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SUPERPROCESSES WITH DEPENDENT SPATIAL MOTION AND GENERAL BRANCHING DENSITIES

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Abstract We construct a class of superprocesses by taking the high density limit of a sequence of interacting-branching particle systems. The spatial motion of the superprocess is determined by a system of interacting diffusions, the branching density is given by an arbitrary bounded non-negative Borel function, and the superprocess is characterized by a martingale problem as a diffusion process with state space $M(\mathbb{R})$, improving and extending considerably the construction of Wang (1997, 1998). It is then proved in a special case that a suitable rescaled process of the superprocess converges to the usual super Brownian motion. An extension to measure-valued branching catalysts is also discussed.

Keywords superprocess, interacting-branching particle system, diffusion process, martingale problem, dual process, rescaled limit, measure-valued catalyst

AMS Subject Classifications Primary 60J80, 60G57; Secondary 60J35.

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1 Introduction

For a given topological space E, let $B(E)$ denote the totality of all bounded Borel functions on E and let $C(E)$ denote its subset comprising of continuous functions. Let $M(E)$ denote the space of finite Borel measures on E endowed with the topology of weak convergence. Write $\langle f, \mu \rangle$ for $\int f d\mu$. For $F \in B(M(E))$ let

$$\frac{\delta F(\mu)}{\delta \mu(x)} = \lim_{r \to 0+} \frac{1}{r}[F(\mu + r\delta_x) - F(\mu)], \qquad x \in E, \tag{1.1}$$

if the limit exists. Let $\delta^2 F(\mu)/\delta\mu(x)\delta\mu(y)$ be defined in the same way with F replaced by $(\delta F/\delta\mu(y))$ on the right hand side. For example, if $F_{m,f}(\mu) = \langle f, \mu^m \rangle$ for $f \in B(E^m)$ and $\mu \in M(E)$, then

$$\frac{\delta F_{m,f}(\mu)}{\delta \mu(x)} = \sum_{i=1}^m \langle \Psi_i(x) f, \mu^{m-1} \rangle, \qquad x \in E, \tag{1.2}$$

where $\Psi_i(x)$ is the operator from $B(E^m)$ to $B(E^{m-1})$ defined by

$$\Psi_i(x) f(x_1, \cdots, x_{m-1}) = f(x_1, \cdots, x_{i-1}, x, x_i, \cdots, x_{m-1}), \quad x_j \in E, \tag{1.3}$$

where $x \in E$ is the ith variable of f on the right hand side.

Now we consider the case where $E = \mathbb{R}$, the one-dimensional Euclidean space. Suppose that $c \in C(\mathbb{R})$ is Lipschitz and $h \in C(\mathbb{R})$ is square-integrable. Let

$$\rho(x) = \int_{\mathbb{R}} h(y - x) h(y) dy, \tag{1.4}$$

and $a(x) = c(x)^2 + \rho(0)$ for $x \in \mathbb{R}$. We assume in addition that ρ is twice continuously differentiable with ρ' and ρ'' bounded, which is satisfied if h is integrable and twice continuously differentiable with h' and h'' bounded. Then

$$\begin{aligned} \mathcal{A}F(\mu) &= \frac{1}{2}\int_{\mathbb{R}} a(x) \frac{d^2}{dx^2} \frac{\delta F(\mu)}{\delta \mu(x)} \mu(dx) \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^2} \rho(x - y) \frac{d^2}{dx dy} \frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)} \mu(dx)\mu(dy) \end{aligned} \tag{1.5}$$

defines an operator $\mathcal{A}$ which acts on a subset of $B(M(\mathbb{R}))$ and generates a diffusion process with state space $M(\mathbb{R})$. Suppose that $\{W(x,t) : x \in \mathbb{R}, t \geq 0\}$ is a Brownian sheet and $\{B_i(t) : t \geq 0\}$, $i = 1, 2, \cdots$, is a family of independent standard Brownian motions which are independent of $\{W(x,t) : x \in \mathbb{R}, t \geq 0\}$. By Lemma 3.1, for any initial conditions $x_i(0) = x_i$, the stochastic equations

$$dx_i(t) = c(x_i(t)) dB_i(t) + \int_{\mathbb{R}} h(y - x_i(t)) W(dy, dt), \qquad t \geq 0, i = 1, 2, \cdots, \tag{1.6}$$

have unique solutions $\{x_i(t) : t \geq 0\}$ and, for each integer $m \geq 1$, $\{(x_1(t), \cdots, x_m(t)) : t \geq 0\}$ is an m-dimensional diffusion process which is generated by the differential operator

$$G^m := \frac{1}{2}\sum_{i=1}^{m} a(x_i)\frac{\partial^2}{\partial x_i^2} + \frac{1}{2}\sum_{i,j=1,i\neq j}^{m} \rho(x_i - x_j)\frac{\partial^2}{\partial x_i \partial x_j}. \tag{1.7}$$

In particular, $\{x_i(t) : t \geq 0\}$ is a one-dimensional diffusion process with generator $G := (a(x)/2)\Delta$. Because of the exchangebility, a diffusion process generated by G^m can be regarded as an interacting particle system or a measure-valued process. Heuristically, $a(\cdot)$ represents the speed of the particles and $\rho(\cdot)$ describes the interaction between them. The diffusion process generated by $\mathcal{A}$ arises as the high density limit of a sequence of interacting particle systems described by (1.6); see Wang (1997, 1998) and section 4 of this paper. For $\sigma \in B(\mathbb{R})^+$, we may also define the operator $\mathcal{B}$ by

$$\mathcal{B}F(\mu) = \frac{1}{2}\int_{\mathbb{R}} \sigma(x)\frac{\delta^2 F(\mu)}{\delta\mu(x)^2}\mu(dx). \tag{1.8}$$

A Markov process generated by $\mathcal{L} := \mathcal{A} + \mathcal{B}$ is naturally called a *superprocess with dependent spatial motion (SDSM)* with parameters (a, ρ, σ), where σ represents the branching density of the process. In the special case where both c and σ are constants, the SDSM was constructed in Wang (1997, 1998) as a diffusion process in $M(\hat{\mathbb{R}})$, where $\hat{\mathbb{R}} = \mathbb{R} \cup \{\partial\}$ is the one-point compactification of $\mathbb{R}$. It was also assumed in Wang (1997, 1998) that h is a symmetric function and that the initial state of the SDSM has compact support in $\mathbb{R}$. Stochastic partial differential equations and local times associated with the SDSM were studied in Dawson et al (2000a, b).

The SDSM contains as special cases several models arising in different circumstances such as the one-dimensional super Brownian motion, the molecular diffusion with turbulent transport and some interacting diffusion systems of McKean-Vlasov type; see e.g. Chow (1976), Dawson (1994), Dawson and Vaillancourt (1995) and Kotelenez (1992, 1995). It is thus of interest to construct the SDSM under reasonably more general conditions and formulate it as a diffusion processes in $M(\mathbb{R})$. This is the main purpose of the present paper. The rest of this paragraph describes the main results of the paper and gives some unsolved problems in the subject. In section 2, we define some function-valued dual process and investigate its connection to the solution of the martingale problem of a SDSM. Duality method plays an important role in the investigation. Although the SDSM could arise as high density limit of a sequence of interacting-branching particle systems with location-dependent killing density σ and binary branching distribution, the construction of such systems seems rather sophisticated and is thus avoided in this work. In section 3, we construct the interacting-branching particle system with uniform killing density and location-dependent branching distribution, which is comparatively easier to treat. The arguments are similar to those in Wang (1998). The high density limit of the interacting-branching particle system is considered in section 4, which gives a solution of the martingale problem of the SDSM in the special case where $\sigma \in C(\mathbb{R})^+$ can be extended into a continuous function on $\hat{\mathbb{R}}$. In section 5, we use the dual process to extend the construction of the SDSM to a general bounded Borel branching density $\sigma \in B(\mathbb{R})^+$. In both sections 4 and 5, we use martingale arguments to show that, if the processes are initially supported by $\mathbb{R}$, they always stay in $M(\mathbb{R})$, which are new results even in the special case considered in Wang (1997, 1998). In section 6, we prove a rescaled limit theorem of the SDSM, which states that a suitable rescaled SDSM converges to the usual super Brownian motion if $c(\cdot)$ is bounded away from zero. This describes another situation where the super Brownian motion arises universally; see also Durrett and Perkins (1998) and Hara and Slade (2000a, b). When $c(\cdot) \equiv 0$, we expect that the same rescaled limit would lead to a measure-valued diffusion process which is the high density limit of a sequence of coalescing-branching particle systems, but there is still a long way to reach a rigorous proof. It suffices to mention that not only the characterization of those high density limits but also that of the coalescing-branching particle systems themselves are still open problems. We refer the reader to Evans and Pitman (1998) and the references therein for some recent work on related models. In section 7, we consider an extension of the construction of the SDSM to the case where σ is of the form $\sigma = \dot{\eta}$ with η belonging to a large class of Radon measures on $\mathbb{R}$, in the lines of Dawson and Fleischmann (1991, 1992). The process is constructed only when $c(\cdot)$ is bounded away from zero and it can be called a *SDSM with measure-valued catalysts*. The transition semigroup of the SDSM with measure-valued catalysts is constructed and characterized using a measure-valued dual process. The derivation is based on some estimates of moments of the dual process. However, the existence of a diffusion realization of the SDSM with measure-valued catalysts is left as another open problem in the subject.

Notation: Recall that $\hat{\mathbb{R}} = \mathbb{R} \cup \{\partial\}$ denotes the one-point compactification of $\mathbb{R}$. Let λ^m denote the Lebesgue measure on $\mathbb{R}^m$. Let $C^2(\mathbb{R}^m)$ be the set of twice continuously differentiable functions on $\mathbb{R}^m$ and let $C^2_\partial(\mathbb{R}^m)$ be the set of functions in $C^2(\mathbb{R}^m)$ which together with their derivatives up to the second order can be extended continuously to $\hat{\mathbb{R}}$. Let $C^2_0(\mathbb{R}^m)$ be the subset of $C^2_\partial(\mathbb{R}^m)$ of functions that together with their derivatives up to the second order *vanish rapidly* at infinity. Let $(T^m_t)_{t\geq 0}$ denote the transition semigroup of the m-dimensional standard Brownian motion and let $(P^m_t)_{t\geq 0}$ denote the transition semigroup generated by the operator G^m. We shall omit the superscript m when it is one. Let $(\hat{P}_t)_{t\geq 0}$ and $\hat{G}$ denote the extensions of $(P_t)_{t\geq 0}$ and G to $\hat{\mathbb{R}}$ with ∂ as a trap. We denote the expectation by the letter of the probability measure if this is specified and simply by $\boldsymbol{E}$ if the measure is not specified.

We remark that, if $|c(x)| \geq \epsilon > 0$ for all $x \in \mathbb{R}$, the semigroup $(P^m_t)_{t>0}$ has density $p^m_t(x,y)$ which satisfies

$$p^m_t(x,y) \leq \text{const} \cdot g^m_{\epsilon t}(x,y), \qquad t > 0, x, y \in \mathbb{R}^m, \tag{1.9}$$

where $g^m_t(x,y)$ denotes the transition density of the m-dimensional standard Brownian motion; see e.g. Friedman (1964, p.24).

2 Function-valued dual processes

In this section, we define a function-valued dual process and investigate its connection to the solution of the martingale problem for the SDSM. Recall the definition of the generator $\mathcal{L} := \mathcal{A} + \mathcal{B}$ given by (1.5) and (1.8) with $\sigma \in B(\mathbb{R})^+$. For $\mu \in M(\mathbb{R})$ and a subset $\mathcal{D}(\mathcal{L})$ of the domain of $\mathcal{L}$, we say an $M(\mathbb{R})$-valued cádlág process $\{X_t : t \geq 0\}$ is a solution of the $(\mathcal{L}, \mathcal{D}(\mathcal{L}), \mu)$-*martingale problem* if $X_0 = \mu$ and

$$F(X_t) - F(X_0) - \int_0^t \mathcal{L}F(X_s)ds, \qquad t \geq 0,$$

is a martingale for each $F \in \mathcal{D}(\mathcal{L})$. Observe that, if $F_{m,f}(\mu) = \langle f, \mu^m \rangle$ for $f \in C^2(\mathbb{R}^m)$, then

$$\begin{aligned}
\mathcal{A}F_{m,f}(\mu) &= \frac{1}{2}\int_{\mathbb{R}^m} \sum_{i=1}^{m} a(x_i) f''_{ii}(x_1, \cdots, x_m)\mu^m(dx_1, \cdots, dx_m) \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^m} \sum_{i,j=1, i\neq j}^{m} \rho(x_i - x_j) f''_{ij}(x_1, \cdots, x_m)\mu^m(dx_1, \cdots, dx_m) \\
&= F_{m,G^m f}(\mu), \qquad (2.1)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{B}F_{m,f}(\mu) &= \frac{1}{2}\sum_{i,j=1, i\neq j}^{m} \int_{\mathbb{R}^{m-1}} \Phi_{ij} f(x_1, \cdots, x_{m-1})\mu^{m-1}(dx_1, \cdots, dx_{m-1}) \\
&= \frac{1}{2}\sum_{i,j=1, i\neq j}^{m} F_{m-1, \Phi_{ij} f}(\mu), \qquad (2.2)
\end{aligned}$$

where Φ_{ij} denotes the operator from $B(E^m)$ to $B(E^{m-1})$ defined by

$$\Phi_{ij} f(x_1, \cdots, x_{m-1}) = \sigma(x_{m-1}) f(x_1, \cdots, x_{m-1}, \cdots, x_{m-1}, \cdots, x_{m-2}), \qquad (2.3)$$

where x_{m-1} is in the places of the ith and the jth variables of f on the right hand side. It follows that

$$\mathcal{L}F_{m,f}(\mu) = F_{m,G^m f}(\mu) + \frac{1}{2}\sum_{i,j=1, i\neq j}^{m} F_{m-1, \Phi_{ij} f}(\mu). \qquad (2.4)$$

Let $\{M_t : t \geq 0\}$ be a nonnegative integer-valued cádlág Markov process with transition intensities $\{q_{i,j}\}$ such that $q_{i,i-1} = -q_{i,i} = i(i-1)/2$ and $q_{i,j} = 0$ for all other pairs (i, j). That is, $\{M_t : t \geq 0\}$ is the well-known Kingman's coalescent process. Let $\tau_0 = 0$ and $\tau_{M_0} = \infty$, and let $\{\tau_k : 1 \leq k \leq M_0 - 1\}$ be the sequence of jump times of $\{M_t : t \geq 0\}$.

Let $\{\Gamma_k : 1 \le k \le M_0 - 1\}$ be a sequence of random operators which are conditionally independent given $\{M_t : t \ge 0\}$ and satisfy

$$\boldsymbol{P}\{\Gamma_k = \Phi_{i,j} | M(\tau_k^-) = l\} = \frac{1}{l(l-1)}, \qquad 1 \le i \ne j \le l, \tag{2.5}$$

where $\Phi_{i,j}$ is defined by (2.3). Let $\boldsymbol{B}$ denote the topological union of $\{B(\mathbb{R}^m) : m = 1, 2, \cdots\}$ endowed with pointwise convergence on each $B(\mathbb{R}^m)$. Then

$$Y_t = P^{M_{\tau_k}}_{t-\tau_k} \Gamma_k P^{M_{\tau_{k-1}}}_{\tau_k - \tau_{k-1}} \Gamma_{k-1} \cdots P^{M_{\tau_1}}_{\tau_2 - \tau_1} \Gamma_1 P^{M_0}_{\tau_1} Y_0, \quad \tau_k \le t < \tau_{k+1}, 0 \le k \le M_0 - 1, \tag{2.6}$$

defines a Markov process $\{Y_t : t \ge 0\}$ taking values from $\boldsymbol{B}$. Clearly, $\{(M_t, Y_t) : t \ge 0\}$ is also a Markov process. To simplify the presentation, we shall suppress the dependence of $\{Y_t : t \ge 0\}$ on σ and let $\boldsymbol{E}^\sigma_{m,f}$ denote the expectation given $M_0 = m$ and $Y_0 = f \in C(\mathbb{R}^m)$, just as we are working with a canonical realization of $\{(M_t, Y_t) : t \ge 0\}$. By (2.6) we have

$$\begin{aligned}
& \boldsymbol{E}^\sigma_{m,f}\Big[\langle Y_t, \mu^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big] \\
= \;& \langle P_t^m f, \mu^m\rangle \\
& + \frac{1}{2}\sum_{i,j=1, i\ne j}^m \int_0^t \boldsymbol{E}^\sigma_{m-1, \Phi_{ij} P_u^m f}\Big[\langle Y_{t-u}, \mu^{M_{t-u}}\rangle \exp\Big\{\frac{1}{2}\int_0^{t-u} M_s(M_s-1)ds\Big\}\Big]du.
\end{aligned} \tag{2.7}$$

Lemma 2.1 *For any $f \in B(\mathbb{R}^m)$ and any integer $m \ge 1$,*

$$\begin{aligned}
& \boldsymbol{E}^\sigma_{m,f}\Big[\langle Y_t, \mu^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big] \\
\le \;& \|f\| \sum_{k=0}^{m-1} 2^{-k} m^k (m-1)^k \|\sigma\|^k \langle 1, \mu\rangle^{m-k},
\end{aligned} \tag{2.8}$$

where $\|\cdot\|$ denotes the supremum norm.

Proof. The left hand side of (2.8) can be decomposed as $\sum_{k=0}^{m-1} A_k$ with

$$A_k = \boldsymbol{E}^\sigma_{m,f}\Big[\langle Y_t, \mu^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\} 1_{\{\tau_k \le t < \tau_{k+1}\}}\Big].$$

Observe that $A_0 = \langle P_t^m f, \mu^m\rangle \le \|f\|\langle 1, \mu\rangle^m$ and

$$\begin{aligned}
A_k \;=\;& \frac{m!(m-1)!}{2^k(m-k)!(m-k-1)!}\int_0^t ds_1 \int_{s_1}^t ds_2 \cdots \\
& \cdot \int_{s_{k-1}}^t \boldsymbol{E}^\sigma_{m,f}\{\langle P^{m-k}_{t-s_k}\Gamma_k \cdots P^{m-1}_{s_2-s_1}\Gamma_1 P^m_{s_1} f, \mu^{m-k}\rangle | \tau_j = s_j : 1 \le j \le k\} ds_k \\
\le\;& \frac{m!(m-1)!}{2^k(m-k)!(m-k-1)!}\int_0^t ds_1 \int_0^t ds_2 \cdots \int_0^t \|f\|\|\sigma\|^k \langle 1, \mu\rangle^{m-k} ds_k \\
\le\;& \frac{m!(m-1)!}{2^k(m-k)!(m-k-1)!}\|f\|\|\sigma\|^k\langle 1, \mu\rangle^{m-k} t^k
\end{aligned}$$

for $1 \leq k \leq m-1$. Then we get the conclusion. $\square$

Lemma 2.2 *Suppose that $\sigma_n \to \sigma$ boundedly and pointwise and $\mu_n \to \mu$ in $M(\mathbb{R})$ as $n \to \infty$. Then, for any $f \in B(\mathbb{R}^m)$ and any integer $m \geq 1$,*

$$\begin{aligned} &\boldsymbol{E}^{\sigma}_{m,f}\Big[\langle Y_t, \mu^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big] \\ =& \lim_{n\to\infty} \boldsymbol{E}^{\sigma_n}_{m,f}\Big[\langle Y_t, \mu_n^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big]. \end{aligned} \tag{2.9}$$

Proof. For $h \in C(\mathbb{R}^2)$ we see by (2.7) that

$$\begin{aligned} &\boldsymbol{E}^{\sigma_n}_{1,\Phi_{12}h}\Big[\langle Y_t, \mu_n^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big] \\ =& \boldsymbol{E}^{\sigma_n}_{1,\Phi_{21}h}\Big[\langle Y_t, \mu_n^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big] \\ =& \int_{\mathbb{R}^2} h(y,y)p_t(x,y)\mu_n(dx)\sigma_n(y)dy. \end{aligned} \tag{2.10}$$

If $f, g \in C(\mathbb{R})^+$ have bounded supports, then we have $f(x)\mu_n(dx) \to f(x)\mu(dx)$ and $g(y)\sigma_n(y)dy \to g(y)\sigma(y)dy$ by weak convergence, so that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} f(x)g(y)p_t(x,y)\mu_n(dx)\sigma_n(y)dy = \int_{\mathbb{R}^2} f(x)g(y)p_t(x,y)\mu(dx)\sigma(y)dy.$$

Since $\{\mu_n\}$ is tight and $\{\sigma_n\}$ is bounded, one can easily see that $\{p_t(x,y)\mu_n(dx)\sigma_n(y)dy\}$ is a tight sequence and hence $p_t(x,y)\mu_n(dx)\sigma_n(y)dy \to p_t(x,y)\mu(dx)\sigma(y)dy$ by weak convergence. Therefore, the value of (2.10) converges as $n \to \infty$ to

$$\begin{aligned} &\boldsymbol{E}^{\sigma}_{1,\Phi_{12}h}\Big[\langle Y_t, \mu^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big] \\ =& \boldsymbol{E}^{\sigma}_{1,\Phi_{21}h}\Big[\langle Y_t, \mu^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big] \\ =& \int_{\mathbb{R}^2} h(y,y)p_t(x,y)\mu(dx)\sigma(y)dy. \end{aligned}$$

Applying bounded convergence theorem to (2.7) we get inductively

$$\begin{aligned} &\boldsymbol{E}^{\sigma}_{m-1,\Phi_{ij}P_t^m f}\Big[\langle Y_t, \mu^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big] \\ =& \lim_{n\to\infty} \boldsymbol{E}^{\sigma_n}_{m-1,\Phi_{ij}P_t^m f}\Big[\langle Y_t, \mu_n^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big] \end{aligned}$$

for $1 \leq i \neq j \leq m$. Then the result follows from (2.7). $\square$

Theorem 2.1 *Let $\mathcal{D}(\mathcal{L})$ be the set of all functions of the form $F_{m,f}(\mu) = \langle f, \mu^m \rangle$ with $f \in C^2(\mathbb{R}^m)$. Suppose that $\{X_t : t \geq 0\}$ is a continuous $M(\mathbb{R})$-valued process and that $\boldsymbol{E}\{\langle 1, X_t \rangle^m\}$ is locally bounded in $t \geq 0$ for each $m \geq 1$. If $\{X_t : t \geq 0\}$ is a solution of the $(\mathcal{L}, \mathcal{D}(\mathcal{L}), \mu)$-martingale problem, then*

$$\boldsymbol{E}\langle f, X_t^m \rangle = \boldsymbol{E}_{m,f}^{\sigma}\Big[\langle Y_t, \mu^{M_t} \rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s - 1)ds\Big\}\Big] \tag{2.11}$$

for any $t \geq 0$, $f \in B(\mathbb{R}^m)$ and integer $m \geq 1$.

Proof. In view of (2.6), the general equality follows by bounded pointwise approximation once it is proved for $f \in C^2(\mathbb{R}^m)$. In this proof, we set $F_\mu(m, f) = F_{m,f}(\mu) = \langle f, \mu^m \rangle$. From the construction (2.6), it is not hard to see that $\{(M_t, Y_t) : t \geq 0\}$ has generator $\mathcal{L}^*$ given by

$$\mathcal{L}^* F_\mu(m, f) = F_\mu(m, G^m f) + \frac{1}{2}\sum_{i,j=1, i\neq j}^{m} [F_\mu(m-1, \varPhi_{ij} f) - F_\mu(m, f)].$$

In view of (2.4) we have

$$\mathcal{L}^* F_\mu(m, f) = \mathcal{L} F_{m,f}(\mu) - \frac{1}{2}m(m-1)F_{m,f}(\mu). \tag{2.12}$$

The following calculations are guided by the relation (2.12). In the sequel, we assume that $\{X_t : t \geq 0\}$ and $\{(M_t, Y_t) : t \geq 0\}$ are defined on the same probability space and are independent of each other. Suppose that for each $n \geq 1$ we have a partition $\varDelta_n := \{0 = t_0 < t_1 < \cdots < t_n = t\}$ of $[0, t]$. Let $\|\varDelta_n\| = \max\{|t_i - t_{i-1}| : 1 \leq i \leq n\}$ and assume $\|\varDelta_n\| \to 0$ as $n \to \infty$. Observe that

$$\begin{aligned}
&\boldsymbol{E}\langle f, X_t^m \rangle - \boldsymbol{E}\Big[\langle Y_t, \mu^{M_t} \rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s - 1)ds\Big\}\Big] \\
=\ & \sum_{i=1}^n \Big(\boldsymbol{E}\Big[\langle Y_{t-t_i}, X_{t_i}^{M_{t-t_i}} \rangle \exp\Big\{\frac{1}{2}\int_0^{t-t_i} M_s(M_s - 1)ds\Big\}\Big] \\
&\quad - \boldsymbol{E}\Big[\langle Y_{t-t_{i-1}}, X_{t_{i-1}}^{M_{t-t_{i-1}}} \rangle \exp\Big\{\frac{1}{2}\int_0^{t-t_{i-1}} M_s(M_s - 1)ds\Big\}\Big]\Big).
\end{aligned} \tag{2.13}$$

By the independence of $\{X_t : t \geq 0\}$ and $\{(M_t, Y_t) : t \geq 0\}$ and the martingale characterization of $\{(M_t, Y_t) : t \geq 0\}$,

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^n & \Big(\boldsymbol{E}\Big[\langle Y_{t-t_i}, X_{t_i}^{M_{t-t_i}} \rangle \exp\Big\{\frac{1}{2}\int_0^{t-t_i} M_s(M_s - 1)ds\Big\}\Big] \\
& - \boldsymbol{E}\Big[\langle Y_{t-t_{i-1}}, X_{t_i}^{M_{t-t_{i-1}}} \rangle \exp\Big\{\frac{1}{2}\int_0^{t-t_i} M_s(M_s - 1)ds\Big\}\Big]\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty}\sum_{i=1}^{n}\boldsymbol{E}\Big(\exp\Big\{\frac{1}{2}\int_0^{t-t_i}M_s(M_s-1)ds\Big\}\boldsymbol{E}\Big[F_{X_{t_i}}(M_{t-t_i},Y_{t-t_i})\\
&\qquad\qquad -F_{X_{t_i}}(M_{t-t_{i-1}},Y_{t-t_{i-1}})\Big|X;\{(M_r,Y_r):0\le r\le t-t_i\}\Big]\Big)\\
&= -\lim_{n\to\infty}\sum_{i=1}^{n}\boldsymbol{E}\Big(\exp\Big\{\frac{1}{2}\int_0^{t-t_i}M_s(M_s-1)ds\Big\}\\
&\qquad\qquad \boldsymbol{E}\Big[\int_{t-t_i}^{t-t_{i-1}}\mathcal{L}^*F_{X_{t_i}}(M_u,Y_u)du\Big|X;\{(M_r,Y_r):0\le r\le t-t_i\}\Big]\Big)\\
&= -\lim_{n\to\infty}\sum_{i=1}^{n}\boldsymbol{E}\Big(\exp\Big\{\frac{1}{2}\int_0^{t-t_i}M_s(M_s-1)ds\Big\}\int_{t-t_i}^{t-t_{i-1}}\mathcal{L}^*F_{X_{t_i}}(M_u,Y_u)du\Big)\\
&= -\lim_{n\to\infty}\int_0^t\sum_{i=1}^{n}\boldsymbol{E}\Big(\exp\Big\{\frac{1}{2}\int_0^{t-t_i}M_s(M_s-1)ds\Big\}\mathcal{L}^*F_{X_{t_i}}(M_{t-u},Y_{t-u})\Big)1_{[t_{i-1},t_i]}(u)du\\
&= -\int_0^t\boldsymbol{E}\Big(\exp\Big\{\frac{1}{2}\int_0^{t-u}M_s(M_s-1)ds\Big\}\mathcal{L}^*F_{X_u}(M_{t-u},Y_{t-u})\Big)du,
\end{aligned}
$$

where the last step holds by the right continuity of $\{X_t : t \geq 0\}$. Using again the independence and the martingale problem for $\{X_t : t \geq 0\}$,

$$
\begin{aligned}
&\lim_{n\to\infty}\sum_{i=1}^{n}\Big(\boldsymbol{E}\Big[\langle Y_{t-t_{i-1}},X_{t_i}^{M_{t-t_{i-1}}}\rangle\exp\Big\{\frac{1}{2}\int_0^{t-t_i}M_s(M_s-1)ds\Big\}\Big]\\
&\qquad\qquad -\boldsymbol{E}\Big[\langle Y_{t-t_{i-1}},X_{t_{i-1}}^{M_{t-t_{i-1}}}\rangle\exp\Big\{\frac{1}{2}\int_0^{t-t_i}M_s(M_s-1)ds\Big\}\Big]\Big)\\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\boldsymbol{E}\Big(\exp\Big\{\frac{1}{2}\int_0^{t-t_i}M_s(M_s-1)ds\Big\}\\
&\qquad\qquad \boldsymbol{E}\Big[F_{M_{t-t_{i-1}},Y_{t-t_{i-1}}}(X_{t_i})-F_{M_{t-t_{i-1}},Y_{t-t_{i-1}}}(X_{t_{i-1}})\Big|M,Y\Big]\Big)\\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\boldsymbol{E}\Big(\exp\Big\{\frac{1}{2}\int_0^{t-t_i}M_s(M_s-1)ds\Big\}\boldsymbol{E}\Big[\int_{t_{i-1}}^{t_i}\mathcal{L}F_{M_{t-t_{i-1}},Y_{t-t_{i-1}}}(X_u)du\Big|M,Y\Big]\Big)\\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\boldsymbol{E}\Big(\exp\Big\{\frac{1}{2}\int_0^{t-t_i}M_s(M_s-1)ds\Big\}\int_{t_{i-1}}^{t_i}\mathcal{L}F_{M_{t-t_{i-1}},Y_{t-t_{i-1}}}(X_u)du\Big)\\
&= \lim_{n\to\infty}\int_0^t\sum_{i=1}^{n}\boldsymbol{E}\Big(\exp\Big\{\frac{1}{2}\int_0^{t-t_i}M_s(M_s-1)ds\Big\}\mathcal{L}F_{M_{t-t_{i-1}},Y_{t-t_{i-1}}}(X_u)\Big)1_{[t_{i-1},t_i]}(u)du\\
&= \int_0^t\boldsymbol{E}\Big(\exp\Big\{\frac{1}{2}\int_0^{t-u}M_s(M_s-1)ds\Big\}\mathcal{L}F_{M_{t-u},Y_{t-u}}(X_u)\Big)du,
\end{aligned}
$$

where we have also used the right continuity of $\{(M_t, Y_t) : t \geq 0\}$ for the last step. Finally, since $\|\Delta_n\| \to 0$ as $n \to \infty$ and $M_t \leq m$ for all $t \geq 0$, we have

$$
\begin{aligned}
&\lim_{n\to\infty} \sum_{i=1}^{n} \Big(\boldsymbol{E}\Big[\langle Y_{t-t_{i-1}}, X_{t_{i-1}}^{M_{t-t_{i-1}}}\rangle \exp\Big\{\frac{1}{2}\int_0^{t-t_i} M_s(M_s-1)ds\Big\}\Big] \\
&\qquad\qquad -\boldsymbol{E}\Big[\langle Y_{t-t_{i-1}}, X_{t_{i-1}}^{M_{t-t_{i-1}}}\rangle \exp\Big\{\frac{1}{2}\int_0^{t-t_{i-1}} M_s(M_s-1)ds\Big\}\Big]\Big) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \boldsymbol{E}\Big(F_{X_{t_{i-1}}}(M_{t-t_{i-1}}, Y_{t-t_{i-1}}) \exp\Big\{\frac{1}{2}\int_0^{t-t_i} M_s(M_s-1)ds\Big\} \\
&\qquad\qquad \Big[1-\exp\Big\{\frac{1}{2}\int_{t-t_i}^{t-t_{i-1}} M_u(M_u-1)du\Big\}\Big]\Big) \\
&= -\lim_{n\to\infty} \sum_{i=1}^{n} \boldsymbol{E}\Big(F_{X_{t_{i-1}}}(M_{t-t_{i-1}}, Y_{t-t_{i-1}}) \exp\Big\{\frac{1}{2}\int_0^{t-t_i} M_s(M_s-1)ds\Big\} \\
&\qquad\qquad \Big[\frac{1}{2}\int_{t-t_i}^{t-t_{i-1}} M_u(M_u-1)du\Big]\Big) \\
&= -\lim_{n\to\infty} \frac{1}{2}\int_0^t \sum_{i=1}^{n} \boldsymbol{E}\Big(F_{X_{t_{i-1}}}(M_{t-t_{i-1}}, Y_{t-t_{i-1}}) \exp\Big\{\frac{1}{2}\int_0^{t-t_i} M_s(M_s-1)ds\Big\} \\
&\qquad\qquad M_{t-u}(M_{t-u}-1)\Big) 1_{[t_{i-1},t_i]}(u)du.
\end{aligned}
$$

Since the semigroups $(P_t^m)_{t\geq 0}$ are strongly Feller and strongly continuous, $\{Y_t : t \geq 0\}$ is continuous in the uniform norm in each open interval between two neighboring jumps of $\{M_t : t \geq 0\}$. Using this, the left continuity of $\{X_t : t \geq 0\}$ and dominated convergence, we see that the above value is equal to

$$
-\frac{1}{2}\int_0^t \boldsymbol{E}\Big(F_{X_u}(M_{t-u}, Y_{t-u}) \exp\Big\{\frac{1}{2}\int_0^{t-u} M_s(M_s-1)ds\Big\} M_{t-u}(M_{t-u}-1)\Big)du.
$$

Combining those together we see that the value of (2.13) is in fact zero and hence (2.11) follows. $\square$

Theorem 2.2 *Let $\mathcal{D}(\mathcal{L})$ be as in Theorem 2.1 and let $\{w_t : t \geq 0\}$ denote the coordinate process of $C([0,\infty), M(\mathbb{R}))$. Suppose that for each $\mu \in M(\mathbb{R})$ there is a probability measure $\boldsymbol{Q}_\mu$ on $C([0,\infty), M(\mathbb{R}))$ such that $\boldsymbol{Q}_\mu\{\langle 1, w_t\rangle^m\}$ is locally bounded in $t \geq 0$ for every $m \geq 1$ and such that $\{w_t : t \geq 0\}$ under $\boldsymbol{Q}_\mu$ is a solution of the $(\mathcal{L}, \mathcal{D}(\mathcal{L}), \mu)$-martingale problem. Then the system $\{\boldsymbol{Q}_\mu : \mu \in M(\mathbb{R})\}$ defines a diffusion process with transition semigroup $(Q_t)_{\geq 0}$ given by*

$$
\int_{M(\mathbb{R})} \langle f, \nu^m\rangle Q_t(\mu, d\nu) = \boldsymbol{E}_{m,f}^{\sigma}\Big[\langle Y_t, \mu^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big]. \qquad (2.14)
$$

Proof. Let $Q_t(\mu,\cdot)$ denote the distribution of w_t under $\boldsymbol{Q}_\mu$. By Theorem 2.1 we have (2.14). Let us assume first that $\sigma(x)\equiv\sigma_0$ for a constant σ_0. In this case, $\{\langle 1,w_t\rangle : t\geq 0\}$ is the Feller diffusion with generator $(\sigma_0/2)xd^2/dx^2$, so that

$$\int_{M(\mathbb{R})} e^{\lambda\langle 1,\nu\rangle}Q_t(\mu,d\nu) = \exp\Big\{\frac{2\langle 1,\mu\rangle\lambda}{2-\sigma_0\lambda t}\Big\}, \qquad t\geq 0, \lambda\geq 0.$$

Then for each $f\in B(\mathbb{R})^+$ the power series

$$\sum_{m=0}^{\infty}\frac{1}{m!}\int_{M(\mathbb{R})}\langle f,\nu\rangle^m Q_t(\mu,d\nu)\lambda^m \tag{2.15}$$

has a positive radius of convergence. By this and Billingsley (1968, p.342) it is not hard to show that $Q_t(\mu,\cdot)$ is the unique probability measure on $M(\mathbb{R})$ satisfying (2.14). Now the result follows from Ethier and Kurtz (1986, p.184). For a non-constant $\sigma\in B(\mathbb{R})^+$, let $\sigma_0=\|\sigma\|$ and observe that

$$\int_{M(\mathbb{R})}\langle f,\nu\rangle^m Q_t(\mu,d\nu) \leq \boldsymbol{E}^{\sigma_0}_{m,f^{\otimes m}}\Big[\langle Y_t,\mu^{M_t}\rangle\exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big]$$

by (2.14) and the construction (2.6) of $\{Y_t : t\geq 0\}$, where $f^{\otimes m}\in B(\mathbb{R}^m)^+$ is defined by $f^{\otimes m}(x_1,\cdots,x_m)=f(x_1)\cdots f(x_m)$. Then the power series (2.15) also has a positive radius of convergence and the result follows as in the case of a constant branching rate. $\square$

3 Interacting-branching particle systems

In this section, we give a formulation of the interacting-branching particle system. We first prove that equations (1.6) have unique solutions. Recall that $c\in C(\mathbb{R})$ is Lipschitz, $h\in C(\mathbb{R})$ is square-integrable and ρ is twice continuously differentiable with ρ' and ρ'' bounded. The following result is an extension of Lemma 1.3 of Wang (1997) where it was assumed that $c(x)\equiv$ const.

Lemma 3.1 *For any initial conditions $x_i(0)=x_i$, equations (1.6) have unique solutions $\{x_i(t) : t\geq 0\}$ and $\{(x_1(t),\cdots,x_m(t)) : t\geq 0\}$ is an m-dimensional diffusion process with generator G^m defined by (1.7).*

Proof. Fix $T>0$ and $i\geq 1$ and define $\{x_i^k(t) : t\geq 0\}$ inductively by $x_i^0(t)\equiv x_i(0)$ and

$$x_i^{k+1}(t) = x_i(0)+\int_0^t c(x_i^k(s))dB_i(s)+\int_0^t\int_{\mathbb{R}} h(y-x_i^k(s))W(dy,ds), \qquad t\geq 0.$$

Let $l(c) \geq 0$ be any Lipschitz constant for $c(\cdot)$. By a martingale inequality we have

$$\begin{aligned}
\boldsymbol{E}\Big\{\sup_{0\leq t\leq T}|x_i^{k+1}(t)-x_i^k(t)|^2\Big\} &\leq 8\int_0^T \boldsymbol{E}\{|c(x_i^k(t))-c(x_i^{k-1}(t))|^2\}dt \\
&\quad +8\int_0^T \boldsymbol{E}\Big\{\int_{\mathbb{R}}|h(y-x_i^k(t))-h(y-x_i^{k-1}(t))|^2dy\Big\}dt \\
&\leq 8l(c)^2\int_0^T \boldsymbol{E}\{|x_i^k(t)-x_i^{k-1}(t)|^2\}dt \\
&\quad +16\int_0^T \boldsymbol{E}\{|\rho(0)-\rho(x_i^k(t)-x_i^{k-1}(t))|\}dt \\
&\leq 8(l(c)^2+\|\rho''\|)\int_0^T \boldsymbol{E}\{|x_i^k(t)-x_i^{k-1}(t)|^2\}dt.
\end{aligned}$$

Using the above inequality inductively we get

$$\boldsymbol{E}\Big\{\sup_{0\leq t\leq T}|x_i^{k+1}(t)-x_i^k(t)|^2\Big\} \leq (\|c\|^2+\rho(0))(l(c)^2+\|\rho''\|)^k(8T)^k/k!,$$

and hence

$$\boldsymbol{P}\Big\{\sup_{0\leq t\leq T}|x_i^{k+1}(t)-x_i^k(t)| > 2^{-k}\Big\} \leq \mathrm{const}\cdot(l(c)^2+\|\rho''\|)^k(8T)^k/k!.$$

By Borel-Cantelli's lemma, $\{x_i^k(t) : 0 \leq t \leq T\}$ converges in the uniform norm with probability one. Since $T > 0$ was arbitrary, $x_i(t) = \lim_{k\to\infty} x_i^k(t)$ defines a continuous martingale $\{x_i(t) : t \geq 0\}$ which is clearly the unique solution of (1.6). It is easy to see that $d\langle x_i\rangle(t) = a(x_i(t))dt$ and $d\langle x_i, x_j\rangle(t) = \rho(x_i(t) - x_j(t))dt$ for $i \neq j$. Then $\{(x_1(t), \cdots, x_m(t)) : t \geq 0\}$ is a diffusion process with generator G^m defined by (1.7). □

Because of the exchangebility, the G^m-diffusion can be regarded as a measure-valued Markov process. Let $N(\mathbb{R})$ denote the space of integer-valued measures on $\mathbb{R}$. For $\theta > 0$, let $M_\theta(\mathbb{R}) = \{\theta^{-1}\sigma : \sigma \in N(\mathbb{R})\}$. Let ζ be the mapping from $\cup_{m=1}^\infty \mathbb{R}^m$ to $M_\theta(\mathbb{R})$ defined by

$$\zeta(x_1,\cdots,x_m) = \frac{1}{\theta}\sum_{i=1}^m \delta_{x_i}, \qquad m \geq 1. \tag{3.1}$$

Lemma 3.2 *For any integers $m, n \geq 1$ and any $f \in C^2(\mathbb{R}^n)$, we have*

$$\begin{aligned}
G^m F_{n,f}(\zeta(x_1,\cdots,x_m)) &= \frac{1}{2\theta^n}\sum_{\alpha=1}^n\sum_{l_1,\cdots,l_n=1}^m a(x_{l_\alpha})f''_{\alpha\alpha}(x_{l_1},\cdots,x_{l_n}) \\
&+ \frac{1}{2\theta^n}\sum_{\alpha,\beta=1,\alpha\neq\beta}^n\ \sum_{l_1,\cdots,l_n=1,l_\alpha=l_\beta}^m c(x_{l_\alpha})c(x_{l_\beta})f''_{\alpha\beta}(x_{l_1},\cdots,x_{l_n}) \\
&+ \frac{1}{2\theta^n}\sum_{\alpha,\beta=1,\alpha\neq\beta}^n\ \sum_{l_1,\cdots,l_n=1}^m \rho(x_{l_\alpha}-x_{l_\beta})f''_{\alpha\beta}(x_{l_1},\cdots,x_{l_n}).
\end{aligned} \tag{3.2}$$

Proof. By (3.1), we have

$$F_{n,f}(\zeta(x_1,\cdots,x_m)) = \frac{1}{\theta^n}\sum_{l_1,\cdots,l_n=1}^{m} f(x_{l_1},\cdots,x_{l_n}). \tag{3.3}$$

Observe that, for $1\le i\le m$,

$$\frac{d^2}{dx_i^2}F_{n,f}(\zeta(x_1,\cdots,x_m)) = \frac{1}{\theta^n}\sum_{\alpha,\beta=1}^{n}\sum_{\{\cdots\}} f''_{\alpha\beta}(x_{l_1},\cdots,x_{l_n}),$$

where $\{\cdots\}=\{$ for all $1\le l_1,\cdots,l_n\le m$ with $l_\alpha=l_\beta=i\}$. Then it is not hard to see that

$$\begin{aligned}
&\sum_{i=1}^{m} c(x_i)^2\frac{d^2}{dx_i^2}F_{n,f}(\zeta(x_1,\cdots,x_m))\\
=\;& \frac{1}{\theta^n}\sum_{\alpha,\beta=1}^{n}\sum_{l_1,\cdots,l_n=1,l_\alpha=l_\beta}^{m} c(x_{l_\alpha})c(x_{l_\beta})f''_{\alpha\beta}(x_{l_1},\cdots,x_{l_n})\\
=\;& \frac{1}{\theta^n}\sum_{\alpha=1}^{n}\sum_{l_1,\cdots,l_n=1}^{m} c(x_{l_\alpha})^2 f''_{\alpha\alpha}(x_{l_1},\cdots,x_{l_n})\\
&+\frac{1}{\theta^n}\sum_{\alpha,\beta=1,\alpha\ne\beta}^{n}\sum_{l_1,\cdots,l_n=1,l_\alpha=l_\beta}^{m} c(x_{l_\alpha})c(x_{l_\beta})f''_{\alpha\beta}(x_{l_1},\cdots,x_{l_n}).
\end{aligned}\tag{3.4}$$

On the other hand, for $1\le i\ne j\le m$,

$$\Big(\frac{d^2}{dx_idx_j}+\frac{d^2}{dx_idx_j}\Big)F_{n,f}(\zeta(x_1,\cdots,x_m)) = \frac{1}{\theta^n}\sum_{\alpha,\beta=1,\alpha\ne\beta}^{n}\sum_{\{\cdots\}} f''_{\alpha\beta}(x_{l_1},\cdots,x_{l_n}),$$

where $\{\cdots\}=\{$ for all $1\le l_1,\cdots,l_n\le m$ with $l_\alpha=i$ and $l_\beta=j\}$. It follows that

$$\begin{aligned}
&\sum_{i,j=1,i\ne j}^{m}\rho(x_i-x_j)\frac{d^2}{dx_idx_j}F_{n,f}(\zeta(x_1,\cdots,x_m))\\
=\;& \frac{1}{\theta^n}\sum_{\alpha,\beta=1,\alpha\ne\beta}^{n}\sum_{l_1,\cdots,l_n=1,l_\alpha\ne l_\beta}^{m}\rho(x_{l_\alpha}-x_{l_\beta})f''_{\alpha\beta}(x_{l_1},\cdots,x_{l_n}).
\end{aligned}$$

Using this and (3.4) with $c(x_i)^2$ replaced by $\rho(0)$,

$$\begin{aligned}
&\sum_{i,j=1}^{m}\rho(x_i-x_j)\frac{d^2}{dx_idx_j}F_{n,f}(\zeta(x_1,\cdots,x_m))\\
=\;& \frac{1}{\theta^n}\sum_{\alpha=1}^{n}\sum_{l_1,\cdots,l_n=1}^{m}\rho(0)f''_{\alpha\alpha}(x_{l_1},\cdots,x_{l_n})\\
&+\frac{1}{\theta^n}\sum_{\alpha,\beta=1,\alpha\ne\beta}^{n}\sum_{l_1,\cdots,l_n=1}^{m}\rho(x_{l_\alpha}-x_{l_\beta})f''_{\alpha\beta}(x_{l_1},\cdots,x_{l_n}).
\end{aligned}\tag{3.5}$$

Then we have the desired result from (3.4) and (3.5). $\square$

Suppose that $X(t) = (x_1(t), \cdots, x_m(t))$ is a Markov process in $\mathbb{R}^m$ generated by G^m. Based on (1.2) and Lemma 3.2, it is easy to show that $\zeta(X(t))$ is a Markov process in $M_\theta(\mathbb{R})$ with generator $\mathcal{A}_\theta$ given by

$$\begin{aligned}\mathcal{A}_\theta F(\mu) &= \frac{1}{2}\int_{\mathbb{R}} a(x)\frac{d^2}{dx^2}\frac{\delta F(\mu)}{\delta\mu(x)}\mu(dx) + \frac{1}{2\theta}\int_{\mathbb{R}^2} c(x)c(y)\frac{d^2}{dxdy}\frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)}\delta_x(dy)\mu(dx) \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^2}\rho(x-y)\frac{d^2}{dxdy}\frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)}\mu(dx)\mu(dy). \end{aligned} \tag{3.6}$$

In particular, if

$$F(\mu) = f(\langle\phi_1,\mu\rangle, \cdots, \langle\phi_n,\mu\rangle), \qquad \mu\in M_\theta(\mathbb{R}), \tag{3.7}$$

for $f\in C^2(\mathbb{R}^n)$ and $\{\phi_i\}\subset C^2(\mathbb{R})$, then

$$\begin{aligned}\mathcal{A}_\theta F(\mu) &= \frac{1}{2}\sum_{i=1}^n f_i'(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_n,\mu\rangle)\langle a\phi_i'',\mu\rangle \\ &+ \frac{1}{2\theta}\sum_{i,j=1}^n f_{ij}''(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_n,\mu\rangle)\langle c^2\phi_i'\phi_j',\mu\rangle \\ &+ \frac{1}{2}\sum_{i,j=1}^n f_{ij}''(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_n,\mu\rangle)\int_{\mathbb{R}^2}\rho(x-y)\phi_i'(x)\phi_j'(y)\mu(dx)\mu(dy).\end{aligned} \tag{3.8}$$

Now we introduce a branching mechanism to the interacting particle system. Suppose that for each $x\in\mathbb{R}$ we have a discrete probability distribution $p(x) = \{p_i(x) : i = 0, 1, \cdots\}$ such that each $p_i(\cdot)$ is a Borel measurable function on $\mathbb{R}$. This serves as the distribution of the offspring number produced by a particle that dies at site $x\in\mathbb{R}$. We assume that

$$p_1(x) = 0, \quad \sum_{i=1}^\infty ip_i(x) = 1, \tag{3.9}$$

and

$$\sigma_p(x) := \sum_{i=1}^\infty i^2 p_i(x) - 1 \tag{3.10}$$

is bounded in $x\in\mathbb{R}$. Let $\Gamma_\theta(\mu, d\nu)$ be the probability kernel on $M_\theta(\mathbb{R})$ defined by

$$\int_{M_\theta(\mathbb{R})} F(\nu)\Gamma_\theta(\mu,d\nu) = \frac{1}{\theta\mu(1)}\sum_{i=1}^{\theta\mu(1)}\sum_{j=0}^\infty p_j(x_i)F\Big(\mu + (j-1)\theta^{-1}\delta_{x_i}\Big), \tag{3.11}$$

where $\mu \in M_\theta(\mathbb{R})$ is given by

$$\mu = \frac{1}{\theta}\sum_{i=1}^{\theta\mu(1)} \delta_{x_i}.$$

For a constant $\gamma > 0$, we define the bounded operator $\mathcal{B}_\theta$ on $B(M_\theta(\mathbb{R}))$ by

$$\mathcal{B}_\theta F(\mu) = \gamma\theta^2[\theta \wedge \mu(1)] \int_{M_\theta(\mathbb{R})} [F(\nu) - F(\mu)]\Gamma_\theta(\mu, d\nu). \tag{3.12}$$

In view of (1.6), $\mathcal{A}_\theta$ generates a Feller Markov process on $M_\theta(\mathbb{R})$, then so does $\mathcal{L}_\theta := \mathcal{A}_\theta + \mathcal{B}_\theta$ by Ethier-Kurtz (1986, p.37). We shall call the process generated by $\mathcal{L}_\theta$ an *interacting-branching particle system* with parameters (a, ρ, γ, p) and unit mass $1/\theta$. Heuristically, each particle in the system has mass $1/\theta$, $a(\cdot)$ represents the migration speed of the particles and $\rho(\cdot)$ describes the interaction between them. The branching times of the system are determined by the killing density $\gamma\theta^2[\theta\wedge\mu(1)]$, where the truncation "$\theta\wedge\mu(1)$" is introduced to make the branching not too fast even when the total mass is large. At each branching time, with equal probability, one particle in the system is randomly chosen, which is killed at its site $x \in \mathbb{R}$ and the offspring are produced at $x \in \mathbb{R}$ according to the distribution $\{p_i(x) : i = 0, 1, \cdots\}$. If F is given by (3.7), then $\mathcal{B}_\theta F(\mu)$ is equal to

$$\frac{\gamma[\theta \wedge \mu(1)]}{2\mu(1)} \sum_{\alpha,\beta=1}^{n} \sum_{j=1}^{\infty} (j-1)^2 \langle p_j f''_{\alpha\beta}(\langle\phi_1, \mu\rangle + \xi_j\phi_1, \cdots, \langle\phi_n, \mu\rangle + \xi_j\phi_n)\phi_\alpha\phi_\beta, \mu\rangle \tag{3.13}$$

for some constant $0 < \xi_j < (j-1)/\theta$. This follows from (3.11) and (3.12) by Taylor's expansion.

4 Continuous branching density

In this section, we shall construct a solution of the martingale problem of the SDSM with continuous branching density by using particle system approximation. Assume that $\sigma \in C(\mathbb{R})$ can be extended continuously to $\hat{\mathbb{R}}$. Let $\mathcal{A}$ and $\mathcal{B}$ be given by (1.5) and (1.8), respectively. Observe that, if

$$F(\mu) = f(\langle\phi_1, \mu\rangle, \cdots, \langle\phi_n, \mu\rangle), \qquad \mu \in M(\mathbb{R}), \tag{4.1}$$

for $f \in C^2(\mathbb{R}^n)$ and $\{\phi_i\} \subset C^2(\mathbb{R})$, then

$$\begin{aligned} \mathcal{A}F(\mu) &= \frac{1}{2}\sum_{i=1}^{n} f'_i(\langle\phi_1, \mu\rangle, \cdots, \langle\phi_n, \mu\rangle)\langle a\phi''_i, \mu\rangle \qquad (4.2)\\ &+ \frac{1}{2}\sum_{i,j=1}^{n} f''_{ij}(\langle\phi_1, \mu\rangle, \cdots, \langle\phi_n, \mu\rangle) \int_{\mathbb{R}^2} \rho(x-y)\phi'_i(x)\phi'_j(y)\mu(dx)\mu(dy), \end{aligned}$$

and

$$\mathcal{B}F(\mu) = \frac{1}{2}\sum_{i,j=1}^{n} f''_{ij}(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_n,\mu\rangle)\langle\sigma\phi_i\phi_j,\mu\rangle. \tag{4.3}$$

Let $\{\theta_k\}$ be any sequence such that $\theta_k \to \infty$ as $k \to \infty$. Suppose that $\{X_t^{(k)} : t \geq 0\}$ is a sequence of cádlág interacting-branching particle systems with parameters $(a, \rho, \gamma_k, p^{(k)})$, unit mass $1/\theta_k$ and initial states $X_0^{(k)} = \mu_k \in M_{\theta_k}(\mathbb{R})$. In an obvious way, we may also regard $\{X_t^{(k)} : t \geq 0\}$ as a process with state space $M(\hat{\mathbb{R}})$. Let σ_k be defined by (3.10) with p_i replaced by $p_i^{(k)}$.

Lemma 4.1 *Suppose that the sequences $\{\gamma_k\sigma_k\}$ and $\{\langle 1,\mu_k\rangle\}$ are bounded. Then $\{X_t^{(k)} : t \geq 0\}$ form a tight sequence in $D([0,\infty), M(\hat{\mathbb{R}}))$.*

Proof. By the assumption (3.9), it is easy to show that $\{\langle 1, X_t^{(k)}\rangle : t \geq 0\}$ is a martingale. Then we have

$$\boldsymbol{P}\Big\{\sup_{t\geq 0}\langle 1, X_t^{(k)}\rangle > \eta\Big\} \leq \frac{\langle 1,\mu_k\rangle}{\eta}$$

for any $\eta > 0$. That is, $\{X_t^{(k)} : t \geq 0\}$ satisfies the compact containment condition of Ethier and Kurtz (1986, p.142). Let $\mathcal{L}_k$ denote the generator of $\{X_t^{(k)} : t \geq 0\}$ and let F be given by (4.1) with $f \in C_0^2(\mathbb{R}^n)$ and with each $\phi_i \in C_\partial^2(\mathbb{R})$ bounded away from zero. Then

$$F(X_t^{(k)}) - F(X_0^{(k)}) - \int_0^t \mathcal{L}_k F(X_s^{(k)})ds, \qquad t \geq 0,$$

is a martingale and the desired tightness follows from the result of Ethier and Kurtz (1986, p.145). □

In the sequel of this section, we assume $\{\phi_i\} \subset C_\partial^2(\mathbb{R})$. In this case, (4.1), (4.2) and (4.3) can be extended to continuous functions on $M(\hat{\mathbb{R}})$. Let $\hat{\mathcal{A}}F(\mu)$ and $\hat{\mathcal{B}}F(\mu)$ be defined respectively by the right hand side of (4.2) and (4.3) and let $\hat{\mathcal{L}}F(\mu) = \hat{\mathcal{A}}F(\mu) + \hat{\mathcal{B}}F(\mu)$, all defined as continuous functions on $M(\hat{\mathbb{R}})$.

Lemma 4.2 *Let $\mathcal{D}(\hat{\mathcal{L}})$ be the totality of all functions of the form (4.1) with $f \in C_0^2(\mathbb{R}^n)$ and with each $\phi_i \in C_\partial^2(\mathbb{R})$ bounded away from zero. Suppose further that $\gamma_k\sigma_k \to \sigma$ uniformly and $\mu_k \to \mu \in M(\hat{\mathbb{R}})$ as $k \to \infty$. Then any limit point $\boldsymbol{Q}_\mu$ of the distributions of $\{X_t^{(k)} : t \geq 0\}$ is supported by $C([0,\infty), M(\hat{\mathbb{R}}))$ under which*

$$F(w_t) - F(w_0) - \int_0^t \hat{\mathcal{L}}F(w_s)ds, \qquad t \geq 0, \tag{4.4}$$

is a martingale for each $F \in \mathcal{D}(\hat{\mathcal{L}})$, where $\{w_t : t \geq 0\}$ denotes the coordinate process of $C([0,\infty), M(\hat{\mathbb{R}}))$.

Proof. We use the notation introduced in the proof of Lemma 4.1. By passing to a subsequence if it is necessary, we may assume that the distribution of $\{X_t^{(k)} : t \geq 0\}$ on $D([0, \infty), M(\hat{\mathbb{R}}))$ converges to $\boldsymbol{Q}_\mu$. Using Skorokhod's representation, we may assume that the processes $\{X_t^{(k)} : t \geq 0\}$ are defined on the same probability space and the sequence converges almost surely to a cádlág process $\{X_t : t \geq 0\}$ with distribution $\boldsymbol{Q}_\mu$ on $D([0, \infty), M(\hat{\mathbb{R}}))$; see e.g. Ethier and Kurtz (1986, p.102). Let $K(X) = \{t \geq 0 : \boldsymbol{P}\{X_t = X_{t\text{-}}\} = 1\}$. By Ethier and Kurtz (1986, p.118), for each $t \in K(X)$ we have a.s. $\lim_{k\to\infty} X_t^{(k)} = X_t$. Recall that f and f''_{ij} are rapidly decreasing and each ϕ_i is bounded away from zero. Since $\gamma_k a_k \to \sigma$ uniformly, for $t \in K(X)$ we have a.s. $\lim_{k\to\infty} \mathcal{L}_k F(X_t^{(k)}) = \hat{\mathcal{L}} F(X_t)$ boundedly by (3.8), (3.13) and the definition of $\hat{\mathcal{L}}$. Suppose that $\{H_i\}_{i=1}^n \subset C(M(\hat{\mathbb{R}}))$ and $\{t_i\}_{i=1}^{n+1} \subset K(X)$ with $0 \leq t_1 < \cdots < t_n < t_{n+1}$. By Ethier and Kurtz (1986, p.31), the set $K(X)$ is at most countable. Then

$$\begin{aligned}
&\boldsymbol{E}\Big\{\Big[F(X_{t_{n+1}}) - F(X_{t_n}) - \int_{t_n}^{t_{n+1}} \hat{\mathcal{L}} F(X_s) ds\Big] \prod_{i=1}^n H_i(X_{t_i})\Big\} \\
= \;& \boldsymbol{E}\Big\{F(X_{t_{n+1}}) \prod_{i=1}^n H_i(X_{t_i})\Big\} - \boldsymbol{E}\Big\{F(X_{t_n}) \prod_{i=1}^n H_i(X_{t_i})\Big\} \\
& \qquad - \int_{t_n}^{t_{n+1}} \boldsymbol{E}\Big\{\hat{\mathcal{L}} F(X_s) \prod_{i=1}^n H_i(X_{t_i})\Big\} ds \\
= \;& \lim_{k\to\infty} \boldsymbol{E}\Big\{F(X_{t_{n+1}}^{(k)}) \prod_{i=1}^n H_i(X_{t_i}^{(k)})\Big\} - \lim_{k\to\infty} \boldsymbol{E}\Big\{F(X_{t_n}^{(k)}) \prod_{i=1}^n H_i(X_{t_i}^{(k)})\Big\} \\
& \qquad - \lim_{k\to\infty} \int_{t_n}^{t_{n+1}} \boldsymbol{E}\Big\{\mathcal{L}_k F(X_s^{(k)}) \prod_{i=1}^n H_i(X_{t_i}^{(k)})\Big\} ds \\
= \;& \lim_{k\to\infty} \boldsymbol{E}\Big\{\Big[F(X_{t_{n+1}}^{(k)}) - F(X_{t_n}^{(k)}) - \int_{t_n}^{t_{n+1}} \mathcal{L}_k F(X_s^{(k)}) ds\Big] \prod_{i=1}^n H_i(X_{t_i}^{(k)})\Big\} \\
= \;& 0.
\end{aligned}$$

By the right continuity of $\{X_t : t \geq 0\}$, the equality

$$\boldsymbol{E}\Big\{\Big[F(X_{t_{n+1}}) - F(X_{t_n}) - \int_{t_n}^{t_{n+1}} \hat{\mathcal{L}} F(X_s) ds\Big] \prod_{i=1}^n H_i(X_{t_i})\Big\} = 0$$

holds without the restriction $\{t_i\}_{i=1}^{n+1} \subset K(X)$. That is,

$$F(X_t) - F(X_0) - \int_0^t \hat{\mathcal{L}} F(X_s) ds, \qquad t \geq 0,$$

is a martingale. As in Wang (1998, pp.783-784) one can show that $\{X_t : t \geq 0\}$ is in fact a.s. continuous. $\square$

Lemma 4.3 *Let $\mathcal{D}(\hat{\mathcal{L}})$ be as in Lemma 4.2. Then for each $\mu \in M(\hat{\mathbb{R}})$, there is a probability measure $\boldsymbol{Q}_\mu$ on $C([0,\infty), M(\hat{\mathbb{R}}))$ under which (4.4) is a martingale for each $F \in \mathcal{D}(\hat{\mathcal{L}})$.*

Proof. It is easy to find $\mu_k \in M_{\theta_k}(\mathbb{R})$ such that $\mu_k \to \mu$ as $k \to \infty$. Then, by Lemma 4.2, it suffices to construct a sequence $(\gamma_k, p^{(k)})$ such that $\gamma_k \sigma_k \to \sigma$ as $k \to \infty$. This is elementary. One choice is described as follows. Let $\gamma_k = 1/\sqrt{k}$ and $\sigma_k = \sqrt{k}(\sigma + 1/\sqrt{k})$. Then the system of equations

$$\begin{cases} p_0^{(k)} + p_2^{(k)} + p_k^{(k)} & = 1, \\ 2p_2^{(k)} + kp_k^{(k)} & = 1, \\ 4p_2^{(k)} + k^2 p_k^{(k)} & = \sigma_k + 1, \end{cases}$$

has the unique solution

$$p_0^{(k)} = \frac{\sigma_k + k - 1}{2k}, \quad p_2^{(k)} = \frac{k - 1 - \sigma_k}{2(k-2)}, \quad p_k^{(k)} = \frac{\sigma_k - 1}{k(k-2)},$$

where each $p_i^{(k)}$ is nonnegative for sufficiently large $k \geq 3$. □

Lemma 4.4 *Let $\boldsymbol{Q}_\mu$ be given by Lemma 4.3. Then for $n \geq 1$, $t \geq 0$ and $\mu \in M(\mathbb{R})$ we have*

$$\boldsymbol{Q}_\mu\{\langle 1, w_t\rangle^n\} \leq \langle 1, \mu\rangle^n + \frac{1}{2}n(n-1)\|\sigma\| \int_0^t \boldsymbol{Q}_\mu\{\langle 1, w_s\rangle^{n-1}\}ds.$$

Consequently, $\boldsymbol{Q}_\mu\{\langle 1, w_t\rangle^n\}$ is a locally bounded function of $t \geq 0$. Let $\mathcal{D}(\hat{\mathcal{L}})$ be the union of all functions of the form (4.1) with $f \in C_0^2(\mathbb{R}^n)$ and $\{\phi_i\} \subset C_\partial^2(\mathbb{R})$ and all functions of the form $F_{m,f}(\mu) = \langle f, \mu^m\rangle$ with $f \in C_\partial^2(\mathbb{R}^m)$. Then (4.4) under $\boldsymbol{Q}_\mu$ is a martingale for each $F \in \mathcal{D}(\hat{\mathcal{L}})$.

Proof. For any $k \geq 1$, take $f_k \in C_0^2(\mathbb{R}))$ such that $f_k(z) = z^n$ for $0 \leq z \leq k$ and $f_k''(z) \leq n(n-1)z^{n-2}$ for all $z \geq 0$. Let $F_k(\mu) = f_k(\langle 1, \mu\rangle)$. Then $\mathcal{A}F_n(\mu) = 0$ and

$$\mathcal{B}F_k(\mu) \leq \frac{1}{2}n(n-1)\|\sigma\|\langle 1, \mu\rangle^{n-1}.$$

Since

$$F_k(X_t) - F_k(X_0) - \int_0^t \mathcal{L}F_k(\langle 1, X_s\rangle)ds, \qquad t \geq 0,$$

is a martingale, we get

$$\begin{aligned} \boldsymbol{Q}_\mu f_k(\langle 1, X_t\rangle^n) &\leq f_k(\langle 1, \mu\rangle) + \frac{1}{2}n(n-1)\|\sigma\| \int_0^t \boldsymbol{Q}_\mu(\langle 1, X_s\rangle^{n-1})ds \\ &\leq \langle 1, \mu\rangle^n + \frac{1}{2}n(n-1)\|\sigma\| \int_0^t \boldsymbol{Q}_\mu(\langle 1, X_s\rangle^{n-1})ds. \end{aligned}$$

Then the desired estimate follows by Fatou's Lemma. The last assertion is an immediate consequence of Lemma 4.3. □

Lemma 4.5 *Let $\boldsymbol{Q}_\mu$ be given by Lemma 4.3. Then for $\mu \in M(\mathbb{R})$ and $\phi \in C^2_\partial(\mathbb{R})$,*

$$M_t(\phi) := \langle \phi, w_t \rangle - \langle \phi, \mu \rangle - \frac{1}{2} \int_0^t \langle a\phi'', w_s \rangle ds, \qquad t \geq 0, \tag{4.5}$$

is a $\boldsymbol{Q}_\mu$-martingale with quadratic variation process

$$\langle M(\phi) \rangle_t = \int_0^t \langle \sigma\phi^2, w_s \rangle ds + \int_0^t ds \int_{\hat{\mathbb{R}}} \langle h(z - \cdot)\phi', w_s \rangle^2 dz. \tag{4.6}$$

Proof. It is easy to check that, if $F_n(\mu) = \langle \phi, \mu \rangle^n$, then

$$\begin{aligned} \hat{\mathcal{L}} F_n(\mu) &= \frac{n}{2} \langle \phi, \mu \rangle^{n-1} \langle a\phi'', \mu \rangle + \frac{n(n-1)}{2} \langle \phi, \mu \rangle^{n-2} \int_{\hat{\mathbb{R}}} \langle h(z - \cdot)\phi', \mu \rangle^2 dz \\ &\quad + \frac{n(n-1)}{2} \langle \phi, \mu \rangle^{n-2} \langle \sigma\phi^2, \mu \rangle. \end{aligned}$$

It follows that both (4.5) and

$$\begin{aligned} M_t^2(\phi) &:= \langle \phi, w_t \rangle^2 - \langle \phi, \mu \rangle^2 - \int_0^t \langle \phi, w_s \rangle \langle a\phi'', w_s \rangle ds \\ &\quad - \int_0^t ds \int_{\hat{\mathbb{R}}} \langle h(z - \cdot)\phi', w_s \rangle^2 dz - \int_0^t \langle \sigma\phi^2, w_s \rangle ds \end{aligned} \tag{4.7}$$

are martingales. By (4.5) and Itô's formula we have

$$\langle \phi, w_t \rangle^2 = \langle \phi, \mu \rangle^2 + \int_0^t \langle \phi, w_s \rangle \langle a\phi'', w_s \rangle ds + 2 \int_0^t \langle \phi, w_s \rangle dM_s(\phi) + \langle M(\phi) \rangle_t. \tag{4.8}$$

Comparing (4.7) and (4.8) we get the conclusion. □

Observe that the martingales $\{M(\phi) : t \geq 0\}$ defined by (4.5) form a system which is linear in $\phi \in C^2_\partial(\mathbb{R})$. Because of the presence of the derivative ϕ' in the variation process (4.6), it seems hard to extend the definition of $\{M(\phi) : t \geq 0\}$ to a general function $\phi \in B(\hat{\mathbb{R}})$. However, following the method of Walsh (1986), one can still define the stochastic integral

$$\int_0^t \int_{\hat{\mathbb{R}}} \phi(s, x) M(ds, dx), \qquad t \geq 0,$$

if both $\phi(s, x)$ and $\phi'(s, x)$ can be extended continuously to $[0, \infty) \times \hat{\mathbb{R}}$. With those in hand, we have the following

Lemma 4.6 *Let $\boldsymbol{Q}_\mu$ be given by Lemma 4.3. Then for any $t \geq 0$ and $\phi \in C^2_\partial(\mathbb{R})$ we have a.s.*

$$\langle \phi, w_t \rangle = \langle \hat{P}_t \phi, \mu \rangle + \int_0^t \int_{\hat{\mathbb{R}}} \hat{P}_{t-s} \phi(x) M(ds, dx).$$

Proof. For any partition $\Delta_n := \{0 = t_0 < t_1 < \cdots < t_n = t\}$ of $[0, t]$, we have

$$\begin{aligned}\langle \phi, w_t\rangle - \langle \hat{P}_t\phi, \mu\rangle &= \sum_{i=1}^{n} \langle \hat{P}_{t-t_i}\phi - \hat{P}_{t-t_{i-1}}\phi, w_{t_i}\rangle \\ &\quad + \sum_{i=1}^{n} [\langle \hat{P}_{t-t_{i-1}}\phi, w_{t_i}\rangle - \langle \hat{P}_{t-t_{i-1}}\phi, w_{t_{i-1}}\rangle].\end{aligned}$$

Let $\|\Delta_n\| = \max\{|t_i - t_{i-1}| : 1 \leq i \leq n\}$ and assume $\|\Delta_n\| \to 0$ as $n \to \infty$. Then

$$\begin{aligned}\lim_{n\to\infty} \sum_{i=1}^{n} \langle \hat{P}_{t-t_i}\phi - \hat{P}_{t-t_{i-1}}\phi, w_{t_i}\rangle &= -\lim_{n\to\infty} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \langle \hat{P}_{t-s}\hat{G}\phi, w_{t_i}\rangle ds \\ &= -\int_0^t \langle \hat{P}_{t-s}\hat{G}\phi, w_s\rangle ds.\end{aligned}$$

Using Lemma 4.5 we have

$$\begin{aligned}&\lim_{n\to\infty} \sum_{i=1}^{n} [\langle \hat{P}_{t-t_{i-1}}\phi, w_{t_i}\rangle - \langle \hat{P}_{t-t_{i-1}}\phi, w_{t_{i-1}}\rangle] \\ =\ & \lim_{n\to\infty} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \int_{\hat{\mathbb{R}}} \hat{P}_{t-t_{i-1}}\phi M(ds, dx) + \lim_{n\to\infty} \frac{1}{2} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \langle a(\hat{P}_{t-t_{i-1}}\phi)'', w_s\rangle ds \\ =\ & \int_0^t \int_{\hat{\mathbb{R}}} \hat{P}_{t-s}\phi M(ds, dx) + \frac{1}{2} \int_0^t \langle a(\hat{P}_{t-s}\phi)'', w_s\rangle ds.\end{aligned}$$

Combining those we get the desired conclusion. □

Theorem 4.1 *Let $\mathcal{D}(\mathcal{L})$ be the union of all functions of the form (4.1) with $f \in C^2(\mathbb{R}^n)$ and $\{\phi_i\} \subset C^2(\mathbb{R})$ and all functions of the form $F_{m,f}(\mu) = \langle f, \mu^m\rangle$ with $f \in C^2(\mathbb{R}^m)$. Let $\{w_t : t \geq 0\}$ denote the coordinate process of $C([0, \infty), M(\mathbb{R}))$. Then for each $\mu \in M(\mathbb{R})$ there is a probability measure $\boldsymbol{Q}_\mu$ on $C([0, \infty), M(\mathbb{R}))$ such that $\boldsymbol{Q}_\mu\{\langle 1, w_t\rangle^m\}$ is locally bounded in $t \geq 0$ for every $m \geq 1$ and such that $\{w_t : t \geq 0\}$ under $\boldsymbol{Q}_\mu$ is a solution of the $(\mathcal{L}, \mathcal{D}(\mathcal{L}), \mu)$-martingale problem.*

Proof. Let $\boldsymbol{Q}_\mu$ be the probability measure on $C([0, \infty), M(\hat{\mathbb{R}}))$ provided by Lemma 4.3. The desired result will follow once it is proved that

$$\boldsymbol{Q}_\mu\{w_t(\{\partial\}) = 0 \text{ for all } t \in [0, u]\} = 1, \qquad u > 0. \tag{4.9}$$

For any $\phi \in C^2_\partial(\mathbb{R})$, we may use Lemma 4.6 to see that

$$M_t^u(\phi) := \langle \hat{P}_{u-t}\phi, w_t\rangle - \langle \hat{P}_t\hat{P}_{u-t}\phi, \mu\rangle = \int_0^t \int_{\hat{\mathbb{R}}} \hat{P}_{u-s}\phi M(ds, dx), \quad t \in [0, u],$$

is a continuous martingale with quadratic variation process

$$\begin{aligned}\langle M^u(\phi)\rangle_t &= \int_0^t \langle\sigma(\hat{P}_{u-s}\phi)^2, w_s\rangle ds + \int_0^t ds \int_{\hat{\mathbb{R}}} \langle h(z-\cdot)\hat{P}_{u-s}(\phi'), w_s\rangle^2 dz \\ &= \int_0^t \langle\sigma(\hat{P}_{u-s}\phi)^2, w_s\rangle ds + \int_0^t ds \int_{\hat{\mathbb{R}}} \langle h(z-\cdot)(\hat{P}_{u-s}\phi)', w_s\rangle^2 dz.\end{aligned}$$

By a martingale inequality we have

$$\begin{aligned}&\boldsymbol{Q}_\mu\Big\{\sup_{0\le t\le u} |\langle\hat{P}_{u-t}\phi, w_t\rangle - \langle\hat{P}_u\phi,\mu\rangle|^2\Big\} \\ \le\ & 4\int_0^u \boldsymbol{Q}_\mu\{\langle\sigma(\hat{P}_{u-s}\phi)^2, w_s\rangle\}ds + 4\int_0^u ds\int_{\hat{\mathbb{R}}} \boldsymbol{Q}_\mu\{\langle h(z-\cdot)\hat{P}_{u-s}(\phi'), w_s\rangle^2\}dz \\ \le\ & 4\int_0^u \langle\sigma(\hat{P}_{u-s}\phi)^2, \mu\hat{P}_s\rangle ds + 4\int_{\hat{\mathbb{R}}} h(z)^2 dz \int_0^u \boldsymbol{Q}_\mu\{\langle 1, w_s\rangle\langle\hat{P}_{u-s}(\phi')^2, w_s\rangle\}ds \\ \le\ & 4\int_0^u \langle\sigma(\hat{P}_{u-s}\phi)^2, \mu\hat{P}_s\rangle ds + 4\|\phi'\|^2\int_{\hat{\mathbb{R}}} h(z)^2 dz\int_0^u \boldsymbol{Q}_\mu\{\langle 1, w_s\rangle^2\}ds.\end{aligned}$$

Choose a sequence $\{\phi_k\}\subset C^2_\partial(\mathbb{R})$ such that $\phi_k(\cdot)\to 1_{\{\partial\}}(\cdot)$ boundedly and $\|\phi_k'\|\to 0$ as $k\to\infty$. Replacing ϕ by ϕ_k in the above and letting $k\to\infty$ we obtain (4.9). □

Combining Theorems 2.2 and 4.1 we get the existence of the SDSM in the case where $\sigma\in C(\mathbb{R})^+$ extends continuously to $\hat{\mathbb{R}}$.

5 Measurable branching density

In this section, we shall use the dual process to extend the construction of the SDSM to a general bounded Borel branching density. Given $\sigma\in B(\mathbb{R})^+$, let $\{(M_t, Y_t): t\ge 0\}$ be defined as in section 2. Choose any sequence of functions $\{\sigma_k\}\subset C(\mathbb{R})^+$ which extends continuously to $\hat{\mathbb{R}}$ and $\sigma_k\to\sigma$ boundedly and pointwise. Suppose that $\{\mu_k\}\subset M(\mathbb{R})$ and $\mu_k\to\mu\in M(\mathbb{R})$ as $k\to\infty$. For each $k\ge 1$, let $\{X_t^{(k)}: t\ge 0\}$ be a SDSM with parameters (a,ρ,σ_k) and initial state $\mu_k\in M(\mathbb{R})$ and let $\boldsymbol{Q}_k$ denote the distribution of $\{X_t^{(k)}: t\ge 0\}$ on $C([0,\infty), M(\mathbb{R}))$.

Lemma 5.1 *Under the above hypotheses, $\{\boldsymbol{Q}_k\}$ is a tight sequence of probability measures on $C([0,\infty), M(\mathbb{R}))$.*

Proof. Since $\{\langle 1, X_t^{(k)}\rangle : t\ge 0\}$ is a martingale, one can see as in the proof of Lemma 4.1 that $\{X_t^{(k)}: t\ge 0\}$ satisfies the compact containment condition of Ethier and Kurtz (1986, p.142). Let $\mathcal{L}_k$ denote the generator of $\{X_t^{(k)}: t\ge 0\}$ and let F be given by (4.1) with $f\in C_0^2(\mathbb{R}^n)$ and with $\{\phi_i\}\subset C^2_\partial(\mathbb{R})$. Then

$$F(X_t^{(k)}) - F(X_0^{(k)}) - \int_0^t \mathcal{L}_k F(X_s^{(k)})ds, \qquad t\ge 0,$$

is a martingale. Since the sequence $\{\sigma_k\}$ is uniformly bounded, the tightness of $\{X_t^{(k)} : t \geq 0\}$ in $C([0,\infty), M(\hat{\mathbb{R}}))$ follows from Lemma 4.4 and the result of Ethier and Kurtz (1986, p.145). We shall prove that any limit point of $\{\boldsymbol{Q}_k\}$ is supported by $C([0,\infty), M(\mathbb{R}))$ so that $\{\boldsymbol{Q}_k\}$ is also tight as probability measures on $C([0,\infty), M(\mathbb{R}))$. Without loss of generality, we may assume $\boldsymbol{Q}_k$ converges as $k \to \infty$ to $\boldsymbol{Q}_\mu$ by weak convergence of probability measures on $C([0,\infty), M(\hat{\mathbb{R}}))$. Let $\phi_n \in C^2(\mathbb{R})^+$ be such that $\phi_n(x) = 0$ when $\|x\| \leq n$ and $\phi_n(x) = 1$ when $\|x\| \geq 2n$ and $\|\phi_n'\| \to 0$ as $n \to \infty$. Fix $u > 0$ and let m_n be such that $\phi_{m_n}(x) \leq 2P_t\phi_n(x)$ for all $0 \leq t \leq u$ and $x \in \mathbb{R}$. For any $\alpha > 0$, the paths $w \in C([0,\infty), M(\hat{\mathbb{R}}))$ satisfying $\sup_{0\leq t\leq u}\langle \phi_{m_n}, w_t\rangle > \alpha$ constitute an open subset of $C([0,\infty), M(\hat{\mathbb{R}}))$. Then, by an equivalent condition for weak convergence,

$$\begin{aligned}
&\boldsymbol{Q}_\mu\Big\{\sup_{0\leq t\leq u} w_t(\{(\partial)\}) > \alpha\Big\} \leq \boldsymbol{Q}_\mu\Big\{\sup_{0\leq t\leq u}\langle \phi_{m_n}, w_t\rangle > \alpha\Big\} \\
\leq\;& \liminf_{k\to\infty}\boldsymbol{Q}_k\Big\{\sup_{0\leq t\leq u}\langle \phi_{m_n}, w_t\rangle > \alpha\Big\} \leq \sup_{k\geq 1}\frac{4}{\alpha^2}\boldsymbol{Q}_k\Big\{\sup_{0\leq t\leq u}\langle \hat{P}_{u-t}\phi_n, w_t\rangle^2\Big\} \\
\leq\;& \sup_{k\geq 1}\frac{8}{\alpha^2}\boldsymbol{Q}_k\Big\{\sup_{0\leq t\leq u}|\langle \hat{P}_{u-t}\phi_n, w_t\rangle - \langle \hat{P}_u\phi_n, \mu_k\rangle|^2\Big\} + \sup_{k\geq 1}\sup_{0\leq t\leq u}\frac{8}{\alpha^2}\langle \hat{P}_u\phi_n, \mu_k\rangle^2.
\end{aligned}$$

As in the proof of Theorem 4.1, one can see that the right hand side goes to zero as $n \to \infty$. Then $\boldsymbol{Q}_\mu$ is supported by $C([0,\infty), M(\mathbb{R}))$. □

Theorem 5.1 *The distribution $Q_t^{(k)}(\mu_k, \cdot)$ of $X_t^{(k)}$ on $M(\mathbb{R})$ converges as $k \to \infty$ to a probability measure $Q_t(\mu, \cdot)$ on $M(\mathbb{R})$ given by*

$$\int_{M(\mathbb{R})}\langle f, \nu^m\rangle Q_t(\mu, d\nu) = \boldsymbol{E}_{m,f}^\sigma\Big[\langle Y_t, \mu^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s - 1)ds\Big\}\Big]. \tag{5.1}$$

Moreover, $(Q_t)_{t\geq 0}$ is a transition semigroup on $M(\mathbb{R})$.

Proof. By Lemma 5.1, $\{Q_t^{(k)}(\mu_k, d\nu)\}$ is a tight sequence of probability measures on $M(\mathbb{R})$. Take any subsequence $\{k_i\}$ so that $Q_t^{(k_i)}(\mu_{k_i}, d\nu)$ converges as $i \to \infty$ to some probability measure $Q_t(\mu, d\nu)$ on $M(\mathbb{R})$. By Lemma 2.1 we have

$$\begin{aligned}
&\int_{M(\mathbb{R})} 1_{[a,\infty)}(\langle 1, \nu\rangle)\langle 1, \nu^m\rangle Q_t^{(k)}(\mu_k, d\nu) \\
\leq\;& \frac{1}{a}\int_{M(\mathbb{R})}\langle 1, \nu^{m+1}\rangle Q_t^{(k)}(\mu_k, d\nu) \\
\leq\;& \frac{1}{a}\sum_{i=0}^m 2^{-i}(m+1)^i m^i\|\sigma_k\|^i\langle 1, \mu_k\rangle^{m-i+1},
\end{aligned}$$

which goes to zero as $a \to \infty$ uniformly in $k \geq 1$. Then for $f \in C(\hat{\mathbb{R}})^+$ we may regard $\{\langle f, \nu^m\rangle Q_t^{(k)}(\mu_k, d\nu)\}$ as a tight sequence of finite measures on $M(\hat{\mathbb{R}})$. By passing to a smaller subsequence $\{k_i\}$ we may assume that $\langle f, \nu^m\rangle Q_t^{(k_i)}(\mu_{k_i}, d\nu)$ converges to a finite measure $K_t(\mu, d\nu)$ on $M(\hat{\mathbb{R}})$. Then we must have $K_t(\mu, d\nu) = \langle f, \nu^m\rangle Q_t(\mu, d\nu)$. By Lemma 2.2 and the proof of Theorem 2.2, $Q_t(\mu, \cdot)$ is uniquely determined by (5.1). Therefore, $Q_t^{(k)}(\mu_k, \cdot)$ converges to $Q_t(\mu, \cdot)$ as $k \to \infty$. From the calculations

$$
\begin{aligned}
&\int_{M(\mathbb{R})} Q_r(\mu, d\eta) \int_{M(\mathbb{R})} \langle f, \nu^m\rangle Q_t(\eta, d\nu) \\
= &\int_{M(\mathbb{R})} \boldsymbol{E}^\sigma_{m,f}\Big[\langle Y_t, \eta^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big] Q_r(\mu, d\eta) \\
= &\ \boldsymbol{E}^\sigma_{m,f}\Big[\int_{M(\mathbb{R})} \langle Y_t, \eta^{M_t}\rangle Q_r(\mu, d\eta) \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big] \\
= &\ \boldsymbol{E}^\sigma_{m,f}\Big[\boldsymbol{E}^\sigma_{M_t,Y_t}\Big(\langle Y_r, \mu^{M_r}\rangle \exp\Big\{\frac{1}{2}\int_0^r M_s(M_s-1)ds\Big\}\Big) \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big] \\
= &\ \boldsymbol{E}^\sigma_{m,f}\Big[\langle Y_{r+t}, \mu^{M_{r+t}}\rangle \exp\Big\{\frac{1}{2}\int_0^{r+t} M_s(M_s-1)ds\Big\}\Big] \\
= &\int_{M(\mathbb{R})} \langle f, \nu^m\rangle Q_{r+t}(\eta, d\nu)
\end{aligned}
$$

we have the Chapman-Kolmogorov equation. □

The existence of a SDSM with a general bounded measurable branching density function $\sigma \in B(\mathbb{R})$ is given by the following

Theorem 5.2 *The sequence $\boldsymbol{Q}_k$ converges as $k \to \infty$ to a probability measure $\boldsymbol{Q}_\mu$ on $C([0,\infty), M(\mathbb{R}))$ under which the coordinate process $\{w_t : t \geq 0\}$ is a diffusion with transition semigroup $(Q_t)_{t\geq 0}$ defined by (5.1). Let $\mathcal{D}(\mathcal{L})$ be the union of all functions of the form (4.1) with $f \in C^2(\mathbb{R}^n)$ and $\{\phi_i\} \subset C^2(\mathbb{R})$ and all functions of the form $F_{m,f}(\mu) = \langle f, \mu^m\rangle$ with $f \in C^2(\mathbb{R}^m)$. Then $\{w_t : t \geq 0\}$ under $\boldsymbol{Q}_\mu$ solves the $(\mathcal{L}, \mathcal{D}(\mathcal{L}), \mu)$-martingale problem.*

Proof. Let $\boldsymbol{Q}_\mu$ be the limit point of any subsequence $\{\boldsymbol{Q}_{k_i}\}$ of $\{\boldsymbol{Q}_k\}$. Using Skorokhod's representation, we may construct processes $\{X_t^{(k_i)} : t \geq 0\}$ and $\{X_t : t \geq 0\}$ with distributions $\boldsymbol{Q}_{k_i}$ and $\boldsymbol{Q}_\mu$ on $C([0,\infty), M(\mathbb{R}))$ such that $\{X_t^{(k_i)} : t \geq 0\}$ converges to $\{X_t : t \geq 0\}$ a.s. when $i \to \infty$; see Ethier and Kurtz (1986, p.102). For any $\{H_j\}_{j=1}^{n+1} \subset C(M(\hat{\mathbb{R}}))$ and $0 \leq t_1 < \cdots < t_n < t_{n+1}$ we may use Theorem 5.1 and dominated convergence to see that

$$
\begin{aligned}
&\boldsymbol{E}\Big\{\prod_{j=1}^n H_j(X_{t_j}) H_{n+1}(X_{t_{n+1}})\Big\} \\
= &\lim_{i\to\infty} \boldsymbol{E}\Big\{\prod_{j=1}^n H_j(X_{t_j}^{(k_i)}) H_{n+1}(X_{t_{n+1}}^{(k_i)})\Big\}
\end{aligned}
$$

$$\begin{aligned} &= \lim_{i\to\infty} \boldsymbol{E}\Big\{\prod_{j=1}^n H_j(X_{t_j}^{(k_i)}) \int_{M(\mathbb{R})} H_{n+1}(\nu) Q_{t_{n+1}-t_n}^{(k_i)}(X_{t_n}^{(k_i)}, d\nu)\Big\} \\ &= \boldsymbol{E}\Big\{\prod_{j=1}^n H_j(X_{t_j}) \int_{M(\mathbb{R})} H_{n+1}(\nu) Q_{t_{n+1}-t_n}(X_{t_n}, d\nu)\Big\}. \end{aligned}$$

Then $\{X_t : t \geq 0\}$ is a Markov process with transition semigroup $(Q_t)_{t\geq 0}$ and actually $\boldsymbol{Q}_k \to \boldsymbol{Q}_\mu$ as $k \to \infty$. The strong Markov property holds since $(Q_t)_{t\geq 0}$ is Feller by (5.1). To see the last assertion, one may simply check that $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ is a restriction of the generator of $(Q_t)_{t\geq 0}$. □

6 Rescaled limits

In this section, we study the rescaled limits of the SDSM constructed in the last section. Given any $\theta > 0$, we defined the operator K_θ on $M(\mathbb{R})$ by $K_\theta\mu(B) = \mu(\{\theta x : x \in B\})$. For a function $h \in B(\mathbb{R})$ we let $h_\theta(x) = h(\theta x)$.

Lemma 6.1 *Suppose that $\{X_t : t \geq 0\}$ is a SDSM with parameters (a, ρ, σ). Let $X_t^\theta = \theta^{-2} K_\theta X_{\theta^2 t}$. Then $\{X_t^\theta : t \geq 0\}$ is a SDSM with parameters $(a_\theta, \rho_\theta, \sigma_\theta)$.*

Proof. We shall compute the generator of $\{X_t^\theta : t \geq 0\}$. Let $F(\mu) = f(\langle\phi, \mu\rangle)$ with $f \in C^2(\mathbb{R})$ and $\phi \in C^2(\mathbb{R})$. Note that $F \circ K_\theta(\mu) = F(K_\theta\mu) = f(\langle\phi_{1/\theta}, \mu\rangle)$. By the theory of transformations of Markov processes, $\{K_\theta X_t : t \geq 0\}$ has generator $\mathcal{L}^\theta$ such that $\mathcal{L}^\theta F(\mu) = \mathcal{L}(F \circ K_\theta)(K_{1/\theta}\mu)$. Since

$$\frac{d}{dx}\phi_{1/\theta}(x) = \frac{1}{\theta}(\phi')_{1/\theta}(x) \quad \text{and} \quad \frac{d^2}{dx^2}\phi_{1/\theta}(x) = \frac{1}{\theta^2}(\phi'')_{1/\theta}(x),$$

it is easy to check that

$$\begin{aligned} \mathcal{L}^\theta F(\mu) &= \frac{1}{2\theta^2} f'(\langle\phi, \mu\rangle)\langle a_\theta\phi'', \mu\rangle \\ &\quad + \frac{1}{2\theta^2} f''(\langle\phi, \mu\rangle) \int_{\mathbb{R}^2} \rho_\theta(x-y)\phi'(x)\phi'(y)\mu(dx)\mu(dy) \\ &\quad + \frac{1}{2} f''(\langle\phi, \mu\rangle)\langle\sigma_\theta\phi^2, \mu\rangle. \end{aligned}$$

Then one may see that $\{\theta^{-2} K_\theta X_t : t \geq 0\}$ has generator $\mathcal{L}_\theta$ such that

$$\begin{aligned} \mathcal{L}_\theta F(\mu) &= \frac{1}{2\theta^2} f'(\langle\phi, \mu\rangle)\langle a_\theta\phi'', \mu\rangle \\ &\quad + \frac{1}{2\theta^2} f''(\langle\phi, \mu\rangle) \int_{\mathbb{R}^2} \rho_\theta(x-y)\phi'(x)\phi'(y)\mu(dx)\mu(dy) \\ &\quad + \frac{1}{2\theta^2} f''(\langle\phi, \mu\rangle)\langle\sigma_\theta\phi^2, \mu\rangle, \end{aligned}$$

and hence $\{X_t^\theta : t \geq 0\}$ has the right generator $\theta^2 \mathcal{L}_\theta$. $\square$

Theorem 6.1 *Suppose that $(\Omega, X_t, \boldsymbol{Q}_\mu)$ is a realization of the SDSM with parameters (a, ρ, σ) with $|c(x)| \geq \epsilon > 0$ for all $x \in \mathbb{R}$. Then there is a $\lambda \times \lambda \times \boldsymbol{Q}_\mu$-measurable function $X_t(\omega, x)$ such that $\boldsymbol{Q}_\mu\{\omega \in \Omega : X_t(\omega, dx)$ is absolutely continuous with respect to the Lebesgue measure with density $X_t(\omega, x)$ for λ-a.e. $t > 0\} = 1$. Moreover, for $\lambda \times \lambda$-a.e. $(t, x) \in [0, \infty) \times \mathbb{R}$ we have*

$$\begin{aligned} \boldsymbol{Q}_\mu\{X_t(x)^2\} &= \int_{\mathbb{R}^2} p_t^2(y, z; x, x)\mu(dx)\mu(dy) \\ &\quad + \int_0^t ds \int_{\mathbb{R}} \mu(dy) \int_{\mathbb{R}} \sigma(z) p_s^2(z, z; x, x) p_{t-s}(y, z) dz. \end{aligned} \tag{6.1}$$

Proof. Recall (1.9). For $r_1 > 0$ and $r_2 > 0$ we use (2.7) and (5.1) to see that

$$\begin{aligned} &\boldsymbol{Q}_\mu\{\langle g^1_{\epsilon r_1}(x, \cdot), X_t\rangle \langle g^1_{\epsilon r_2}(x, \cdot), X_t\rangle\} = \boldsymbol{Q}_\mu\{\langle g^1_{\epsilon r_1}(x, \cdot) \otimes g^1_{\epsilon r_2}(x, \cdot), X_t^2\rangle\} \\ &= \langle P_t^2 g^1_{\epsilon r_1}(x, \cdot) \otimes g^1_{\epsilon r_2}(x, \cdot), \mu^2\rangle + \int_0^t \langle P_{t-s}\phi_{12} P_s^2 g^1_{\epsilon r_1}(x, \cdot) \otimes g^1_{\epsilon r_2}(x, \cdot), \mu^2\rangle ds \\ &= \int_{\mathbb{R}^2} P_t^2 g^1_{\epsilon r_1}(x, \cdot) \otimes g^1_{\epsilon r_2}(x, \cdot)(y, z)\mu(dy)\mu(dz) \\ &\quad + \int_0^t ds \int_{\mathbb{R}} \mu(dy) \int_{\mathbb{R}} \sigma(z) P_s^2 g^1_{\epsilon r_1}(x, \cdot) \otimes g^1_{\epsilon r_2}(x, \cdot)(z, z) p_{t-s}(y, z) dz. \end{aligned}$$

Observe that

$$P_t^2 g^1_{\epsilon r_1}(x, \cdot) \otimes g^1_{\epsilon r_2}(x, \cdot)(y, z) = \int_{\mathbb{R}^2} g^1_{\epsilon r_1}(x, z_1) g^1_{\epsilon r_2}(x, z_2) p_t^2(y, z; z_1, z_2) dz_1 dz_2$$

converges to $p_t^2(y, z; x, x)$ boundedly as $r_1 \to 0$ and $r_2 \to 0$. Note also that

$$\begin{aligned} &\int_{\mathbb{R}} \sigma(z) P_s^2 g^1_{\epsilon r_1}(x, \cdot) \otimes g^1_{\epsilon r_2}(x, \cdot)(z, z) p_{t-s}(y, z) dz \\ &\leq \mathrm{const} \cdot \|\sigma\| \frac{1}{\sqrt{s}} \int_{\mathbb{R}} T_{\epsilon s} g^1_{\epsilon r_1}(x; \cdot)(z) g^1_{\epsilon(t-s)}(y, z) dz \\ &\leq \mathrm{const} \cdot \|\sigma\| \frac{1}{\sqrt{s}} g^1_{\epsilon(t+r_1)}(y, x) \\ &\leq \mathrm{const} \cdot \|\sigma\| \frac{1}{\sqrt{st}}. \end{aligned}$$

By dominated convergence theorem we get

$$\begin{aligned} &\lim_{r_1, r_2 \to 0} \boldsymbol{Q}_\mu\{\langle g^1_{\epsilon r_1}(x, \cdot), X_t\rangle \langle g^1_{\epsilon r_2}(x, \cdot), X_t\rangle\} \\ &= \int_{\mathbb{R}^2} p_t^2(y, z; x, x)\mu(dy)\mu(dz) \\ &\quad + \int_0^t ds \int_{\mathbb{R}} \mu(dy) \int_{\mathbb{R}} \sigma(z) p_t^2(z, z; x, x) p_{t-s}(y, z) dz. \end{aligned}$$

Then it is easy to check that

$$\lim_{r_1,r_2\to 0}\int_0^T dt\int_{\mathbb{R}}\boldsymbol{Q}_\mu\{\langle g^1_{\epsilon r_1}(x,\cdot)-g^1_{\epsilon r_2}(x,\cdot),X_t\rangle^2\}dx=0$$

for each $T>0$, so there is a $\lambda\times\lambda\times\boldsymbol{Q}_\mu$-measurable function $X_t(\omega,x)$ satisfying (6.1) and

$$\lim_{r\to 0}\int_{\mathbb{R}} g^1_{\epsilon r}(x,y)X_t(\omega,dy)=X_t(\omega,x) \tag{6.2}$$

in $L^2(\lambda\times\lambda\times\boldsymbol{Q}_\mu)$. For any square integrable $\phi\in C(\mathbb{R})$,

$$\begin{aligned}
&\int_0^T\boldsymbol{Q}_\mu\Big\{\Big|\langle\phi,X_t\rangle-\int_{\mathbb{R}}\phi(x)X_t(x)dx\Big|^2\Big\}dt\\
\leq\;& 2\int_0^T\boldsymbol{Q}_\mu\left\{\langle\phi-T_{\epsilon r}\phi,X_t\rangle^2\right\}dt\\
&\quad +2\int_0^T\boldsymbol{Q}_\mu\Big\{\Big|\langle T_{\epsilon r}\phi,X_t\rangle-\int_{\mathbb{R}}\phi(x)X_t(x)dx\Big|^2\Big\}dt,
\end{aligned}\tag{6.3}$$

and by Schwarz inequality,

$$\begin{aligned}
&\boldsymbol{Q}_\mu\Big\{\Big|\langle T_{\epsilon r}\phi,X_t\rangle-\int_{\mathbb{R}}\phi(x)X_t(x)dx\Big|^2\Big\}\\
=\;&\boldsymbol{Q}_\mu\Big\{\Big|\int_{\mathbb{R}}X_t(dx)\int_{\mathbb{R}}\phi(x)g^1_{\epsilon r}(y,x)dx-\int_{\mathbb{R}}\phi(x)X_t(x)dx\Big|^2\Big\}\\
=\;&\boldsymbol{Q}_\mu\Big\{\Big|\int_{\mathbb{R}}[\langle g^1_{\epsilon r}(\cdot,x),X_t\rangle-X_t(x)]\phi(x)dx\Big|^2\Big\}\\
\leq\;&\int_{\mathbb{R}}\boldsymbol{Q}_\mu\left\{|\langle g^1_{\epsilon r}(\cdot,x),X_t\rangle-X_t(x)|^2\right\}dx\int_{\mathbb{R}}\phi(x)^2dx.
\end{aligned}$$

By this and (6.2) we get

$$\lim_{r\to 0}\int_0^T\boldsymbol{Q}_\mu\Big\{\Big|\langle T_{\epsilon r}\phi,X_t\rangle-\int_{\mathbb{R}}\phi(x)X_t(x)dx\Big|^2\Big\}dt=0.$$

On the other hand, using (2.8) and (5.1) one may see that

$$\lim_{r\to 0}\boldsymbol{Q}_\mu\{\langle\phi-T_{\epsilon r}\phi,X_t\rangle^2\}\leq\lim_{r\to 0}\|\phi-T_{\epsilon r}\phi\|^2\boldsymbol{Q}_\mu\{\langle 1,X_t\rangle^2\}=0.$$

Then letting $r\to 0$ in (6.3) we have

$$\int_0^T\boldsymbol{Q}_\mu\Big\{\Big|\langle\phi,X_t\rangle-\int_{\mathbb{R}}\phi(x)X_t(x)dx\Big|^2\Big\}dt=0,$$

completing the proof. □

By Theorem 6.1, for $\lambda\times\lambda$-a.e. $(t,x)\in[0,\infty)\times\mathbb{R}$ we have

$$\begin{aligned}\boldsymbol{Q}_\mu\{X_t(x)^2\} &\leq \text{const}\cdot\Big[\frac{1}{\sqrt{t}}\langle 1,\mu\rangle\int_{\mathbb{R}} g^1_{\epsilon t}(x,y)\mu(dy)\\ &\qquad +\int_0^t\frac{ds}{\sqrt{s}}\int_{\mathbb{R}}\mu(dy)\int_{\mathbb{R}}\|\sigma\|g^1_{\epsilon s}(z,x)g^1_{\epsilon(t-s)}(z,x)dz\Big]\\ &\leq \text{const}\cdot\Big[\frac{1}{\sqrt{t}}\langle 1,\mu\rangle+\sqrt{t}\|\sigma\|\Big]\int_{\mathbb{R}} g^1_{\epsilon t}(x,y)\mu(dy). \end{aligned}\tag{6.4}$$

Theorem 6.2 *Suppose that $\{X_t : t\geq 0\}$ is a SDSM with parameters (a,ρ,σ) with $|c(x)|\geq\epsilon>0$ for all $x\in\mathbb{R}$. Let $X^\theta_t=\theta^{-2}K_\theta X_{\theta^2 t}$. Assume $a(x)\to a_\partial$, $\sigma(x)\to\sigma_\partial$ and $\rho(x)\to 0$ as $|x|\to\infty$. Then the conditional distribution of $\{X^\theta_t : t\geq 0\}$ given $X^\theta_0=\mu\in M(\mathbb{R})$ converges as $\theta\to\infty$ to that of a super Brownian motion with underlying generator $(a_\partial/2)\Delta$ and uniform branching density σ_∂.*

Proof. Since $\|\sigma_\theta\|=\|\sigma\|$ and $X^\theta_0=\mu$, as in the proof of Lemma 5.1 one can see that the family $\{X^\theta_t : t\geq 0\}$ is tight in $C([0,\infty),M(\mathbb{R}))$. Choose any sequence $\theta_k\to\infty$ such that the distribution of $\{X^{\theta_k}_t : t\geq 0\}$ converges to some probability measure $\boldsymbol{Q}_\mu$ on $C([0,\infty),M(\mathbb{R}))$. We shall prove that $\boldsymbol{Q}_\mu$ is the solution of the martingale problem for the super Brownian motion so that actually the distribution of $\{X^\theta_t : t\geq 0\}$ converges to $\boldsymbol{Q}_\mu$ as $\theta\to\infty$. By Skorokhod's representation, we can construct processes $\{X^{(k)}_t : t\geq 0\}$ and $\{X^{(0)}_t : t\geq 0\}$ such that $\{X^{(k)}_t : t\geq 0\}$ and $\{X^{\theta_k}_t : t\geq 0\}$ have identical distributions, $\{X^{(0)}_t : t\geq 0\}$ has the distribution $\boldsymbol{Q}_\mu$ and $\{X^{(k)}_t : t\geq 0\}$ converges a.s. to $\{X^{(0)}_t : t\geq 0\}$ in $C([0,\infty),M(\mathbb{R}))$. Let $F(\mu)=f(\langle\phi,\mu\rangle)$ with $f\in C^2(\mathbb{R})$ and $\phi\in C^2(\mathbb{R})$. Then for each $k\geq 0$,

$$F(X^{(k)}_t)-F(X^{(k)}_0)-\int_0^t\mathcal{L}_kF(X^{(k)}_s)ds,\qquad t\geq 0,\tag{6.5}$$

is a martingale, where $\mathcal{L}_k$ is given by

$$\begin{aligned}\mathcal{L}_kF(\mu) &= \frac{1}{2}f'(\langle\phi,\mu\rangle)\langle a_{\theta_k}\phi'',\mu\rangle+\frac{1}{2}f''(\langle\phi,\mu\rangle)\langle\sigma_{\theta_k}\phi^2,\mu\rangle\\ &\quad +\frac{1}{2}f''(\langle\phi,\mu\rangle)\int_{\mathbb{R}^2}\rho_{\theta_k}(x-y)\phi'(x)\phi'(y)\mu(dx)\mu(dy).\end{aligned}$$

Observe that

$$\begin{aligned}&\int_0^t\boldsymbol{E}\{|f'(\langle\phi,X^{(k)}_s\rangle)|\langle|a_{\theta_k}-a_\partial|\phi'',X^{(k)}_s\rangle\}ds\\ \leq\ & \|f'\|\|\phi''\|\int_0^t\boldsymbol{E}\{\langle|a_{\theta_k}-a_\partial|,X^{(k)}_s\rangle\}ds\\ \leq\ & \|f'\|\|\phi''\|\int_0^t\langle P_s|a_{\theta_k}-a_\partial|,\mu\rangle ds\\ \leq\ & \|f'\|\|\phi''\|\int_0^t ds\int_{\mathbb{R}}\mu(dx)\int_{\mathbb{R}}|a_{\theta_k}(y)-a_\partial|p_s(x,y)dy.\end{aligned}$$

Then we have

$$\lim_{k\to\infty}\int_0^t \boldsymbol{E}\{|f'(\langle\phi, X_s^{(k)}\rangle)|\langle|a_{\theta_k}-a_\partial|\phi'', X_s^{(k)}\rangle\}ds = 0. \tag{6.6}$$

In the same way, one sees that

$$\lim_{k\to\infty}\int_0^t \boldsymbol{E}\{|f''(\langle\phi, X_s^{(k)}\rangle)|\langle|\sigma_{\theta_k}-\sigma_\partial|\phi^2, X_s^{(k)}\rangle\}ds = 0. \tag{6.7}$$

Using the density process of $\{X_t^{(k)} : t\geq 0\}$ we have the following estimates

$$\begin{aligned}
&\boldsymbol{E}\Big|f''(\langle\phi, X_s^{(k)}\rangle)\int_{\mathbb{R}^2}\rho_{\theta_k}(x-y)\phi'(x)\phi'(y)X_s^{(k)}(dx)X_s^{(k)}(dy)\Big| \\
\leq\ & \|f''\|\int_{\mathbb{R}^2}|\rho_{\theta_k}(x-y)||\phi'(x)\phi'(y)|\boldsymbol{E}\{X_s^{(k)}(x)X_s^{(k)}(y)\}dxdy \\
\leq\ & \|f''\|\int_{\mathbb{R}^2}|\rho_{\theta_k}(x-y)||\phi'(x)\phi'(y)|\boldsymbol{E}\{X_s^{(k)}(x)^2\}^{1/2}\boldsymbol{E}\{X_s^{(k)}(y)^2\}^{1/2}dxdy \\
\leq\ & \|f''\|\Big(\int_{\mathbb{R}^2}|\rho_{\theta_k}(x-y)|^2|\phi'(x)\phi'(y)|^2dxdy\int_{\mathbb{R}^2}\boldsymbol{E}\{X_s^{(k)}(x)^2\}\boldsymbol{E}\{X_s^{(k)}(y)^2\}dxdy\Big)^{1/2} \\
\leq\ & \|f''\|\Big(\int_{\mathbb{R}^2}|\rho_{\theta_k}(x-y)|^2|\phi'(x)\phi'(y)|^2dxdy\Big)^{1/2}\int_{\mathbb{R}}\boldsymbol{E}\{X_s^{(k)}(x)^2\}dx.
\end{aligned}$$

By (6.4), for any fixed $t\geq 0$,

$$\int_0^t ds\int_{\mathbb{R}}\boldsymbol{E}\{X_s^{(k)}(x)^2\}dx$$

is uniformly bounded in $k\geq 1$. Since $\rho_{\theta_k}(x-y)\to 0$ for $\lambda\times\lambda$-a.e. $(x,y)\in\mathbb{R}^2$ and since $\|\rho_{\theta_k}\| = \|\rho\|$, we have

$$\lim_{k\to\infty}\int_{\mathbb{R}^2}|\rho_{\theta_k}(x-y)|^2|\phi'(x)\phi'(y)|^2dxdy = 0$$

when $\phi'\in L^2(\lambda)$. Then

$$\lim_{k\to\infty}\boldsymbol{E}\Big|f''(\langle\phi, X_s^{(k)}\rangle)\int_{\mathbb{R}^2}\rho_{\theta_k}(x-y)\phi'(x)\phi'(y)X_s^{(k)}(dx)X_s^{(k)}(dy)\Big| = 0. \tag{6.8}$$

Using (6.6),(6.7), (6.8) and the martingale property of (6.5) ones sees in a similar way as in the proof of Lemma 4.2 that

$$F(X_t^{(0)}) - F(X_0^{(0)}) - \int_0^t \mathcal{L}_0F(X_s^{(0)})ds, \qquad t\geq 0,$$

is a martingale, where $\mathcal{L}_0$ is given by

$$\mathcal{L}_0F(\mu) = \frac{1}{2}a_\partial f'(\langle\phi,\mu\rangle)\langle\phi'',\mu\rangle + \frac{1}{2}\sigma_\partial f''(\langle\phi,\mu\rangle)\langle\phi^2,\mu\rangle.$$

This clearly implies that $\{X_t^{(0)} : t\geq 0\}$ is a solution of the martingale problem of the super Brownian motion. □

7 Measure-valued catalysts

In this section, we assume $|c(x)| \geq \epsilon > 0$ for all $x \in \mathbb{R}$ and give construction for a class of SDSM with measure-valued catalysts. We start from the construction of a class of measure-valued dual processes. Let $M_B(\mathbb{R})$ denote the space of Radon measures ζ on $\mathbb{R}$ to which there correspond constants $b(\zeta) > 0$ and $l(\zeta) > 0$ such that

$$\zeta([x, x + l(\zeta)]) \leq b(\zeta)l(\zeta), \qquad x \in \mathbb{R}. \tag{7.1}$$

Clearly, $M_B(\mathbb{R})$ contains all finite measures and all Radon measures which are absolutely continuous with respect to the Lebesgue measure with bounded densities. Let $M_B(\mathbb{R}^m)$ denote the space of Radon measures ν on $\mathbb{R}^m$ such that

$$\nu(dx_1, \cdots, dx_m) = f(x_1, \cdots, x_m)dx_1, \cdots, dx_{m-1}\zeta(dx_m) \tag{7.2}$$

for some $f \in C(\mathbb{R}^m)$ and $\zeta \in M_B(\mathbb{R})$. We endow $M_B(\mathbb{R}^m)$ with the topology of vague convergence. Let $M_A(\mathbb{R}^m)$ denote the subspace of $M_B(\mathbb{R}^m)$ comprising of measures which are absolutely continuous with respect to the Lebesgue measure and have bounded densities. For $f \in C(\mathbb{R}^m)$, we define $\lambda_f^m \in M_A(\mathbb{R}^m)$ by $\lambda_f^m(dx) = f(x)dx$. Let $\boldsymbol{M}$ be the topological union of $\{M_B(\mathbb{R}^m) : m = 1, 2, \cdots\}$.

Lemma 7.1 *If $\zeta \in M_B(\mathbb{R})$ satisfies (7.1), then*

$$\int_{\mathbb{R}} p_t(x, y)\zeta(dy) \leq h(\epsilon, \zeta; t)/\sqrt{t}, \qquad t > 0, x \in \mathbb{R},$$

where

$$h(\epsilon, \zeta; t) = const \cdot b(\zeta)\Big[2l(\zeta) + \sqrt{2\pi\epsilon t}\Big], \qquad t > 0.$$

Proof. Using (1.9) and (7.1) we have

$$\begin{aligned}
\int_{\mathbb{R}} p_t(x, y)\zeta(dy) &\leq \text{const} \cdot \int_{\mathbb{R}} g_{\epsilon t}(x, y)\zeta(dy) \\
&\leq \text{const} \cdot \frac{2b(\zeta)l(\zeta)}{\sqrt{2\pi\epsilon t}} \sum_{k=0}^{\infty} \exp\Big\{ - \frac{k^2 l(\zeta)^2}{2\epsilon t}\Big\} \\
&\leq \text{const} \cdot \frac{b(\zeta)}{\sqrt{2\pi\epsilon t}}\Big[2l(\zeta) + \int_{\mathbb{R}} \exp\Big\{ - \frac{y^2}{2\epsilon t}\Big\}dy\Big] \\
&\leq \text{const} \cdot \frac{b(\zeta)}{\sqrt{2\pi\epsilon t}}\Big[2l(\zeta) + \sqrt{2\pi\epsilon t}\Big],
\end{aligned}$$

giving the desired inequality. □

Fix $\eta \in M_B(\mathbb{R})$ and let Φ_{ij} be the mapping from $M_A(\mathbb{R}^m)$ to $M_B(\mathbb{R}^{m-1})$ defined by

$$\begin{aligned}&\Phi_{ij}\mu(dx_1,\cdots,dx_{m-1})\\ =\ &\mu'(x_1,\cdots,x_{m-1},\cdots,x_{m-1},\cdots,x_{m-2})dx_1\cdots dx_{m-2}\eta(dx_{m-1}),\end{aligned}\tag{7.3}$$

where μ' denotes the Radon-Nikodym derivative of μ with respect to the m-dimensional Lebesgue measure, and x_{m-1} is in the places of the ith and the jth variables of μ' on the right hand side. We may also regard $(P_t^m)_{t>0}$ as operators on $M_B(\mathbb{R}^m)$ determined by

$$P_t^m\nu(dx) = \int_{\mathbb{R}^m} p_t^m(x,y)\nu(dy)dx, \qquad t>0, x\in\mathbb{R}^m. \tag{7.4}$$

By Lemma 7.1 one can show that each P_t^m maps $M_B(\mathbb{R}^m)$ to $M_A(\mathbb{R}^m)$ and, for $f\in C(\mathbb{R}^m)$,

$$P_t^m\lambda_f^m(dx) = P_t^m f(x)dx, \qquad t>0, x\in\mathbb{R}^m. \tag{7.5}$$

Let $\{M_t : t\geq 0\}$ and $\{\Gamma_k : 1\leq k\leq M_0-1\}$ be defined as in section 2. Then

$$Z_t = P_{t-\tau_k}^{M_{\tau_k}}\Gamma_k P_{\tau_k-\tau_{k-1}}^{M_{\tau_{k-1}}}\Gamma_{k-1}\cdots P_{\tau_2-\tau_1}^{M_{\tau_1}}\Gamma_1 P_{\tau_1}^{M_0}Z_0, \quad \tau_k\leq t<\tau_{k+1}, 0\leq k\leq M_0-1, \tag{7.6}$$

defines a Markov process $\{Z_t : t\geq 0\}$ taking values from $\boldsymbol{M}$. Of course, $\{(M_t, Z_t) : t\geq 0\}$ is also a Markov process. We shall suppress the dependence of $\{Z_t : t\geq 0\}$ on η and let $\boldsymbol{E}_{m,\nu}^\eta$ denote the expectation given $M_0 = m$ and $Z_0 = \nu\in M_B(\mathbb{R}^m)$. Observe that by (7.4) and (7.6) we have

$$\begin{aligned}&\boldsymbol{E}_{m,\nu}^\eta\Big[\langle Z_t', \mu^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big]\\ =\ &\langle (P_t^m\nu)', \mu^m\rangle\\ +\ &\frac{1}{2}\sum_{i,j=1,i\neq j}^m \int_0^t \boldsymbol{E}_{m-1,\Phi_{ij}P_u^m\nu}^\eta\Big[\langle Z_{t-u}', \mu^{M_{t-u}}\rangle \exp\Big\{\frac{1}{2}\int_0^{t-u} M_s(M_s-1)ds\Big\}\Big]du.\end{aligned}\tag{7.7}$$

Lemma 7.2 *Let $\eta\in M_B(\mathbb{R})$. For any integer $k\geq 1$, define $\eta_k\in M_A(\mathbb{R})$ by*

$$\eta_k(dx) = kl(\eta)^{-1}\eta((il(\eta)/k, (i+1)l(\eta)/k])dx, \qquad x\in (il(\eta)/k, (i+1)l(\eta)/k],$$

where $i = \cdots,-2,-1,0,1,2,\cdots$. Then $\eta_k\to\eta$ by weak convergence as $k\to\infty$ and

$$\eta_k([x, x+l(\eta)]) \leq 2b(\eta)l(\eta), \qquad x\in\mathbb{R}.$$

Proof. The convergence $\eta_k\to\eta$ as $k\to\infty$ is clear. For any $x\in\mathbb{R}$ there is an integer i such that

$$[x, x+l(\eta)] \subset (il(\eta)/k, (i+1)l(\eta)/k + l(\eta)].$$

Therefore, we have

$$\begin{aligned}\eta_k([x, x + l(\eta)]) &\le \eta_k((il(\eta)/k, (i+1)l(\eta)/k + l(\eta)]) \\ &= \eta((il(\eta)/k, (i+1)l(\eta)/k + l(\eta)]) \\ &\le \eta((il(\eta)/k, il(\eta)/k + 2l(\eta)]) \\ &\le 2b(\eta)l(\eta),\end{aligned}$$

as desired. □

Lemma 7.3 *If $\eta \in M_B(\mathbb{R})$ and if $\nu \in M_B(\mathbb{R}^m)$ is given by (7.2), then*

$$\begin{aligned}&\boldsymbol{E}^{\eta}_{m,\nu}\Big[\langle Z'_t, \mu^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big] \\ \le\ & \|f\|h(\epsilon,\zeta;t)\Big[\langle 1,\mu\rangle^m/\sqrt{t} + \sum_{k=1}^{m-1} 2^k m^k (m-1)^k h(\epsilon,\eta;t)^k \langle 1,\mu\rangle^{m-k} t^{k/2}\Big]. \qquad (7.8)\end{aligned}$$

(Note that the left hand side of (7.8) is well defined since $Z_t \in M_A(\mathbb{R})$ a.s. for each $t > 0$ by (7.6).)

Proof. The left hand side of (7.8) can be decomposed as $\sum_{k=0}^{m-1} A_k$ with

$$A_k = \boldsymbol{E}^{\eta}_{m,\nu}\Big[\langle Z'_t, \mu^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\} 1_{\{\tau_k \le t < \tau_{k+1}\}}\Big].$$

By (7.2) and Lemma 7.1,

$$A_0 = \langle (P^m_t \nu)', \mu^m\rangle \le \|f\|h(\epsilon,\zeta;t)\langle 1,\mu\rangle^m/\sqrt{t}.$$

By the construction (7.6) we have

$$\begin{aligned}A_k &= \frac{m!(m-1)!}{2^k(m-k)!(m-k-1)!}\int_0^t ds_1 \int_{s_1}^t ds_2 \cdots \int_{s_{k-1}}^t \\ &\qquad \boldsymbol{E}^{\eta}_{m,\nu}\{\langle (P^{m-k}_{t-s_k}\Gamma_k \cdots P^{m-1}_{s_2-s_1}\Gamma_1 P^m_{s_1}\nu)', \mu^{m-k}\rangle | \tau_j = s_j : 1 \le j \le k\} ds_k\end{aligned}$$

for $1 \le k \le m-1$. Observe that

$$\int_{s_{k-1}}^t \frac{ds_k}{\sqrt{t-s_k}\sqrt{s_k - s_{k-1}}} \le \frac{2\sqrt{2}}{\sqrt{t-s_{k-1}}}\int_{(t+s_{k-1})/2}^t \frac{ds_k}{\sqrt{t-s_k}} \le \frac{4\sqrt{t}}{\sqrt{t-s_{k-1}}}. \qquad (7.9)$$

By (7.5) we have $P^{m-k}_s \lambda^{m-k}_h \le \lambda^{m-k}_{\|h\|}$ for $h \in C(\mathbb{R}^{m-k})$. Then using (7.9) and Lemma 7.1 inductively we get

$$\begin{aligned}A_k &\le \frac{m!(m-1)!\|f\|}{2^k(m-k)!(m-k-1)!}\int_0^t ds_1 \int_{s_1}^t ds_2 \cdots \int_{s_{k-1}}^t \\ &\qquad \frac{h(\epsilon,\zeta;t)h(\epsilon,\eta;t)^k \langle 1,\mu\rangle^{m-k}}{\sqrt{t-s_k}\cdots\sqrt{s_2-s_1}\sqrt{s_1}} ds_k \\ &\le \frac{2^k m!(m-1)!\|f\|}{(m-k)!(m-k-1)!} h(\epsilon,\zeta;t)h(\epsilon,\eta;t)^k \langle 1,\mu\rangle^{m-k} t^{k/2} \\ &\le 2^k m^k (m-1)^k \|f\| h(\epsilon,\zeta;t)h(\epsilon,\eta;t)^k \langle 1,\mu\rangle^{m-k} t^{k/2}.\end{aligned}$$

Returning to the decomposition we get the desired estimate. $\square$

Lemma 7.4 *Suppose $\eta \in M_B(\mathbb{R})$ and define $\eta_k \in M_A(\mathbb{R})$ as in Lemma 7.2. Assume that $\mu_k \to \mu$ weakly as $k \to \infty$. Then we have*

$$\begin{aligned} & \boldsymbol{E}^{\eta}_{m,\nu}\Big[\langle Z'_t, \mu^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big] \\ = & \lim_{k\to\infty} \boldsymbol{E}^{\eta_k}_{m,\nu}\Big[\langle Z'_t, \mu_k^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big]. \end{aligned}$$

Proof. Based on (7.7), the desired result follows by a similar argument as in the proof of Lemma 2.2. $\square$

Let $\eta \in M_B(\mathbb{R})$ and let η_k be defined as in Lemma 7.2. Let σ_k denote the density of η_k with respect to the Lebesgue measure and let $\{X_t^{(k)} : t \geq 0\}$ be a SDSM with parameters (a, ρ, σ_k) and initial state $\mu_k \in M(\mathbb{R})$. Assume that $\mu_k \to \mu$ weakly as $k \to \infty$. Then we have the following

Theorem 7.1 *The distribution $Q_t^{(k)}(\mu_k, \cdot)$ of $X_t^{(k)}$ on $M(\mathbb{R})$ converges as $k \to \infty$ to a probability measure $Q_t(\mu, \cdot)$ on $M(\mathbb{R})$ given by*

$$\int_{M(\mathbb{R})} \langle f, \nu^m\rangle Q_t(\mu, d\nu) = \boldsymbol{E}^{\eta}_{m,\lambda_f^m}\Big[\langle Z'_t, \mu^{M_t}\rangle \exp\Big\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\Big\}\Big]. \tag{7.10}$$

Moreover, $(Q_t)_{t\geq 0}$ is a transition semigroup on $M(\mathbb{R})$.

Proof. With Lemmas 7.3 and 7.4, this is similar to the proof of Theorem 5.1. $\square$

A Markov process with transition semigroup defined by (7.10) is the so-called SDSM with measure-valued catalysts.

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