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MARTINGALES ON RANDOM SETS AND THE STRONG MARTINGALE PROPERTY

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Abstract Let X be a process defined on an optional random set. The paper develops two different conditions on X guaranteeing that it is the restriction of a uniformly integrable martingale. In each case, it is supposed that X is the restriction to Λ of some special semimartingale Z with canonical decomposition $Z = M + A$. The first condition, which is both necessary and sufficient, is an absolute continuity condition on A. Under additional hypotheses, the existence of a martingale extension can be characterized by a strong martingale property of X on Λ. Uniqueness of the extension is also considered.

Keywords Martingale, random set, strong martingale property

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0. Introduction

Let Λ be an optional random set and let $X_t(\omega)$ be defined for $(t,\omega) \in \Lambda$. We consider the following and some of its extensions.

(0.1) Problem. *Find necessary and sufficient conditions on X guaranteeing that it is the restriction to Λ of a globally defined, right continuous uniformly integrable martingale.*

For an example where this formulation may be natural, consider a process $(Y_t)_{t\geq 0}$ with values in a manifold. Given a coordinate patch V, let $\Lambda := \{(t,\omega) : Y_t(\omega) \in V\}$ and let $X_t(\omega)$ denote a real component of $Y_t(\omega)$ for $(t,\omega) \in \Lambda$. A second natural example is provided by $X = f \circ W$, where W is a Markov process in a state space E and f is a function defined on a subset S of E, Λ denoting in this case $\{(t,\omega) : W_t(\omega) \in S\}$.

The solution is obvious if there is an increasing sequence of stopping times T_n which are complete sections of Λ (*i.e.*, $[\![T_n]\!] \subset \Lambda$ and $\mathbf{P}\{T_n < \infty\} = 1$) such that $\Lambda \subset \cup_n [\![0, T_n]\!]$. A number of other cases may now be found in the literature. The case of an optional right-open interval of the form $[\![0, \zeta[\![$ was discussed first by Maisonneuve [Ma77] for continuous martingales, and by Sharpe [Sh92] in the general case. See also [Ya82], [Zh82].

In sections 2 and 3, we give a complete solution for the discrete parameter problem under mild conditions on X. The continuous parameter case, treated in sections 4 and 5, involves considerable additional complication. Roughly speaking, in the discrete parameter case the condition is that X have a "strong martingale property" on Λ (defined in (3.4)), but in the continuous parameter context, an example is given in section 5 to show that this condition is not sufficient. Theorem (4.1), one of the main results of the paper, assumes X is the restriction of a special semimartingale $Z = M + A$ (with A predictable and of integrable total variation) and gives a necessary and sufficient condition in terms of absolute continuity of A with respect to C, the predictable compensator of unit mass at the part of the end of Λ not contained in Λ. Theorems (5.3) and (5.10) gives some conditions under which the strong martingale property is sufficient to imply existence of a martingale extension. The proofs are based on a reduction to the special case treated in [Sh92], which is discussed along with a number of extensions in section 1.

In view of the obvious case discussed in the first paragraph, it should be kept in mind that these results are primarily of interest in cases where Λ is "sectionally challenged."

1. Setup

Throughout this paper, we suppose given a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and a filtration $(\mathcal{F}_t)$ with t either a positive integer index or a positive real index. In the latter case, $(\mathcal{F}_t)$ is assumed to satisfy the usual right continuity and completeness hypotheses. Expectation of a random variable X with respect to $\mathbf{P}$ is denoted by $\mathbf{P}X$ rather than EX.

The optional (resp., predictable) σ-algebra $\mathcal{O}$ (resp, $\mathcal{P}$) with respect to $(\mathcal{F}_t)$ is that generated by the right (resp., left) continuous processes adapted to $(\mathcal{F}_t)$. We assume given a random set $\Lambda \in \mathcal{O}$ and a process X defined on Λ (*i.e.*, $X_t(\omega)$ is defined only for $(t,\omega) \in \Lambda$) satisfying at minimum:

(1.1) $\quad X$ is the restriction to Λ of some (right continuous) special semimartingale.

Recall that a semimartingale Z is *special* in case it admits a decomposition $Z = Z_0 + M + A$ with $M_0 = A_0 = 0$, M a local martingale and A predictable and of locally finite variation, or equivalently, $Z_t^* := \sup_{s\leq t} |Z_s|$ is locally integrable. See [DM80, VII.23]. The special semimartingale in (1.1ii) is of course not unique, but we reserve the notation $Z = M + A$ for the canonical decomposition of a suitably chosen special semimartingale Z extending X, with M a martingale and A predictable, $A_0 = 0$ and of locally integrable variation. We denote by $\mathcal{S}^1(\mathcal{F}_\cdot)$ the class of special semimartingales $Z = M + A$ over the filtration $(\mathcal{F}_t)$ such that M is a uniformly integrable martingale and A is of integrable total variation.

The reader is referred to [DM75] and [DM80] for a detailed discussion of the definitions and results used below, but a brief review and clarification of notation may be in order.

- Given a random time ζ, let ε_ζ denote the random measure putting unit mass at ζ on $\{\zeta < \infty\}$.
- Given a random measure ν on $\mathbf{R}^+$ and a positive measurable process W, $W * \nu$ denotes the random measure having density W with respect to ν.
- The left limit X_{0-} is defined to be 0 in all cases, even if X is not defined at 0.
- We use the term *predictable compensator* instead of dual predictable projection. The predictable compensator B^p of an process B of integrable total variation is the unique predictable process B^p such that $\mathbf{P} \int H_t \, dB_t = \mathbf{P} \int H_t \, dB_t^p$ for every bounded predictable process H. If B is adapted, this is to say that $B - B^p$ is a martingale.
- By the optional projection ${}^oW$ of a positive measurable process W we mean the unique optional process satisfying $\mathbf{P}\, {}^oW_T 1_{\{T<\infty\}} = \mathbf{P}\, W_T 1_{\{T<\infty\}}$ for all stopping times T.

The fundamental martingale extension result, to which all other cases will be reduced, is the following extension of [Sh92, (4.8)].

(1.2) Proposition. *Let ζ denote a stopping time over $(\mathcal{F}_t)$ and suppose $\Lambda \in \mathcal{O}$ satisfies $[\![0,\zeta[\![\subset \Lambda \subset [\![0,\zeta]\!]$. Let $\Omega_0 := \{\omega : \zeta(\omega) \notin \Lambda(\omega), 0 < \zeta(\omega) < \infty\}$ so we may write $\Lambda = [\![0,\zeta[\![\cup \{(\omega, \zeta(\omega)) : \omega \notin \Omega_0\}$. Let C denote the predictable compensator of $1_{\Omega_0} * \varepsilon_\zeta$. Let X be defined on Λ and suppose X satisfies (1.1). Define the process Z by*

$$Z_t(\omega) := \begin{cases} X_t(\omega) & \text{for } (t,\omega) \in \Lambda, \\ X_{\zeta-}(\omega) & \text{for } t \geq \zeta \text{ on } \Omega_0, \\ 0 & \text{for } t \geq 0 \text{ on } \Lambda_0^c, \\ X_\zeta(\omega) & \text{for } t > \zeta \text{ on } \Omega_0^c. \end{cases}$$

Suppose Z is a semimartingale in $\mathcal{S}^1(\mathcal{F}_\cdot)$, having canonical decomposition $Z = M + A$. Then X is the restriction to Λ of a uniformly integrable martingale if and only if $A \ll C$. In this case, let H denote a predictable version of dA/dC, and set

$$F := (X_{\zeta-} - H_\zeta) 1_{\Omega_0} + X_\zeta 1_{\Omega_0^c} 1_{\{\zeta<\infty\}} 1_{\Lambda_0}.$$

Then $\bar{X} := X 1_\Lambda + F 1_{\Lambda^c}$ is a uniformly integrable martingale extending X. Note that $\bar{X}_\infty = Z_\infty - H_\zeta 1_{\Omega_0} 1_{\{0<\zeta<\infty\}}$.

Proof. The case proved in [Sh92] assumed the stronger hypotheses (i) $\zeta > 0$ a.s. and $\Lambda = [\![0,\zeta[\![$; and (ii) $\mathbf{P} \sup_t |M_t| < \infty$. We first show how to relax condition (ii) to the get the same result under the weaker condition (ii') M is uniformly integrable. (We continue to assume (i) for the moment, so that we have the simplifications $\Omega_0 = \{0 < \zeta < \infty\}$, $Z_t = X_t 1_{\{t<\zeta\}} + X_{\zeta-} 1_{\{\zeta<\infty\}}$ and $F = (X_{\zeta-} - H_\zeta) 1_{\{\zeta<\infty\}} = (Z_{\zeta-} - H_\zeta) 1_{\{\zeta<\infty\}}$.) Let $T_n := \inf\{t : |M_t| \geq n\}$ so that the stopping times $T_n \uparrow \infty$ a.s. Since $\mathbf{P} \sup_t |M_t^{T_n}| \leq n + \mathbf{P}|M_{T_n}| \leq n + \mathbf{P}|M_\infty|$, each of the stopped processes M^{T_n} is a martingale of class $\mathcal{H}^1$. Let X^n denote the restriction of $Z^n := Z^{T_n}$ to Λ, let $M^n := M^{T_n}$ and $A^n := A^{T_n}$. Note the following points.

(a) $Z^n = Z_0 + M^n + A^n$ is a semimartingale of class $\mathcal{H}^1$;

(b) on $\{T_n \geq \zeta\} \cap \{\zeta < \infty\}$, $Z^n_{\zeta-} = Z_{\zeta-} = X_{\zeta-} = X^n_{\zeta-}$;
(c) on $\{T_n < \zeta < \infty\}$, $Z^n_{\zeta-} = Z_{T_n} = X^n_{\zeta-}$;
(d) as $Z \in \mathcal{S}^1$, Z_∞ exists, $Z_\infty = Z_0 + M_\infty + A_\infty \in L^1$, and $Z_{\zeta-} = Z_0 + M_{\zeta-} + A_{\zeta-} \in L^1$;
(e) $Z^n_{\zeta-} \to Z_{\zeta-}$ a.s. on $\{0 < \zeta < \infty\}$;
(f) in view of the preceding observations, $|Z^n_{\zeta-}| \leq |Z_{T_n}| + |Z_{\zeta-}|$ for all n on $\{\zeta > 0\}$.

The theorem applied to X^n (as the restriction of Z^n) shows that X^n extends to the martingale $\bar{X}^n$ of class $\mathcal{H}^1$ determined by $\bar{X}^n_\infty := Z^n_{\zeta-} - H^n_\zeta 1_{\{\zeta<\infty\}}$, H^n denoting a predictable version of dA^n/dC. Let μ_n denote the $\mathbf{P}$-measure generated by $A - A^n$ and λ the $\mathbf{P}$-measure generated by C (*i.e.*, $\lambda(Y) := \mathbf{P}\int Y_t\, dC_t$ for Y bounded, predictable). Then $H^n - H = d\mu_n/d\lambda$, which implies $|H^n - H| = d|\mu_n|/d\lambda$. Clearly, $|\mu_1| \geq |\mu_2| \geq \dots$, and $|\mu_n|(1) = \mathbf{P}\int_0^\infty |dA_t - A^n_t| \to 0$. It follows that $|H^n - H| \to 0$ a.e. (λ) and in $L^1(\lambda)$. Consequently, $\mathbf{P}|H^n_\zeta 1_{\{0<\zeta<\infty\}} - H_\zeta 1_{\{0<\zeta<\infty\}}| = \mathbf{P}\int_0^\infty |H^n_t - H_t|\, dC_t = \int |H^n - H|\, d\lambda \to 0$. That is, $H^n_\zeta 1_{\{0<\zeta<\infty\}} \to H_\zeta 1_{\{0<\zeta<\infty\}}$ in $L^1(\mathbf{P})$. As we observed above in (e), $Z^n_{\zeta-} \to Z_{\zeta-}$ a.s. on $\{0 < \zeta < \infty\}$, and in fact by (f), the $Z^n_{\zeta-}1_{\{0<\zeta<\infty\}}$ are dominated by the uniformly integrable family $(|Z_{T_n}| + |Z_{\zeta-}|)1_{\{0<\zeta<\infty\}}$. It follows that $Z^n_{\zeta-}1_{\{0<\zeta<\infty\}} \to Z_{\zeta-}1_{\{0<\zeta<\infty\}}$ in $L^1(\mathbf{P})$. We may therefore choose a sequence n_k tending to infinity so rapidly that $F_n := (Z^n_{\zeta-} - H^n_\zeta)1_{\{0<\zeta<\infty\}}$ converges a.s. and in $L^1(\mathbf{P})$ to $F := (Z_{\zeta-} - H_\zeta)1_{\{0<\zeta<\infty\}}$ as $n \to \infty$ along (n_k), and in particular, F_{n_k} is a uniformly integrable family. It follows immediately that the uniformly integrable martingale $\bar{X}$ with final value F satisfies $\bar{X}_t = \lim_k \bar{X}^{n_k}_t$ a.s. and in L^1 and so $\bar{X}$ extends X. This shows that the condition $M \in \mathcal{H}^1$ in [Sh92, Theorem (4.8)] may be replaced by the weaker condition (ii') on M.

We now reduce the rest of the problem by a simple artifice to the known case where $\Lambda = [\![0,\zeta[\![$, $\zeta > 0$ a.s., and M is uniformly integrable. We first extend Λ if necessary so that $(0,\omega) \in \Lambda$ for all ω, defining $X_0(\omega) := 0$ on the extension. In this way, we may assume $\{\zeta = 0\} \subset \Omega_0^c$ while not affecting the definitions of Z and F. Define the stopping time ζ' by

$$\zeta'(\omega) := \begin{cases} \zeta(\omega) + 1 & \text{if } \omega \notin \Omega_0; \\ \zeta(\omega) & \text{if } \omega \in \Omega_0. \end{cases}$$

Note that since $\{\zeta = 0\} \subset \Omega_0^c$, $\zeta' > 0$ everywhere. Extend X on Λ to X' on $\Lambda' := [\![0,\zeta'[\![$ by setting $X' := X1_\Lambda + X_\zeta 1_{\Lambda'\setminus\Lambda}$. Clearly $Z = X'$ on Λ', and $Z'_t = X'_{\zeta'-}$ for $t \geq \zeta'$. Let C' denote the predictable compensator of $\varepsilon_{\zeta'} = 1_{\Omega_0} * \varepsilon_\zeta + 1_{\Omega_0^c} * \varepsilon_{\zeta+1}$, so that $C' = C + D$, where D is carried by the predictable set $]\!]\zeta,\infty[\![$. Since Z stops at ζ, so do M and A and so the condition $A \ll C$ is equivalent to $A \ll C'$. Moreover, in this case, $dA/dC = dA/dC'$ on $]\!]0,\zeta]\!]$. Suppose now that $A \ll C$ and let $H := dA/dC = dA/dC'$ with $H \in \mathcal{P}$ vanishing on $]\!]\zeta,\infty[\![$. The conditions of [Sh92, (4.8)] are now satisfied by the data Λ', X', Z, A, C', and we may conclude that X' has a martingale extension given by $\bar{X} := X'1_{\Lambda'} + (X'_{\zeta'-} - H_{\zeta'})1_{[\![\zeta',\infty[\![}$. Since $H = 0$ on $]\!]\zeta,\infty[\![$, it follows that

$$\bar{X}_t = \begin{cases} X_t & \text{for } t < \zeta, \\ X_{\zeta-} - H_\zeta & \text{for } t \geq \zeta \text{ on } \Omega_0, \\ X_\zeta & \text{for } t \geq \zeta \text{ on } \Omega_0^c, \end{cases}$$

is the desired extension of X. Conversely, if X has a martingale extension, then so does X', and an application of the converse direction of [Sh92, (4.8)] shows that $A \ll C'$, which is as we showed above equivalent to $A \ll C$. $\square$

(1.3) Remark. In the statement and proof of Proposition (1.2), we took the simplest path to extending Λ and X so that $\Lambda_0 \subset \Lambda$, by giving X the value 0 on the part of $[\![0]\!]$ not in Λ. In fact, we could have chosen any integrable $\mathcal{F}_0$-measurable random variable J by making simple changes to the definitions of F and Z. We allow for such a modification in (1.4) below.

It was shown in [Sh92] that under the hypotheses $\zeta > 0$ a.s., $\Lambda =]\!]0, \zeta[\![$ and $Z \in \mathcal{H}^1$, the extension $\bar{X}$ of X is unique among extensions which stop at ζ and satisfy $\bar{X}_\zeta \in \mathcal{F}_{\zeta-}$. We adapt the proof of this result to get a uniqueness result under the broader conditions of the preceding theorem as modified by (1.3).

(1.4) Proposition. *Let ζ, Λ, Ω_0, X be as in* (1.2) *and fix $J \in L^1(\mathcal{F}_0)$ extending X_0 on Λ_0 as in* (1.3)*. Then the process $\bar{X}$ constructed in* (1.2) *is the unique uniformly integrable martingale extending X and satisfying*

(i) $\bar{X}_\infty \in \mathcal{F}_\zeta$;
(ii) $\bar{X}_0(\omega) = J(\omega)$ *for* $\omega \notin \Lambda_0$;
(iii) $\bar{X}_\infty 1_{\Omega_0}$ *is measurable with respect to the trace of $\mathcal{F}_{\zeta-}$ on Ω_0.*

Proof. By taking differences, we may assume $X = 0$ on $\Lambda \cup [\![0]\!]$, with uniqueness equivalent to showing $\bar{X}_\infty = 0$. We may assume by (ii) that $\Lambda_0 = \Omega$. Hypothesis (i) implies of course that $\bar{X}$ stops at ζ. As $X_{\zeta-} = 0$ on $\Omega_0^c \cap \{\zeta < \infty\}$, this proves $\bar{X}_t = 0$ for all $t \geq \zeta$ on $\Omega_0^c \cap \{\zeta < \infty\}$, hence that $\bar{X}_\infty = 0$ on Ω_0^c. It follows that $\bar{X}_t$ is a step process with a single jump of size $\bar{X}_\infty$ at ζ provided $0 < \zeta < \infty$ and $\zeta \notin \Lambda$. By (iii), there exists a predictable process Y such that $\bar{X} = Y 1_{\Omega_0} * \varepsilon_\zeta$. As $\mathbf{P}|\bar{X}_\infty| < \infty$, we have $\mathbf{P}|Y_\zeta| 1_{\Omega_0} < \infty$. Let C denote the predictable compensator of $1_{\Omega_0} * \varepsilon_\zeta$. Then $\mathbf{P} \int |Y_t| \, dC_t < \infty$. As $\bar{X}$ is a martingale, the predictable compensator of $Y 1_{\Omega_0} * \varepsilon_\zeta$ vanishes, so $Y * C = 0$. Write $Y = Y^+ - Y^-$ to see that this implies $Y^+ * C = Y^- * C$, hence $Y^+ = Y^-$ a.e. with respect to the $\mathbf{P}$-measure on $\mathcal{P}$ given by $U \to \mathbf{P} \int U_t \, dC_t$, hence $|Y| * C = 0$, hence $|Y| 1_{\Omega_0} * \varepsilon_\zeta = 0$. The latter condition is equivalent to $|\bar{X}_\infty| = 0$, completing the proof. $\square$

The stopping argument employed in the first paragraph of the proof of (1.2) can be modified to give the following local version of (1.2).

(1.5) Proposition. *Assume the same general hypotheses as* (1.2)*, but relaxed so that the canonical decomposition of the special semimartingale $Z = Z_0 + M + A$ has the properties (a) M is a local martingale; (b) there is an increasing sequence T_n of stopping times such that for each n, M^{T_n} is a uniformly integrable martingale, $\mathbf{P} \int_0^{T_n} |dA_s| < \infty$ and $A^{T_n} \ll C$. Then X extends to a local martingale $\bar{X}$, and the stopping times T_n reduce $\bar{X}$ to a uniformly integrable martingale.*

Proof. By the same artifice employed in the proof of (1.2), we may reduce the problem to the case $\Omega_0 = \Omega$. Let X^n denote the restriction to Λ of $Z^{T_n} = Z_0 + M^{T_n} + A^{T_n}$, a special semimartingale in $\mathcal{S}^1(\mathcal{F}_\cdot)$, hence satisfying the conditions of (1.2) with respect to X^n on Λ. Let H^n be a predictable version of dA^{T_n}/dC, which may be assumed to vanish outside the predictable interval $]\!]0, T_n]\!]$. By (1.2), $\bar{X}^n_\infty := X^n_{\zeta-} - H^n_\zeta 1_{\{\zeta < \infty\}}$ determines a uniformly integrable martingale extending X^n. By construction, $H^n_\zeta = 0$ on $\{T_n < \zeta\}$, so we may write $\bar{X}^n_\infty = X^n_{\zeta-} - H^n_\zeta 1_{\{\zeta \leq T_n, \zeta < \infty\}}$. Recall that $\bar{X}^n$ stops at ζ for all n, and $\bar{X}^n_{\zeta-} \in \mathcal{F}_{\zeta-}$. Now compare $\bar{X}^{n+1}$ and $\bar{X}^n$. For finite t,

$$\bar{X}_t^{n+1} = \begin{cases} X_t^{n+1} & \text{for } t < \zeta \\ X_{\zeta-}^{n+1} - H_\zeta^{n+1} & \text{for } t \geq \zeta \end{cases}$$
$$= \begin{cases} X_{t\wedge T_{n+1}} & \text{for all } t \text{ on } \{T_{n+1} < \zeta\} \\ X_t & \text{for } t < \zeta \text{ on } \{T_{n+1} \geq \zeta\} \\ X_{\zeta-} - H_\zeta^{n+1} & \text{for } t \geq \zeta \text{ on } \{T_{n+1} \geq \zeta\} \end{cases}$$

The corresponding expansion of $\bar{X}_t^n$, together with the observation that $H_t^{n+1} = H_t^n$ on $[\![0, T_n]\!]$ yields for $t \leq T_n$

(i) on $\{T_{n+1} < \zeta\}$, $\bar{X}_t^{n+1} = X_{t\wedge T_{n+1}} = X_{t\wedge T_n}$, $\bar{X}_t^n = X_{t\wedge T_n}$;
(ii) on $\{T_n < \zeta \leq T_{n+1}\}$, $\bar{X}_t^{n+1} = X_t$, $\bar{X}_t^n = X_t$;
(iii) on $\{T_n \geq \zeta\}$, for $t < \zeta$, $\bar{X}_t^{n+1} = X_t$, $\bar{X}_t^n = X_t$;
(iv) on $\{T_n \geq \zeta\}$, for $t \geq \zeta$ (and $t \leq T_n$), $\bar{X}_t^{n+1} = X_{\zeta-} - H_\zeta^{n+1}$, $\bar{X}_t^n = X_{\zeta-} - H_\zeta^n$.

Thus, in all cases, $\bar{X}_t^{n+1} = \bar{X}_t^n$ for $t \leq T_n$. This consistency condition means there is a local martingale $\bar{X}$ such that $\bar{X}^{T_n} = \bar{X}^n$, as claimed. $\square$

2. A special discrete parameter case

We begin with a special case—essentially the discrete parameter version of (1.2) but with features of (1.5).

Fix a filtration $(\mathcal{F}_n)_{n\geq 0}$ and let Λ be a discrete parameter random set satisfying

(2.1i) for every $n \geq 0$, Λ_n (the section of Λ at time n) is in $\mathcal{F}_n$;
(2.1ii) $\Lambda_0 \supset \Lambda_1 \supset \dots$.

Let $G(\omega) := \sup\{n : \omega \in \Lambda_n\}$ (with $\sup \emptyset := 0$) and $L(\omega) := \inf\{n : \omega \notin \Lambda_n\} = G(\omega) + 1$. Though G is not in general a stopping time over $(\mathcal{F}_n)$, $\{L \leq n\} = \Lambda_n^c$ shows that L is a stopping time over $(\mathcal{F}_n)$.

Suppose given a process X defined on Λ such that for every n, X_n is integrable on its domain of definition. Let $J \in L^1(\mathcal{F}_0)$. We shall first extend Λ so that $\Lambda_0 = \Omega$, setting $X_0(\omega) := J(\omega)$ for $\omega \in \Lambda_0^c$. Thus we shall always assume throughout this section that

(2.2) $\Lambda_0 = \Omega$.

In particular, under (2.2), G is truly the end of Λ. We define the σ-algebra of events prior to G by its collection of measurable functions: $\mathcal{F}_G := \{K_G 1_{\{G<\infty\}} + \Phi 1_{\{G=\infty\}} : (K_n) \text{ adapted to } (\mathcal{F}_n), \Phi \in \mathcal{F}_\infty\}$, so that $\mathcal{F}_G$ may be identified with the usual (discrete parameter) definition of $\mathcal{F}_{L-}$ by its random variables: $\{Y_L 1_{\{L<\infty\}} + \Phi 1_{\{L=\infty\}} : Y \text{ predictable}, \Phi \in \mathcal{F}_\infty\}$.

Define processes Z, A, C and D by

(2.3)
$$\begin{aligned} Z_n &:= X_{n\wedge G} \quad (n \geq 0); \\ A_0 &:= C_0 := 0; \\ A_n - A_{n-1} &:= \mathbf{P}\{Z_n - Z_{n-1} \mid \mathcal{F}_{n-1}\} := \mathbf{P}\{X_{n\wedge G} - X_{(n-1)\wedge G} \mid \mathcal{F}_{n-1}\}; \quad (n \geq 1) \\ C_n - C_{n-1} &:= \mathbf{P}\{n \geq L \mid \mathcal{F}_{n-1}\} = 1_{\{n>L\}} + 1_{\{n\leq L\}} \mathbf{P}\{n = L \mid \mathcal{F}_{n-1}\}; \quad (n \geq 1) \\ D_n &:= 1_{\{n\geq L\}} \quad (n \geq 0). \end{aligned}$$

That is, Z extends X by stopping at the end of Λ. Writing $Z_n = X_n 1_{\{G>n\}} + X_G 1_{\{G\leq n\}}$ shows that $Z_n \in \mathcal{F}_n$. Clearly $|Z_n| \leq \max_{k\leq n\wedge G} |X_k|$, so $\mathbf{P}|Z_n| \leq \sum_{k\leq n} \mathbf{P}|X_k| < \infty$. (The expectations in the preceding expression are taken only over the domains of definition of the X_k.) The conditional expectation defining A is therefore meaningful, and A is the unique predictable (*i.e.*, $A_n \in \mathcal{F}_{n-1}$ for $n \geq 1$) process with $A_0 = 0$ such that $Z - A$ is a martingale. The adapted process D starts at 0 and jumps up by 1 at $L(\geq 1)$, and C is its predictable compensator, so that $D - C$ is a martingale.

Note that since Z and D both stop at the stopping time L, so do A and C.

(2.4) Theorem. *Under* (2.1), (2.2) *and the notation established above, suppose* $A \ll C$*—i.e., for every* $k \geq 1$, $A_k = A_{k-1}$ *on* $\{C_k = C_{k-1}\}$. *Letting* $0/0 := 0$, *one may define a predictable process* H *unambiguously by*

$$H_k := (A_k - A_{k-1})/(C_k - C_{k-1}), \qquad k \geq 1.$$

Then

$$\bar{X}_n := Z_n - H_L 1_{\{L \leq n\}} \tag{2.5}$$

defines a martingale extension of X. *It is the unique martingale extending* X, *stopping at* L, *and with* $\bar{X}_L \in \mathcal{F}_G$.

Proof. We reduce this to (1.2) by making the obvious extension to continuous time, replacing the integer index n with the interval $[n, n+1[$. More specifically, define $\mathcal{F}'_t := \mathcal{F}_n$ for $t \in [n, n+1[$, $\Lambda' := \cup_n [n, n+1[\times \Lambda_n$, $X'_t(\omega) := X_n(\omega)$ in case $(t, \omega) \in \Lambda'$ with $t \in [n, n+1[$. Similarly extend Z, D, A and B to give step processes Z', D', A' and B'. The stopping times $T_n := n$ reduce Z, M and A in the sense of (1.5). Then (1.2) and (1.5) applied to the primed processes shows at once that $\bar{X}$ is a martingale and the extension is unique by (1.4) because $\mathcal{F}_G$ may be identified with $\mathcal{F}_{L-}$. $\square$

(2.6) Remark. Proposition (1.2) implies that the condition $A \ll C$ in also necessary in order that X have a martingale extension. Thus, any conditions equivalent to X having a martingale extension are equivalent to the condition $A \ll C$.

(2.7) Remark. If we had imposed a stronger hypothesis on X, requiring that $Z \in \mathcal{S}^1(\mathcal{F}_{\cdot})$ (*i.e.*, M is uniformly integrable and A has integrable variation), then the triple of conditions $\bar{X}$ is uniformly integrable, $\bar{X}_L \in \mathcal{F}_G$, $\bar{X}$ stops at L is equivalent to the pair of conditions $\bar{X}$ is uniformly integrable, $\bar{X}_\infty \in \mathcal{F}_G$, for under the latter pair, $\bar{X}_\infty \in \mathcal{F}_L$ so $\bar{X}$ necessarily stops at L.

(2.8) Remark. The extension of X defined by (2.5) actually stops at G in a number of important cases. Let L_p denote the predictable part of L, defined in [Sh92, §2] as the largest predictable stopping with graph contained in $[\![L]\!]$. It is easy to see, for example, that $L_p = 1$ on $\{L = 1\} = \{G = 0\}$. The details are given in greater generality in (4.8) and the surrounding discussion.

3. The general discrete parameter case

Let Λ denote an arbitrary optional random set. That is, $\Lambda \subset \{0, 1, \dots\} \times \Omega$ satisfies (2.1i) but not necessarily (2.1ii). We suppose also that X is defined on Λ and optional in the sense that for each n, X_n is measurable with respect to the trace of $\mathcal{F}_n$ on Λ_n.

For $m \leq n$, let

$$W_{m,n} := \mathbf{P}\{\Lambda_n \mid \mathcal{F}_m\}, \qquad \Gamma_{m,n} := \{W_{m,n} = 1\}.$$

It is easy to see that $\Gamma_{m,n}$ is the largest (modulo null sets) $\mathcal{F}_m$-measurable subset of Λ_n. Note that $m \to 1 - W_{m,n}$ is a positive martingale, and therefore

$$\Gamma_{0,n} \subset \Gamma_{1,n} \subset \cdots \subset \Gamma_{n,n} = \Lambda_n.$$

(3.1) Definition. *X has the simple martingale property on Λ provided, for every pair $m < n$,*

$$\mathbf{P}\{X_n \mid \mathcal{F}_m\} = X_m \qquad \text{on } \Lambda_m \cap \Gamma_{m,n}.$$

Note that the conditional expectation in the line above makes sense, for as we pointed out above, $\Lambda_m \cap \Gamma_{m,n} \subset \Lambda_n$, the domain of definition of X_n.

A stronger version of (3.1) will be required. Given stopping times $S \leq T$, define $\Lambda_S := \{\omega : (S(\omega), \omega) \in \Lambda\}$ and $\Gamma_{S,T} := \{\mathbf{P}\{\Lambda_T \mid \mathcal{F}_S\} = 1\}$. Then $\Lambda_S \cap \Gamma_{S,T}$ determines the part of the graph of S which is in Λ and on which is is almost certain that T is in Λ. The set $\Gamma_{S,T}$ may also be described as the largest $\mathcal{F}_S$-measurable set contained in Λ_T. These definitions apply equally in discrete and continuous parameter cases. The following is phrased in continuous parameter terms for later use.

(3.2) Lemma. *Let T be a stopping time and let Y_t^T be a right continuous version of the martingale $\mathbf{P}\{\Lambda_T^c \mid \mathcal{F}_t\}$ and $\zeta_T := \inf\{t : Y_t^T = 0\}$. Then for every stopping time $S \leq T$, $\mathbf{P}\{\Lambda_T \mid \mathcal{F}_S\} = 1$ if and only if $S \geq \zeta_T$, and hence $\Gamma_{S,T} = \{S = T \in \Lambda\} \cup \{\zeta_T \leq S < T\}$.*

Proof. Since Y^T is a right continuous, positive martingale, $Y_S^T = 0$ if and only if $S \geq \zeta_T$. $\square$

(3.3) Lemma. *For $S \leq T$ stopping times, $\Lambda_S \cap \Gamma_{S,T} \subset \Lambda_T$.*

Proof. By definition of $\Gamma_{S,T}$, $\mathbf{P}(\Lambda_S \cap \Gamma_{S,T} \cap \Lambda_T^c) = \mathbf{P}(\mathbf{P}\{\Lambda_T^c \mid \mathcal{F}_S\}; \Lambda_S \cap \Gamma_{S,T}) = 0$. $\square$

(3.4) Definition. *X has the strong martingale property on Λ provided, for every pair $S \leq T$ of stopping times,*

$$\mathbf{P}\{X_T \mid \mathcal{F}_S\} = X_S \qquad \text{on } \Lambda_S \cap \Gamma_{S,T}. \tag{3.5}$$

Note that the conditional expectation in (3.5) makes sense because of (3.3). Unlike ordinary uniformly integrable martingales, where the simple and strong martingale properties are equivalent (optional sampling theorem) the same is not the case for general Λ. For example, in a coin tossing model, define Λ by $\Lambda_0 := \Omega$, $\Lambda_1 :=$ heads on first toss, $\Lambda_2 :=$ tails on first toss. It is easy to check that the sets $\Gamma_{m,n} = \emptyset$ for $m < n$, and consequently an arbitrary adapted X defined on Λ has the simple martingale property. However, if we define stopping times $D_0 := 0$, $D_1(\omega) := \inf\{n : (n, \omega) \in \Lambda\}$ (which takes values either 1 or 2) then the strong martingale property is plainly not valid for every adapted X on Λ—eg, $X_1 = X_2 = 1$, $X_0 = 0$. The following result indicates an important special case where the simple and strong martingale properties are equivalent.

(3.6) Theorem. *Let Λ be an optional random set satisfying* (2.1ii) *(ie, $\Lambda_0 \supset \Lambda_1 \supset \dots,$) let X be defined on Λ, adapted and with X_n integrable on Λ_n for every n. Let Z, A and C be defined as in* (2.3). *Then the following are equivalent.*

(i) *X is the restriction to Λ of a martingale;*
(ii) $A \ll C$;
(iii) *for every $n \geq 1$, $A_n = A_{n-1}$ on $\{C_n = C_{n-1}, n \leq L\}$;*
(iv) *for every $n \geq 1$, $\mathbf{P}\{(Z_n - Z_{n-1})1_{\{L \geq n\}} \mid \mathcal{F}_{n-1}\} = 0$ on $\{\mathbf{P}\{L = n \mid \mathcal{F}_{n-1}\} = 0\}$;*
(v) *for every $n \geq 1$, $\mathbf{P}\{(X_n - X_{n-1})1_{\{L > n\}} \mid \mathcal{F}_{n-1}\} = 0$ on $\{\mathbf{P}\{L = n \mid \mathcal{F}_{n-1}\} = 0\}$;*
(vi) *X satisfies (3.1) for pairs of the form $n-1, n$;*
(vii) *X has the simple martingale property on Λ.*

If, in addition, $Z \in \mathcal{S}^1(\mathcal{F}_\cdot)$ (or equivalently, X is the restriction to Λ of a uniformly integrable martingale), then each of the conditions above is equivalent to

(viii) *X has the strong martingale property on Λ.*

Proof. Properties (i) and (ii) are equivalent by (2.4), and (ii) is clearly equivalent to (iii) since A and C stop at L. The equivalence of (iii) and (iv) then follows by definition of Z, A and C and the fact that $\{L \geq n\} \in \mathcal{F}_{n-1}$. For equivalence of (iv) and (v), note that $Z_n = Z_{n-1}$ on $\{L = n\}$. It is clear that (i) $\implies$ (vii) $\implies$ (vi), so the proof of the first assertion will be complete once we prove (vi) $\implies$ (v). Assume (vi) holds and let $n \geq 1$. Then

$$\mathbf{P}\{(X_n - X_{n-1})1_{\Gamma_{n-1,n} \cap \Lambda_{n-1}} \mid \mathcal{F}_{n-1}\} = 0.$$

But, under (2.1ii), $\Lambda_k = \{L > k\}$, and so

$$\begin{aligned}\Gamma_{n-1,n} \cap \Lambda_{n-1} &= \{\mathbf{P}\{L > n \mid \mathcal{F}_{n-1}\} = 1\} \cap \{L \geq n\} \\ &= \{(1_{\{L \geq n\}} - \mathbf{P}\{L = n\} \mid \mathcal{F}_{n-1}\}) = 1\} \cap \{L \geq n\} \\ &= \{\mathbf{P}\{L = n \mid \mathcal{F}_{n-1}\} = 0\} \cap \{L \geq n\}.\end{aligned}$$

On this latter set ($\in \mathcal{F}_{n-1}$), $L > n$ a.s., from which (v) follows. Finally, is X is the restriction of a uniformly integrable martingale, it suffices to observe that (i) $\implies$ (viii) $\implies$ (vii). $\square$

We turn now to the case of a general optional (discrete parameter) random set Λ. The idea here is to reduce the problem to (3.6) by a time change argument. For $n \geq 0$, let $D_n := \inf\{k \geq n : k \in \Lambda\}$. Then D_n is an increasing sequence of stopping times tending to infinity, and the graph $[\![D_n]\!]$ of D_n is a subset of Λ. As in the special case, let $G(\omega) := \sup\{n : (n,\omega) \in \Lambda\}$ denote the end of Λ, and $L := G + 1$.

We assume given a process X defined on Λ. We shall generally be assuming X is the restriction of a semimartingale $Z \in \mathcal{S}^1(\mathcal{F}_\cdot)$. Define $\hat{\mathcal{F}}_n := \mathcal{F}_{D_n}$ and define the random set $\hat{\Lambda} := \{(n,\omega) : D_n(\omega) < \infty\} = \{(n,\omega) : n < L(\omega)\}$. Clearly $\hat{\Lambda}$ is optional with respect to $(\hat{\mathcal{F}}_n)$. Note that $\hat{\Lambda}_0 \supset \hat{\Lambda}_1 \supset \dots$, so $\hat{\Lambda}$ satisfies (2.1ii) with respect to $(\hat{\mathcal{F}}_n)$. Observe to that G is still the end of $\hat{\Lambda}$. Next define a process $\hat{X}_n$ adapted to $(\hat{\mathcal{F}}_n)$ on the random set $\hat{\Lambda}$ by

$$\hat{X}_n(\omega) := X_{D_n(\omega)}(\omega), \qquad (n,\omega) \in \hat{\Lambda}. \tag{3.7}$$

(3.8) Proposition. *Let X be defined on Λ and satisfy:*

(i) *X is the restriction to Λ of a semimartingale $Z \in \mathcal{S}^1(\mathcal{F}_\cdot)$;*
(ii) *X has the strong martingale property on Λ.*

Then $\hat{X}$ is the restriction to $\hat{\Lambda}$ of a semimartingale $\hat{Z} \in \mathcal{S}^1(\hat{\mathcal{F}}_\cdot)$, and $\hat{X}$ has the strong martingale property on $\hat{\Lambda}$.

Proof. Write as usual $Z = M + A$, with M a uniformly integrable martingale over $(\mathcal{F}_\cdot)$ and A predictable with $\mathbf{P}\sum |\Delta A_n| < \infty$. Let $\hat{Z}_n := Z(D_n)$ (even on $\{D_n = \infty\}$.) We have then $\hat{Z} = \hat{M} + \hat{A}$, and clearly $\hat{M}$ is a uniformly integrable martingale over $(\hat{\mathcal{F}}_\cdot)$. Though $\hat{A}$ is not in general predictable over $(\hat{\mathcal{F}}_\cdot)$, it is adapted and clearly has integrable total variation. Therefore $\hat{Z} \in \mathcal{S}^1(\hat{\mathcal{F}}_\cdot)$. Then $\hat{X}$ is the restriction of $\hat{Z} \in \mathcal{S}^1(\mathcal{F}_\cdot)$ to $\hat{\Lambda}$, and in particular $\hat{X}_n$ is integrable on $\hat{\Lambda}_n = \{n < L\}$. By (3.6), to complete the proof, it will suffice to prove that for each fixed $n \geq 1$, $\mathbf{P}\{\hat{X}_n \mid \hat{\mathcal{F}}_{n-1}\} = \hat{X}_{n-1}$ on $\{\mathbf{P}\{\hat{\Lambda}_n \mid \hat{\mathcal{F}}_{n-1}\} = 1\} \cap \hat{\Lambda}_{n-1}$. Let $\hat{N} := \{\mathbf{P}\{L = n \mid \hat{\mathcal{F}}_{n-1}\} = 0\} \in \hat{\mathcal{F}}_{n-1}$. Note also that on $\{D_{n-1} = k\}$ with $k > n-1$, $D_n = D_{n-1}$ and so $\hat{X}_n = \hat{X}_{n-1}$. Thus for arbitrary $\hat{U} \in \hat{\mathcal{F}}_{n-1}$,

$$\begin{aligned}\mathbf{P}\{(\hat{X}_n - \hat{X}_{n-1})1_{\{L>n\}}1_{\hat{N}}1_{\hat{U}}\} &= \mathbf{P}\{(\hat{X}_n - \hat{X}_{n-1})1_{\{L>n\}}1_{\hat{N}}1_{\hat{U}}1_{\{D_{n-1}=n-1\}}\} \\ &= \mathbf{P}\{(X_{D_n} - X_{n-1})1_{\{L>n\}}1_{\hat{N}}1_{\hat{U}}1_{\Lambda_{n-1}}\}.\end{aligned}$$

The set $V := \hat{N} \cap \hat{U} \cap \Lambda_{n-1} \in \mathcal{F}_{n-1}$, and $\{L > n\} = \{G \geq n\} = \{D_n < \infty\}$ so $V \cap \{L > n\} \in \mathcal{F}_{n-1}$ is contained in $\Lambda_{n-1} \cap \Gamma_{n-1,D_n}$. The strong martingale property of X then shows that the last displayed term vanishes. □

(3.9) Corollary. *Let X and Λ satisfy the hypotheses of* (3.8). *Then X extends to a uniformly integrable martingale $\bar{X}$.*

Proof. The process $\hat{X}$ constructed in (3.8) satisfies the conditions of (3.6) relative to $\hat{\Lambda}$ and the filtration $(\hat{\mathcal{F}}_n)$. Let $\hat{X}_\infty \in L^1$ denote its final value. Then we have $\mathbf{P}\{\hat{X}_\infty \mid \mathcal{F}_{D_n}\} = X_{D_n}$ on $\{D_n < \infty\}$ for every n. Let $\bar{X}_n := \mathbf{P}\{\hat{X}_\infty \mid \mathcal{F}_n\}$. Once we show that $\bar{X}_n = X_n$ on Λ_n, $\bar{X}$ will be the desired extension of X. But, on Λ_n, $X_n = X_{D_n} = \mathbf{P}\{\hat{X}_\infty \mid \mathcal{F}_{D_n}\}$, so for every set $S \in \mathcal{F}_n \subset \mathcal{F}_{D_n}$ with $S \subset \Lambda_n$, $\mathbf{P}\{X_n 1_S\} = \mathbf{P}\{\hat{X}_\infty 1_S\}$. As $S \in \mathcal{F}_n$ is an arbitrary subset of Λ_n, and X_n is $\mathcal{F}_n$-measurable on Λ_n, this proves $X_n = \bar{X}_n$ on Λ_n. □

(3.10) Corollary. *Under the hypotheses of* (3.8) *and* (3.9), *if Y is a uniformly integrable martingale extending X and if $Y_\infty \in \mathcal{F}_G$, then $Y = \bar{X}$, as constructed in* (3.9).

Proof. With notation as in the proof of (3.8), Y_{D_n} is a uniformly integrable martingale over $(\hat{\mathcal{F}}_n)$ defined for all n (not just $n < \hat{L}$), and as $Y_\infty \in \mathcal{F}_G = \hat{\mathcal{F}}_G$, (2.4) shows that $Y_\infty = \hat{X}_\infty$. □

4. Continuous parameter case

Fix an optional set $\Lambda \subset \mathbf{R}^+ \times \Omega$ and suppose X is defined on Λ and satisfies (1.1). The main result of this section is the following.

(4.1) Theorem. *Let $L := \sup \Lambda$ (with $\sup \emptyset := 0$) denote the end of Λ, $\Omega_0 := \{\omega : 0 < L(\omega) < \infty, L(\omega) \notin \Lambda(\omega)\}$, and let C denote the predictable compensator of $1_{\Omega_0} * \varepsilon_L$. Suppose there exists a semimartingale $Z = M + A$ in $\mathcal{S}^1(\mathcal{F}_\cdot)$ extending X, such that $A \ll C$ and $Z_\infty = Z_{L-}(= X_{L-})$ on Ω_0, $Z_\infty = Z_L(= X_L)$ on $\Omega_0^c \cap \{L < \infty\} \cap \Lambda_L$. Then X extends to a unique uniformly integrable martingale $\bar{X}$ such that $\bar{X} \in \mathcal{F}_L$, $\bar{X}_\infty 1_{\Omega_0}$ is measurable with respect to the trace of $\mathcal{F}_{L-}$ on Ω_0, and $\bar{X}_t = 0$ for all $t \geq 0$ on $\{L = 0\} \cap \Lambda_0^c$.*

Before beginning the proof of (4.1), we make some preliminary reductions that will simplify the proof.

Reduction 1. Λ may be assumed right closed. Indeed, if Λ is not right closed and we let Λ' denote its closure from the right, then Λ' is also optional, and if we define X' on Λ' as (say) the lim sup of X values from the right, then we have $Z = X'$ on Λ', and as the end of Λ' is L, the condition $A \ll C$ has the same force whether we deal with X' on Λ' or X on Λ. From now until the end of the proof, Λ will be assumed right closed.

Reduction 2. We may assume $L(\omega) = 0$ if and only if $\Lambda(\omega) = \emptyset$. Extend the original Λ to be a subset of $[-1, \infty[\times\Omega$, adjoining $\{(t,\omega) : -1 \le t < 0, \omega \in \Lambda_0\}$ to the original Λ. Let $\mathcal{F}_t := \mathcal{F}_0$ for $t \in [-1, 0[$, and define X on the extended Λ by $X_t(\omega) = X_0(\omega)$ for $(t,\omega) \in \Lambda \cap [\![-1, 0[\![$. The existence of a martingale extension of the original X is clearly not affected by this extension. In addition, if we extend M and A back to time -1 by setting $A_t := 0$ and $M_t := M_0$ for $-t \le t < 0$, then the new X continues to be the restriction to the new Λ of the new Z. The new random measure C is carried by $]\!]0, \infty[\![$, as is the new A. Thus we affect neither the hypotheses nor the conclusions of the theorem by changing the time domain in this way. However, in the proofs, it is awkward to have a time index starting at -1, so we relabel the time axis to start at 0. Shifting time by 1 does not affect affect the hypotheses nor the conclusions. The net effect is that Λ may be assumed to satisfy $L(\omega) = 0$ if and only if $\Lambda(\omega) = \emptyset$.

We follow the development as in discrete case as far as possible. Let

$$D_t := \inf\{s > t : s \in \Lambda\}; \qquad g_s := \sup\{t < s : t \in \Lambda\}.$$

Because Λ is right closed, $D_t \in \Lambda$ for every $t \ge 0$ for every $t < L$. The process $t \to D_t$ is right continuous, and its left continuous inverse is given by $s \to g_s$. Each D_t is a stopping time over $(\mathcal{F}_t)$. Let $\hat{\mathcal{F}}_t := \mathcal{F}_{D_t}$, and note that L is a stopping time over $(\hat{\mathcal{F}}_t)$, for $\{L > t\} = \{D_t < \infty\} \in \hat{\mathcal{F}}_t$. Define the random set $\hat{\Lambda} := [\![0, L[\![\cup ([\![L]\!] \cap \Lambda)$, so that $\hat{\Lambda}$ is optional relative to $\hat{\mathcal{F}}_t$. Define $\hat{X}$ on $\hat{\Lambda}$ by

$$\hat{X}_t := \begin{cases} X(D_t) & \text{for } t < L, \\ X_{L-} & \text{for } t \ge L \text{ on } \Omega_0, \\ X_L & \text{for } t \ge L \text{ on } \Omega_0^c \cap \{L > 0\}. \end{cases}$$

The following result is evident.

(4.2) Lemma. *With Λ modified in accordance with reductions 1 and 2, the range of the map $t \to D_t$ (in $[0, \infty[$) is $\Lambda \setminus \Lambda^i$, where $\Lambda^i := \{t \in \Lambda : t > 0, t = g_t < D_t\}$—i.e., the points in Λ which are accumulation points of Λ from the left but not from the right.*

In outline the proof will involve showing first that $\hat{X}_t$ extends to a martingale with final value $\bar{X}_\infty$. We will then let $\bar{X}$ be a right continuous version of the martingale $\bar{X}_t := \mathbf{P}\{\bar{X}_\infty \mid \mathcal{F}_t\}$, and show that the hypotheses imply that $\bar{X}$ extends X. In preparation for these arguments we need some results which use ideas surrounding the change of variable formula in the form given, say, in [Sh88, p379].

(4.3) Lemma. *Let $\hat{T}$ be a stopping time over $(\hat{\mathcal{F}}_t)$. Then $\{\hat{T} < g_s\} \in \mathcal{F}_s$.*

Proof. First of all, note that $D_t < s$ if and only if $t < g_s$. It is also clear that $g_s \in \mathcal{F}_s$. Therefore

$$\{\hat{T} < g_t\} = \cup_{r<t, r \text{ rational}} \{\hat{T} < r < g_s\},$$

and for every fixed $r < t$, we have $\{\hat{T} < r < g_s\} = \{\hat{T} < r\} \cap \{r < g_s\} = \{\hat{T} < r\} \cap \{D_r < s\} \in \mathcal{F}_s$ since $\{\hat{T} < r\} \in \mathcal{F}_{D_r}$. $\square$

(4.4) Lemma. *Let $\hat{H}$ be predictable over $(\hat{\mathcal{F}}_t)$ with $\hat{H}_0 = 0$. Then $\hat{H}(g_s)$ is predictable over $(\mathcal{F}_s)$.*

Proof. It suffices to check this in case $\hat{H} = 1_{]\!]\hat{T},\infty[\![}$ with $\hat{T}$ a stopping time over $(\hat{\mathcal{F}}_t)$, as such processes generate the predictable processes over $(\hat{\mathcal{F}}_t)$ vanishing at 0. For $\hat{H}$ of this form, $\hat{H} \circ g_s = 1_{]\!]\hat{T},\infty[\![}(g_s) = 1_{\{\hat{T}<g_s\}}$, and the latter is left continuous in s and adapted to $(\mathcal{F}_s)$ by the preceding lemma. $\square$

Proof of Theorem (4.1). We work with reductions 1 and 2 in force. Let $\hat{Z}_t := Z(D_t)$, $\hat{M}_t := M(D_t)$, $\hat{A}_t := A(D_t)$, all of which make sense for $0 \le t \le \infty$, and let $\hat{A}^p$ denote the predictable (relative to $(\hat{\mathcal{F}}_t)$) compensator of $\hat{A}_t$. We show first that $\hat{A}^p$ is absolutely continuous with respect to $\hat{C}$, the predictable (relative to $(\hat{\mathcal{F}}_t)$) compensator of $1_{\{0<L<\infty, L\notin\Lambda\}} * \varepsilon_L$. Let $\hat{H}$ be bounded and predictable over $(\hat{\mathcal{F}}_t)$ with $\hat{H}_0 = 0$. Then $\hat{H}(g_s)$ is predictable over $(\mathcal{F}_s)$ by (4.4), and by the change of variable formula,

$$\text{(4.5)} \qquad \mathbf{P}\int_{]\!]0,\infty[\![} \hat{H}_t \, d\hat{A}_t = \mathbf{P}\int_{]\!]0,\infty]\!]} \hat{H}(g_s) 1_{\{0<g_s<\infty\}} \, dA_s.$$

Now suppose in addition that $\mathbf{P}\int_0^\infty |\hat{H}_t| \, d\hat{C}_t = 0$, or equivalently, $\mathbf{P}\,|\hat{H}_L| 1_{\{0<L<\infty, L\notin\Lambda\}} = 0$. Then $t \to |\hat{H}(g_t)|$ also vanishes a.s. at $t = L$ on $\{0 < L < \infty, L \notin \Lambda\}$ since, on that set, $g_L = L$. That is, $\mathbf{P}\int |\hat{H}(g_s)| 1_{\{0<g_s<\infty\}} \, dC_s = 0$. The condition $A \ll C$ then implies that the right side of (4.5) vanishes, and consequently, $\mathbf{P}\int_{]\!]0,L]\!]} \hat{H}_t \, d\hat{A}_t^p = \mathbf{P}\int_{]\!]0,L]\!]} \hat{H}_t \, d\hat{A}_t = 0$. This proves that $\hat{A}^p \ll \hat{C}$. Observe now that $\hat{Z} = \hat{M}_t + \hat{A}_t$ is a semimartingale over $(\hat{\mathcal{F}}_t)$, and it is in $\mathcal{S}^1(\hat{\mathcal{F}}_.)$ because $\hat{M}$ is a uniformly integrable martingale over $(\hat{\mathcal{F}}_t)$ and $\hat{A}_t$ is optional over $(\hat{\mathcal{F}}_t)$ and of integrable total variation. The canonical decomposition of $\hat{Z}$ is then $(\hat{M}_t + \hat{A}_t - \hat{A}_t^p) + \hat{A}_t^p$. Observe too that $\hat{Z}_t = Z_\infty$ for $t \ge L$, and that on Ω_0, $D_t < L$ for all $t < L$ so that $\hat{Z}_{L-} = \lim_{t\uparrow\uparrow L} \hat{Z}_t = Z_{L-} = Z_\infty$ by hypothesis. Therefore $\hat{Z}_t = X_{L-}$ for all $t \ge L$ on Ω_0. Clearly $\hat{X}_t = X(D_t)$ for $t < L$, and on $\Omega_0^c \cap \{0 < L < \infty\}$, $\hat{X}_L = \hat{Z}_L = Z_\infty = X_L$ by the hypotheses on Z. We have now shown that $\hat{X}$ is the restriction to $\hat{\Lambda}$ of $\hat{Z}$, and that the conditions of (1.2) with respect to the process $\hat{X}$ on $\hat{\Lambda}$ are satisfied. Therefore, by (1.2), $\hat{X}$ extends to a uniformly integrable martingale (with respect to $(\hat{\mathcal{F}}_t)$) whose final value we shall denote by $\bar{X}_\infty$. In fact, by (1.2), if we let $\hat{H} \in \hat{\mathcal{P}}$ be a version of $d\hat{A}^p/d\hat{C}$, then we may take

$$\text{(4.6)} \qquad \bar{X}_\infty = Z_\infty - \hat{H}_L 1_{\Omega_0 \cap \{0<L<\infty\}}.$$

Let $\bar{X}_t$ be a right continuous version of $\mathbf{P}\{\bar{X}_\infty \mid \mathcal{F}_t\}$. Clearly $X_{D_t} = \bar{X}_{D_t}$ a.s on $\{t < L\}$ for every $t \ge 0$. It follows that $X_s = \bar{X}_s$ for all $s \in \Lambda$ in the range of the map $t \to D_t$, so by (4.2), $\bar{X} = X$ on $\Lambda \setminus \Lambda^i$. Let $\tilde{\Lambda} := \{\bar{X} \ne X\} \subset \Lambda^i$. Clearly $\tilde{\Lambda}$ is optional, though Λ^i need not be. As $\tilde{\Lambda}$ has countable sections, we may express $\tilde{\Lambda} = \cup_n [\![T_n]\!]$, where the stopping times T_n have disjoint graphs. In order to prove $\tilde{\Lambda}$ is evanescent (which implies that $\bar{X}$ extends X) it suffices to show $T_n = \infty$ a.s. for every n. Fix n and let T denote T_n, so that T is a stopping time with $[\![T]\!] \subset \Lambda^i$. In particular, since $L = D_T \in \Lambda$ is not possible on Ω_0, $D_T < L$ on $\{T < L\} \cap \Omega_0$. Let $K_t := 1_{]\!]T,D_T]\!]}(t) Y_t$, where Y is an arbitrary bounded predictable process. Then $\mathbf{P}\int |K_t| \, dC_t = \mathbf{P}|K_L| 1_{\Omega_0} = 0$ since $\mathbf{P}\{T < L \le D_T\} \cap \Omega_0 = 0$. In view of the hypothesis $A \ll C$, we have $\mathbf{P}\int K_t \, dA_t = 0$, and as $Y \in \mathrm{b}\mathcal{P}$ is arbitrary, this shows that dA does not charge the interval $]\!]T, D_T]\!]$, so $A(D_T) = A_T$. It follows that $Z_T = \mathbf{P}\{Z(D_T) \mid \mathcal{F}_T\}$. Putting this together with (4.6) and the fact that $\bar{X}$ is a uniformly integrable martingale, we find

$$\begin{aligned}
\bar{X}_T = \mathbf{P}\{\bar{X}(D_T) \mid \mathcal{F}_T\} &= \mathbf{P}\{Z(D_T) - \hat{H}_L 1_{\Omega_0 \cap \{D_T = \infty\}} \mid \mathcal{F}_T\} \\
&= Z_T - \mathbf{P}\{\hat{H}_L 1_{\Omega_0 \cap \{D_T=\infty\}} \mid \mathcal{F}_T\} \\
&= X_T - \mathbf{P}\{\hat{H}_L 1_{\Omega_0 \cap \{D_T=\infty\}} \mid \mathcal{F}_T\} \quad \text{on } \{T < \infty\}.
\end{aligned}$$

However, on Ω_0, if $T < \infty$ then $T < L$ and so $D_T < \infty$, and consequently $\bar{X}_T = X_T$ a.s. on $\{T < \infty\}$. This proves $T = \infty$ a.s., finishing the existence part of the theorem.

For uniqueness, suppose Y is a another uniformly integrable martingale extending X and satisfying (a) $Y_\infty \in \mathcal{F}_L$; (b) $Y_\infty 1_{\Omega_0 \cap \{0<L<\infty\}}$ is measurable with respect to the trace of F_{L-} on Ω_0; (c) $Y_\infty = 0$ on $\{L = 0\}$. Subtracting Y from $\bar{X}$, we see that uniqueness is equivalent to showing that $\bar{X} = 0$ if $X = 0$ on Λ. Let $\hat{X}_t := \bar{X}(D_t)$. Then $\hat{X}_t$ is a uniformly integrable martingale over $(\hat{\mathcal{F}}_\cdot)$ extending 0 on $\hat{\Lambda}$, stopping at L, and satisfying $\hat{X}_\infty \in \mathcal{F}_L$, $\hat{X}_\infty 1_{\Omega_0 \cap \{0<L<\infty\}}$ measurable with respect to the trace of $\mathcal{F}_{L-}$ on $\Omega_0 \cap \{0 < L < \infty\}$. The condition $\hat{X}_\infty \in \mathcal{F}_L$ implies $\hat{X}_\infty \in \hat{\mathcal{F}}_L$, for L is a stopping time over $\hat{\mathcal{F}}_\cdot$ and so the test is $\hat{X}_\infty 1_{\{L \le t\}} \in \hat{\mathcal{F}}_t$ for every $t \ge 0$. By definition of $\hat{\mathcal{F}}_t$, this is the same as $\hat{X}_\infty 1_{\{L \le t\} \cap \{D_t \le s\}} \in \mathcal{F}_s$ for all $t, s \ge 0$. However, on $\{L \le t\}$, $D_t = \infty$, so $\hat{X}_\infty \in \hat{\mathcal{F}}_L$. The σ-algebra $\mathcal{F}_{L-}$ is generated by events of the form $W \cap \{L > t\}$ with $W \in \mathcal{F}_t$ and $t \ge 0$. As $\mathcal{F}_t \subset \hat{\mathcal{F}}_t$, this proves $\mathcal{F}_{L-} \subset \hat{\mathcal{F}}_{L-}$. Now apply the uniqueness result (1.4) to $\hat{X}$ to see $\hat{X}_\infty = 0$. $\square$

The extension of X defined by (4.1) takes a simpler form some particular cases which we now describe. We work under the hypotheses of (4.1), together with reductions 1 and 2 and the notation developed in its proof. The end L of Λ is a stopping time over $(\hat{\mathcal{F}}_t)$. Decompose $[\![L]\!] = [\![L_0]\!] \cup [\![L_1]\!]$, where $[\![L_0]\!] := [\![L]\!] \cap]\!]0, \infty[\![\cap \Lambda^c$ and $[\![L_1]\!] := [\![L]\!] \setminus [\![L_0]\!]$. It is clear that L_0 and L_1 are stopping times over $(\hat{\mathcal{F}}_t)$. It is also clear by definition of Ω_0 that $\Omega_0 = \{L_0 < \infty\}$. Let L_p denote the predictable part of L_0, defined in [Sh92, §2] as the largest predictable stopping time (over $(\hat{\mathcal{F}}_t)$) with graph contained in $[\![L_0]\!]$. Choose stopping times T_n announcing L_p, and observe that for every bounded left continuous predictable (with respect to $(\hat{\mathcal{F}}_t)$) process $\hat{Y}$, making use of the fact that $\hat{A}^p$ is carried by $]\!]0, L]\!]$ in the third equality,

$$\begin{aligned}
\mathbf{P}\, \hat{Y}_{L_p} \hat{H}_{L_p} 1_{\{L_p < \infty\}} &= \mathbf{P} \int \hat{Y}_t 1_{[\![L_p, \infty[\![}(t) \hat{H}_t \, d\hat{C}_t \\
&= \mathbf{P} \int \hat{Y}_t 1_{[\![\zeta_p, \infty[\![}(t) \, d\hat{A}^p_t \\
&= \mathbf{P}\, \hat{Y}_L (\hat{A}^p_\infty - \hat{A}^p_{L_p-}) \\
&= \lim_n \mathbf{P}\, \hat{Y}_{T_n} (\hat{A}^p_\infty - \hat{A}^p_{T_n}) \\
&= \lim_n \mathbf{P}\, \hat{Y}_{T_n} (\hat{Z}_\infty - \hat{Z}_{T_n}) \\
&= \mathbf{P}\, \hat{Y}_{L_p} (\hat{Z}_\infty - \hat{Z}_{L_p-}).
\end{aligned}$$

Now write the last term as $\mathbf{P}\, \hat{Y}_{L_p} (\hat{Z}_\infty - \hat{Z}_{L_p-}) 1_{\Omega_0^c} 1_{\{L_p<\infty\}} + \mathbf{P}\, \hat{Y}_{L_p} (\hat{Z}_\infty - \hat{Z}_{L_p-}) 1_{\Omega_0} 1_{\{L_p<\infty\}}$, and note that on $\{L_p < \infty\} \subset \Omega_0$, $\hat{Z}_\infty = \hat{Z}_{L-} = \hat{Z}_{L_p-}$, so the second term vanishes. On the other hand $\Omega_0^c \cap \{L_p < \infty\} = \emptyset$, so the first term also vanishes and so we are led to the identity, for every $Y \in \mathrm{b}\hat{\mathcal{P}}$,

$$\mathbf{P}\, Y_{L_p} H_{L_p} 1_{\{L_p<\infty\}} = 0. \tag{4.7}$$

This proves $\hat{H}_{L_p} = 0$ on $\{L_p < \infty\}$. This may be restated as follows.

(4.8) Corollary. *Under the hypotheses of Theorem* (4.1), $\bar{X}_\infty = X_{L-}$ *on* $\{L_p < \infty\}$.

In the special case $\Lambda = [\![0, \zeta[\![$ with $\zeta > 0$ a.s., if we denote by ζ_p the predictable part of ζ (*i.e.*, $[\![\zeta_p]\!]$ is the largest predictable subset of $[\![\zeta]\!]$) then (4.8) shows $\bar{X}_\infty = X_{\zeta_p -}$ on $\{\zeta_p < \infty\}$.

(4.9) Remark. Theorem (4.1) is utterly worthless if Λ contains its end L, for in this case $C = 0$, and the condition $A \ll C$ implies $A = 0$, so the theorem amounts to "X has an extension to a uniformly integrable martingale if it is the restriction to Λ of a uniformly integrable martingale."

5. The strong martingale property, continuous parameter case

Given a stopping times $S \le T$, define Λ_T and $\Gamma_{S,T}$ as in section 3, and define the strong martingale property of X on Λ as in (3.4). Under the hypotheses of Theorem (4.1), X may be regarded as the restriction to Λ of a uniformly integrable martingale, and consequently X has the strong martingale property on Λ.

We investigate in this section conditions under which the strong martingale property implies the existence of a martingale extension of X on Λ. We give first an example to show that in general there can be no equivalence between the absolute continuity condition $A \ll C$ of (4.1) and a strong martingale property such as holds in the discrete parameter case, (3.6). Consider the following example from [Sh92]. Let B denote linear Brownian motion, let ζ be an exponential time with parameter 1 independent of B, and let $\Lambda := [\![0, \zeta[\![$. Let $S \le T$ be stopping times over $(\mathcal{F}_t)$, the natural filtration (suitably completed) for $(B_t)_{t<\zeta}$. We show $\Lambda_S \cap \Gamma_{S,T} = \{S = T < \zeta\}$. Since ζ is independent of B and has exponential distribution, $\mathbf{P}\{S < \zeta < T \mid \mathcal{F}_S\} > 0$ on $\{S < T, S < \zeta\}$, and consequently $\mathbf{P}\{S < T < \zeta \mid \mathcal{F}_S\} < 1$ a.s. on $\{S < T < \zeta\}$. By definition of $\Gamma_{S,T}$, it follows that $\Lambda_S \cap \Gamma_{S,T} \subset \{S = T < \zeta\}$. Because of this, the strong martingale property holds trivially for every X on Λ satisfying obvious integrability conditions. However, as shown in [Sh92], $B_t^2 1_{\{t<\zeta\}}$ has a martingale extension, while $|B_t| 1_{\{t<\zeta\}}$ does not. In other words, the strong martingale property need not by itself imply the existence of a martingale extension.

Before beginning a discussion of sufficiency of the strong martingale property, here is a preliminary result which reduces the work needed to verify it. Recall the notation of (3.3) and (3.4), where for a given stopping time T, Y_t^T denotes a right continuous version of $\mathbf{P}\{\Lambda_T^c \mid \mathcal{F}_t\}$ and $\zeta_T := \inf\{t : Y_t^T = 0\}$.

(5.1) Proposition. *X has the strong martingale property on Λ provided* (3.5) *holds for all pairs of stopping times $S \le T$ such that $[\![T]\!] \subset \Lambda$ and $[\![S]\!] \subset [\![\zeta_T, T[\![\cap \Lambda$.*

Proof. We must verify (3.5) for an arbitrary pair $S \le T$ of stopping times. Define the stopping time T' by $[\![T']\!] = [\![T]\!] \cap \Lambda$, so that $T' \ge S$ and $[\![T']\!] \subset \Lambda$. Clearly $\Lambda_T = \Lambda_{T'}$, so $\Gamma_{S,T} = \Gamma_{S,T'}$. It follows that (3.5) holds for the pair S, T if and only if it holds for the pair S, T', and it therefore suffices to verify (3.5) assuming $[\![T]\!] \subset \Lambda$. Define the stopping time S' by $[\![S']\!] = [\![S]\!] \cap [\![\zeta_T, T]\!] \cap \Lambda$ and let $S'' := S' \wedge T$. Then the stopping time $S'' \le T$ and

$[\![S'']\!] \subset \Lambda$. By hypothesis, (3.5) holds for the pair S'', T. This property is equivalent to the equality, for every bounded right continuous adapted process W,

$$\mathbf{P}\{X_T W_{S''}; \Lambda_{S''} \cap \Gamma_{S'',T}\} = \mathbf{P}\{X_{S''} W_{S''}; \Lambda_{S''} \cap \Gamma_{S'',T}\}. \tag{5.2}$$

For $\omega \in \Lambda_{S''} \cap \Gamma_{S'',T}$, either $\omega \in \{S \geq \zeta_T\}$ and $S(\omega) = S''(\omega)$ and $\omega \in \Lambda_S \cap \Gamma_{S,T}$, or $\omega \in \{S < \zeta_{\Lambda_T}\}$ and $S''(\omega) = T(\omega)$ and by (3.2), $\omega \notin \Lambda_S \cap \Gamma_{S,T}$. Thus we have

$$\begin{aligned}
\Lambda_{S''} \cap \Gamma_{S'',T} &= \Lambda_{S''} \cap \Gamma_{S'',T} \cap \{S \geq \zeta_T\} \cup \Lambda_{S''} \cap \Gamma_{S'',T} \cap \{S < \zeta_T\} \\
&= \Lambda_S \cap \Gamma_{S,T} \cap \{S \geq \zeta_T\} \cup \Lambda_T \cap \Gamma_{T,T} \cap \{S < \zeta_{\Lambda_T}\} \\
&= \Lambda_S \cap \Gamma_{S,T} \cap \{S \geq \zeta_T\} \cup \Lambda_T \cap \{S < \zeta_{\Lambda_T}\}.
\end{aligned}$$

Considering (5.2) separately on the sets $\{S \geq \zeta_T\}$, $\{S < \zeta_T\}$ (both in $\mathcal{F}_{S''}$ since $S \leq S''$) gives

$$\begin{aligned}
\mathbf{P}\{X_T W_{S''}; \Lambda_{S''} \cap \Gamma_{S'',T}, S \geq \zeta_T\} &= \mathbf{P}\{X_{S''} W_{S''}; \Lambda_{S''} \cap \Gamma_{S'',T}, S \geq \zeta_T\} \\
\mathbf{P}\{X_T W_{S''}; \Lambda_{S''} \cap \Gamma_{S'',T}, S < \zeta_T\} &= \mathbf{P}\{X_{S''} W_{S''}; \Lambda_{S''} \cap \Gamma_{S'',T}, S < \zeta_T\}
\end{aligned}$$

Making the reductions from the previous display makes the second equation a triviality, and the first becomes

$$\mathbf{P}\{X_T W_S; \Lambda_S \cap \Gamma_{S,T}, S \geq \zeta_T\} = \mathbf{P}\{X_S W_S; \Lambda_S \cap \Gamma_{S,T}, S \geq \zeta_T\}.$$

By (3.2), the term $S \geq \zeta_T$ may now be omitted, giving

$$\mathbf{P}\{X_T W_S; \Lambda_S \cap \Gamma_{S,T}\} = \mathbf{P}\{X_S W_S; \Lambda_S \cap \Gamma_{S,T}\}.$$

This proves that the pair S, T satisfies (3.5). $\square$

For sufficiency of the strong martingale property, we begin with the case $]\!]0, \zeta[\![\subset \Lambda \subset [\![0, \zeta]\!]$, as in (1.2).

(5.3) Theorem. *Let X be defined on Λ, $]\!]0, \zeta[\![\subset \Lambda \subset [\![0, \zeta]\!]$, and suppose $[\![\zeta]\!] \backslash \Lambda$ is contained in a predictable set K with countable sections, and having no finite limit points other than ζ. Assume also that X is the restriction to Λ of a semimartingale $Z \in \mathcal{S}^1(\mathcal{F}_.)$. Then X extends to a uniformly integrable martingale if and only if X has the strong martingale property on Λ.*

Proof. Assume X has the strong martingale property on Λ. Let $\Omega_0 := \{\omega : \zeta(\omega) \notin \Lambda(\omega)\}$, and let C denote the predictable compensator of $1_{\Omega_0} * \varepsilon_\zeta$, as usual. With the conditions imposed on K, the predicable compensator of $1_{\Omega_0} * \varepsilon_\zeta$ is carried by K. Let T_n denote the time of the n^{th} jump of C so that the predictable stopping times T_n have disjoint graphs and increase with n to a limit $T_\infty \geq \zeta$. Denote by $K_1 := \cup_n [\![T_n]\!] \in \mathcal{P}$ the (discrete) support of C, so $K_1 \subset K \subset [\![0, \zeta]\!]$. Define $d_t(\omega) := \inf\{s > t : s \in K_1(\omega)\}$, so that for every $t \geq 0$, $d_t > t$, and $d_t \in K_1$ if $d_t < \infty$. For every stopping time $S \leq \zeta$, d_S being the debut of the predictable set $K_1 \cap]\!]S, \infty[\![$ (which contains its debut) is a predictable stopping time with graph in K_1. Fix a stopping time $S \leq \zeta$ and let τ_k be a sequence of stopping times announcing d_S. We may assume $\tau_k \geq S$. Fix k and let $T := \tau_k \wedge \zeta$. Then, by construction, $]\!]S, T]\!] \cap K_1$ is evanescent, and therefore

$$0 = \mathbf{P} \int 1_{]\!]S,T]\!]}(t)\, dC_t = \mathbf{P} \int 1_{]\!]S,T]\!]}(t)\, 1_{\Omega_0}(t) \varepsilon_\zeta(dt) = \mathbf{P}(\{S < \zeta \leq T\} \cap \Omega_0). \tag{5.4}$$

Then

$$\Gamma_{S,T} \cap \Lambda_S = \{\mathbf{P}\{\Lambda_T \mid \mathcal{F}_S\} = 1\} \cap \Lambda_S = \{\mathbf{P}\{\Lambda_T^c \mid \mathcal{F}_S\} = 0\} \cap \Lambda_S.$$

Write $\Lambda_T^c = (\Lambda_T^c \cap \Omega_0) \cup (\Lambda_T^c \cap \Omega_0^c)$ and note that $(\Lambda_T^c \cap \Omega_0) \cap \Lambda_S = \{S < \zeta \leq T\} \cap \Omega_0$ is null by (5.4). In addition, $(\Lambda_T^c \cap \Omega_0^c) \cap \Lambda_S = \{S \leq \zeta < T\} \cap \Omega_0^c$ is also null since $T \leq \zeta$, and so $\Gamma_{S,T} \cap \Lambda_S = \Lambda_S$. We modify Z if necessary so that Z stops at ζ, and $Z_t = Z_{\zeta-}$ for $t \geq \zeta$ on Ω_0. This does not change the condition $Z \in \mathcal{S}^1(\mathcal{F}_\cdot)$. Let $Z = M + A$ denote the canonical decomposition of Z. By the strong martingale property of X for the pair S, T, we find $\mathbf{P}\{(Z_T - Z_S)1_{\Lambda_S} \mid \mathcal{F}_S\} = 0$, hence that $\mathbf{P}\{(A_T - A_S)1_{\Lambda_S} \mid \mathcal{F}_S\} = 0$. Now let $k \to \infty$, so that $T = \tau_k \wedge \zeta$ increases to $d_S \wedge \zeta$, strictly from below if $d_S \leq \zeta$. It follows that $\mathbf{P}\{(Z(d_S-) - Z_S)1_{\Lambda_S} \mid \mathcal{F}_S\} = 0$. Thus

$$\mathbf{P}\{(A(d_S-) - A_S)1_{\{d_S \leq \zeta\}} + (A_\zeta - A_S)1_{\{d_S > \zeta\}} \mid \mathcal{F}_S\}1_{\Lambda_S} = 0. \tag{5.5}$$

We shall prove that (5.5) implies that A is carried by K_1, which will show (by Theorem (4.1)) that X extends to a uniformly integrable martingale. Let $T_0 := 0$ and fix $n \geq 1$. Given a stopping time S with $[\![S]\!] \subset [\![T_{n-1}, T_n[\![$, note that $d_{S\wedge\zeta} = T_n$ on $\{S < T_n\}$. Substituting in (5.5) yields

$$\mathbf{P}\{(A(T_n-) - A_{S\wedge\zeta})1_{\{T_n < \infty\}} + (A_\zeta - A_{S\wedge\zeta})1_{\{T_n = \infty\}} \mid \mathcal{F}_{S\wedge\zeta}\}1_{\Lambda_{S\wedge\zeta}} = 0.$$

It is clear that $\Lambda + S \wedge \zeta = \Lambda_S \cup (\Lambda_\zeta \cap \{S \geq \zeta\})$. The set $\{S \geq \zeta\} \in \mathcal{F}_{S\wedge\zeta}$, and on $\{S \geq \zeta\}$, $T_n = \infty$ and as A stops at ζ, the integrand vanishes. Thus we may replace $1_{\Lambda_{S\wedge\zeta}}$ in the last display with 1_{Λ_S}. As the trace of $\mathcal{F}_{S\wedge\zeta}$ on Λ_S is clearly the same as the trace of $\mathcal{F}_S$ on Λ_S, and considering that $\Lambda_S \subset \{S \leq \zeta\}$, the formula may be rewritten more simply as

$$\mathbf{P}\{A(T_n-) - A_S \mid \mathcal{F}_S\} = 0 \quad \text{on } \Lambda_S. \tag{5.6}$$

Define a predictable process V_t of integrable variation by

$$V_t := \begin{cases} 0 & \text{for } t < T_{n-1} \\ A_t - A(T_{n-1}) & \text{for } T_{n-1} \leq t < T_n \\ A(T_n-) - A(T_{n-1}) & \text{for } t \geq T_n. \end{cases}$$

That is, $V = 1_{]\!]T_{n-1}, T_n[\![} * A$. Then V has potential $U_t := \mathbf{P}\{V_\infty - V_t \mid \mathcal{F}_t\}$. Since V is carried by $[\![0, \zeta]\!]$, $U = 0$ on $[\![\zeta, \infty]\!]$. For any stopping time S with $[\![S]\!] \subset [\![T_{n-1}, T_n[\![$, $U_S = \mathbf{P}\{A(T_n-) - A_S \mid \mathcal{F}_S\}$ which vanishes on $\Lambda_S \supset \{S < \zeta\}$, by (5.6). This proves $U = 0$ on $[\![T_{n-1}, T_n[\![$. Because V is predictable and is carried by $[\![T_{n-1}, T_n[\![$, we may conclude that $V = 0$—that is, A does not charge $]\!]T_{n-1}, T_n[\![$, and as n is arbitrary, this shows that A is carried by K_1, as claimed. □

(5.7) Remark. The strong martingale property was used only for pairs $S \leq T$ with graphs contained in $[\![T_{n-1}, T_n[\![\cap [\![0, \zeta]\!]$. In addition, the proof showed that for such a pair of stopping times, $\Lambda_S \cap \Gamma_{S,T} = \Lambda_S$.

(5.8) Hypotheses and Notation. The following will remain in force for the rest of the section. We are given a process X on an optional random set Λ. Suppose:

(5.8i) Λ is right closed;

(5.8ii) L denotes the end of Λ, $[\![L_0]\!] := [\![L]\!] \cap \Lambda^c \cap]\!]0, \infty[\![$ and $\Omega_0 := \{L_0 < \infty\}$;

(5.8iii) X is the restriction to Λ of a special semimartingale $Z = M + A \in \mathcal{S}^1(\mathcal{F}_\cdot)$ such that $Z_\infty = Z_{L-}$ on Ω_0 and $Z_\infty = Z_L$ on $\Omega_0^c \cap \{L < \infty\}$;

(5.8iv) $D_t := \inf\{s > t : s \in \Lambda\}$;

(5.8v) $\hat{\mathcal{F}}_t := \mathcal{F}_t$, $\hat{\Lambda} := [\![0, L[\![\cup ([\![L]\!] \cap \Lambda)$, $\hat{Z}_t := Z(D_t)$, $\hat{X}$ denotes the restriction of $\hat{Z}$ to $\hat{\Lambda}$;

(5.8vi) Λ^i denotes the points in Λ that are isolated to the right but not to the left.

(5.9) Lemma. *Let $\hat{T}$ be a stopping time over $(\hat{\mathcal{F}}_t)$. Then $D_{\hat{T}}$ is a stopping time over $(\mathcal{F}_t)$ and $\mathcal{F}_{D_{\hat{T}}} = \hat{\mathcal{F}}_{\hat{T}}$.*

Proof. We show first that $D_{\hat{T}}$ is a stopping time over $(\mathcal{F}_t)$. By standard approximation arguments for stopping times, It suffices to give a proof for $\hat{T}$ taking discrete values. Such $\hat{T}$ may be constructed as a countable infimum of times of the form $\hat{T}$ taking just two values, t_0 (on a set $B \in \hat{\mathcal{F}}_{t_0}$) and ∞ (on B^c.) It suffices to prove $D_{\hat{T}}$ is a stopping time over $(\mathcal{F}_t)$ for such $\hat{T}$. We have in this case

$$D_{\hat{T}} = \begin{cases} D_{t_0} & \text{on } B \in \hat{\mathcal{F}}_{t_0} = \mathcal{F}_{D_{t_0}} \\ \infty & \text{on } B^c. \end{cases}$$

As D_{t_0} is a stopping time over $(\mathcal{F}_t)$, $D_{\hat{T}}$ must also be a stopping time over $(\mathcal{F}_t)$, as it is the debut of the $(\mathcal{F}_t)$-optional set $[\![D_{t_0}, \infty [\![\cap B$. We turn now to proving $\mathcal{F}_{D_{\hat{T}}} = \hat{\mathcal{F}}_{\hat{T}}$. Assume first that $\hat{T}$ takes just two values, as above. Then $\hat{\mathcal{F}}_{\hat{T}} = \{F \in F_\infty : F \cap B \in \hat{\mathcal{F}}_{t_0}\}$, while $\mathcal{F}_{D_{\hat{T}}} = \{F \in \mathcal{F}_\infty : F \cap B \in \mathcal{F}_{D_{t_0}}\}$, which is clearly equal to $\hat{\mathcal{F}}_{\hat{T}}$. The general case now follows by a straightforward argument from the following pair of observations. (a) Let S_1, S_2 be stopping times. Then $\mathcal{F}_{S_1 \wedge S_2} = \mathcal{F}_{S_1} \cap \mathcal{F}_{S_2}$. [The inclusion $\mathcal{F}_{S_1 \wedge S_2} \subset \mathcal{F}_{S_1} \cap \mathcal{F}_{S_2}$ is clear. For the other direction, suppose $B \in \mathcal{F}_{S_1} \cap \mathcal{F}_{S_2}$. Then $B \cap \{S_1 \wedge S_2 \leq t\} = ((B \cap \{S_1 \leq S_2\}) \cap \{S_1 \leq t\}) \cup ((B \cap \{S_2 \leq S_1\}) \cap \{S_2 \leq t\})$, which is obviously in $\mathcal{F}_t$.] (b) Let S_n be a decreasing sequence of stopping times with limit S. Then $\mathcal{F}_S = \cap_n \mathcal{F}_{S_n}$. □

The conditions in the following theorem are unfortunately very strong. Note that hypothesis (iii) in particular is satisfied if Λ^c is right closed.

(5.10) Theorem. *Let X and Λ satisfy the conditions (5.8), and suppose in addition*

(i) *$[\![L_0]\!]$ is contained in a predictable set $K \subset \Lambda$ with countable sections having no limit point before L;*
(ii) *for all $t \in K$, $D_t = t$;*
(iii) *$\Lambda \setminus \Lambda^i$ contains the graph on no stopping time.*

Then X extends to a uniformly integrable martingale if and only if X has the strong martingale property on Λ.

Proof. Only one direction requires proof. Assume X has the strong martingale property on Λ. We make the same minor modifications to X and Λ as in (4.1), so that we may assume $L(\omega) = 0$ if and only if $\Lambda(\omega) = \emptyset$. The given condition on K implies, as in the proof of (5.3), that C, the predictable compensator of $1_{\Omega_0} * \varepsilon_L = \varepsilon_{L_0}$, is a sum of jumps at predictable stopping times T_n have disjoint graphs and increasing with n to a limit $T_\infty \geq L$.

Let $\hat{C}$ denote the predictable (relative to $(\hat{\mathcal{F}}_t)$) compensator of ε_{L_0}. As any predictable stopping time over $(\mathcal{F}_t)$ is also a predictable stopping time over $(\hat{\mathcal{F}}_t)$, $\mathcal{P} \subset \hat{\mathcal{P}}$—the predictable processes over $(\hat{\mathcal{F}}_t)$. For each $n \geq 1$, C does not charge the predictable interval $]\!] T_{n-1}, T_n [\![$, so $\mathbf{P}\{T_{n-1} < L_0 < T_n\} = 0$, and consequently $\hat{C}$ does not charge $]\!] T_{n-1}, T_n [\![$. Note however that since $[\![T_n]\!] \subset K$, $D(T_n) = T_n$ by the hypotheses on K, and therefore $\hat{\mathcal{F}}_{T_n} = \mathcal{F}_{T_n}$ by Lemma (5.9). It follows that $\hat{C} = C$, for $\Delta\hat{C}(T_n) = \mathbf{P}\{L = T_n < \infty, L \notin \Lambda \mid \hat{\mathcal{F}}_{T_n}\} = \mathbf{P}\{L = T_n < \infty, L \notin \Lambda \mid \mathcal{F}_{T_n}\} = \Delta C(T_n)$. This proves that the times T_n are identical whether computed relative to $(\mathcal{F}_t)$ or to $(\hat{\mathcal{F}}_t)$. To finish the proof, it will suffice to check that $\hat{X}$ has the strong martingale property. In view of the remark (5.7), it suffices to check the strong martingale property for pairs of stopping times $\hat{S} \leq \hat{T}$ with graphs in $[\![T_{n-1}, T_n [\![\cap [\![0, L]\!]$ for some $n \geq 1$. As remarked in (5.7), for such stopping times one has $\hat{\Lambda}_{\hat{S}} \cap \hat{\Gamma}_{\hat{S},\hat{T}} = \hat{\Lambda}_{\hat{S}}$. By

(5.9), $D_{\hat{S}} \le D_{\hat{T}}$ is a pair of stopping times over $(\mathcal{F}_t)$. Because of the hypothesis $D_{T_n} = T_n$, $\hat{T} < T_n$ implies $D_{\hat{T}} < D_{T_n} = T_n$, and consequently $D_{\hat{S}}$ and $D_{\hat{T}}$ have graphs contained in $[\![T_{n-1}, T_n[\![\cap [\![0, L]\!]$. The strong martingale property of X on Λ gives $\mathbf{P}\{X(D_{\hat{T}}) \mid \mathcal{F}_{D_{\hat{S}}}\} = X(D_{\hat{S}})$ on $\Lambda_{D_{\hat{S}}} \cap \Gamma_{D(\hat{S}),D(\hat{T})}$. Lemma (5.9) then shows that on the same set, $\mathbf{P}\{\hat{X}_{\hat{T}} \mid \hat{\mathcal{F}}_{\hat{S}}\} = \hat{X}_{\hat{S}}$. But $\hat{\Lambda}_{\hat{S}} = \{\hat{S} < L\} = \{D_{\hat{S}} \in \Lambda\}$, and so by (5.9), $\hat{\Lambda}_{\hat{S}} \cap \hat{\Gamma}_{\hat{S},\hat{T}} = \hat{\Lambda}_{\hat{S}} \cap \{\mathbf{P}\{\Lambda_{D(\hat{T})} \mid \hat{\mathcal{F}}_{\hat{S}}\} = 1\} = \Lambda_{D_{\hat{S}}} \cap \{\mathbf{P}\{\Lambda_{D(\hat{T})} \mid \mathcal{F}_{D(\hat{S})}\} = 1\} = \Lambda_{D_{\hat{S}}} \cap \Gamma_{D(\hat{S}),D(\hat{T})}$. This proves that $\hat{X}$ has the strong martingale property on $[\![0, L[\![$. By (5.3), the restriction of $\hat{X}$ to $[\![0, L[\![$ has an extension to a uniformly integrable martingale whose final value we denote by $\bar{X}_\infty$. Let $\bar{X}$ be a right continuous version of the martingale $\mathbf{P}\{\bar{X}_\infty \mid \mathcal{F}_t\}$, so that we have $\bar{X}_{D_t} = X_{D_t}$ for all $t < L$. It follows that $X_t = \bar{X}_t$ for all t in the range of the map $t \to D_t$, hence for all $t \in \Lambda$ except possibly for $t \in \Lambda^i$. However, in view of hypothesis (iii), $\{X \neq \bar{X}\} \cap \Lambda^i$ must be evanescent, hence $X = \bar{X}$ on Λ. $\square$

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