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Sharp asymptotic behavior for wetting models in (1+1)–dimension

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Abstract

We consider continuous and discrete (1+1)–dimensional wetting models which undergo a localization/delocalization phase transition. Using a simple approach based on Renewal Theory we determine the precise asymptotic behavior of the partition function, from which we obtain the scaling limits of the models and an explicit construction of the infinite volume measure in all regimes, including the critical one.

Key words: Wetting Transition, Critical Wetting, δ–Pinning Model, Renewal Theory, Fluctuation Theory for Random Walks

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1 Introduction

1.1 Definition of the model

The building blocks of our model are a σ–finite measure μ on $\mathbb{R}$ (the *single site a priori measure*) and a function $V : \mathbb{R} \mapsto \mathbb{R} \cup \{+\infty\}$ (the *potential*). We allow two possible choices of μ:

- **Continuous set–up:** $\mu = \mathrm{d}x$ is the Lebesgue measure on $\mathbb{R}$. In this case we require that $\exp(-V(\cdot))$ be bounded and continuous.
- **Discrete set–up:** μ is the counting measure on $\mathbb{Z}$.

In both settings we assume that $V(0) < \infty$ and that

$$\kappa := \int_{\mathbb{R}} e^{-V(y)} \, \mu(\mathrm{d}y) < \infty .$$

Additional assumptions on $V(\cdot)$ will be stated in the next subsection.

For $\varepsilon \geq 0$, $N \in \mathbb{N}$ our model is defined by the following probability measure on $(\mathbb{R}^+)^N := [0,\infty)^N$:

$$\mathbf{P}^a_{\varepsilon,N}(\mathrm{d}x) := \frac{1}{Z^a_{\varepsilon,N}} \exp(-H^a_N(x)) \prod_{i=1}^{N} \big(1_{(x_i>0)} \, \mu(\mathrm{d}x_i) + \varepsilon \delta_0(\mathrm{d}x_i) \big) , \tag{1.1}$$

where $Z^a_{\varepsilon,N}$ is the normalizing constant (*partition function*), a is a label that stands for f (*free*) or c (*constrained*) and the corresponding Hamiltonians are defined by

$$H^{\mathrm{f}}_N(x) := \sum_{i=0}^{N-1} V(x_{i+1} - x_i) \qquad x_0 := 0 ,$$

$$H^{\mathrm{c}}_N(x) := \sum_{i=0}^{N} V(x_{i+1} - x_i) \qquad x_0 := x_{N+1} := 0 .$$

We interpret $\mathbf{P}^a_{\varepsilon,N}$ as an effective model for a (1+1)–dimensional interface above an impenetrable wall that, when $\varepsilon > 0$, attracts it (see Figure 1 ad the relative caption).

Remark 1.1. We stress that the assumption $V(0) < \infty$ has been made only for simplicity and can be removed at the price of some heavier notation. Also notice that in the definition (1.1) of our model we have chosen to impose the positivity constraint ($x_i > 0$) instead of the more customary nonnegativity one ($x_i \geq 0$). Of course this makes a real difference only in the discrete set–up, and also in this case the difference is unessential, since choosing the nonnegativity constraint amounts to switching $\varepsilon \to 1 + \varepsilon$ in the r.h.s. of (1.1). However dealing with the positivity constraint allows to analyze the continuous and discrete set–ups in a unified way, and this is the reason of our choice.

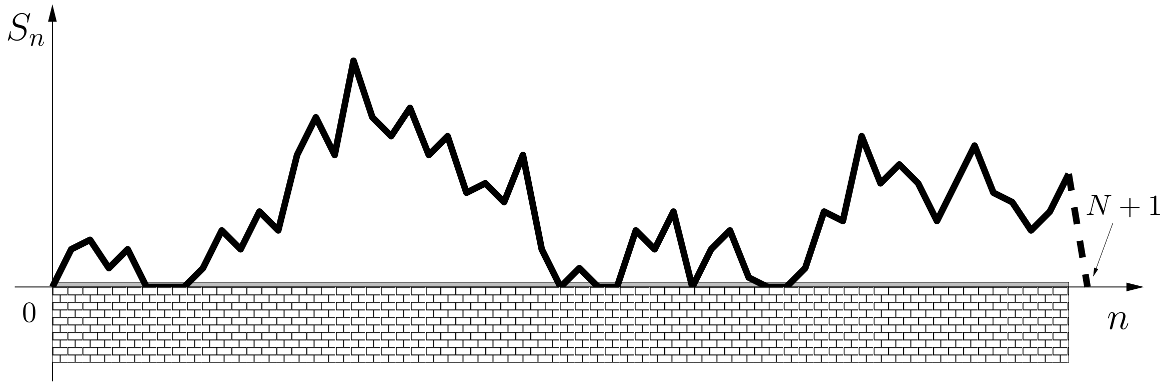

Figure 1: A trajectory of the walk (or interface) above the wall. The last step may be constrained or free, according to the model. The walk is rewarded when it enters the gray *thin* layer close to the wall. The competition between the *energetic gain* coming from this reward and the *entropic repulsion* due to the presence of the wall leads to a nontrivial behavior. Such a model has been proposed at several instances for the study of interfaces and polymers, we refer to [20] for details and references.

1.2 A random walk viewpoint

We introduce a sequence $(Y_i)_{i\in\mathbb{N}}$ of IID random variables with law $\mathbf{P}$ such that

$$\mathbf{P}(Y_1 \in \mathrm{d}x) \ := \ \frac{1}{\kappa} \exp(-V(x))\, \mu(\mathrm{d}x)\,,$$

and we denote by $(S_n)_{n\geq 0}$ the associated random walk: $S_0 := 0$, $S_n := Y_1 + \ldots + Y_n$.
The basic assumption we make on the potential $V(\cdot)$ is the following one:

(H) The *truncated variance* $t \mapsto \mathcal{V}(t) := \mathbf{E}\big[|Y_1|^2 \mathbf{1}_{\{|Y_1|\leq t\}}\big]$ is slowly varying at infinity and $\mathbf{E}\big[Y_1\big] = 0$.

We recall that a function $L(\cdot)$ is said to be slowly varying at infinity if for every $c > 0$ one has $L(ct)/L(t) \to 1$ as $t \to \infty$. This entails that $L(x)/x^\alpha \to 0$ as $x \to \infty$, for every $\alpha > 0$, cf. [4, Prop. 1.3.6]. Notice that if the truncated variance is slowly varying, then we have $\mathbf{P}\big[|Y_1| \geq t\big] \leq \mathcal{V}(t)/t^2$ for large t, cf. [4, Th. 8.3.1], hence all the moments of Y_1 of order less than 2 are automatically finite.

Of course assumption (H) holds whenever Y_1 is centered and has a *finite variance*, that is when $\mathbf{E}\big[Y_1\big] = 0$ and $\mathbf{E}\big[|Y_1|^2\big] =: \sigma^2 < \infty$. Allowing the truncated variance to be slowly varying turns out to be a very natural generalization: indeed assumption (H) is a *necessary and sufficient* condition for $(S_n)_{n\geq 0}$ to be in the domain of attraction (without centering) of the Gaussian law, see Appendix A.1 for more on this issue (cf. also [4, Th. 8.3.1]).

Now let us look more closely to our model. For $\varepsilon = 0$ we have the following random walk interpretation: $\mathbf{P}^{\mathrm{f}}_{0,N}$ is just the law of $(S_1, \ldots, S_N)$ under the positivity constraint $\{S_1 > 0, S_2 > 0, \ldots, S_N > 0\}$, while $\mathbf{P}^{\mathrm{c}}_{0,N}$ is the law of the same random vector under the further constraint $\{S_{N+1} = 0\}$. Then by the weak convergence toward Brownian meander and Brownian excursion we have that the $\mathbf{P}^a_{0,N}$–typical height of the interface in the bulk is of order $\sqrt{N}$, hence *very far* from the interface (*delocalized regime*). On the other hand, when $\varepsilon > 0$ the interface receives an ε–reward each time it touches the wall and intuitively one expects that if ε is large enough this attractive effect should be able to beat the entropic repulsion, leading to a *localized regime*. As we are going to see, this scenario is correct.

1.3 The phase diagram and the scaling limits

The model we are considering has been studied in [16] in the discrete set–up, for the special choice $\exp(-V(x)) = \frac{v}{2}\,\mathbf{1}_{(|x|=1)} + r\,\mathbf{1}_{(x=0)}$, with $v > 0$, $r \geq 0$ and $v + r = 1$, and more recently in [9] in the continuous set–up with finite variance (that is when $\mathbf{E}\big[|Y_1|^2\big] < \infty$). In both cases it has been proven that:

(i) There is a phase transition at $\varepsilon = \varepsilon_c > 0$, between a *delocalized regime* ($\varepsilon \leq \varepsilon_c$) in which the interface is repelled by the wall and a *localized regime* ($\varepsilon > \varepsilon_c$) in which the interface sticks close to the wall. A convenient definition of (de)localization may be given for instance in terms of the free energy, that is by looking at the Laplace asymptotic behavior of the partition function, cf. [9, § 2.2].

(ii) More quantitatively, Brownian scaling limits hold, inducing a further distinction in the delocalized regime. More precisely, the linearly interpolated diffusive rescaling of $\mathbf{P}^a_{\varepsilon,N}$ converges in distribution as $N \to \infty$:

- when $\varepsilon < \varepsilon_c$ (*strictly delocalized regime*), to the Brownian meander if $a = \mathrm{f}$ or to the normalized Brownian excursion if $a = \mathrm{c}$;
- when $\varepsilon = \varepsilon_c$ (*critical regime*), to the reflecting Brownian motion if $a = \mathrm{f}$ or to the reflecting Brownian bridge if $a = \mathrm{c}$;
- when $\varepsilon > \varepsilon_c$ (*localized regime*), to the law concentrated on the function taking the constant value 0 for both $a = \mathrm{f}$ and $a = \mathrm{c}$.

The proof of these results in [16] has been obtained by exploiting some very peculiar properties enjoyed by walks with increments in $\{-1, 0, +1\}$. On the other hand, the more general approach adopted in [9] is based on bounds on the asymptotic behavior of the partition function $Z^a_{\varepsilon,N}$ as $N \to \infty$.

1.4 Outline of the results

The purpose of this note is to present a simple approach based on Renewal Theory, which is applicable in complete generality in both the continuous and discrete cases, that allows to determine the *precise asymptotic behavior* of the partition function $Z^a_{\varepsilon,N}$ in all regimes. This yields a considerable simplification of several steps in [9] and allows the extension of the above results (i) and (ii) to the general continuous and discrete set–ups in a straightforward way.

Another important byproduct of our approach, and possibly the main result presented here, is the infinite volume limit of our model, that is the weak convergence of $\mathbf{P}^a_{\varepsilon,N}$ *without rescaling*, as a probability measure on $(\mathbb{R}^+)^{\mathbb{N}}$. This issue has been already considered in [9] but only for the localized regime. Here we show that the weak limit as $N \to \infty$ of $\mathbf{P}^a_{\varepsilon,N}$ exists in complete generality, namely in both the continuous and discrete cases, for both $a = \mathrm{f}$ and $a = \mathrm{c}$ and in all the regimes (that is for all values of ε), cf. Theorem 4.1. This will come with an explicit description of the limit measure, whose properties differ considerably in the strictly delocalized, critical and localized regimes, in complete analogy to the above mentioned scaling limits.

The exposition is organized as follows:

- In Section 2 we describe a Renewal Theory approach to our model.
- This will lead to the determination of the *precise* asymptotic behavior of the partition function (Section 3), to be compared to [9, Lemma 3].
- These results, in turn, are the key to proving the existence of the infinite volume limit in Section 4 in all regimes.
- Finally, in Section 5 we give the main ingredients to extend the proof of the scaling limits given in [9] to our general setting.

2 A Renewal Theory viewpoint

In this section we make explicit the link with Renewal Theory, showing that a suitable modification of the constrained partition function $Z^{\mathrm{c}}_{\varepsilon,N}$ can be interpreted as the (generalized) Green function associated with a probability measure $q(\cdot)$ that we define below. We also give a simple relation linking $Z^{\mathrm{f}}_{\varepsilon,N}$ and $Z^{\mathrm{c}}_{\varepsilon,N}$. Throughout the paper, we write for positive sequences $(a_n)_n$ and $(b_n)_n$:

$$a_n \sim b_n \iff \lim_{n\to\infty} a_n/b_n = 1.$$

2.1 The constrained case

Let us consider first the $\varepsilon = 0$ case. We claim that

$$\frac{Z^{\mathrm{c}}_{0,N}}{\kappa^{N+1}} \sim \frac{C}{\sqrt{2\pi}} \frac{L(N)}{N^{3/2}} \qquad (N \to \infty), \tag{2.1}$$

where C is a positive constant and $L(\cdot)$ is a slowly varying function. The proof of this relation is deferred to Appendix A, where C and $L(\cdot)$ are given explicitly. We stress that in the case of finite variance we have $L(\cdot) \equiv \sigma^{-1}$.

We set for convenience $Z^{\mathrm{c}}_{\varepsilon,0} := \exp(-V(0))$. We recall that $L(x)/x^\alpha \to 0$ as $x \to \infty$ for every $\alpha > 0$, because $L(\cdot)$ is slowly varying. Then from equation (2.1) it follows that $\gamma := \sum_{n\geq 1} Z^{\mathrm{c}}_{0,n-1}/\kappa^n < \infty$, hence we can define a probability distribution $q(\cdot)$ on $\mathbb{N}$ by setting

$$q(n) := \frac{1}{\gamma} \frac{Z^{\mathrm{c}}_{0,n-1}}{\kappa^n} \qquad (n \geq 1),$$

and we have for some positive constant c_q:

$$q(n) \sim c_q \frac{L(n)}{n^{3/2}} \qquad (n \to \infty). \tag{2.2}$$

Next we pass to the $\varepsilon > 0$ case. It is convenient to switch from the parameter ε to $\delta := \gamma\varepsilon$ and to make a change of scale, by setting for $N \geq 1$

$$\widetilde{Z}^{\mathrm{c}}_{\delta,N} := \frac{(\delta/\gamma)\, Z^{\mathrm{c}}_{(\delta/\gamma),N-1}}{\kappa^N} \qquad \widetilde{Z}^{\mathrm{f}}_{\delta,N} := \frac{(\delta/\gamma)\, Z^{\mathrm{f}}_{(\delta/\gamma),N}}{\kappa^N}. \tag{2.3}$$

Then from the very definition of the partition function $Z^c_{\varepsilon,N}$ it follows that $\widetilde{Z}^{\mathrm{c}}_{\delta,N+1}$ satisfies the following recurrence relation (cf. [9, Lemma 2]):

$$\widetilde{Z}^{\mathrm{c}}_{\delta,0} := 1, \qquad \widetilde{Z}^{\mathrm{c}}_{\delta,N} = \delta \sum_{t=1}^{N} q(t)\, \widetilde{Z}^{\mathrm{c}}_{\delta,N-t}, \quad (N \geq 1). \tag{2.4}$$

This is nothing but the *(generalized) renewal equation* driven by $\delta\, q(\cdot)$, see [3], and it is easily checked that its unique solution is given by

$$\widetilde{Z}^{\mathrm{c}}_{\delta,N} = \sum_{k=0}^{\infty} \delta^k\, q^{k*}(N), \tag{2.5}$$

where q^{k*} denotes the k–fold convolution of q with itself (by convention $q^{0*}(n) := \mathbf{1}_{(n=0)}$). Notice that the infinite sum in the right hand side of (2.5) is in fact a sum from 0 to N.

It will be shown in the next section (but it is somewhat clear from (2.5)) that the asymptotic behavior of $\widetilde{Z}^{\mathrm{c}}_{\delta,N}$ is radically different according to whether $\delta \lesseqgtr 1$, and since $\delta = \gamma\varepsilon$ it is natural to set (cf. equation (4) in [9, § 1.2])

$$\varepsilon_c := \frac{1}{\gamma} = \frac{1}{\sum_{n\geq 1} Z^{\mathrm{c}}_{0,n-1}/\kappa^n}. \tag{2.6}$$

We will see that the three regimes $\varepsilon < \varepsilon_c$, $\varepsilon = \varepsilon_c$ and $\varepsilon > \varepsilon_c$ correspond indeed to the strictly delocalized, critical and localized regimes mentioned in the introduction. It is also interesting to observe that in the discrete set–up equation (2.6) gives the explicit formula $\varepsilon_c = 1/\mathbf{P}(\overline{H}_1 = 0)$, where $\overline{H}_1$ is the first weak descending ladder height of the random walk $(S_n)_{n\geq 0}$, see Appendix A.3.

Remark 2.1. We point out that this approach can be generalized via the so–called Markov Renewal Theory [3], allowing to study *periodically inhomogeneous* models, that is the case in which ε is substituted by $\varepsilon_i \in \mathbb{R}$ and $\varepsilon_i = \varepsilon_{i+T}$ for some $T \in \mathbb{N}$ and every i. This has been recently worked out in [8], in the context of models of *copolymers with adsorption*.

2.2 The free case.

From the definition of $Z^{\mathrm{f}}_{\varepsilon,N}$, conditioning on the last epoch when the interface touches the wall, we have the following simple relation for the modified free partition function (cf. equation (18) in [9, § 2.3]):

$$\widetilde{Z}^{\mathrm{f}}_{\delta,N} = \sum_{t=0}^{N} \widetilde{Z}^{\mathrm{c}}_{\delta,t}\, P(N-t), \tag{2.7}$$

where $P(n)$ is the probability that the unperturbed random walk $(S_i)_i$ stays positive up to epoch n:

$$P(n) := \mathbf{P}(S_i > 0,\, i = 1, \ldots, n)\,, \qquad P(0) := 1\,. \tag{2.8}$$

It is worth recalling that if we set

$$L'(n) := \sqrt{n}\, P(n)\,, \tag{2.9}$$

then assumption (H) yields that $L'(\cdot)$ is slowly varying at infinity, cf. for instance [11]. We note that, by the standard theory of stability [4], this is equivalent to saying that the first descending ladder epoch $\overline{T}_1$ (see Appendix A.3) is in the domain of attraction of the positive stable law of index 1/2, like in the simple random walk case.

3 Sharp asymptotic behavior of the partition function

In this section we specialize the theory of the renewal equation [3] to our *heavy–tailed setting*, see (2.2), in order to find the asymptotic behavior of $\widetilde{Z}^{\mathrm{c}}_{\delta,N}$ and $\widetilde{Z}^{\mathrm{f}}_{\delta,N}$. The result is radically different in the three regimes $\delta \lesseqgtr 1$, that we consider separately.

3.1 The strictly delocalized regime ($\delta < 1$).

The following lemma gives the asymptotic behavior of the partition function in the strictly delocalized regime. To get some intuition we observe that $\widetilde{Z}^{\mathrm{c}}_{\delta,N}$ when $\delta < 1$ is the Green function of a renewal process with *defective* interarrival distribution $\delta\, q(\cdot)$, as it follows from (2.5).

Lemma 3.1. *If $\delta < 1$ and relations* (2.2) *and* (2.9) *hold, then*

$$\widetilde{Z}^{\mathrm{c}}_{\delta,N} \sim \frac{\delta\, c_q}{(1-\delta)^2}\, \frac{L(N)}{N^{3/2}} \qquad (N \to \infty)\,, \tag{3.1}$$

$$\widetilde{Z}^{\mathrm{f}}_{\delta,N} \sim \frac{1}{1-\delta}\, \frac{L'(N)}{N^{1/2}} \qquad (N \to \infty)\,. \tag{3.2}$$

Proof. From (2.5) we have that

$$\frac{n^{3/2}}{L(n)}\, \widetilde{Z}^{\mathrm{c}}_{\delta,n} = \sum_{k=1}^{\infty} \delta^k\, \frac{n^{3/2}}{L(n)}\, q^{k*}(n)\,. \tag{3.3}$$

We claim that

$$\lim_{n\to\infty} \frac{n^{3/2}}{L(n)}\, q^{k*}(n) = k\, c_q, \qquad \forall\, k \geq 1. \tag{3.4}$$

We argue by induction, the case $k = 1$ being true by (2.2). Suppose that we have proven (3.4) for $k = 1, \ldots, m$, then we have:

$$\frac{n^{3/2}}{L(n)}\, q^{(m+1)*}(n) = \left[\sum_{i=1}^{\lfloor n/2 \rfloor} + \sum_{\lfloor n/2 \rfloor+1}^{n-1} \right] \frac{n^{3/2}}{L(n)}\, q^{m*}(i)\, q(n-i) =$$

$$= \sum_{i=1}^{\lfloor n/2 \rfloor} q^{m*}(i) \left(\frac{n^{3/2}}{L(n)}\, q(n-i) \right) + \sum_{i=1}^{\lceil n/2 \rceil - 1} \left(\frac{n^{3/2}}{L(n)}\, q^{m*}(n-i) \right) q(i),$$

and by dominated convergence the claim follows.

By the uniform convergence property of slowly varying sequences [4, Th. 1.2.1] we have that $L(xt)/L(t) \to 1$ as $t \to \infty$ *uniformly in* $x \in [\alpha, 1/\alpha]$, for every $\alpha > 0$. In particular, there exists a positive constant c_1 such that $L(xt) \le c_1 L(t)$ for all $x \in [1/2, 1]$ and for all $t \ge 1$. Then, setting $c := 5/2 + \log_2(c_1)$, we claim that there exists $C > 0$ such that

$$q^{k*}(n) \le C\, k^c \frac{L(n)}{n^{3/2}}, \qquad \forall k, n \in \mathbb{N}. \tag{3.5}$$

Again, we argue by induction: by (2.2) the case $k = 1$ holds for some positive constant C. If (3.5) holds for all $k < 2m$, $m \in \mathbb{N}$, then we get:

$$\begin{aligned} q^{2m*}(n) &\le 2 \sum_{i=1}^{\lfloor n/2 \rfloor} q^{m*}(i)\, q^{m*}(n-i) \le 2 \sum_{i=1}^{\lfloor n/2 \rfloor} q^{m*}(i)\, C\, m^c \frac{L(n-i)}{(n-i)^{3/2}} \\ &\le 2\, C\, m^c \frac{c_1 L(n)}{(n/2)^{3/2}} \sum_{i=1}^{\lfloor n/2 \rfloor} q^{m*}(i) \le C\, m^c\, (2^{5/2} c_1) \frac{L(n)}{n^{3/2}} = C\,(2m)^c \frac{L(n)}{n^{3/2}}. \end{aligned}$$

The case $k = 2m + 1$ follows similarly and consequently (3.5) holds. Therefore we can apply dominated convergence in (3.3) (we recall that $\delta \in (0, 1)$), getting

$$\frac{n^{3/2}}{L(n)} \widetilde{Z}^{\mathrm{c}}_{\delta,n} = \sum_{k=1}^{\infty} \frac{n^{3/2}}{L(n)} q^{k*}(n)\, \delta^k \xrightarrow{n\to\infty} \sum_{k=1}^{\infty} k\, c_q\, \delta^k = \frac{\delta}{(1-\delta)^2}\, c_q.$$

and (3.1) is proven. Finally equation (3.2) follows by (2.7) and (2.9) applying again dominated convergence:

$$\frac{n^{1/2}}{L'(n)} \widetilde{Z}^{\mathrm{f}}_{\delta,n} = \sum_{t=0}^{n} \widetilde{Z}^{\mathrm{c}}_{\delta,t} \frac{n^{1/2}}{L'(n)} P(n-t) \xrightarrow{n\to\infty} \sum_{t=0}^{\infty} \widetilde{Z}^{\mathrm{c}}_{\delta,t} = \sum_{k=0}^{\infty} \delta^k = \frac{1}{1-\delta},$$

where in the equality before the last we have used (2.5). □

3.2 The critical case ($\delta = 1$).

We treat now the case $\delta = 1$.

Lemma 3.2. *If* $\delta = 1$ *and relations* (2.2) *and* (2.9) *hold, then*

$$\widetilde{Z}^{\mathrm{c}}_{1,N} \sim \frac{1}{2\pi} \frac{1}{c_q\, L(N)\sqrt{N}} \qquad (N \to \infty), \tag{3.6}$$

$$\widetilde{Z}^{\mathrm{f}}_{1,N} \sim \frac{L'(N)}{2\, c_q\, L(N)} \qquad (N \to \infty). \tag{3.7}$$

Proof. When $\delta = 1$ it is clear from (2.5) that $\widetilde{Z}^{\mathrm{c}}_{1,N}$ is the Green function of the renewal process with step distribution $q(\cdot)$. More explicitly, if we set

$$\xi_k := T_1 + \cdots + T_k, \qquad (T_i)_i \text{ IID}, \qquad \mathbb{P}(T_i = n) = q(n), \quad n \in \mathbb{N}, \tag{3.8}$$

then the law of ξ_k is $q^{k*}(\cdot)$ and it is immediate to check that $\widetilde{Z}^{\mathrm{c}}_{1,N} = \mathbb{P}(\exists k: \ \xi_k = N)$. Then the asymptotic behavior (3.6) is a result of Doney's [10, Th. B].

To prove (3.7), we split the sum in (2.7) in three parts:

$$\frac{L(n)}{L'(n)} \widetilde{Z}^{\mathrm{f}}_{1,n} = \left(\sum_{t=0}^{\lfloor \alpha n \rfloor} + \sum_{t=\lfloor \alpha n \rfloor + 1}^{\lfloor (1-\alpha) n \rfloor - 1} + \sum_{t=\lfloor (1-\alpha) n \rfloor}^{n} \right) \left(\frac{L(n)}{L'(n)} \widetilde{Z}^{\mathrm{c}}_{1,t} \, P(n-t) \right). \tag{3.9}$$

Combining the asymptotic relations (2.9) and (3.6) with the uniform convergence property of slowly varying sequences mentioned before equation (3.5), it is easy to check that the second sum above converges as $n \to \infty$ to the integral

$$\frac{1}{2\pi \, c_q} \int_{\alpha}^{1-\alpha} \frac{\mathrm{d}y}{\sqrt{y(1-y)}} \,.$$

On the other hand the fact that $\sum_{i=1}^k L(i)/\sqrt{i} \sim 2L(k)\sqrt{k}$ as $k \to \infty$, cf. [4, Prop. 1.5.8], entails that the limits as $n \to \infty$ of the first and third sums in (3.9) are vanishing as $\alpha \to 0$, and equation (3.7) follows. □

3.3 The localized case ($\delta > 1$).

Let $\delta > 1$. By continuity there exists $\textsc{f}_\delta > 0$ such that

$$\delta \sum_{t=1}^{\infty} q(t) \, \exp(-\textsc{f}_\delta \, t) \; = \; 1 .$$

We set $q_\delta(t) := \delta \, q(t) \, \exp(-\textsc{f}_\delta t)$, $t \in \mathbb{N}$, so that $q_\delta(\cdot)$ is a probability measure on $\mathbb{N}$. Notice that $\mu_\delta := \sum_t t \, q_\delta(t) < \infty$.

Lemma 3.3. *If $\delta > 1$ then*

$$\widetilde{Z}^{\mathrm{c}}_{\delta,N} \sim \frac{1}{\mu_\delta} \exp(N \, \textsc{f}_\delta) \qquad (N \to \infty) \,, \tag{3.10}$$

$$\widetilde{Z}^{\mathrm{f}}_{\delta,N} \sim \left(\frac{1}{\mu_\delta} \sum_{t=0}^{\infty} e^{-\textsc{f}_\delta t} \, P(t) \right) \exp(N \, \textsc{f}_\delta) \qquad (N \to \infty) \,. \tag{3.11}$$

Proof. From (2.5) it is immediately seen that

$$e^{-\textsc{f}_\delta N} \, \widetilde{Z}^{\mathrm{c}}_{\delta,N} \; = \; \sum_{k=0}^{\infty} {q_\delta}^{k*}(N) \,.$$

Arguing as in the proof of Lemma 3.2 we have that the r.h.s. above is the Green function of the renewal process with step distribution $q_\delta(\cdot)$. However this distribution has *finite mean* μ_δ and therefore equation (3.10) is nothing but the standard Renewal Theorem [3].

Finally, to prove (3.11) we resort to (2.7):

$$\exp(-\textsc{f}_\delta \, N) \, \widetilde{Z}^{\mathrm{f}}_{\delta,N} \; = \; \sum_{t=0}^{N} \left(e^{-\textsc{f}_\delta (N-t)} \widetilde{Z}^{\mathrm{c}}_{\delta, N-t} \right) e^{-\textsc{f}_\delta t} P(t) \; \xrightarrow{N \to \infty} \; \frac{1}{\mu_\delta} \sum_{t=0}^{\infty} e^{-\textsc{f}_\delta t} \, P(t) \,,$$

having applied (3.10) and dominated convergence. □

4 Infinite volume measures

Now we apply the asymptotic results obtained in the preceding section to the issue of the infinite volume limit. Although $\mathbf{P}^{\mathrm{f}}_{\varepsilon,N}$ and $\mathbf{P}^{\mathrm{c}}_{\varepsilon,N}$ have been defined as measures on $(\mathbb{R}^+)^N$, it is convenient to extend them to $(\mathbb{R}^+)^{\mathbb{N}}$ in an arbitrary way (for example by multiplying them by $\prod_{i=N+1}^{\infty} \delta_0(\mathrm{d}x_i)$). Our main result is the following theorem.

Theorem 4.1. *For every $\varepsilon \geq 0$ both $\mathbf{P}^{\mathrm{f}}_{\varepsilon,N}$ and $\mathbf{P}^{\mathrm{c}}_{\varepsilon,N}$ as measures on $(\mathbb{R}^+)^{\mathbb{N}}$ converge weakly as $N \to \infty$ to the same limit $\mathbf{P}_\varepsilon$, law of an irreducible Markov chain which is:*

1. *positive recurrent if $\varepsilon > \varepsilon_c$ (localized regime)*
2. *transient if $\varepsilon < \varepsilon_c$ (strictly delocalized regime)*
3. *null recurrent if $\varepsilon = \varepsilon_c$ (critical regime)*

Let us introduce the times $(\tau_k)_{k\geq 0}$ at which the interfaces touches the wall:

$$\tau_0 := 0 \qquad\qquad \tau_j := \inf\{n > \tau_{j-1} : x_n = 0\} \qquad\qquad x \in (\mathbb{R}^+)^{\mathbb{N}},$$

and the excursions $(e_k(\cdot))_{k\geq 0}$ of the interface above the wall:

$$e_k(i) := \{x_{\tau_k+i} : i = 0, \ldots, \tau_{k+1} - \tau_k\} \qquad\qquad x \in (\mathbb{R}^+)^{\mathbb{N}}.$$

We also set $\iota_N := \sup\{k : \tau_k \leq N\}$. The law of $(\tau_k)_{k\leq \iota_N}$ under $\mathbf{P}^a_{\varepsilon,N}$ can be viewed as a probability measure $p^a_{\varepsilon,N}$ on the class $\mathcal{A}_N$ of subsets of $\{1,\ldots,N\}$: indeed for $A \in \mathcal{A}_N$, writing

$$A = \{t_1, \ldots, t_{|A|}\}, \qquad 0 =: t_0 < t_1 < \cdots < t_{|A|} \leq N, \tag{4.1}$$

we can set

$$p^a_{\varepsilon,N}(A) := \mathbf{P}^a_{\varepsilon,N}(\tau_i = t_i,\ i \leq \iota_N). \tag{4.2}$$

From the inclusion of $\mathcal{A}_N$ into $\{0,1\}^{\mathbb{N}}$, the family of all subsets of $\mathbb{N}$, $p^a_{\varepsilon,N}$ can be viewed as a measure on $\{0,1\}^{\mathbb{N}}$.

The fundamental observation, that follows from the very definition (1.1) of our model, is that under $\mathbf{P}^a_{\varepsilon,N}$ and *conditionally on* $\{\iota_N, (\tau_k)_{k\leq\iota_N}\}$ the excursions $(e_k)_{0\leq k\leq \iota_N-1}$ are *independent* and their (conditional) laws are the same as under the random walk measure $\mathbf{P}$. Therefore, if we can prove that $p^a_{\varepsilon,N}$ has a weak limit on $\{0,1\}^{\mathbb{N}}$, then it is easily checked that the measure $\mathbf{P}^a_{\varepsilon,N}$ on $(\mathbb{R}^+)^{\mathbb{N}}$ converges weakly too, and the limiting measure $\mathbf{P}_\varepsilon$ is constructed by pasting the excursion over the limit zero set. If the limit zero set has a finite number of zeros, the last excursion has infinite length and has to be drawn according the law of the *random walk* $(S, \mathbf{P})$ *conditioned to stay positive*, as it is defined in [2]. Therefore the analysis is completed by the following result.

Proposition 4.2. *For every $\varepsilon \geq 0$ both $p^{\mathrm{f}}_{\varepsilon,N}$ and $p^{\mathrm{c}}_{\varepsilon,N}$ as measures on $\{0,1\}^{\mathbb{N}}$ converge to the same limit p_ε, under which $(\tau_k)_{k\geq 0}$ is:*

1. *for $\varepsilon > \varepsilon_c$ a renewal process with interarrival probability $q_\delta(\cdot)$ (see §3.3).*

2. *for $\varepsilon < \varepsilon_c$ a terminating renewal process with defective interarrival probability $\delta\, q(\cdot)$*

3. *for $\varepsilon = \varepsilon_c$ a renewal process with interarrival probability $q(\cdot)$.*

Proof. For all $\epsilon \geq 0$ and $a = \mathrm{f}, \mathrm{c}$, we have by the Markov property:

$$\mathbf{P}^a_{\varepsilon,N}\left(\tau_1 = k_1, \tau_2 = k_2, \ldots, \tau_j = k_j\right) \;=\; \left[\prod_{i=1}^{j} \delta\, q(k_i - k_{i-1})\right] \frac{\widetilde{Z}^a_{\delta,\, N+\mathbf{1}_{(a=\mathrm{c})}-k_j}}{\widetilde{Z}^a_{\delta,\, N+\mathbf{1}_{(a=\mathrm{c})}}},$$

for all $0 =: k_0 < k_1 < \cdots < k_j \leq N$ (the factor $\mathbf{1}_{(a=\mathrm{c})}$ in the partition functions is due to definition (2.3)). Letting $N \to \infty$, we obtain the thesis by Lemmas 3.1-3.2-3.3. □

4.1 More on the strictly delocalized regime

If $\varepsilon < \varepsilon_c$, i.e. $\delta < 1$, then under $\mathbf{P}_\varepsilon$ the number of returns to 0 and the last return to 0 are a.s. finite random variables. Their distributions are given in the following proposition, whose proof is a straightforward consequence of Proposition 4.2.

Proposition 4.3. *Let $\mathcal{N} := \#\{i \in \mathbb{N} : x_i = 0\}$ and $\mathcal{L} := \sup\{i \in \mathbb{N} : x_i = 0\}$. Then for $\varepsilon < \varepsilon_c$:*

$$\mathbf{P}_\varepsilon(\mathcal{N} = k) \;=\; (1-\delta)\,\delta^k, \qquad k = 0, 1, \ldots \tag{4.3}$$

$$\mathbf{P}_\varepsilon(\mathcal{L} = k) \;=\; (1-\delta)\,\widetilde{Z}^{\mathrm{c}}_{\delta,k}, \qquad k = 0, 1, \ldots \tag{4.4}$$

5 Scaling limits

We finally turn to the scaling limits of our model. We denote by $\left(X^N_t\right)_{t\in[0,1]}$ the linear interpolation of $\left(S_{i/N} \cdot L(N)/\sqrt{N}\right)_{i=0,\ldots,N}$, the choice of the norming sequence being the natural one, see Appendix A.1. We are interested in the weak convergence in $C([0,1])$ of the law of $\left(X^N_t\right)_{t\in[0,1]}$ under $\mathbf{P}^a_{\varepsilon,N}$. This problem has been solved in [9] in the finite variance continuous set–up, but for $\varepsilon \neq \varepsilon_c$ the techniques can be adapted in a straightforward way to treat the general continuous and discrete settings considered here.

Consequently we focus on the critical case $\varepsilon = \varepsilon_c$. In fact in this regime the result proven in [9] for $a = \mathrm{c}$ is not optimal, the reason being that the authors were not aware of Doney's result [10, Th. B] which yields (3.6). In this section we show that the sharp asymptotic relations (3.6) and (3.7) allow to simplify significantly the arguments in [9], proving the following

Theorem 5.1. *If $\delta = 1$ then the process $\left(X^N_t\right)_{t\in[0,1]}$ under $\mathbf{P}^a_{\varepsilon,N}$ converges in distribution to the reflecting Brownian motion on $[0,1]$ for $a = \mathrm{f}$ and to the reflecting Brownian bridge on $[0,1]$ for $a = \mathrm{c}$.*

In the preceding section we have shown that, under the measure $\mathbf{P}^a_{\varepsilon,N}$, there is a remarkable decoupling between the *zero level set* (that is the set of points where the interface touches the wall) and the excursions of the interface above the wall. Namely, conditionally on the zero level set, the excursions are an independent family and their (conditional) laws are the same as under the initial measure $\mathbf{P}$. Since our basic assumption (H) entails that the measure $\mathbf{P}$ is attracted to the Gaussian law, it is not a surprise that the law of the rescaled excursion under $\mathbf{P}$ converges weakly to the law of the *Brownian excursion*. The proof of this fact in the continuous set–up can be found for instance in [9] (for the case of finite variance, but it can be easily adapted to the general case). On the other hand, it appears that in the literature there is no general proof of this fact for the discrete set–up (for the case of walks with increments in $\{\pm 1, 0\}$ one can use the result in [17]). However a proof can be given exploiting the local limit theorem recently obtained in [5] and [6]: the details (in a more general setting) are carried out in [7].

Therefore we focus our attention to the law of the rescaled zero level set. More precisely we introduce $\mathcal{A}_N^a$, a random subset of $[0,1]$, by setting $\mathbb{P}(\mathcal{A}_N^a = A/N) = p^a_{\varepsilon_c,N}(A)$, $a = \mathrm{f}, \mathrm{c}$, for $A \subseteq \{0,\ldots,N\}$ (recall the notation introduced in (4.2)). If we can prove the weak convergence of the zero set $\mathcal{A}_N^a$, then, in view of the convergence of the excursions mentioned above, the weak convergence of the full measure $\mathbf{P}^a_{\varepsilon,N}$ follows arguing like in Section 8 of [9]. Therefore Theorem 5.1 is a consequence of the following proposition, first proven in [9, Prop. 10] (in the finite variance continuous case and with an additional assumption for $a = \mathrm{c}$).

Proposition 5.2. *As $N \to \infty$ we have that:*

(i) $\mathcal{A}_N^{\mathrm{f}}$ *converges in law to* $\{t \in [0,1] : B(t) = 0\}$,

(ii) $\mathcal{A}_N^{\mathrm{c}}$ *converges in law to* $\{t \in [0,1] : \beta(t) = 0\}$,

where B is a standard Brownian motion and β is a Brownian bridge over $[0,1]$.

The basic notions about the convergence in law of random sets are recalled in Appendix B (for more details see [14, § 3] and [18]).

It is convenient to introduce a simpler random set $\mathcal{A}_N$, to which the random sets $\mathcal{A}_N^{\mathrm{f}}$ and $\mathcal{A}_N^{\mathrm{c}}$ are strictly linked. Namely we consider again the renewal process $\xi_k = T_1 + \cdots + T_k$, where $(T_i)_i$ is IID and $\mathbb{P}(T = n) = q(n)$, $n \in \mathbb{N}$, and we set $\mathcal{A}_N := \{\xi_k/N : k \in \mathbb{N}\} \cap [0,1]$. Then the asymptotic relation (2.2) for $q(\cdot)$ implies the following basic result, first proven in [9, Lemma 5] for the case in which $L(\cdot)$ is a constant (but the proof extends to our general set–up in a straightforward way).

Lemma 5.3. *The sequence $(\mathcal{A}_N)_N$ converges in law to* $\{t \in [0,1] : B(t) = 0\}$.

We point out that the proof of this result given in [9] uses in an essential way the theory of *regenerative sets* and their connection with *subordinators* (we refer to [14] for more on this subject). In view of the importance of this result, in Appendix B we sketch an alternative and more direct proof, which is built on the basic relation (3.6).

Proof of Proposition 5.2. Let us first consider the free case $a = \mathrm{f}$. It is easy to see that the laws of $\mathcal{A}_N^{\mathrm{f}}$ and $\mathcal{A}_N$ are equivalent, more precisely for every bounded measurable functional Φ we have

$$\mathbb{E}\Big[\Phi(\mathcal{A}_N^{\mathrm{f}})\Big] = \mathbb{E}\Big[\Phi(\mathcal{A}_N)\, f_N^{\mathrm{f}}(\sup \mathcal{A}_N)\Big], \qquad f_N^{\mathrm{f}}(t) := \frac{P\left(N\,(1-t)\right)}{\widetilde{Z}^{\mathrm{f}}_{1,N} \cdot Q\left(N\,(1-t)\right)}, \quad t \in [0,1],$$

where $Q(n) := \sum_{t=n+1}^{\infty} q(t)$. The asymptotic behavior of $q(\cdot)$ being given by (2.2), it follows from [4, Prop. 1.5.10] that $Q(n) \sim 2c_q L(n)/\sqrt{n}$ as $n \to \infty$. Hence by (2.9) and (3.7) one sees that $\lim_{N\to\infty} f_N^{\mathrm{f}}(t) = 1$ uniformly in $t \in [0,\gamma]$, for every $\gamma \in (0,1)$, and then (i) is an easy consequence of Lemma 5.3.

We turn now to $a = \mathrm{c}$. Here it is more convenient to study the Radon-Nikodym derivative of the law of $\mathcal{A}_N^{\mathrm{c}} \cap [0,1/2]$ w.r.t. the law of $\mathcal{A}_N \cap [0,1/2]$. This time the Radon–Nikodym derivative is given by (cf. [9, Proof of (32), Step 1])

$$\mathbb{E}\left[\Phi(\mathcal{A}_N^{\mathrm{c}} \cap [0,1/2])\right] = \mathbb{E}\Big[\Phi(\mathcal{A}_N \cap [0,1/2])\; f_N^{\mathrm{c}}\Big(\sup\big(\mathcal{A}_N \cap [0,1/2]\big)\Big)\Big] ,$$

$$f_N^{\mathrm{c}}(t) := \frac{\sum_{n=0}^{N/2} \widetilde{Z}_{1,n}^{\mathrm{c}}\, q(N+1-Nt-n)}{\widetilde{Z}_{1,N+1}^{\mathrm{c}}\, Q(\lfloor N/2 \rfloor - Nt)} , \quad t \in [0,1/2] .$$

By (2.2), (2.9) and (3.6) we see that:

$$\lim_{N\to\infty} f_N^{\mathrm{c}}(t) = \frac{\int_0^{1/2} y^{-1/2}\,(1-t-y)^{-3/2}\,\mathrm{d}y}{2\,(1/2-t)^{-1/2}} = \frac{1}{\sqrt{2}}\,\frac{1}{1-t} ,$$

uniformly in $t \in [0,\gamma]$, for any $\gamma \in (0,1/2)$. Then (ii) follows from Lemma 5.3 by the same arguments used in [9, Proof of (32), Step 3]. □

A An asymptotic relation

We are going to prove that relation (2.1) holds true.

A.1 A Local Limit Theorem

We recall that, by our basic assumption (H), one has the weak convergence

$$\text{under } \mathbf{P}: \qquad \frac{L(n)}{\sqrt{n}}\, S_n \;\Rightarrow\; \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\,\mathrm{d}x \qquad (n\to\infty) , \tag{A.1}$$

where $L(\cdot)$ is a slowly varying function satisfying the relation $L(x) \sim 1/\sqrt{\mathcal{V}(\sqrt{x}/L(x))}$ as $x \to \infty$. More explicitly, this function can be defined as $L(x) := \sqrt{x}/g^{-1}(x)$, where $g(\cdot)$ is any *increasing* function such that $g(x) \sim x^2/\mathcal{V}(x)$ as $n \to \infty$ (the existence of such $g(\cdot)$ is guaranteed by [4, Th. 1.5.3], where an explicit definition is given).

We point out that equation (A.1) expresses the most general instance in which a random walk is attracted (without centering) to the Gaussian law, which in turn happens if and only if condition (H) holds, cf. [13, §IX.8 & §XVII.5]. Of course in the special case $\sigma^2 := \mathbf{E}\big[|Y_1|^2\big] < \infty$ we have $L(t) \equiv \sigma^{-1}$ by the Central Limit Theorem.

Let us denote by $f_n(x)$ the density (resp. the mass function) of S_n under $\mathbf{P}$, in the continuous set–up (resp. in the discrete set–up). Then the Local Limit Theorem for Densities, cf. [15, §46], (resp. Gnedenko's Local Limit Theorem, cf. [4, §8.4]) yields the asymptotic relation

$$f_n(0) \sim \frac{1}{\sqrt{2\pi}}\,\frac{L(n)}{\sqrt{n}} \qquad (n\to\infty) . \tag{A.2}$$

A.2 The continuous case

We follow the proof given in [9] in the case of finite variance. Introducing the set $\mathcal{C}_n := \{x_1 > 0, \ldots, x_n > 0\}$, the very definition of $Z^{c}_{0,n}$ gives

$$Z^{c}_{0,n} \;=\; \int_{\mathcal{C}_n} e^{-H^{c}_n(x_1,\ldots,x_n)}\,\mathrm{d}x_1\cdots\mathrm{d}x_n\,. \tag{A.3}$$

On the other hand for the density of S_{n+1} under $\mathbf{P}$ we have

$$f_{n+1}(0) \;=\; \int_{\mathbb{R}^n} \frac{e^{-H^{c}_n(x_1,\ldots,x_n)}}{\kappa^{n+1}}\,\mathrm{d}x_1\cdots\mathrm{d}x_n\,.$$

Next we introduce the linear transformation $T_n : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$T_n(x_1,\ldots,x_n) \;:=\; (x_2-x_1,\, x_3-x_1,\, \ldots,\, x_n-x_1,\, -x_1)\,.$$

Notice that T_n preserves Lebesgue measure, that $(T_n)^{n+1}$ is the identity map on $\mathbb{R}^n$ and that H^{c}_n is invariant along the orbits of T_n, namely $H^{c}_n(T_n(x)) = H^{c}_n(x)$. Moreover, the $n+1$ sets $\{(T_n)^k(\mathcal{C}_n)\,,\ k=0,\ldots,n\}$ are *disjoint* and their union differs from the whole $\mathbb{R}^n$ only by a set of zero Lebesgue measure. These considerations yield

$$f_{n+1}(0) \;=\; \sum_{k=0}^{n} \int_{(T_n)^k(\mathcal{C}_n)} \frac{e^{-H^{c}_n(x)}}{\kappa^{n+1}}\,\mathrm{d}x \;=\; (n+1)\int_{\mathcal{C}_n} \frac{e^{-H^{c}_n(x)}}{\kappa^{n+1}}\,\mathrm{d}x\,,$$

and comparing with (A.3) we get

$$\frac{Z^{c}_{0,n}}{\kappa^{n+1}} \;=\; \frac{1}{n+1}\,f_{n+1}(0)\,.$$

Therefore it suffices to apply (A.2) to show that relation (2.1) holds with $C=1$.

A.3 The discrete case

For $S=(S_n)_{n\geq 0}\in\mathbb{R}^{\mathbb{N}}$ we introduce the *weak descending* ladder epochs

$$\overline{T}_0 := 0 \qquad\qquad \overline{T}_{k+1} := \inf\{n > T_k :\ S_n \leq S_{T_k}\}$$

and the corresponding ladder heights $\overline{H}_k := -S_{\overline{T}_k}$ (if $\overline{T}_k = +\infty$ we set $\overline{H}_k := +\infty$). The reason why these quantities are of interest to us is that the definition of $Z^{c}_{0,n}$ yields

$$\frac{Z^{c}_{0,n}}{\kappa^{n+1}} \;=\; \mathbf{P}\big(\overline{T}_1 = n+1,\, \overline{H}_1 = 0\big)\,. \tag{A.4}$$

Notice in particular that by (2.6) we have $\varepsilon_c^{-1} = \sum_{n\geq 0} Z^{c}_{0,n}/\kappa^{n+1} = \mathbf{P}(\overline{H}_1 = 0)$.

Now we are going to use a fundamental combinatorial identity discovered by Alili and Doney, that for $k,\, n\in\mathbb{N}$ and $x\geq 0$ reads as

$$\mathbf{P}\big(\overline{T}_k = n,\, \overline{H}_k = x\big) \;=\; \frac{k}{n}\,\mathbf{P}\big(\overline{H}_{k-1} \leq x < \overline{H}_k,\, S_n = x\big)\,,$$

cf. [1, Eq. (3)] (the interchange between $<$ and $\leq$ with respect to that formula is due to the fact that they consider *strong* instead of weak ladder variables). Plugging this identity into (A.4) and arguing as in the step (i) of the proof of Proposition 6 in [1], we get

$$\frac{Z^{\mathrm{c}}_{0,n}}{\kappa^{n+1}} = \frac{1}{n+1}\mathbf{P}\big(\overline{H}_1 > 0,\, S_n = 0\big) \sim \frac{1}{n}\mathbf{P}\big(\overline{H}_1 > 0\big)\,\mathbf{P}\big(S_n = 0\big) \sim \frac{\mathbf{P}\big(\overline{H}_1 > 0\big)}{\sqrt{2\pi}}\,\frac{L(n)}{n^{3/2}}\,,$$

where we have applied (A.2), and equation (2.1) is proved with $C = \mathbf{P}(\overline{H}_1 > 0)$.

B The critical zero level set

We want to sketch here an alternative proof of Lemma 5.3. For the purpose of this section it is convenient to consider the random set $\mathcal{A}_N$ of Section 5 on the whole positive real line instead of the interval $[0,1]$. More precisely, introducing the renewal process $\xi_k = T_1 + \cdots + T_k$, where $(T_i)_{i\in\mathbb{N}}$ is IID and $\mathbb{P}(T_1 = n) = q(n)$, we set $\mathcal{A}_N := \{\xi_k/N : k \in \mathbb{N}\}$.

Let us first recall some basic facts on the convergence of closed sets. We denote by $\mathcal{F}$ the family of all closed sets of $\mathbb{R}^+$, and we endow it with the topology of Matheron, cf. [18] and [14, § 3], which in our setting can be conveniently described as follows. For $F \in \mathcal{F}$ and $t \in \mathbb{R}^+$ we set $d_t(F) := \inf\big(F \cap (t,\infty)\big)$. Notice that $t \mapsto d_t(F)$ is a right–continuous function and that the set F can be actually identified with the function $d_{(\cdot)}(F)$, because $F = \{t \in \mathbb{R}^+ : d_{t-}(F) = t\}$. Then in terms of $d_{(\cdot)}(F)$ the Matheron topology is the standard Skorohod topology on càdlàg functions taking values in $\overline{\mathbb{R}^+} := \mathbb{R}^+ \cup \{+\infty\}$. We point out that with this topology the space $\mathcal{F}$ is metrizable, separable and *compact*, hence in particular Polish. Moreover the Borel σ–field on $\mathcal{F}$ coincides with the σ–field generated by the maps $\{d_t(\cdot),\, t \in \mathbb{R}^+\}$.

Let us denote respectively by $\mathbb{P}_N$ and $\mathbb{P}^{(BM)}$ the laws of the random closed sets $\mathcal{A}_N$ and $\{t \in \mathbb{R}^+ : B(t) = 0\}$, where $B(\cdot)$ is a standard Brownian motion. These laws are probability measure on $\mathcal{F}$, and our goal is to prove that $\mathbb{P}_N$ converges weakly to $\mathbb{P}^{(BM)}$ as $N \to \infty$. Thanks to the compactness of $\mathcal{F}$, we can focus on the convergence of the *marginal distributions*. More precisely, it is sufficient to show that for every $n \in \mathbb{N}$ and for all $t_1 < \ldots < t_n \in \mathbb{R}^+$ one has the weak convergence of the image laws on $(\mathbb{R}^+)^n$:

$$\mathbb{P}_N \circ (d_{t_1}, \ldots, d_{t_n})^{-1} \implies \mathbb{P}^{(BM)} \circ (d_{t_1}, \ldots, d_{t_n})^{-1} \qquad (N \to \infty)\,, \tag{B.1}$$

and the weak convergence $\mathbb{P}_N \Rightarrow \mathbb{P}^{(BM)}$ follows, because the distributions of $(d_{t_1}, \ldots, d_{t_n})$ determine laws on $\mathcal{F}$.

The validity of (B.1) can be obtained by direct computation. For simplicity we will only consider the case $n = 1$, the general case follows along the same line. We recall that for any $t > 0$, the law of d_t under $\mathbb{P}^{(BM)}$ is given by

$$\mathbb{P}^{(BM)}\big(d_t \in \mathrm{d}y\big) = \frac{t^{1/2}}{\pi\, y(y-t)^{1/2}}\,\mathbf{1}_{(y>t)}\,\mathrm{d}y =: \rho_t(y)\,\mathrm{d}y\,,$$

cf. [19], hence we have to show that for every $x \in \mathbb{R}^+$

$$\lim_{N\to\infty} \mathbb{P}_N\big(d_t \geq x\big) = \int_x^\infty \rho_t(y)\,\mathrm{d}y\,.$$

Using the Markov property for the renewal process (ξ_k) we get

$$\begin{aligned}\mathbb{P}_N\big(d_t \geq x\big) &= \sum_{k\in\mathbb{N}} \mathbb{P}\big(\xi_k \leq Nt\,,\ \xi_{k+1} \geq Nx\big)\\ &= \sum_{i=1}^{\lfloor Nt\rfloor} \sum_{j=\lceil Nx\rceil}^{\infty} \left(\sum_{k\in\mathbb{N}} \mathbb{P}\big(\xi_k = i\big)\right) \mathbb{P}\big(\xi_1 = j-i\big)\\ &= \sum_{i=1}^{\lfloor Nt\rfloor} \left(\sum_{k\in\mathbb{N}} q^{*k}(i)\right) \sum_{j=\lceil Nx\rceil}^{\infty} q(j-i) = \sum_{i=1}^{\lfloor Nt\rfloor} \widetilde{Z}^{\mathrm{c}}_{1,i}\, Q\big(\lceil Nx\rceil - i - 1\big)\,,\end{aligned}$$

where we have applied (2.5). We recall the notation $Q(n) := \sum_{k=n+1}^{\infty} q(k)$, introduced in the proof of Proposition 5.2, and the fact that $Q(n) \sim 2c_q L(n)/\sqrt{n}$ as $n \to \infty$, as it follows from (2.2) applying [4, Prop. 1.5.10]. Since the asymptotic behavior of the constrained partition function in the critical case is given by (3.6), we obtain

$$\mathbb{P}_N\big(d_t \geq x\big) \;\sim\; \sum_{i=1}^{\lfloor Nt\rfloor} \frac{1}{2\pi}\,\frac{1}{c_q\, L(i)\,\sqrt{i}}\,\frac{2c_q L\big(\lceil Nx\rceil - i - 1\big)}{\sqrt{\lceil Nx\rceil - i - 1}} \qquad (N\to\infty)\,.$$

Now using the fact that $L(ct)/L(t) \to 1$ as $t \to \infty$ *uniformly* in $t \in [\alpha, 1/\alpha]$, for every $\alpha > 0$ (cf. [4, Th. 1.2.1]), and the convergence of the Riemann sums to the corresponding integral we easily get

$$\exists \lim_{N\to\infty} \mathbb{P}_N\big(d_t \geq x\big) \;=\; \frac{1}{\pi}\int_0^t \mathrm{d}s\, \frac{1}{\sqrt{s}}\,\frac{1}{\sqrt{x-s}} \;=\; \int_x^{\infty} \mathrm{d}z\, \rho_t(z)\,,$$

that is what was to be proven.

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