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Branching Random Walk with Catalysts

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Abstract

Shnerb et al. (2000), (2001) studied the following system of interacting particles on $\mathbb{Z}^d$: There are two kinds of particles, called A-particles and B-particles. The A-particles perform continuous time simple random walks, independently of each other. The jumprate of each A-particle is D_A. The B-particles perform continuous time simple random walks with jumprate D_B, but in addition they die at rate δ and a B-particle at x at time s splits into two particles at x during the next ds time units with a probability $\beta N_A(x,s)ds + o(ds)$, where $N_A(x,s)$ ($N_B(x,s)$) denotes the number of A-particles (respectively B-particles) at x at time s. Conditionally on the A-system, the jumps, deaths and splittings of different B-particles are independent. Thus the B-particles perform a branching random walk, but with a birth rate of new particles which is proportional to the number of A-particles which coincide with the appropriate B-particles. One starts the process with all the $N_A(x,0)$, $x \in \mathbb{Z}^d$, as independent Poisson variables with mean μ_A, and the $N_B(x,0)$, $x \in \mathbb{Z}^d$, independent of the A-system, translation invariant and with mean μ_B.

Shnerb et al. (2000) made the interesting discovery that in dimension 1 and 2 the expectation $\mathbb{E}\{N_B(x,t)\}$ tends to infinity, *no matter what the values of* $\delta, \beta, D_A, D_B, \mu_A, \mu_B \in (0,\infty)$ *are.* We shall show here that nevertheless *there is a phase transition in all dimensions*, that is, the system becomes (locally) extinct for large δ but it survives for β large and δ small.

Keywords and phrases: Branching random walk, survival, extinction.

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1. Introduction and statement of result.

We investigate the survival/extinction of the system of A and B-particles described in the abstract. (A somewhat more formal description of this system is given in Section 2.) This is essentially the system introduced in Shnerb et al. (2000), which they view, among other interpretations, as a model for interacting molecules or for individuals carrying specific genotypes. We have only changed the notation slightly and made more explicit assumptions on the initial distributions than Shnerb et al. (2000). Shnerb et al. (2000) indicates that in dimension 1 or 2 the B-particles "survive" for *all* choices of the parameters $\delta, \beta, D_A, D_B, \mu_A, \mu_B > 0$. However, they deal with some form of continuum limit of the system and we found it difficult to interpret what their claim means for the system described in the abstract. For the purpose of this paper we shall say that the B-particles *survive* if

$$\limsup_{t\to\infty} \mathbb{P}\{N_B(\mathbf{0}, t) > 0\} > 0, \tag{1.1}$$

where $\mathbb{P}$ is the annealed probability law, i.e., the law governing the combined system of both types of particles. We shall see that in all dimensions there are choices of $\delta, \beta, D_A, D_B, \mu_A, \mu_B > 0$ for which the B-particles do *not* survive in the sense of (1.1). A much weaker sense of survival is that

$$\limsup_{t\to\infty} \mathbb{E}N_B(\mathbf{0}, t) > 0. \tag{1.2}$$

Our first theorem confirms the discovery of Shnerb et al. (2000) that even more than (1.2) holds in dimension 1 or 2 for all positive parameter values. Note that $\mathbb{E}$ denotes expectation with respect to $\mathbb{P}$, so that this theorem deals with the annealed expectation.

Theorem 1. *If $d = 1$ or 2, then for all $\delta, \beta, D_A, D_B, \mu_A, \mu_B > 0$*

$$\mathbb{E}N_B(\mathbf{0}, t) \to \infty \text{ faster than exponentially in } t. \tag{1.3}$$

Despite this result, it is *not* true that (1.1) holds for all $\delta, \beta, D_A, D_B, \mu_A, \mu_B > 0$. In fact our principal result is the following theorem, which deals with the quenched expectation (i.e., in a fixed realization of the catalyst system). Here

$$\mathcal{F}_A := \sigma\text{-field generated by } \{N_A(x, s) : x \in \mathbb{Z}^d, s \ge 0\}.$$

Theorem 2. *For all $\beta, D_A, D_B, \mu_A, \mu_B > 0$ and for all dimensions d, there exists a $\delta_0 < \infty$ such that for $\delta \ge \delta_0$ it holds*

$$\mathbb{E}\{N_B(\mathbf{0}, t)|\mathcal{F}_A\} \le e^{-\delta t/2} \text{ for all large } t \text{ a.s.}, \tag{1.4}$$

and consequently for all x

$$N_B(x, t) = 0 \text{ for all large } t \text{ a.s.} \tag{1.5}$$

In dimensions $d \ge 3$ we can even show that the unconditional (annealed) expectation of $N_B(x, t)$ tends to 0 when the birthrate β is small and the deathrate δ is large.

Theorem 3. *If $d \ge 3$, then there exists $\beta_0 > 0$ and $\delta_0 < \infty$ (depending on D_A, D_B, μ_A, μ_B) such that for $\beta \le \beta_0, \delta \ge \delta_0$*

$$\mathbb{E}N_B(\mathbf{0}, t) \le \mu_B e^{-\delta t/2}. \tag{1.6}$$

Consequently, for $\beta \le \beta_0, \delta \ge \delta_0$ and all $x \in \mathbb{Z}^d$,

$$N_B(x, t) = 0 \text{ for all large } t \text{ a.s.} \tag{1.7}$$

Remark 1. What can we say now about the phase diagram of survival/extinction of the B-particles ? For which values of the parameters $\beta, \delta, D_A, D_B, \mu_A, \mu_B$ does survival in the sense of (1.1) occur ? Let us concentrate on the dependence of this phenomenon on the values of β and δ for some fixed strictly positive values of D_A, D_B, μ_A, μ_B. We then see that in all dimensions, for fixed $\beta > 0$, there is no survival for δ large enough. Since survival is less likely as δ becomes larger, there is for fixed β a critical value $\delta_c(\beta) < \infty$ such that (1.1) holds for $\delta < \delta_c$, but not for $\delta > \delta_c$. (We have not excluded the possibility that $\delta_c = 0$ for some values of β.) The following argument shows that for fixed δ there is survival for large β. Let $Z_0 = N_B(\mathbf{0}, 0)$ be the number of particles at the origin at time 0, and let Z_n be the the number of B-particles at time n which have been at $\mathbf{0}$ since the time they were born (necessarily at $\mathbf{0}$). $\{Z_n\}_{n\geq 0}$ is a branching process in a stationary ergodic random environment. (See Athreya and Ney (1972), Section VI.5 for a description of such processes.) The environment is determined by the values of $N_A(\mathbf{0}, \cdot)$. The probability that a particle which is alive in this process at time k is still alive at time $k+1$ is at least $\exp[-\delta - D_B]$. The additional term D_B in the deathrate here represents the rate at which a particle jumps away from x; after such a jump a particle and its descendants are not counted in Z_n anymore. Also

$$\mathbb{E}\{Z_{k+1}|Z_i,\, i \leq k, \mathcal{F}_A\} \geq Z_k \exp[-\delta - D_B + \beta] \\ \times I[\text{some } A\text{-particle stays at } \mathbf{0} \text{ during the whole time interval } [k, k+1)].$$

It is now easy to check that for large β the conditions of Theorem VI.5.3 in Athreya and Ney (1972) hold, so that there is a strictly positive probability that the process $\{Z_n\}$ survives forever. Thus (1.1) holds for large β. In fact, Theorem 5.3 of Tanny (1977) now shows that even

$$\mathbb{P}\{N_B(\mathbf{0}, t) \to \infty\} > 0. \tag{1.8}$$

We see from the above that

$$\delta_c(\beta) \to \infty \text{ as } \beta \to \infty. \tag{1.9}$$

We repeat that we have not excluded the possibility that $\delta_c = 0$ for some values of β. In fact, we do not know how $\delta_c(\beta)$ behaves for small β.

Remark 2. Professor Sorin Solomon has pointed out to us that the B-particles may survive if β is large enough (compared to the other parameters), even if one starts with only one A-particle and one B-particle in the entire system. Indeed one can look at the process of the B-particles which "follow the A-particle." More precisely, let the A-particle start at the origin and assume it jumps at the times $s_1 < s_2 < \dots$ to the positions $x_1, x_2, \dots$, respectively. Take $s_0 = 0, x_0 = \mathbf{0}$. Then consider the B-particles which are at x_k during $[\frac{1}{2}s_k + \frac{1}{2}s_{k+1}, s_{k+1})$ and jump to x_{k+1} during $[s_{k+1}, \frac{1}{2}s_{k+1} + \frac{1}{2}s_{k+2})$. The number of these B-particles at the successive times s_k is a branching process with ransom environments. One can show that for large β this process has a positive probability of survival (see Athreya and Ney (1972), Theorem VI.5.3).

Remark 3. It is not difficult to deal with the boundary cases where exactly one of D_A or D_B equals 0. The arguments for these cases are independent of the dimension.

Case i) $D_A = 0, D_B > 0$. In this case

$$\liminf_{t\to\infty} \mathbb{P}\{N_B(\mathbf{0}, t) > 0\} > 0. \tag{1.10}$$

To see this, take K_1 a large integer so that $\beta K_1 > \delta + D_B$ and let x be a site with $N_A(x, 0) \geq K_1$. Now condition on the A-process. Since the A-particles don't move when $D_A = 0$, there will be $N_A(x, 0) \geq K_1$ A-particles at x at all times. Now let $Z(x, t)$ denote the number of B-particles at time t which have been at x since the time they were born (or since time 0 for the particles which start at x). Then $\{Z(x, t)\}_{t\geq 0}$ is an ordinary binary splitting branching process, in which a particle splits at rate $\beta N_A(x, 0) \geq \beta K_1$, and a particle dies at rate $\delta + D_B$ (see Remark 1 for explanation of the term D_B in the deathrate here). Since the birthrate exceeds the deathrate, by our choice of K_1, the process $Z(x, \cdot)$ is supercritical and

$$\mathbb{P}\{Z(x, t) > 0 \text{ for all } t | \mathcal{F}_A\} > 0$$

(see Athreya and Ney (1972), Section 3.4). (1.10) now follows from the fact that $\mathbb{P}\{N_A(\mathbf{0},0) > K_1\} > 0$.

Case ii) $D_A > 0, D_B = 0$. In this case the B-process becomes extinct when δ is large enough, that is, (1.4) and (1.5) hold for large δ. To see this, consider the process $Z(\mathbf{0},\cdot)$ defined above, conditioned on the A-process. Since the B-particles don't move in this case, $Z(\mathbf{0},t)$ is simply equal to $N_B(\mathbf{0},t)$. In this case, this is a time inhomogeneous binary splitting branching process with splitting rate $\beta N_A(\mathbf{0},t)$ at time t and death rate δ. By equation V.6.2 in Harris (1963), we have

$$\begin{aligned}\mathbb{E}\{N_B(\mathbf{0},t)|\mathcal{F}_A, N_B(\mathbf{0},0)\} &= \mathbb{E}\{Z(\mathbf{0},t)|\mathcal{F}_A, N_B(\mathbf{0},0)\}\\ &= N_B(\mathbf{0},0)\exp\Big[\int_0^t \big(\beta N_A(\mathbf{0},s)-\delta\big)ds\Big]. \qquad (1.11)\end{aligned}$$

Now the A-process is stationary and ergodic (in time) (see Derman (1955), Theorem 2) and therefore

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t N_A(\mathbf{0},s)ds = \mathbb{E}N_A(\mathbf{0},0) = \mu_A \text{ a.s.}$$

Thus for $\delta > 2\beta\mu_A$ the right hand side of (1.11)is almost surely no more than $e^{-\delta t/2}$ for all large t. Thus, (1.4) holds. We shall prove at the end of Section 4 that this also implies (1.5).

Remark 4. Both Theorem 2 and Theorem 3 depend on properties of the random walk performed by the B-particles. It seems that most results of this paper remain valid if we let the particles perform random walks which are not simple, but satisfy strong moment conditions. However, one cannot do away with all moment conditions. For instance, one can show that if the A-particles move according to a simple random walk and the B-particles perform a random walk which jumps from x to $x+y$ with probability $q(y) := K(1+\|y\|)^{-(d+1+\varepsilon)}$ for some $\varepsilon > 0$ and normalization constant K which makes $q(\cdot)$ into a probability distribution, then for some β_0 and any $\beta \ge \beta_0, \delta > 0$ the B-particles survive, even in the sense that

$$N_B(\mathbf{0},t) \to \infty \text{ in probability as } t \to \infty. \qquad (1.12)$$

Remark 5. A brief discussion of the existence of the system of A and B-particles which has the properties listed in the abstract can be found in the next section.

Remark 6. There exist a large number of papers on branching systems with catalysts, or branching random walks in random environments. See for instance the following papers and some of their references: Carmona and Molchanov (1994), Dawson and Fleischmann (2000), Gärtner and F. den Hollander (2003), Gärtner, König and Molchanov (2000), Klenke (2000a, 2000b) and Molchanov (1994). It appears that all of these papers have a somewhat different setup, or investigate a different problem from those in the present paper. To mention some differences, many of the previous papers have considered either an environment which remains constant over time (equivalent to non-moving cvatalysts), or an environment whose states at different times are independent. Often the catalyst only influences the branching *rate*, but not the branching law. In addition the branching law is frequently a critical one. The closest paper to ours is probably Gärtner and den Hollander (2003). It does consider moving catalysts which do influence the branching laws, but it concentrates on asymptotics as $t \to \infty$ of moments of the annealed system.

It seems that the question of survival of a branching system with moving catalysts in the quenched situation has not been investigated before.

The remainder of the paper is organized as follows: In Section 2 we discuss the construction of the system and give some formulae for the conditional expectation of $N_B(x,t)$, given the A-sytem. Theorem 1 is proven in Section 3, by using as a lower bound for $N_B(\mathbf{0},t)$ the number of B-particles all of whose ancestors (including themselves) have stayed at $\mathbf{0}$ during all of $[0,t]$. This forms a branching process in a stationary random environment, and it is not hard to give a lower bound on its expectation.

Theorem 3 is proven in Section 4. There we decompose the expectation of the number of particles according to the path followed by their ancestors (see (4.2)). This reduces the problem to estimating the expectation of $\exp[\beta L(\lambda, \pi)]$, where $L(\lambda, \pi)$ denotes the amount of time during $[0, t]$ when a given B-particle λ is on a given path π. In dimension d, the probability that λ is at $\pi(s)$ is $O\big((1+s)^{-d/2}\big)$. This rather crude bound is enough in dimension 3 or higher to control the right hand side of (4.2).

Theorem 2 has by far the most involved proof here. We first prove (1.4) for a discrete time approximation to our model in Section 5. This discrete time model is easier to handle than the original continuous time model. We give an outline of the proof in the beginning of Section 5. The proof for the continuous time model has the same structure as for the discrete time model, and is given in Section 6.

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2. Construction of the process and estimates for its expectation.

We briefly discuss here a construction of the process

$$\{N_A(x,t), N_B(x,t)\}_{x\in\mathbb{Z}^d, t\geq 0} \tag{2.1}$$

and then give an expression for $E\{N_B(x,t)|\mathcal{F}_A\}$. We shall use a particular construction of the process (2.1), because this will allow us to establish monotonicity in the birthrate (see (2.5) below) by means of coupling. It also leads us naturally to formulae for certain expectations which will be needed in the later proofs. However, we believe that there is only one process that agrees with the description given in the abstract. More precisely, we believe that the finite dimensional distributions of $N_A(x_i,t_i), N_B(x_i,t_i), 1 \leq i \leq k$, for any choice of the x_i, t_i are unique. We have not investigated this question of uniqueness of the process, but briefly comment on this some more immediately after the proof of Lemma 2.

Throughout the rest of this paper K_i will denote a constant which is strictly positive and finite. The precise values of these constants are unimportant and the same symbol K_i may stand for different constants in different formulae. We further use C_i for some other strictly positive and finite constants which remain the same throughout the paper. Both the K_i and the C_i are independent of t.

First we note that the construction of the A-process for all time is trivial, since the A-particles by themselves merely perform independent continuous time simple random walks. We shall take the paths of each of the A-particles right continuous. In order to construct the process in (2.1) we must therefore construct the B-process conditionally on a fixed realization of the A-process. Now, conditionally on the A-configuration, each of the B-particles present at time 0 is the progenitor of a branching random walk with deathrate δ and with splitting rate $\beta N_A(x,t)$ per particle at the site x at time t. The branching random walks generated by different starting particles are independent. The construction therefore breaks into two parts. First we must construct the branching random walk generated by a single particle. Once this is done, there is no difficulty in constructing the branching random walks for all the initial B-particles together. Because these branching random walks are independent one simply uses a product space with each component corresponding to the branching random walk generated by one particle. One then defines $N_B(x,t)$ as the total number of B-particles at x at time t in all these branching random walks together. As a second step we must show that this $N_B(x,t)$ is almost surely finite for all (x,t). Once this is done, the resulting process $\{N_A(x,t), N_B(x,t)\}$ is a version of the desired process. We remark that the construction of a branching random walk starting with one particle has been carried out in the time-homogeneous case in a long article Ikeda, Nagasawa and Watanabe (1968a,1968b), and has been discussed further in Savits (1969).

It will be convenient to carry out the first step for more general birthrates. For each $x \in \mathbb{Z}^d$, let $b(x, \cdot) : [0, \infty) \to [0, \infty]$ be a locally integrable function. We want to construct a branching random walk which has birthrate $b(x, t)$ per particle at (x, t). The process starts with one particle at some site. We shall denote this progenitor by σ. We represent the process of descendants of σ at time t as a collection of particles which are indexed by $(p + 1)$-tuples $\langle \sigma, i_1, \dots, i_p \rangle$ with $p \geq 0$. The usual interpretation of these $(p + 1)$-tuples is that $\langle \sigma, i_1, \dots, i_p \rangle$ is the i_p-th child of the i_{p-1}-th child $\cdots$ of the i_1-th child of $\langle \sigma \rangle := \sigma$. However, this interpretation has no significance for us, and we shall in fact drop it in the coupling construction leading to (2.5) below. As usual, not all possible particles $\langle \sigma, j_1, \dots, j_p \rangle$ will occur in our process. Only the particles which are alive at some time are realized in the process (see Harris (1963), Sect. VI.2, and Jagers (1975), Sect. 1.2, for more details). For the construction we choose for each possible $(p + 1)$-tuple $\langle \sigma, j_1, \dots, j_p \rangle$ (with $j_i = 1, 2, \dots$) and for each $x \in \mathbb{Z}^d$, several Poisson processes, denoted by $\mathcal{P}(x, \sigma, j_1, j_2, \dots, j_p, \mathcal{E})$. $\mathcal{E}$ denotes one of the possible changes which a particle can undergo. There is one process for $\mathcal{E} = \pm e_i$, which correspond to the particle making a jump $\pm e_i$, $1 \leq i \leq d$. These $2d$ processes each have the constant rate $D_B/(2d)$. There is a process with $\mathcal{E} = death$ which has the constant rate δ and which corresponds to the death (without offspring) of the particle $\langle \sigma, j_1, \dots, j_p \rangle$. Finally there is a process with $\mathcal{E} = split$ which corresponds to the particle $\langle \sigma, j_1, \dots, j_p \rangle$ splitting into the two particles $\langle \sigma, j_1, \dots, j_p, 1 \rangle$ and $\langle \sigma, j_1, \dots, j_p, 2 \rangle$. This last process, $\mathcal{P}(x, \sigma, j_1, \dots, j_p, \text{split})$ has the nonconstant rate $b(x, \cdot)$. The family tree of the branching random walk is now constructed by "following the instructions of the Poisson processes". More specifically, we begin with one particle σ alive at time 0. If a particle $\langle \sigma, i_1, \dots, i_p \rangle$ is alive and at x at time t, then it stays at x till the next point $t' \geq t$ in one of the Poisson processes $\mathcal{P}(x, \sigma, i_1, \dots, i_p, \mathcal{E})$ associated to the site x and the particle $\langle \sigma, i_1, \dots, i_p \rangle$. If this first point is a point of the process $\mathcal{P}(x, \sigma, i_1, \dots, i_p, \pm e_i)$, then the particle $\langle \sigma, i_1, \dots, i_p \rangle$ jumps at time t' to the position $x \pm e_i$ and then waits there till the next point in one of the processes $\mathcal{P}(x \pm e_i, \sigma, i_1, \dots, i_p, \mathcal{E})$. If the first point during $[t, \infty)$ in the Poisson processes $\mathcal{P}(x, \sigma, i_1, \dots, i_p, \mathcal{E})$ is a point of $\mathcal{P}(x, \sigma, i_1, \dots, i_p, \text{death})$, then the particle $\langle \sigma, i_1, \dots, i_p \rangle$ dies at that time and has no further offspring in the family tree. Finally, if the first point in the Poisson processes at x is a point of $\mathcal{P}(x, \sigma, i_1, \dots, i_p, \text{split})$, then the particle $\langle \sigma, i_1, \dots, i_p \rangle$ itself disappears, but it is replaced by the two particles $\langle \sigma, i_1, \dots, i_p, 1 \rangle$ and $\langle \sigma, i_1, \dots, i_p, 2 \rangle$ at position x. We shall occasionaly denote the process constructed in the above manner by $W = W_\sigma$.

We say that *no explosion occurs during* $[0, t]$ in W_σ, if "following the instructions" gives only finitely many changes in the family tree of W_σ during $[0, t]$. *No explosion occurs* means that no explosion occurs during any finite interval $[0, t]$. The preceding construction gives a well defined process W_σ which is a strong Markov process with the correct rates for jumping, death and splitting, provided that almost surely no explosion occurs in W_σ. We shall soon prove that no explosion occurs in any W_σ in our case, which has $b(x, t) = \beta N_A(x, t)$. We shall even show that in the full process of (2.1) there are almost surely never infinitely many particles at one site.

Before we proceed we make the following simple, but important monotonicity observation. Suppose that $b'(\cdot)$ and $b''(\cdot)$ are both locally integrable functions from $[0, \infty)$ to $[0, \infty]$ satisfying

$$b'(x, t) \leq b''(x, t) \text{ for all } x \in \mathbb{Z}^d \text{ and } t \leq T. \tag{2.2}$$

We can then choose Poisson processes $\mathcal{P}'(x, \sigma, j_1, \dots, j_p, \mathcal{E})$ and $\mathcal{P}''(x, \sigma, j_1, \dots, j_p, \mathcal{E})$ such that

$$\mathcal{P}'(x, \sigma, j_1, \dots, j_p, \text{split}) \cap [0, T] \subset \mathcal{P}''(x, \sigma, j_1, \dots, j_p, \text{split}) \cap [0, T] \tag{2.3}$$

and

$$\mathcal{P}'(x, \sigma, j_1, \dots, j_p, \mathcal{E}) \cap [0, T] = \mathcal{P}''(x, \sigma, j_1, \dots, j_p, \mathcal{E}) \cap [0, T] \quad \text{for all } x, j_1, \dots, j_p \text{ and } \mathcal{E} \neq \text{split}. \tag{2.4}$$

(Here we abused notation somewhat by viewing a Poisson process $\mathcal{P}$ simply as the set of times at which it has a jump.) We now want to use the Poisson processes $\mathcal{P}'$ and $\mathcal{P}''$ to couple two branching random walks with birthrates per particle equal to b' and b'', respectively (but with the same constant deathrate per particle and jumprate as before). To do this, we construct the process W''_σ in the above manner, but with the $\mathcal{P}''$ processes replacing the processes $\mathcal{P}$. However, the process W'_σ corresponding to b' is not obtained by merely replacing $\mathcal{P}$ by $\mathcal{P}'$. We also have to change the indexing of our particles and use the $\mathcal{P}'$ process which has the changed index. Specifically, suppose we already obtained some particle $\langle \sigma, i_1, \dots, i_p\rangle$ which belongs to both processes, and this particle splits into the particles $\langle \sigma, i_1, \dots, i_p, 1\rangle$ and $\langle \sigma, i_1, \dots, i_p, 2\rangle$ at some position x and time t in W''_σ. This means that t is a point of $\mathcal{P}''(x, i_1, \dots, i_p, \text{split})$. If t is also a point of $\mathcal{P}'(x, i_1, \dots, i_p, \text{split})$, then of course the particle $\langle \sigma, i_1, \dots, i_p\rangle$ splits also at time t in W'_σ into the particles $\langle \sigma, i_1, \dots, i_p, 1\rangle$ and $\langle \sigma, i_1, \dots, i_p, 2\rangle$. However, if t is not a point of $\mathcal{P}'(x, i_1, \dots, i_p)$, we will still rename the particle $\langle \sigma, i_1, \dots, i_p\rangle$ of W'_σ to $\langle \sigma, i_1, \dots, i_p, 1\rangle$ in W'_σ and it will start using the processes $\mathcal{P}'(x, i_1, \dots, i_p, 1, \mathcal{E})$ from time t until it dies, splits or is renamed again. W'_σ will never have a particle $\langle \sigma, i_1, \dots, i_p, 2\rangle$ in this scenario. Particles which exist in both processes (when using the above renaming procedure) will die or jump together in both processes (by virtue of (2.4)). This renaming of particles in W'_σ has the effect that for both W'_σ and W''_σ we always use a Poisson process with the same index at the same point of space-time. It is clear that under this construction

$$W''_\sigma \text{ has at least as many particles alive at } x, t \text{ as } W'_\sigma, \text{ for all } x \text{ and all } t \le T. \tag{2.5}$$

In fact, we have an even stronger monotonicity property. To explain this we introduce *the piece till time T of the path associated to a B-particle ρ* in a process W. Consider first a particle ρ which is alive at time T in W. For such a ρ, there was at each time $s \in [0, T]$ a unique B-particle which was the ancestor of ρ and this ancestor had a position, $x(s, \rho)$ say (note that this ancestor may equal ρ). By the piece till T of the path associated to ρ we then mean the path $\{x(s, \rho)\}_{0 \le s \le T}$. If ρ is a particle which lived at some time in W, but died or was replaced at time $T' \le T$, then the piece till T of path associated to ρ is taken to be the path $\{x(s, \rho)\}_{0 \le s \le T'}$. It follows from our construction that for the processes W'_σ and W''_σ corresponding to b' and b'' which satisfy (2.2)

$$\text{the piece till } T \text{ of any path in } W'_\sigma \text{ is also a piece till } T \text{ of a path in } W''. \tag{2.6}$$

In particular, if no explosion occurs in W''_σ during $[0, T]$, then the same holds for W'_σ. It is also the case that W'_σ is a process with the same distribution as a process with birthrate b' per particle, because the switching from $\mathcal{P}(x, i_1, \dots, i_p, \mathcal{E})$ to $\mathcal{P}(x, i_1, \dots, i_p, 1, \mathcal{E})$ at a certain time t which is a splitting time for W''_σ but not for W'_σ, has no influence on the *distribution of* the Poisson processes during $[t, \infty)$.

We shall need some formulae for the expected size of a branching random walk in the time inhomogeneous case.

Lemma 1. *If $b(x, s) = b$ for all x and $s \le t$, then, for any given progenitor σ,*

$$\mathbb{E}\{\textit{number of descendants of } \sigma \textit{ alive at time } t \textit{ in } W_\sigma\} = e^{(b-\delta)t}. \tag{2.7}$$

If, for some fixed x the splitting rate $b(x, \cdot)$ is piecewise linear and right continuous on $[0, t]$, then, for any given progenitor σ at x at time 0,

$$\begin{aligned}&\mathbb{E}\{\textit{number of descendants of } \sigma \textit{ alive at time } t \textit{ in } W_\sigma \\ &\qquad\qquad \textit{whose associated path stayed at } x \textit{ for all of } [0, t]\} \\ &= \exp\Big[\int_0^t b(x, s)ds - (\delta + D_B)t\Big].\end{aligned} \tag{2.8}$$

Proof. First assume that the splitting rate is equal to a constant b for all x, $s \le t$. Then the birthrate is independent of position so that we can ignore the positions of the particles. For each particle corresponding Poisson events with $\mathcal{E}$ = death or split occur at rates δ and b, respectively. Thus (2.7) is just a special case of the formula for the mean of a continuous time branching process which can be found in Harris (1963), equation V.6.2.

Next consider (2.8) and assume that $b(x, s) = b$ for all $s \le t$, but just for the single x under consideration. Then the number of descendants of σ whose associated path stayed at x all the time is a time homogeneous branching process with splitting rate b and deathrate $\delta + D_B$, for the same reasons as in Remark 1. Just as in the argument for (2.7) we find that expectation in the left hand side of (2.8) equals $\exp[(b - \delta - D_B)t]$ in this situation. More generally, the expected number of descendants of Z particles whose associated path stays at x during all of $[0, t]$ is $Z \exp[(b - \delta - D_B)t]$. Finally, assume that $b(x, t) = b_i$ for $s_i \le s < s_{i+1}$ for some sequence of times $s_0 = 0 < s_1 < \cdots < s_m = t$. Let $Z(s)$ be the number of descendants of σ alive at time s in W_σ whose associated path stayed at x for all of $[0, s]$. Then the preceding argument gives

$$\begin{aligned} E\{Z(s_{i+1})|Z(u),\, u \le s_i\} &= Z(s_i) \exp[(b_i - \delta - D_B)(s_{i+1} - s_i)] \\ &= Z(s_i) \exp\Big[\int_{s_i}^{s_{i+1}} b(x, s) ds - (\delta + D_B)(s_{i+1} - s_i)\Big]. \end{aligned}$$

(2.8) follows by induction on m in this case. ■

We claim that in order to show that almost surely no explosion occurs during any finite time interval, it is enough to show that almost surely only finitely many particles are born during any finite interval $[0, t]$. Indeed, if there are only finitely many births, then there are also only finitely many deaths. Moreover, if only finitely many particles are born, then following the instructions can lead to meeting infinitely many Poisson points with $\mathcal{E}$ equal to a jump only if there is a fixed particle $\langle \sigma, i_1, \dots, i_p \rangle$ which follows a path which makes infinitely many jumps during $[0, t]$. However, the rates of the processes $\mathcal{P}(x, \sigma, i_1, \dots, i_p, \pm e_i),\ x \in \mathbb{Z}^d$, are all the same, and therefore the probability that the particle $\langle \sigma, i_1, \dots, i_p \rangle$ makes infinitely many jumps during $[0, t]$ is zero. This proves our claim. It follows that the probability (in a given configuration of the A-system) that there are infinitely many instructions to follow during $[0, t]$ for the descendants of a given σ can be strictly positive only if

$$\lim_{N \to \infty} \mathbb{P}\{\text{some particle } \langle \sigma, i_1, \dots, i_N \rangle \text{ is born during } [0, t] | \mathcal{F}_A\} > 0. \tag{2.9}$$

In turn, (2.9) can only be the case if

$$\begin{aligned} \mathbb{E}\{\text{number of particles } \langle \sigma, i_1, \dots, i_N \rangle \text{ (with some } N \\ \text{and } i_1, \dots, i_N) \text{ born during } [0, t] | \mathcal{F}_A\} = \infty. \end{aligned} \tag{2.10}$$

The next lemma shows that (2.10) fails for almost all realizations of the A-process.

In this proof we find it useful to write P_A for the probability measure governing the A-system only. Similarly P_B will be short for the conditional probablity measure governing the B-system, given the A-configuration. Actually, until the construction of the process (2.1) is completed, we have to regard P_B as the distribution of all the Poisson processes $\mathcal{P}(x, \sigma, i_0, \dots, i_p, \mathcal{E})$ and of $\{N_B(x, 0), x \in \mathbb{Z}^d\}$. As before, $\mathbb{P}$ will be used for the probability measure for the combined A and B-system (the so-called annealed probability measure). E_A, E_B and $\mathbb{E}$ denote the expectations with respect to P_A, P_B and $\mathbb{P}$, respectively.

Lemma 2.

$$\mathbb{E}\{\textit{number of descendants born during } [0, t] \textit{ of any particle starting at } \mathbf{0}\} < \infty. \tag{2.11}$$

Remark 7. For convenience the proof has been written only in the case where the A and B-particles perform a simple random walk. However, after minor changes the proof of this lemma works for any choice of the random walks for the particles.

Proof. If there are no particles at $\mathbf{0}$ at time 0, then the number in braces in (2.11) vanishes. Otherwise, let σ be a generic particle at $\mathbf{0}$ at time 0. Any descendant $\langle \sigma, i_1, \dots, i_p \rangle$ of σ born during $[0, t]$ either is still alive at time t, or dies without offspring, or splits during $[0, t]$. For brevity we write ρ for the particle $\langle \sigma, i_1, \dots, i_p \rangle$. Consider first the case that ρ is still alive at time t. Then the piece till t of the path associated to ρ (see the lines before (2.6) for definition) is almost surely given by a sequence $0 < s_1(\rho) < \cdots < s_\ell(\rho) < t$ of jumptimes and a sequence $x_1, x_2, \dots, x_\ell$ of positions right after the jumps such that

$$\begin{aligned} x(s,\rho) &= \mathbf{0} \text{ for } 0 \le s < s_1, \\ x(s,\rho) &= x_k \text{ for } s_k \le s < s_{k+1},\ k+1 \le \ell, \\ x(s,\rho) &= x_\ell \text{ for } s_\ell \le s \le t \end{aligned} \tag{2.12}$$

(ℓ depends on σ and t). Let us first estimate only the number of particles still alive at $[0, t]$. We decompose the number of descendants alive at time t of a particle starting at $\mathbf{0}$ according to the path associated to these descendants. We then get the following contribution to the left hand side of (2.11):

$$\begin{aligned} \sum_{\ell \ge 0} \sum_{x_1,\dots,x_\ell} \int_{0<s_1<\cdots<s_\ell<t} & \mathbb{E}\{\text{number of particles } \rho \text{ alive at } t \text{ which} \\ & \quad \text{descend from a particle at } \mathbf{0} \text{ at time 0 and whose} \\ & \quad \text{associated path has jumptimes in } ds_1, ds_2, \dots, ds_\ell \\ & \quad \text{and positions } x_1, x_2, \dots, x_\ell \text{ after the jumps}\}. \end{aligned} \tag{2.13}$$

We shall first condition on the A-system in the last expectation. If the A-system is fixed, then the splittingrate for the B-particles at a site x at time s is $\beta N_A(x, s)$. It is not hard to show that this is a right continuous, piecewise linear function of s, because only finitely many A-particles visit a given site in a bounded time interval. (This is a simple special case of (2.26) below.) Let $q(\cdot)$ be the distribution of a jump of simple random walk, that is

$$q(\pm e_i) = \frac{1}{2d},\ 1 \le i \le d, \text{ and } q(y) = 0 \text{ otherwise.} \tag{2.14}$$

Then the probability that a given B-particle $\langle \sigma, i_1, \dots, i_p \rangle$ alive at x' at time s makes a jump from x' to x'' during the next small interval of length ds is the probability of a point in $\mathcal{P}(x', \sigma, i_1, \dots, i_p, x'' - x')$ during the next ds units of time, i.e., $D_B q(x'' - x')ds + o(ds)$. Define further, with $s_0 = 0, x_0 = \mathbf{0}$,

$$J(\{s_i, x_i\}) = \sum_{i=1}^{\ell} \int_{s_{i-1}}^{s_i} N_A(x_{i-1}, u)du + \int_{s_\ell}^{t} N_A(x_\ell, u)du. \tag{2.15}$$

β times this expression can be thought of as the total birthrate along the path given by the s_i and x_i. A small extension of the argument for Lemma 1 then allows us to write

$$\begin{aligned} & E_B\{\text{number of particles } \rho \text{ which descend from a particle} \\ & \quad \text{at } \mathbf{0} \text{ at time 0 and whose associated path has jumptimes in} \\ & \quad ds_1, ds_2, \dots, ds_\ell \text{ and positions } x_1, x_2, \dots, x_\ell \text{ after the jumps} | \mathcal{F}_A\} \\ & = \mu_B \exp[-(\delta + D_B)t + \beta J(\{s_i, x_i\})] \prod_{i=1}^{\ell} [D_B q(x_i - x_{i-1})] ds_1 \dots ds_\ell. \end{aligned} \tag{2.16}$$

Substitution of this into (2.13) gives

$$\begin{aligned} & \mathbb{E}\{\text{number of particles } \rho \text{ which descend from a particle at } \mathbf{0} \\ & \quad \text{and are alive at time } t\} \\ & = \mu_B \sum_{\ell \ge 0} \sum_{x_1,\dots,x_\ell} \exp[-(\delta + D_B)t] \prod_{i=1}^{\ell} [D_B q(x_i - x_{i-1})] \\ & \qquad \times \int_{0<s_1<\cdots<s_\ell<t} E_A \exp[\beta J(\{s_i, x_i\})] ds_1 \dots ds_\ell. \end{aligned} \tag{2.17}$$

Essentially the same argument gives

$$\mathbb{E}\{\text{number of particles } \rho \text{ which descend from a particle at } \mathbf{0}$$
$$\text{and which die without offspring or which split before time } t\}$$
$$= \mu_B \sum_{\ell \geq 0} \sum_{x_1,\dots,x_\ell} \prod_{i=1}^{\ell} [D_B q(x_i - x_{i-1})] \int_0^t ds_{\ell+1} \exp[-(\delta + D_B)s_{\ell+1}]$$
$$\times \int_{0<s_1<\dots<s_\ell<s_{\ell+1}} E_A\Big\{\big(\delta + \beta N_A(x_{s_\ell}, s_{\ell+1})\big) \exp[\beta \widehat{J}(\{s_i, x_i\})]\Big\} ds_1 \dots ds_\ell, \tag{2.18}$$

where

$$\widehat{J}(\{s_i, x_i\}) = \sum_{i=1}^{\ell} \int_{s_{i-1}}^{s_i} N_A(x_{i-1}, u)du + \int_{s_\ell}^{s_{\ell+1}} N_A(x_\ell, u)du.$$

($s_{\ell+1}$ here represents the time at which a particle dies without offspring or splits).

For the time being we concentrate on (2.17). We must work on $E_A \exp[\beta J(\{s_i, x_i\})]$. Note that $J(\{s_i, x_i\})$ represents the total time spent by all A-particles on the path which is at x_{i-1} during $[s_{i-1}, s_i)$ and at x_ℓ during $[s_\ell, t)$. Let us denote this path by $\pi = \pi(s_i, x_i)$. Then $J(\{s_i, x_i\})$ can be written as the sum over all A-particles λ present at time 0, of the time spent by λ on π during $[0, t]$. Denote this time spent by λ on π by $L(\lambda) = L(\lambda, \pi)$. Define further

$$\mathcal{F}_{A,0} := \sigma\text{-field generated by all } N_A(x, 0),\ x \in \mathbb{Z}^d. \tag{2.19}$$

Then, for given π, and conditionally on $\mathcal{F}_{A,0}$, the random variables $L(\lambda, \pi)$ for different λ are independent. Thus

$$E_A\{\exp[\beta J(\{s_i, x_i\})] | \mathcal{F}_{A,0}\} = \prod_\lambda E_A\{\exp[\beta L(\lambda, \pi)] | \mathcal{F}_{A,0}\}. \tag{2.20}$$

Let

$$\kappa(\lambda) = \kappa(\lambda, \pi) = \inf\{s : \lambda \text{ is on } \pi \text{ at time } s\}$$

($\kappa = \infty$ if the set on the right is empty). Since λ cannot spend any time on π before $\kappa(\lambda, \pi)$, a first entry decomposition gives

$$E_A\{\exp[\beta L(\lambda, \pi)] | \mathcal{F}_{A,0}\}$$
$$= P_A\{\kappa(\lambda, \pi) > t | \mathcal{F}_{A,0}\} + \int_0^t P_A\{\kappa(\lambda, \pi) \in du | \mathcal{F}_{A,0}\} E_A\{\exp[\beta L(\lambda', \pi'(u))] | \mathcal{F}_{A,0}\}, \tag{2.21}$$

where λ' is a particle at $\pi(u) = \pi'(0)$ at time 0, and $\pi'(u)$ is the path obtained from π by a time shift of size u. More precisely, $\pi'(u)$ is a path defined on the time interval $[0, t-u)$; moreover, if $s_k < u \leq s_{k+1}$, then $\pi'(u)$ is at x_k at times in $[0, s_{k+1} - u)$, at x_j at times in $[s_j - u, s_{j+1} - u)$ for $k < j < \ell$ and at x_ℓ for times in $[s_\ell - u, t - u)$. We now write for any path π' of length $t - u$.

$$L(\lambda', \pi'(u)) = \int_0^{t-u} I[\lambda' \text{ is on } \pi' \text{ at time } v] dv.$$

This leads to the relation

$$E_A\{\exp[\beta L(\lambda', \pi'(u))] | \mathcal{F}_{A,0}\}$$
$$= \sum_{p=0}^{\infty} \frac{\beta^p}{p!} E_A\{[L(\lambda', \pi'(u))]^p | \mathcal{F}_{A,0}\}$$
$$= \sum_{p=0}^{\infty} \beta^p \int_{0<v_1<v_2<\dots<v_p \leq t-u} P_A\{\lambda' \text{ is on } \pi' \text{ at times } v_1, \dots, v_p | \mathcal{F}_{A,0}\} dv_1 \dots dv_p. \tag{2.22}$$

We shall work on a good bound for $E_A\{[L(\lambda', \pi'(u))]^p|\mathcal{F}_{A,0}\}$ in a later lemma. For now we merely use the trivial bound

$$L(\lambda', \pi'(u)) \le t - u \le t. \tag{2.23}$$

Substituting this bound into (2.21) and (2.20) gives

$$E_A\{\exp[\beta L(\lambda,\pi)]\mathcal{F}_{A,0}\} \le P_A\{\kappa(\lambda,\pi) > t|\mathcal{F}_{A,0}\} + P_A\{\kappa(\lambda,\pi) \le t|\mathcal{F}_{A,0}\}e^{\beta t}$$

and

$$\begin{aligned}
&E_A\{\exp[\beta J(\{s_i, x_i\})]|\mathcal{F}_{A,0}\} \\
&\le \prod_\lambda \big[1 - P_A\{\kappa(\lambda,\pi) \le t|\mathcal{F}_{A,0}\} + P_A\{\kappa(\lambda,\pi) \le t|\mathcal{F}_{A,0}\}e^{\beta t}\big] \\
&\le \exp\Big[\sum_\lambda P_A\{\kappa(\lambda,\pi) \le t|\mathcal{F}_{A,0}\}(e^{\beta t} - 1)\Big].
\end{aligned} \tag{2.24}$$

Now, let $\{S_s\}_{s\ge 0}$ be a continuous time simple random walk with jumprate D_A, starting at $\mathbf{0}$. Each A-particle performs (a translate of) such a random walk. Then for any A-particle λ starting at a given site x and any given path π,

$$P_A\{\kappa(\lambda,\pi) \le t|\mathcal{F}_{A,0}\} = P\{x + S_s \text{ is on } \pi \text{ for some } s \le t\}.$$

Moreover, since the number of A-particles at x at time 0 has a Poisson distribution with mean μ_A,

$$\begin{aligned}
&E_A\{\exp\Big[\sum_\lambda P_A\{\kappa(\lambda,\pi) \le t|\mathcal{F}_{A,0}\}(e^{\beta t} - 1)\Big]\} \\
&= \exp\Big[\mu_A \sum_x \Big(\exp\big[P\{x + S_s \text{ is on } \pi \text{ for some } s \le t\}(e^{\beta t} - 1)\big] - 1\Big)\Big] \\
&\le \exp\big[\Gamma \sum_x P\{x + S_s \text{ is on } \pi \text{ for some } s \le t\}\big],
\end{aligned} \tag{2.25}$$

where $\Gamma = \Gamma(t) = \mu_A e^{\beta t}\exp[e^{\beta t}]$.

To complete the estimate of (2.17) we shall prove that if π has ℓ jumptimes, then

$$\sum_x P\{x + S_s \text{ is on } \pi \text{ for some } s \le t\} \le (tD_A + \ell). \tag{2.26}$$

Together with (2.17) and (2.24) this will imply that

$$\begin{aligned}
&\mathbb{E}\{\text{number of descendants of any particle starting at } \mathbf{0} \text{ which are alive at } t\} \\
&\le \mu_B \sum_{\ell\ge 0}\sum_{x_1,\dots,x_\ell} \exp[-(\delta + D_B)t]\prod_{i=1}^{\ell}[D_B q(x_i - x_{i-1})] \\
&\qquad\qquad \int_{0<s_1<\dots<s_\ell<t} \exp[\Gamma(tD_A + \ell)]ds_1\dots ds_\ell \\
&= \mu_B \sum_{\ell\ge 0} \frac{[e^\Gamma t D_B]^\ell}{\ell!}\exp[\Gamma t D_A - (\delta + D_B)t] \\
&= \mu_B \exp[\big(e^\Gamma D_B + \Gamma D_A - \delta - D_B\big)t] < \infty.
\end{aligned} \tag{2.27}$$

It remains to prove (2.26). To this end, note that the expression in (2.26) equals

$$E\{\text{number of distinct particles which coincide with } \pi \text{ at some time during } [0,t]\},$$

where we now start with one particle at each site x and the particles move with the same random walk as the A-particles. A particle may coincide with π for the first time in an interval $[s, s+ds)$ because π has a jump to the position of the particle during this interval, or because the particle jumps to the position of π during this interval. If π jumps only at s_i to x_i for $1 \le i \le \ell$ during $[0,t)$, then the contribution of the former kind of jumps to the left hand side of (2.26) is at most

$$E\Big\{\sum_{i=1}^{\ell}(\text{number of particles present at } x_i \text{ at time } s_i)\Big\}$$
$$= \sum_{i=1}^{\ell}\sum_x P\{x + S_{s_i} = x_i\} = \ell. \tag{2.28}$$

Similarly, the expected number of particles which jump to a given position y during $[s, s+ds)$ equals

$$\sum_{x,z\in\mathbb{Z}^d} P\{x + S_s = z\}q(y-z)D_A ds = D_A ds.$$

Therefore the contribution to the left hand side of (2.26) which is due to jumps of some particle to the position of π is at most

$$\int_0^t D_A ds = tD_A.$$

Adding this to the contribution (2.28) gives (2.26).

Thus we proved that the expression in (2.17) is finite. To show that (2.18) is also finite, we merely observe that $\widehat{J}(\{s_i, x_i\}) \le J(\{s_i, x_i\})$ and that

$$E_A\Big[N_A(x_{s_\ell}, s_{\ell+1})\exp[\beta\widehat{J}(\{s_i,x_i\})]\Big]$$
$$\le \Big\{E_A\big[N_A(x_{s_\ell}, s_{\ell+1})\big]^2\Big\}^{1/2}\Big\{E_A \exp[2\beta\widehat{J}(\{s_i,x_i\})]\Big\}^{1/2}$$
$$\le \Big\{E_A\big[N_A(x_{s_\ell}, s_{\ell+1})\big]^2\Big\}^{1/2} E_A \exp[2\beta\widehat{J}(\{s_i,x_i\})]$$
$$= [\mu_A^2 + \mu_A]^{1/2} E_A \exp[2\beta\widehat{J}(\{s_i,x_i\})].$$

We leave further details to the reader. ■

As explained before, this lemma shows that almost surely, none of the branching processes of descendants of one particle explodes. This shows that our procedure indeed constructs a branching random random walk with the correct rates for the descendants of a single particle. In fact, we think that given the A-system and some B-particle at time 0, any other construction of the process of descendants of σ must give a process with the same finite dimensional distributions as W_σ. This is based on the Corollary on p. 273 of Ikeda, Nagasawa and Watanabe (1968a, 1968b). The required independence properties of the A-particles, and conditionally on $\mathcal{F}_A$ of the B-particles, therefore make the process in (2.1) unique in the sense of finite dimensional distributions. We should make two comments to this. Firstly, a Markov branching process with property C of Ikeda, Nagasawa and Watanabe (1968a, 1968b) is automatically a minimal process, i.e., it is trapped in a special point after the first explosion (see Theorem 1.1 and property C1 in this paper). Secondly, Ikeda, Nagasawa and Watanabe (1968a,1968b) deals with a compact space for the positions of the particles and the time-homogeneous case only. To apply their theory we have to first go over to the space-time process, so that the possible states of the particles have the form (x,t) with $x \in \mathbb{Z}^d$ and $t \in [0,\infty)$. In order to have the particles move in a compact space we then have to let our particles move in the one point compactification of $\mathbb{Z}^d \times [0,\infty)$, with the point at infinity taken as a trap. We shall not pursue this matter of uniqueness of the process, and *assume from now on that we work with the process constructed from the Poisson processes* $\mathcal{P}(x, \sigma, j_1, \dots, j_p, \mathcal{E})$ *by the above method.*

In order to show that the proces which we constructed is a "good" process we next show that almost surely

$$N_A(x,t) < \infty \text{ and } N_B(x,t) < \infty \text{ for all } x, t. \tag{2.29}$$

The finiteness of the $N_A(x,t)$ for all $x \in \mathbb{Z}^d$ and *integral* t is clear from the fact that $E_A N_A(x,t) \equiv \mu_A < \infty$. It then also follows for arbitrary $t \geq 0$ from the maximal inequality

$$P_A\{N_A(x,k) \geq \frac{1}{2} N e^{-D_A}\} \geq \frac{1}{2} P_A\{\sup_{k-1 \leq s \leq k} N_A(x,s) \geq N\}, \tag{2.30}$$

which holds for all $N \geq$ some N_0. One easy way to see (2.30) is that when $N_A(x,\tau) \geq N$ at some stopping time τ, then the number of particles which stay at x during $[\tau, \tau+1]$ is stochastically larger than a random variable with a binomial distribution corresponding to N trials with success probability $\exp[-D_A]$. Thus

$$P_A\{N_A(x,k) \geq \frac{1}{2} N e^{-D_A} | \sup_{k-1 \leq s \leq k} N_A(x,s) \geq N\} \to 1$$

as $N \to \infty$.

The finiteness of the $N_B(x,t)$ is proven in the next lemma.

Lemma 3. *Almost surely* $[\mathbb{P}]$ $N_B(x,t) < \infty$ *for all* $x \in \mathbb{Z}^d, t \geq 0$.

Proof. For the sake of argument take $x = \mathbf{0}$ and fix $t \geq 0$. Each B-particle at $\mathbf{0}$ at time t must be a descendant of a particle starting at some site x. Thus,

$$\begin{aligned} N_B(\mathbf{0},t) = \sum_x &(\text{number of particles at } \mathbf{0} \text{ at time } t \text{ which descend} \\ &\text{from a particle at } x \text{ at time } 0). \end{aligned}$$

By translation invariance we now have

$$\begin{aligned} \mathbb{E} N_B(\mathbf{0},t) &= \sum_x \mathbb{E}(\text{number of particles at } \mathbf{0} \text{ at time } t \text{ which descend} \\ &\qquad \text{from a particle at } x \text{ at time } 0) \\ &= \sum_x \mathbb{E}(\text{number of particles at } -x \text{ at time } t \text{ which descend} \\ &\qquad \text{from a particle at } \mathbf{0} \text{ at time } 0) \\ &= \mathbb{E}(\text{number of descendants of some particle at } \mathbf{0} \text{ which} \\ &\qquad \text{are alive at time } t). \end{aligned} \tag{2.31}$$

The right hand side here was shown to be finite in Lemma 2. Thus almost surely $N_B(\mathbf{0},k)$ is finite for all integral k, and by translation invariance the same is true for all $N_B(x,k)$. The finiteness of all $N_B(x,t)$, $x \in \mathbb{Z}^d, t \geq 0$, now follows in essentially the same way as for the N_A. Indeed, we now have analogously to (2.30)

$$P_B\{\sup_{k-1 \leq s \leq k} N_B(x,s) \geq N\} \leq 2 P_B\{N_B(x,k) \geq \frac{1}{2} N e^{-\delta - D_B}\}, \tag{2.32}$$

for large N and uniformly in the A-configuration, because the probability for a B-particle alive at x at time τ to stay alive at x during $[\tau, \tau+1]$ is $\exp[-\delta - D_B]$.

■

We end this section with a proof that for estimating $\mathbb{E}\{N_B(\mathbf{0},t)|\mathcal{F}_A\}$ we may restrict ourselves to a particle system with particles restricted to the cube

$$\mathcal{C}(C_1 t\log t) := [-C_1 t\log t, C_1 t\log t]^d \tag{2.33}$$

for $s \le t$ and a suitable constant C_1, and with $N_A(x,s)$ bounded by $\log t$. This estimate will not be used until Section 6, but we derive it here because it uses the same methods as the proof of Lemma 2. In accordance with (2.33) we use the notation

$$\mathcal{C}(r) = [-r,r]^d \cap \mathbb{Z}^d. \tag{2.34}$$

It is also convenient to have the A-system defined for negative times as well. Since the A-system is stationary in time such an extension exists, and in this extension $N_A(x,t)$ still has a Poisson ditribution with mean μ_A for all x,t, and all the A-particles perform independent simple random walks with jumprate D_A.

Lemma 4. *There exist constants $0 < C_1, C_2 < \infty$ such that almost surely for all large t*

$$\begin{aligned}&\textit{there are no A-particles which visit both } \mathcal{C}(\tfrac{1}{2}C_1 t\log t) \textit{ and}\\ &\qquad\qquad \textit{the complement of } \mathcal{C}(C_1 t\log t) \textit{ during } [0,t],\end{aligned} \tag{2.35}$$

$$\sup\{N_A(x,s) : x \in \mathcal{C}(C_1 t\log t), -t \le s \le t\} \le \log t, \tag{2.36}$$

$$\textit{for all } x \notin \mathcal{C}(C_1 t\log t),\ \sup\{N_A(x,s) : -t \le s \le t\} \le \log \|x\|, \tag{2.37}$$

and for any A-configuration which satisfies (2.35)-(2.37) it holds that

$$\begin{aligned}&E_B\{\textit{number of B-particles which are alive at } \mathbf{0} \textit{ at time } t \textit{ and whose associated}\\ &\qquad \textit{path has been outside } \mathcal{C}(\tfrac{1}{2}C_1 t\log t) \textit{ during } [0,t] \textit{ or coincided at some}\\ &\qquad \textit{time in } [0,t] \textit{ with an A-particle which has been outside } \mathcal{C}(C_1 t\log t)\}\\ &\le C_2 e^{-\delta t}.\end{aligned} \tag{2.38}$$

Proof. We need an estimate on the fluctuations of $\{S_s\}$. We shall write $\|a\|$ for $\|a\|_\infty = \max_{1\le s\le d}|a(s)|$ when $a = (a(1),\dots,a(d)$. First note a standard large deviation estimate: if $X_1, X_2, \dots$ are i.i.d. bounded, mean zero, d-dimensional random vectors, then

$$P\Big\{\Big\|\sum_{i=1}^n X_i\Big\| \ge x\Big\} \le d\exp\big[-K_1\frac{x^2}{n+x}\big], \quad x \ge 0, \tag{2.39}$$

for some constant K_1 depending on the common distribution of the X_i (apply for instance Chow and Teicher (1988), Exercise 4.3.14 to one coordinate at a time). Now, if X_i denotes the i-th jump of the random walk $\{S_s\}_{s\ge 0}$ and $N(t)$ the number of such jumps during $[0,t]$, then, for another constant $K_2 > 0$

$$\begin{aligned}P\{\|S_t\| \ge x\} &= \sum_{n=0}^\infty P\{N(t)=n\}P\Big\{\Big\|\sum_{i=1}^n X_i\Big\| \ge x\Big\}\\ &\le \sum_{n=0}^\infty e^{-tD_A}\frac{(tD_A)^n}{n!} d\exp\big[-K_1\frac{x^2}{n+x}\big]\\ &\le 2d\exp\big[-K_2\frac{x^2}{t+|x|}\big]\end{aligned} \tag{2.40}$$

(check this separately for $x \le 2tD_A$ and for $x > 2tD_A$). In particular,

$$P\{\|S_s\| \le K_3\sqrt{s+1}\} \ge \frac{1}{2}, \quad s \ge 0, \tag{2.41}$$

for yet another constant K_3. Finally, by the maximal inequality (Billingsley (1986), Theorem 22.5),

$$\begin{aligned} &P\{\sup_{0\le s_1,s_2\le t} \|S_{s_1} - S_{s_2}\| \ge x\} \\ &\le P\{\sup_{s\le t} P\{\|S_s\| \ge x/2\} \\ &\le 4\sup_{s\le t} P\{\|S_s\| \ge x/8\} \le 8d\exp\Big[-\frac{K_2x^2}{64(t+|x|)}\Big]. \end{aligned} \tag{2.42}$$

With the estimate (2.42) in hand we turn to the proof of (2.35). Note that the proof we are about to present remains valid no matter which strictly positive value we take for C_1. Let us estimate

$$\begin{aligned} &P\{\exists \text{ an } A\text{-particle which visits both } \mathcal{C}(\frac{1}{2}C_1k\log k) \\ &\quad\text{and the complement of } \mathcal{C}(\frac{3}{4}C_1k\log k) \text{ during } [0,k]\}. \end{aligned} \tag{2.43}$$

Suppose there is particle starting at x for which the event in (2.43) occurs. We consider two cases:
Case (i) $x \in \mathcal{C}(2C_1k\log k)$. In this case we use that the probability of visiting both the complement of $\mathcal{C}(\frac{3}{4}C_1k\log k)$ and $\mathcal{C}(\frac{1}{2}C_1k\log k)$ for a given particle at time 0 is at most

$$P\{\sup_{s_1\le s_2\le t} \|S_{s_1} - S_{s_2}\| \ge \frac{1}{4}C_1k\log k\} \le 8d\exp[-K_4k\log k], \tag{2.44}$$

by virtue of (2.42). The particles starting at some $x \in \mathcal{C}(2C_1k\log k)$ therefore contribute, for large k at most

$$\sum_{x\in\mathcal{C}(2C_1k\log k)} E_A\{N_A(x,0)\}8d\exp[-K_4k\log k] \le \exp[-\frac{K_4}{2}k\log k]$$

to the probability in (2.43) for large k.
Case (ii) $x \notin \mathcal{C}(2C_1k\log k)$. Any such x has at least one coordinate $> 2C_1k\log k$ in absolute value. For the sake of argument let $x_1 > 2C_1k\log k$. If a particle starting at such an x reaches $\mathcal{C}(\frac{1}{2}C_1k\log k)$ before time k, then the fluctuation in its path up till time k is at least $(x_1 - C_1/2)k\log k \ge (x_1/2)$. The probability of such a fluctuation is at most $8d\exp[-K_4x_1]$, again by virtue of (2.42). Consequently the particles starting at points x in case (ii) contribute at most

$$16d^2 \sum_{x:x_1>2C_1k\log k} \exp[-K_4x_1] \le \exp[-\frac{K_4}{2}k\log k]$$

to the probability in (2.43). The statement (2.35) now follows from these estimates and a straightforward Borel-Cantelli argument.

The statement (2.36) is much easier to prove. Again the value of $C_1 > 0$ has no influence on the argument. By stationarity, each $N_A(x,t)$ has a Poisson distribution of mean μ_A and generating function

$$E_Ae^{\theta N_A(x,t)} = \exp[\mu_A(e^\theta - 1)]. \tag{2.45}$$

This, together with (2.30), implies that for any $\theta > 0$

$$\begin{aligned}
&\sum_{x\in\mathcal{C}(C_1 t\log t)} P_A\{\sup_{|s|\le t} N_A(x,s) \ge \log\lfloor t\rfloor\} \\
&\le 2 \sum_{x\in\mathcal{C}(C_1 t\log t)} \sum_{|k|\le\lceil t\rceil} P_A\{N_A(x,k) \ge \frac{1}{2}e^{-D_A}\log\lfloor t\rfloor\} \\
&\le K_5 t[t\log t]^d \exp[-\frac{\theta}{2}e^{-D_A}\log\lfloor t\rfloor + \mu_A(e^\theta - 1)]. \qquad (2.46)
\end{aligned}$$

By taking $\theta = \log\log t$ we see that this is

$$O\Big(\exp[-K_6 \log t \log\log t]\Big)$$

as $t \to \infty$. (2.36) is now immediate by Borel-Cantelli again.

We leave the proof of (2.37) to the reader. The proof is essentially the same as for (2.36) and it is valid for any value of $C_1 > 0$.

Finally we turn to (2.38). It is here were we have to take C_1 large. We begin with an estimate for the number of descendants of some fixed particle σ which starts at a location y. Assume that the A-configuration satisfies (2.35)-(2.37). As before, the process of descendants of σ is denoted by W_σ. We further define

$$\mathcal{F}_{B,0} = \sigma\text{-field generated by } \{N_B(x,0) : x \in \mathbb{Z}^d\}.$$

We shall prove that for a suitable constant K_7 and for t large enough, and uniformly in the starting position y of σ

$$\begin{aligned}
&\mathbb{E}\{\text{number of particles alive in } W_\sigma \text{ at time } t \text{ such that the piece till } t \\
&\quad\text{of its associated path intersects } \mathcal{C}(\frac{1}{8}C_1 t\log t) \text{ and the complement of} \\
&\quad\mathcal{C}(\frac{1}{2}C_1 t\log t) \text{ during } [0,t] | \mathcal{F}_A \vee \mathcal{F}_{B,0}\} \\
&\quad\le e^{-\delta t}\exp\big[-K_7\|y\|\big]. \qquad (2.47)
\end{aligned}$$

To see this we use a decomposition with respect to the associated paths of the particles. As in (2.13) the left hand side of (2.47) can be written as

$$\begin{aligned}
&\sum_{\ell\ge 0}\sum_{x_1,\dots,x_\ell}\int_{0<s_1<\cdots<s_\ell<t} \mathbb{E}\{\text{number of particles which descend from } \sigma \\
&\quad\text{and whose associated path has jumptimes in } ds_1, ds_2, \dots, ds_\ell \\
&\quad\text{and positions } x_1, x_2, \dots, x_\ell \text{ after the jumps} | \mathcal{F}_A \vee \mathcal{F}_{B,0}\}, \qquad (2.48)
\end{aligned}$$

where the sum over the x_i runs over all ℓ-tuples with at least one x_i in $\mathcal{C}(\frac{1}{8}C_1 t\log t)$ and at least one $x_j \notin \mathcal{C}(\frac{1}{2}C_1 t\log t)$. Set $x_0 = y, s_0 = 0$ and

$$r = \max\{\|x_i\| : 0 \le i \le \ell\}. \qquad (2.49)$$

Then

$$r \ge \frac{1}{2}C_1 t\log t.$$

Then along the path which jumps to x_k at time s_k, $k \le \ell$, the birthrate per particle is never more than

$$\sup_{s\le t}\sup_{\|z\|\le r} \beta N_A(z,s) \le \beta \log r,$$

by virtue of (2.36) and (2.37). Therefore, by (2.6), the expectation appearing in (2.48) can only be increased if we replace W_σ by a branching random walk, $\widehat{W}_\sigma$ say, starting with σ and with birthrate per particle at site x and time s equal to

$$\begin{aligned} &\beta \log r \text{ if } \|x\| \le r,\ s \le t, \text{ and} \\ &\beta N_A(x,s) \text{ if } \|x\| > r \text{ or } s > t. \end{aligned} \tag{2.50}$$

Now our construction of the branching random walk is such that particles of $\widehat{W}_\sigma$ whose associated path lies in the ball $\{x : \|x\| \le r\}$ are not influenced by the birthrate at any site outside this ball. In other words, if we restrict ourselves to particles whose associated path lies in $\{x : \|x\| \le r\}$, then

$$\begin{aligned} &\mathbb{E}\{\text{number of particles in } \widehat{W}_\sigma \text{ which descend from } \sigma \text{ and whose} \\ &\quad \text{associated path has jumptimes in } ds_1, ds_2, \dots, ds_\ell \text{ and} \\ &\quad \text{positions } x_1, x_2, \dots, x_\ell \text{ after the jumps} | \mathcal{F}_A \vee \mathcal{F}_{B,0}\} \end{aligned}$$

is unchanged if we replace $\widehat{W}_\sigma$ by yet another branching random walk, $\overline{W}_\sigma$ say, starting from σ, but with the constant birthrate per particle of $\beta \log r$ at all sites and all times. However, we know from Lemma 1, and the arguments for (2.16), that

$$\begin{aligned} &\mathbb{E}\{\text{number of particles in } \overline{W}_\sigma \text{ which descend from } \sigma \text{ and whose} \\ &\quad \text{associated path has jumptimes in } ds_1, ds_2, \dots, ds_\ell \text{ and} \\ &\quad \text{positions } x_1, x_2, \dots, x_\ell \text{ after the jumps} | \mathcal{F}_A \vee \mathcal{F}_{B,0}\} \\ &\quad = \exp[-(D_B+\delta)t + \beta t \log r] \prod_{i=1}^{\ell} [D_B q(x_i - x_{i-1})] ds_1 \dots ds_\ell. \end{aligned}$$

It follows that the expression in (2.48) is bounded by

$$\sum_{\ell \ge 0} \sum_{x_1,\dots,x_\ell} \exp[-(D_B+\delta)t + \beta t \log r] \int_{0<s_1<\dots<s_\ell<t} \prod_{i=1}^{\ell} [D_B q(x_i - x_{i-1})] ds_1 \dots ds_\ell,$$

where the sum runs only over paths $x_1, \dots, x_\ell$ with some x_i in $\mathcal{C}(\frac{1}{8}C_1 t \log t)$ and some $x_j \notin \mathcal{C}(\frac{1}{2}C_1 t \log t)$, and r is given by (2.49). The last expression is simply an expectation over random walk paths starting at y. More precisely, it is at most

$$\begin{aligned} &e^{-\delta t} E\Big\{ \exp\big[\beta t \log[\sup_{s \le t} \|y + S_s\|]\big] \\ &\qquad \times I[y + S_{s_1} \in \mathcal{C}(\frac{1}{8}C_1 t \log t) \text{ and } y + S_{s_2} \notin \mathcal{C}(\frac{1}{2}C_1 t \log t) \text{ for some } s_1, s_2 \le t]\Big\} \\ &\le e^{-\delta t + \beta t \log(\|y\|+1)} E\Big\{ \exp\big[\beta t \log[\sup_{s \le t} \|S_s\|]\big] \\ &\qquad \times I[y + S_{s_1} \in \mathcal{C}(\frac{1}{8}C_1 t \log t) \text{ and } y + S_{s_2} \notin \mathcal{C}(\frac{1}{2}C_1 t \log t) \text{ for some } s_1, s_2 \le t]\Big\}. \end{aligned} \tag{2.51}$$

To estimate the right hand side here we set

$$\overline{F}(z) = P\{\sup_{s \le t} \|S_s\| > z\}.$$

By (2.42) we have for $z \ge t$ that $\overline{F}(z) \le 8d \exp[-K_8 z]$. Now take

$$C_1 \ge \frac{64\beta}{K_8}. \tag{2.52}$$

First consider sites y with $y \notin \mathcal{C}(\frac{1}{4}C_1 t \log t)$. For such $y = (y(1), \dots, y(d))$

$$\begin{aligned}
&E\Big\{ \exp\big[\beta t \log[\sup_{s\le t} \|S_s\|]\big] \\
&\qquad \times I[y + S_{s_1} \in \mathcal{C}(\frac{1}{8}C_1 t \log t) \text{ and } y + S_{s_2} \notin \mathcal{C}(\frac{1}{2}C_1 t \log t) \text{ for some } s_1, s_2 \le t]\Big\} \\
&\le E\Big\{ \exp\big[\beta t \log[\sup_{s\le t} \|S_s\|]\big] I[\sup_{s\le t} \|S_s\| \ge \max_i |y(i)| - \frac{1}{8}C_1 t \log t] > \|y\|/2\Big\} \\
&\le -\int_{(\|y\|/2, \infty)} e^{\beta t \log z} d\overline{F}(z) \\
&= e^{\beta t \log(\|y\|/2)} \overline{F}(\|y\|/2) + \int_{\|y\|/2}^{\infty} e^{\beta t \log z} \overline{F}(z) dz \\
&\le \exp[-\frac{1}{8}K_8 \|y\|].
\end{aligned} \tag{2.53}$$

Use of this bound in (2.51) shows that (2.47) holds for $y \notin \mathcal{C}(\frac{1}{4}C_1 t \log t)$. Next take $y \in \mathcal{C}(\frac{1}{4}C_1 t \log t)$. Now

$$\begin{aligned}
&E\Big\{ \exp\big[\beta t \log[\sup_{s\le t} \|S_s\|]\big] \\
&\qquad \times I[y + S_{s_1} \in \mathcal{C}(\frac{1}{8}C_1 t \log t) \text{ and } y + S_{s_2} \notin \mathcal{C}(\frac{1}{2}C_1 t \log t) \text{ for some } s_1, s_2 \le t]\Big\} \\
&\le E\Big\{ \exp\big[\beta t \log[\sup_{s\le t} \|S_s\|]\big] I[\sup_{s\le t} \|S_s\| \ge \frac{1}{4}C_1 t \log t]\Big\},
\end{aligned}$$

and the proof of (2.47) can be completed as in (2.53).

Thus, (2.47) holds in all cases. We now take the expectation over the $N_B(x, 0)$ and sum over all $y \in \mathbb{Z}^d$ to obtain for large t

$$\begin{aligned}
&E_B\{\text{number of } B\text{-particles which are alive at } \mathbf{0} \text{ at time } t \\
&\qquad \text{and whose associated path has been outside } \mathcal{C}(\frac{1}{2}C_1 t \log t) \text{ during } [0, t]\} \\
&\le C_2 e^{-\delta t}
\end{aligned} \tag{2.54}$$

for some constant C_2. However, when the A-system satisfies (2.35), then any B-particle which stays inside $\mathcal{C}(\frac{1}{2}C_1 t \log t)$ during $[0, t]$ cannot meet any A-particle which also visits $\mathcal{C}(C_1 t \log t)$ during $[0, t]$. Thus, under (2.35), the left hand side of (2.38) and (2.54) are the same and the lemma follows. ■

3. The expectation of N_B increases faster than exponentially in dimensions 1 and 2. Here we shall prove Theorem 1. In fact we shall prove a slightly stronger result.

Proof of Theorem 1. Assume that there is some B-particle, σ say, at the origin at time 0. Consider the subprocess of those B-particles which are descendants of σ and such that they and all their ancestors back to time 0 have never moved away from the origin. Conditionaly on the A-system, this is a branching process with birthrate $\beta N_A(\mathbf{0}, s)$ at time s and constant death rate $\delta + D_B$ (compare Remark 1). From Lemma 1 we see that the conditional expectation, given $\mathcal{F}_A$ and $N_B(\mathbf{0}, 0)$, of the number of B-particles at $\mathbf{0}$ at time t in this subprocess is

$$N_B(\mathbf{0}, 0) \exp\Big\{ \int_0^t [\beta N_A(\mathbf{0}, s) - (\delta + D_B)] ds \Big\}.$$

Consequently

$$\mathbb{E} N_B(\mathbf{0}, t) \geq \mu_B e^{-(\delta + D_B)t} E_A \Big\{ \exp\big\{\beta \int_0^t N_A(\mathbf{0}, s) ds\big\} \Big\}.$$

It therefore suffices for (1.3) to prove that

$$E_A \Big\{ \exp\big\{\beta \int_0^t N_A(\mathbf{0}, s) ds\big\} \Big\} \text{ goes to } \infty \text{ faster than exponentially in } t. \tag{3.1}$$

Now consider the A-particles which start somewhere at time 0 and reach the origin at some time during $[0, t/2]$. Let us call the first arrival of such a particle at the origin a *new arrival* and let there be $\nu = \nu_t$ such arrivals and assume that they occur at the times $0 \leq t_1 \leq t_2 \leq \cdots \leq t_\nu \leq t/2$. Write λ_i for the particle which arrives at time t_i and let T_i be the amount of time λ_i spends at the origin during $[0, t]$, which is the same as the amount of time spent at the origin during $[t_i, t]$. By definition all the particles λ_i are distinct. Therefore, conditionally on the t_i, the random variables $T_i, i = 1, 2, \dots,$ are independent. Moreover, the conditional distribution of T_i is the distribution of the time spent at the origin during $[t_i, t]$ by an A-particle which is at the origin at time t_i. This distribution is the same as that of the time spent at the origin during $[0, t - t_i] \supset [0, t/2]$ by a particle starting at the origin. Let us write $G = G_t$ for the distribution of the total time spent at the origin during $[0, t/2]$ by an A-particle starting at the origin. Then, conditionally on the t_i, and the number ν of t_i, $\sum_i T_i$ is stochastically larger than the sum of ν i.i.d. variables each with the distribution G. Now

$$\int_0^t N_A(\mathbf{0}, s) ds \geq \sum_i (\text{time spent by } \lambda_i \text{ at } \mathbf{0} \text{ during } [0, t]),$$

so that also $\int_0^t N_A(\mathbf{0}, s) ds$ is stochastically larger than the sum of ν independent random variables with distribution G. In particular,

$$\begin{aligned} &E_A \Big\{ \exp\big\{\beta \int_0^t N_A(\mathbf{0}, s) ds\big\} \Big\} \\ &\geq E_A \Big\{ \big[\int e^{\beta u} G(du) \big]^\nu \Big\} \\ &\geq \Big[\int e^{\beta u} G(du) \Big]^{E_A \nu} \text{ (by Jensen's inequality).} \end{aligned} \tag{3.2}$$

Now recall that each A-particle performs a simple random walk. Since simple random walk is recurrent in dimension 1 and 2, each A-particle does eventually reach the origin, so that $\nu_t \to \infty$ almost surely as $t \to \infty$. Consequently also

$$E_A \nu_t \to \infty \text{ as } t \to \infty. \tag{3.3}$$

In order to prove (3.1) we now prove that

$$\int e^{\beta u} G_t(du) \geq e^{\beta \eta t/2} \tag{3.4}$$

for some constant $\eta > 0$. Clearly (3.2)-(3.4) will imply (3.1) and hence (1.3).

To obtain the lower bound for $\int \exp(\beta u) G(du)$, consider a particle which starts at the origin and whose position at time s is given by S_s with $\{S_s\}$ as in the lines following (2.24). Then the time axis is partitioned into alternating intervals with $S_s = \mathbf{0}$ and with $S_s \neq \mathbf{0}$. More precisely, there exist $a_1 = 0 < b_1 < a_2 < b_2 < \dots$ such that

$$S_s \begin{cases} = \mathbf{0} & \text{if } a_i \le s < b_i \text{ for some } i \\ \neq \mathbf{0} & \text{if } b_i \le s < a_{i+1} \text{ for some } i. \end{cases}$$

The amount of time spent at the origin during $[0, t/2]$ by the particle is then at least

$$\sum_{i \le \rho} [b_i - a_i],$$

where

$$\rho := \max\{i : b_i \le t/2\}.$$

Moreover, $b_i - a_i$, $i \ge 0$, $a_{j+1} - b_j$, $j \ge 0$ are all independent and the $b_i - a_i$ have an exponential distribution with mean $1/D_A$. It particular, by the weak law of large numbers, there exists some M such that

$$P\Big\{\sum_{i=1}^m [b_i - a_i] \in [\frac{m}{2D_A}, \frac{2m}{D_A}]\Big\} \ge \frac{1}{2} \text{ for } m \ge M. \tag{3.5}$$

Thus, for any fixed $\eta, K > 0$ with

$$0 < (3KD_A + 4)\eta < \frac{1}{2} \tag{3.6}$$

and t large, we have

$$\begin{aligned} &P\{\text{time spent at the origin during } [0, t/2] \text{ by the particle } \ge \eta t\} \\ &\ge P\{\rho \ge 2D_A \eta t \text{ and } \sum_{i \le 2D_A \eta t} (b_i - a_i) \ge \eta t\} \\ &\ge P\{(a_{i+1} - b_i) \le K \text{ for all } i \le 2D_A \eta t\} P\{\eta t \le \sum_{i \le 2D_A \eta t} (b_i - a_i) \le 4\eta t\} \\ &\ge \Big[P\{a_2 - b_1 \le K\}\Big]^{2D_A \eta t} \frac{1}{2}. \end{aligned} \tag{3.7}$$

The one but last inequality in (3.7) holds because

$$(a_{i+1} - b_i) \le K \text{ for all } i \le 2D_A \eta t$$

together with

$$\sum_{i \le 2D_A \eta t} (b_i - a_i) \le 4\eta t$$

implies that

$$b_{\lceil 2D_A \eta t \rceil} \le K\big[2D_A \eta t + 1\big] + 4\eta t < \frac{t}{2},$$

and consequently $\rho \ge 2D_A \eta t$. We have also used the independence of the $b_i - a_i$ and the $a_{j+1} - b_j$ in this inequality. The last inequality in (3.7) follows from (3.5).

(3.7) implies that

$$\int e^{\beta u} G(du) \ge \frac{1}{2} \exp[\beta \eta t] \Big[P\{a_2 - b_1 \le K\}\Big]^{2D_A \eta t}.$$

To satisfy (3.6) we take

$$K = K(\eta) = \Big\lfloor \frac{1 - 10\eta}{6D_A \eta} \Big\rfloor.$$

By the recurrence of $\{S_s\}$, the variables $b_{i+1} - a_i$ are all finite with probability 1 and consequently $P\{a_2 - b_1 \le K(\eta)\} \to 1$ as $\eta \downarrow 0$. We can therefore choose η so small that (3.4) holds for large t. ■

4. An upper bound for the expectation of N_B in dimensions ≥ 3.

In the preceding section we derived a lower bound for $\mathbb{E}N_B(\mathbf{0},t)$. In this section we shall use an upper bound for this expectation to derive Theorem 3.

Proof of Theorem 3. By (2.31)

$$\begin{aligned}\mathbb{E}N_B(\mathbf{0},t) = \mathbb{E}(&\text{number of descendants of some particle at } \mathbf{0}\\ &\text{which are alive at time } t).\end{aligned} \tag{4.1}$$

By (2.13)-(2.20) we then have (in the notation of Lemma 2)

$$\begin{aligned}\mathbb{E}N_B(\mathbf{0},t) =& \mu_B \sum_{\ell\geq 0}\sum_{x_1,\dots,x_\ell} \exp[-(\delta+D_B)t]\prod_{i=1}^{\ell}[D_B q(x_i-x_{i-1})]\\ &\times \int\limits_{0<s_1<\dots<s_\ell<t} ds_1\dots ds_\ell E_A\Big\{\prod_\lambda E_A\{\exp[\beta L(\lambda,\pi)]|\mathcal{F}_{A,0}\}\Big\}.\end{aligned} \tag{4.2}$$

Furthermore, (2.21) and (2.22) say

$$\begin{aligned}E_A\{&\exp[\beta L(\lambda,\pi)]\mathcal{F}_{A,0}\} = P_A\{\kappa(\lambda,\pi)>t|\mathcal{F}_{A,0}\}\\ &+\int_0^t P_A\{\kappa(\lambda,\pi)\in du|\mathcal{F}_{A,0}\}\\ &\times\sum_{p=0}^\infty \beta^p \int\limits_{0<v_1<v_2<\dots<v_p\leq t-u} P_A\{\lambda' \text{ is on } \pi' \text{ at times } v_1,\dots,v_p|\mathcal{F}_{A,0}\}dv_1\dots dv_p.\end{aligned} \tag{4.3}$$

For brevity denote the position of π' at time v_j by z_j and set $v_0=0, z_0=\mathbf{0}$. Note the simple inequality

$$\sup_z P\{S_s=z\}\leq \sup_z P\{S_{\lfloor s\rfloor}=z\} = \sup_z P\Big\{\sum_{k=1}^{\lfloor s\rfloor}[S_k-S_{k-1}]=z\Big\}.$$

Together with the local central limit theorem (see Spitzer (1976), Proposition 7.9 and the remark following it) this implies that there exists a constant K_1 such that

$$\begin{aligned}P\{\lambda' &\text{ is at } z_j \text{ at time } v_j|\lambda' \text{ is at } z_\ell \text{ at time } v_\ell \text{ for } \ell\leq j-1\}\\ &= P\{S_{v_j-v_{j-1}}=z_j-z_{j-1}\}\leq \frac{K_1}{[v_j-v_{j-1}+1]^{d/2}}.\end{aligned}$$

We can therefore bound the last sum in the right hand side of (4.3) by

$$\begin{aligned}&\sum_{p=0}^\infty \beta^p \int\limits_{0<v_1<v_2<\dots<v_p\leq t-u} \prod_{j=1}^p P\{S_{v_j-v_{j-1}}=z_j-z_{j-1}\}dv_1\dots dv_p\\ &\leq \sum_{p=0}^\infty \beta^p \int\limits_{0<v_1<v_2<\dots<v_p\leq t-u} \prod_{j=1}^p \frac{K_1}{[v_j-v_{j-1}+1]^{d/2}}dv_1\dots dv_p\\ &\leq \sum_{p=0}^\infty \beta^p\Big[\int_0^\infty \frac{K_1}{(v+1)^{d/2}}dv\Big]^p.\end{aligned} \tag{4.4}$$

If we define β_0 by

$$2\beta_0 K_1 \int_0^\infty \frac{dv}{(v+1)^{d/2}} = 1,$$

then $\beta_0 > 0$ (recall that $d \geq 3$ now). Moreover the right hand side of (4.4) will be ≤ 2 for $\beta \leq \beta_0$. Substitution of this bound into (4.3) gives

$$\begin{aligned} &E_A\{\exp[\beta L(\lambda,\pi)]|\mathcal{F}_{A,0}\} \\ &\qquad \leq P\{\kappa(\lambda,\pi) > t|\mathcal{F}_{A,0}\} + P\{\kappa(\lambda,\pi) \leq t|\mathcal{F}_{A,0}\}2. \end{aligned}$$

As in (2.24) and (2.25) (with $e^{\beta t}$ replaced by 2) this gives

$$\begin{aligned} &E_A\Big\{\prod_\lambda E_A\{\exp[\beta L(\lambda,\pi)]\big|\mathcal{F}_{A,0}\}\Big\} \\ &\leq \exp\big[K_2 \sum_x P\{x + S_s \text{ is on } \pi \text{ for some } s \leq t\}\big] \\ &\leq \exp[K_2(tD_A + \ell)], \end{aligned}$$

for some constant K_2 which does not depend on t this time (use (2.26)). Finally, (4.2) shows that

$$\begin{aligned} \mathbb{E}N_B(\mathbf{0},t) &\leq \mu_B \sum_{\ell\geq 0} \frac{(D_Bt)^\ell}{\ell!} \exp\big[-(\delta+D_B)t + K_2(tD_A+\ell)\big] \text{ (compare (2.27))} \\ &= \mu_B \exp\big[\big(D_B e^{K_2} + K_2 D_A - \delta - D_B\big)t\big]. \end{aligned}$$

This implies (1.6) for $\beta \leq \beta_0, \delta \geq \delta_0$ if we take $\delta_0 = 2(D_B e^{K_2} + K_2 D_A - D_B)$.

Finally we show that (1.6) implies (1.7). Clearly (1.6) and the Borel-Cantelli lemma show that for each fixed $x \in \mathbb{Z}^d$, almost surely $N_B(x,k) = 0$ for all large integers k. But if $\mathcal{G}(t)$ denotes the σ-field generated by the A-system for all times and the B-system up till time t, then one easily sees that for $k-1 < t \leq k$ it holds that

$$P\{N_B(x,k) \geq 1|\mathcal{G}(t)\} \geq e^{-\delta-D_B} \text{ a.e. on } \{N_B(x,t) \geq 1\}.$$

Indeed, no matter what the configuration at time t is, as long as there is at least one B-particle at x at time t, there is a conditional probability of at least $\exp[-(\delta+D_B)(k-t)] \geq \exp[-\delta - D_B]$ that this particle survives and does not move till time k (see the argument for (2.32)). The same argument shows that

$$P\{N_B(x,\lceil\tau(t)\rceil) \geq 1\big|\mathcal{G}(\tau(t)\} \geq e^{-\delta-D_B} \text{a.e. on } \{\tau(t) < \infty\},$$

where $\tau(t) = \inf\{s \geq t : N_B(x,s) \geq 1\}$. But then the martingale convergence theorem implies that $N_B(x,k) \geq 1$ for infinitely many k almost everywhere on the event $\{N_B(x,t) \geq 1 \text{ for arbitrarily large } t\}$ (see Breiman (1968), Exercise 5.6.9). Thus (1.7) must hold. ■

5. A discrete time approximation.

We begin with an **outline of the proof of a discrete time version of Theorem 2.** This section can be viewed as a warm-up exercise for the full proof of Theorem 2. The principal ideas for the full proof are all present here and will come out clearer because we make the simplification that both types of particles perform discrete time random walks. To avoid periodicity problems we shall allow these random walks to stand still. That is, an A-particle at x during $[k-1,k)$ will at time k stay at x with probability $1 - D_A$, and jump to $x \pm e_i$ with probability $D_A q(\pm e_i) = D_A/(2d)$ (with $q(\cdot)$ as in (2.14)). (We take our path right continuous, so that at time k the particle is at its new position. We also assume that $0 < D_A, D_B < 1$ now.) All A-particles move independently of each other. As before we write $N_A(x,k)$ and $N_B(x,k)$ for the number of A and B-particles at x at time k, respectively. Assume now that the A-configuration is given.

A B-particle which is at x during $[k-1,k)$ will then produce i B-particles at time k with probability α_i. These probabilities satisfy

$$\alpha_i \geq 0, \quad \sum_{i=0}^{\infty} \alpha_i = 1 \text{ and } \sum_{i=0}^{\infty} i\alpha_i = e^{-\delta+\beta N_A(x,k-1)}. \tag{5.1}$$

Thus the expected number of B-particles at time k produced by a B-particle at x at time $k-1$ is $\exp[-\delta+\beta N_A(x,k-1)]$, which imitates the situation in the true model (compare Lemma 1). The new B-particles, if any, will then independently stay at x with probability $1-D_B$, or make a jump of size $\pm e_i$ with probability $D_B q(\pm e_i) = D_B/(2d)$. They then stay at their new location for the whole interval $[k,k+1)$. Only the expectations $\sum_i i\alpha_i$ will play a role in this section, and not the values of the α_i themselves.

The proof of (1.4) rests on showing that for almost all A-configurations, the conditional expectation $E\{N_B(\mathbf{0},t)|\mathcal{F}_A\}$ does not grow faster than exponentially in t. Then the process must die out if the death rate δ is made greater than the rate at which $E\{N_B(\mathbf{0},t)|\mathcal{F}_A\}$ grows. It is easy to give a fairly explicit expression for this last conditional expectation (see (5.2) below). This expression shows that for our purposes it is enough to show that $\sum_{\ell=0}^{t-1} N_A(x_\ell,\ell)$ does not grow faster than Kt for some constant K, along most paths $(x_0,x_1,\dots,x_t)$ in $[-t,t]^d$. Here we call $(x_0,x_1,\dots,x_t)$ a path if $x_{\ell+1}-x_\ell \in \{\mathbf{0},\pm e_i,\ 1\leq i\leq d\}$ for each ℓ. Of course $E_A\big\{\sum_{\ell=0}^{t-1} N_A(x_\ell,\ell)\big\} = \mu_A t$ for any path $(x_0,\dots,x_t)$ so that we have to establish some kind of large deviation estimate for the A-system. We have to show that there are "few paths" in space-time with $\sum_{\ell=0}^{t-1} N_A(x_\ell,\ell)$ much larger than its expectation $\mu_A t$. Note that $E\{N_B(\mathbf{0},t)|\mathcal{F}_A\}$ contains a sum of exponentials like $\exp[\beta\sum_{\ell=0}^{t-1} N_A(x_\ell,\ell)]$, so it is quite posible for $\sum_{\ell=0}^{t-1} N_A(x_\ell,\ell)$ *typically* to be of order t and still to have $E\{N_B(\mathbf{0},t)\}$ grow faster than exponentially (as asserted in $d=1,2$ by Theorem 1).

Now to control $\sum_{\ell=0}^{t-1} N_A(x_\ell,\ell)$ we break up this sum into subsums; the r-th subsum contains the terms for which $\gamma_r\mu_A C_0^{dr} < N_A(x_\ell,\ell) \leq \gamma_{r+1}\mu_A C_0^{d(r+1)}$. Here γ_r is more or less constant, and C_0 is some large integer, so we should think of the sizes of the terms in the subsums as growing exponentially in r. We have to show (at least) that there are only $o(tC_0^{-dr})$ terms in the r-th subsum. To this end we partition space-time, $\mathbb{Z}^d\times\mathbb{Z}$, into the cubes

$$\mathcal{B}_r(\mathbf{i},k) := \Big(\prod_{s=1}^{d}[i(s)\Delta_r,(i(s)+1)\Delta_r\Big)\times[k\Delta_r,(k+1)\Delta_r),$$

with $\Delta_r = C_0^{6r}$ and $\mathbf{i}=(i(1),\dots,i(d))\in\mathbb{Z}^d, k\in\mathbb{Z}$. We call these cubes *$r$-blocks*. We divide the r-blocks into bad ones and good ones. Roughly speaking, an r-block is bad if it contains a vertex $(x,k)\in\mathbb{Z}^d\times\mathbb{Z}$ for which $N_A(x,k) > \gamma_r\mu_A C_0^{dr}$, but the actual definition of a bad r-block is somewhat more complicated. For any path $\pi=(x_0,\dots,x_t)$ of length t we then define $\phi_r(\pi)$ as the number of bad r-blocks which intersect the space-time path $\big((x_0,0),(x_1,1),\dots,(x_t,t)\big)$. We further define $\Phi_r = \sup\{\phi_r(\pi):\pi\subset[-t,t]^d\}$. It is then not hard to see that $\sum_{\ell=0}^{t-1} N_A(x_\ell,\ell) \leq \gamma_1\mu_A C_0^d t + \sum_{r\geq 1}\Phi_r\gamma_{r+1}\mu_A C_0^{r(d+6)+d}$ for any path $\pi\subset[-t,t]^d$. Thus our problem reduces to finding a good estimate for Φ_r. The first observation is that we can take $\Phi_r=0$ for $r\geq R$, where R is the smallest integer with $C_0^R \geq [\log t]^{1/d}$. This is so, because $N_A(x,k)\leq \log t$ for all $(x,k)\in[-t,t]^d\times[0,t]$ (outside an event of negligable probability). This rests on a coarse estimate for the Poisson distribution (recall that each $N_A(x,k)$ has a Poisson distribution with mean μ_A). The principal step in the proof is a recursive inequality $\Phi_r \leq C_0^{6(d+1)}\Phi_{r+1}+$ a random term. The first term in the right hand side is an upper bound for the number of bad r-cubes inside bad $(r+1)$-cubes, which simply comes from the fact that any $(r+1)$-cube contains exactly $C_0^{6(d+1)}$ r-cubes. The random term in the right hand side is the supremum over π of the number of bad r-cubes which intersect π and which lie in a good $(r+1)$-cube. To control this, we show, roughly speaking, that the collection of indicator functions of good $(r+1)$-cubes which contain some bad r-cube is bounded above by an independent collection of zero-one valued random variables with a small probability of taking the value 1. This small probability is basically the probability of finding a bad r-cube inside a good $(r+1)$-cube, which is estimated by routine large deviation estimates for a random walk. Finally some percolation estimates then show that with high probability there are very few good $(r+1)$-cubes which contain any bad r-cubes. This gives the desired recursive bound on Φ_r.

Iteration of the recursive bound finally estimates Φ_r in terms of Φ_R, which is zero, and controllable quantities. This results in a bound on $\sum_{\ell=0}^{t-1} N_A(x_\ell, \ell)$ which is linear in t.

We now turn to the details of the proof. In this section t will be an integer. We call $(x_0, \dots, x_t)$ a *path* if $x_\ell - x_{\ell-1}$ equals $\mathbf{0}$ or $\pm e_i$ for $1 \le \ell \le t$. We define the *piece till time t of the associated path* (or just the associated path, for short) of a particle ρ, alive at time t, to be the path $\{x(\ell,\rho)\}_{0\le \ell \le t}$, where $x(\ell,\rho)$ is the position of the ancestor of ρ alive at time ℓ. We shall use the notation q_A and q_B to denote the "jump probabilities" for the A and B-particles respectively, that is

$$q_A(y) = \begin{cases} 1 - D_A & \text{if } y = \mathbf{0} \\ D_A/(2d) & \text{if } y = \pm e_i,\ 1 \le i \le d, \end{cases}$$

and similarly when A is replaced by B. $\{\widetilde{S}_k\}_{k\ge 0}$ will be a typical random walk as performed by the A-particles, that is $\widetilde{S}_k$ will be the sum of k independent random variables, each with the distribution q_A.

It is then easy to see that for any given B-particle σ at x_0 at time 0, and for any path $(x_0, x_1, \dots, x_t)$, the expected number of descendants of σ alive at time t and whose associated path is $(x_0, \dots, x_t)$, equals

$$\Big[\prod_{i=1}^t q_B(x_i - x_{i-1})\Big] \exp[-t\delta + \beta \sum_{q=0}^{t-1} N_A(x_q, q)]. \tag{5.2}$$

The initial configuration is chosen in the same way as before, that is, all the $N_A(x,0), x \in \mathbb{Z}^d$, are independent Poisson variables with mean μ_A, and the $N_B(y,0)$, $y \in \mathbb{Z}^d$, are independent of the A-system, translation invariant, and with mean μ_B. Finally, $\mathbb{P}, \mathbb{E}, \mathcal{F}_A, P_A, P_B, E_A$ and E_B have the same meaning as before.

We shall prove the following result for this system:

Theorem 2-Discrete. *For all $\beta, D_A, D_B, \mu_A, \mu_B > 0$ and for all dimensions d, there exists a $\delta_0 < \infty$ such that for $\delta \ge \delta_0$ and for almost all $[P_A]$ A-configurations, it holds*

$$\mathbb{E}\{N_B(\mathbf{0}, t) | \mathcal{F}_A\} \le \mu_B e^{-\delta t/2} \text{ for all large } t, \tag{5.3}$$

and for all x

$$N_B(x,t) = 0 \text{ for all large } t \text{ a.e. } [P_B] \tag{5.4}$$

(now with N_A and N_B referring to the discrete time system and t restricted to integer values).

We do need a fair amount of new notation. We fix $\gamma_0 > 0$ such that

$$\gamma_0 \prod_{j=1}^\infty [1 - 2^{-j/4}] \ge 2 \text{ and } \mu_A\Big\{\gamma_0 \prod_{j=1}^\infty [1 - 2^{-j/4}] - e + 1\Big\} > 3d + 6. \tag{5.5}$$

We next fix a large integer $C_0 \ge 2$, so that for all $r \ge 1$

$$-C_0^{-r/2} + \Big(1 + \frac{C_3 (r \log C_0)^d}{C_0^r}\Big)(e^{C_0^{-r/2}} - 1)\big(1 - C_0^{-r/4}\big) \le -\frac{1}{2} C_0^{-3r/4}, \tag{5.6}$$

and

$$3^{d+1} C_0^{6(d+1)(r+1)} \exp\big[-\mu_A C_0^{(d-\frac{3}{4})r}\big] \le 1, \tag{5.7}$$

where C_3 is the constant of Lemma 6 below (this C_3 depends only on the distribution q_A and the dimension d, but not on C_0). We shall use that (5.5) and $C_0 \ge 2$ imply that

$$\gamma_0 \prod_{j=1}^\infty \Big[1 - \frac{1}{C_0^{j/4}}\Big] \ge 2 \tag{5.8}$$

and

$$\mu_A\Big\{\gamma_0 \prod_{j=1}^{\infty}\Big[1-\frac{1}{C_0^{j/4}}\Big]-e+1\Big\} > 3d+6. \tag{5.9}$$

We next take $\gamma_1 = \gamma_0$ and for $r > 0$

$$\gamma_{r+1} = \gamma_0 \prod_{j=1}^{r}\Big[1-\frac{1}{C_0^{j/4}}\Big]. \tag{5.10}$$

We set

$$\Delta_r = C_0^{6r}.$$

For $\mathbf{i} = (i(1), i(2), \dots, i(d)) \in \mathbb{Z}^d$ and $k \in \mathbb{Z}$, we take

$$\mathcal{B}_r(\mathbf{i}, k) = \Big(\prod_{s=1}^{d}[i(s)\Delta_r, (i(s)+1)\Delta_r)\Big) \times [k\Delta_r, (k+1)\Delta_r).$$

FIGURE 1. Relative location of the sets $\mathcal{B}_r, \widetilde{\mathcal{B}}_r^+, \widetilde{\mathcal{B}}_r$ and $\mathcal{V}_r$ for $d = 1$ ($\widetilde{\mathcal{B}}_r^+$ is defined just before Lemma 7). These sets are "left closed, right open", that is, the solid segments are in the sets, but the dashed segements are not. The space and time directions are along the horizontal and vertical axes, respectively. $\mathcal{V}_r$ is the line segment which constitutes the bottom of $\widetilde{\mathcal{B}}_r$.

The last coordinate is interpreted as the time coordinate here, so that $\mathcal{B}_r$ is a block in space-time. We call these blocks r-*blocks*. We also need somewhat larger space-time blocks, which contain the r-blocks (see Figure 1). We define these as

$$\widetilde{\mathcal{B}}_r(\mathbf{i}, k) = \Big(\prod_{s=1}^{d}[(i(s)-3)\Delta_r, (i(s)+4)\Delta_r)\Big) \times [(k-1)\Delta_r, (k+1)\Delta_r).$$

The *pedestal* of $\mathcal{B}_r(\mathbf{i})$ is defined as

$$\mathcal{V}_r(\mathbf{i}, k) = \Big(\prod_{s=1}^{d}[(i(s)-3)\Delta_r, (i(s)+4)\Delta_r)\Big) \times \{(k-1)\Delta_r\}.$$

This is again a set in space-time, but with the time taking only the value $(k-1)\Delta_r$. It is the face of $\widetilde{\mathcal{B}}_r(\mathbf{i}, k)$ with the lowest time coordinate.

For $x = (x(1), \dots, x(d)) \in \mathbb{Z}^d$ we introduce the notation

$$\mathcal{Q}_r(x) = \prod_{s=1}^{d} [x(s), x(s) + C_0^r).$$

Note that the edge size of the cube $\mathcal{Q}_r$ is only C_0^r and not Δ_r. We also introduce the random variables

$$U_r(x,v) = \sum_{y \in \mathcal{Q}_r(x)} N_A(y,v) = \sum_{\substack{y: x(s) \le y(s) < x(s) + C_0^r \\ 1 \le s \le d}} N_A(y,v).$$

An r-block $\mathcal{B}_r(\mathbf{i}, k)$ is called *good* if

$$U_r(x,v) \le \gamma_r \mu_A C_0^{dr} \text{ for all } (x,v) \text{ for which } \mathcal{Q}_r(x) \times \{v\} \subset \widetilde{\mathcal{B}}_r(\mathbf{i}, k). \tag{5.11}$$

A *bad r-block* is one that is not good. Similarly, the pedestal $\mathcal{V}_r(\mathbf{i}, k)$ is called good if

$$U_r(x,v) \le \gamma_r \mu_A C_0^{dr} \text{ for all } (x,v) \text{ for which } \mathcal{Q}_r(x) \times \{v\} \subset \mathcal{V}_r(\mathbf{i}, k). \tag{5.12}$$

If $\pi = (x_0, x_1, \dots, x_t)$ is a path on $\mathbb{Z}^d$ of $(t+1)$ vertices contained in $[-t,t]^d$, then we define

$$\begin{aligned} \phi_r(\pi) = {} & \text{number of bad } r\text{-blocks which intersect} \\ & \qquad \text{the space-time path } \big((x_0,0), (x_1,1), \dots, (x_t,t)\big), \end{aligned} \tag{5.13}$$

and finally

$$\Phi_r = \sup_\pi \phi_r(\pi). \tag{5.14}$$

Here and in the rest of this proof π runs through all paths $(x_0, \dots, x_t)$ on $\mathbb{Z}^d$ of $(t+1)$ vertices and contained in $[-t,t]^d$. Many of these quantities depend on t, but we will suppress the t in the notation. The space-time path corresponding to $\pi = (x_0, \dots, x_t)$ is $\widehat{\pi} := ((x_0,0), (x_1,1), \dots, (x_t,t))$. As before we extend the A-system so that it is defined for all times in $\mathbb{Z}$ and is stationary in time.

The reason for introducing ϕ and Φ is the following lemma, which gives a simple bound for $E_B N_B(\mathbf{0}, t)$.

Lemma 5. *Let $R = R(t)$ be such that*

$$C_0^R \ge [\log t]^{1/d} > C_0^{R-1}. \tag{5.15}$$

Then, for all large t,

$$\begin{aligned} & P_A\{\Phi_r > 0 \textit{ for any } r \ge R\} \\ & \le P_A\{\textit{some } r\textit{-block which intersects } [-t,t]^d \times [0,t] \textit{ is bad and has } r \ge R\} \\ & \le \frac{1}{t^2}. \end{aligned} \tag{5.16}$$

Moreover, for any A-configuration with $\Phi_r = 0$ for all $r \ge R(t)$, it holds for any path $\pi = (x_0, \dots, x_t) \subset [-t,t]^d$

$$\begin{aligned} & E_B\{\textit{number of } B\textit{-particles alive at time } t \textit{ with associated path } \pi | \mathcal{F}_A\} \\ & \le \mu_B \Big[\prod_{i=1}^t q_B(x_i - x_{i-1}) \Big] \exp\Big\{ -t\delta + \beta t \gamma_1 \mu_A C_0^d + \beta \sum_{r=1}^{R-1} \gamma_{r+1} \mu_A C_0^{r(d+6)+d} \Phi_r \Big\}. \end{aligned} \tag{5.17}$$

Proof. The first statement follows from a straighforward estimate for the Poisson distribution. Indeed, for any r-block $\mathcal{B}_r(\mathbf{i},k)$ we have

$$P_A\{\mathcal{B}_r(\mathbf{i},k) \text{ is bad}\} \le \sum_{(\mathbf{i},k)} P_A\{U_r(x,\ell) > \gamma_r \mu_A C_0^{dr}\}, \tag{5.18}$$

where $\sum_{(\mathbf{i},k)}$ denotes the sum over all (x,ℓ) for which $\mathcal{Q}_r(x)\times\{\ell\}$ is contained in $\widetilde{\mathcal{B}}_r(\mathbf{i},k)$. The number of such summands is at most $K_1\Delta_r^{d+1} = K_1 C_0^{6r(d+1)}$, for some constant $K_1 = K_1(d)$. On the other hand, for fixed (x,ℓ), $U_r(x,\ell)$ has a Poisson distribution with mean $\mu_A C_0^{dr}$, so that every summand in (5.18) is, by standard large deviation estimates (compare (2.45), (2.46)), at most

$$\exp\big[-\theta_0\gamma_r\mu_A C_0^{dr} + \mu_A C_0^{dr}(e^{\theta_0}-1)\big] \le \exp[-(d+6)C_0^{dr}], \tag{5.19}$$

where we took $\theta_0 > 0$ so that

$$\mu_A[\theta_0\gamma_r - (e^{\theta_0}-1)] > d+6$$

(e.g. $\theta_0 = 1$ is possible, by our choice of the γ; see (5.9)). In particular for large t and $r \ge R(t)$, the right hand side of (5.18) is at most

$$K_1 C_0^{6r(d+1)} \exp[-(d+6)C_0^{dr}] \le \exp[-(d+5)C_0^{dr}] \le \frac{1}{t^{(d+4)}}\exp[-C_0^{dr}].$$

Thus, for $r \ge R$, the probability that any particular r-block is bad is at most $t^{-(d+4)}\exp[-C_0^{dr}]$. Any r-block which intersects $\widehat{\pi}$ must intersect $[-t,t]^d\times[-t,t]$, and there are at most $(2t+1)^{d+1}$ such blocks. Thus the left hand side of (5.16) is bounded by $\sum_{r\ge R}(2t+1)^{d+1}t^{-(d+4)}\exp[-C_0^{dr}]$, and (5.16) follows.

Now fix an A-configuration with $\Phi_r = 0$ for all $r \ge R$. In addition let $\pi = (x_0,\dots,x_t)$ be a path in $[-t,t]^d$. Then the left hand side of (5.17) is given by μ_B times (5.2), so we must estimate

$$\sum_{q=0}^{t-1} N_A(x_q,q). \tag{5.20}$$

We decompose this sum according to the size of the $N_A(x_q,q)$. The $N_A(x_q,q)$ which are $\le \gamma_1\mu_A C_0^d$ contribute at most $t\gamma_1\mu_A C_0^d$. If $r \ge 1$ and

$$\gamma_r\mu_A C_0^{dr} < N_A(x_q,q) \le \gamma_{r+1}\mu_A C_0^{d(r+1)}, \tag{5.21}$$

then

$$U_r(x_q - z, q) = \sum_{y\in\mathcal{Q}_r(x_q-z)} N_A(y,q) \ge N_A(x_q,q) > \gamma_r\mu_A C_0^{dr},$$

for any choice of $z \in [0, C_0^r)^d$. For some such z, $\mathcal{Q}_r(x-z)\times\{q\}$ is contained in some r-block $\mathcal{B}(\mathbf{i},m)$. Then $\mathcal{B}_r(\mathbf{i},m)$ is bad and contains the point (x_q,q). Moreover, this r-block intersects the space-time path $\widehat{\pi}$ at least in (x_q,q). Consequently, the contribution to (5.20) of all its summands which satisfy (5.21) is at most

$$\begin{aligned}&\gamma_{r+1}\mu_A C_0^{d(r+1)}(\text{number of vertices on any space-time path in any } r\text{-block})\phi_r(\pi)\\ &\le \gamma_{r+1}\mu_A C_0^{d(r+1)} C_0^{6r}\Phi_r.\end{aligned}$$

In addition, there are no summands in (5.20) which satisfy (5.21) with $r \ge R$. Thus, (5.20) is bounded by

$$t\gamma_1\mu_A C_0^d + \sum_{r=1}^{R-1}\gamma_{r+1}\mu_A C_0^{r(d+6)+d}\Phi_r.$$

(5.17) now follows from (5.2). ∎

The estimate (5.17) makes it clear that we should find bounds for the Φ_r. This is the aim of the next few lemmas, which derive a recursive inequality for the Φ_r. We begin with a technical lemma. This is the first step in showing that if the pedestal of some $(r+1)$-block $\mathcal{B}_{r+1}(\mathbf{i},k)$ is good, then there is only a small probability that $\mathcal{B}_{r+1}(\mathbf{i},k)$ contains a bad r-block. Because a discrete time random walk which starts outside the pedestal of $\mathcal{B}_{r+1}(\mathbf{i},k)$ at time $(k-1)\Delta_{r+1}$ cannot enter $\mathcal{B}_{r+1}(\mathbf{i},k)$ at any later time, the estimate will remain valid (as Lemma 7 will show), even if we condition on the $(r+1)$-blocks $\mathcal{B}_{r+1}(\mathbf{j},\ell)$ with $\ell < k$, or $\ell = k$ but $\mathbf{j}$ not too close to $\mathbf{i}$. This will allow us to treat the good $(r+1)$-blocks as being independent.

Lemma 6. *There exist a constant $C_3 = C_3(d, D_A)$, which is independent of C_0, such that if $\mathcal{V}_{r+1}(\mathbf{i}, k)$ is good, and $\Delta_{r+1} - \Delta_r \le u \le 2\Delta_{r+1}$, then for $r \ge 1$, $y \in \mathbb{Z}^d$,*

$$\sum_{z:(z,(k-1)\Delta_{r+1}) \in \mathcal{V}_{r+1}(\mathbf{i},k)} N_A(z, (k-1)\Delta_{r+1}) P\big\{z + \widetilde{S}_u \in \mathcal{Q}_r(y)\big\} \le \gamma_{r+1}\mu_A C_0^{dr}\Big[1 + \frac{C_3(r \log C_0)^d}{C_0^{2r}}\Big]. \tag{5.22}$$

Proof. Note that the set $\{z \in \mathbb{Z}^d : (z, (k-1)\Delta_{r+1}) \in \mathcal{V}_{r+1}(\mathbf{i}, k)\}$ is a disjoint union of $7^d C_0^{5d(r+1)}$ blocks of the form

$$\mathcal{M}(\mathbf{j}) := \mathcal{Q}_{r+1}(C_0^{r+1}\mathbf{j}) = \prod_{s=1}^d [j(s)C_0^{r+1}, (j(s)+1)C_0^{r+1}).$$

Let $\Lambda = \Lambda(\mathbf{i}, r)$ be the set of $\mathbf{j} \in \mathbb{Z}^d$ with

$$\mathcal{M}(\mathbf{j}) \subset \{z \in \mathbb{Z}^d : (z, (k-1)\Delta_{r+1}) \in \mathcal{V}_{r+1}(\mathbf{i}, k)\}.$$

Also, for each $\mathbf{j} \in \Lambda$ let $z_{\mathbf{j}} \in \mathcal{M}(\mathbf{j})$ be such that

$$P\big\{z_{\mathbf{j}} + \widetilde{S}_u \in \mathcal{Q}_r(y)\big\} = \max_{z \in \mathcal{M}(\mathbf{j})} P\big\{z + \widetilde{S}_u \in \mathcal{Q}_r(y)\big\}.$$

Then the left hand side of (5.22) equals

$$\begin{aligned}&\sum_{\mathbf{j}\in\Lambda}\sum_{z\in\mathcal{M}(\mathbf{j})} N_A(z, (k-1)\Delta_{r+1}) P\big\{z + \widetilde{S}_u \in \mathcal{Q}_r(y)\big\}\\ &\le \sum_{\mathbf{j}\in\Lambda}\sum_{z\in\mathcal{M}(\mathbf{j})} N_A(z, (k-1)\Delta_{r+1}) P\big\{z_{\mathbf{j}} + \widetilde{S}_u \in \mathcal{Q}_r(y)\big\}.\end{aligned}$$

Since $\mathcal{V}_{r+1}(\mathbf{i}, k)$ is assumed to be good, we have

$$\begin{aligned}\sum_{z\in\mathcal{M}(\mathbf{j})} N_A(z, (k-1)\Delta_{r+1}) &= U_{r+1}(C_0^{r+1}\mathbf{j}, (k-1)\Delta_{r+1})\\ &\le \gamma_{r+1}\mu_A C_0^{d(r+1)} = \sum_{z\in\mathcal{M}(\mathbf{j})} \gamma_{r+1}\mu_A.\end{aligned}$$

We can therefore continue to obtain that the left hand side of (5.22) is at most

$$\begin{aligned}&\sum_{\mathbf{j}\in\Lambda}\sum_{z\in\mathcal{M}(\mathbf{j})} \gamma_{r+1}\mu_A P\big\{z_{\mathbf{j}} + \widetilde{S}_u \in \mathcal{Q}_r(y)\big\}\\ &\le \sum_{\mathbf{j}\in\Lambda}\sum_{z\in\mathcal{M}(\mathbf{j})} \gamma_{r+1}\mu_A P\big\{z + \widetilde{S}_u \in \mathcal{Q}_r(y)\big\}\\ &\quad + \sum_{\mathbf{j}\in\Lambda}\sum_{z\in\mathcal{M}(\mathbf{j})} \gamma_{r+1}\mu_A \big|P\big\{z_{\mathbf{j}} + \widetilde{S}_u \in \mathcal{Q}_r(y)\big\} - P\big\{z + \widetilde{S}_u \in \mathcal{Q}_r(y)\big\}\big|.\end{aligned} \tag{5.23}$$

The first multiple sum in the right hand side of (5.23) is at most

$$\begin{aligned}&\sum_{z\in\mathbb{Z}^d}\gamma_{r+1}\mu_A \sum_{w\in\mathcal{Q}_r(y-z)} P\{\widetilde{S}_u = w\} = \sum_{w\in\mathbb{Z}^d} P\{\widetilde{S}_u = w\} \sum_{z\in\mathcal{Q}_r(y-w)} \gamma_{r+1}\mu_A\\ &\le \sum_{w\in\mathbb{Z}^d} P\{\widetilde{S}_u = w\}\gamma_{r+1}\mu_A C_0^{dr} = \gamma_{r+1}\mu_A C_0^{dr}.\end{aligned} \tag{5.24}$$

On the other hand, we have for any $z \in \mathcal{M}(\mathbf{j})$ that $\|z - z_{\mathbf{j}}\| \le C_0^{r+1}$ and

$$\begin{aligned}
&\big|P\{z_{\mathbf{j}} + \widetilde{S}_u \in \mathcal{Q}_r(y) - P\{z + \widetilde{S}_u \in \mathcal{Q}_r(y)\}\big| \\
&\le \sum_{w \in \mathcal{Q}_r(y)} \big|P\{z_{\mathbf{j}} + \widetilde{S}_u = w\} - P\{z + \widetilde{S}_u = w\}\| \\
&\le \sum_{v \in \mathcal{Q}_r(y-z)} \sup_{w:\|w-v\| \le C_0^{r+1}} \big|P\{\widetilde{S}_u = v\} - P\{\widetilde{S}_u = w\}\big|.
\end{aligned}$$

It follows that the second multiple sum in the right hand side of (5.23) is bounded in absolute value by

$$\begin{aligned}
&\sum_{z \in \mathbb{Z}^d} \gamma_{r+1}\mu_A \sum_{v \in \mathcal{Q}_r(y-z)} \sup_{w:\|w-v\| \le C_0^{r+1}} \big|P\{\widetilde{S}_u = v\} - P\{\widetilde{S}_u = w\}\big| \\
&= \gamma_{r+1}\mu_A \sum_{v \in \mathbb{Z}^d} \sum_{z \in \mathcal{Q}_r(y-v)} \sup_{w:\|w-v\| \le C_0^{r+1}} \big|P\{\widetilde{S}_u = v\} - P\{\widetilde{S}_u = w\}\big| \\
&= \gamma_{r+1}\mu_A C_0^{dr} \sum_{v \in \mathbb{Z}^d} \sup_{w:\|w-v\| \le C_0^{r+1}} \big|P\{\widetilde{S}_u = v\} - P\{\widetilde{S}_u = w\}\big|. \qquad (5.25)
\end{aligned}$$

Even though one can do better, we shall be content with proving the crude estimate

$$\sum_{v \in \mathbb{Z}^d} \sup_{w:\|w-v\| \le C_0^{r+1}} \big|P\{\widetilde{S}_u = v\} - P\{\widetilde{S}_u = w\}\big| \le K_2 C_0^{r+1} \frac{(\log u)^d}{\sqrt{u}} \qquad (5.26)$$

for some constant K_2 which depends only on $q_A(\cdot)$ and d, but not on C_0, and $u \ge 2$. To prove this bound we define for $\theta = (\theta(1), \dots, \theta(d)) \in \mathbb{R}^d$

$$\widetilde{\psi}(\theta) = \sum_{v \in \mathbb{Z}^d} q_A(v) e^{i\theta \cdot v},$$

where $\theta \cdot v := \sum_{s=1}^d \theta(s)v(s)$. Then $Ee^{i\theta \cdot \widetilde{S}_u} = [\widetilde{\psi}(\theta)]^u$ and

$$\begin{aligned}
&\big|P\{\widetilde{S}_u = v\} - P\{\widetilde{S}_u = w\}\big| \\
&= \Big|\frac{1}{(2\pi)^d} \int_{\theta \in [-\pi,\pi]^d} \big[e^{-i\theta \cdot v} - e^{-i\theta \cdot w}\big][\widetilde{\psi}(\theta)]^u d\theta\Big| \\
&\le \frac{d}{(2\pi)^d} \int_{\theta \in [-\pi,\pi]^d} \|v - w\| \|\theta\| |\widetilde{\psi}(\theta)|^u d\theta \\
&\le K_3 \frac{\|v-w\|}{u^{(d+1)/2}}, \qquad (5.27)
\end{aligned}$$

where again K_3 (and $K_4 - K_7$ below) depend only on $q_A(\cdot)$ and d, but not on C_0 (see Spitzer (1976), Proofs of Propositions 7.9 and 12.1 for similar estimates). We use this estimate for $v \in [-2\sqrt{u}\log u, 2\sqrt{u}\log u]^d$. This leads to a contribution to (5.26) of

$$[2\sqrt{u}\log u + 1]^d K_4 C_0^{r+1} u^{-(d+1)/2} \le K_5 C_0^{r+1} \frac{(\log u)^d}{\sqrt{u}}. \qquad (5.28)$$

The remaining terms in (5.26) contribute at most

$$2P\{\widetilde{S}_u \notin [-2\sqrt{u}\log u + C_0^{r+1}, 2\sqrt{u}\log u - C_0^{r+1}]^d\} \le K_6 \exp[-K_7(\log u)^2], \qquad (5.29)$$

by the fact that

$$u \geq \Delta_{r+1} - \Delta_r \geq \frac{1}{2} C_0^{6r+6}, \tag{5.30}$$

and hence $\sqrt{u} \geq (1/2) C_0^{3r+3} \geq 2C_0^{r+1}$, and by Bernstein's inequality (Chow and Teicher (1988), Exercise 4.3.14). (5.26) follows from (5.28) and (5.29).

Finally we see from (5.23) - (5.26) that the left hand side of (5.22) is at most

$$\gamma_{r+1}\mu_A C_0^{dr}\Big[1 + K_2 C_0^{r+1}\frac{(\log u)^d}{\sqrt{u}}\Big] \leq \gamma_{r+1}\mu_A C_0^{dr}\Big[1 + C_3\frac{(r\log C_0)^d}{C_0^{2r+2}}\Big].$$

(5.22) follows. ■

We need the following σ-fields:

$$\begin{aligned}
&\mathcal{H}_{r+1}(\mathbf{i},k) := \sigma\text{-field generated by}\\
&\Big\{N_A(x,\ell) : \{x \in \prod_{s=1}^d [(i(s)-3)\Delta_{r+1}, (i(s)+4)\Delta_{r+1}) \text{ and } -t \leq \ell \leq (k-1)\Delta_{r+1}\}\\
&\qquad \text{or } \{x \in [-t-4C_0^r, t+4C_0^r]^d \setminus \prod_{s=1}^d [(i(s)-5)\Delta_{r+1}, (i(s)+6)\Delta_{r+1})\\
&\qquad \text{and } -t \leq \ell \leq (k+1)\Delta_{r+1}\}\Big\}
\end{aligned} \tag{5.31}$$

FIGURE 2. $\mathcal{H}_{r+1}(\mathbf{i},k)$ is generated by the $N_A(x,\ell)$ with (x,ℓ) located in the gray region. $\mathcal{K}_{r+1}$ is generated by the $N_A(x,\ell)$ with (x,ℓ) on the fat line segment at the bottom of $\widetilde{\mathcal{B}}_{r+1}$. (The illustration is for $d=1$.)

(see Figure 2),

$$\begin{aligned}
&\mathcal{K}_{r+1}(\mathbf{i},k) := \sigma\text{-field generated by}\\
&\quad\{N_A(x,(k-1)\Delta_{r+1}) : x \in \prod_{s=1}^d [(i(s)-3)\Delta_{r+1}, (i(s)+4)\Delta_{r+1})\}\\
&\quad = \sigma\text{-field generated by } \{N_A(x,(k-1)\Delta_{r+1}) : (x,(k-1)\Delta_{r+1}) \in \mathcal{V}_{r+1}(\mathbf{i},k)\}.
\end{aligned} \tag{5.32}$$

Note that $\mathcal{K}_{r+1} \subset \mathcal{H}_{r+1}$. For the purpose of the next two lemmas only we also want to consider the "central part in the space direction $\times$ the upper part in the time direction" of $\widetilde{\mathcal{B}}_{r+1}(\mathbf{i}, k)$. This is defined as

$$\widetilde{\mathcal{B}}^+_{r+1}(\mathbf{i}, k) := \prod_{s=1}^{d} [(i(s) - 1)\Delta_{r+1}, (i(s) + 2)\Delta_{r+1}) \times [k\Delta_{r+1} - \Delta_r, (k + 1)\Delta_{r+1})$$

(see Figure 1; note that $r + 1$ is replaced by r in the figure). Note that $\mathcal{B}_{r+1}(\mathbf{i}, k) \subset \widetilde{\mathcal{B}}^+_{r+1}(\mathbf{i}, k) \subset \widetilde{\mathcal{B}}_{r+1}(\mathbf{i}, k)$. With the exception of the tildes over the $\mathcal{B}$, we will use a tilde over a symbol to indicate that it is related to the discrete time process of this section. For instance $\widetilde{\rho}$, introduced in the next lemma, is the analogue for the discrete time system of a quantity ρ to be used for the continuous time system in the next section.

Lemma 7. *Let*

$$\widetilde{\rho}_{r+1} = 3^{d+1} C_0^{6(d+1)(r+1)} \exp\big[-\frac{1}{2}\gamma_r \mu_A C_0^{(d-\frac{3}{4})r}\big], \ r \geq 1. \tag{5.33}$$

Then for $r \geq 1$, on the event $\{\mathcal{V}_{r+1}(\mathbf{i}, k)$ is good$\}$,

$$\begin{aligned} &P_A\{\mathcal{B}_{r+1}(\mathbf{i}, k) \text{ contains some bad } \mathcal{B}_r(\mathbf{j}, q) | \mathcal{H}_{r+1}(\mathbf{i}, k)\} \\ &= P_A\{\mathcal{B}_{r+1}(\mathbf{i}, k) \text{ contains some bad } \mathcal{B}_r(\mathbf{j}, q) | \mathcal{K}_{r+1}(\mathbf{i}, k)\} \\ &\leq \widetilde{\rho}_{r+1}. \end{aligned} \tag{5.34}$$

Proof. The event

$$\mathcal{A}(\mathbf{i}, k) := \{\mathcal{B}_{r+1}(\mathbf{i}, k) \text{ contains some bad } \mathcal{B}_r(\mathbf{j}, q)\}$$

is defined in terms of the $N_A(y, m)$ with

$$\begin{aligned} (y, m) &\in \Big(\prod_{s=1}^{d} [i(s)\Delta_{r+1} - 3\Delta_r, (i(s) + 1)\Delta_{r+1} + 3\Delta_r) \Big) \\ &\qquad\qquad\qquad \times [k\Delta_{r+1} - \Delta_r, (k + 1)\Delta_{r+1}) \\ &\subset \Big(\prod_{s=1}^{d} [(i(s) - 1)\Delta_{r+1}, (i(s) + 2)\Delta_{r+1}) \Big) \times [k\Delta_{r+1} - \Delta_r, (k + 1)\Delta_{r+1}) \\ &= \widetilde{\mathcal{B}}^+_{r+1}(\mathbf{i}, k). \end{aligned} \tag{5.35}$$

An A-particle present at time $(k - 1)\Delta_{r+1}$ cannot move over a distance greater than $2\Delta_{r+1}$ during $[(k - 1)\Delta_{r+1}, (k+1)\Delta_{r+1})$. Such a particle cannot reach $\widetilde{\mathcal{B}}^+_{r+1}(\mathbf{i}, k)$ unless it is located in $\prod_{s=1}^d [(i(s)-3)\Delta_{r+1}, (i(s)+4)\Delta_{r+1})$ at time $(k - 1)\Delta_{r+1}$. In other words, only particles from the pedestal of $\mathcal{B}_{r+1}(\mathbf{i}, k)$ can reach $\widetilde{\mathcal{B}}^+_{r+1}(\mathbf{i}, k)$. For the same reason particles in the pedestal $\mathcal{V}_{r+1}(\mathbf{i}, k)$ cannot reach any (x, ℓ) with $x \in [-t - 3C_0^r, t + 3C_0^r]^d \setminus \prod_{s=1}^d [(i(s) - 5)\Delta_{r+1}, (i(s) + 6)\Delta_{r+1})$ and $-t \leq \ell < (k + 1)\Delta_{r+1}$. It follows that the event $\mathcal{A}(\mathbf{i}, k)$ is determined entirely by the particles in the pedestal $\mathcal{V}_{r+1}(\mathbf{i}, k)$ and their paths during $[(k-1)\Delta_{r+1}, (k+1)\Delta_{r+1})$. When the $N_A(y, m)$ with $(y, m) \in \mathcal{V}_{r+1}(\mathbf{i}, k)$ are given, then none of the $N_A(x, \ell)$ with (x, ℓ) outside this pedestal but occuring in the right hand side of (5.31) have any influence on the $N_A(y, m)$ with $(y, m) \in \widetilde{\mathcal{B}}^+_{r+1}(\mathbf{i}, k)$ (nor do they contain information about these $N_A(y, m)$). Therefore, the conditional probability of any event defined in terms of the $N_A(y, m)$, with $(y, m) \in \widetilde{\mathcal{B}}^+_{r+1}(\mathbf{i}, k)$, given $\mathcal{H}_{r+1}(\mathbf{i}, k)$ or given only $\mathcal{K}_{r+1}(\mathbf{i}, k)$, are the same. This proves the equality in (5.34).

We turn to the inequality in (5.34). For $\mathcal{A}(\mathbf{i}, k)$ to occur, some $\mathcal{B}_r$ contained in $\mathcal{B}_{r+1}(\mathbf{i}, k)$ has to be bad, that is,

$$\begin{aligned} U_r(y, m) > \gamma_r \mu_A C_0^{dr} &\text{ for some } (y, m) \text{ for which } \mathcal{Q}_r(y) \times \{m\} \subset \widetilde{\mathcal{B}}_r(\mathbf{j}, q) \\ &\text{for some } (\mathbf{j}, q) \text{ with } \mathcal{B}_r(\mathbf{j}, q) \subset \mathcal{B}_{r+1}(\mathbf{i}, k). \end{aligned}$$

This means that we must have

$$U_r(y,m) > \gamma_r \mu_A C_0^{dr} \text{ for some } (y,m) \in \widetilde{\mathcal{B}}^+_{r+1}(\mathbf{i},k). \tag{5.36}$$

There are at most $[3\Delta_{r+1}]^{d+1}$ such choices for (y,m). For each such (y,m) we have

$$\Delta_{r+1} - \Delta_r \le m - (k-1)\Delta_{r+1} \le 2\Delta_{r+1}. \tag{5.37}$$

Moreover, as observed in the beginning of this proof, all the particles counted in one of the above $U_r(y,m)$ have to come from particles in the pedestal $\mathcal{V}_{r+1}(\mathbf{i},k)$. Fix a $(y,m) \in \widetilde{\mathcal{B}}^+_{r+1}(\mathbf{i},k)$ and set

$$u = m - (k-1)\Delta_{r+1}.$$

Then $U_r(y,m)$ counts the number of particles from the pedestal $\mathcal{V}_{r+1}(\mathbf{i},k)$ which are in $\mathcal{Q}_r(y)$ at time m. A particle at $(z,(k-1)\Delta_{r+1})$ in this pedestal has probability

$$\widetilde{p}(y-z,u) := P\{z + \widetilde{S}_u \in \mathcal{Q}_r(y)\}$$

of being counted in $U_r(y,m)$. Moreover, all particles in the pedestal $\mathcal{V}_{r+1}(\mathbf{i},k)$ move independently. Thus given all the $N_A(z,(k-1)\Delta_{r+1})$ with $(z,(k-1)\Delta_{r+1})$ in the pedestal, the conditional moment generating function of $U_r(y,m)$ is

$$\begin{aligned} &E\{\exp[\theta U_r(y,m)] | \mathcal{K}_{r+1}(\mathbf{i},k)\} \\ &= \prod_{(z,(k-1)\Delta_{r+1}) \in \mathcal{V}_{r+1}(\mathbf{i},k)} \big[1 - \widetilde{p}(y-z,u) + \widetilde{p}(y-z,u)e^\theta\big]^{N_A(z,(k-1)\Delta_{r+1})} \\ &\le \exp\Big[\sum_{(z,(k-1)\Delta_{r+1}) \in \mathcal{V}_{r+1}(\mathbf{i},k)} N_A(z,(k-1)\Delta_{r+1})\widetilde{p}(y-z,u)(e^\theta - 1)\Big]. \end{aligned} \tag{5.38}$$

We take

$$\theta = C_0^{-r/2}.$$

We then obtain from Lemma 6 and the choices (5.6) for C_0 and (5.10) for the γ that on $\{\mathcal{V}_{r+1}(\mathbf{i},k)$ is good$\}$

$$\begin{aligned} &P_A\{U_r(y,m) > \gamma_r\mu_A C_0^{dr} | \mathcal{K}_{r+1}(\mathbf{i},k)\} \\ &\le \exp\Big[-\theta\gamma_r\mu_A C_0^{dr} + \sum_{(z,(k-1)\Delta_{r+1}) \in \mathcal{V}_{r+1}(\mathbf{i},k)} N_A(z,(k-1)\Delta_{r+1})\widetilde{p}(y-z,u)(e^\theta - 1)\Big] \\ &\le \exp\Big[-\theta\gamma_r\mu_A C_0^{dr} + \gamma_{r+1}\mu_A C_0^{dr}\big[1 + \frac{C_3(r\log C_0)^d}{C_0^{2r}}\big](e^\theta - 1)\Big] \\ &\le \exp\big[-\frac{1}{2}\gamma_r\mu_A C_0^{(d-\frac{3}{4})r}\big]. \end{aligned} \tag{5.39}$$

As we already pointed out, $\mathcal{A}(\mathbf{i},k)$ can occur only if (5.36) occurs for one of at most $[3\Delta_{r+1}]^{d+1}$ possible choices for (y,m). Thus the inequality in (5.34) holds. ■

We finally can prove our recurrence relation for the Φ_r. For a path $\pi = (x_0, x_1, \dots, x_t)$ we define

$$\begin{aligned} \psi_{r+1}(\pi) = {} & \text{number of } (r+1)\text{-blocks } \mathcal{B}_{r+1}(\mathbf{i},k) \text{ which intersect the space-time path } \widehat{\pi} \\ & \text{and which have a good pedestal but contain a bad } r\text{-block}. \end{aligned} \tag{5.40}$$

Also

$$\Psi_r = \sup_\pi \psi_r(\pi).$$

We remind the reader that ϕ_r and Φ_r were defined in (5.13) and (5.14).

Lemma 8. *For any path $\pi \subset [-t,t]^d$ and $r \geq 1$ it holds*

$$\phi_r(\pi) \leq C_0^{6(d+1)}\Phi_{r+1} + C_0^{6(d+1)}\psi_{r+1}(\pi) \tag{5.41}$$

and

$$\Phi_r \leq C_0^{6(d+1)}\Phi_{r+1} + C_0^{6(d+1)}\Psi_{r+1}. \tag{5.42}$$

Moreover, there exist some constants $C_4 = C_4(d)$ and $\kappa_0 = \kappa(d)$, such that for $\kappa \geq \kappa_0$, t sufficiently large, and $1 \leq r \leq R-1$,

$$P_A\big\{\Psi_{r+1} \geq \kappa \frac{t}{\Delta_{r+1}}[\widetilde{\rho}_{r+1}]^{1/(d+1)}\big\} \leq \exp\Big[-tC_4\kappa \exp\big[-\frac{1}{2(d+1)}\gamma_r\mu_A C_0^{(d-\frac{3}{4})r}\big]\Big]. \tag{5.43}$$

Proof. The inequality (5.41) is immediate from the definitions of ϕ, Φ and ψ. Indeed, if $\pi = (x_0, \dots, x_t)$, then any bad r-block $\mathcal{B}_r(\mathbf{j}, q)$ which intersects $\widehat{\pi} = ((x_0, 0), (x_1, 1), \dots, (x_t, t))$ belongs to a unique $(r+1)$-block, $\mathcal{B}_{r+1}(\mathbf{i}, k)$ say. The latter must intersect $\widehat{\pi}$, and it may either be bad, or good. There are no more than Φ_{r+1} bad $(r+1)$-blocks intersecting $\widehat{\pi}$ and each one contains exactly C_0^{6d+6} r-blocks. Thus the first term in the right hand side of (5.41) is an upper bound for the number of r-blocks which intersect $\widehat{\pi}$ and are contained in a bad $(r+1)$-block. The number of bad r-blocks which intersect $\widehat{\pi}$ and which are contained in a good $(r+1)$-block is bounded by the second term in the right hand side of (5.41), because the pedestal of any good block is itself good, by definition.

(5.42) follows from (5.41) by taking the sup over π.

The important estimate is therefore (5.43). Roughly speaking we shall obtain this by proving that the collection of pairs $(\mathbf{i}, k)$ for which the $(r+1)$-block $\mathcal{B}_{r+1}(\mathbf{i}, k)$ has a good pedestal but also contains a bad r-block, lies stochastically below an independent percolation system in which each site $(\mathbf{i}, k)$ is open with probability $\widetilde{\rho}_{r+1}$. As stated this is not correct. We need to restrict ourselves to certain subclasses of pairs $(\mathbf{i}, k)$. To state our claim precisely we define

$$\begin{aligned} Y(\mathbf{i}, k) = I[&\mathcal{B}_{r+1}(\mathbf{i}, k) \text{ intersects } [-t,t]^d \times [0,t],\\ &\text{has a good pedestal and contains a bad } r\text{-block}],\end{aligned}$$

and let $Z(\mathbf{i}, k)$ be a a system of independent random variables with

$$P\{Z(\mathbf{i}, k) = 1\} = 1 - P\{Z(\mathbf{i}, k) = 0\} = \widetilde{\rho}_{r+1}.$$

Let $\mathbf{a} = (a(1), \dots, a(d), b)$ be a $(d+1)$-vector with each $a(s) \in \{0, 1, \dots, 11\}$, $1 \leq s \leq d$, and $b \in \{0, 1\}$. Our claim is that for fixed $\mathbf{a}, b$, the system of $Y(\mathbf{i}, k)$ with $i(s) \equiv a(s) \mod 12$ for $1 \leq s \leq d$, $k \equiv b \mod 2$, lies stochastically below the system of $Z(\mathbf{i}, k)$, for the same pairs $(\mathbf{i}, k)$. Here $(\mathbf{i}, k)$ is further restricted to pairs $(\mathbf{i}, k)$ for which $\mathcal{B}_{r+1}(\mathbf{i}, k)$ intersects $[-t,t]^d \times [0,t]$. We prove this claim by means of Lemma 7. Indeed, let us write $(\mathbf{i}, k) \equiv (\mathbf{a}, b)$ $(\mathbf{i} \equiv \mathbf{a})$ if $i(s) \equiv a(s) \mod 12$ for $1 \leq s \leq d$ and $k \equiv b \mod 2$ (respectively, $i(s) \equiv a(s) \mod 12$ for $1 \leq s \leq d$). For fixed $\mathbf{a} \in \{0, 1, \dots, 11\}^d$ and $b \in \{0, 1\}$, and for $(\mathbf{i}, k) \equiv (\mathbf{a}, b)$, let us define

$$\begin{aligned} \mathcal{L}'(\mathbf{i}, k) = \sigma&\text{-field generated by } Y(\mathbf{j}, \ell) \text{ with } 0 \leq \ell \leq k-2 \text{ and}\\ &(\mathbf{j}, \ell) \equiv (\mathbf{i}, k) \equiv (\mathbf{a}, b), \text{ or } \ell = k, \mathbf{j} \equiv \mathbf{i} \equiv \mathbf{a} \text{ and } \mathbf{j}\\ &\text{precedes } \mathbf{i} \text{ in the lexicographical ordering of } \mathbb{Z}^d, \end{aligned} \tag{5.44}$$

and

$$\mathcal{L}(\mathbf{i}, k) = \mathcal{L}'(\mathbf{i}, k) \vee \mathcal{K}_{r+1}(\mathbf{i}, k).$$

Thus $Y(\mathbf{j}, k)$ appears in the right hand side of (5.44) only if $j(s) \leq i(s) - 12$ for some $1 \leq s \leq d$. Moreover, $Y(\mathbf{j}, \ell)$ depends only on the $N_A(y, m)$ with $(y, m) \in \widetilde{\mathcal{B}}_{r+1}^{+}(\mathbf{j}, \ell) \cup \mathcal{V}_{r+1}(\mathbf{j}, \ell)$. Then

$$\mathcal{K}_{r+1}(\mathbf{i}, k) \subset \mathcal{L}(\mathbf{i}, k) \subset \mathcal{H}_{r+1}(\mathbf{i}, k),$$

so that Lemma 7 shows

$$P\{Y(\mathbf{i},k)=1|\mathcal{L}(\mathbf{i},k)\} = P\{Y(\mathbf{i},k)=1|\mathcal{K}_{r+1}(\mathbf{i},k)\} \le \widetilde{\rho}_{r+1}. \tag{5.45}$$

Note that Lemma 7 only states the last inequality on the event $\{\mathcal{V}_{r+1}(\mathbf{i},k)$ is good$\}$. But on the complementary event $\{\mathcal{V}_{r+1}(\mathbf{i},k)$ is bad$\}$, $Y(\mathbf{i},k)=0$ by definition, so that the inequality in (5.45) always holds. Now successively determine the $Y(\mathbf{i},k)$ with $(\mathbf{i},k)\equiv(\mathbf{a},b)$ in the ordering in which $(\mathbf{i}',k')$ precedes $(\mathbf{i}'',k'')$ when $k'\le k''-1$ or $k'=k''$ and $\mathbf{i}'$ precedes $\mathbf{i}''$ in the lexicographical ordering of $\mathbb{Z}^d$. Then (5.45) says that at each stage the conditional probability that $Y(\mathbf{i},k)=1$ given the Y's which have been determined already, is at most $\widetilde{\rho}_{r+1}$. Thus the $Y(\mathbf{i},k)$ with $(\mathbf{i},k)\equiv(\mathbf{a},b)$ can be successively coupled with the $Z(\mathbf{i},k)$ for the same indices, such that $Y(\mathbf{i},k)\le Z(\mathbf{i},k)$. This proves our claim.

Now the left hand side of (5.43) equals the probability that for some path $\pi=(x_0,\dots,x_t)\subset[-t,t]^d$ there are at least $\kappa t[\widetilde{\rho}_{r+1}]^{1/(d+1)}/\Delta_{r+1}$ $(r+1)$-blocks $\mathcal{B}_{r+1}(\mathbf{i},k)$ which intersect $\widehat{\pi}$ and with $Y(\mathbf{i},k)=1$. By partitioning the possible pairs $(\mathbf{i},k)$ into $2\cdot(12)^d$ equivalence classes, we see that this last probability is at most

$$\begin{aligned}
&\sum_{\mathbf{a},b} P_A\{\exists\ \pi=(x_0,\dots,x_t) \text{ such that there are at least}\\
&\qquad 2^{-1}(12)^{-d}\kappa\frac{t}{\Delta_{r+1}}[\widetilde{\rho}_{r+1}]^{1/(d+1)} \quad (r+1)\text{-blocks } \mathcal{B}_{r+1}(\mathbf{i},k) \text{ which}\\
&\qquad \text{intersect } \widehat{\pi} \text{ and with } Y(\mathbf{i},k)=1, \text{ and } (\mathbf{i},k)\equiv(\mathbf{a},b)\}\\
&\le \sum_{\mathbf{a},b} P\{\exists\ \pi=(x_0,\dots,x_t) \text{ such that there are at least}\\
&\qquad 2^{-1}(12)^{-d}\kappa\frac{t}{\Delta_{r+1}}[\widetilde{\rho}_{r+1}]^{1/(d+1)} \quad (r+1)\text{-blocks } \mathcal{B}_{r+1}(\mathbf{i},k)\text{which}\\
&\qquad \text{intersect } \widehat{\pi} \text{ and with } Z(\mathbf{i},k)=1, \text{ and } (\mathbf{i},k)\equiv(\mathbf{a},b)\}.
\end{aligned} \tag{5.46}$$

Thus, we merely have to estimate each of the summands in the right hand side here for the independent system of $Z(\mathbf{i},k)$.

To this end, choose an integer $\widetilde{\nu}$ such that

$$[\widetilde{\rho}_{r+1}]^{-1/(d+1)} \le \widetilde{\nu} \le 2[\widetilde{\rho}_{r+1}]^{-1/(d+1)} \tag{5.47}$$

and form the blocks

$$\mathcal{D}(\mathbf{j},q) := \Big(\prod_{s=1}^d [\widetilde{\nu}j(s)\Delta_{r+1}, \widetilde{\nu}(j(s)+1)\Delta_{r+1})\Big) \times [q\widetilde{\nu}\Delta_{r+1},(q+1)\widetilde{\nu}\Delta_{r+1}).$$

(Note that (5.7)and (5.8) guarantee that $\widetilde{\rho}_{r+1}\le 1$, so that $\widetilde{\nu}$ is well defined.) Each of these block is a disjoint union of $\widetilde{\nu}^{d+1}$ $(r+1)$-blocks. We claim that a space-time path $\widehat{\pi}=((x_0,0),\dots,(x_t,t))$ intersects at most

$$\widetilde{\lambda} := 2^d\Big(\frac{t}{\widetilde{\nu}\Delta_{r+1}}+1\Big) \tag{5.48}$$

such blocks. To see this, note that during the time interval $[\ell\widetilde{\nu}\Delta_{r+1},(\ell+1)\widetilde{\nu}\Delta_{r+1})$ the space-time path intersects at most 2^d blocks $\mathcal{D}(\mathbf{j},q)$ (which must have $q=\ell$). Indeed, if $\widehat{\pi}$ intersects $\mathcal{D}(\mathbf{j},q)$, at some time, then it takes more than $\widetilde{\nu}\Delta_{r+1}$ steps to get to any block $\mathcal{D}(\mathbf{j}',q)$ with $|j(s)-j'(s)|>1$ and this cannot be done during $[\ell\widetilde{\nu}\Delta_{r+1},(\ell+1)\widetilde{\nu}\Delta_{r+1})$. Thus, $\widehat{\pi}$ can only intersect pairs $\mathcal{D}(\mathbf{j},q),\mathcal{D}(\mathbf{j}',q)$ with $|j(s)-j(s')|\le 1$, $1\le s\le d$. This establishes our claim.

We should now replace $\widetilde{\lambda}$ by $\lfloor\widetilde{\lambda}\rfloor$, but for brevity we continue to write $\widetilde{\lambda}$ instead of $\lfloor\widetilde{\lambda}\rfloor$ in the remainder of this proof. If the blocks intersecting $\widehat{\pi}$ are $\mathcal{D}(\mathbf{j}_0,0),\dots,\mathcal{D}(\mathbf{j}_{\widetilde{\lambda}-1},\widetilde{\lambda}-1)$, then we must have $|j_{\ell+1}(s)-j_\ell(s)|\le 1$

for $0 \le \ell \le \widetilde{\lambda} - 2$, $1 \le s \le d$. In addition we must have $|j_0(s)| \le t/(\widetilde{\nu}\Delta_{r+1}) + 1$ for $1 \le s \le d$. Therefore, there are at most

$$\widetilde{\lambda}^d \exp\big[K_8\widetilde{\lambda}\big] \tag{5.49}$$

possibilities for the choice of $\mathcal{D}(\mathbf{j}_0, 0), \dots, \mathcal{D}(\mathbf{j}_{\widetilde{\lambda}-1}, \widetilde{\lambda} - 1)$ as π varies over all paths in $[-t, t]^d$ ($K_8 = K_8(d)$). Now let us fix such a choice for $\mathcal{D}(\mathbf{j}_0, 0), \dots, \mathcal{D}(\mathbf{j}_{\widetilde{\lambda}-1}, \widetilde{\lambda} - 1)$ and let $\widehat{\pi}$ be any space-time path contained in

$$\bigcup_{q=0}^{\widetilde{\lambda}-1} \mathcal{D}(\mathbf{j}_q, q). \tag{5.50}$$

Note that if $\mathcal{B}_{r+1}(\mathbf{i}, k)$ intersects $\widehat{\pi}$, then $\mathcal{B}_{r+1}(\mathbf{i}, k)$ is contained in the union in (5.50). Therefore, the probability in the right hand side of (5.46) is at most

$$\begin{aligned}\sum_{\mathcal{D}(\mathbf{j}_0,0),\dots,\mathcal{D}(\mathbf{j}_{\widetilde{\lambda}-1},\widetilde{\lambda}-1)} & P\{\text{the union (5.50) contains at least} \\ & 2^{-1}(12)^{-d}\kappa\frac{t}{\Delta_{r+1}}[\widetilde{\rho}_{r+1}]^{1/(d+1)} \qquad (r+1)\text{-blocks } \mathcal{B}_{r+1}(\mathbf{i}, k) \\ & \text{with } Z(\mathbf{i}, k) = 1\}. \end{aligned} \tag{5.51}$$

The next, and almost last, step is to estimate the probability which appears as summand in (5.51). This, however, is easy. Indeed, the union in (5.50) contains exactly $\widetilde{\lambda}\widetilde{\nu}^{d+1}$ $(r + 1)$-blocks $\mathcal{B}_{r+1}(\mathbf{i}, k)$ with $\widetilde{\lambda}\widetilde{\nu}^{d+1}$ corresponding variables $Z(\mathbf{i}, k)$ which can take on the values 0 or 1, and which are independent with $P\{Z(\mathbf{i}, k) = 1\} = \widetilde{\rho}_{r+1}$. Thus the number of these Z's which equal 1 has a binomial distribution with parameters $\widetilde{\lambda}\widetilde{\nu}^{d+1}$ (for the number of trials) and $\widetilde{\rho}_{r+1}$ (for the success probability). Therefore, if $T = T(\widetilde{\lambda}\widetilde{\nu}^{d+1}, \widetilde{\rho}_{r+1})$ is a random variable with this binomial distribution, then each of the summands in (5.51) is bounded by

$$P\{T \ge 2^{-1}(12)^{-d}\kappa\frac{t}{\Delta_{r+1}}[\widetilde{\rho}_{r+1}]^{1/(d+1)}\}. \tag{5.52}$$

But

$$E\{T\} = \widetilde{\lambda}\widetilde{\nu}^{d+1}\widetilde{\rho}_{r+1} \in [\widetilde{\lambda}, 2^{d+1}\widetilde{\lambda}] \text{ (by (5.47)).}$$

Also,

$$\begin{aligned}\widetilde{\lambda} = 2^d\Big(\frac{t}{\widetilde{\nu}C_0^{6(r+1)}} + 1\Big) &\ge 2^d t\frac{[\widetilde{\rho}_{r+1}]^{1/(d+1)}}{2C_0^{6(r+1)}} \text{ (by (5.47))} \\ &= t\frac{3\cdot 2^d}{2}\exp\big[-\frac{1}{2(d+1)}\gamma_r\mu_A C_0^{(d-\frac{3}{4})r}\big] \text{ (by (5.33)).}\end{aligned}$$

Since $C_0^{(d-3/4)r} \le [\log t]^{1-3/(4d)}$ for $r \le R - 1$ (see (5.15) for R), we see that

$$\widetilde{\lambda} \text{ and } E\{T\} \to \infty \text{ as } t \to \infty, \text{ uniformly in } r \le R - 1. \tag{5.53}$$

Similarly

$$E\{T\} \le \frac{4^{d+1}t}{\widetilde{\nu}\Delta_{r+1}} \le 2^{-1}2^{-1}(12)^{-d}\kappa\frac{t}{\Delta_{r+1}}[\widetilde{\rho}_{r+1}]^{1/(d+1)},$$

provided $\kappa \ge$ some κ_0 which depends on d only (see (5.47)) and $t \ge$ some t_0. By standard exponential bounds (e.g. Bernstein's inequality, Chow and Teicher (1988), Exercise 4.3.14) the probability in (5.52) is, for $\kappa \ge \kappa_0$ and $t \ge t_0$, at most

$$K_9\exp\big[-K_{10}\kappa\frac{t}{\Delta_{r+1}}[\widetilde{\rho}_{r+1}]^{1/(d+1)}\big].$$

This bounds each summand in (5.51), and the number of summands is no more than the expression (5.49). Therefore, the right hand side of (5.46) is bounded by

$$\begin{aligned}
&2(12\widetilde{\lambda})^d \exp\big[K_8\widetilde{\lambda}\big]K_9 \exp[-K_{10}\kappa \frac{t}{\Delta_{r+1}}[\widetilde{\rho}_{r+1}]^{1/(d+1)}] \\
&\le 2K_9(12\widetilde{\lambda})^d \exp\big[K_8\widetilde{\lambda} - K_{10}\kappa 2^{-d-1}\widetilde{\lambda}\big] \text{ (see (5.48))} \\
&\le \exp\Big[-tC_4\kappa \exp\big[-\frac{1}{2(d+1)}\gamma_r\mu_A C_0^{(d-\frac{3}{4})r}\big]\Big],
\end{aligned}$$

provided κ_0 is raised, if necessary, to $2^{d+2}K_8/K_{10}$, and $\kappa \ge \kappa_0$ and $t \ge$ some t_0. ■

Proof of Theorem 2-Discrete. We want to apply (5.42) successively for $r = R-1, R-2, \dots, 1$. (5.43) shows that if we take $\kappa = \kappa_0$, then

$$\begin{aligned}
&P_A\{\Psi_{r+1} \ge \kappa_0 \frac{t}{\Delta_{r+1}}[\widetilde{\rho}_{r+1}]^{1/(d+1)} \text{ for some } 1 \le r \le R-1\} \\
&\le \sum_{r=1}^{R-1} \exp\Big[-tC_4\kappa_0 \exp\big[-\frac{1}{2(d+1)}\gamma_r\mu_A C_0^{(d-\frac{3}{4})r}\big]\Big]. \qquad (5.54)
\end{aligned}$$

Now note that $\gamma_r \le \gamma_0$ for all r, and that

$$C_0^{(d-\frac{3}{4})r} \le [\log t]^{1-\frac{3}{4d}}$$

for $r \le R-1$. Therefore, there exists some t_1 such that the right hand side of (5.54) is for $t \ge t_1$ at most $\exp[-\sqrt{t}]$. Since this is summable over t, the Borel-Cantelli lemma shows that almost surely for all large t

$$\Psi_r \le \kappa_0 \frac{t}{\Delta_{r+1}}[\widetilde{\rho}_{r+1}]^{1/(d+1)} \text{ for all } 1 \le r \le R-1, \qquad (5.55)$$

and we may restrict ourselves to sample points and values of t for which (5.55) is valid. By (5.16) we may further assume that

$$\Phi_r = 0 \text{ for all } r \ge R. \qquad (5.56)$$

For such a sample point and t we have from (5.42) that for any $1 \le r \le R-1$,

$$\begin{aligned}
\Phi_r &\le C_0^{6(d+1)}\Phi_{r+1} + C_0^{6(d+1)}\Psi_{r+1} \\
&\le C_0^{6(d+1)}\kappa_0 \frac{t}{\Delta_{r+1}}[\widetilde{\rho}_{r+1}]^{1/(d+1)} + C_0^{6(d+1)}\Phi_{r+1} \\
&\le C_0^{6(d+1)}\kappa_0 \frac{t}{\Delta_{r+1}}[\widetilde{\rho}_{r+1}]^{1/(d+1)} + C_0^{12(d+1)}\kappa_0 \frac{t}{\Delta_{r+2}}[\widetilde{\rho}_{r+2}]^{1/(d+1)} + C_0^{12(d+1)}\Phi_{r+2} \\
&\le \cdots \le \sum_{j=1}^{R-r} C_0^{6j(d+1)}\kappa_0 \frac{t}{\Delta_{r+j}}[\widetilde{\rho}_{r+j}]^{1/(d+1)} + C_0^{6(R-r)(d+1)}\Phi_R \\
&= t\kappa_0 \sum_{j=1}^{R-r} C_0^{6j(d+1)-6(r+j)} 3C_0^{6(r+j)} \exp\big[-\frac{1}{2(d+1)}\gamma_{r+j-1}\mu_A C_0^{(d-\frac{3}{4})(r+j-1)}\big] \\
&\le K_{11}t\kappa_0 C_0^{6(d+1)} \exp\big[-K_{12}C_0^{r/4}\big]. \qquad (5.57)
\end{aligned}$$

Finally we use Lemma 5. By (5.17) and (5.57) we have for any sample point for which (5.55) and (5.56) are valid, that

$$\begin{aligned}&E_B\{\text{number of } B\text{-particles alive at time } t \text{ with associated path } \pi|\mathcal{F}_A\}\\ &\le \mu_B\Big[\prod_{i=1}^t q_B(x_i-x_{i-1})\Big]\exp\Big\{-t\delta+\beta t\gamma_1\mu_A C_0^d\\ &\qquad +\beta\sum_{r=1}^{R-1} C_0^{r(d+6)+d}\gamma_{r+1}\mu_A K_{11}t\kappa_0 C_0^{6(d+1)}\exp\big[-K_{12}C_0^{r/4}\big]\Big\}.\end{aligned}$$

Now note that $\gamma_{r+1}\le\gamma_0$, so that

$$\sum_{r=1}^{R-1} C_0^{r(d+6)+d}\gamma_{r+1}\mu_A K_{11}t\kappa_0 C_0^{6(d+1)}\exp\big[-K_{12}C_0^{r/4}\big]\le tK_{13}$$

for some constant K_{13}. Thus

$$\begin{aligned}&E_B\{\text{number of } B\text{-particles alive at time } t \text{ with associated path } \pi|\mathcal{F}_A\}\\ &\le \mu_B\Big[\prod_{i=1}^t q_B(x_i-x_{i-1})\Big]\exp\big[-t\delta+t\beta\gamma_1\mu_A C_0^d+t\beta K_{13}\big]\end{aligned}\tag{5.58}$$

for large t.

To conclude the proof we observe that the last estimate holds as long as the path π is contained in $[-t,t]^d$. We can therefore sum over all such paths which end at $\mathbf{0}$ at time t to obtain for any A-configuration which satisfies (5.55) and (5.56) that

$$\begin{aligned}&E_B\{\text{number of } B\text{-particles alive at } \mathbf{0} \text{ time } t \text{ whose associated}\\ &\qquad\qquad\text{path lies in } [-t,t]^d|\mathcal{F}_A\}\\ &\le \mu_B\exp\big[-t\delta+t\beta\gamma_1\mu_A C_0^d+t\beta K_{13}\big].\end{aligned}\tag{5.59}$$

for large t. However, the associated path of any particle which is at $\mathbf{0}$ at time t cannot have left $[-t,t]^d$ during $[0,t]$, for then the path would not be able to come back to $\mathbf{0}$ by time t. Thus (5.59) actually says that

$$\begin{aligned}&E_B\{\text{number of } B\text{-particles alive at } \mathbf{0} \text{ at time } t|\mathcal{F}_A\}\\ &\qquad\le \mu_B\exp\big[-t\delta+t\beta\gamma_1\mu_A C_0^d+t\beta K_{13}\big].\end{aligned}$$

If we take $\delta_0=2\beta\gamma_1\mu_A C_0^d+2\beta K_{13}$ then (5.3) holds for all $\delta\ge\delta_0$. (5.4) follows from (5.3) as in the proof of (1.7) at the end of Section 4. ∎

6. Proof of Theorem 2 in continuous time.

In this section we shall prove Theorem 2 as stated in the Introduction. We shall do this by imitating the steps for Theorem 2-Discrete. There is now an added complication, due to the fact that the A-particles can move over a large distance in a fixed time interval, instead of having bounded displacements as in Section 5. This will be dealt with in Lemma 10 by a new estimate on the number of blocks which contain particles which had a large displacement. (Roughly speaking we show that there are few $(r+1)$ blocks which contain particles which moved over a distance of order Δ_{r+1} in Δ_{r+1} time units.) We shall use most of the quantities of Section 5, but decorate them with a superscript c to indicate that they refer to the continuous time model.

As in the previous sections $N_A(x,v)$ and $N_B(x,v)$ denote the number of A-particles and B-particles at (x,v) (v does not have to be an integer now).

A *path* $\pi = (x_0, x_1, \dots, x_m)$ is now a sequence of vertices of $\mathbb{Z}^d$ with $x_{j+1} - x_j = \pm e_i, 1 \le i \le d$. It no longer corresponds to a unique space-time path $\widehat{\pi}$. A *space-time path* $\widehat{\pi}$ is now specified by giving the successive positions x_i and jumptimes s_i. For $s_1 < s_2 \dots$ we shall sometimes denote this path by $\widehat{\pi}(\{s_i, x_i\})$. We make the convention that $s_0 = 0$. In addition we are here only discussing space-time paths over the time interval $[0, t]$, so we tacitly take $s_m \le t$. $\widehat{\pi}(\{s_i, x_i\})$ is then the path which is at postion x_i during $[s_i, s_{i+1})$ for $i < m$ and at position x_m during $[s_m, t]$. If it is important that the path has exactly m jumptimes, then we shall write $\widehat{\pi}(\{s_i, x_i\}_{i \le m})$. Throughout, we shall also assume that the positions of our paths are contained in

$$\mathcal{C}(C_1 t \log t) = [-C_1 t \log t, C_1 t \log t]^d$$

with C_1 the constant of Lemma 4. We shall be particularly interested in the following class of paths with exactly ℓ jumps:

$$\Xi(\ell, t) = \{\widehat{\pi}(\{s_i, x_i\}_{0 \le i \le \ell}) \text{ with } 0 < s_1 < \dots < s_\ell < t \text{ and } x_i \in \mathcal{C}(C_1 t \log t)\}. \tag{6.1}$$

For a particle ρ alive at time t we abbreviate "the piece till t of the path associated to ρ" to "the associated path of ρ" (see the definition just after (2.5)). This path is a space-time path.

Analogously to Section 5 we fix γ_0 such that

$$\gamma_0 \prod_{j=1}^{\infty} [1 - 2^{-j/4}] \ge 2 \text{ and } \mu_A \Big\{ \frac{1}{2} e^{-D_A} \gamma_0 \prod_{j=1}^{\infty} [1 - 2^{-j/4}] - e + 1 \Big\} > 3d + 6. \tag{6.2}$$

Then we fix an integer $C_0 \ge 2$, so large that for all $r \ge 1$,

$$-C_0^{-r/2} + \Big(1 + \frac{C_8 (r \log C_0)^d}{C_0^r}\Big)(e^{C_0^{-r/2}} - 1)\big(1 - C_0^{-r/4}\big) \le -\frac{1}{2} C_0^{-3r/4}, \tag{6.3}$$

and

$$\chi_{r+1}(C_5, C_6) \le 1, \tag{6.4}$$

where C_8 is the constant of Lemma 12 below, and χ_{r+1} is given in Lemma 10 below. Note that C_8 does not depend on C_0. Also C_5, C_6, which are fixed as in the last statement of Lemma 10, do depend on γ_0 (which is fixed now), but not on C_0, so that we can indeed fulfill (6.3) and (6.4) by taking C_0 large. γ_r is defined as in (5.10). As in Section 5, (6.2) together with $C_0 \ge 2$, implies (5.8) and

$$\mu_A \Big\{ \frac{1}{2} e^{-D_A} \gamma_0 \prod_{j=1}^{\infty} \Big[1 - \frac{1}{C_0^{j/4}}\Big] - e + 1 \Big\} > 3d + 6. \tag{6.5}$$

As before we take

$$\Delta_r = C_0^{6r},$$

$$\mathcal{B}_r(\mathbf{i}, k) := \prod_{s=1}^{d} [i(s)\Delta_r, (i(s)+1)\Delta_r) \times [k\Delta_r, (k+1)\Delta_r),$$

$$\widetilde{\mathcal{B}}_r(\mathbf{i}, k) := \prod_{s=1}^{d} [(i(s)-3)\Delta_r, (i(s)+4)\Delta_r) \times [(k-1)\Delta_r, (k+1)\Delta_r),$$

$$\widetilde{\mathcal{B}}_r^{+}(\mathbf{i}, k) := \prod_{s=1}^{d} [(i(s)-1)\Delta_r, (i(s)+2)\Delta_r) \times [k\Delta_r - \Delta_{r-1}, (k+1)\Delta_r),$$

and finally the pedestal of $\mathcal{B}_r(\mathbf{i},k)$ is defined as

$$\mathcal{V}_r(\mathbf{i},k) = \prod_{s=1}^{d} [(i(s)-3)\Delta_r, (i(s)+4)\Delta_r) \times \{(k-1)\Delta_r\}.$$

Next we define as before

$$\mathcal{Q}_r(x) = \prod_{s=1}^{d} [x(s), x(s) + C_0^r)$$

and for any time v, even one that is not an integer,

$$U_r^c(x,v) = \sum_{y \in \mathcal{Q}_r(x)} N_A(y,v).$$

The definitions (5.11) and (5.12) of a *good block* and a *good pedestal* are as before, but now v is not restricted to the integers: The r-block $\mathcal{B}_r(\mathbf{i},k)$ is called *good* if

$$U_r^c(x,v) \le \gamma_r \mu_A C_0^{dr} \text{ for all } (x,v) \text{ for which } \mathcal{Q}_r(x) \times \{v\} \subset \widetilde{\mathcal{B}}_r(\mathbf{i},k). \tag{6.6}$$

A *bad r-block* is one that is not good. The pedestal $\mathcal{V}_r(\mathbf{i},k)$ is called good if

$$U_r^c(x,v) \le \gamma_r \mu_A C_0^{dr} \text{ for all } (x,v) \text{ for which } \mathcal{Q}_r(x) \times \{v\} \subset \mathcal{V}_r(\mathbf{i},k). \tag{6.7}$$

For a space-time path $\widehat{\pi}(\{s_i, x_i\})$ we define

$$\phi_r^c(\widehat{\pi}) = \text{ number of bad } r\text{-blocks which intersect the space-time path } \widehat{\pi}. \tag{6.8}$$

Further

$$\Phi_r^c(\ell) = \sup_{\widehat{\pi} \in \Xi(\ell,t)} \phi_r^c(\widehat{\pi}). \tag{6.9}$$

We call an r-block $\mathcal{B}_r(\mathbf{i},k)$ *contaminated* if there are A-particles at some space-time point (x,v) in $\widetilde{\mathcal{B}}_r^+(\mathbf{i},k)$ which have also visited some point (y,v') with $y \notin \prod_{s=1}^d [(i(s)-3)\Delta_r, (i(s)+4)\Delta_r)$ and $(k-1)\Delta_r \le v' < v$. This last definition can also be phrased in terms of the following quanities: if $(x,v) \in \mathcal{B}_r(\mathbf{i},k)$, then define

$$w_r^c(x,v,\mathbf{i},k) := (\text{number of } A\text{-particles at } (x,v) \text{ which were in } \prod_{s=1}^{d} [(i(s)-3)\Delta_r, (i(s)+4)\Delta_r) \text{ during the whole interval } [(k-1)\Delta_r, v).$$

Then $\mathcal{B}_r(\mathbf{i},k)$ is contaminated if and only if $w_r^c(x,v,\mathbf{i},k) < N_A(x,v)$ for some $(x,v) \in \widetilde{\mathcal{B}}_r^+(\mathbf{i},k)$.

In analogy with $U_r^c(x,k)$ we define

$$W_r^c(x,v,\mathbf{i},k) = \sum_{y \in \mathcal{Q}_r(x)} w_{r+1}^c(y,v,\mathbf{i},k).$$

Note the subscript $r+1$ on the w_{r+1}^c here. This is not a typographical error. We want the sum over a cube of edgelength C_0^r of the number of particles which have stayed inside a cube of edgelength $7\Delta_{r+1}$ for a while. A partial analogue of a bad r-block is an r-block $\mathcal{B}_r(\mathbf{j},q)$ which has $W_r^c(x,v,\mathbf{i},k) > \gamma_r \mu_A C_0^{dr}$ for some (x,v) for which $\mathcal{Q}_r(x) \times \{v\} \subset \widetilde{\mathcal{B}}_r(\mathbf{j},q)$, and where $\mathcal{B}_{r+1}(\mathbf{i},k)$ is the unique $(r+1)$-block containing $\mathcal{B}_r(\mathbf{j},q)$. We shall call such an r-block *inferior.* Note that $W_r^c(x,v) \le U_r^c(x,v)$ and hence an inferior block is also a bad block.

We can now define two new quantities, which together take the place of Ψ in the last section:

$$\Omega_{r+1}^c(\ell) = \sup_{\widehat{\pi} \in \Xi(\ell,t)} (\text{number of good } (r+1)\text{-blocks } \mathcal{B}_{r+1}(\mathbf{i},k) \\ \text{which are also contaminated and intersect } \widehat{\pi})$$

and

$$\Theta_{r+1}^c(\ell) = \sup_{\widehat{\pi} \in \Xi(\ell,t)} (\text{number of good } (r+1)\text{-blocks } \mathcal{B}_{r+1}(\mathbf{i},k) \text{ which} \\ \text{contain some inferior } r\text{-block and which intersect } \widehat{\pi}).$$

We can now imitate many of the steps from the last section. Lemma 5 is replaced by the following lemma.

Lemma 9. *Let $R = R(t)$ be such that*

$$C_0^R \ge [\log t]^{1/d} > C_0^{R-1}. \tag{6.10}$$

Then, for all large t,

$$P_A\{\Phi_r^c(\ell) > 0 \textit{ for any } r \ge R \textit{ and any } \ell \ge 0\} \le \frac{1}{t^2}. \tag{6.11}$$

Moreover, for any A-configuration with $\Phi_r^c(\ell) = 0$ for all $r \ge R(t)$, it holds for any path $\widehat{\pi}(\{s_i, x_i\}) \in \Xi(\ell,t)$

$$\begin{aligned} &E_B\{\textit{number of particles alive at } t \textit{ whose associated path has jumptimes} \\ &\qquad \textit{in } ds_1, ds_2, \dots, ds_\ell \textit{ and positions } x_1, x_2, \dots, x_\ell \textit{ after the jumps}|\mathcal{F}_A\} \\ &\le \mu_B \prod_{i=1}^{\ell} [D_B q(x_i - x_{i-1})] \\ &\quad \times \exp\Big\{ - (\delta + D_B)t + \beta t \gamma_1 \mu_A C_0^d + \beta \sum_{r=1}^{R-1} \gamma_{r+1} \mu_A C_0^{r(d+6)+d} \Phi_r^c(\ell) \Big\} ds_1 \dots ds_\ell. \end{aligned} \tag{6.12}$$

Proof. $\Phi_r^c(\ell) > 0$ for some ℓ can occur only if there is some bad r-block $\mathcal{B}_r(\mathbf{i},k)$ which intersects $\mathcal{C}(C_1 t \log t) \times [0,t]$. In turn, this requires that $U_r^c(x,v) > \gamma_r \mu_A C_0^{dr}$ for some (x,v) for which $\mathcal{Q}_r(x) \times \{v\} \subset \widetilde{B}_r(\mathbf{i},k)$. In particular, (x,v) must have x within distance $4C_0^{6r}$ of $\mathcal{C}(C_1 t \log t)$ and $v \in [-\Delta_r, t + \Delta_r)$. Thus

$$\begin{aligned} &P_A\{\Phi_r^c(\ell) > 0 \text{ for some } r \ge R \text{ and } \ell \ge 0\} \\ &\le \sum_{r \ge R} P_A\{U_r^c(x,v) > \gamma_r \mu_A C_0^{dr} \text{ for some } (x,v) \text{ with } x \text{ within} \\ &\qquad \text{distance } 4C_0^{6r} \text{ from } \mathcal{C}(C_1 t \log t) \text{ and } -\Delta_r \le v < t + \Delta_r\}. \end{aligned} \tag{6.13}$$

Now let $k \le v \le (k+1)$ for some integer k. The probability that an A-particle at a site y at time v is still at y at time $(k+1)$ is at least e^{-D_A}. Therefore, if

$$U_r^c(x,v) = \sum_{y \in \mathcal{Q}_r(x)} N_A(y,v) > \gamma_r \mu_A C_0^{dr},$$

then there is a conditional probability of at least some $K_1 > 0$, that there are at least

$$\frac{1}{2} e^{-D_A} \gamma_r \mu_A C_0^{dr}$$

particles in $\mathcal{Q}_r(x)$ at time $(k+1)$ (compare (2.30) and the lines following it). We conclude that the r-th summand in the right hand side of (6.13) is for $\theta_0 = 1$, t large, and $r \geq R$ at most

$$\sum_{-\Delta_r \leq k \leq t+\Delta_r} \quad \sum_{\substack{x \text{ within distance} \\ 4C_0^{6r} \text{ from } \mathcal{C}(C_1 t \log t)}} \frac{1}{K_1}$$
$$\times P_A\Big\{ \sum_{y \in \mathcal{Q}_r(x)} N_A(y,(k+1)) > \frac{1}{2} e^{-D_A} \gamma_r \mu_A C_0^{dr} \Big\}$$
$$\leq K_2(t+\Delta_r)[t \log t + C_0^{6r}]^d \exp[-\theta_0 \frac{1}{2} e^{-D_A} \gamma_r \mu_A C_0^{dr} + \mu_A C_0^{dr}(e^{\theta_0}-1)]$$
$$\leq K_3[t \log t + C_0^{6r}]^{d+1} \exp[-(3d+6)C_0^{dr}] \text{ (see (6.5))}$$
$$\leq \frac{K_4}{t^3} \exp[-C_0^{dr}]$$

(compare (5.19)). Summing over $r \geq R$ now yields the desired (6.11).

With regard to (6.12), let $\widehat{\pi} \in \Xi(\ell, t)$ have jumptimes s_i, and positions x_i after the jumps for $1 \leq i \leq \ell$. Observe that as in (2.16)

$$E_B\{\text{number of particles alive at } t \text{ whose associated path has jumptimes}$$
$$\text{in } ds_1, ds_2, \dots, ds_\ell \text{ and positions } x_1, x_2, \dots, x_\ell \text{ after the jumps} | \mathcal{F}_A\}$$
$$= \mu_B \exp[-(\delta + D_B)t + \beta J(\{s_i, x_i\}_{i \leq \ell})] \prod_{i=1}^{\ell} [D_B q(x_i - x_{i-1})] ds_1 \dots ds_\ell. \tag{6.14}$$

Here, as in (2.15),

$$J(\{s_i, x_i\}_{i \leq \ell}) = \sum_{i=1}^{\ell} \int_{s_{i-1}}^{s_i} N_A(x_{i-1}, u) du + \int_{s_\ell}^{t} N_A(x_\ell, u) du. \tag{6.15}$$

To estimate this we define the set of times when the appropriate N_A lies in the interval $(\gamma_r \mu_A C_0^{dr}, \gamma_{r+1} \mu_A C_0^{d(r+1)}]$:

$$\Lambda_r = \Lambda_r(\widehat{\pi}) = \bigcup_{i=1}^{\ell} \{u \in [s_{i-1}, s_i) : \gamma_r \mu_A C_0^{dr} < N_A(x_{i-1}, u) \leq \gamma_{r+1} \mu_A C_0^{d(r+1)}\}$$
$$\cup \{u \in [s_\ell, t] : \gamma_r \mu_A C_0^{dr} < N_A(x_\ell, u) \leq \gamma_{r+1} \mu_A C_0^{d(r+1)}\}.$$

Then, analogously to the estimates for (5.20),

$$J(\{s_i, x_i\}_{i \leq \ell}) \leq t \gamma_1 \mu_A C_0^d + \sum_{r=1}^{\infty} \gamma_{r+1} \mu_A C_0^{d(r+1)} |\Lambda_r|, \tag{6.16}$$

where $|\Lambda_r|$ denotes the Lebesgue measure of Λ_r. (The first term in the right hand side comes from the contributions of times u when the appropriate N_A is $\leq \gamma_1 \mu_A C_0^d$.) Now if $\gamma_r \mu_A C_0^{dr} < N_A(x_{i-1}, u) \leq \gamma_{r+1} \mu_A C_0^{d(r+1)}$, then there is some $z \in [0, C_0^r)^d$ such that $\mathcal{Q}_r(x_{i-1} - z) \times \{u\}$ is contained in some $B_r(\mathbf{i}, m)$, which contains (x_{i-1}, u) and which is bad (compare the lines following (5.21)). Therefore, (x_{i-1}, u) belongs to one of the bad r-cubes which intersect $\widehat{\pi}$. There are at most $\Phi_r^c(\ell)$ such cubes and any space-time path spends at most Δ_r units of time in a given bad r-block. This shows that

$$|\Lambda_r| \leq \Delta_r \Phi_r(\ell) = C_0^{6r} \Phi_r^c(\ell).$$

(6.12) now follows from (6.14)-(6.16) and the assumption $\Phi_r^c(\ell) = 0$ for $r \geq R$. ∎

Our next task is to estimate the distribution of $\Omega^c_{r+1}(\ell)$. This estimate is new and has no analogue in the discrete model of Section 5. Indeed, in that model it is not possible for a particle which is outside $\prod_{s=1}^d[(i(s)-3)\Delta_{r+1},(i(s)+4)\Delta_{r+1})$ at some time $v' \in [(k-1)\Delta_{r+1},(k+1)\Delta_{r+1})$ to reach $\mathcal{B}_{r+1}(\mathbf{i},k)$ during $[v',(k+1)\Delta_{r+1})$. Thus there are no $(r+1)$-boxes in the discrete model which can be called contaminated. Here we shall not immediately estimate the distribution of the full $\Omega^c_{r+1}(\ell)$ but only of $\Omega^c_{r+1}(\ell,\mathbf{a},b)$, which is defined just as $\Omega^c_{r+1}(\ell)$, except that it only counts good blocks $\mathcal{B}^c_{r+1}(\mathbf{i},k)$ which intersect $\widehat{\pi}$, are contaminated and have $(\mathbf{i},k) \equiv (a,b)$, as defined in the lines before (5.45). Since $\Omega^c_{r+1}(\ell)$ is bounded by the finite sum $\sum_{(\mathbf{a},b)} \Omega^c_{r+1}(\ell,\mathbf{a},b)$, it will be easy to obtain a bound for the distribution of $\Omega^c_{r+1}(\ell)$ from the bounds for $\Omega^c_{r+1}(\ell,\mathbf{a},b)$. For the time being let $(\mathbf{a},b)$ be fixed.

To produce an estimate for $\Omega^c_{r+1}(\ell,\mathbf{a},b)$ we use a time discretization for the paths of the A-particles. Instead of letting each A-particle perform a translated copy of the random walk $\{S_u\}$, we let each A-particle perform a random walk which can only jump at the times k/n, $k \in \mathbb{Z}$, and at each such time it moves an amount y which has the distribution

$$q^n_A(y) := \begin{cases} 1-\frac{D_A}{n} & \text{if } y=\mathbf{0} \\ \frac{D_A}{2dn} & \text{if } y=\pm e_i,\ 1\le i\le d. \end{cases}$$

Here n is a positive integer, which we shall later let go to infinity. Let $\{S^n_u\}_{u\ge 0}$ be a random walk with this transition probability. It is clear that $\{S^n_u\}_{0\le u\le t}$ converges weakly (as $n\to\infty$) in the Skorokhod space $D(\mathbb{Z}^d,[0,t])$ to $\{S_u\}_{0\le u\le t}$. Consequently, the path of any given A-particle with the discretized motion converges weakly to the path under the true continuous time random walk. We have similar convergence for the paths of any finite number of independent A-particles.

We now use a special construction for the paths of the A-particles in the present discretized model with transition probabilities q^n_A. For the sake of definiteness we take $b=0$; as we point out below, only a trivial modification is needed for the case $b=1$. For each (x,s) in space-time, with $x\in\mathbb{Z}^d, s\in(1/n)\mathbb{Z}$, and for each integer $q\ge 0$ we let $\{S^n_u(x,s,q)\}_{u\ge 0}$ be an independent copy of the random walk $\{S^n_u\}_{u\ge 0}$. All these random walks are independent of each other. In addition we attach to each A-particle σ in the system a uniform $[0,1]$ random variable $U(\sigma)$. All these uniform random variables will be independent of each other and of the random walks. Their only use will be to order some particles in some definite order (but the order itself has no significance; any order would do). Finally, let

$$\mathcal{W}_{r+1}(\mathbf{i}) = \partial \prod_{s=1}^d [(i(s)-3)\Delta_{r+1},(i(s)+4)\Delta_{r+1}-1],$$

where ∂ denotes the topological boundary. Notice that the path of any particle which is outside $\prod_{s=1}^d[(i(s)-3)\Delta_{r+1},(i(s)+4)\Delta_{r+1})$ at some time $v'\in[(k-1)\Delta_{r+1},(k+1)\Delta_{r+1})$ which also visits $\widetilde{\mathcal{B}}^+_{r+1}(\mathbf{i},k)$ at some time $v>v'$, must contain a piece $(x_\xi,x_{\xi+1},\dots,x_\zeta)$ which crosses from $\mathcal{W}_{r+1}(\mathbf{i})$ to $\partial\prod_{s=1}^d[(i(s)-1)\Delta_{r+1},(i(s)+2)\Delta_{r+1}-1]$. More specifically, there must be a piece $(x_\xi,x_{\xi+1},\dots,x_\zeta)$ which satisfies

$$x_\xi\in\mathcal{W}_{r+1}(\mathbf{i}),\quad x_\zeta\in\partial\prod_{s=1}^d[(i(s)-1)\Delta_{r+1},(i(s)+2)\Delta_{r+1}-1]$$
$$\text{and } x_\kappa\in\prod_{s=1}^d[(i(s)-3)\Delta_{r+1}+1,(i(s)+4)\Delta_{r+1}-2] \text{ for } \xi<\kappa<\zeta, \tag{6.17}$$

and which is traversed during $[v',v]\subset[(k-1)\Delta_{r+1},(k+1)\Delta_{r+1})$. Moreover, v' and v can be taken as integer multiples of $1/n$, because particles in the discretized sytem only move at such times. (This is a path from $\mathcal{W}_{r+1}(\mathbf{i})$ to $\partial\prod_{s=1}^d[(i(s)-1)\Delta_{r+1},(i(s)+2)\Delta_{r+1}-1]$ in the "interior" of $\mathcal{W}_{r+1}(\mathbf{i})$. Note that $\mathcal{W}_{r+1}(\mathbf{i})$ is the surface of a cube in space which contains $\partial\prod_{s=1}^d[(i(s)-1)\Delta_{r+1},(i(s)+2)\Delta_{r+1}-1]$ in its interior. In turn, the latter is the surface of a cube which contains $\prod_{s=1}^d[i(s)\Delta_{r+1},(i(s)+1)\Delta_{r+1})$ in its interior.)

The last observation is the motivation for the following construction. Assume that the paths of all A-particles till time $(k-1)\Delta_{r+1}$ with k even have already been constructed in some way. In the case $k=0$ this simply means that we begin with a mean μ_A Poisson system of A-particles at time $-\Delta_{r+1}$. (The only change which is needed for the case $b=1$ is that we work with odd k's and start with a Poisson system at time $-2\Delta_{r+1}$ in that case.) At each point $(x,(k-1)\Delta_{r+1})$ (in space-time) order all particles σ present so that their associated uniform variables $U(\sigma)$ are increasing. To the q-th particle in this order associate the path $\{x+S^n_u(x,(k-1)\Delta_{r+1},q)\}_{u\geq 0}$. This particle then moves to $x+S^n_1(x,(k-1)\Delta_{r+1},q)$ at time $(k-1)\Delta_{r+1}+1/n$. The *index* associated with this particle is then taken to be $(x,(k-1)\Delta_{r+1},q,1)$. The last coordinate 1 here indicates that one step was taken since the last choice of an associated random walk. Assume we have constructed the paths of all particles up to and including time $v\in[(k-1)\Delta_{r+1},(k+1)\Delta_{r+1})$ (with v a multiple of $1/n$) and that each particle has an index. To construct the paths $1/n$ time units further, we look for each $y\in\mathbb{Z}^d$ at all particles at (y,v). If y does not belong to

$$\bigcup_{\mathbf{j}\equiv\mathbf{a}}\mathcal{W}_{r+1}(\mathbf{j}),\tag{6.18}$$

and a particle at (y,v) has index (z,v',q,g), then this particle moves to $y+S^n_{g+1}(z,v',q)$ and its new index is $(z,v',q,g+1)$. In other words it moves one step further in the random walk it is presently asociated with, and the last component of its index increases by 1, to indicate the number of steps the particle has taken according to its present associated random walk. (The first two components z,v' indicate at which space-time point the particle started using this random walk and the third component specifies which of the random walks from the point (z,v') the particle is using.) If on the other hand, y lies in the union (6.18), then all particles at this point are again ranked according to increasing values of their uniform random variables and the particle with rank q' will move to $y+S^n_1(y,v,q')$ at time $v+1/n$. Its index will then be $(y,v,q',1)$. We continue this procedure till all positions at time $(k+1)\Delta_{r+1}$ have been determined. We then start anew with k replaced by $k+1$. That is, we order all particles at one site $(x,(k+1)\Delta_{r+1})$ and move the q-th particle at that site to $x+S^n_1(x,(k+1)\Delta_{r+1}),q)$ and give it the index $(x,(k+1)\Delta_{r+1},q,1)$, and so on.

Basically, the above procedure switches each particle to a new random walk every time the particle visits the set (6.18). It is clear that in the above construction all the A-particles perform independent random walks with transition probability q^n_A. Thus this gives us a construction of the discretized A-system.

Lemma 10. *For $0<C_5,C_6<\infty$ let*

$$\chi_{r+1}(C_5,C_6)=C_5\exp[-C_6C_0^{6(r+1)}].$$

Also define

$$\begin{aligned}Y(\mathbf{i},k)=&I[\mathcal{B}_{r+1}(\mathbf{i},k)\textit{ is good, but also contaminated,}\\&\textit{and intersects }[-C_1t\log t,C_1t\log t]^d\times[0,t],\end{aligned}$$

and let $Z(\mathbf{i},k)$ be a system of independent random variables with

$$P\{Z(\mathbf{i},k)=1\}=1-P\{Z(\mathbf{i},k)=0\}=\chi_{r+1}(C_5,C_6).\tag{6.19}$$

Then one can fix $0<C_5,C_6<\infty$, with C_5,C_6 depending on d, γ_0,μ_A and D_A only (but not on C_0 or r), such that for fixed $(\mathbf{a},b)$, the collection

$$\begin{aligned}\{Y(\mathbf{i},k):&(\mathbf{i},k)\textit{ such that }(\mathbf{i},k)\equiv(\mathbf{a},b)\textit{ and }\mathcal{B}_{r+1}(\mathbf{i},k)\\&\textit{intersects }[-C_1t\log t,C_1t\log t]^d\times[0,t]\}\end{aligned}\tag{6.20}$$

lies stochastically below the collection

$$\begin{aligned}\{Z(\mathbf{i},k):&(\mathbf{i},k)\textit{ such that }(\mathbf{i},k)\equiv(\mathbf{a},b)\textit{ and }\mathcal{B}_{r+1}(\mathbf{i},k)\\&\textit{intersects }[-C_1t\log t,C_1t\log t]^d\times[0,t]\}.\end{aligned}\tag{6.21}$$

Proof. We shall prove the analogue of the claim of the lemma for the discretized system with bounds which are uniform in n. More precisely, let the A-particles move according to copies of (translates of) $\{S_u^n\}$, and let $N_A^n(x,u)$ be the number of A-particles at (x,u) in this system. We then say that $\mathcal{B}_{r+1}(\mathbf{i},k)$ is good (at level n) if

$$\sum_{y\in\mathcal{Q}_r(x)} N_A^n(y,v) \le \gamma_r\mu_A C_0^{dr} \text{ for all } (x,v) \text{ for which } \mathcal{Q}_r(x)\times\{v\}\subset \widetilde{\mathcal{B}}_{r+1}(\mathbf{i},k). \tag{6.22}$$

This is of course simply the previous definition (6.6) with N_A replaced by N_A^n. Similarly, the definition of a contaminated block needs no change, except that the A-particles are now assumed to move according to $\{S_u^n\}$. Finally,

$$\begin{aligned} Y^n(\mathbf{i},k) =& I\Big[\mathcal{B}_{r+1}(\mathbf{i},k) \text{ is good, but also contaminated at level } n,\\ &\text{and intersects } [-C_1t\log t, C_1t\log t]^d\times[0,t]\Big], \end{aligned}$$

One can now check that $Y^n(\mathbf{i},k)$ converges weakly to $Y(\mathbf{i},k)$ as $n\to\infty$. This would be a simple consequence of the continuous mapping theorem (see Billingsley (1968), Theorem 5.2) if there were only finitely many A-particles in the system, because $Y(\mathbf{i},k)$ would be an almost surely continuous function of the paths of these A-particles. But we have already seen that only finitely many A-particles enter a bounded region in space-time (either as a special case of (2.26) or by the proof of (2.35)). This same proof can be used to show that

$$\begin{aligned} P\{&\text{in the discretized system in which particles move according to } \{S_u^n\}\\ &\text{there is some particle which visits } [-C_1t\log t - 4C_0^{6(r+1)}, C_1t\log t + 4C_0^{6(r+1)}]\\ &\text{and the complement of } [-M,M]^d \text{ during } [-t,t]\} \end{aligned}$$

tends to 0, uniformly in n, as $M\to\infty$. From this one sees that one can remove all particles outside $[-M,M]^d$ from the system with only a small probability that $Y(\mathbf{i},k)$ or $Y^n(\mathbf{i},k)$ are changed if M is large. Thus $Y(\mathbf{i},k)$ and $Y^n(\mathbf{i},k)$ can be approximated in probability by functions of only finitely many A-particles, so that indeed $Y^n(\mathbf{i},k)$ converges weakly to $Y(\mathbf{i},k)$. The same argument shows that the joint distribution of the family of $Y(\mathbf{i},k)$ in (6.20) is the limit of the joint distribution of the corresponding family of $Y^n(\mathbf{i},k)$. We leave the tedious details to the reader. The result of this weak convergence is that it suffices to prove the lemma for the Y^n instead of Y, as long as the estimates are uniform in n.

We now prove the lemma with Y^n in the place of Y. For the sake of definiteness let $b=0$. We construct the discretized A-system as described before the lemma by means of the random walks $\{S_u^n(x,s,q)\}$. Assume that some $\mathcal{B}_{r+1}(\mathbf{i},k)$ with $(\mathbf{i},k)\equiv(a,b)$ is contaminated. As pointed out for (6.17) this implies that the path of some A-particle must contain a piece which satisfies (6.17). If this piece starts at $x_\xi\in\mathcal{W}_{r+1}(\mathbf{i})$ at time v, then our construction is such that the piece $(x_\xi,\dots,x_\zeta)$ has to consist of the first $\zeta-\xi-1$ positions of $x_\xi+S_u^n(x_\xi,v,q)$ for some q, because the x's in this piece do not lie in the union (6.18). That implies that there is some $x\in\mathcal{W}_{r+1}(\mathbf{i})$, some q, and some $v\in[(k-1)\Delta_{r+1},(k+1)\Delta_{r+1})$ for which

$$S_1^n(x,v,q)\neq\mathbf{0} \text{ and } \max_{u\le 2\Delta_{r+1}}\|S_u^n(x,v,q)\|\ge 2\Delta_{r+1}. \tag{6.23}$$

As pointed out for (6.17) we can restrict v to the integer multiples of $1/n$ here. Now $(x,v)\in\mathcal{W}_{r+1}(\mathbf{i})\times\{v\}$ is contained in $\widetilde{\mathcal{B}}_{r+1}(\mathbf{i},k)$. If $\mathcal{B}_{r+1}(\mathbf{i},k)$ is good at level n, then $N_A^n(x,v)\le\gamma_{r+1}\mu_A C_0^{d(r+1)}\le K_5C_0^{d(r+1)}$ ($K_5=\gamma_0\mu_A$ will do). In this case (6.23) must occur with a $q\le K_5C_0^{d(r+1)}$, because there are at most $K_5C_0^{d(r+1)}$ particles at (x,v). It follows that

$$\begin{aligned} &\big\{\mathcal{B}_{r+1}(\mathbf{i},k) \text{ is good, but also contaminated, at level } n\big\}\\ &\subset \mathcal{E}_{r+1}^n(\mathbf{i},k)\\ &:= \bigcup_{\substack{x\in\mathcal{W}_{r+1}(\mathbf{i}),\ nv\in\mathbb{Z}\\ v\in[(k-1)\Delta_{r+1},(k+1)\Delta_{r+1})\\ q\le K_5C_0^{d(r+1)}}} \{S_1^n(x,v,q)\neq\mathbf{0},\ \max_{u\le 2\Delta_{r+1}}\|S_u^n(x,v,q)\|\ge 2\Delta_{r+1}\}. \end{aligned} \tag{6.24}$$

In other words,

$$\{Y^n(\mathbf{i},k)=1\}\subset \mathcal{E}^n_{r+1}(\mathbf{i},k). \tag{6.25}$$

Now the sets $\mathcal{W}_{r+1}(\mathbf{i})\times[(k-1)\Delta_{r+1},(k+1)\Delta_{r+1})$ for different $(\mathbf{i},k)\equiv(\mathbf{a},b)$ are disjoint, and therefore the corresponding events $\mathcal{E}^n_{r+1}(\mathbf{i},k)$ are independent. Moreover, $\mathcal{E}^n_{r+1}(\mathbf{i},k)$ is defined in (6.24) as a union over at most

$$|\mathcal{W}_{r+1}(\mathbf{i})|K_5C_0^{d(r+1)}2\Delta_{r+1}n\le K_6\Delta^d_{r+1}C_0^{d(r+1)}n=K_6C_0^{7d(r+1)}n$$

events of the form (6.23). Thus, once again by Bernstein's inequality,

$$\begin{aligned}P\{\mathcal{E}^n_{r+1}(\mathbf{i},k)\}&\le K_6nC_0^{7d(r+1)}P\{S^n_1\neq\mathbf{0},\max_{u\le 2\Delta_{r+1}}\|S^n_u\|\ge 2\Delta_{r+1}\}\\&\le K_7nC_0^{7d(r+1)}\frac{1}{n}\exp\left[-K_8\Delta_{r+1}\right]\le C_5\exp\left[-C_6C_0^{6(r+1)}\right]\end{aligned}$$

for suitable C_5,C_6 We can therefore extend our probability space such that on it are defined random variables $Z^n(\mathbf{i},k)$ which satisfy (6.19) and such that the collection (6.21) is independent when Z is replaced by Z^n in (6.19) and (6.21), and in addition are such that $Y^n(i,k)\le Z^n(i,k)$ (we take $Z^n(\mathbf{i},k)$ as the indicator function of a suitable set containing $\mathcal{E}^n_{r+1}(\mathbf{i},k)$). It follows that for every positive increasing function f of all its arguments,

$$Ef(Y^n(\mathbf{i},k))\le Ef(Z^n(\mathbf{i},k)) \tag{6.26}$$

where $(\mathbf{i},k)$ ranges over the pairs appearing in (6.19), (6.21). Actually the right hand side does not depend on n. If $Z(\mathbf{i},k)$ is an independent collection of random variables as in (6.21) and satisfying (6.19) then the right hand side of (6.26) equals $Ef(Z(\mathbf{i},k))$. By letting $n\to\infty$ we obtain that

$$Ef(Y(\mathbf{i},k))\le Ef(Z(\mathbf{i},k)). \tag{6.27}$$

In fact Theorem 11 of Strassen (1965) gives that (6.27) implies the existence of a coupling of the families in (6.20) and (6.21) such that $Y(\mathbf{i},k)\le Z(\mathbf{i},k)$. Since we shall only need (6.27) we skip the details. ■

For the remainder we fix C_5,C_6 such that the properties stated in Lemma 10 hold. The discretized system will not be used anymore in the sequel.

Lemma 11. *There exist a constant $C_7=C_7(d)$ and constants κ_0,t_0 (independent of r,ℓ) such that for $1\le r\le R(t)$, $\kappa\ge\kappa_0$, $t\ge t_0$ and any $\ell\ge 0$*

$$P_A\big\{\Omega^c_{r+1}(\ell)\ge\kappa(t+\ell)\exp[-C_0^{r/2}]\big\}\le\exp\big[-C_7\kappa(t+\ell)\exp[-C_0^{r/2}]\big]. \tag{6.28}$$

Proof. As we already mentioned, we merely prove that (6.28) holds with $\Omega^c_{r+1}(\ell,\mathbf{a},b)$ instead of $\Omega^c_{r+1}(\ell)$. The estimate (6.28) as stated then follows by combining these estimates for all possible choices of $(\mathbf{a},b)$ and adjusting the constants.

The present proof is an imitation of the proof of (5.43). Analogously to (5.47), we fix ν such that

$$[\chi_{r+1}]^{-1/(d+1)}\le\nu\le 2[\chi_{r+1}]^{-1/(d+1)}$$

(note that $\chi_{r+1}\le 1$, by (6.4)). We now define

$$\mathcal{D}^c(\mathbf{j},q)=\prod_{s=1}^d[\nu j(s)\Delta_{r+1},\nu(j(s)+1)\Delta_{r+1})\times[q\nu\Delta_{r+1},(q+1)\nu\Delta_{r+1}). \tag{6.29}$$

It is still true that each $\mathcal{D}^c(\mathbf{j},q)$ is the disjoint union of ν^{d+1} $(r+1)$-blocks. However, the argument following (5.48) to show that at most $\widetilde{\lambda}$ such blocks intersect a space-time path $\widehat{\pi}$ breaks down. Instead of this estimate, we now show that for $\ell\ge0$ at most

$$\lambda(\ell):=3^d\Big(\frac{t+\ell}{\nu\Delta_{r+1}}+2\Big) \tag{6.30}$$

blocks $\mathcal{D}^c(\mathbf{j}, q)$ can intersect a space-time path $\widehat{\pi} \in \Xi(\ell, t)$ with jumptimes $s_1 < \cdots < s_\ell < t$ and positions $x_1, \dots, x_\ell$ (see (6.1) for Ξ). To derive this bound, fix $\widehat{\pi} \in \Xi(\ell, t)$, and let σ_q be the number of boxes $\mathcal{D}^c(\mathbf{j}, q)$ which intersect $\widehat{\pi}$. Such a box can intersect $\widehat{\pi}$ only if $0 \le q \le t/(\nu\Delta_{r+1})$. If there are σ such boxes with a given q which intersect $\widehat{\pi}$, then the piece of π from time $q\nu\Delta_{r+1}$ through time $(q+1)\nu\Delta_{r+1} - 1$ must connect σ disjoint cubes of edgelength $\nu\Delta_{r+1}$. This requires at least $(\lfloor 3^{-d}\sigma \rfloor - 1)\nu\Delta_{r+1}$ jumps of $\widehat{\pi}$ during $[q\nu\Delta_{r+1}, (q+1)n\nu\Delta_{r+1})$. Since $\widehat{\pi} \in \Xi(\ell, t)$ this requires

$$\sum_{0 \le q \le t/(\nu\Delta_{r+1})} [3^{-d}\sigma_q - 2] \le \ell/(\nu\Delta_{r+1}).$$

Consequently, the total number of boxes $\mathcal{D}^c(\mathbf{j}, q)$ which intersect $\widehat{\pi}$ is at most

$$\begin{aligned}\sum_{0 \le q \le t/(\nu\Delta_{r+1})} \sigma_q &= 3^d \sum_{0 \le q \le t/(\nu\Delta_{r+1})} [3^{-d}\sigma_q - 2 + 2] \\ &\le 3^d \ell/(\nu\Delta_{r+1}) + 3^d \sum_{0 \le q \le t/(\nu\Delta_{r+1})} 2 \\ &\le 3^d(\ell + t)/(\nu\Delta_{r+1}) + 3^d 2.\end{aligned}$$

This proves the bound (6.30).

From here on the proof of (6.28) is essentially the same as that of (5.43) from (5.48) on, so we shall be brief. As in (5.46) and (5.51) we have

$$\begin{aligned}&P_A\{\Omega^c_{r+1}(\ell, \mathbf{a}, b) \ge 2^{-1}(12)^{-d}\kappa(t+\ell)\exp[-C_0^{r/2}]\} \\ &\le \sum_{\mathcal{D}(\mathbf{j}_0,0),\dots,\mathcal{D}(\mathbf{j}_{\lambda-1},\lambda-1)} P\Big\{\bigcup_{q=0}^{\lambda-1} \mathcal{D}^c(\mathbf{j}_q, q) \text{ contains at least} \\ &\qquad 2^{-1}(12)^{-d}\kappa(t+\ell)\exp[-C_0^{r/2}] \quad (r+1)\text{-blocks } \mathcal{B}_{r+1}(\mathbf{i}, k) \text{ with } Z(\mathbf{i}, k) = 1\Big\}.\end{aligned} \tag{6.31}$$

Here $(\mathcal{D}^c(\mathbf{j}_0, 0), \dots, \mathcal{D}^c(\mathbf{j}_{\lambda-1}, \lambda-1))$ runs over the possible collections of blocks $\mathcal{D}^c$ which intersect a space-time path $\widehat{\pi} \in \Xi(\ell, t)$. For some constant K_9 which depends on d only, there are at most

$$[2C_1 t \log t + 1]^d \exp[K_9\lambda]$$

collections of this form. If we fix such a collection $\mathcal{D}^c(\mathbf{j}_0, 0), \dots, \mathcal{D}^c(\mathbf{j}_{\lambda-1}, \lambda-1)$, then the probability that

$$\bigcup_{q=0}^{\lambda-1} \mathcal{D}(\mathbf{j}_q, q)$$

contains at least $2^{-1}(12)^{-d}\kappa(t+\ell)\exp[-C_0^{r/2}]$ $(r+1)$-blocks $\mathcal{B}_{r+1}(\mathbf{i}, k)$ with $(\mathbf{i}, k) \equiv (\mathbf{a}, b)$ and with $Z(\mathbf{i}, k) = 1$, is bounded by

$$P\{T \ge 2^{-1}(12)^{-d}\kappa(t+\ell)\exp[-C_0^{r/2}], \tag{6.32}$$

where T has a binomial distribution corresponding to $\lambda\nu^{d+1}$ trials with success probability χ_{r+1}. By Bernstein's inequality there exists some constant $C_7 = C_7(d)$ such that for any $\lambda \ge 2ET$ it holds

$$P\{T \ge \lambda\} \le \exp[-C_7 4(12)^d\lambda]. \tag{6.33}$$

We now fix $\kappa_0 \ge 1$ such that for all $r \ge 1$

$$\begin{aligned}\kappa_0 \exp[-C_0^{r/2}] &\ge \big[16 \cdot (36)^d + \frac{4K_9 3^d}{C_7}\big][\chi_{r+1}]^{1/(d+1)} \\ &= \big[16 \cdot (36)^d + \frac{4K_9 3^d}{C_7}\big] C_5^{1/(d+1)} \exp\big[-\frac{C_6}{d+1} C_0^{6(r+1)}\big],\end{aligned} \tag{6.34}$$

Then, for $\kappa \ge \kappa_0$, and $1 \le r \le R(t) - 1$,

$$2^{-1}(12)^{-d}\kappa(t+\ell)\exp[-C_0^{r/2}] \ge 4\lambda \ge 2\lambda\nu^{d+1}\chi_{r+1} = 2E\{T\}$$

for all $\ell \ge 0$, provided $t \ge$ some t_1 (independent of $r \ge 1, \ell \ge 0$). Thus, by (6.33), the probability in (6.32) is, at most

$$\exp\big[-2C_7\kappa(t+\ell)\exp[-C_0^{r/2}]\big]. \tag{6.35}$$

This is also a bound for each of the summands in the right hand side of (6.31). It follows that the left hand side of (6.28) is for $t \ge$ some t_2 (independent of $r \ge 1$, $\ell \ge 0$) bounded by

$$\begin{aligned}&2\cdot 12^d(2C_1 t\log t+1)^d\exp\big[K_9\lambda - 2C_7\kappa(t+\ell)\exp[-C_0^{r/2}]\big]\\ &\le \exp\big[-C_7\kappa(t+\ell)\exp[-C_0^{r/2}]\big].\end{aligned}$$

■

We must next estimate the tail of the distribution of $\Theta^c_{r+1}(\ell)$. This will replace the estimate for Ψ_{r+1} in (5.43). First we note the following analogue of Lemma 6.

Lemma 12. *There exists a constant C_8, which is independent of C_0, such that if $\mathcal{V}_{r+1}(\mathbf{i},k)$ is good, and $(\Delta_{r+1}-\Delta_r) \le u \le 2\Delta_{r+1}$, then for $r \ge 1$*

$$\begin{aligned}\sum_{z:(z,(k-1)\Delta_{r+1})\in\mathcal{V}_{r+1}(\mathbf{i},k)} & N_A(z,(k-1)\Delta_{r+1})P\{z+S_u\in\mathcal{Q}_r(y)\}\\ &\le \gamma_{r+1}\mu_A C_0^{dr}\Big[1+\frac{C_8(r\log C_0)^d}{C_0^{2r}}\Big]\end{aligned} \tag{6.36}$$

for $r \ge 1, y \in \mathbb{Z}^d$.

Proof. The proof proceeds exactly as in Lemma 6 with $\{\widetilde{S}_u\}$ replaced by $\{S_u\}$ till we come to (5.26), which needs to be proven for nonintegral u as well now. To prove this inequality in this generality, note that just as in (5.27)

$$\begin{aligned}\big|P\{S_u=v\}-P\{S_u=w\}\big| &\le \frac{d}{(2\pi)^d}\int_{\theta\in[-\pi,\pi]^d}\|v-w\|\|\theta\||Ee^{i\theta\cdot S_u}|d\theta\\ &\le \frac{d}{(2\pi)^d}\int_{\theta\in[-\pi,\pi]^d}\|v-w\|\|\theta\||\psi(\theta)|^{\lfloor u\rfloor}d\theta\end{aligned} \tag{6.37}$$

with

$$\psi(\theta) = Ee^{i\theta\cdot S_1} \le \exp[-K_{10}\|\theta\|^2] \text{ for } \theta(i)\in[-\pi,\pi],\ 1\le i\le d.$$

In the last inequality in (6.37) we used that S_u is the sum of the independent variables $S_1 - S_0, S_2 - S_1, \dots, S_{\lfloor u\rfloor} - S_{\lfloor u\rfloor -1}, S_u - S_{\lfloor u\rfloor}$. The rest is as in Lemma 6.

■

We need a replacement for the σ-field $\mathcal{H}_{r+1}(\mathbf{i},k)$. We define this as follows:

$$\begin{aligned}\mathcal{H}^c_{r+1}(\mathbf{i},k) :=&\ \sigma\text{-field generated by the paths of all particles through}\\ &\text{time } (k-1)\Delta_{r+1} \text{ plus the paths till time } (k+1)\Delta_{r+1}\\ &\text{of all particles which are not in } \mathcal{V}_{r+1}(\mathbf{i},k) \text{ at time } (k-1)\Delta_{r+1}.\end{aligned}$$

$\mathcal{K}_{r+1}(\mathbf{i},k)$ is defined exactly as in (5.32). Note that all the $N_A(x,(k-1)\Delta_{r+1})$ with $(x,(k-1)\Delta_{r+1}) \in \mathcal{V}_{r+1}(\mathbf{i},k)$ are $\mathcal{K}_{r+1}(\mathbf{i},k)$-measurable, and that $\mathcal{K}_{r+1}(\mathbf{i},k) \subset \mathcal{H}^c_{r+1}(\mathbf{i},k)$. Lemma 7 needs essentially no change; we only need to replace "bad" by "inferior" and to replace the σ-fields.

Lemma 13. *Let*

$$\rho_{r+1} = C_0^{6(d+1)(r+1)+dr} \exp\big[-\frac{1}{2}\gamma_r\mu_A C_0^{(d-\frac{3}{4})r}\big].$$

Then there exists a constant $C_9 = C_9(D_A, \gamma_0\mu_A)$ *such that for* $1 \le r \le R(t)-1$, *on the event* $\{\mathcal{V}_{r+1}(\mathbf{i},k)$ *is good*$\}$,

$$\begin{aligned} &P_A\{\mathcal{B}_{r+1}(\mathbf{i},k) \text{ contains some inferior } \mathcal{B}_r(\mathbf{j},q)|\mathcal{H}^c_{r+1}(\mathbf{i},k)\}\\ &= P_A\{\mathcal{B}_{r+1}(\mathbf{i},k) \text{ contains some inferior } \mathcal{B}_r(\mathbf{j},q)|\mathcal{K}_{r+1}(\mathbf{i},k)\}\\ &\le C_9\rho_{r+1}. \end{aligned} \tag{6.38}$$

Proof. Whether an r-block $\mathcal{B}_r(\mathbf{j},q)$ which is contained in $\mathcal{B}_{r+1}(\mathbf{i},k)$ is inferior or not, depends only on the $w^c_{r+1}(x,v,\mathbf{i},k)$ with $(x,v) \in \widetilde{\mathcal{B}}_r(\mathbf{j},q)$. All such (x,v) lie in $\widetilde{\mathcal{B}}^+_{r+1}(\mathbf{i},k)$. Once all paths up to and including time $(k-1)\Delta_{r+1}$ are given, any such $w^c_{r+1}(x,v,\mathbf{i},k)$ depends only on the paths of particles which stay in $\prod_{s=1}^d[(i(s)-3)\Delta_{r+1},(i(s)+4)\Delta_{r+1})$ during the whole interval $[(k-1)\Delta_{r+1},v)$. Clearly this excludes the particles which are outside $\mathcal{V}_{r+1}(\mathbf{i},k)$ at time $(k-1)\Delta_{r+1}$. Therefore, the paths of the latter particles after time $(k-1)\Delta_{r+1}$ do not influence the $w^c_{r+1}(x,v,\mathbf{i},k)$ with (x,v) in $\widetilde{\mathcal{B}}^+_{r+1}(\mathbf{i},k)$. Also the paths of any particles through time $(k-1)\Delta_{r+1}$ do not influence the distribution of the $w^c_{r+1}(x,v,\mathbf{i},k)$ with (x,v) in $\widetilde{\mathcal{B}}^+_{r+1}(\mathbf{i},k)$ (as long as the $N^n_A(x,(k-1)\Delta_{r+1})$ with $(x,(k-1)\Delta_{r+1}) \in \mathcal{V}_{r+1}(\mathbf{i},k)$ are fixed). Therefore the two conditional probabilities in (6.38) are the same.

The fact that the two conditional probabilities in (6.38) are bounded by $C_9\rho_{r+1}$ is proven almost exactly as in Lemma 7. Very much as in (5.36) the conditional probabilities in (6.38) are bounded by the probability that

$$W^c_r(x,v,\mathbf{i},k) > \gamma_r\mu_A C_0^{dr} \text{ for some } (x,v) \text{ for which } \mathcal{Q}_r(y)\times\{v\} \subset \widetilde{\mathcal{B}}^+_{r+1}(\mathbf{i},k). \tag{6.39}$$

Here v necessarily satisfies

$$\frac{1}{2}\Delta_{r+1} \le \Delta_{r+1} - \Delta_r \le u := v-(k-1)\Delta_{r+1} \le 2\Delta_{r+1}. \tag{6.40}$$

We claim that we may restrict ourselves to v's which are an integer multiple of $K_{11}C_0^{-dr}$ for some small constant $K_{11} > 0$, at a cost of a factor 2 in the probability. By this we mean that the probability of the event in (6.39) is at most twice the probability of this event with the added requirement that v is a multiple of $K_{11}C_0^{-dr}$. This follows from an argument like the one following (6.13). In fact, if $W^c_r(y,\tau,\mathbf{i},k) > \gamma_r\mu_A C_0^{dr}$ at some stopping time $\tau \in [mK_{11}C_0^{-dr},(m+1)K_{11}C_0^{-dr}]$, then the conditional probability that also

$$W^c_r(y,(m+1)K_{11}C_0^{-dr},\mathbf{i},k) > \gamma_r\mu_A C_0^{dr}$$

is at least

$$\begin{aligned}&\big[P_A\{\text{a given particle stands still for } K_{11}C_0^{-dr} \text{ time units}\}\big]^{\gamma_r\mu_A C_0^{dr}+1}\\ &\qquad = \exp[-D_A K_{11}C_0^{-dr}(\gamma_r\mu_A C_0^{dr}+1)] \ge \frac{1}{2},\end{aligned}$$

provided $K_{11} = K_{11}(D_A,\gamma_0\mu_A) > 0$ is taken small enough. This proves our claim.

It follows that the conditional probabilities in (6.38) are bounded by twice the sum over at most

$$\Big[\frac{2\Delta_{r+1}}{K_{11}C_0^{-dr}}+1\Big][3\Delta_{r+1}]^d \le K_{12}C_0^{6(r+1)(d+1)+dr}$$

terms of the form

$$P_A\{W^c_r(y,v,\mathbf{i},k) > \gamma_r\mu_A C_0^{dr}|\mathcal{K}_{r+1}(\mathbf{i},k)\}. \tag{6.41}$$

This last probability is estimated as in (5.38) and the following lines. We merely have to replace $\widetilde{p}(y-z,u)$ there by

$$p(y-z,u) := P\big\{z+S_u \in \mathcal{Q}_r(y)\big\},$$

to use Lemma 12 in place of Lemma 6, and to take into account that the requirement (5.37) is replaced by (6.40). ∎

Finally we come to the version of Lemma 8 which we need for the continuous model.

Lemma 14. *For $\ell \geq 0$, $r \geq 1$ and any path $\widehat{\pi} \in \Xi(\ell, t)$ it holds*

$$\phi_r^c(\widehat{\pi}) \leq C_0^{6(d+1)} \Phi_{r+1}^c(\ell) + C_0^{6(d+1)} \Omega_{r+1}^c(\ell) + C_0^{6(d+1)} \Theta_{r+1}^c(\ell) \tag{6.42}$$

and

$$\Phi_r^c(\ell) \leq C_0^{6(d+1)} \Phi_{r+1}^c(\ell) + C_0^{6(d+1)} \Omega_{r+1}^c(\ell) + C_0^{6(d+1)} \Theta_{r+1}^c(\ell). \tag{6.43}$$

Moreover, there exist some constants C_{10}, κ_0, such that for $\kappa \geq \kappa_0$, t sufficiently large, and $\ell \geq 0$, $1 \leq r \leq R(t) - 1$,

$$P_A\{\Theta_{r+1}^c(\ell) \geq \kappa \frac{t+\ell}{\Delta_{r+1}} [\rho_{r+1}]^{1/(d+1)}\} \\ \leq \exp\Big[- (t+\ell) C_{10} \kappa \exp\big[- \frac{1}{2(d+1)} \gamma_r \mu_A C_0^{(d-\frac{3}{4})r} \big] \Big]. \tag{6.44}$$

Proof. The proof of (6.42) is again easy. Let $\widehat{\pi}$ be a path in $\Xi(\ell, t)$. Let $\mathcal{B}_r(\mathbf{j}, q)$ be a bad r-block which intersects $\widehat{\pi}$, and let $\mathcal{B}_r(\mathbf{j}, q) \subset \mathcal{B}_{r+1}(\mathbf{i}, k)$. Then are several possibilities. First it may be that $\mathcal{B}_{r+1}(\mathbf{i}, k)$ is bad. There are at most $\Phi_{r+1}^c(\ell)$ such $\mathcal{B}_{r+1}(\mathbf{i}, k)$ (note that they have to intersect $\widehat{\pi}$), and such $(r+1)$-blocks contain at most $C_0^{6(d+1)}$ r-blocks. These are counted by the first term in the right hand side of (6.42). The next possibility is that $\mathcal{B}_{r+1}(\mathbf{i}, k)$ is good, but that $\mathcal{B}_{r+1}(\mathbf{i}, k)$ is contaminated. There are at most $\Omega_{r+1}^c(\ell)$ good $(r+1)$-blocks which are also contaminated and intersect $\widehat{\pi}$. Each contains at most $C_0^{6(d+1)}$ r-blocks. These are counted by the second term in the right hand side of (6.42). The last possibility is that $\mathcal{B}_{r+1}(\mathbf{i}, k)$ is good, and that $\mathcal{B}_{r+1}(\mathbf{i}, k)$ is not contaminated. In particular, $w_{r+1}^c(x, v, \mathbf{i}, k) = N_A(x, v)$ for all $(x, v) \in \widetilde{\mathcal{B}}_{r+1}^+(\mathbf{i}, k)$. A fortiori, this must hold for all (x, v) for which $\mathcal{Q}_r(x) \times \{v\} \subset \widetilde{\mathcal{B}}_r(\mathbf{j}, q)$. But then $W_r^c(x, v, \mathbf{i}, k) = U_r(x, v)$ for such (x, v). Since we assumed that $\mathcal{B}_r(\mathbf{j}, q)$ is bad, there exists such an (x, v) with $W_r^c(x, v, \mathbf{i}, k) = U_r(x, v) > \gamma_r \mu_A C_0^{dr}$. Thus $\mathcal{B}_r(\mathbf{j}, q)$ is in fact inferior, and hence $\mathcal{B}_{r+1}(\mathbf{i}, k)$ is taken into account in $\Theta_{r+1}^c(\ell)$. Therefore $\mathcal{B}_r(\mathbf{j}, q)$ is counted in the last term on the right of (6.42). This proves (6.42) and (6.43) follows by taking the sup over $\widehat{\pi} \in \Xi(\ell, t)$.

The proof of (6.44) is essentially the same as that of (5.43) or (6.28). We merely have to take into account that the number of distinct boxes $\mathcal{D}(\mathbf{j}, q)$ (defined as in (6.29)) which can intersect $\widehat{\pi}$ is bounded only by (6.30). We leave the details to the reader. ∎

We next combine our estimates in the same way as we did in the proof of Theorem2-Discrete.

Lemma 15. *There exist some t_0 and constants K_{13}, K_{14} such that for all $t \geq t_0$,*

$$P_A\{\Phi_r^c(\ell) \geq K_{14} \kappa_0 (t+\ell) \exp[-K_{13} C_0^{r/4}] \text{ for some } r \geq 1, \ell \geq 0\} \leq \frac{2}{t^2}. \tag{6.45}$$

In addition, there exists a constant C_{11} such that for $t \geq t_0$

$$P_A\big\{t \gamma_1 \mu_A C_0^d + \sum_{r=1}^{R-1} \gamma_{r+1} \mu_A C_0^{r(d+6)+d} \Phi_r^c(\ell) \geq C_{11}(t+\ell) \text{ for some } \ell \geq 0\big\} \leq \frac{2}{t^2}. \tag{6.46}$$

Proof. Let $\gamma = \lim_{r \to \infty} \gamma_r$. Then by (6.28) and (6.44), for $1 \leq r < R$ and $t \geq t_0$

$$P\{\Omega_{r+1}^c(\ell) \geq \kappa_0 (t+\ell) \exp[-C_0^{r/2}] \text{ or} \\ \Theta_{r+1}^c(\ell) \geq \kappa_0 (t+\ell) C_0^{dr/(d+1)} \exp[-\frac{1}{2(d+1)} \gamma \mu_A C_0^{(d-\frac{3}{4})r} \text{ for some } \ell \geq 0\} \\ \leq \sum_{\ell \geq 0} \exp\big[- C_7 \kappa_0 (t+\ell) \exp[-C_0^{r/2}]\big] \\ + \sum_{\ell \geq 0} \exp\Big[- (t+\ell) C_{10} \kappa_0 \exp\big[- \frac{1}{2(d+1)} \gamma_0 \mu_A C_0^{(d-\frac{3}{4})r} \big]\Big] \\ \leq \frac{1}{t^3}, \tag{6.47}$$

provided we choose t_0 large enough (t_0 can be taken independent of $r \in [1, R-1]$, but depends on C_0). Summing this over $r \le R-1 \le [d \log C_0]^{-1} \log\log t$, we see that outside a set of probability $O(t^{-3} \log\log t)$, none of the events in (6.47) with $1 \le r < R$ occur. In view of (6.11) this shows that outside a set of probability $\le 2t^{-2}$ none of the events in (6.47) occur, and in addition $\Phi_r^c(\ell) = 0$ for all $r \ge R$ and all $\ell \ge 0$.

Now consider a sample point for which none of the events in (6.47) occur and for which $\Phi_r^c(\ell) = 0$ for $r \ge R, \ell \ge 0$. It suffices for (6.45) to show that for such a sample point the event in the left hand side of (6.45) does not occur. But this follows as in (5.57). Indeed at such a sample point we have, by virtue of (6.43),

$$\begin{aligned}\Phi_r^c(\ell) &\le C_0^{6(d+1)}\Phi_{r+1}^c(\ell) + C_0^{6(d+1)}\kappa_0(t+\ell)\Big[\exp[-C_0^{r/2}] \\ &\quad + C_0^{dr/(d+1)}\exp[-\frac{1}{2(d+1)}\gamma\mu_A C_0^{(d-\frac{3}{4})r}]\Big] \\ &\le C_0^{6(d+1)}\Phi_{r+1}^c(\ell) + \kappa_0(t+\ell)\exp[-K_{13}C_0^{r/4}] \\ &\le \cdots \le \kappa_0(t+\ell)\sum_{j=0}^{R-r-1} C_0^{6j(d+1)}\exp[-K_{13}C_0^{(r+j)/4}] \\ &\le K_{14}\kappa_0(t+\ell)\exp[-K_{13}C_0^{r/4}].\end{aligned}$$

Thus (6.45) holds for suitable K_i.

Finally, (6.46) now obvious. ■

Proof of Theorem 2. It is now easy to complete the proof of Theorem 2. By the Borel-Cantelli lemma the event in the left hand side of (6.46) almost surely occurs for only finitely many integers t. Moreover, $\Phi_r^c(\ell)$ is nondecreasing in t. By (6.46) and (6.12) we therefore have almost surely $[P_A]$ for all large t that

$$\begin{aligned}&E_B\{\text{number of } B\text{-particles at } \mathbf{0} \text{ alive at time } t \\ &\qquad \text{whose associated path lies in } \mathcal{C}(C_1 t\log t)\times[0,t]\} \\ &\le \sum_{\ell\ge 0}\mu_B\frac{[D_B t]^\ell}{\ell!}\exp[-(\delta+D_B)t + 2\beta C_{11}(t+\ell)] \\ &= \mu_B\exp[(-\delta - D_B + 2\beta C_{11} + D_B e^{2\beta C_{11}})t].\end{aligned}$$

Now take $\delta_0 = 3[2\beta C_{11} + D_B e^{2\beta C_{11}}]$. Then this last inequality and (2.38) show that for $\delta \ge \delta_0$, we have almost surely $[P_A]$ for all large t

$$E_B\{\text{number of } B\text{-particles at } \mathbf{0} \text{ alive at time } t\} \le e^{-\delta t/2}.$$

As at the end of Section 4, this implies that almost surely for all large t, there are no B-particles alive at $\mathbf{0}$ at time t. ■

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