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Limit Theorems for Self-normalized Large Deviation [1]

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Abstract. Let $X, X_1, X_2, \cdots$ be i.i.d. random variables with zero mean and £nite variance σ^2. It is well known that a £nite exponential moment assumption is necessary to study limit theorems for large deviation for the standardized partial sums. In this paper, limit theorems for large deviation for self-normalized sums are derived only under £nite moment conditions. In particular, we show that, if $EX^4 < \infty$, then

$$\frac{P(S_n/V_n \geq x)}{1-\Phi(x)} = \exp\left\{-\frac{x^3 EX^3}{3\sqrt{n}\sigma^3}\right\}\left[1 + O\Big(\frac{1+x}{\sqrt{n}}\Big)\right],$$

for $x \geq 0$ and $x = O(n^{1/6})$, where $S_n = \sum_{i=1}^n X_i$ and $V_n = (\sum_{i=1}^n X_i^2)^{1/2}$.

Key Words and Phrases: Cramér large deviation, limit theorem, self-normalized sum.

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1 Introduction and main results

Let $X, X_1, X_2, \cdots$, be a sequence of non-degenerate independent and identically distributed (i.i.d.) random variables with zero mean. Set

$$S_n = \sum_{i=1}^n X_i, \quad V_n^2 = \sum_{i=1}^n X_i^2, \quad n \geq 1.$$

The self-normalized version of the classical central limit theorem states that, as $n \to \infty$,

$$\sup_x \left| P(S_n \geq xV_n) - \left\{1 - \Phi(x)\right\} \right| \to 0,$$

if and only if the distribution of X is in the domain of attraction of the normal law, where $\Phi(x)$ denotes the standard normal distribution function. This beautiful self-normalized central limit theorem was conjectured by Logan, Mallows, Rice and Shepp (1973), and latterly proved by Gine, Götze and Mason (1997). For a short summary of developments that have eventually led to Gine, Götze and Mason (1997), we refer to the Introduction of the latter paper.

The self-normalized central limit theorem is useful when x is not too large or when the error is well estimated. There are two approaches for estimating the error of the normal approximation. One approach is to investigate the absolute error in the self-normalized central limit theorem via Berry-Esseen bounds or Edgeworth expansions. This has been done by many researchers. For details, we refer to Slavova (1985), Hall (1988) and Bentkus and Götze (1996) for the Berry-Esseen bounds, Wang and Jing (1999) for an exponential nonuniform Berry-Esseen bound, Hall (1987) as well as Hall and Jing (1995) for Edgeworth expansions. See also van Zwet (1984), Friedrich (1989), Bentkus, Bloznelis and Götze (1996), Bentkus, Götze and van Zwet (1997), Putter and van Zwet (1998) and Wang, Jing and Zhao (2000). Another approach is to estimate the relative error $P(S_n \geq xV_n)/(1 - \Phi(x))$. In this direction, Jing, Shao and Wang (2003) re£ned Shao (1999), Wang and Jing (1999) as well as Chistyakov and Götze (2003), and obtained the following result: if $0 < \sigma^2 = EX^2 < \infty$, then there exists an absolute constant $A > 0$ such that

$$\exp\left\{-A\,(1+x)^2\tilde{\Delta}_{n,x}\right\} \leq \frac{P\big(S_n \geq xV_n\big)}{1 - \Phi(x)} \leq \exp\left\{A\,(1+x)^2\tilde{\Delta}_{n,x}\right\}, \tag{1.1}$$

for all $x \geq 0$ satisfying $\tilde{\Delta}_{n,x} \leq 1/A$, where

$$\tilde{\Delta}_{n,x} = \sigma^{-2}EX^2 I_{|X| \geq \sqrt{n}\sigma/(1+x)} + (1+x)\sigma^{-3}n^{-1/2}E|X|^3 I_{|X| \leq \sqrt{n}\sigma/(1+x)}.$$

Jing, Shao and Wang (2003) actually established (1.1) for independent random variables that are not necessarily identically distributed. It follows from (1.1) that if $E|X|^3 < \infty$, then

$$\frac{P(S_n \geq xV_n)}{1-\Phi(x)} \;=\; 1 + O(1)\,(1+x)^3\, n^{-1/2}\, \sigma^{-3}\, E|X|^3, \tag{1.2}$$

for $0 \leq x \leq An^{1/6}\sigma/(E|X|^3)^{1/3}$.

Result (1.2) is useful in statistics because it provides not only the relative error but also a Berry-Esseen type rate of convergence. Indeed, as a direct consequence of this result, it has been shown in Jing, Shao and Wang (2003) that bootstrapped studentized t-statistics possess large deviation properties in the region $0 \leq x \leq o(n^{1/6})$ under only a £nite third moment condition. However, (1.1) as well as (1.2) does not capture the term with $n^{-1/2}$ explicitly. This short has limited further applications of the self-normaized large deviation.

In this paper we investigate the limit theorems for self-normalized large deviation. Under £nite moment conditions, a leading term with $n^{-1/2}$ in (1.1) and (1.2) is obtained explicitly.

THEOREM 1.1. *Assume that $EX^4 < \infty$. Then, for $x \geq 0$ and $x = O(n^{1/6})$,*

$$\frac{P\big(S_n \geq xV_n\big)}{1-\Phi(x)} \;=\; \exp\Big\{-\frac{x^3EX^3}{3\sqrt{n}\sigma^3}\Big\}\,\Big[1+O\Big(\frac{1+x}{\sqrt{n}}\Big)\Big], \tag{1.3}$$

$$\frac{P\big(S_n \leq -xV_n\big)}{\Phi(-x)} \;=\; \exp\Big\{\frac{x^3EX^3}{3\sqrt{n}\sigma^3}\Big\}\,\Big[1+O\Big(\frac{1+x}{\sqrt{n}}\Big)\Big]. \tag{1.4}$$

If in addition $EX^3 = 0$, then, for $|x| = O\big(n^{1/6}\big)$,

$$P\big(S_n \leq xV_n\big) - \Phi(x) = O\big(n^{-1/2}e^{-x^2/2}\big). \tag{1.5}$$

Write $\mathcal{L}_{n,x} = (1+x)\rho_n + \Delta_{n,x}$, where $\rho_n = E|X|^3/(\sqrt{n}\sigma^3)$ and

$$\Delta_{n,x} = (1+x)^3\sigma^{-3}n^{-1/2}E|X|^3 I_{\{|X|>\sqrt{n}\sigma/(1+x)\}} + (1+x)^4\sigma^{-4}n^{-1}EX^4 I_{\{|X|\leq\sqrt{n}\sigma/(1+x)\}}.$$

Furthermore, we obtain the following bounds which re£ne (1.1) under £nite third moment condition in the region $0 \leq x \leq c\rho_n^{-1/2}$, where c is an absolute constant.

THEOREM 1.2. *Assume that $E|X|^3 < \infty$. Then, there exists a positive absolute constant c such that, for $0 \leq x \leq c\,\rho_n^{-1/2}$,*

$$\exp\Big(-\frac{x^3EX^3}{3\sqrt{n}\sigma^3} - A\mathcal{L}_{n,x}\Big) \leq \frac{P\big(S_n \geq xV_n\big)}{1-\Phi(x)} \leq \exp\Big(-\frac{x^3EX^3}{3\sqrt{n}\sigma^3} + A\mathcal{L}_{n,x}\Big). \tag{1.6}$$

where A is an absolute positive constant.

REMARK 1.1. Similar results to those in Theorem 1.1 hold for the standardized mean under much stronger conditions. For instance, it follows from Section 5.8 of Petrov (1995) that, for $x \ge 0$ and $x = O(n^{1/6})$,

$$\begin{aligned} \frac{P\Big(S_n \ge x\sqrt{n}\sigma\Big)}{1-\Phi(x)} &= \exp\left\{\frac{x^3 EX^3}{6\sqrt{n}\sigma^3}\right\}\left[1+O\Big(\frac{1+x}{\sqrt{n}}\Big)\right], \\ \frac{P\Big(S_n \le -x\sqrt{n}\sigma\Big)}{\Phi(-x)} &= \exp\left\{-\frac{x^3 EX^3}{6\sqrt{n}\sigma^3}\right\}\left[1+O\Big(\frac{1+x}{\sqrt{n}}\Big)\right]. \end{aligned}$$

hold only when Cramér's condition is satis£ed, i.e., $Ee^{tX} < \infty$ for t being in a neighborhood of zero. We also notice that there are different formulae for the self-normalized and standardized cases.

REMARK 1.2. It is readily seen that $\mathcal{L}_{n,x_n} = o(x_n^3/\sqrt{n})$ for $x_n \to \infty$ and $x_n = O(\rho_n^{-1/2})$. Hence $-\frac{x^3 EX^3}{3\sqrt{n}\sigma^3}$ in (1.6) provides a leading term in this case. However it remains an open problem for more re£ned results.

This paper is organized as follows. The proofs of main results will be given in Section 3. Next section we present two auxiliary theorems that will be used in the proofs of main results. The proofs of these auxiliary theorems will be postponed to Sections 4 and 5 respectively. Without loss of generality, throughout the paper, we assume $\sigma^2 = EX^2 = 1$ and denote by $A, A_1, A_2, \cdots$ and $c, C, C_1, C_2, \cdots$ absolute positive constants, which may be different at each occurrence. If a constant depends on a parameter, say u, then we write $A(u)$. In addition to the notation for $\mathcal{L}_{n,x}$ and $\Delta_{n,x}$ de£ned in Theorem 1.2, we always let

$$\rho_n = n^{-1/2} E|X|^3 \quad \text{and} \quad \Psi_n(x) = \Big[1-\Phi(x)\Big] \exp\Big(-\frac{x^3 EX^3}{3\sqrt{n}}\Big).$$

2 Two auxiliary theorems

Throughout the section we assume that $X, X_1, X_2, \cdots$, are i.i.d. random variables satisfying $EX = 0$, $EX^2 = 1$ and $E|X|^3 < \infty$. Two theorems in this section are established under quite general setting, which will be interesting in themselves. The proofs of these two theorems will be given in Sections 4 and 5 respectively.

THEOREM 2.1. *Let $h = x/B_n$, where B_n is a sequence of positive constants with*

$$|B_n^2 - n| \;\; \leq \;\; C_1\, nEX^2 I_{\{|X|\geq \sqrt{n}/(1+x)\}}. \tag{2.1}$$

Suppose that $\eta_j := \eta_n(x, X_j), 1 \leq j \leq n$, satisfy the conditions:

$$\begin{aligned}
\Big| Ee^{h\eta_j} - 1 - \frac{x^2}{2B_n^2} + \frac{x^3}{3B_n^3} EX^3 \Big| \;\; &\leq \;\; C_2\, n^{-1} \Delta_{n,x}, &(2.2)\\
\Big| E\eta_j e^{h\eta_j} - \frac{x}{B_n} \Big| \;\; &\leq \;\; C_3\, x^2\, n^{-1/2}\, \rho_n, &(2.3)\\
\Big| E\eta_j^2 e^{h\eta_j} - 1 \Big| \;\; &\leq \;\; C_4\, x\, \rho_n, &(2.4)\\
E|\eta_j|^3 e^{h\eta_j} \;\; &\leq \;\; C_5\, E|X|^3. &(2.5)
\end{aligned}$$

Then, for $2 \leq x \leq c\rho_n^{-1}$ with c sufficiently small and for any $|b| \leq x^2/4$,

$$P\Big(\sum_{j=1}^{n} \eta_j \geq (x + bx^{-1}) B_n \Big) \leq (1 + 2|b|x^{-2}) e^{-b} \Psi_n(x)\, \exp(A\, \mathcal{L}_{n,x}); \tag{2.6}$$

for $2 \leq x \leq c_1 \rho_n^{-1/2}$ with c_1 sufficiently small,

$$P\Big(\sum_{j=1}^{n} \eta_j \geq x\, B_n \Big) \geq \; \Psi_n(x)\, \exp(-A\, \mathcal{L}_{n,x}). \tag{2.7}$$

THEOREM 2.2. *Write, for $1 \leq m \leq n$,*

$$T_m = \frac{1}{\sqrt{n}} \sum_{j=1}^{m} \zeta_j \quad \textit{and} \quad \Lambda_{n,m} = \frac{1}{n^2} \sum_{k=1}^{m-1} \sum_{j=k+1}^{n} \psi_{k,j},$$

where $\zeta_j := \zeta_n(x, X_j)$ and $\psi_{k,j} := \psi_n(x, X_k, X_j)$ satisfy the conditions:

$$\begin{aligned}
\Big| E\zeta_j^2 - 1 + \frac{x}{\sqrt{n}} EX^3 \Big| \;\; &\leq \;\; C_6\, \Delta_{n,x}/(1+x)^2, &(2.8)\\
\Big| E\zeta_j^3 - EX^3 \Big| \;\; &\leq \;\; C_7\, \sqrt{n}\, \Delta_{n,x}/(1+x)^3, &(2.9)\\
E\zeta_j \;=\; 0, \quad |\zeta_j| \;\; &\leq \;\; C_8\, \sqrt{n}/(1+x), \quad E\zeta_j^4 \;\leq\; C_9\, EX^4 I_{\{|X|\leq \sqrt{n}/(1+x)\}}, &(2.10)\\
E(\psi_{k,j}|X_k) \;\; &= \;\; E(\psi_{k,j}|X_j) = 0, \quad for\ k \neq j, &(2.11)\\
E|\psi_{k,j}| \;\; &\leq \;\; C_{10}\, |x|, \quad E|\psi_{k,j}|^{3/2} \;\leq\; C_{11}\, |x|^{3/2} \big(E|X|^3\big)^2. &(2.12)
\end{aligned}$$

Then, for $2 \leq x \leq c\, \rho_n^{-1}$ with c sufficiently small, and for any constants sequence $\lambda_n(x)$ satisfying $|\lambda_n(x)| \leq C\, \Delta_{n,x}/(1+x)$,

$$P\Big(T_n + \Lambda_{n,n} \geq x + \lambda_n(x) \Big) \;\leq\; \Psi_n(x) \left(1 + A\mathcal{L}_{n,x}\right) +\; A_1 \big(x\rho_n\big)^{3/2}. \tag{2.13}$$

3 Proofs of main results

Proof of Theorem 1.1. Note that $\mathcal{L}_{n,x} \asymp (1+x)/\sqrt{n}$ for $x \geq 0$ and $x = O(n^{1/6})$. Theorem 1.1 follows immediately from Theorem 1.2. We omit the details.

Proof of Theorem 1.2. Without loss of generality, assume $x \geq 2$. If $0 \leq x < 2$, the results are direct consequences of the Berry-Esseen bound (cf. Bentkus and Götze (1996))

$$\Big|P(S_n \geq xV_n) - \big\{1 - \Phi(x)\big\}\Big| \leq A\,\rho_n.$$

We first provide four lemmas. For simplicity of presentation, define $\tau = \sqrt{n}/(1+x)$ and assume $2 \leq x \leq c\,\rho_n^{-1/2}$ with c sufficiently small throughout the section except where we point out.

LEMMA 3.1. *We have,*

$$P\Big(S_n - \frac{x}{2\sqrt{n}}(V_n^2 - n) \geq x\sqrt{n}\Big) \;\; \geq \;\; \Psi_n(x)\,\exp(-A\,\mathcal{L}_{n,x}); \tag{3.1}$$

and for $2 \leq x \leq c\,\rho_n^{-1}$ with c sufficiently small and for arbitrary $|\delta| \leq x^2/4$,

$$P\Big(S_n - \frac{x}{2\sqrt{n}}(V_n^2 - n) \geq \big(x + \delta x^{-1}\big)\sqrt{n}\Big) \;\leq\; (1 + 2|\delta|x^{-2})e^{-\delta}\,\Psi_n(x)\,\exp(A\,\mathcal{L}_{n,x}). \tag{3.2}$$

Proof. We first prove (3.2). Let $h = x/\sqrt{n}$ and $\eta_j = X_j - \frac{x}{2\sqrt{n}}(X_j^2 - 1)$. It follows from (4.12)-(4.14) in Wang and Jing (1999) that

$$\begin{aligned}
\Big|E\eta_1 e^{h\eta_1} - \frac{x}{\sqrt{n}}\Big| \;&\leq\; 16\,x^2\,n^{-1/2}\,\rho_n, \\
\Big|E\eta_1^2 e^{h\eta_1} - 1\Big| \;&\leq\; 30\,x\,\rho_n, \\
E|\eta_1|^3 e^{h\eta_1} \;&\leq\; 30\,E|X|^3.
\end{aligned}$$

Thus (3.2) follows immediately from (2.6) in Theorem 2.1 with $B_n^2 = n$ if we prove

$$\Big|Ee^{h\eta_1} - 1 - \frac{x^2}{2n} + \frac{x^3}{3n^{3/2}}EX^3\Big| \;\leq\; C\,n^{-1}\Delta_{n,x}\,. \tag{3.3}$$

Without loss of generality, assume $x \leq \rho_n^{-1}/16$. By noting $E|X|^3 \geq (EX^2)^{3/2} = 1$, we get $h \leq 1/4$. This implies that $h\eta_1 = h^2/2 + hX_1 - (hX_1)^2/2 \leq 1$ and $|h\eta_1|^k e^{h\eta_1} \leq \sup_{s\leq 1}|s|^k e^s \leq e$ for $k = 0, 1, \ldots, 4$. Therefore, using Taylor's expansion

$$\Big|e^x - \sum_{j=0}^{k}\frac{x^j}{j!}\Big| \leq \frac{|x|^{k+1}}{(k+1)!}e^{x\vee 0}, \quad \text{for } k \geq 1,$$

we obtain that

$$
\begin{aligned}
Ee^{h\eta_1} &= E\Big[1 + h\eta_1 + \tfrac{1}{2}\,(h\eta_1)^2 + \tfrac{1}{6}\,(h\eta_1)^3 + \tfrac{\theta}{24}\,(h\eta_1)^4\Big] I_{(|X_1|\le\tau)} \\
&\quad + E\Big(1 + h\eta_1 + \tfrac{\theta_1}{2}\Big) I_{(|X_1|>\tau)} \\
&= 1 + \tfrac{1}{2}\,E(h\eta_1)^2 I_{(|X_1|\le\tau)} + \tfrac{1}{6}\,E(h\eta_1)^3 I_{(|X_1|\le\tau)} \\
&\quad + \tfrac{\theta_1}{2}\,P(|X_1|>\tau) + \tfrac{\theta}{24}\,E(h\eta_1)^4 I_{(|X_1|\le\tau)}, \qquad (3.4)
\end{aligned}
$$

where $|\theta| \le e$ and $|\theta_1| \le e$. Since $EX^2 = 1$, it is readily seen that

$$
\begin{aligned}
\Big|E(h\eta_1)^2 I_{(|X_1|\le\tau)} - h^2 + E(hX)^3\Big| &\le 8\,n^{-1}\Delta_{n,x}, \\
\Big|E(h\eta_1)^3 I_{(|X_1|\le\tau)} - E(hX)^3\Big| &\le 32\,n^{-1}\Delta_{n,x}, \\
E(h\eta_1)^4 I_{(|X_1|\le\tau)} &\le 16\,n^{-1}\Delta_{n,x}.
\end{aligned}
$$

Taking these estimates back into (3.4), we obtain (3.3), and hence (3.2).

The proof of (3.1) is similar by using (2.7). We omit the details. The proof of Lemma 3.1 is now complete. □

The next lemma is from Lemma 6.4 in Jing, Shao and Wang (2003).

LEMMA 3.2. *Let $\{\xi_i, 1 \le i \le n\}$ be a sequence of independent random variables with $E\xi_i = 0$ and $E\xi_i^2 < \infty$. Then*

$$
P\Big[\sum_{i=1}^n \xi_i \ge a\Big\{4D_n + \big(\sum_{i=1}^n \xi_i^2\big)^{1/2}\Big\}\Big] \le 8e^{-a^2/2} \qquad (3.5)
$$

where $D_n = \big(\sum_{i=1}^n E\xi_i^2\big)^{1/2}$.

In Lemmas 3.3-3.4, we use the notation:

$$
\begin{aligned}
&\bar{X}_i = X_i I_{\{|X_i|\le\tau\}}, \qquad \bar{S}_n = \sum_{i=1}^n \bar{X}_i, \qquad \bar{V}_n^2 = \sum_{i=1}^n \bar{X}_i^2, \qquad B_n^2 = \sum_{i=1}^n E\bar{X}_i^2 \\
&S_n^{(i)} = S_n - X_i, \qquad V_n^{(i)} = (V_n^2 - X_i^2)^{1/2}.
\end{aligned}
$$

LEMMA 3.3. *We have*

$$
P\big(S_n \ge xV_n\big) \le A\,\Psi_n(x)\,\exp(A\,\mathcal{L}_{n,x}). \qquad (3.6)
$$

Proof. Let $\Omega_n = (1 - x^{-1}/2, 1 + x^{-1}/2)$. Recalling $2 \le x \le c\rho_n^{-1/2}$ with c suf£ciently small, it follows from (2.27)-(2.29) in Shao (1999) that

$$\begin{aligned}
&P\big(S_n \ge xV_n,\ V_n^2/n \notin \Omega_n\big) \\
&\quad \le\ P\big(S_n \ge xV_n, V_n^2 \ge 9n\big) + P\big\{S_n \ge xV_n, n(1 + x^{-1}/2) \le V_n^2 \le 9n\big\} \\
&\qquad + P\big\{S_n \ge xV_n, V_n^2 \le n(1 - x^{-1}/2)\big\} \\
&\quad \le\ 4\ \exp\big(-x^2/2 - x/8 + Ax^3\rho_n\big) \\
&\quad \le\ A\,\Psi_n(x).
\end{aligned}$$

This, together with (3.2), implies that

$$\begin{aligned}
&P\big(S_n \ge xV_n\big) = P\big(S_n \ge xV_n,\ V_n^2/n \in \Omega_n\big) + P\big(S_n \ge xV_n,\ V_n^2/n \notin \Omega_n\big) \\
&\quad \le P\left\{S_n - \frac{x}{2\sqrt{n}}(V_n^2 - n) \ge (x - x^{-1}/4)\sqrt{n}\right\} + P\big(S_n \ge xV_n,\ V_n^2/n \notin \Omega_n\big) \\
&\quad \le A\Psi_n(x)\exp(A\,\mathcal{L}_{n,x}),
\end{aligned}$$

as required. The proof of Lemma 3.3 is complete. □

LEMMA 3.4. *We have*

$$P\big(S_n \ge xV_n\big) \ \le\ \Psi_n(x)\ \exp(A\,\mathcal{L}_{n,x}) + A_1e^{-3x^2}; \tag{3.7}$$

$$P\big(S_n \ge xV_n\big) \ \le\ \Psi_n(x)\ \exp(A\,\mathcal{L}_{n,x}) + A_1\,(x\rho_n)^{3/2}. \tag{3.8}$$

Proof. As in Wang and Jing (1999), for $x \ge 2$,

$$\begin{aligned}
P(S_n \ge xV_n) \ &\le\ P(\bar{S}_n \ge x\bar{V}_n) + \sum_{i=1}^{n} P\big[S_n^{(i)} \ge (x^2 - 1)^{1/2}V_n^{(i)}\big]P(|X_i| > \tau). \\
&=\ K_n + I_n, \qquad \text{say.}
\end{aligned} \tag{3.9}$$

It follows easily from Lemma 3.3 that for all i

$$P\big[S_n^{(i)} \ge (x^2 - 1)^{1/2}V_n^{(i)}\big] \le A\Psi_n(x)\ \exp(A\,\mathcal{L}_{n,x}).$$

This, together with $P(|X_i| > \tau) \le n^{-1}\Delta_{n,x}$, implies that

$$I_n \ \le\ A\,\Delta_{n,x}\,\Psi_n(x)\ \exp(A\,\mathcal{L}_{n,x}) \tag{3.10}$$

In view of (3.9) and (3.10), the inequalities (3.7) and (3.8) will follow if we prove

$$\begin{aligned} K_n &\leq \Psi_n(x)\, \exp(A\,\mathcal{L}_{n,x}) + A_1 e^{-3x^2}, & (3.11)\\ K_n &\leq \Psi_n(x)\, \exp(A\,\mathcal{L}_{n,x}) + A_1\,(x\rho_n)^{3/2}, & (3.12)\end{aligned}$$

for $2 \leq x \leq c\rho_n^{-1}$ with c suf£ciently small.

We £rst prove (3.11). Let $D_n^2 = \sum_{i=1}^n E\bar{X}_i^4$ and $\xi_i = \bar{X}_i^2 - E\bar{X}_i^2$. By the inequality $(1+y)^{1/2} \geq 1 + y/2 - y^2$ for any $y \geq -1$ and Lemma 3.2,

$$\begin{aligned} K_n &= P\Big[\bar{S}_n \geq x\Big\{B_n^2 + \sum_{i=1}^n (\bar{X}_i^2 - E\bar{X}_i^2)\Big\}^{1/2}\Big] \\ &\leq P\Big[\bar{S}_n \geq xB_n\Big\{1 + \frac{1}{2B_n^2}\sum_{i=1}^n \xi_i - \frac{1}{B_n^4}\Big(\sum_{i=1}^n \xi_i\Big)^2\Big\}\Big] \\ &\leq P\Big[\Big|\sum_{i=1}^n \xi_i\Big| \geq \sqrt{6}x\Big\{4D_n + \Big(\sum_{i=1}^n \xi_i^2\Big)^{1/2}\Big\}\Big] \\ &\quad + P\Big[\bar{S}_n \geq xB_n\Big\{1 + \frac{1}{2B_n^2}\sum_{i=1}^n \xi_i - \frac{1}{B_n^4}\Big(\sum_{i=1}^n \xi_i\Big)^2\Big\}, \\ &\qquad\qquad \Big|\sum_{i=1}^n \xi_i\Big| \leq \sqrt{6}x\Big\{4D_n + \Big(\sum_{i=1}^n \xi_i^2\Big)^{1/2}\Big\}\Big] \\ &\leq 8e^{-3x^2} + K_{n,1}, \qquad (3.13)\end{aligned}$$

where, after some algebra (see, Jing, Shao and Wang (2003), for example),

$$K_{n,1} \leq P\Big(\sum_{j=1}^n \eta_j \geq xB_n\Big),$$

with $\eta_i = \bar{X}_i - 2^{-1}xB_n^{-1}(\bar{X}_i^2 - E\bar{X}_i^2) + 24x^3B_n^{-3}(\bar{X}_i^4 + 16E\bar{X}_i^4)$. As in the proof of Lemma 3.1, tedious but elementary calculations show that the inequalities (2.1)-(2.5) hold true for $B_n^2 = \sum_{i=1}^n E\bar{X}_i^2$ and the η_i de£ned above. Therefore it follows from (3.2) in Theorem 2.1 that

$$K_{n,1} \leq \Psi_n(x)\, \exp(A\,\mathcal{L}_{n,x}), \qquad (3.14)$$

for $2 \leq x \leq c\rho_n^{-1}$ with c suf£ciently small. Take this estimate back into (3.13), we get the desired (3.11).

We next prove (3.12). Note that

$$\begin{aligned}
(\bar V_n^2 - n)^2 &= \sum_{j=1}^n (\bar X_i^2 - 1)^2 + \sum_{k\neq j} (\bar X_k^2 - 1)(\bar X_j^2 - 1)\\
&= \sum_{j=1}^n (\bar X_i^2 - 1)^2 + \sum_{k\neq j} (\bar X_k^2 - E\bar X_k^2)(\bar X_j^2 - E\bar X_j^2)\\
&\qquad + (n-1)(E\bar X_j^2 - 1)\sum_{j=1}^n (2\bar X_j^2 - 1 - E\bar X_j^2).
\end{aligned}$$

By the inequality $(1+y)^{1/2} \geq 1 + y/2 - y^2$ for any $y \geq -1$ again, we have

$$\begin{aligned}
K_n &= P\Big[\bar S_n \geq x\sqrt{n}\Big\{1 + \frac{1}{n}\sum_{i=1}^n (\bar X_i^2 - 1)\Big\}^{1/2}\Big]\\
&\leq P\Big[\bar S_n \geq x\sqrt{n}\Big\{1 + \frac{1}{2n}(\bar V_n^2 - n) - \frac{1}{n^2}(\bar V_n^2 - n)^2\Big\}\Big]\\
&= P\Big(\frac{1}{\sqrt{n}}\sum_{j=1}^n \zeta_j + \frac{1}{n^2}\sum_{k=1}^{n-1}\sum_{j=k+1}^n \psi_{k,j} \geq x - \lambda_n(x)\Big), \qquad (3.15)
\end{aligned}$$

where

$$\begin{aligned}
\zeta_j^* &= \bar X_j - \frac{x}{2\sqrt{n}}(\bar X_j^2 - 1) + \frac{x}{n^{3/2}}(\bar X_j^2 - 1)^2\\
&\quad + \frac{(n-1)x}{n^{3/2}}(E\bar X_1^2 - 1)\sum_{j=1}^n (2\bar X_j^2 - 1 - E\bar X_j^2),\\
\zeta_j &= \zeta_j^* - E\zeta_j^*\\
&= \bar X_j - E\bar X_j - \frac{x}{2\sqrt{n}}\Big(1 + \frac{2}{n} + \frac{2(n-1)}{n}EX^2 I_{(|X|\geq\tau)}\Big)(\bar X_j^2 - E\bar X_j^2)\\
&\quad + \frac{x}{n^{3/2}}(\bar X_j^4 - E\bar X_j^4),\\
\psi_{k,j} &= 2x(\bar X_k^2 - E\bar X_k^2)(\bar X_j^2 - E\bar X_j^2),\\
\lambda_n(x) &= \sqrt{n}E\zeta_1^*\\
&= \sqrt{n}EXI_{(|X|\geq\tau)} - \frac{x}{2}EX^2 I_{(|X|\geq\tau)} + \frac{x}{n}E(\bar X_1^2 - 1)^2 - \frac{(n-1)x}{n}\Big(EX^2 I_{(|X|\geq\tau)}\Big)^2.
\end{aligned}$$

It is easy to see that ζ_j satisfy the conditions (2.8)- (2.10), $\psi_{k,j}$ satisfy the conditions (2.11)-(2.12) in Theorem 2.2 and $|\lambda_n(x)| \leq 3xn^{-1} + 3(1+x)^{-1}\Delta_{n,x} \leq 6(1+x)^{-1}\Delta_{n,x}$. Hence, in view of (3.15), (3.12) follows immediately Theorem 2.2. This also completes the proof of Lemma 3.4. $\square$

After these preliminaries, we are now ready to prove Theorem 1.2.

As is well-known (see Wang and Jing (1999) for example),

$$P(S_n \geq xV_n) \ \geq \ P\Big(S_n - \frac{x}{2\sqrt{n}}(V_n^2 - n) \geq x\sqrt{n}\Big).$$

The left hand inequality of (1.6) follows from Lemma 3.1 immediately.

To prove the right hand inequality of (1.6), we use Lemmas 3.4. If $\Psi_n(x) \geq (x\rho_n)^{1/2}$, then by (3.12),

$$P(S_n \geq xV_n) \leq \Psi_n(x) \, \exp(A\,\mathcal{L}_{n,x})\Big(1 + A_1 x\rho_n\Big) \leq \Psi_n(x) \, \exp\Big(A_2\mathcal{L}_{n,x}\Big). \qquad (3.16)$$

Recall we may assume that $2 \leq x \leq \rho_n^{-1}/16$. It is readily seen that

$$\big\{1 - \Phi(x)\big\}^{-3} \exp\left\{\frac{x^3 EX^3}{\sqrt{n}}\right\} \leq (2\pi)^{3/2} x^3 e^{2x^2}.$$

This implies that, when $\Psi_n(x) \leq (x\rho_n)^{1/2}$,

$$e^{-3x^2} \leq A\big\{1 - \Phi(x)\big\}^3 \exp\left\{-\frac{x^3 EX^3}{\sqrt{n}}\right\} \leq A\, x\rho_n\, \Psi_n(x).$$

Therefore, by (3.7),

$$P(S_n \geq xV_n) \leq \Psi_n(x) \, \exp(A\,\mathcal{L}_{n,x})\Big(1 + A_1 x\rho_n\Big) \leq \Psi_n(x) \, \exp(A\,\mathcal{L}_{n,x}). \qquad (3.17)$$

Collecting the estimates (3.16) and (3.17), we get the right hand inequality of (1.6). The proof of Theorem 1.2 is now complete.

4 Proof of Theorem 2.1

The proof of Theorem 2.1 is based on the conjugate method. To employ the method, let $\xi_1, \cdots, \xi_n$ be independent random variables with ξ_j having distribution function $V_j(u)$ de£ned by

$$V_j(u) = E\Big\{e^{h\eta_j} I(\eta_j \leq u)\Big\}\Big/ Ee^{h\eta_j}, \qquad \text{for} \quad j = 1, \cdots, n,$$

Also de£ne $M_n^2(h) = \sum_{j=1}^n Var(\xi_j)$,

$$G_n(t) = P\Big\{\frac{\sum_{j=1}^n (\xi_j - E\xi_j)}{M_n(h)} \leq t\Big\} \quad \text{and} \quad R_n(h) = \frac{(x + bx^{-1})B_n - \sum_{j=1}^n E\xi_j}{M_n(h)}.$$

Since $EX^2 = 1$, we have $E|X|^3 \geq 1$ and there exists a positive constant c_0 such that $EX^2 I_{(|X|\geq\tau)} \leq 1/(4C_1)$ for $2 \leq x \leq c_0\,\rho_n^{-1}$. Let $c_1 = 2^8 \max\{1, C_1, \ldots, C_5\}$ and $c_2 = \min\{c_0, c_1^{-1}\}$. Taking account of the conditions (2.1)-(2.5), it can be shown by a elementary method that, for $2 \leq x \leq c_2\,\rho_n^{-1}$,

$$
\begin{aligned}
Ee^{h\eta_j} &= 1 + \frac{x^2}{2n} - \frac{EX^3}{3}\Big(\frac{x}{\sqrt{n}}\Big)^3 + O_1 n^{-1}\Delta_{n,x} \\
&= \exp\left\{\frac{x^2}{2n} - \frac{EX^3}{3}\Big(\frac{x}{\sqrt{n}}\Big)^3 + O_2 n^{-1}\Delta_{n,x}\right\}, \qquad (4.1)\\
E\xi_j &= E\eta_j e^{h\eta_j}\Big/Ee^{h\eta_j} \;=\; \frac{x}{\sqrt{n}} + O_3\,x^2\,n^{-1/2}\,\rho_n, \qquad (4.2)\\
Var(\xi_j) &= E\eta_j^2 e^{h\eta_j}\Big/Ee^{h\eta_j} - (E\xi_j)^2 \;=\; 1 + O_4\,x\,\rho_n, \qquad (4.3)\\
E|\xi_j|^3 &\leq E|\eta_j|^3 e^{h\eta_j}\Big/Ee^{h\eta_j} \;\leq\; 2\,C_5\,n^{1/2}\,\rho_n, \qquad (4.4)
\end{aligned}
$$

where $|O_j| \leq 1/(2c_2)$, for $j = 1, \ldots, 4$.

Using (4.3)-(4.4) and (2.1), we get, for $2 \leq x \leq c_2\,\rho_n^{-1}$,

$$
\begin{aligned}
n/2 &\leq M_n^2(h) \;=\; n + O_4\,n\,x\,\rho_n \;\leq\; 3n/2, \qquad (4.5)\\
|R_n(h)| &= \Big|\frac{(x + bx^{-1})B_n - \sum_{j=1}^n E\xi_j}{M_n(h)}\Big| \\
&\leq (3/2)|b|x^{-1} + 2(C_1 + |O_3|)\,x^2\,\rho_n, \qquad (4.6)\\
|hM_n(h) - x| &\leq \frac{1}{x}\left|h^2 M_n^2(h) - x^2\right| \\
&\leq x\left\{\Big|\frac{M_n^2(h)}{n} - 1\Big| + \Big|\frac{1}{B_n^2} - \frac{1}{n}\Big|\,M_n^2(h)\right\} \\
&\leq (4C_1 + |O_4|)\,x^2\,\rho_n. \qquad (4.7)
\end{aligned}
$$

Let $A_n(h) = R_n(h) + hM_n(h)$, $c_3 = \frac{1}{4}(6C_1 + 2|Q_3| + |O_3|)^{-1}$ and $c_4 = \min\{c_2, c_3\}$. It follows from (4.6)-(4.7) that, for $2 \leq x \leq c_4\,\rho_n^{-1}$,

$$
\begin{aligned}
|A_n(h) - x| &\leq |R_n(h)| + |hM_n(h) - x| \\
&\leq (3/2)|b|x^{-1} + \big(6C_1 + 2|O_3| + |O_4|\big)\,x^2\,\rho_n \qquad (4.8)
\end{aligned}
$$

and (recall $|b| \leq x^2/4$),

$$
x/2 \;\leq\; A_n(h) \;\leq\; 3x/2. \qquad (4.9)
$$

After these preliminaries, we next give the proof of Theorem2.1.

Write $\delta_n(x) = (x + bx^{-1})B_n$. By the conjugate method,

$$\begin{aligned}
P\Big(\sum_{j=1}^{n}\eta_j > \delta_n(x)\Big) &= \Big(\prod_{j=1}^{n} Ee^{h\eta_j}\Big)\int_{\delta_n(x)}^{\infty} e^{-hu}dP\Big(\sum_{j=1}^{n}\xi_j \le u\Big), \\
&= \Big(\prod_{j=1}^{n} Ee^{h\eta_j}\Big)\int_{0}^{\infty} e^{-h\delta_n(x)-hM_n(h)v}dG_n\big\{v + R_n(h)\big\}, \\
&= \Big(\prod_{j=1}^{n} Ee^{h\eta_j}\Big)\, e^{-x^2-b}\Big(\int_{0}^{\infty} e^{-hM_n(h)v}d\Big[G_n\big\{v+R_n(h)\big\} - \Phi\big\{v+R_n(h)\big\}\Big] \\
&\qquad + \int_0^\infty e^{-hM_n(h)v}d\Phi\big\{v+R_n(h)\big\}\Big) \\
&:= I_0(h)\, e^{-x^2-b}\Big(I_1(h) + I_2(h)\Big). \qquad (4.10)
\end{aligned}$$

It follows from (4.1) that, for $2 \le x \le c_4\,\rho_n^{-1}$,

$$\begin{aligned}
\exp\Big\{\frac{x^2}{2} - \frac{x^3}{3\sqrt{n}}EX^3 - A\Delta_{n,x}\Big\} &\le I_0(h) = \Big(Ee^{h\eta_j}\Big)^n \\
&\le \exp\Big\{\frac{x^2}{2} - \frac{x^3}{3\sqrt{n}}EX^3 + A\Delta_{n,x}\Big\}. \qquad (4.11)
\end{aligned}$$

Next we estimate $I_2(h)$. We have

$$\begin{aligned}
I_2(h) &= \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-hM_n(h)v - \frac{1}{2}(v+R_n(h))^2}dv \\
&= \frac{e^{-R_n^2(h)/2}}{\sqrt{2\pi}}\int_0^\infty e^{-(hM_n(h)+R_n(h))v-\frac{1}{2}v^2}dv \\
&:= \frac{e^{-R_n^2(h)/2}}{\sqrt{2\pi}}\, I_3(h). \qquad (4.12)
\end{aligned}$$

Write $\psi(x) = \big\{1-\Phi(x)\big\}/\Phi'(x) = e^{x^2/2}\int_x^\infty e^{-y^2/2}dy$. Clearly, $\psi\{A_n(h)\} = I_3(h)$, and for $x \ge 2$,

$$\frac{1}{2x} \le \psi(x) \le \frac{1}{x} \quad \text{and} \quad |\psi'(x)| = |x\psi(x) - 1| \le x^{-2}.$$

These estimates, together with (4.8) and (4.9), imply that for $2 \le x \le c_4\,\rho_n^{-1}$,

$$\begin{aligned}
I_3(h) &= \psi(x) + \psi'(\theta)\,\big\{A_n(h) - x\big\}, \qquad \text{[where } \theta \in (x/2, 3x/2)\text{]} \\
&= \psi(x)\,\big[\,O_5 + O_6\,x\,\rho_n\,\big],
\end{aligned}$$

where $|O_5 - 1| \le 2|b|x^{-2}$ and $|O_6| \le A$. Therefore, for $2 \le x \le c_4\,\rho_n^{-1}$,

$$I_2(h) \;\;=\;\; e^{x^2/2}\left\{1-\Phi(x)\right\}e^{-R_n^2(x)/2}\Big(O_5 + O_6\,x\,\rho_n\Big). \tag{4.13}$$

As for $I_1(h)$, by (4.4)-(4.5), integration by parts and Berry-Esseen theorem, we get

$$|I_1(h)| \le 2\sup_v |G_n(v) - \Phi(v)| \le 4M_n^{-3}(h)\sum_{j=1}^n E|\xi_j|^3 \le 16\,C_5\,\rho_n.$$

This implies that for $x \ge 2$,

$$I_1(h) \;\;=\;\; O_7\,x\,\rho_n\,e^{x^2/2}\left\{1-\Phi(x)\right\}, \tag{4.14}$$

where $|O_7| \le A$.

It follows easily from (4.10)-(4.11) and (4.13)-(4.14) that for $2 \le x \le c_4\,\rho_n^{-1}$,

$$\begin{aligned} P\Big(\sum_{j=1}^n \eta_j \ge \delta_n(x)\Big) \;&\le\; (1+2|b|x^{-2})e^{-b}\Psi_n(x)\exp\Big(A\,\Delta_{n,x}\Big)\,(1+A_1 x\rho_n) \\ &\le\; (1+2|b|x^{-2})e^{-b}\Psi_n(x)\exp\Big(A\,\mathcal{L}_{n,x}\Big). \end{aligned}$$

This proves (2.6). Similarly, by letting $b = 0$, it follows from (4.10)-(4.11) and (4.13)-(4.14) that

$$\begin{aligned} & P\Big(\sum_{j=1}^n \eta_j \ge x\,B_n\Big) \\ \ge\;\; & \Psi_n(x)\exp\left\{-A\Delta_{n,x} - R_n^2(h)/2\right\}\Big[1 - \Big\{|O_6| + |O_7|e^{R_n^2(h)/2}\Big\}\;x\,\rho_n\Big] \\ \ge\;\; & \Psi_n(x)\exp\left\{-A\Delta_{n,x}\right\}\Big[1 - A_1\,x\,\rho_n\Big] \\ \ge\;\; & \Psi_n(x)\exp\left\{-A_2\,\mathcal{L}_{n,x}\right\}, \end{aligned}$$

for $2 \le x \le c_4\,\rho_n^{-1/2}$, where we have used the fact that $|R_n(h)| \le 1$ by (4.6), and also

$$R_n^2(h) \le 4(C_1 + |O_3|)^2 x^4 n^{-1}(E|X|^3)^2 \le 8(C_1 + |O_3|)^2\Delta_{n,x}$$

since $\big(E|X|^3\big)^2 \le 2\big\{E|X|^3 I_{(|x|>\tau)}\big\}^2 + 2EX^4 I_{(|x|\le\tau)}$. This proves (2.7) and hence complete the proof of Theorem 2.1.

5 Proof of Theorem 2.2

The idea for the proof of Theorem 2.2 is similar to Proposition 5.4 in Jing, Shao and Wang (2002), but we need some different details. Throughout this section, we use the following notations: $g(t,x) \;=\; Ee^{it\zeta_1/\sqrt{n}}$ and

$$\Omega_n(x,t) \;=\; \Big\{(x,t) : 1+x \le \frac{1}{3}(1+C_6)^{-1}/\rho_n, |t| \le \frac{1}{6}(1+C_7)^{-1}/\rho_n\Big\}.$$

The proof of Theorem 2.2 is based on the following lemmas.

LEMMA 5.1. *If* $(x,t) \in \Omega_n(x,t)$*, then*

$$\begin{aligned} \big|g(t,x)\big| &\le e^{-t^2/6n}, && (5.1)\\ \Big|g^n(t,x) - e^{-\frac{t^2}{2}}\Big| &\le A\,(1+x)\,\rho_n\,(t^2+|t|^3)\,e^{-t^2/6}, && (5.2) \end{aligned}$$

$$\Big|g^n(t,x) - e^{-\frac{t^2}{2}}\Big[1+n\big\{g(t,x)-1\big\}+\frac{t^2}{2}\Big]\Big| \;\le\; A\,(1+x)^2\,\rho_n^2\,(|t|^3+|t|^8)\,e^{-t^2/6}. \quad (5.3)$$

Proof. It follows from Taylor's expansion of e^{ix} that

$$\Big|g(t,x) - 1 + \frac{t^2E\zeta_1^2}{2n}\Big| \;\le\; \frac{1}{6}\,|t|^3\,n^{-3/2}\,E|\zeta_1|^3. \quad (5.4)$$

In view of (2.8) and (2.9), we have that

$$|E\zeta_1^2 - 1| \le (1+C_6)(1+x)\rho_n \;\text{ and }\; E|\zeta_1|^3 \le (1+C_7)E|X|^3. \quad (5.5)$$

Taking these estimates back into (5.4), we obtain for $(x,t) \in \Omega_n(x,t)$,

$$\begin{aligned} \big|g(t,x)\big| &\le 1 - \frac{t^2}{2n} + \frac{t^2}{2n}\big|E\zeta_1^2-1\big| + \frac{1}{6}\,|t|^3\,n^{-3/2}\,E|\zeta_1|^3\\ &\le 1 - \frac{t^2}{2n} + \frac{t^2}{6n} + \frac{t^2}{6n} \;\le\; e^{-t^2/6n}. \end{aligned}$$

This proves (5.1).

Using (5.1), we have that for $(x,t) \in \Omega_n(x,t)$ and $|t|^3 \ge (1+C_7)^{-1}/\rho_n$,

$$\Big|g^n(t,x) - e^{-\frac{t^2}{2}}\Big| \;\le\; 2e^{-t^2/6} \;\le\; 2(1+C_7)\rho_n|t|^3e^{-t^2/6}. \quad (5.6)$$

On the other hand, by using (5.5), we get that for $(x,t) \in \Omega_n(x,t)$ and $|t|^3 \le (1+C_7)^{-1}/\rho_n$,

$$g(t,x) = 1 - r_1(t,x), \tag{5.7}$$

where

$$r_1(t,x) = \frac{t^2}{2n} E\zeta_1^2 + \eta \frac{|t|^3}{6n^{3/2}} E|\zeta_1|^3, \quad |\eta| \le 1,$$

having $|r_1(t,x)| \le 1/4$ and

$$\big|r_1(t,x)\big|^2 \le \frac{t^4}{2n^2}\big(E|\zeta_1|^3\big)^{4/3} + \frac{|t|^6}{18n^3}\big(E|\zeta_1|^3\big)^2 \le \frac{5}{9}(1+C_7)|t|^3 n^{-1}\rho_n. \tag{5.8}$$

Therefore, it follows from $\ln(1+z) = z + \eta_1 z^2$, whenever $|z| < 1/2$, where $|\eta_1| \le 1$, that for $(x,t) \in \Omega_n(x,t)$ and $|t|^3 \le (1+C_7)^{-1}/\rho_n$,

$$\begin{aligned} \ln g(t,x) &= -\frac{t^2}{2n} E\zeta_1^2 + \theta\, |t|^3\, n^{-1}\, \rho_n, \quad |\theta| \le 1 + C_7, & (5.9)\\ \ln g^n(t,x) &= -\frac{t^2}{2} E\zeta_1^2 + \theta\, |t|^3 \rho_n \;=\; -\frac{t^2}{2} + r(t,x), & (5.10)\end{aligned}$$

where, by using (5.5) again,

$$|r(t,x)| \;\le\; \frac{t^2}{2}|E\zeta_1^2 - 1| + |\theta|\, |t|^3 \rho_n \;\le\; t^2/3.$$

Also, we have

$$|r(t,x)| \;\le\; \frac{t^2}{2}|E\zeta_1^2 - 1| + |\theta|\, |t|^3 \rho_n \;\le A(t^2 + |t|^3)(1+x)\rho_n.$$

These estimates, together with $|e^z - 1| \le |z| e^{|z|}$, imply that for $(x,t) \in \Omega_n(x,t)$ and $|t|^3 \le (1+C_7)^{-1}/\rho_n$,

$$\Big|g^n(t,x) - e^{-\frac{t^2}{2}}\Big| \;\le\; e^{-\frac{t^2}{2}}\Big|e^{r(t,x)} - 1\Big| \;\le\; A(1+x)\rho_n(t^2 + |t|^3)e^{-t^2/6}. \tag{5.11}$$

Now, (5.2) follows from (5.6) and (5.11).

If we instead the estimate of $r_1(t,x)$ in (5.8) by

$$|r_1(t,x)|^2 \le \frac{t^4}{2n^2}\big(E\zeta_1^2\big)^2 + \frac{|t|^6}{18n^3}\big(E|\zeta_1|^3\big)^2 \le 2\, n^{-2}\, t^4\,,$$

we can rewrite (5.9) and (5.10) as

$$\begin{aligned} \ln g(t,x) &= g(t,x) - 1 + \theta_1\, n^{-2}\, t^4, \quad |\theta_1| \le 2,\\ \ln g^n(t,x) &= n\big\{g(t,x) - 1\big\} + \theta\, n^{-1}\, t^4. \end{aligned}$$

Therefore, by using (5.5) and (5.7), we obtain for $(x,t) \in \Omega_n(x,t)$,

$$\begin{aligned}\Big|\ln g^n(t,x) + \frac{t^2}{2}\Big| &\le \Big|n\big\{g(t,x)-1\big\} + \frac{t^2}{2}\Big| + 2n^{-1}t^4 \\ &\le \frac{t^2}{2}|E\zeta_1^2 - 1| + \frac{|t|^3}{6\sqrt{n}}E|\zeta_1|^3 + 2n^{-1}t^4 \ \le\ t^2/3\end{aligned}$$

(recalling $|t| \le \frac{1}{6}\rho_n^{-1} \le \sqrt{n}/6$). Also, we have

$$\begin{aligned}\Big|\ln g^n(t,x) + \frac{t^2}{2}\Big| &\le \frac{t^2}{2}|E\zeta_1^2 - 1| + \frac{|t|^3}{6\sqrt{n}}E|\zeta_1|^3 + 2n^{-1}t^4 \\ &\le A(1+x)\rho_n(t^2+t^4).\end{aligned}$$

Now, by using $|e^z - 1 - z| \le \frac{z^2}{2}e^{|z|}$, we obtain that for $(x,t) \in \Omega_n(x,t)$,

$$\begin{aligned}g^n(t,x) - e^{-\frac{t^2}{2}} &= e^{-\frac{t^2}{2}}\left\{e^{\ln g^n(t,x) + \frac{t^2}{2}} - 1\right\} \\ &= e^{-\frac{t^2}{2}}\left[n\big\{g(t,x)-1\big\} + \frac{t^2}{2} + r^*(t,x)\right],\end{aligned}$$

where

$$\begin{aligned}|r^*(t,x)| &\le 2n^{-1}t^4 + \frac{1}{2}\Big|\ln g^n(t,x) + \frac{t^2}{2}\Big|^2 \exp\Big\{\Big|\ln g^n(t,x) + \frac{t^2}{2}\Big|\Big\} \\ &\le A\,(1+x)^2\,\rho_n^2\,(|t|^4 + |t|^8)\,e^{t^2/3}.\end{aligned}$$

This implies (5.3). The proof of Lemma 5.1 is now complete. □

LEMMA 5.2. *If $(x,t) \in \Omega_n(x,t)$, then*

$$\Big|E\Lambda_{n,n}e^{itT_n}\Big| \le Ax\rho_n^2\,t^2e^{-t^2/6}, \tag{5.12}$$

$$\Big|Ee^{it(T_n+\Lambda_{n,n})} - g^n(t,x)\Big| \le Ax\rho_n^2t^2e^{-t^2/6} + A_1x^{3/2}\rho_n^2|t|^{3/2} \tag{5.13}$$

and for any $1 \le m \le n$,

$$\Big|Ee^{it(T_n+\Lambda_{n,n})}\Big| \le \big(1 + A\,x\,|t|\big)e^{-(m-2)t^2/(12n)} + A_1\,m\,n^{-1}\,x^{3/2}\,\rho_n^2\,|t|^{3/2}. \tag{5.14}$$

Proof. It follows from (2.11), (2.12), (5.5) and Holder's inequality that

$$\begin{aligned}\Big|E\psi_{1,2}e^{it(\zeta_1+\zeta_2)/\sqrt{n}}\Big| &= \Big|E\psi_{1,2}\Big(e^{it\zeta_1/\sqrt{n}} - 1\Big)\Big(e^{it\zeta_2/\sqrt{n}} - 1\Big)\Big| \\ &\le \Big(\frac{|t|}{\sqrt{n}}\Big)^2 E\left\{|\psi_{1,2}|\zeta_1||\zeta_2|\right\} \\ &\le \frac{t^2}{n}\left\{E|\psi_{1,2}|^{3/2}\right\}^{2/3}\left\{E|\zeta_1|^3\right\}^{2/3} \le\ Ax\rho_n^2t^2.\end{aligned}$$

Therefore, it follows from independence of ζ_j and Lemma 5.1 that

$$\left|E\Lambda_{n,n}e^{itT_n}\right| \le \left|E\psi_{1,2}e^{it(\zeta_1+\zeta_2)/\sqrt{n}}\right| |g(t,x)|^{n-2} \le Ax\rho_n^2 t^2 e^{-t^2/6}.$$

This proves (5.12).

To prove (5.13) and (5.14), put

$$\Lambda_{n,m}^* = \Lambda_{n,n} - \Lambda_{n,m} = \frac{1}{n^2}\sum_{k=m+1}^{n-1}\sum_{j=k+1}^{n}\psi_{k,j} \quad \text{and} \quad \Lambda_{n,m}^* = 0, \quad m \ge n.$$

By (2.12) and $|e^{iz} - 1 - iz| \le 2|z|^{3/2}$,

$$\begin{aligned}&\left|Ee^{it(T_n+\Lambda_{n,n})} - Ee^{it(T_n+\Lambda_{n,m}^*)} - itE\Lambda_{n,m}e^{it(T_n+\Lambda_{n,m}^*)}\right| \\ &\quad\le\ 2|t|^{3/2}E|\Lambda_{n,m}|^{3/2} \le A|t|^{3/2}mn^{-2}E|\psi_{1,2}|^{3/2} \le Amn^{-1}x^{3/2}\rho_n^2|t|^{3/2}. \qquad (5.15)\end{aligned}$$

Therefore, (5.13) follows easily from (5.12) and (5.15) with $m = n$. In view of independence of ζ_j, on the other hand, (5.15) implies that for any $1 \le m \le n$,

$$\begin{aligned}\left|Ee^{it(T_n+\Lambda_{n,n})}\right| &\le |g^m(t,x)| + Ax|t|\left|g^{m-2}(t,x)\right| + Amn^{-1}x^{3/2}\rho_n^2\,|t|^{3/2} \\ &\le \big(1 + Ax|t|\big)e^{-(m-2)t^2/(6n)} + Amn^{-1}x^{3/2}\rho_n^2\,|t|^{3/2}\end{aligned}$$

where we have used the estimate (recalling (2.12)):

$$\begin{aligned}E|\Lambda_{n,m}e^{it(T_n+\Lambda_{n,m}^*)}| &= E\Big|\frac{1}{n^2}\sum_{k=1}^{m}\sum_{j=k+1}^{n}\psi_{k,j}e^{it(T_n+\Lambda_{n,m}^*)}\Big| \\ &\le |g(t,x)|^{m-2}E|\psi_{1,2}|\ \le\ A\,x\,|g(t,x)|^{m-2}.\end{aligned}$$

This gives (5.14). The proof of Lemma 5.2 is now complete. □

LEMMA 5.3. *Let F be a distribution function with the characteristic function f. Then for all $y \in R$ and $T > 0$ it holds that*

$$\lim_{z\downarrow y}F(z) \ \le\ \frac{1}{2} + V.P.\int_{-T}^{T}\exp(-iyt)\frac{1}{T}K\big(\frac{t}{T}\big)f(t)dt. \qquad (5.16)$$

$$\lim_{z\uparrow y}F(z) \ \ge\ \frac{1}{2} - V.P.\int_{-T}^{T}\exp(-iyt)\frac{1}{T}K\big(-\frac{t}{T}\big)f(t)dt, \qquad (5.17)$$

where

$$V.P.\int_{-T}^{T} = \lim_{h\downarrow 0}\Big(\int_{-T}^{-h} + \int_{h}^{T}\Big),$$

and $2K(s) = K_1(s) + iK_2(s)/(\pi s)$,

$$K_1(s) = 1 - |s|, \quad K_2(s) = \pi s(1-|s|)cot\, \pi s + |s|, \quad for\ |s| < 1,$$

and $K(s) \equiv 0$ *for* $|s| \geq 1$.

Proof of Lemma 5.3 can be found in Prawitz (1972).

LEMMA 5.4. *It holds that if* $0 \leq x \leq c\rho_n^{-1}$ *with* $c \leq \frac{1}{6}(1+C_6)^{-1}$, *then for any* $y \in R$,

$$|I^+(y)|,\ |I^-(y)| \ \leq \ A\rho_n e^{-y^2/2} + A_1(1+x^{3/2})\rho_n^2, \tag{5.18}$$

where $K_1(s)$ *is de£ned as in Lemma 5.3,*

$$\begin{aligned} I^+(y) &= \frac{1}{T}\int_{-T}^{T} e^{-iyt} K_1\big(\frac{t}{T}\big) Ee^{it(T_n+\Lambda_{n,n})}dt, \\ I^-(y) &= \frac{1}{T}\int_{-T}^{T} e^{-iyt} K_1\big(-\frac{t}{T}\big) Ee^{it(T_n+\Lambda_{n,n})}dt, \qquad T = \frac{1}{6}(1+C_7)^{-1}/\rho_n. \end{aligned}$$

Proof. We only prove (5.18) for $I^+(y)$. Without loss of generality, we assume $\rho_n \leq 12^{-3}$. This assumption implies that $1 + x \leq \frac{1}{3}(1+C_6)^{-1}/\rho_n$ for $0 \leq x \leq c\rho_n^{-1}$ with $c \leq \frac{1}{6}(1+C_6)^{-1}$. Let $I^+ = I^+(y)$ and $T_1 = \rho_n^{-1/3}$. We have that

$$I^+ = \int_{-T_1}^{T_1} \cdots + \int_{T_1 \leq |t| \leq T} \cdots \ := \ I_1 + I_2.$$

It is easy to see that $[12nt^{-2}\log|t|] \leq n-2$ if $|t| \geq 12$ and $n \geq 6$. Hence, recalling $|T_1| \geq 12$ and $\sqrt{n} \geq 12/E|X|^3 \geq 12$, by (5.14) with $m = [12nt^{-2}\log|t|] + 2$,

$$|I_2| \leq \frac{1}{T}\int_{T_1 \leq |t| \leq T} \big|Ee^{it(S_n+\Lambda_{n,n})}\big|dt \leq A(1+x^{3/2})\rho_n^2. \tag{5.19}$$

Noting $K_1(s) = 1 - |s|$, for $|s| < 1$, we obtain $|I_1| \leq |I_{11}| + |I_{12}|$, where

$$I_{11} = \frac{1}{T}\int_{-T_1}^{T_1} e^{-iyt} Ee^{it(S_n+\Lambda_{n,n})}dt, \qquad I_{12} = \frac{2}{T^2}\int_0^{T_1} t\big|Ee^{it(S_n+\Lambda_{n,n})}\big|dt.$$

It follows from (5.14) with $m = [12nt^{-2}\log|t|] + 2$ again that

$$|I_{12}| \leq \frac{2}{T^2}\Big\{\int_0^{12} tdt + \int_{12}^{T_1} t\big|Ee^{it(S_n+\Lambda_{n,n})}\big|dt\Big\} \leq A(1+x^{3/2})\rho_n^2. \tag{5.20}$$

On the other hand, noting that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iyt-t^2/2}dt = e^{-y^2/2},$$

it follows from Lemmas (5.1)-(5.2) (i.e, (5.2) and (5.13)) that

$$\begin{aligned}
|I_{11}| &\le \frac{1}{T}\Big|\int_{-\infty}^{\infty} e^{-iyt-t^2/2}dt\Big| + \frac{1}{T}\int_{|t|\ge T_1} e^{-t^2/2}dt + \frac{1}{T}\int_{-T_1}^{T_1}\Big|Ee^{it(S_n+\Lambda_{n,n})} - e^{-t^2/2}\Big|dt \\
&\le A\rho_n e^{-y^2/2} + A_1(1+x^{3/2})\rho_n^2.
\end{aligned}$$

Collecting all these estimates, we conclude the proof of Lemma 5.4. □

LEMMA 5.5. *The integral*

$$\begin{aligned}
J^+(y) &= \frac{i}{\pi}V.P.\int_{-T}^{T} e^{-iyt}K_2\Big(\frac{t}{T}\Big)Ee^{it(S_n+\Lambda_{n,n})}\frac{dt}{t}, \\
J^-(y) &= \frac{i}{\pi}V.P.\int_{-T}^{T} e^{-iyt}K_2\Big(-\frac{t}{T}\Big)Ee^{it(S_n+\Lambda_{n,n})}\frac{dt}{t}, \qquad T = \frac{1}{6}(1+C_7)^{-1}/\rho_n,
\end{aligned}$$

satisfy that, for any $y \in R$ and $0 \le x \le c\rho_n^{-1}$ with $c \le \frac{1}{6}(1+C_6)^{-1}$,

$$\Big|J^+(y) + 1 - 2\Phi(y) - 2\mathcal{L}_n(y)\Big| \le A(1+x^{3/2})\rho_n^{3/2}, \tag{5.21}$$

$$\Big|J^-(y) + 1 - 2\Phi(y) - 2\mathcal{L}_n(y)\Big| \le A(1+x^{3/2})\rho_n^{3/2}, \tag{5.22}$$

where $K_2(s)$ is de£ned as in Lemma 5.3 and

$$\mathcal{L}_n(y) = n\Big\{E\Phi\Big(y - \frac{\zeta_1}{\sqrt{n}}\Big) - \Phi(y)\Big\} - \frac{1}{2}\Phi^{(2)}(y).$$

Proof. We only prove (5.21). Similar to the proof of Lemma 5.4, we assume $\rho_n \le 12^{-3}$, which implies $1 + x \le \frac{1}{3}(1+C_6)^{-1}/\rho_n$ for $0 \le x \le c\rho_n^{-1}$ with $c \le \frac{1}{6}(1+C_6)^{-1}$. Write $J^+ = J^+(y)$ and $f_n(t,x) = \Big[1 + n\big\{g(t,x) - 1\big\} + \frac{t^2}{2}\Big]e^{-t^2/2}$. The J^+ can be rewrite as

$$J^+ = J_{11} + J_{12} + J_{13} + J_2,$$

where

$$\begin{aligned}
J_{11} &= \frac{i}{\pi}V.P.\int_{-T_1}^{T_1} e^{-iyt}f_n(t,x)\frac{dt}{t}, \\
J_{12} &= \frac{i}{\pi}V.P.\int_{-T_1}^{T_1} e^{-iyt}\Big\{Ee^{it(S_n+\Lambda_{n,n})} - f_n(t,x)\Big\}\frac{dt}{t}, \\
J_{13} &= \frac{i}{\pi}V.P.\int_{-T_1}^{T_1} e^{-iyt}\Big\{K_2\big(\frac{t}{T}\big) - 1\Big\}Ee^{it(S_n+\Lambda_{n,n})}\frac{dt}{t}, \\
J_2 &= \frac{i}{\pi}V.P.\int_{T_1\le|t|\le T} e^{-iyt}K_2\big(\frac{t}{T}\big)Ee^{it(S_n+\Lambda_{n,n})}\frac{dt}{t}
\end{aligned}$$

and $T_1 = \rho_n^{-1/3}$. Similar to (5.19), it follows that $|J_2| \le A(1 + x^{3/2})\rho_n^2$. By using (5.3) and (5.13), we have

$$
\begin{aligned}
|J_{12}| &\le \int_{-T_1}^{T_1} \Big| Ee^{it(S_n + \Lambda_{n,n})} - g^n(t,x)\Big| \frac{dt}{|t|} + \int_{-T_1}^{T_1} \Big| g^n(t,x) - f_n(t,x)\Big| \frac{dt}{|t|} \\
&\le A\Big\{(1+x)^2\rho_n^2 + x^{3/2}\rho_n^{3/2}\Big\}.
\end{aligned}
$$

Noting that $|K_2(s) - 1| \le As^2$, for $|s| \le 1/2$ (cf., e.g., Lemma 2.1 in Bentkus (1994), similar to (5.20), it can be easily shown that

$$
|J_{13}| \le AT^{-2}\int_{-T_1}^{T_1} |t|\Big| Ee^{it(S_n + \Lambda_{n,n})}\Big| dt \le A(1 + x^{3/2})\rho_n^2.
$$

On the other hand, simple calculation shows that

$$
\frac{i}{2\pi} V.P. \int_{-\infty}^{\infty} e^{-iyt} f_n(t,x)\frac{dt}{t} = -\frac{1}{2} + \Phi(y) + \mathcal{L}_n(y).
$$

Therefore, it follows from all these estimates and $x \le c\rho_n^{-1}$ that

$$
\begin{aligned}
&\Big| J^+ + 1 - 2\Phi(y) - 2\mathcal{L}_n(y)\Big| \\
&\quad\le \Big| J_{11} + 1 - 2\Phi(y) - 2\mathcal{L}_n(y)\Big| + |J_{12}| + |J_{13}| + |J_2| \\
&\quad\le \int_{|t|\ge T_1} \frac{1}{|t|}\Big| f_n(t,x)\Big| dt + A\Big\{(1+x)^2\rho_n^2 + x^{3/2}\rho_n^{3/2}\Big\} \\
&\quad\le A\Big\{(1+x)^2\rho_n^2 + x^{3/2}\rho_n^{3/2}\Big\} \\
&\quad\le A\,(1 + x^{3/2})\,\rho_n^{3/2}.
\end{aligned}
$$

This also completes the proof of Lemma 5.5. □

LEMMA 5.6. *For any* $|y| \le A(1+x)$,

$$
\Big|\mathcal{L}_n(y) - \frac{EX^3}{\sqrt{2\pi n}}\Big(\frac{y^2}{6} - \frac{yx}{2}\Big)e^{-y^2/2}\Big| \le A\left\{\rho_n + \Delta_{n,x}/(1+x)\right\} e^{-y^2/2}, \tag{5.23}
$$

where $\mathcal{L}_n(y)$ *is de£ned as in Lemma 5.5. For* $0 \le x \le c\rho_n^{-1}$ *with* c *suf£ciently small, and* $y_0 = x + \lambda_n(x)$*, where* $|\lambda_n(x)| \le C\,\Delta_{n,x}/(1+x)$,

$$
\Big| e^{-y_0^2/2} - e^{-x^2/2}\Big| \le A\,\rho_n, \tag{5.24}
$$

$$
\Big| \Phi(y_0) - \Phi(x)\Big| \le A\left\{\rho_n + \Delta_{n,x}/(1+x)\right\} e^{-x^2/2} + A_1\rho_n^2, \tag{5.25}
$$

$$
\Big|\mathcal{L}_n(y_0) + \frac{EX^3}{3\sqrt{n}}\frac{x^2}{\sqrt{2\pi}}e^{-x^2/2}\Big| \le A\left\{\rho_n + \Delta_{n,x}/(1+x)\right\} e^{-x^2/2} + A_1(1+x^2)\rho_n^2. \tag{5.26}
$$

Proof. We £rst note that

$$\Phi^{(2)}(y) = -\frac{y}{\sqrt{2\pi}} e^{-y^2/2}, \quad \Phi^{(3)}(y) = \frac{y^2-1}{\sqrt{2\pi}} e^{-y^2/2} \tag{5.27}$$

and

$$|\Phi^{(4)}(y)| \le A(1+|y|^3)e^{-y^2/2}. \tag{5.28}$$

Using (5.28), $|y| \le A(1+x)$ and $|\zeta_1| \le A\sqrt{n}/(1+x)$, it can be easily seen that for $|\theta| \le 1$,

$$\Big|\Phi^{(4)}\big(y+\theta\frac{|\zeta_1|}{\sqrt{n}}\big)\Big| \le A(1+|y|^3)\exp\Big\{-\frac{y^2}{2} + \frac{|y||\zeta_1|}{\sqrt{n}}\Big\} \le A(1+x)^3 e^{-y^2/2}.$$

This, together with Taylor's expansion and $E\zeta_1^4 \le C_9 EX^4 I_{(|X|\le\tau)}$, implies that

$$\begin{aligned} Q_n &:= \Big|E\Phi\Big(y - \frac{\zeta_1}{\sqrt{n}}\Big) - \Phi(y) - \frac{E\zeta_1^2}{2n}\Phi^{(2)}(y) + \frac{E\zeta_1^3}{6n^{3/2}}\Phi^{(3)}(y)\Big| \\ &\le \frac{1}{24n^2}\, E\,\zeta_1^4\,\Phi^{(4)}\Big(y+\theta\frac{|\zeta_1|}{\sqrt{n}}\Big) \qquad (|\theta|\le 1) \\ &\le A\,EX^4 I_{(|X|\le\tau)}(1+x)^3 n^{-2} e^{-y^2/2}. \end{aligned}$$

Therefore, taking account of (2.8)-(2.9) and (5.27), we have

$$\begin{aligned} &\Big|\mathcal{L}_n(y) - \frac{EX^3}{\sqrt{2\pi n}}\Big(\frac{y^2}{6} - \frac{yx}{2}\Big)e^{-y^2/2}\Big| \\ &\quad\le \Big|\mathcal{L}_n(y) - \frac{xEX^3}{2\sqrt{n}}\Phi^{(2)}(y) - \frac{EX^3}{6\sqrt{n}}\Phi^{(3)}(y)\Big| + \frac{\rho_n}{6\sqrt{2\pi}}e^{-y^2/2} \\ &\quad\le nQ_n + \frac{1}{2}|\Phi^{(2)}(y)|\Big|E\zeta_1^2 - 1 - \frac{\alpha x}{\sqrt{n}}EX^3\Big| \\ &\qquad + \frac{1}{6n^{1/2}}|\Phi^{(3)}(y)|\Big|E\zeta_1^3 - \beta EX^3\Big| + \frac{\rho_n}{6\sqrt{2\pi}}e^{-y^2/2} \\ &\quad\le A\big\{\rho_n + \Delta_{n,x}/(1+x)\big\}e^{-y^2/2}. \end{aligned}$$

This concludes (5.23).

We next prove (5.24)-(5.26). It can be easily seen that $\Delta_{n,x} \le (1+x)^3\rho_n^{-1}$, and for $x \le c\rho_n^{-1}$ with c small enough,

$$\begin{aligned} |y_0^2 - x^2| &\le 2x|\lambda_n(x)| + \lambda_n^2(x) \le (2C + C^2)\Delta_{n,x} \le x^2/3, \\ -(x - |y_0 - x|)^2 &\le -x^2 + 2x|\lambda_n(x)| \le -x^2 + 2C\,\Delta_{n,x} \le -3x^2/4. \end{aligned}$$

These estimates imply that

$$\begin{aligned}
\Big|e^{-y_0^2/2} - e^{-x^2/2}\Big| &\le \frac{1}{2}|y_0^2 - x^2| e^{-x^2/2 + |y_0^2 - x^2|/2} \\
&\le \frac{A}{2}\rho_n (1+x)^3 e^{-x^2/3} \le A\rho_n, && (5.29) \\
\Big|\Phi(y_0) - \Phi(x)\Big| &\le |y_0 - x|\,\Phi'(x) + \frac{(y_0-x)^2}{2}\Big|\Phi^{(2)}(x + \theta|y_0 - x|)\Big| \quad (|\theta| \le 1) \\
&\le A\frac{\Delta_{n,x}}{1+x} e^{-x^2/2} + \frac{\rho_n^2}{2}(1+x)^7 e^{-(x - |y_0 - x|)^2/2} \\
&\le A\frac{\Delta_{n,x}}{1+x} e^{-x^2/2} + A_1\rho_n^2. && (5.30)
\end{aligned}$$

This proves (5.24) and (5.25). On the other hand, we have

$$\Big| -\frac{y_0^2}{6} + \frac{y_0 x}{2} - \frac{1}{3}x^2\Big| \le \frac{1}{6}\left|y_0^2 - x^2\right| + \frac{x}{2}\left|y_0 - x\right| \le A\,\Delta_{n,x}.$$

This, together with (5.27)-(5.24), implies that

$$\begin{aligned}
\Big|\mathcal{L}_n(y_0) + \frac{EX^3}{3\sqrt{n}}\frac{x^2}{\sqrt{2\pi}} e^{-x^2/2}\Big| &\le A\rho_n\Delta_{n,x} e^{-y_0^2/2} + A_1\big\{\rho_n + \Delta_{n,x}/(1+x)\big\} e^{-y_0^2/2} \\
&\le A\big\{\rho_n + \Delta_{n,x}/(1+x)\big\} e^{-x^2/2} + A_1(1+x^2)\rho_n^2.
\end{aligned}$$

This gives (5.26). The proof of Lemma 5.6 is now complete. □

After these lemmas, we are now ready to prove Theorem 2.2.

By using (5.17) and Lemmas 5.4-5.5, we obtain for $2 \le x \le c\rho_n^{-1}$ with c suf£ciently small,

$$\begin{aligned}
P\Big(T_n + \Delta_{n,n} \ge x + \lambda_n(x)\Big) &\le \frac{1}{2}\Big[I^-(y_0) + 1 - J^-(y_0)\Big] \\
&\le 1 - \Phi(y_0) + \mathcal{L}_n(y_0) + A\rho_n e^{-y_0^2/2} + A_1 x^{3/2}\rho_n^{3/2},
\end{aligned}$$

where $y_0 = x + \lambda_n(x)$. Furthermore, it follows from Lemma 5.6 that

$$\begin{aligned}
&P\Big(T_n + \Delta_{n,n} \ge x + \lambda_n(x)\Big) \\
&\quad\le 1 - \Phi(x) - \frac{EX^3}{3\sqrt{n}}\frac{x^2}{\sqrt{2\pi}} e^{-x^2/2} + A\big(\rho_n + \Delta_{n,x}/x\big) e^{-x^2/2} + A_1\, x^{3/2}\rho_n^{3/2}. \quad (5.31)
\end{aligned}$$

Using the well-known inequality

$$\frac{1}{\sqrt{2\pi}}\Big(\frac{1}{x} - \frac{1}{x^3}\Big) e^{-x^2/2} \le 1 - \Phi(x) \le \frac{1}{\sqrt{2\pi}x} e^{-x^2/2}, \quad x > 0,$$

we have for $x \geq 2$,

$$\Big| x^3 \big\{ 1 - \Phi(x) \big\} - \frac{x^2}{\sqrt{2\pi}} e^{-x^2/2} \Big| \leq \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

and $e^{-x^2/2} \leq Ax \big\{ 1 - \Phi(x) \big\}$. Taking these estimates back into (5.31), we get for $2 \leq x \leq c\rho_n^{-1}$ with c suf£ciently small,

$$P\Big(T_n + \Delta_{n,n} \geq x + \lambda_n(x) \Big) \quad \leq \quad \big\{ 1 - \Phi(x) \big\} \Big[1 - \frac{x^3 EX^3}{3\sqrt{n}} + A\mathcal{L}_{n,x} \Big] + \; A_1 \, x^{3/2} \rho_n^{3/2}.$$

Now we conclude the result if we prove

$$1 - \frac{x^3 EX^3}{3\sqrt{n}} + A\mathcal{L}_{n,x} \quad \leq \quad \exp\Big\{ -\frac{x^3 EX^3}{3\sqrt{n}} \Big\} \Big[1 + A_1 \, \mathcal{L}_{n,x} \Big]. \tag{5.32}$$

In fact, (5.32) is obvious for $EX^3 < 0$. In the case that $EX^3 > 0$, if $\frac{x^3 EX^3}{3\sqrt{n}} \leq 2$, then

$$\begin{aligned} 1 - \frac{x^3 EX^3}{3\sqrt{n}} + A\,\mathcal{L}_{n,x} \quad & \leq \quad \exp\Big\{ -\frac{x^3 EX^3}{3\sqrt{n}} \Big\} + A\,\mathcal{L}_{n,x} \\ & \leq \quad \exp\Big\{ -\frac{x^3 EX^3}{3\sqrt{n}} \Big\} \Big[1 + Ae^2 \, \mathcal{L}_{n,x} \Big]. \end{aligned}$$

On the other hand, it follows easily from the de£nition of $\Delta_{n,x}$ that for $2 \leq x \leq c\rho_n^{-1}$,

$$A(x\rho_n + \Delta_{n,x}) \leq (1 + x^3 EX^3/\sqrt{n})/9,$$

by choosing c suf£ciently small. Therefore, if $\frac{x^3 EX^3}{3\sqrt{n}} \geq 2$, then

$$1 - \frac{x^3 EX^3}{3\sqrt{n}} + A\,\mathcal{L}_{n,x} \leq 0 \leq \exp\Big\{ -\frac{x^3 EX^3}{3\sqrt{n}} \Big\} \Big[1 + A\,\mathcal{L}_{n,x} \Big].$$

Collecting all these estimates, we get the desired (5.32). This also completes the proof of Theorem 2.2.

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