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WEAK CONVERGENCE FOR THE ROW SUMS OF A TRIANGULAR ARRAY OF EMPIRICAL PROCESSES INDEXED BY A MANAGEABLE TRIANGULAR ARRAY OF FUNCTIONS

Miguel A. Arcones
Department of Mathematical Sciences
State University of New York
Binghamton, NY 13902
arcones@math.binghamton.edu
http://math.binghamton.edu/arcones/index.html

Abstract: We study the weak convergence for the row sums of a general triangular array of empirical processes indexed by a manageable class of functions converging to an arbitrary limit. As particular cases, we consider random series processes and normalized sums of i.i.d. random processes with Gaussian and stable limits. An application to linear regression is presented. In this application, the limit of the row sum of a triangular array of empirical process is the mixture of a Gaussian process with a random series process.

Keywords: Empirical processes, triangular arrays, manageable classes.

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1. Introduction. Let $(S_{n,j}, \mathcal{S}_{n,j})$, $1 \le j \le k_n$, be measurable spaces, where $\{k_n\}_{n=1}^{\infty}$ is a sequence of positive integers converging to infinity. Let $\{X_{n,j} : 1 \le j \le k_n\}$ be $S_{n,j}$–valued independent r.v.'s defined on $(\prod_{j=1}^{k_n} S_{n,j}, \prod_{j=1}^{k_n} \mathcal{S}_{n,j})$. Let $f_{n,j}(\cdot, t) : S_{n,j} \to I\!R$ be a measurable function for each $1 \le j \le k_n$ and each $t \in T$. Let $c_n(t)$ be a real number. Let

$$Z_n(t) := \left(\sum_{j=1}^{k_n} f_{n,j}(X_{n,j}, t) \right) - c_n(t). \tag{1.1}$$

We study the weak convergence of the sequence of stochastic processes $\{Z_n(t) : t \in T\}$. Observe that $Z_n(t)$ is a sum of independent random variables minus a shift. As usual, we will use the definition of weak convergence of stochastic processes in Hoffmann–Jørgensen (1991).

As a particular case, we consider normalized sums of i.i.d. random processes. Let $\{X_j\}_{j=1}^{\infty}$ be a sequence of i.i.d.r.v.'s with values in a measurable space $(S, \mathcal{S})$, let X be a copy of X_1, let $f(\cdot, t) : S \to I\!R$ be a measurable function for each $t \in T$, let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive numbers converging to infinity and let $c_n(t)$ be a real number. The sequence of processes

$$\left\{ Z_n(t) := \left(a_n^{-1} \sum_{j=1}^{n} f(X_j, t) \right) - c_n(t) : t \in T \right\},\ n \ge 1, \tag{1.2}$$

is a particular case of the sequence of processes in (1.1).

Let $\{X_j\}_{j=1}^{\infty}$ be a sequence of independent r.v.'s. with values in $(S_j, \mathcal{S}_j)$. Let $f_j(\cdot, t) : S_j \to I\!R$ be a measurable function for each $1 \le j$ and each $t \in T$. Define

$$\{Z_n(t) := \sum_{j=1}^{n} f_j(X_j, t) : t \in T\}, \tag{1.3}$$

This sequence of stochastic processes is another particular case of the processes in (1.1). We call the process in (1.3) a random series process.

We present weak limit theorems for sums of general triangular arrays of independent random variables with an arbitrary limit distribution. Usually limit theorems for sums of triangular arrays of independent r.v.'s are studied for infinitesimal arrays (see for example Gnedenko and Kolmogorov, 1968). For infinitesimal arrays, the limit distribution is infinitely divisible. In general, random series are not infinitely divisible. The considered set–up allows to have limit distributions which are a mixture of an infinitely divisible distribution and a random series. In Section 3, an application of the presented limit theorems is given. In this example, the limit distribution of certain triangular of empirical processes is a mixture of a Gaussian processes and a random series process.

In Section 2, we prove the weak convergence of the process $\{Z_n(t) : t \in T\}$, as in (1.1), for classes of functions satisfying a uniform bound on packing numbers. Given a set $K \subset I\!R^n$, the packing number $D(u, K)$ is defined by

$$D(u, K) := \sup\{m : \text{there exists } v_1, \dots, v_m \in K \text{ such that } |v_i - v_j| > u, \text{ for } i \ne j\}, \tag{1.4}$$

where $|v|$ is the Euclidean norm. The interest of this concept hinges on the following maximal inequality:

$$E[\sup_{v \in K} |\sum_{j=1}^{n} \epsilon_j (v^{(j)} - v_0^{(j)})|] \le 9 \int_0^D (\log D(u, K))^{1/2}\, du, \tag{1.5}$$

for any $K \in I\!\!R^n$ and any $v_0 \in K$, where $\{\epsilon_j\}$ is a sequence of Rademacher r.v.'s, $v = (v^{(1)}, \dots, v^{(n)})$ and $D = \sup_{v \in K} |v|$ (see Theorem II.3.1 in Marcus and Pisier, 1981; see also Pollard, 1990, Theorem 3.5). We consider triangular arrays of functions satisfying the following condition:

DEFINITION 1.1. *Given a triangular array of sets $\{S_{n,j} : 1 \le j \le k_n, 1 \le n\}$, a parameter set T and functions $\{f_{n,j}(\cdot, t) : 1 \le j \le k_n, 1 \le n, t \in T\}$, $f_{n,j}(\cdot, t)$ is defined on $S_{n,j}$, we say that the triangular array of functions $\{f_{n,j}(\cdot, t) : 1 \le j \le k_n, 1 \le n, t \in T\}$ is manageable with respect to the envelope functions $\{F_{n,j}(\cdot) : 1 \le j \le k_n, 1 \le n\}$, where $F_{n,j}$ is a function defined on $S_{n,j}$ such that $\sup_{t \in T} |f_{n,j}(x_{n,j}, t)| \le F_{n,j}(x_{n,j})$ for each $x_{n,j} \in S_{n,j}$, if the function $M(u)$, defined on $(0,1)$ by*

$$M(u) := \sup_{n, \tau_{n,j}, x_{n,j}} D\left(u \left(\sum_{j=1}^{k_n} \tau_{n,j}^2 F_{n,j}^2(x_{n,j}) \right)^{1/2}, \mathcal{G}_n(x_{n,1}, \dots, x_{n,k_n}, \tau_{n,1}, \dots, \tau_{n,k_n}) \right),$$

where

$$\mathcal{G}_n(x_{n,1}, \dots, x_{n,k_n}, \tau_{n,1}, \dots, \tau_{n,k_n}) = \{(\tau_{n,1} f_{n,1}(x_{n,1}, t), \dots, \tau_{n,k_n} f_{n,k_n}(x_{n,k_n}, t)) \in I\!\!R^{k_n} : t \in T\}.$$

and the the sup *is taken over $n \ge 1$, $\tau_{n,1}, \dots, \tau_{n,k_n} \in \{0,1\}$ and $x_{n,1} \in S_{n,1}, \dots, x_{n,k_n} \in S_{n,k_n}$, satisfies that $\int_0^1 (\log M(u))^{1/2}\, du < \infty$.*

The last definition is a slight modification of Definition 7.9 in Pollard (1990). The difference between his definition and ours is that he allows $\tau_{n,1}, \dots, \tau_{n,k_n} \ge 0$. Definition 1.1 is a generalization to the triangular array case of the concept of VC subgraph classes, which has been studied by several authors (see for example Vapnik and Červonenkis, 1971, 1981; Dudley, 1978, 1984; Giné and Zinn 1984, 1986; Pollard 1984, 1990; and Alexander, 1987a, 1987b). We refer to Pollard (1990) for ways to check Definition 1.1. Observe that by (1.5), for a manageable class and a sequence of Rademacher r.v.'s $\{\epsilon_j\}_{j=1}^{\infty}$,

$$E[\sup_{t \in T} |\sum_{j=1}^{k_n} \epsilon_j \tau_{n,j} (f_{n,j}(x_{n,j}, t) - f_{n,j}(x_{n,j}, t_0))|] \le 9 \int_0^1 (\log M(u))^{1/2}\, du \left(\sum_{j=1}^{k_n} \tau_{n,j}^2 F_{n,j}^2(x_{n,j}) \right)^{1/2}, \tag{1.6}$$

for each $\tau_{n,1}, \dots, \tau_{n,k_n} \in \{0,1\}$, each $x_{n,1} \in S_{n,1}, \dots, x_{n,k_n} \in S_{n,k_n}$ and each $t_0 \in T$. This inequality will allow us to obtain the pertinent weak limit theorems.

Triangular arrays of empirical processes have ben considered by several authors. Alexander (1987a) and Pollard (1990, Theorem 10.6) consider triangular arrays of empirical processes whose limit distribution is a Gaussian process. More work in triangular arrays, mostly in their relation with partial–sum processes, can be found in Arcones, Gaenssler and Ziegler (1992); Gaenssler and Ziegler (1994); Gaenssler (1994); and Ziegler (1997).

Triangular arrays of empirical process with a Gaussian limit appear in statistics very often (see for example Pollard, 1984, 1990; Le Cam, 1986; Kim and Pollard, 1990; and Arcones, 1994). For an application to M–estimation of the presented results see Arcones (1996). In this reference, the convergence of M–estimators to a stable limit distribution is considered. These results are not possible without the contribution in this paper.

2. Weak convergence of row sums of a triangular array of empirical processes indexed by a manageable triangular array of functions. In this section, we give sufficient conditions for the weak convergence of the stochastic processes in (1.1) when the class of functions $\{f_{n,j}(x_{n,j},t) : 1 \le j \le k_n,\ t \in T\}$ is manageable with respect to some triangular array of envelope functions $\{F_{n,j}(x_{n,j}) : 1 \le j \le k_n\}$. Every $F_{n,j}(x_{n,j})$ is bigger than or equal to $\sup_{t\in T}|f_{n,j}(x_{n,j},t)|$, but it is not necessarily the smallest r.v. satisfying this property. We call a finite partition π of T to a map $\pi : T \to T$ such that $\pi(\pi(t)) = \pi(t)$ for each $t \in T$, and the cardinality of $\{\pi(t) : t \in T\}$ is finite.

THEOREM 2.1. *With the above notation, let $b > 0$, suppose that:*

(i) The finite dimensional distributions of $\{Z_n(t) := \left(\sum_{j=1}^{k_n} f_{n,j}(X_{n,j},t)\right) - c_n(t) : t \in T\}$ converge to those of $\{Z(t) : t \in T\}$.

(ii) The triangular array of functions $\{f_{n,j}(\cdot,t) : 1 \le j \le k_n, 1 \le n, t \in T\}$ is manageable with respect to the envelope functions $\{F_{n,j}(\cdot) : 1 \le j \le k_n, 1 \le n\}$.

(iii) For each $t \in T$, $\sup_{n\ge 1}\sum_{j=1}^{k_n}\Pr\{|f_{n,j}(X_{n,j},t)| \ge 2^{-1}b\} < \infty$.

(iv) For each $\eta > 0$, there exists a finite partition π of T such that

$$\limsup_{n\to\infty}\sum_{j=1}^{k_n}\Pr{}^*\{\sup_{t\in T}|f_{n,j}(X_{n,j},t) - f_{n,j}(X_{n,j},\pi(t))| \ge \eta\} \le \eta.$$

(v) $\sup_{n\ge 1} E[\sum_{j=1}^{k_n} F_{n,j}^2(X_{n,j}) I_{F_{n,j}(X_{n,j})\le b}] < \infty$.

(vi) For each $\eta > 0$, there exists a finite partition π of T such that

$$\limsup_{n\to\infty}\sup_{t\in T}\sum_{j=1}^{k_n} E[(f_{n,j}(X_{n,j},t) - f_{n,j}(X_{n,j},\pi(t)))^2 I_{F_{n,j}(X_{n,j})\le b}] \le \eta.$$

(vii) For each $\eta > 0$, there exists a finite partition π of T such that

$$\limsup_{n\to\infty}\sup_{t\in T}|E[S_n(t,b) - S_n(\pi(t),b)] - c_n(t) + c_n(\pi(t))| \le \eta,$$

where $S_n(t,b) = \sum_{j=1}^{k_n} f_{n,j}(X_{n,j},t) I_{F_{n,j}(X_{n,j})\le b}$.

Then,

$$\{Z_n(t) : t \in T\} \xrightarrow{w} \{Z(t) : t \in T\}.$$

PROOF. By Theorem 2.1 in Arcones (1998), it suffices to show that for each $\eta > 0$, there exists a finite partition π of T such that

$$\limsup_{n\to\infty}\Pr\{\sup_{t\in T}|\sum_{j=1}^{k_n}\epsilon_j(f_{n,j}(X_{n,j},t) - f_{n,j}(X_{n,j},\pi(t)))I_{F_{n,j}(X_{n,j})\le b}| \ge \eta\} \le 4\eta. \tag{2.1}$$

Take $a > 0$, $\delta > 0$, $\tau > 0$ and a finite partition π of T, in this order, such that

(2.2) $$\eta^{-1} \limsup_{n\to\infty} E[\sum_{j=1}^{k_n} F_{n,j}^2(X_{n,j}) I_{F_{n,j}(X_{n,j})\le b}] < a,$$

$$36a^{1/2} \int_0^{2^{-1/2}a^{-1/2}\delta} (\log M(u))^{1/2}\, du \le \eta^2,$$

$$144\tau \limsup_{n\to\infty} E[(\sum_{j=1}^{k_n} F_{n,j}^2(X_{n,j}) I_{F_{n,j}(X_{n,j})\le b})^{1/2}] \int_0^1 (\log M(u))^{1/2}\, du < \eta\delta^2,$$

$$\limsup_{n\to\infty} \sup_{t\in T} \sum_{j=1}^{k_n} E[(f_{n,j}(X_{n,j},t) - f_{n,j}(X_{n,j},\pi(t)))^2 I_{F_{n,j}(X_{n,j})\le b}] < \delta^2$$

and

$$\limsup_{n\to\infty} \delta^{-2} 8b^2 \sum_{j=1}^{k_n} \Pr\{\sup_{t\in T} |f_{n,j}(X_{n,j},t) - f_{n,j}(X_{n,j},\pi(t))| \ge \tau\} \le \eta.$$

Observe that by taking a refinement of partitions, we can get a partition so that both conditions (iv) and (vi) hold simultaneously. We have that

$$\Pr\{\sup_{t\in T} |\sum_{j=1}^{k_n} \epsilon_j (f_{n,j}(X_{n,j},t) - f_{n,j}(X_{n,j},\pi(t))) I_{F_{n,j}(X_{n,j})\le b}| \ge \eta\}$$

$$\le \Pr\{\sum_{j=1}^{k_n} F_{n,j}^2(X_{n,j}) I_{F_{n,j}(X_{n,j})\le b} \ge a\}$$

$$+ \Pr\{\sup_{t\in T} \sum_{j=1}^{k_n} (f_{n,j}(X_{n,j},t) - f_{n,j}(X_{n,j},\pi(t)))^2 I_{F_{n,j}(X_{n,j})\le b} \ge 2\delta^2\}$$

$$+ \Pr\{A \cap \{\sup_{t\in T} |\sum_{j=1}^{k_n} \epsilon_j (f_{n,j}(X_{n,j},t) - f_{n,j}(X_{n,j},\pi(t))) I_{F_{n,j}(X_{n,j})\le b}| \ge \eta\}\}$$

$$=: I + II + III,$$

where

$$A := \left\{ \sum_{j=1}^{k_n} F_{n,j}^2(X_{n,j}) I_{F_{n,j}(X_{n,j})\le b} < a, \right.$$

$$\left. \sup_{t\in T} \sum_{j=1}^{k_n} (f_{n,j}(X_{n,j},t) - f_{n,j}(X_{n,j},\pi(t)))^2 I_{F_{n,j}(X_{n,j})\le b} < 2\delta^2 \right\}.$$

By (2.2),

$$I \le a^{-1} E[\sum_{j=1}^{k_n} F_{n,j}^2(X_{n,j}) I_{F_{n,j}(X_{n,j})\le b}] \le \eta,$$

for n large enough. We have that

$$II \le \Pr\{\sup_{t\in T} |\sum_{j=1}^{k_n} (f_{n,j}(X_{n,j},t) - f_{n,j}(X_{n,j},\pi(t)))^2 I_{F_{n,j}(X_{n,j})\le b}$$

$$-E[(f_{n,j}(X_{n,j},t)-f_{n,j}(X_{n,j},\pi(t)))^2 I_{F_{n,j}(X_{n,j})\le b}])|\ge\delta^2\}$$

$$\le 2\delta^{-2}E[\sup_{t\in T}|\sum_{j=1}^{k_n}\epsilon_j(f_{n,j}(X_{n,j},t)-f_{n,j}(X_{n,j},\pi(t)))^2 I_{F_{n,j}(X_{n,j})\le b}|]$$

$$\le 8b^2\delta^{-2}\sum_{j=1}^{k_n}\Pr\{\sup_{t\in T}|f_{n,j}(X_{n,j},t)-f_{n,j}(X_{n,j},\pi(t))|\ge\tau\}$$

$$+2\delta^{-2}E[\sup_{t\in T}|\sum_{j=1}^{k_n}\epsilon_j(f_{n,j}(X_{n,j},t)-f_{n,j}(X_{n,j},\pi(t)))^2 I_{F_{n,j}(X_{n,j})\le b,\ \sup_{t\in T}|f_{n,j}(X_{n,j},t)-f_{n,j}(X_{n,j},\pi(t))|<\tau}|]$$

$$\le\eta$$

$$+2\delta^{-2}E[\sup_{t\in T}|\sum_{j=1}^{k_n}\epsilon_j(f_{n,j}(X_{n,j},t)-f_{n,j}(X_{n,j},\pi(t)))^2 I_{F_{n,j}(X_{n,j})\le b,\ \sup_{t\in T}|f_{n,j}(X_{n,j},t)-f_{n,j}(X_{n,j},\pi(t))|<\tau}|].$$

By (4.19) in Ledoux and Talagrand (1991) and (1.6),

$$2\delta^{-2}E[\sup_{t\in T}|\sum_{j=1}^{k_n}\epsilon_j(f_{n,j}(X_{n,j},t)-f_{n,j}(X_{n,j},\pi(t)))^2 I_{F_{n,j}(X_{n,j})\le b,\ \sup_{t\in T}|f_{n,j}(X_{n,j},t)-f_{n,j}(X_{n,j},\pi(t))|<\tau}|].$$

$$\le 8\tau\delta^{-2}E[\sup_{t\in T}|\sum_{j=1}^{k_n}\epsilon_j(f_{n,j}(X_{n,j},t)-f_{n,j}(X_{n,j},\pi(t))I_{F_{n,j}(X_{n,j})\le b}|]$$

$$\le 16\tau\delta^{-2}E[\sup_{t\in T}|\sum_{j=1}^{k_n}\epsilon_j(f_{n,j}(X_{n,j},t)-f_{n,j}(X_{n,j},t_0))I_{F_{n,j}(X_{n,j})\le b}|]$$

$$\le 144\tau\delta^{-2}E[\int_0^{D_n}(\log D(u,\mathcal{F}_n))^{1/2}\,du],$$

where $t_0\in T$,

$$\mathcal{F}_n:=\{(f_{n,1}(X_{n,1},t)I_{F_{n,1}(X_{n,1})\le b},\ldots,f_{n,k_n}(X_{n,k_n},t)I_{F_{n,k_n}(X_{n,k_n})\le b})\in\mathbb{R}^{k_n}:t\in T\}$$

and

$$D_n^2:=\sup_{t\in T}\sum_{j=1}^{k_n}f_{n,j}^2(X_{n,j},t)I_{F_{n,j}(X_{n,j})\le b}\le\sum_{j=1}^{k_n}F_{n,j}^2(X_{n,j})I_{F_{n,j}(X_{n,j})\le b}.$$

By (ii),

$$D(u,\mathcal{F}_n)\le M(u(\sum_{j=1}^{k_n}F_{n,j}^2(X_{n,j})I_{F_{n,j}(X_{n,j})\le b})^{-1/2}).$$

So,

$$2\delta^{-2}E[\sup_{t\in T}|\sum_{j=1}^{k_n}\epsilon_j((f_{n,j}(X_{n,j},t)-f_{n,j}(X_{n,j},\pi(t)))^2 I_{F_{n,j}(X_{n,j})\le b}|]$$

$$\le 144\tau\delta^{-2}\int_0^1(\log M(u))^{1/2}\,du\ E[(\sum_{j=1}^{k_n}F_{n,j}^2(X_{n,j})I_{F_{n,j}(X_{n,j})\le b})^{1/2}]\le\eta,$$

for n large enough.

By (1.5),

$$III \le 9\eta^{-1} E[I_A \int_0^{D'_n} (\log D(u, \mathcal{F}'_n))^{1/2}\, du],$$

where

$$\mathcal{F}'_n := \{(f_{n,1}(X_{n,1}, t) - f_{n,1}(X_{n,1}, \pi(t)), \dots, f_{n,k_n}(X_{n,k_n}, t) - f_{n,k_n}(X_{n,k_n}, \pi(t))) \in I\!R^{k_n} : t \in T\}$$

and

$$D'^2_n := \sup_{t\in T} \sum_{j=1}^{k_n} (f_{n,j}(X_{n,j}, t) - f_{n,j}(X_{n,j}, \pi(t)))^2 I_{F_{n,j}(X_{n,j}) \le b}.$$

In A, $D'^2_n \le 2\delta^2$. We have that

$$(\log D(u, \mathcal{F}'_n))^{1/2} \le 2^{1/2} (\log D(2^{-1}u, \mathcal{F}_n))^{1/2} \le 2^{1/2} (\log M(2^{-1} a^{-1/2} u))^{1/2}.$$

So,

$$III \le 18\eta^{-1} \int_0^{2^{1/2}\delta} (\log M(2^{-1}a^{-1/2}u))^{1/2}\, du$$

$$\le 36 a^{1/2} \eta^{-1} \int_0^{2^{-1/2}a^{-1/2}\delta} (\log M(u))^{1/2}\, du \le \eta$$

From all these estimations, (2.1) follows. □

Condition (i) in previous theorem is a necessary condition. Condition (iii) in Theorem 2.1 is a very weak condition. Under some regularity conditions, conditions (iv) and (vii) in Theorem 2.1 are also necessary (see Theorem 2.2 in Arcones, 1998). Observe that by the Hoffmann–Jørgensen inequality, (v) is equivalent to

$$\sup_{n\ge 1} E[|\sum_{j=1}^{k_n} \epsilon_j F_{n,j}(X_{n,j}) I_{F_{n,j}(X_{n,j})\le b}|] < \infty.$$

So, using the second moment, we are not imposing a stronger condition. Under some regularity conditions, the following condition is also necessary: for each $\eta > 0$, there exists a finite partition π of T such that

$$\limsup_{n\to\infty} E^*[\sup_{t\in T} |S_n(t, b) - S_n(\pi(t), b) - E[S_n(t, b) - S_n(\pi(t), b)]|^2] \le \eta.$$

So, for each $\eta > 0$, there exists a finite partition π of T such that

$$\limsup_{n\to\infty} \sup_{t\in T} \sum_{j=1}^{k_n} \mathrm{Var}((f_{n,j}(X_{n,j}, t) - f_{n,j}(X_{n,j}, \pi(t))) I_{F_{n,j}(X_{n,j})\le b}) \le \eta.$$

This means that condition (vi) in the previous theorem is close to be a necessary condition. It is a necessary condition when the r.v.'s are symmetric (under some regularity conditions).

Next, we consider the case of random series processes.

Theorem 2.2. *With the notation in (1.3), let $b > 0$. Suppose that:*

(i) For each $t \in T$, $\sum_{j=1}^n f_j(X_j,t)$ converges in distribution.

(ii) The triangular array of functions $\{f_j(\cdot,t) : 1 \le j \le n, t \in T\}$ is manageable with respect to the envelope functions $\{F_j(\cdot) : 1 \le j \le n\}$.

(iii) For each $\eta > 0$, there exists a finite partition π of T such that

$$\sum_{j=1}^{\infty} \Pr{}^*\{\sup_{t\in T} |f_j(X_j,t) - f_j(X_j,\pi(t))| \ge \eta\} \le \eta.$$

(iv) $\sum_{j=1}^{\infty} E[F_j^2(X_j) I_{F_j(X_j)\le b}] < \infty$.

(v) For each $\eta > 0$, there exists a finite partition π of T such that

$$\sup_{t\in T} \sum_{j=1}^{\infty} E[(f_j(X_j,t) - f_j(X_j,\pi(t)))^2 I_{F_j(X_j)\le b}] \le \eta.$$

(vi) For each $\eta > 0$, there exists a finite partition π of T such that

$$\sup_{t\in T} |\sum_{j=1}^{\infty} E[(f_j(X_j,t) - f_j(X_j,\pi(t))) I_{F_j(X_j)\le b}]| \le \eta.$$

Then, $\{\sum_{j=1}^n f_j(X_j,t) : t \in T\}$ converges weakly.

Proof. We apply Theorem 2.1. Conditions (i), (ii), (iv) and (v) in Theorem 2.1 are assumed. Condition (iii) follows from the three series theorem. As to condition (vi), we have to prove that for each $\eta > 0$, there exists a finite partition π of t such that

$$\sup_{t\in T} \sum_{j=1}^{\infty} E[(f_j(X_j,t) - f_j(X_j,\pi(t)))^2 I_{F_j(X_j)\le b}] \le \eta.$$

Take $m < \infty$ such that

$$\sum_{j=m+1}^{\infty} E[F_j^2(X_j) I_{F_j(X_j)\le b}] \le 2^{-2}\eta. \tag{2.3}$$

Take a finite partition π of T such that

$$\sum_{j=1}^{\infty} \Pr{}^*\{\sup_{t\in T} |f_j(X_j,t) - f_j(X_j,\pi(t))| \ge m^{-1/2} 2^{-1} \eta^{1/2}\} \le 2^{-2} b^{-2} \eta.$$

Then,

$$\sup_{t\in T} \sum_{j=1}^{m} E[(f_j(X_j,t) - f_j(X_j,\pi(t))^2 I_{F_j(X_j)\le b}] \le 2^{-1}\eta.$$

By (2.3),

$$\sup_{t\in T} \sum_{j=m+1}^{\infty} E[(f_j(X_j,t) - f_j(X_j,\pi(t))^2 I_{F_j(X_j)\le b}] \le 2^{-1}\eta.$$

Hence, the claim follows. □

By the Ito–Nisio theorem in $l_\infty(T)$ the convergence of $\{\sum_{j=1}^n f_j(X_j,t) : t \in T\}$ in the previous theorem holds outer almost surely. A proof of this fact can be found in Proposition A.13 in van der Vaart and Wellner (1996). We must notice that this proposition in van der Vaart and Wellner (1996) is wrong: it is not true that the outer a.s. convergence of $\{\sum_{j=1}^n f_j(X_j,t) : t \in T\}$ implies the weak convergence of this sequence (see the remark after Theorem 2.3 in Arcones, 1998).

The proof of Theorem 2.1, above, and Theorem 2.5 in Arcones (1998) give that:

THEOREM 2.3. *With the above notation, let $b > 0$, suppose that:*

(i) For each $\eta > 0$,

$$\sum_{j=1}^{k_n} \Pr\{F_{n,j}(X_{n,j}) \geq \eta\} \to 0.$$

(ii) For each $s,t \in T$, the following limit exists

$$\lim_{n\to\infty} \sum_{j=1}^{k_n} \mathrm{Cov}(f_{n,j}(X_{n,j},s) I_{|f_{n,j}(X_{n,j},s)|\leq b}, f_{n,j}(X_{n,j},t) I_{|f_{n,j}(X_{n,j},t)|\leq b}).$$

(iii) The triangular array of functions $\{f_{n,j}(\cdot,t) : 1 \leq j \leq k_n, 1 \leq n, t \in T\}$ is manageable with respect to the envelope functions $\{F_{n,j}(\cdot) : 1 \leq j \leq k_n, 1 \leq n\}$.

(iv) $\sup_{n\geq 1} E[\sum_{j=1}^{k_n} F_{n,j}^2(X_{n,j}) I_{F_{n,j}(X_{n,j})\leq b}] < \infty$.

(v) For each $\eta > 0$, there exists a finite partition π of T such that

$$\limsup_{n\to\infty} \sup_{t\in T} \sum_{j=1}^{k_n} E[(f_{n,j}(X_{n,j},t) - f_{n,j}(X_{n,j},\pi(t)))^2 I_{|f_{n,j}(X_{n,j},t) - f_{n,j}(X_{n,j},\pi(t))|\leq b}] \leq \eta.$$

Then,

$$\{Z_n(t) := \sum_{j=1}^{k_n} (f_{n,j}(X_{n,j},t) - E[f_{n,j}(X_{n,j},t) I_{|f_{n,j}(X_{n,j},t)|\leq b}]) : t \in T\} \xrightarrow{w} \{Z(t) : t \in T\},$$

where $\{Z(t) : t \in T\}$ is a mean–zero Gaussian process with covariance given by

$$E[Z(s)Z(t)] = \lim_{n\to\infty} \sum_{j=1}^{k_n} \mathrm{Cov}(f_{n,j}(X_{n,j},s) I_{|f_{n,j}(X_{n,j},s)|\leq b}, f_{n,j}(X_{n,j},t) I_{|f_{n,j}(X_{n,j},t)|\leq b}).$$

Under regularity conditions (i) and (ii) in Theorem 2.3 are necessary conditions for the weak convergence of $\{Z_n(t) : t \in T\}$ to a Gaussian process. Last corollary is related with Theorem 10.6 in Pollard (1990) (see also theorems 2.2 and 2.7 in Alexander, 1987a). Observe that under condition (i), (v) is equivalent to:

(v)' For each $\eta > 0$, there exists a finite partition π of T such that

$$\limsup_{n\to\infty} \sup_{t\in T} \sum_{j=1}^{k_n} E[(f_{n,j}(X_{n,j},t) - f_{n,j}(X_{n,j},\pi(t)))^2 I_{F_{n,j}(X_{n,j})\leq b}] \leq \eta.$$

The following corollary follows directly from Theorem 2.3.

COROLLARY 2.4. *Let* $\{X_j\}_{j=1}^{\infty}$ *be a sequence of i.i.d.r.v.'s with values in a measurable space* $(S, \mathcal{S})$, *let* $f(\cdot, t) : S \to I\!R$ *be a measurable function for each* $t \in T$, *let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence of positive numbers regularly varying of order* $1/2$, *let* $F(x)$ *be a measurable function such that* $F(X) \geq \sup_{t \in T} |f(X, t)|$ *and let* $b > 0$. *Suppose that:*

(i) For each $\eta > 0$,

$$n \Pr\{F(X) \geq a_n \eta\} \to 0.$$

(ii) For each $s, t \in T$, *the following limit exists*

$$\lim_{n \to \infty} n a_n^{-2} \mathrm{Cov}(f(X, s) I_{|f(X,s)| \leq b a_n}, f(X, t) I_{|f(X,t)| \leq b a_n}).$$

(iii) The triangular array of functions $\{(f(x_1, t), \ldots, f(x_n, t)) : 1 \leq n, t \in T\}$ *is manageable with respect to the envelope functions* $\{(F(x_1), \ldots, F(x_n)) : 1 \leq n\}$.

(iv) $\sup_{n \geq 1} n a_n^{-2} E[F^2(X) I_{F(X) \leq b a_n}] < \infty$.

(v) For each $\eta > 0$, *there exists a finite partition* π *of* T *such that*

$$\limsup_{n \to \infty} \sup_{t \in T} n a_n^{-1} E[(f(X, t) - f(X, \pi(t)))^2 I_{|f(X,t) - f(X, \pi(t))| \leq a_n}] \leq \eta.$$

Then,

$$\{a_n^{-1} \sum_{j=1}^{n} (f(X_j, t) - E[f(X_j, t)]) : t \in T\} \xrightarrow{w} \{Z(t) : t \in T\},$$

where $\{Z(t) : t \in T\}$ *is a mean–zero Gaussian process with*

$$E[Z(s)Z(t)] = \lim_{n \to \infty} n a_n^{-2} \mathrm{Cov}(f(X, s) I_{|f(X,s)| \leq a_n}, f(X, t) I_{|f(X,t)| \leq a_n}),$$

for each $s, t \in T$.

When $a_n = n^{1/2}$, Alexander (1987b) obtained necessary and sufficient conditions for the CLT of empirical processes indexed by VC classes.

Next, we consider the weak convergence of $\{Z_n(t) : t \in T\}$ to an infinitely divisible process without Gaussian part.

THEOREM 2.5. *With the notation corresponding to the processes in (1.1), let* $b > 0$, *suppose that:*

(i) The finite dimensional distributions of $\{Z_n(t) : t \in T\}$ *converge.*

(ii) For each $\eta > 0$,

$$\max_{1 \leq j \leq k_n} \sup_{t \in T} \Pr\{|f_{n,j}(X_{n,j}, t)| \geq \eta\} \to 0,$$

(iii) For each $\eta > 0$, *there exists a finite partition* π *of* T *such that*

$$\limsup_{n \to \infty} \sum_{j=1}^{k_n} \Pr{}^*\{\sup_{t \in T} |f_{n,j}(X_{n,j}, t) - f_{n,j}(X_{n,j}, \pi(t))| \geq \eta\} \leq \eta.$$

(iv) The triangular array of functions $\{f_{n,j}(\cdot, t) : 1 \leq j \leq k_n, 1 \leq n, t \in T\}$ *is manageable with respect to the envelope functions* $\{F_{n,j}(\cdot, t) : 1 \leq j \leq k_n, 1 \leq n\}$.

(v)

$$\lim_{\delta\to 0}\limsup_{n\to\infty} E[\sum_{j=1}^{k_n} F_{n,j}^2(X_{n,j}) I_{F_{n,j}(X_{n,j})\le\delta}] = 0.$$

(vi) For each $\eta > 0$, there exists a finite partition π of T such that

$$\limsup_{n\to\infty}\sup_{t\in T} |E[S_n(t,b) - S_n(\pi(t),b)] - c_n(t) + c_n(\pi(t))| \le \eta.$$

Then,

$$\{Z_n(t) : t \in T\} \xrightarrow{w} \{Z(t) : t \in T\}.$$

PROOF. By Theorem 2.9 in Arcones (1998), we have to prove that

$$\lim_{\delta\to 0}\limsup_{n\to\infty} E^*[\sup_{t\in T} |S_n(t,\delta) - E[S_n(t,\delta)]|] = 0.$$

By (1.6) and conditions (iv) and (v),

$$E^*[\sup_{t\in T} |S_n(t,\delta) - E[S_n(t,\delta)]|]$$

$$\le 2E[\sup_{t\in T} |\sum_{j=1}^{k_n} \epsilon_j f_{n,j}(X_{n,j},t) I_{F_{n,j}(X_{n,j})\le\delta}|]$$

$$\le 2E[|\sum_{j=1}^{k_n} \epsilon_j f_{n,j}(X_{n,j},t_0) I_{F_{n,j}(X_{n,j})\le\delta}|] + 18\int_0^1 (\log M(u))^{1/2}\,du\ E[(\sum_{j=1}^{k_n} F_{n,j}^2(X_{n,j}) I_{F_{n,j}(X_{n,j})\le\delta})^{1/2}]$$

$$\le (2 + 18\int_0^1 (\log M(u))^{1/2}\,du)(E[\sum_{j=1}^{k_n} F_{n,j}^2(X_{n,j}) I_{F_{n,j}(X_{n,j})\le\delta}])^{1/2} \to 0,$$

where $t_0 \in T$. So, the claim follows. $\square$

As to the case of stable limits, we have that:

COROLLARY 2.6. *With the notation corresponding to the processes in (1.2), let $1 < \alpha < 2$ and let $b > 0$, suppose that:*

(i) $a_n \nearrow \infty$ and a_n is regularly varying of order α^{-1}.

(ii) For each $\lambda_1, \dots, \lambda_m \in I\!R$ and each $t_1, \dots, t_m \in T$, there exists a finite constant $N(\lambda_1, \dots, \lambda_m, t_1, \dots, t_m)$ such that

$$\lim_{n\to\infty} n\Pr\{\sum_{l=1}^{m} \lambda_l f(X, t_l) \ge u a_n\} = \alpha^{-1} u^{-\alpha} N(\lambda_1, \dots, \lambda_m, t_1, \dots, t_m),$$

for each $u > 0$.

(iii) For each $\eta > 0$, there exists a finite partition π of T such that

$$\limsup_{n\to\infty} n\Pr\{\sup_{t\in T} |f(X,t) - f(X,\pi(t))| \ge a_n\eta\} \le \eta.$$

(iv) The triangular array of functions $\{(f(x_1,t), \dots, f(x_n,t)) : 1 \le n, t \in T\}$ is manageable with respect to the envelope functions $\{(F(x_1), \dots, F(x_n)) : 1 \le n\}$.

(v) $\sup_{n\geq 1} n \Pr\{F(X) \geq b a_n\} < \infty$.

Then, the sequence of stochastic processes

$$\{Z_n(t) := a_n^{-1} \sum_{j=1}^{n} (f(X_j,t) - E[f(X,t)]) : t \in T\}, \ n \geq 1,$$

converges weakly.

PROOF. We apply Theorem 2.5. It is easy to see that conditions (i)–(iv) in this theorem are satisfied. By Lemma 2.7 in Arcones (1998),

$$\lim_{\delta\to 0} \limsup_{n\to\infty} n a_n^{-2} E[F^2(X) I_{F(X)\leq \delta a_n}] = 0$$

and condition (iv) follows. Condition (vi) in Theorem 2.5 follows similarly. □

Observe that (ii) and (iii) are necessary conditions for the weak convergence of $\{Z_n(t) : t \in T\}$. Also note that if $F(x) = \sup_{t\in T} |f(x,t)|$, then (v) is implied by (ii) and (iii). Last corollary is related with the work in Romo (1993). Among several differences, Romo (1993) only considered the case $a_n = n^{1/\alpha}$. Observe that it is not clear from the work in Romo (1993) when the sequence of functions $\{n\mathcal{L}(n^{1/\alpha}\delta_{X_1}) I_{\|x\|_{\mathcal{F}}\geq\delta}\}_{n=1}^{\infty}$ is tight (see Theorem 2.1 in the cited reference). Instead, Corollary 2.6 has conditions ready to use. The proof of the following corollary is similar to that of the last corollary and it is omitted.

COROLLARY 2.7. *Let $b > 0$. Under the notation corresponding to the processes in (1.2), suppose that:*

(i) $a_n \nearrow \infty$ and a_n is regularly varying of order 1.

(ii) For each $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ and each $t_1, \ldots, t_m \in T$, there exists a finite constant $N(\lambda_1, \ldots, \lambda_m, t_1, \ldots, t_m)$ such that

$$\lim_{n\to\infty} n \Pr\{\sum_{l=1}^{m} \lambda_l f(X, t_l) \geq u a_n\} = u^{-1} N(\lambda_1, \ldots, \lambda_m, t_1, \ldots, t_m),$$

for each $u > 0$.

(iii) For each $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ and each $t_1, \ldots, t_m \in T$, the following limit exists

$$\lim_{n\to\infty} n a_n^{-1} E[\sum_{l=1}^{m} \lambda_l f(X, t_l) I_{|\sum_{l=1}^{m} \lambda_l f(X,t_l)|\leq b a_n}] - \sum_{l=1}^{m} \lambda_l c_n(t_l).$$

(iv) For each $\eta > 0$, there exists a finite partition π of T such that

$$\limsup_{n\to\infty} n \Pr\{\sup_{t\in T} |f(X,t) - f(X,\pi(t))| \geq a_n \eta\} \leq \eta.$$

(v) The triangular array of functions $\{(f(x_1,t), \ldots, f(x_n,t)) : 1 \leq n, t \in T\}$ is manageable with respect to the envelope functions $\{(F(x_1), \ldots, F(x_n)) : 1 \leq n\}$.

(vi) $\sup_{n\geq 1} n \Pr\{F(X) \geq b a_n\} < \infty$.

(vii) For each $\eta > 0$, there exists a finite partition π of T such that

$$\lim_{n\to\infty} \sup_{t\in T} |n a_n^{-1} E[(f(X,t) - f(X,\pi(t))) I_{F(X)\leq a_n}] - c_n(t) + c_n(\pi(t))| \leq \eta.$$

Then, the sequence of stochastic processes

$$\{(a_n^{-1}\sum_{j=1}^{n} f(X_j,t)) - c_n(t) : t \in T\},\ n \geq 1,$$

converges weakly.

3. An application to linear regression. In this section, we give an application of Theorem 2.1 to linear regression. We consider the simple linear regression model without a constant term, that is we assume that: $Y_{n,j} = z_{n,j}\theta_0 + U_j$, $1 \leq j \leq n$ where $\{U_j\}_{j=1}^{\infty}$ is a sequence of i.i.d.r.v.'s; $\{z_{n,j} : 1 \leq j \leq n\}$ are real numbers and $\theta_0 \in I\!R^d$ is a parameter to be estimated. As it is well known, this model represents a linear relation between the variables y and z, where U_j is an error term. $z_{n,j}$ is called the regressor or predictor variable, and usually it can be chosen arbitrarily. Y_j is called the response variable. The problem is to estimate θ_0 from the data $(z_{n,1}, Y_{n,1}), \ldots, (z_{n,n}, Y_{n,n})$.

The usual estimator of θ_0 is the least squares (LS) estimator (see for example Draper and Smith, 1981). The problem with the least squares estimator is that it is not robust. A common alternative to the LS estimator is the least absolute deviations (LAD) estimator. The LAD estimator $\hat{\theta}_n$ is defined as

$$\sum_{i=1}^{n} |Y_i - z_i'\hat{\theta}_n| = \inf_{\theta \in I\!R^d} \sum_{i=1}^{n} |Y_i - z_i'\theta|.$$

A nice discussion on these estimators is in Portnoy and Koenker (1997).

In this section, we obtain the asymptotic distribution of the LAD estimator for a particular choice of the regressor variables $z_{n,1}, \ldots, z_{n,n}$.

We will need the following lemma:

LEMMA 3.1. Let Θ be a Borel subset of $I\!R^d$. Let $\{G_n(\eta) : \eta \in \Theta\}$ be a sequence of stochastic processes. Let $\{\hat{\eta_n}\}$ be a sequence of $I\!R^d$–valued random variables. Suppose that:

(i) $G_n(\hat{\eta}_n) \leq \inf_{\eta \in \Theta} G_n(\eta) + o_P(1)$.

(ii) $\hat{\eta}_n = 0_P(1)$.

(iii) There exists a stochastic process $\{G(\eta) : \eta \in I\!R^d\}$ such that for each $M < \infty$,

$$\{G_n(\eta) : |\eta| \leq M\}$$

converges weakly to $\{G(\eta) : |\eta| \leq M\}$.

(v) With probability one, the stochastic process $\{G(\eta) : \eta \in I\!R^d\}$ has a unique minimum at $\tilde{\eta}$; and for each $\delta > 0$ and for each $M < \infty$ with $|\tilde{\eta}| \leq M$,

$$\inf_{\substack{|\eta| \leq M \\ |\eta - \tilde{\eta}| > \delta}} G(\eta) > G(\tilde{\eta}).$$

Then, $\hat{\eta}_n \xrightarrow{d} \tilde{\eta}$.

The proof of the previous lemma is omitted. Similar results have been used by many authors.

THEOREM 3.2. Suppose that the following conditions are satisfied:

(i) $F_U(0) = 2^{-1}$, $F_U(u)$ is differentiable at $u = 0$ and $F'_U(0) > 0$, where $F_U(u) = \Pr\{U \le u\}$.
(ii) $E[|U|] < \infty$.
(iii) For each $j \ge 1$, $\lim_{n\to\infty} y_{n,j} =: y_j$ exists, where $y_{n,j} = a_n^{-1} z_{n,j}$ and $a_n^2 = \sum_{j=1}^n z_{n,j}^2$.
(iv) $\lim_{m\to\infty} \limsup_{n\to\infty} \max_{m\le j\le n} |y_{n,j}| = 0$.
(v) $\sum_{j=1}^\infty |y_j| < \infty$.
Then, $a_n(\hat\theta_n - \theta_0)$ converges in distribution to $\tilde\eta$, where $\tilde\eta$ is the value of η that minimizes

$$G(\eta) := \sigma g \eta + \sigma^2 F'_U(0)\eta^2 + \sum_{j=1}^\infty (|U_j - y_j\eta| - |U_j|),$$

where $\sigma^2 = 1 - \sum_{j=1}^\infty y_j^2$ and g is a standard normal r.v. independent of the sequence $\{U_j\}$.

Proof. We apply Lemma 3.1 with $G_n(\eta) = \sum_{j=1}^n (|U_j - y_{n,j}\eta| - |U_j|)$ and $\hat\eta_n = a_n(\hat\theta_n - \theta_0)$. Observe that since $\hat\theta_n$ is the value that minimizes $\sum_{j=1}^n (|Y_j - z_{n,j}\theta| - |U_j|) = \sum_{j=1}^n (|U_j - z_{n,j}(\theta - \theta_0)| - |U_j|)$, $a_n(\hat\theta_n - \theta_0)$ is the value that minimizes $\sum_{j=1}^n (|U_j - y_{n,j}\eta| - |U_j|)$.

First, we prove by using Theorem 2.1 that for each $M < \infty$, $\{G_n(\eta) : |\eta| \le M\}$ converges weakly to $\{G(\eta) : |\eta| \le M\}$.

The set of $\mathbb{R}^n$, $\{(U_1 - y_{n,1}\eta, \ldots, U_n - y_{n,n}\eta) : \eta \in \mathbb{R}\}$ lies in a linear space of dimension one. So, it has pseudodimension 1 (in the sense of Pollard, 1990). Since $|U_j - y_{n,j}\eta| = \max(U_j - y_{n,j}\eta, 0) + \max(-U_j + y_{n,j}\eta, 0)$ and these operations maintain the pesudodimension bounded, we conclude the manageability of the triangular array $\{(U_1 - y_{n,1}\eta, \ldots, U_n - y_{n,n}\eta) : \eta \in \mathbb{R}\}$.

To prove convergence of the finite dimensional distributions, we need to prove that for each $\eta_1, \ldots, \eta_p, \lambda_1 \ldots, \lambda_p \in \mathbb{R}$,

$$\log E[\exp(it \sum_{j=1}^n \sum_{k=1}^p \lambda_k(|U_j - y_{n,j}\eta_k| - |U_j|))] \tag{3.1}$$

$$\to \sum_{j=1}^\infty \log E[\exp(it \sum_{k=1}^p \lambda_k(|U_j - y_j\eta_k| - |U_j|))] + it \sum_{k=1}^p \sigma^2\eta_k^2 F'_U(0) - 2^{-1}t^2(\sum_{k=1}^p \lambda_k\eta_k)^2\sigma^2.$$

We have that

$$\log E[\exp(it \sum_{j=1}^n \sum_{k=1}^p \lambda_k(|U_j - y_{n,j}\eta_k| - |U_j|))]$$
$$= \sum_{j=1}^m \log E[\exp(it \sum_{k=1}^p \lambda_k(|U_j - y_{n,j}\eta_k| - |U_j|))]$$
$$+ \sum_{j=m+1}^n \log E[\exp(it \sum_{k=1}^p \lambda_k(|U_j - y_{n,j}\eta_k| - |U_j|))].$$

By condition (iii)

$$\sum_{j=1}^m \log E[\exp(it \sum_{k=1}^p \lambda_k(|U_j - y_{n,j}\eta_k| - |U_j|))] \to \sum_{j=1}^m \log E[\exp(it \sum_{k=1}^p \lambda_k(|U_j - y_j\eta_k| - |U_j|))],$$

which if m large enough is approximately equal to

$$\sum_{j=1}^{\infty} \log E[\exp(it \sum_{k=1}^{p} \lambda_k(|U_j - y_j\eta_k| - |U_j|))].$$

It is easy to see that

$$E[|U+t| - |U|] = t^2 F_U'(0) + o(t^2) \tag{3.2}$$

and

$$E[(|U+t| - |U|)^2] = t^2 + o(t^2), \tag{3.3}$$

as $t \to 0$. Now, using (3.2) and (3.3),

$$\sum_{j=m+1}^{n} \log E[\exp(it \sum_{k=1}^{p} \lambda_k(|U_j - y_{n,j}\eta_k| - |U_j|))]$$

$$\approx \sum_{j=m+1}^{n} E[\exp(it \sum_{k=1}^{p} \lambda_k(|U_j - y_{n,j}\eta_k| - |U_j|)) - 1]$$

$$\approx \sum_{j=m+1}^{n} E[it \sum_{k=1}^{p} \lambda_k(|U_j - y_{n,j}\eta_k| - |U_j|) + 2^{-1}(it \sum_{k=1}^{p} \lambda_k(|U_j - y_{n,j}\eta_k| - |U_j|)^2]$$

$$\to it \sum_{k=1}^{p} \sigma^2 \eta_k^2 F_U'(0) - 2^{-1} t^2 (\sum_{k=1}^{p} \lambda_k \eta_k)^2 \sigma^2.$$

The checking of the rest of the conditions in Theorem 2.1 is trivial.

Let $\xi_n = |\sum_{j=1}^{n} y_{n,j}\mathrm{sign}(U_j)| + 3\sum_{j=1}^{\infty} |y_j|$. Let $\xi_0 = |\sigma g + \sum_{j=1}^{\infty} y_j \mathrm{sign}(U_j)| + 3\sum_{j=1}^{\infty} |y_j|$. It is easy to see that ξ_n converges in distribution to ξ_0 and that this convergence is jointly with the weak convergence of G_n to G. Hence, $G_n(\sigma^{-2}(F_U(0))^{-1}\xi_n)$ convergence in distribution to $G(\sigma^{-2}(F_U(0))^{-1}\xi_0)$. We have that

$$\sigma^{-2}(F_U(0))^{-1} G(\sigma^{-2}(F_U(0))^{-1}\xi_0) \geq -\sigma|g||\xi_0| + \xi_0^2 - |\xi_0| \sum_{j=1}^{\infty} |y_j|$$

$$\geq (\sigma|g| + 2\sum_{j=1}^{\infty} |y_j|) \sum_{j=1}^{\infty} |y_j|.$$

We also have that $G_n(-\sigma^{-2}(F_U(0))^{-1}\xi_n)$ converges in distribution to a r.v. wich can be bound by below by $\sigma^2 F_U(0)(\sigma|g| + 2\sum_{j=1}^{\infty} |y_j|) \sum_{j=1}^{\infty} |y_j|$. Now, $G_n(0) = 0$, $G_n(\sigma^{-2}(F_U(0))^{-1}\xi_n)$ and $G_n(-\sigma^{-2}(F_U(0))^{-1}\xi_n)$ converging to positive r.v.'s. and the convexity of G_n imply that $|\eta_n| \leq \sigma^{-2}(F_U(0))^{-1}|\xi_n|$ for n large. Hence, $|\eta_n| = O_P(1)$.

We have that

$$G(\eta) \geq -\sigma|g||\eta| + \sigma^2\eta^2 F_U'(0) - |\eta| \sum_{j=1}^{\infty} |y_j|.$$

So, $\lim_{|\eta| \to \infty} G(\eta) = \infty$. Hence, in order to check condition (iv) in Lemma 3.1, it suffice to prove that for each $a, b \in Q$, $a < b$,

$$P\{G'(\eta) = 0, \text{ for each } \eta \in (a, b)\} = 0.$$

Now, if $G'(\eta) = 0$, for each $\eta \in (a,b)$, then $y_j^{-1}U_j \in (a,b)$ for infinitely many j's. By the Lemma of Borel–Cantelli,

$$P\{y_j^{-1}U_j \in (a,b) \text{ for infinitely many } j'\text{s}\} = 0.$$

Therefore, Lemma 3.1 applies. □

There are possible choices of $\{z_{n,j}\}$ satisfying the conditions in Theorem 3.2. For example, if n is even, let $z_{n,j} = 2^{n-j}$, for $1 \le j \le 2^{-1}n$; and let $z_{n,j} = n^{-1/2}2^n$, for $2^{-1}n+1 \le j \le n$. If n is odd, let $z_{n,j} = 2^{n-j}$, for $1 \le j \le 2^{-1}(n+1)$; and let $z_{n,j} = (n+1)^{-1/2}2^n$, for $2^{-1}(n+3) \le j \le n$. Then, if n is even, $a_n^2 = (5/3)4^n - 3^{-1}2^n$; and if n is odd, $a_n^2 = (5/3)4^n - 3^{-1}2^{n-1}$; $y_j = (3/5)^{1/2}2^{-j}$ and $\sigma^2 = 1/5$.

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