho)$. Smaller u can be dealt with by increasing C in the statement of lemma. We have for $u \geq 2$

$$\begin{aligned} \mathbf{E}(G^{\lambda}(k+u+1,l) + G^{\tilde{\lambda}}(m-k-u,n-l)) &= m+n+2\sqrt{l(k+u+1)} \\ &\quad + 2\sqrt{(n-l)(m-k-u)} + 1 \\ &\leq ((1-\varrho)^2+\varrho^2)t + 1 + 2\sqrt{\varrho^2 s}\,\sqrt{(1-\varrho)^2 s + u + \sqrt{\frac{l}{k+u}}} \\ &\quad + 2\sqrt{\varrho^2(t-s)+1}\,\sqrt{(1-\varrho)^2(t-s) - u + 1} \\ &\leq ((1-\varrho)^2+\varrho^2)t + C(\varrho) + 2\sqrt{\varrho^2 s}\,\sqrt{(1-\varrho)^2 s + u} \\ &\quad + 2\sqrt{\varrho^2(t-s)}\,\sqrt{(1-\varrho)^2(t-s) - (u-1)} \\ &\leq t + C(\varrho) + \frac{\varrho}{1-\varrho}u - \frac{\varrho}{1-\varrho}(u-1) - \frac{1}{4}\,\frac{\varrho}{(1-\varrho)^3}\,\frac{(u-1)^2}{t-s} \\ &\leq t - C_1(\varrho)\,\frac{u^2}{t} + C_2(\varrho). \end{aligned}$$

If $u = \lfloor at^{2/3} \rfloor$, then we can choose constants $M = M(\varrho)$ and $C_1 = C(\varrho)$ such that for all $a > M$ and $t \geq 1$,

$$\mathbf{E}(G^{\lambda}(k+u+1,l) + G^{\tilde{\lambda}}(m-k-u,n-l)) \leq t - C_1 a^2 t^{1/3}. \tag{7.6}$$

Smaller a can be dealt with by increasing the constant C in the statement of the lemma. Now note that, using (7.4), we get

$$\begin{aligned}
\mathbf{P}(E) &\leq \mathbf{P}\{A_0(t) - t \leq G^{\lambda}(k+u+1,l) + G^{\tilde{\lambda}}(m-k-u,n-l) - t\} \\
&\leq \mathbf{P}(A_0(t) - t \leq -\frac{1}{2}C_1 a^2 t^{1/3}) \\
&\quad + \mathbf{P}(G^{\lambda}(k+u+1,l) + G^{\tilde{\lambda}}(m-k-u,n-l) - t \geq -\frac{1}{2}C_1 a^2 t^{1/3}) \\
&\leq \mathbf{P}(A_0(t) - t \leq -\frac{1}{2}C_1 a^2 t^{1/3}) + C_2 a^{-4} \leq C a^{-3\alpha}.
\end{aligned}$$

For the last line we used (7.6), the fact that

$$\mathbf{Var}(G^{\lambda}(k+u+1,l) + G^{\tilde{\lambda}}(m-k-u,n-l)) \leq C t^{2/3},$$

(notice that the choice (7.5) places these coordinates in the G's on the respective characteristics, see (2.6)), and Corollary 7.2.

$\square$

We now prove the last theorems under assumption (2.11).

Proof of Theorem 2.4. The statement follows from the observation that

$$A_0(t) \leq \hat{G}(t) \leq G^{\rho}(t)$$

(if there is less weight, the paths get shorter), Corollary 7.2 and Theorem 5.8. $\square$

Proof of Theorem 2.5. For the first inequality, we introduce the process $G^{\mathcal{W}=0}$ which uses the same weights as G^{ρ}, except on the j-axis, where all weights are 0 (so $\omega_{0j}^{\mathcal{W}=0} = 0$). It is not hard to see that

$$\hat{Z}_l(t) \leq Z_l^{\mathcal{W}=0}(t),$$

simply because the right-most maximal path for $G^{\mathcal{W}=0}$ stays at least as long on the i-axis as a maximal path for $\hat{G}$, and it can coalesce with, but never cross a maximal path for $\hat{G}$. So we get

$$\mathbf{P}\{\hat{Z}_l(t) \geq k + at^{2/3}\} \leq \mathbf{P}\{Z_l^{\mathcal{W}=0}(t) \geq k + at^{2/3}\}. \tag{7.7}$$

To find an upper bound for the probability above, we will in fact develop a bound for $Z_0^{\mathcal{W}=0}(t)$. Then we observe that if $Z_l^{\mathcal{W}=0}(t)$ is large but $Z_0^{\mathcal{W}=0}(t)$ is not, then $Z_l^0(t)$ is large for an appropriate coupled system. But this last quantity we control with Lemma 7.4.

To get started, note that, as in the proof of the previous lemma,

$$\{Z_0^{\mathcal{W}=0}(t) > u\} = \{G^{\mathcal{W}=0}(t) = \sup_{z>u}(U_z^{\varrho} + A_z(t))\}. \tag{7.8}$$

Now define a stationary process G^λ, with $\lambda > \varrho$, whose origin is placed at $(u, 0)$. It uses as weights on the i-axis

$$\omega_{i0}^\lambda = \frac{1-\varrho}{1-\lambda}\omega_{u+i+1,0}.$$

On the vertical line $i = u$, G^λ uses independent $\mathrm{Exp}(\lambda)$ weights. This construction guarantees that for $i \geq 0$

$$U_{u+i+1}^\varrho - U_{u+1}^\varrho \leq G^\lambda(i, 0).$$

Also, for $z > u$,

$$A_z(t) \leq G^\lambda(m-u, n) - G^\lambda(z-u-1, 0).$$

This implies that

$$\sup_{z>u}(U_z^\varrho + A_z(t)) \leq U_{u+1}^\varrho + G^\lambda(m-u, n).$$

This means that, using (7.8),

$$\{Z_0^{\mathcal{W}=0}(t) > u\} \subset \{G^{\mathcal{W}=0}(t) \leq U_{u+1}^\varrho + G^\lambda(m-u, n)\}. \tag{7.9}$$

Again we have that for the optimal λ,

$$\mathbf{E}(G^\lambda(m-u, n)) = (\sqrt{m-u} + \sqrt{n})^2,$$

which leads to

$$\begin{aligned}
\mathbf{E}(U_{u+1}^\varrho + G^\lambda(m-u, n)) &\leq \frac{u+1}{1-\varrho} + m + n - u + 2\sqrt{n}\sqrt{m-u} \\
&\leq (1-\varrho)^2 t + \varrho^2 t + \frac{\varrho u}{1-\varrho} + C_1(\varrho) + 2\sqrt{\varrho^2 t}\sqrt{(1-\varrho)^2 t - (u-1)} \\
&\leq t + C_2(\varrho) - \frac{1}{4}\frac{\varrho}{(1-\varrho)^3}\frac{(u-1)^2}{t}.
\end{aligned}$$

Just as in the proof Lemma 7.4, we see that if $u = \lfloor bt^{2/3} \rfloor$, we can choose constants $M = M(\varrho)$ and $C_1 = C_1(\varrho)$ such that for all $b > M$ and $t \geq 1$,

$$\mathbf{E}(U_{u+1}^\varrho + G^\lambda(m-u, n)) \leq t - C_1 b^2 t^{1/3}.$$

Note that with (7.9)

$$\begin{aligned}
\mathbf{P}(Z_0^{\mathcal{W}=0}(t) > bt^{2/3}) \leq{} & \mathbf{P}(G^{\mathcal{W}=0}(t) - t \leq -\frac{1}{2}C_1 b^2 t^{1/3}) \\
& + \mathbf{P}(U_{u+1}^\varrho + G^\lambda(m-u, n) - t \geq -\frac{1}{2}C_1 b^2 t^{1/3}).
\end{aligned}$$

Now we can use the fact that

$$\mathbf{Var}(U_{u+1}^\varrho + G^\lambda(m-u, n)) = O(u + t^{2/3})$$

(again, the optimal choice for λ has placed the coordinates in G on the characteristics w.r.t. λ), and Theorem 2.4 to conclude that for $b > M$

$$\mathbf{P}\{Z_0^{\mathcal{W}=0}(t) > bt^{2/3}\} \leq Cb^{-3\alpha}. \tag{7.10}$$

For $b \le M$ we can increase C.

Now that we have control over $Z_0^{\mathcal{W}=0}(t)$, we return to bound the right-hand side of (7.7). A little picture reveals that if $Z_l^{\mathcal{W}=0}(t) \ge k + at^{2/3}$ and $Z_0^{\mathcal{W}=0}(t) \le at^{2/3}/2$, then the maximal path from $(at^{2/3}/2, 1)$ to (m, n) [without weights on the axes] must pass to the right of $(k + at^{2/3}, l)$. Taking $(at^{2/3}/2, 0)$ as the new origin we see that this last event has smaller probability than the event $\{Z_l^0(t) \ge k + at^{2/3}/2\}$. An application of Lemma 7.4 to this probability together with (7.10) give an upper bound for (7.7). This proves the first inequality of Theorem 2.5.

The second inequality of Theorem 2.5 is a corollary of the first via a transposition. Assume $k - at^{2/3} \ge 0$, otherwise this statement is trivial. Also take $a > 2\frac{(1-\varrho)^2}{\varrho^2}$ for one can always increase C if this is not the case. Fix $\widetilde{s}$ such that $k - at^{2/3} = (1-\varrho)^2\widetilde{s}$ and then put

$$k' = \lfloor (1-\varrho)^2\widetilde{s} \rfloor \quad \text{and} \quad l' = \lfloor \varrho^2\widetilde{s} \rfloor .$$

With these definitions

$$l \ge l' + \varrho^2 s - 1 - \frac{\varrho^2}{(1-\varrho)^2} \cdot (1-\varrho)^2\widetilde{s} \;\ge\; l' + \frac{\varrho^2}{(1-\varrho)^2} \cdot at^{2/3} - 1.$$

Define $\hat{Y}_{k'}^T$ to be the highest point of the left-most maximal path on the vertical line $i = k'$. As the left-most maximal path is North-East, we have

$$\mathbf{P}\{\hat{Y}_l(t) \le k - at^{2/3}\} = \mathbf{P}\{\hat{Y}_{k'}^T \ge l\} \le \mathbf{P}\{\hat{Y}_{k'}^T \ge l' + \frac{\varrho^2}{(1-\varrho)^2} \cdot at^{2/3} - 1\}.$$

If

$$\widetilde{a} = a\varrho^2/(1-\varrho)^2 - 1 > 1,$$

then the right hand-side is bounded by $\mathbf{P}\{\hat{Y}_{k'}^T \ge l' + \widetilde{a}t^{2/3}\}$. The transposed array $\{\widetilde{\omega}_{ij}\} = \{\omega_{ji}\}$ satisfies assumption (2.11) with respect to the parameter $1 - \varrho$. Moreover, $\hat{Y}_{k'}^T$ becomes the right-most point of the right-most maximal path on the horizontal line $i = k'$ in the transposed picture. The first part of the Theorem with $1 - \varrho$, $\widetilde{s} \le t$ and $\widetilde{a}$ then completes the proof because $\widetilde{a}^{-3\alpha} < [\frac{1}{2}(\widetilde{a} + 1)]^{-3\alpha} = C'(\varrho, \alpha) \cdot a^{-3\alpha}$. □

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