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Sectorial Local Non-Determinism and the Geometry of the Brownian Sheet*

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Abstract

We prove the following results about the images and multiple points of an N-parameter, d-dimensional Brownian sheet $B = \{B(t)\}_{t\in\mathbb{R}_+^N}$:

(1) If $\dim_{\mathrm{H}} F \le d/2$, then $B(F)$ is almost surely a Salem set.

(2) If $N \le d/2$, then with probability one

$$\dim_{\mathrm{H}} B(F) = 2\dim_{\mathrm{H}} F \qquad \text{for all Borel sets } F \subset \mathbb{R}_+^N,$$

where "$\dim_{\mathrm{H}}$" could be everywhere replaced by the "Hausdorff," "packing," "upper Minkowski," or "lower Minkowski dimension."

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(3) Let M_k be the set of k-multiple points of B. If $N \leq d/2$ and $Nk > (k-1)d/2$, then $\dim_{\mathrm{H}} M_k = \dim_{\mathrm{P}} M_k = 2Nk - (k-1)d$ a.s.

The Hausdorff dimension aspect of **(2)** was proved earlier; see Mountford (1989) and Lin (1999). The latter references use two different methods; ours of **(2)** are more elementary, and reminiscent of the earlier arguments of Monrad and Pitt (1987) that were designed for studying fractional Brownian motion.
If $N > d/2$ then **(2)** fails to hold. In that case, we establish uniform-dimensional properties for the $(N, 1)$-Brownian sheet that extend the results of Kaufman (1989) for 1-dimensional Brownian motion.
Our innovation is in our use of the *sectorial local nondeterminism* of the Brownian sheet (Khoshnevisan and Xiao, 2004).

Key words: Brownian sheet, sectorial local nondeterminism, image, Salem sets, multiple points, Hausdorff dimension, packing dimension

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1 Introduction

Let $B = \{B(t)\}_{t\in\mathbb{R}_+^N}$ denote the (N,d)-Brownian sheet. That is, B is the N-parameter Gaussian random field with values in $\mathbb{R}^d$; its mean-function is zero, and its covariance function is given by the following:

$$\operatorname{Cov}\left(B_i(s), B_j(t)\right) = \begin{cases} \prod_{k=1}^N \min(s_k, t_k), & \text{if } 1 \le i = j \le d, \\ 0, & \text{otherwise.} \end{cases} \tag{1.1}$$

We have written $B(t)$ in vector form as $(B_1(t), \ldots, B_d(t))$, as is customary.

The (N,d)-Brownian sheet is one of the two natural multiparameter extensions of the ordinary Brownian motion in $\mathbb{R}^d$. The other one is Lévy's N-parameter Brownian motion or, more generally, (N,d)-fractional Brownian motion (fBm) of index $H \in (0,1)$.

It has been long known that fractional Brownian motion is locally nondeterministic (LND, see Pitt, 1978) whereas the Brownian sheet B is not. As a result, two distinct classes of methods have been developed; one to study fractional Brownian motion, and the other, Brownian sheet. Despite this, it has recently been shown that the Brownian sheet satisfies the following "sectorial" local nondeterminism (Khoshnevisan and Xiao, 2004):

Lemma 1.1 (Sectorial LND) *Let B_0 be an $(N,1)$-Brownian sheet. Then for all positive real number a, integers $n \ge 1$, and all $u, v, t^1, \ldots, t^n \in [a,\infty)^N$, we have*

$$\operatorname{Var}\left(B_0(u) \mid B_0(t^1), \ldots, B_0(t^n)\right) \ge \frac{a^{N-1}}{2} \sum_{k=1}^N \min_{0\le j\le n} \left|u_k - t_k^j\right|, \tag{1.2}$$

$$\begin{aligned} &\operatorname{Var}\left(B_0(u) - B_0(v) \mid B_0(t^1), \ldots, B_0(t^n)\right) \\ &\ge \frac{a^{N-1}}{2} \sum_{k=1}^N \min\left(\min_{0\le j\le n} \left|u_k - t_k^j\right| + \min_{0\le j\le n} \left|v_k - t_k^j\right| \;,\; |u_k - v_k| \right), \end{aligned} \tag{1.3}$$

where $t_k^0 = 0$ for every $k = 1, \ldots, N$.

Khoshnevisan and Xiao (2004) have applied the sectorial LND of the Brownian sheet to study the distributional properties of the level set

$$B^{-1}(x) := \left\{t \in (0,\infty)^N : B(t) = x\right\}, \quad {}^\forall x \in \mathbb{R}^d. \tag{1.4}$$

Also, they use sectorial LND of the sheet to study the continuity of the local times of B on a fixed Borel set $F \subset (0,\infty)^N$. Khoshnevisan and Xiao have suggested that, for many problems, the previously-different treatments of the Brownian sheet and fractional Brownian motion can be unified, and generalized so that they do not rely on many of the special properties of the sheet or fBm.

The present paper is a continuation of Khoshnevisan and Xiao (2004). Our main purpose is to describe how to apply sectorial LND in order to study the geometry of the surface of the Brownian sheet. In some cases, our arguments have analogues for fBm; in other cases, our derivations can be applied to prove new results about fBm; see the proofs of Theorems 3.3 and 3.6, for instance.

First we consider the Fourier dimension of the image $B(F)$ for a general (N,d)-Brownian sheet, where $F \subset (0,\infty)^N$ is a fixed Borel set. It is well known that

$$\dim_{\mathrm{H}} B(F) = \min\left(d\,, 2\dim_{\mathrm{H}} F\right) \quad \text{a.s.} \tag{1.5}$$

where $\dim_{\mathrm{H}}$ denotes Hausdorff dimension. If $N > d/2$ and we replace $\dim_{\mathrm{H}}$ by the packing dimension $\dim_{\mathrm{P}}$, then (1.5) can fail; see Talagrand and Xiao (1996). In rough terms, this is because when $N > d/2$, $\dim_{\mathrm{P}} B(F)$ is not determined by $\dim_{\mathrm{P}} F$. It turns out that, in that case, $\dim_{\mathrm{P}} B(F)$ is determined by the packing dimension profile of F defined by Falconer and Howroyd (1997); see Xiao (1997) for details.

Clearly, two distinct cases come up in (1.5): $\dim_{\mathrm{H}} F > d/2$ or $\dim_{\mathrm{H}} F \le d/2$. In the first case, Khoshnevisan and Xiao (2004) have shown that $B(F)$ a.s. has interior points. This verifies an earlier conjecture of Mountford (1989a). Presently, we treat the second case, and prove that for all non-random Borel sets $F \subset (0,\infty)^N$ with $\dim_{\mathrm{H}} F \le d/2$, the image $B(F)$ is almost surely a Salem set with Fourier dimension $2\dim_{\mathrm{H}} F$. That is,

Theorem 1.2 *If $F \subset (0,\infty)^N$ is a non-random Borel set with $\dim_{\mathrm{H}} F \le d/2$, then $\dim_{\mathrm{F}} B(F) = 2\dim_{\mathrm{H}} F$ almost surely.*

When $N = 1$, B denotes the ordinary Brownian motion in $\mathbb{R}^d$, and the latter result is due to Kahane (1985a, 1985b), where he also established a similar result for fractional Brownian motion. However, Kahane's proof does not seem to extend readily to the Brownian sheet case. We will appeal to sectorial LND to accomplish this task.

Note that the exceptional null-set in (1.5) depends on F. One might ask whether the so-called uniform Hausdorff dimension result is valid. That is, we wish to know whether there exists a single null set outside which (1.5) holds simultaneously for all Borel sets $F \subset (0,\infty)^N$. Of course, this can not be true when $N > d/2$. For instance, consider F to be the zero set $B^{-1}(0)$. The following establishes this uniform dimension result in the non-trivial case that $N \le d/2$. Its proof can be found in Section 2 below.

Theorem 1.3 *Choose and fix positive integers $N \le d/2$. Then,*

$$\mathbb{P}\left\{\dim_{\mathrm{H}} B(F) = 2\dim_{\mathrm{H}} F \quad \textit{for all Borel sets } F \subset (0,\infty)^N\right\} = 1, \tag{1.6}$$

where "$\underline{\dim}_{\mathrm{H}}$" can be everywhere replaced, consistently, by any one of the following: "$\dim_{\mathrm{H}}$"; "$\dim_{\mathrm{P}}$"; "$\overline{\dim}_{\mathrm{M}}$"; or "$\overline{\dim}_{\mathrm{m}}$."

Starting with the pioneering work of Kaufman (1968) for planar Brownian motion, a number of authors have established uniform dimension results for stochastic processes; see Xiao (2004) for a survey of such results for Markov processes and their applications. When $N \le d/2$, the uniform dimension result for the Brownian sheet was first proved by Mountford (1989b). Mountford's proof is based on special properties of the sheet. Lin (1999) has extended the result of Mountford (1989b) to (N,d,α)-stable sheets [see Ehm (1981) for the definition] by using a "stopping time" argument for the upper bound, and by estimating the moments of sojourn times for the lower bound. In Section 3 we provide a relatively elementary proof of Theorem 1.3 which uses our notion of sectorial LND. Our proof is reminiscent of the earlier arguments of Kaufman (1968) and Monrad and Pitt (1987).

As we have mentioned, Theorem 1.3 does not hold when $N > d/2$. In Section 3 we derive weaker uniform dimension properties for the $(N, 1)$-Brownian sheet; see Theorems 3.3 and 3.6. Our results are extensions of the results of Kaufman (1989) for one-dimensional Brownian motion.

In Section 4 we determine the Hausdorff and packing dimensions of the set M_k of k-multiple points of the (N, d)-Brownian sheet. Thus, we complete an earlier attempt by Chen (1994).

In the above we have mentioned various concepts of fractal dimensions such as Hausdorff, packing and box-counting dimensions, and packing dimension profiles. Xiao (2004) contains a brief introduction on their definitions and properties. We refer to Falconer (1990) for further information on Hausdorff and box-counting dimensions and to Taylor and Tricot (1985) for information on packing measures and packing dimension.

Throughout this paper, $B = \{B(t)\}_{t\in\mathbb{R}_+^N}$ denotes an (N, d)-Brownian sheet. Sometimes we refer to B as an "(N, d)-sheet," or alternatively a "sheet." We use $|\cdot|$ to denote the Euclidean norm in $\mathbb{R}^m$ irrespective of the value of the integer $m \ge 1$. We denote the m-dimensional Lebesgue measure by λ_m. Unspecified positive and finite constants will be denoted by c which may have different values from line to line. Specific constants in Section i will be denoted by $c_{i,1}, c_{i,2}$ and so on. Finally, we denote the closed ball of radius $r > 0$ about $x \in \mathbb{R}^d$ consistently by

$$U(x, r) := \left\{ y \in \mathbb{R}^d : \ |x - y| \le r \right\}. \tag{1.7}$$

2 Salem sets

In this section we continue the line of research of Kahane (1985a, 1985b, 1993) and study the asymptotic properties of the Fourier transforms of the image measures under the mapping $t \mapsto B(t)$, where B is the (N, d)-Brownian sheet. In particular, we will show that, for every non-random Borel set $F \subset (0, \infty)^N$ such that $\dim_{\mathrm{H}} F \le d/2$, $B(F)$ is almost surely a Salem set.

Let ν be a Borel probability measure on $\mathbb{R}^d$. We say that ν is an *M_β-measure* if its Fourier transform $\widehat{\nu}$ possesses the following property:

$$\widehat{\nu}(\xi) = o(|\xi|^{-\beta}) \quad \text{as} \ \ |\xi| \to \infty. \tag{2.1}$$

Note that if $\beta > d/2$, then certainly $\widehat{\nu}$ is square-integrable on $\mathbb{R}^d$. This, and the Plancherel theorem, together imply that ν is supported by a set of positive d-dimensional Lebesgue measure. From this perspective, our main interest is in studying sets that carry only finite M_β-measures with $\beta \le d/2$.

We define the *Fourier dimension* of ν as

$$\dim_{\mathrm{F}} \nu := \sup \left\{ \alpha \in [0, d] : \ \nu \text{ is an } \ M_{\alpha/2}\text{-measure } \right\}. \tag{2.2}$$

Then it is easy to verify that

$$\dim_{\mathrm{F}} \nu = \liminf_{|\xi|\to\infty} \frac{-2\log|\widehat{\nu}(\xi)|}{\log|\xi|} \wedge d. \tag{2.3}$$

Define the *Fourier dimension* of a Borel set $E \subset \mathbb{R}^d$ as

$$\dim_{\mathrm{F}} E = \sup_{\nu \in \mathcal{P}(E)} \dim_{\mathrm{F}} \nu, \tag{2.4}$$

where $\mathcal{P}(\mathbf{E})$ is the collection of all probability measures on E for any Borel set E. The Fourier dimension bears a relation to the Hausdorff dimension of E. First, recall that for every $0 < \alpha < d$ the α-dimensional Riesz energy of a Borel probability measure ν on $\mathbb{R}^d$ is a constant multiple of $\int_{\mathbb{R}^d} |\widehat{\nu}(\xi)|^2 |\xi|^{\alpha-d}\, d\xi$ (Kahane, 1985a; Ch. 10). Therefore, the Frostman theorem implies that for every Borel set $E \subset \mathbb{R}^d$,

$$\dim_{\mathrm{F}} E \le \dim_{\mathrm{H}} E. \tag{2.5}$$

Moreover, this inequality is often strict, as observed in Kahane (1985, p. 250) that the Hausdorff dimension of $E \subset \mathbb{R}^d$ does not change when E is embedded in $\mathbb{R}^{d+1}$, while the Fourier dimension of E now considered as a subset of $\mathbb{R}^{d+1}$ will be 0. Another interesting example is the standard, ternary Cantor set C on the line. Then, a theorem of Rajchman [see Kahane and Salem (1994, p. 59) or Zygmund (1959, p. 345)] suggests that $\dim_{\mathrm{F}} C = 0$, whereas a celebrated theorem of Hausdorff states that $\dim_{\mathrm{H}} C = \log 2/\log 3$.

In accordance with the existing literature we say that a Borel set E is a *Salem set* if (2.5) is an equality; i.e., if $\dim_{\mathrm{F}} E = \dim_{\mathrm{H}} E$. Such sets are of importance in studying the problem of uniqueness and multiplicity for trigonometric series; see Zygmund (1959, Chapter 9) and Kahane and Salem (1994) for more information.

Let $B_0 := \{B_0(t)\}_{t \in \mathbb{R}^N_+}$ denote the N-parameter Brownian sheet in $\mathbb{R}$. For all $n \ge 2$ and $t^1, \ldots, t^n, s^1, \ldots, s^n \in (0, \infty)^N$, we will write $\mathbf{s} := (s^1, \ldots, s^n)$, $\mathbf{t} := (t^1, \ldots, t^n)$, and

$$\Psi(\mathbf{s}, \mathbf{t}) := \mathrm{Var}\left(\sum_{j=1}^n \left(B_0(t^j) - B_0(s^j) \right) \right). \tag{2.6}$$

For $\mathbf{s} \in (0, \infty)^{nN}$ and $r > 0$, we define

$$\begin{aligned} \mathcal{O}(\mathbf{s}, r) &:= \bigcup_{i_1=1}^n \cdots \bigcup_{i_N=1}^n \bigcap_{k=1}^N \left\{ u \in (0,\infty)^N : \left| u_k - s_k^{i_k} \right| \le r \right\}, \quad \text{and} \\ \mathcal{G}(\mathbf{s}, r) &:= \left\{ \mathbf{t} = (t^1, \ldots, t^n) : t^j \in \mathcal{O}(\mathbf{s}, r) \text{ for all } 1 \le j \le n \right\}. \end{aligned} \tag{2.7}$$

We point out that $\mathcal{O}(\mathbf{s}, \mathbf{t})$ is a finite union of hyper-cubes whose sides are parallel to the axes. Moreover, there are no more than n^N of these hyper-cubes in $\mathcal{O}(\mathbf{s}, \mathbf{t})$.

The following lemma is essential for the proof of Theorem 1.2.

Lemma 2.1 *Let $\varepsilon \in (0, T)$ be fixed. There exists a positive constant $c_{2,1}$ such that $\Psi(\mathbf{s}, \mathbf{t}) \ge c_{2,1}\, r$ for all $r \in (0, \varepsilon)$ and all $\mathbf{s}, \mathbf{t} \in [\varepsilon, T]^{nN}$ with $\mathbf{t} \notin \mathcal{G}(\mathbf{s}, r)$.*

Proof O ur proof follows the proof of Proposition 4.2 of Khoshnevisan and Xiao (2004); see Lemma 1.1 of the present paper.

Since $\mathbf{t} \notin \mathcal{G}(\mathbf{s}, r)$, there exist $j_0 \in \{1, \ldots, n\}$ and $k_0 \in \{1, \ldots, N\}$ such that

$$\min_{1 \le \ell \le n} |t_{k_0}^{j_0} - s_{k_0}^{\ell}| > r. \tag{2.8}$$

The pair (j_0, k_0) is held fixed for the remainder of the proof.

For all $u \geq 0$ and $1 \leq k \leq N$, define

$$X_k(u) := \frac{B_0(\overbrace{\varepsilon, \ldots, \varepsilon}^{k-1 \text{ terms}}, \varepsilon + u, \overbrace{\varepsilon, \ldots, \varepsilon}^{N-k \text{ terms}}) - B_0(\overbrace{\varepsilon, \ldots, \varepsilon}^{N \text{ times}})}{\varepsilon^{(N-1)/2}}. \tag{2.9}$$

Clearly, the process $\{X_k(u)\}_{u \geq 0}$ is centered and Gaussian. In fact, a direct computation of its covariance proves that X_k is standard Brownian motion.

For all $t \in [\varepsilon, T]^N$, we decompose the rectangle $[0, t]$ into the following disjoint union:

$$[0, t] = [0, \varepsilon]^N \cup \bigcup_{k=1}^{N} D_k(t_k) \cup \Delta(\varepsilon, t), \tag{2.10}$$

where $D_k(t_k) := \{s \in [0, T]^N : 0 \leq s_i \leq \varepsilon \text{ if } i \neq k, \text{ and } \varepsilon < s_k \leq t_k\}$, and $\Delta(\varepsilon, t)$ can be written as a union of $2^N - N - 1$ sub-rectangles of $[0, t]$. Then we have the following decomposition for B_0: For all $t \in [\varepsilon, T]^N$,

$$B_0(t) = B_0(\varepsilon, \ldots, \varepsilon) + \varepsilon^{(N-1)/2} \sum_{k=1}^{N} X_k(t_k - \varepsilon) + B'(\varepsilon, t). \tag{2.11}$$

Here, $B'(\varepsilon, t) := \int_{\Delta(\varepsilon,t)} dB_0(s)$. Since all the processes on the right-hand side of (2.11) are defined as increments of B_0 over disjoint sets, they are independent. Therefore

$$\begin{aligned} \Psi(\mathbf{s}, \mathbf{t}) &\geq \varepsilon^{N-1} \sum_{k=1}^{N} \operatorname{Var}\left[\sum_{j=1}^{n} \left(X_k(t_k^j - \varepsilon) - X_k(s_k^j - \varepsilon)\right)\right] \\ &\geq \varepsilon^{N-1} \operatorname{Var}\left[\sum_{j=1}^{n} \left(X_{k_0}(t_{k_0}^j - \varepsilon) - X_{k_0}(s_{k_0}^j - \varepsilon)\right)\right]. \end{aligned} \tag{2.12}$$

Because $\{X_{k_0}(u)\}_{u \geq 0}$ is standard Brownian motion and $|t_{k_0}^{j_0} - s_{k_0}^j| \geq r$ for all $j = 1, \ldots, n$, we can apply Equation (8) of Kahane (1985a, p. 266, Eq. (8)) to conclude that

$$\operatorname{Var}\left[\sum_{j=1}^{n} \left(X_{k_0}(t_{k_0}^j - \varepsilon) - X_{k_0}(s_{k_0}^j - \varepsilon)\right)\right] \geq c_{2,2} r. \tag{2.13}$$

[To obtain this, we set Kahane's parameter as follows: $\gamma = n = 1$; his p is our n; his t_j is our $t_{k_0}^j - \varepsilon$; s_ℓ is our $s_{k_0}^\ell - \varepsilon$; and his ε is our r.] Our lemma follows from (2.12) and (2.13). □

Now we consider the (N, d)-Brownian sheet B. For any Borel probability measure μ on $\mathbb{R}_+^N$, we let μ_B denote the image-measure of μ under the mapping $t \mapsto B(t)$. The Fourier transform of μ_B can be written as follows:

$$\widehat{\mu_B}(\xi) = \int_{\mathbb{R}_+^N} e^{i\xi \cdot B(t)} \, \mu(dt), \quad {}^\forall \xi \in \mathbb{R}^d. \tag{2.14}$$

The following theorem describes the asymptotic behavior of $\widehat{\mu_B}(\xi)$ as $|\xi| \to \infty$.

Theorem 2.2 *Let $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing function satisfying the "doubling property." That is, $\tau(2r) \le c_{2,3}\, \tau(r)$ for all $r \ge 0$. Choose and fix a Borel probability measure μ on $[\varepsilon, T]^N$ such that*

$$\mu(U(x,r)) \le c_{2,4}\, \tau(2r), \qquad \forall x \in \mathbb{R}^N_+,\ r \ge 0. \tag{2.15}$$

Then there exists a finite, positive constant ϱ such that

$$\limsup_{|\xi|\to\infty} \frac{|\widehat{\mu_B}(\xi)|}{\sqrt{\tau\left(1/|\xi|^2\right)\log^{\varrho}|\xi|}} < \infty \qquad a.s. \tag{2.16}$$

Proof S ince the components $B_1, \ldots, B_d$ of the Brownian sheet B are independent copies of $B_0 = \{B_0(t)\}_{t\in\mathbb{R}^N_+}$, we see from (2.14) that for any positive integer $n \ge 1$,

$$\begin{aligned} \|\widehat{\mu_B}(\xi)\|_{2n}^{2n} &= \mathbb{E}\left[\int_{\mathbb{R}^{2nN}_+} e^{i\xi\cdot\sum_{j=1}^n (B(t^j)-B(s^j))}\, \mu^{\otimes n}(d\mathbf{s})\, \mu^{\otimes n}(d\mathbf{t})\right] \\ &= \int_{\mathbb{R}^{2nN}_+} e^{-|\xi|^2\Psi(\mathbf{s},\mathbf{t})/2}\, \mu^{\otimes n}(d\mathbf{s})\, \mu^{\otimes n}(d\mathbf{t}), \end{aligned} \tag{2.17}$$

where $\|\cdot\|_n$ denotes the $L^n(\mathbb{P})$ norm and $\mu^{\otimes n}(d\mathbf{s}) := \mu(ds^1)\times\cdots\times\mu(ds^n)$.
Let $\mathbf{s} \in [\varepsilon, T]^{nN}$ be fixed and we integrate $[\mu^{\otimes n}(d\mathbf{t})]$ first. Write

$$\begin{aligned} &\int_{\mathbb{R}^{nN}_+} e^{-|\xi|^2\Psi(\mathbf{s},\mathbf{t})/2}\, \mu^{\otimes n}(d\mathbf{t}) \\ &\quad = \int_{\mathcal{G}(\mathbf{s},r)} e^{-|\xi|^2\Psi(\mathbf{s},\mathbf{t})/2}\, \mu^{\otimes n}(d\mathbf{t}) \\ &\qquad + \sum_{k=1}^\infty \int_{\mathcal{G}(\mathbf{s},r2^k)\setminus\mathcal{G}(\mathbf{s},r2^{k-1})} e^{-|\xi|^2\Psi(\mathbf{s},\mathbf{t})/2}\, \mu^{\otimes n}(d\mathbf{t}). \end{aligned} \tag{2.18}$$

By (2.15), we always have

$$\int_{\mathcal{G}(\mathbf{s},r)} e^{-|\xi|^2\Psi(\mathbf{s},\mathbf{t})/2}\, \mu^{\otimes n}(d\mathbf{t}) \le \left(c_{2,4}\, n^N \tau(2r)\right)^n. \tag{2.19}$$

Choose and fix some $\xi \in \mathbb{R}^d\setminus\{0\}$, and consider $r := |\xi|^{-2}$. It follows from Lemma 2.1, the doubling property of function τ, and (2.15) that

$$\begin{aligned} &\int_{\mathcal{G}(\mathbf{s},r2^k)\setminus\mathcal{G}(\mathbf{s},r2^{k-1})} e^{-|\xi|^2\Psi(\mathbf{s},\mathbf{t})/2}\, \mu^{\otimes n}(d\mathbf{t}) \\ &\quad \le e^{-c_{2,1}|\xi|^2 r 2^{k-2}} \left(c_{2,4}\, n^N \tau(2^{k+1}r)\right)^n \\ &\quad \le \left(c_{2,4}\, n^N \tau(2r)\right)^n e^{-c_{2,6}2^k} c_{2,3}^{kn}. \end{aligned} \tag{2.20}$$

But $1 + \sum_{k=1}^\infty \exp(c_{2,6}2^k) c_{2,3}^{kn} \le c_{2,7}^n\, n^{\rho n}$ with $\rho := \log c_{2,3}$. Therefore, (2.19) and (2.20) together imply the following bound for the integral of (2.18):

$$\int_{\mathbb{R}^{nN}_+} e^{-|\xi|^2\Psi(\mathbf{s},\mathbf{t})/2}\, \mu^{\otimes n}(d\mathbf{t}) \le c_{2,8}^n\, n^{(N+\rho)n}\left[\tau(2/|\xi|^2)\right]^n. \tag{2.21}$$

Integrate both sides of (2.21) $[\mu^{\otimes n}(d\mathbf{s})]$ to find that

$$\|\widehat{\mu_B}(\xi)\|_{2n}^{2n} \le c_{2,9}^n n^{(N+\rho)n} \left[\tau(2/|\xi|^2)\right]^n. \tag{2.22}$$

This and the Stirling formula together imply the existence of an $a>0$ such that

$$\sup_{\xi\in\mathbb{R}^d} \mathbb{E}\left[\exp\left(a\left|\frac{\widehat{\mu_B}(\xi)}{\sqrt{\tau(2/|\xi|^2)}}\right|^{2/(N+\rho)}\right)\right] \le 2. \tag{2.23}$$

The Markov inequality implies then that for all $b>0$,

$$\sum_{m\in\mathbf{Z}^d} \mathbb{P}\left\{|\widehat{\mu_B}(m)| \ge b\sqrt{\tau(2/|m|^2)\log^{N+\rho}|m|}\right\} \le 2\sum_{m\in\mathbf{Z}^d} |m|^{-ab^{2/(N+\rho)}}, \tag{2.24}$$

which is finite as long as we picked $b > \big(d/a\big)^{(N+\rho)/2}$. By the Borel–Cantelli lemma,

$$\limsup_{|m|\to\infty} \frac{|\widehat{\mu_B}(m)|}{\sqrt{\tau(2/|m|^2)\log^{N+\rho}|m|}} < \infty \qquad \text{a.s.} \tag{2.25}$$

Therefore (2.16) follows, with $\varrho := N+\rho$, from (2.25) and Lemma 1 of Kahane (1985a, p. 252). This finishes the proof of Theorem 2.2. □

We are ready to present our proof of Theorem 1.2.

Proof [Proof of Theorem 1.2] In accord with (1.5) and (2.5),

$$\dim_{\mathrm{F}} B(F) \le \dim_{\mathrm{H}} B(F) = 2\dim_{\mathrm{H}} F \qquad \text{a.s.,} \tag{2.26}$$

for every Borel set $F\subseteq\mathbb{R}_+^N$ that satisfies $\dim_{\mathrm{H}} F \le d/2$.

To prove the converse inequality, it suffices to demonstrate that if $\dim_{\mathrm{H}} F \le d/2$ then $\dim_{\mathrm{F}} B(F) \ge 2\gamma$ a.s. for all $\gamma\in(0,\dim_{\mathrm{H}} F)$. Without loss of generality, we may and will assume $F\subset(0,\infty)^N$ is compact. See Theorem 4.10 of Falconer (1990) for the reasoning. Hence we may further assume that $F\subset[\varepsilon,\, T]^N$ for some positive constants $\varepsilon<T$.

Frostman's lemma implies that there is a probability measure μ on F such that $\mu(U(x,r)) \le c_{2,11}\, r^\gamma$ for all $x\in\mathbb{R}_+^N$ and $r>0$; see (1.7) for notation. Let μ_B denote the image measure of μ under B, and appeal to Theorem 2.2 to find that $\dim_{\mathrm{F}}\mu_B \ge 2\gamma$ a.s. Because $\gamma\in(0,\dim_{\mathrm{H}} F)$ is arbitrary, it follows that $\dim_{\mathrm{F}} B(F) \ge \dim_{\mathrm{F}}\mu_B \ge 2\dim_{\mathrm{H}} F$. This bound complements (2.26), whence follows our proof. □

3 Uniform dimension results for the images

In this section we prove Theorem 1.3, and present a weak uniform dimension property of the $(N,1)$-sheet.

Our proof of Theorem 1.3 is reminiscent of the method of Kaufman (1968) designed for the planar Brownian motion. See also the techniques of Monrad and Pitt (1987) for N-parameter fBm in $\mathbb{R}^d$. The following lemma constitutes the key step in our proof; it will come in handy also in Section 4 below. Throughout, we write

$$\mathcal{F}_n := 4^{-n}\{1, 2, \ldots, 4^n\}^N \qquad \forall n \geq 1. \tag{3.1}$$

Lemma 3.1 *Choose and fix $N \leq d/2$, $\varepsilon, \delta \in (0,1)$, and $\beta \in (1-\delta, 1)$. Then with probability 1, for all large enough n, there do not exist more than $2^{n\delta d}$ distinct points of the form $t^j \in \mathcal{F}_n$ such that*

$$\left|B(t^i) - B(t^j)\right| < 3 \cdot 2^{-n\beta} \qquad \forall i \neq j. \tag{3.2}$$

Proof T hroughout this proof define

$$\Omega(u,v) := |B(u) - B(v)| \qquad \forall u, v \in \mathbb{R}^N_+. \tag{3.3}$$

Let A_n be the event that there do exist more than $2^{n\delta d}$ distinct points of the form $4^{-n}k^j$ such that (3.2) holds. Let N_n be the number of n-tuples of distinct $t^1, \ldots, t^n \in \mathcal{F}_n$ such that (3.2) holds; i.e.,

$$N_n := \sum_{\substack{t^1,\ldots,t^n \in \mathcal{F}_n \\ \text{all distinct}}} \cdots\cdots \sum \mathbf{1}_{\{\Omega(t^i,t^j) < 3\cdot 2^{-n\beta}\}}. \tag{3.4}$$

Because $A_n \subseteq \left\{N_n \geq \binom{[2^{n\delta d}+1]}{n}\right\}$, Markov's inequality implies that

$$\mathbb{P}(A_n) \leq \mathbb{E}(N_n) \times \binom{[2^{n\delta d}+1]}{n}^{-1}. \tag{3.5}$$

Thus, we estimate

$$\mathbb{E}(N_n) = \sum_{\substack{t^1,\ldots,t^n \in \mathcal{F}_n \\ \text{all distinct}}} \cdots\cdots \sum \mathbb{P}\left\{\max_{1 \leq i \neq j \leq n} \Omega(t^i, t^j) < 3 \cdot 2^{-n\beta}\right\}. \tag{3.6}$$

Let us fix $n-1$ distinct points $t^1, \ldots, t^{n-1} \in \mathcal{F}_n$, and first estimate the following sum:

$$\sum_{t^n \in \mathcal{F}_n \setminus \{t^1,\ldots,t^{n-1}\}} \mathbb{P}\left\{\max_{1 \leq i \neq j \leq n} \Omega(t^i, t^j) < 3 \cdot 2^{-n\beta}\right\}. \tag{3.7}$$

For all fixed $t^1, \ldots, t^{n-1} \in \mathcal{F}_n$ we can find at most $(n-1)^N$ points of the form $\tau^u = (\tau^u_1, \ldots, \tau^u_N) \in [\varepsilon, 1]^N$ such that for every $\ell = 1, \ldots, N$,

$$\tau^u_\ell = t^j_\ell \qquad \text{for some } j = 1, \ldots, n-1. \tag{3.8}$$

Let us denote the collection of these τ^u's by $\Gamma_n := \{\tau^u\}_{u \in U(n)}$. Clearly, $t^1, \ldots, t^{n-1}$ are all in Γ_n, and $\#U(n) \leq (n-1)^N$.

It follows from Lemma 1.1 that for every $t^n \notin \Gamma_n$, there exists $\tau^{u_n} \in \Gamma_n$ such that

$$\operatorname{Var}\left(B_0(t^n) \mid B_0(t^1), \ldots, B_0(t^{n-1})\right) \geq c_{3,1}\, |t^n - \tau^{u_n}|. \tag{3.9}$$

But the coordinate processes $B_1, \ldots, B_d$ are i.i.d. copies of B_0. Being Gaussian, the conditional density function of $B(t^n)$ given $B(t^1), \ldots, B(t^{n-1})$ is therefore bounded above by $\{c_{3,1}|t^n - \tau^{u_n}|\}^{-d/2}$. Consequently, (3.9) implies that

$$\begin{aligned} &\mathbb{P}\left\{\max_{1\le i\ne j\le n} \Omega(t^i, t^j) < 3 \cdot 2^{-n\beta}\right\} \\ &\le \mathbb{P}\left\{\max_{1\le i\ne j\le n-1} \Omega(t^i, t^j) < 3 \cdot 2^{-n\beta}\right\} \cdot \left(\frac{3 \cdot 2^{-n\beta}}{c_{3,1}^{1/2}\ |t^n - \tau^{u_n}|^{1/2}}\right)^d. \end{aligned} \tag{3.10}$$

This has content only when $t^n \notin \Gamma_n$. If $t^n \in \Gamma_n$, then instead we use the obvious bound,

$$\mathbb{P}\left\{\max_{1\le i\ne j\le n} \Omega(t^i, t^j) < 3 \cdot 2^{-n\beta}\right\} \le \mathbb{P}\left\{\max_{1\le i\ne j\le n-1} \Omega(t^i, t^j) < 3 \cdot 2^{-n\beta}\right\}. \tag{3.11}$$

The most conservative combination of (3.10) and (3.11) yields

$$\begin{aligned} &\sum_{t^n \in \mathcal{F}_n \setminus \{t^1, \ldots, t^{n-1}\}} \mathbb{P}\left\{\max_{1\le i\ne j\le n} \Omega(t^i, t^j) < 3 \cdot 2^{-n\beta}\right\} \\ &\le \mathbb{P}\left\{\max_{1\le i\ne j\le n-1} \Omega(t^i, t^j) < 3 \cdot 2^{-n\beta}\right\} \\ &\quad \times \left[\sum_{t^n \notin \Gamma_n} c_{3,2} \left(\frac{3 \cdot 2^{-n\beta}}{|t^n - \tau^{u_n}|^{1/2}}\right)^d + (n-1)^N\right]. \end{aligned} \tag{3.12}$$

Note that

$$\begin{aligned} \sum_{t^n \notin \Gamma_n} \left(\frac{3 \cdot 2^{-n\beta}}{|t^n - \tau^{u_n}|^{1/2}}\right)^d &\le \sum_{\tau^u \in \Gamma_n} \sum_{t^n \ne \tau^u} \left(\frac{3 \cdot 2^{-n\beta}}{|t^n - \tau^{u}|^{1/2}}\right)^d \\ &\le c_{3,3} (n-1)^{N+1} 2^{n(1-\beta)d}. \end{aligned} \tag{3.13}$$

The last inequality is due to the fact that if $N \le d/2$ then for all fixed τ^u,

$$\sum_{t^n \ne \tau^u} \frac{1}{|t^n - \tau^u|^{d/2}} \le c\, n\, 2^{nd}. \tag{3.14}$$

Plug (3.13) into (3.12) to obtain

$$\begin{aligned} &\sum_{t^n \in \mathcal{F}_n \setminus \{t^1, \ldots, t^{n-1}\}} \mathbb{P}\left\{\max_{1\le i\ne j\le n} \Omega(t^i, t^j) < 3 \cdot 2^{-n\beta}\right\} \\ &\le \mathbb{P}\left\{\max_{i\ne j\le n-1} \Omega(t^i, t^j) < 3 \cdot 2^{-n\beta}\right\} \times c_{3,4} (n-1)^{N+1} 2^{n(1-\beta)d}. \end{aligned} \tag{3.15}$$

We apply induction, and sum the latter over $t^{n-1}, \ldots, t^1$, in this order. Thanks to (3.6), this proves that

$$\mathbb{E}(N_n) \le c_{3,5}^n\, [(n-1)!]^{N+1}\, 2^{n^2(1-\beta)d}. \tag{3.16}$$

By (3.5) and (3.16), we can bound $\mathbb{P}(A_n)$ as follows:

$$\mathbb{P}(A_n) \leq c_{3,6}^n \,(n-1)^{n(N+2)}\, 2^{n^2(1-\beta-\delta)d}. \tag{3.17}$$

We have used also the elementary inequality,

$$\binom{[2^{n\delta d}+1]}{n} \geq \left(\frac{2^{n\delta d}+1}{n}\right)^n \geq \frac{2^{n^2\delta d}}{n^n}. \tag{3.18}$$

Because $0 < 1-\beta < \delta$, (3.17) implies that $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$. According to the Borel–Cantelli Lemma, $\mathbb{P}(\limsup_n A_n) = 0$. This finishes the proof of our lemma. $\square$

Recall (3.1). For $n = 1, 2, \ldots$ and $\mathbf{k} := (\mathbf{k_1}, \ldots, \mathbf{k_N}) \in \mathbf{4^n}\mathfrak{F}_\mathbf{n}$ define

$$I_\mathbf{k}^n := \bigcap_{i=1}^{N} \left\{ t \in [0,1]^N : (k_i - 1)4^{-n} \leq t_i \leq k_i 4^{-n} \right\}. \tag{3.19}$$

Each $I_\mathbf{k}^n$ is then a hyper-cube of side-length 4^{-n}, and its sides are parallel to the axes.

According to Theorem 2.4 of Orey and Pruitt (1973), the Brownian sheet has the same uniform modulus of continuity as Brownian motion, as long as we stay away from the axes. In particular, for all $\varepsilon \in (0,1)$, we have,

$$\lim_{\eta \to 0} \eta^{\theta-(1/2)} \sup_{\substack{s,t \in [\varepsilon,1]^N: \\ |s-t| \leq \eta}} |\Omega(s,t)| = 0 \quad \text{a.s.} \quad {}^\forall\theta \in (0, 1/2); \tag{3.20}$$

see (3.3) for the definition of Ω. Consequently, for all $\beta, \varepsilon \in (0,1)$, the following holds with probability one:

$$\max_{k \in 4^N \mathfrak{F}_n} \sup_{t \in I_\mathbf{k}^n \cap [\varepsilon,1]^N} |B(t) - B(4^{-n}\mathbf{k})| \leq \mathbf{2} \cdot \mathbf{2}^{-\mathbf{n}\beta} \qquad \text{for all } n \text{ large}. \tag{3.21}$$

This and Lemma 3.1 together imply our next lemma.

Lemma 3.2 *Choose and fix $\varepsilon, \delta \in (0,1)$ and $\beta \in (1-\delta, 1)$. Then a.s.,*

$$\sup_{x \in \mathbb{R}^d} \sum_{\mathbf{k} \in \{\mathbf{1}, \ldots, \mathbf{4^n}\}^\mathbf{N}} \mathbf{1}_{\{B^{-1}\left(U(x, 2^{-n\beta})\right) \cap [\varepsilon,1]^N \cap I_\mathbf{k}^n \neq \emptyset\}} \leq 2^{n\delta d}, \tag{3.22}$$

for all n sufficiently large.

Now we are ready to prove Theorem 1.3.

Proof [Proof of Theorem 1.3] Because of the σ-stability of Hausdorff and packing dimensions and the scaling probability of B, we only need to verify (1.6) for all Borel sets $F \subseteq [\varepsilon, 1]^N$. [This argument is sometimes called *regularization.*]

The modulus of continuity of the Brownian sheet (3.20), and Theorem 6 of Kahane (1985a, p. 139) together imply that outside a single null set,

$$\mathbb{P}\left\{\dim_{\mathrm{H}} B(F) \leq 2\dim_{\mathrm{H}} F \quad \text{for every Borel set} \quad F \subseteq [0,1]^N\right\} = 1. \tag{3.23}$$

Next we derive the same bound, but where "$\dim_{\mathrm{H}}$" is replaced everywhere by "$\overline{\dim}_{\mathrm{M}}$," "$\overline{\dim}_{\mathrm{m}}$," and/or "$\dim_{\mathrm{P}}$."

For any bounded Euclidean set K let $N_K(r)$ be the metric entropy of K at $r > 0$; i.e., $N_K(r)$ is the minimum number of balls of radius $r > 0$ needed to cover K. Recall that $\overline{\dim}_{\mathrm{M}} K = \limsup_{r\to 0} \log N_K(r)/|\log r|$, whereas $\overline{\dim}_{\mathrm{m}} K = \liminf_{r\to 0} \log N_K(r)/|\log r|$. Now choose and fix some $\varepsilon \in (0,1)$, and a (possibly-random) compact set $F \subseteq [\varepsilon\,,1]^N$. Given any radius-$r$ ball $U \subset [\varepsilon\,,1]^N$ and any $\eta \in (0,1/2)$, the diameter of $B(U)$ is at most r^η; consult (3.20). This proves that outside a single null set, the following holds for all Borel sets $F \subseteq [\varepsilon\,,1]^N$ and all $r, \eta \in (0,1/2)$.

$$N_{B(F)}(r^\eta) \le c_{3,7} N_F(r). \tag{3.24}$$

From this, we can readily deduce the following outside a single null set: For all Borel sets $F \subseteq [\varepsilon\,,1]^N$, $\overline{\dim}_{\mathrm{M}} B(F) \le \eta^{-1}\overline{\dim}_{\mathrm{M}} F$ and $\overline{\dim}_{\mathrm{m}} B(F) \le \eta^{-1}\overline{\dim}_{\mathrm{m}} F$. Let $\eta \uparrow 1/2$ and then $\varepsilon \downarrow 0$ to find that (3.23) holds also when "$\dim_{\mathrm{H}}$" is replaced by either "$\overline{\dim}_{\mathrm{M}}$" or "$\overline{\dim}_{\mathrm{m}}$." It also holds for "$\dim_{\mathrm{P}}$" by regularization of $\overline{\dim}_{\mathrm{M}}$.

To prove the lower bounds it suffices to verify that outside a single null set,

$$\dim_{\mathrm{H}} B^{-1}(E) \le \frac{1}{2}\dim_{\mathrm{H}} E, \tag{3.25}$$

for every Borel set $E \subseteq \mathbb{R}^d$, where "$\dim_{\mathrm{H}}$" could be any one of "$\dim_{\mathrm{H}}$," "$\overline{\dim}_{\mathrm{m}}$," "$\overline{\dim}_{\mathrm{M}}$," or "$\dim_{\mathrm{P}}$." Indeed, we can then select $E := B(F)$ and derive the lower bounds by noticing that $B^{-1}(B(F)) \supseteq F$. Equivalently, we seek to prove that for all $\varepsilon \in (0,1)$, the following holds a.s., simultaneously for all Borel sets $E \subset \mathbb{R}^d$:

$$\dim_{\mathrm{H}} \left\{ t \in [\varepsilon,\, 1]^N : B(t) \in E \right\} \le \frac{1}{2}\dim_{\mathrm{H}} E \tag{3.26}$$

First we prove this for $\dim_{\mathrm{H}} = \dim_{\mathrm{H}}$. By regularization, it suffices to consider only compact sets $E \subset \mathbb{R}^d$.

Let $\alpha > \dim_{\mathrm{H}} E$ be fixed (but possibly random); also choose and fix $\varepsilon, \delta \in (0,1)$ and $\beta \in (1-\delta, 1)$. Then we can find balls $U(x_1, r_1), U(x_2, r_2), \ldots$ that cover E, and

$$\sum_{\ell=1}^\infty r_\ell^\alpha < \infty. \tag{3.27}$$

Thanks to Lemma 3.2, outside a single null set, we have: for all ℓ large, $[\varepsilon\,,1]^N \cap B^{-1}(U(x_\ell, r_\ell))$ is a union of at most $r_\ell^{-\delta d/\beta}$-many balls of radius r_ℓ^2. [For $r_\ell := 2^{-n}$, this is precisely Lemma 3.2. For the general case, use monotonicity and the fact that δ and β can be changed a little without changing the content of the lemma.] Hence we have obtained a covering of $B^{-1}(E)$. Let $s := \frac{1}{2}\alpha + (2\beta)^{-1}(\delta d)$ and appeal to (3.27) to find that $\sum_\ell r_\ell^{-\delta d/\beta} r_\ell^{2s} < \infty$. This proves that $\dim_{\mathrm{H}} B^{-1}(E) \le \frac{1}{2}\alpha + (2\beta)^{-1}(\delta d)$. Let $\delta \downarrow 0$ and $\alpha \downarrow \dim_{\mathrm{H}} E$ to obtain (3.26) for Hausdorff dimension.

As regards the other three dimensions, we note that by Lemma 3.2, for all n large and all Borel sets $E \subset \mathbb{R}^d$,

$$N_{B^{-1}(E)\cap[\varepsilon\,,1]^N}(4^{-n}) \le 2^{n\delta d} + N_E(2^{-n\beta}). \tag{3.28}$$

Here, as before, $\delta \in (0,1)$ and $\beta \in (1-\delta, 1)$ are fixed. Take the base-4 logarithm of the preceding display, divide it by n, and then apply a standard monotonicity argument to obtain the following:

$$\begin{aligned}
\overline{\dim}_{\mathrm{M}}\left(B^{-1}(E) \cap [\varepsilon, 1]^N\right) &\leq \max\left(\frac{\delta d}{2}, \frac{\beta}{2}\overline{\dim}_{\mathrm{M}} E\right), \\
\overline{\dim}_{\mathrm{m}}\left(B^{-1}(E) \cap [\varepsilon, 1]^N\right) &\leq \max\left(\frac{\delta d}{2}, \frac{\beta}{2}\overline{\dim}_{\mathrm{m}} E\right).
\end{aligned} \tag{3.29}$$

Let $\beta \uparrow 1$ and $\delta \downarrow 0$ to deduce (3.26) for $\overline{\dim}_{\mathrm{M}}$ and $\overline{\dim}_{\mathrm{m}}$. Regularization and the said inequality for $\overline{\dim}_{\mathrm{M}}$ results in (3.26) for $\dim_{\mathrm{P}}$. This completes our proof. □

When $N > d/2$, both (1.6) no longer holds. In fact, when $N > d/2$, the level sets of B have dimension $N - (d/2) > 0$ (Khoshnevisan, 2002; Corollary 2.1.2, p. 474). Therefore, (1.6) is obviously false for $F := B^{-1}(0)$.

In the following, we prove two weaker forms of uniform result for the images of the $(N, 1)$-Brownian sheet B_0; see Theorems 3.3 and 3.6 below. They extend the results of Kaufman (1989) for one-dimensional Brownian motion.

Theorem 3.3 *With probability 1 for every Borel set $F \subseteq (0,1]^N$,*

$$\dim_{\mathrm{H}} B_0(F + t) = \min(1, 2\dim_{\mathrm{H}} F) \quad \textit{for almost all } t \in [0,1]^N. \tag{3.30}$$

Define

$$H_R(x) := R\mathbf{1}_{[-1,1]}(Rx) \quad \forall x \in \mathbb{R},\, R > 0. \tag{3.31}$$

Also define

$$I_R(x, y) := \int_{[0,1]^N} H_R\left(B_0(x+t) - B_0(y+t)\right) dt \quad \forall R > 0,\, x, y \in [\varepsilon, 1]^N. \tag{3.32}$$

The following lemma is the key to our proof of Theorem 3.3. Sectorial LND plays an important role in its proof.

Lemma 3.4 *For all $x, y \in [\varepsilon, 1]^N$, $R > 1$ and integers $p = 1, 2, \ldots$,*

$$\|I_R(x, y)\|_p^p \leq c_{3,8}^p (p!)^N |y - x|^{-p/2}. \tag{3.33}$$

Proof T he pth moment of $I_R(x, y)$ is equal to

$$R^p \int \cdots \int_{[0,1]^{Np}} \mathbb{P}\left\{\max_{1 \leq i \leq p} \left|B_0(x + t^i) - B_0(y + t^i)\right| < R^{-1}\right\} dt^1 \cdots dt^p. \tag{3.34}$$

We will estimate the above integral by integrating in the order $dt^p, dt^{p-1}, \ldots, dt^1$. First let $t^1, \ldots, t^{p-1} \in [0,1]^N$ be fixed and assume, without loss of generality, that all coordinates of $t^1, \ldots, t^{p-1}$ are distinct. In analogy with (3.3) define

$$\Omega_i := B_0(x + t^i) - B_0(y + t^i) \qquad \forall i = 1, \ldots, p. \tag{3.35}$$

We begin by estimating the conditional probabilities

$$\mathcal{P}(t^p) := \mathbb{P}\left\{ |\Omega_p| < R^{-1} \,\middle|\, \max_{1\le i\le p-1} |\Omega_i| < R^{-1} \right\}. \tag{3.36}$$

Because B_0 is sectorially LND, we have

$$\begin{aligned}
&\operatorname{Var}\left(\Omega_p \,|\, \Omega_i,\ 1\le i\le p-1\right)\\
&\ge \operatorname{Var}\left(\Omega_p \,\middle|\, B_0(x+t^i), B_0(y+t^i), 1\le i\le p-1\right)\\
&\ge c_{3,9}\sum_{k=1}^N \min\left\{\upsilon_k+\bar{\upsilon}_k\ ,\ |x_k-y_k|\right\},
\end{aligned} \tag{3.37}$$

where $c_{3,9}>0$ is a constant which depends on ε [we have used the fact that $x_k+t_k^p\ge\varepsilon$ for every $1\le k\le N$] and

$$\begin{aligned}
\upsilon_k &:= \min_{0\le i\le p-1}\left(|t_k^p-t_k^i|,\ |x_k+t_k^p-y_k-t_k^i|\right),\\
\bar{\upsilon}_k &:= \min_{0\le i\le p-1}\left(|t_k^p-t_k^i|,\ |y_k+t_k^p-x_k-t_k^i|\right),
\end{aligned} \tag{3.38}$$

where $t_k^0=0$ for every $k=1,\ldots,N$.

Observe that for every $1\le k\le N$, we have

$$\upsilon_k+\bar{\upsilon}_k \ge \min_{\substack{0\le i\le p-1\\ \ell=1,2,3}} \left|t_k^p - z_k^{i,\ell}\right|, \tag{3.39}$$

where $z_k^{i,1}=t_k^i$, $z_k^{i,2}=t_k^i+y_k-x_k$ and $z_k^{i,3}=t_k^i+x_k-y_k$ for $k=1,\ldots,N$.

It follows from (3.37) and (3.39) that

$$\begin{aligned}
&\operatorname{Var}\left(\Omega_p \,|\, \Omega_i,\ 1\le i\le p-1\right)\\
&\qquad\ge c_{3,10}\sum_{k=1}^N \min\left\{\min_{\substack{0\le i\le p-1\\ \ell=1,2,3}}\left|t_k^p-z_k^{i,\ell}\right|\ ,\ |x_k-y_k|\right\}.
\end{aligned} \tag{3.40}$$

Therefore, we have

$$\mathcal{P}(t^p)\le c_{3,11}R^{-1}\left[\sum_{k=1}^N \min\left\{\min_{\substack{0\le i\le p-1\\ \ell=1,2,3}}\left|t_k^p-z_k^{i,\ell}\right|\ ,\ |x_k-y_k|\right\}\right]^{-1/2}. \tag{3.41}$$

We note that the points $t^1,\ldots,t^{p-1}$ introduce a natural partition of $[0,1]^N$. More precisely, let $\pi_1,\ldots,\pi_N$ be N permutations of $\{1,\ldots,p-1\}$ such that for every $k=1,\ldots,N$,

$$t_k^{\pi_k(1)} < t_k^{\pi_k(2)} < \ldots < t_k^{\pi_k(p-1)}. \tag{3.42}$$

For convenience, we define also $t_k^{\pi_k(0)}:=0$ and $t_k^{\pi_k(p)}:=1$ for all $1\le k\le N$.

For every $\mathbf{j} = (j_1, \ldots, j_N) \in \{1, \ldots, p-1\}^N$, let $\tau^{\mathbf{j}} = (t_1^{\pi_1(j_1)}, \ldots, t_N^{\pi_N(j_N)})$ be the "center" of the rectangle

$$I_{\mathbf{j}} := \prod_{k=1}^{N} \left[t_k^{\pi_k(j_k)} - \frac{t_k^{\pi_k(j_k)} - t_k^{\pi_k(j_k-1)}}{2} \; , \; t_k^{\pi_k(j_k)} + \frac{t_k^{\pi_k(j_k+1)} - t_k^{\pi_k(j_k)}}{2} \right), \tag{3.43}$$

with the convention being that whenever $j_k = 1$, the left-end point of the interval is 0; and whenever $j_k = p-1$, the interval is closed and its right-end is 1. Thus the rectangles $\{I_{\mathbf{j}}\}_{\mathbf{j} \in \{\mathbf{1}, \ldots, \mathbf{p-1}\}^{\mathbf{N}}}$ form a partition of $[0,1]^N$.

For every $t^p \in [0,1]^N$, there is a unique $\mathbf{j} \in \{1, \ldots, p-1\}^N$ such that $t^p \in I_{\mathbf{j}}$. Moreover, there exists a point $s^{\mathbf{j}}$ (depending on t^p) such that for every $k = 1, \ldots, N$, the k-th coordinate of $s^{\mathbf{j}}$ satisfies

$$s_k^{\mathbf{j}} \in \left\{ t_k^{\pi_k(j_k)}, \, t_k^{\pi_k(j_k-1)} + |x_k - y_k|, \, t_k^{\pi_k(j_k+1)} - |x_k - y_k| \right\}, \tag{3.44}$$

[If $j_1 = 1$, then we should also include t_k^p in the right hand side of (3.44). Since this does not affect the rest of the proof, we omit it for convenience] and

$$\min_{\substack{0 \le i \le p-1 \\ \ell = 1,2,3}} \left| t_k^p - z_k^{i,\ell} \right| = \left| t_k^p - s_k^{\mathbf{j}} \right| \tag{3.45}$$

for every $k = 1, \ldots, N$. Hence, for every $t^p \in I_{\mathbf{j}}$, (3.41) can be rewritten as

$$\mathcal{P}(t^p) \le c_{3,11} \, R^{-1} \left[\sum_{k=1}^{N} \min \left\{ |t_k^p - s_k^{\mathbf{j}}| \; , \; |x_k - y_k| \right\} \right]^{-1/2}. \tag{3.46}$$

Note that, as t^p varies in $I_{\mathbf{j}}$, there are at most 3^N corresponding points $s^{\mathbf{j}}$. Define

$$I_{\mathbf{j}}^{\mathcal{G}} := \left\{ t^p \in I_{\mathbf{j}} : \text{ for every } s^{\mathbf{j}} \text{ we have } |x_k - y_k| \le |t_k^p - s_k^{\mathbf{j}}| \text{ for all } k = 1, \ldots, N \right\}$$

as the set of "$\mathcal{G}$ood" points, and $I_{\mathbf{j}}^{\mathcal{B}} := I_{\mathbf{j}} \setminus I_{\mathbf{j}}^{\mathcal{G}}$ be the collection of "$\mathcal{B}$ad points." For every $t^p \in I_{\mathbf{j}}^{\mathcal{G}}$, (3.46) yields

$$\mathcal{P}(t^p) \le c_{3,11} \, R^{-1} \left[\sum_{k=1}^{N} |x_k - y_k| \right]^{-1/2} \le c_{3,11} \, R^{-1} \, |y - x|^{-1/2}. \tag{3.47}$$

If $t^p \in I_{\mathbf{j}}^{\mathcal{B}}$, then for $s^{\mathbf{j}}$ satisfying (3.46) we have $|t_k^p - s_k^{\mathbf{j}}| < |x_k - y_k|$ for some $k \in \{1, \ldots, N\}$. We denote the collection of those indices by $\mathcal{U}$. Then, for every $k \notin \mathcal{U}$, $|x_k - y_k| \le |t_k^p - s_k^{\mathbf{j}}|$, and we have

$$\mathcal{P}(t^p) \le c_{3,11} \, R^{-1} \left[\sum_{k \in \mathcal{U}} \left| t_k^p - s_k^{\mathbf{j}} \right| + \sum_{k \notin \mathcal{U}} |x_k - y_k| \right]^{-1/2}. \tag{3.48}$$

It follows from (3.47) and (3.48) that $\int_{I_{\mathbf{j}}} \mathcal{P}(t^p) \, dt^p$ is at most

$$\begin{aligned} &\int_{I_{\mathbf{j}}^{\mathcal{G}}} c_{3,11} \, R^{-1} \, |y - x|^{-1/2} \, dt^p \\ &\quad + \int_{I_{\mathbf{j}}^{\mathcal{B}}} c_{3,11} \, R^{-1} \left[\sum_{k \in \mathcal{U}} \left| t_k^p - s_k^{\mathbf{j}} \right| + \sum_{k \notin \mathcal{U}} |x_k - y_k| \right]^{-1/2} dt^p \\ &\le c_{3,12} \, R^{-1} \, |y - x|^{-1/2}, \end{aligned} \tag{3.49}$$

where the last inequality follows from the facts that $|x-y| \le \sqrt{N}$ and the integral over $I_{\mathbf{j}}^{\mathcal{B}}$ is bounded by $c\,R^{-1}$. Hence, we have

$$\int_{[0,1]^N} \mathcal{P}(t^p)\,dt^p = \sum_{\mathbf{j}} \int_{I_{\mathbf{j}}} \mathcal{P}(t^p)\,dt^p \le c_{3,12}\,p^N\,R^{-1}\,|y-x|^{-1/2}. \tag{3.50}$$

Continue integrating $dt^{p-1},\ldots,dt^1$ in (3.34) in the same way, we finally obtain (3.33) as desired.
□

Remark 3.5 For later use in the proof of Theorem 3.6, we remark that the method of the proof of Lemma 3.4 can be used also to prove that

$$\begin{aligned}\int\cdots\int_{[0,1]^{2Np}} &\mathbb{P}^2\left\{\max_{1\le j\le 2p}|B_0(x+t^j)-B_0(y+t^j)| \le 2^{-7n/8}\right\}\,d\mathbf{t}\\ &\le c_{3,13}^p\,[(2p)!]^N\left(2^{-7np/2}n^{2p}+2^{-7np/2}\,|x-y|^{-2p}\right).\end{aligned} \tag{3.51}$$

In fact, by taking $R := 2^{7n/8}$ in (3.41), we obtain

$$\mathcal{P}^2(t^{2p}) \le 2^{-7n/4}c_{3,14}\left[\sum_{k=1}^N \min\left\{\min_{\substack{0\le i\le 2p-1\\ \ell=1,2,3}}\left|t_k^{2p}-z_k^{i,\ell}\right|\ ,\ |x_k-y_k|\right\}\right]^{-1}. \tag{3.52}$$

Based on (3.52) and the argument in the proof of Lemma 3.4, we follow through (3.47), (3.48), and (3.49). This leads us to (3.51).

With the help of Lemma 3.4, we can modify the proof of Theorem 1 in Kaufman (1989) to prove our Theorem 3.3.

Proof [Proof of Theorem 3.3] Almost surely, $\dim_{\mathrm{H}} B_0(F+t) \le \min\left(1, 2\dim_{\mathrm{H}} F\right)$ for all Borel sets F and all $t\in[0,1]^N$. Thus, we need to prove only the lower bound.

We first demonstrate that there exists a constant $c_{3,15}$ and an a.s.-finite random variable $n_0 = n_0(\omega)$ such that almost surely for all $n > n_0(\omega)$,

$$I_{2^n}(x,y) \le c_{3,15}n^N|y-x|^{-1/2} \quad \forall x,y\in[\varepsilon,\,1]^N. \tag{3.53}$$

Consider the set $Q_n \subseteq [0,1]^N$ defined by

$$Q_n := \left\{8^{-n}\mathbf{k}:\ k_j = 0,1,\ldots,8^n,\ \forall j=1,\ldots,N\right\}. \tag{3.54}$$

Then $\#Q_n = (8^n+1)^N$. So the number of pairs $x,y\in Q_n$ is at most $c\,8^{2Nn}$. Hence for $u>1$, Lemma 3.4 implies that

$$\begin{aligned}&\mathbb{P}\left\{I_{2^n}(x,y) > u\,n^N\,|y-x|^{-1/2}\ \text{for some}\ x,y\in Q_n\cap[\varepsilon\,,1]^N\right\}\\ &\quad\le 8^{2Nn}\,c_{3,7}^p\,(p!)^N\,(un^N)^{-p}.\end{aligned} \tag{3.55}$$

By choosing $p := n$, $u := c_{3,7}\, 8^{2N}$, and owing to Stirling's formula, we know that the probabilities in (3.55) are summable. Therefore, by the Borel-Cantelli lemma, a.s. for all n large enough,

$$I_{2^n}(x,y) \le c_{3,16} n^N |y-x|^{-1/2} \qquad \forall x, y \in Q_n \cap [\varepsilon,\, 1]^N. \tag{3.56}$$

Now we are ready to prove (3.53). This is a trivial task unless $n^{2N}\, 4^{-n} < |y-x|$, which we assume is the case. For $x, y \in [\varepsilon\,, 1]^N$, we can find $\bar{x}$ and $\bar{y} \in Q_{n-1} \cap [\varepsilon,\, 1]^N$ so that $|x-\bar{x}| \le \sqrt{N}\, 8^{-n}$ and $|y-\bar{y}| \le \sqrt{N}\, 8^{-n}$, respectively. By the modulus of continuity of B_0, we see that $I_{2^n}(x,y) \le I_{2^{n-1}}(\bar{x},\bar{y})$ for all n large enough. On the other hand, by (3.56) and the assumption $n^{2N}\, 4^{-n} < |y-x|$, we have

$$I_{2^{n-1}}(\bar{x},\bar{y}) \le (n-1)^N |\bar{x}-\bar{y}|^{-1/2} \le c_{3,15} n^N |y-x|^{-1/2}. \tag{3.57}$$

Equation (3.53) follows.

For any Borel set $F \subset (0,1]^N$ and all $\gamma \in (0, \dim_{\mathrm{H}} F)$, we choose $\eta \in (0, 1 \wedge 2\gamma)$. Then F carries a probability measure μ such that

$$\mu(S) \le c_{3,17}\, (\operatorname{diam} S)^\gamma \quad \text{ for all measurable sets } S \subset (0,1]^N. \tag{3.58}$$

By Theorem 4.10 in Falconer (1990), we may and will assume μ is supported on a compact subset of F. Hence (3.53) is applicable.

Let ν_t be the image measure of μ under the mapping $x \mapsto B_0(x+t)$ $(x, t \in (0,1]^N)$. By Frostman's Theorem, in order to prove $\dim_{\mathrm{H}} B_0(F+t) \ge \eta$, it suffices to prove that

$$\iint_{\mathbb{R}^2} \frac{\nu_t(du)\, \nu_t(dv)}{|u-v|^\eta} < \infty. \tag{3.59}$$

Now we follow Kaufman (1989), and note that the left-hand side is equal to

$$\begin{aligned} &\iint \frac{\mu(dx)\, \mu(dy)}{|B_0(x+t) - B_0(y+t)|^\eta} \\ &= \eta \int_0^\infty \iint H_R\left(B_0(x+t) - B_0(y+t)\right)\, R^{\eta-2}\, \mu(dx)\, \mu(dy)\, dR \\ &\le 1 + \int_1^\infty \iint H_R\left(B_0(x+t) - B_0(y+t)\right)\, R^{\eta-2}\, \mu(dx)\, \mu(dy)\, dR. \end{aligned} \tag{3.60}$$

To prove that the last integral is finite for almost all $t \in [0,1]^N$, we integrate it over $[0,1]^N$ and prove that

$$\iint \int_1^\infty I_R(x,y)\, R^{\eta-2}\, dR\, \mu(dx)\, \mu(dy) < \infty. \tag{3.61}$$

We split the above integral over $D = \{(x,y) : |x-y| \le R^{-2}\}$ and its complement, and denote them by J_1 and J_2, respectively. Since $(\mu \times \mu)(D) \le c_{3,17}\, R^{-2\gamma}$ and $\eta \in (0, 2\gamma)$, we have

$$J_1 \le c_{3,17} \int_1^\infty R^{-2\gamma+\eta-1}\, dR < \infty. \tag{3.62}$$

On the other hand, $|x-y|^{-1/2} < R$ for all $(x,y) \in D^c$. Moreover, by (3.53), $I_R(x,y) < c(\omega)\,(\log R)^N |x-y|^{-1/2}$. It follows that

$$\begin{aligned} J_2 &\le c_{3,18}(\omega) \iint \frac{\mu(dx)\,\mu(dy)}{|x-y|^{1/2}} \int_{|x-y|^{-1/2}}^{\infty} R^{\eta-2}\,(\log R)^N\,dR \\ &< c_{3,19}(\omega) \iint \frac{\log^N(1/|y-x|)}{|x-y|^{\eta/2}}\,\mu(dx)\,\mu(dy) < \infty, \end{aligned} \tag{3.63}$$

where the last inequality follows from (3.58). Combining (3.62) and (3.63) gives (3.61). This completes the proof of Theorem 3.3. □

Theorem 3.6 *With probability 1: $\lambda_1(B_0(F+t)) > 0$ for almost all $t \in (0,\infty)^N$, for every Borel set $F \subset (0,1]^N$ with $\dim_{\rm H} F > 1/2$.*

Proof S ince $\dim_{\rm H} F > 1/2$, there exists a Borel probability measure μ on F such that

$$\iint \frac{\mu(ds)\,\mu(dt)}{|s-t|^{1/2}} < \infty. \tag{3.64}$$

Again, we will assume that μ is supported by a compact subset of F.

Let ν_t denote the image-measure of μ, as it did in the proof of Theorem 3.3. It suffices to prove that

$$\int_{[0,1]^N} \int_{\mathbb{R}} |\widehat{\nu_t}(u)|^2\,du\,dt < \infty, \qquad \text{a.s.}, \tag{3.65}$$

where exception null set does not depend on μ. Here, $\widehat{\nu_t}$ denotes the Fourier transform of ν_t; i.e.,

$$\widehat{\nu_t}(u) := \int_{\mathbb{R}_+^N} e^{iuB_0(x+t)}\mu(dx). \tag{3.66}$$

Note that we only need to consider $x \in [0,1]^N$ in (3.66) because the support of μ is contained in $[0,1]^N$.

We choose and fix a smooth, even function $\psi : \mathbb{R} \to \mathbb{R}_+$ such that $\psi(s) = 1$ when $1 \le |s| \le 2$ and $\psi(s) = 0$ outside $1/2 < |s| < 5/2$. Then $\int_{|u|>1} |\widehat{\nu_t}(u)|^2\,du$ is bounded above by

$$\begin{aligned} &\sum_{n=0}^{\infty} \int_{\mathbb{R}} \psi(2^{-n}u)\,|\widehat{\nu_t}(u)|^2\,du \\ &\quad = \sum_{n=0}^{\infty} 2^n \int_{\mathbb{R}_+^{2N}} \widehat{\psi}\left(2^n B_0(x+t) - 2^n B_0(y+t)\right)\,\mu(dx)\,\mu(dy). \end{aligned} \tag{3.67}$$

Consequently, it suffices to show that

$$\sum_{n=0}^{\infty} 2^n \int_{[0,1]^N} \int_{\mathbb{R}_+^{2N}} \widehat{\psi}\left(2^n B_0(x+t) - 2^n B_0(y+t)\right)\,\mu(dx)\,\mu(dy)\,dt < \infty. \tag{3.68}$$

To this end, we define for all $x,y \in [0,1]^N$ and $n \ge 1$,

$$J(x,y,n) := \int_{[0,1]^N} \widehat{\psi}\left(2^n B_0(x+t) - 2^n B_0(y+t)\right)\,dt. \tag{3.69}$$

Lemma 3.7 *There exists a positive and finite constant α such that, with probability 1, for all n large,*

$$\sup_{4^{-n}n^2 \le |y-x| \le \sqrt{N}} \sqrt{|y-x|}\ |J(x,y,n)| \le c_{3,20} n^{2N}(2+\alpha)^{-n}. \tag{3.70}$$

Proof F or all integer $p \ge 1$,

$$\begin{aligned} \|J(x,y,n)\|_{2p}^{2p} &= \mathbb{E}\left[\int_{[0,1]^{2Np}} \prod_{j=1}^{2p} \widehat{\psi}\big(2^n B_0(x+t^j) - 2^n B_0(y+t^j)\big)\, d\mathbf{t}\right] \\ &= \mathbb{E}\left[\int_{S_n} \prod_{j=1}^{2p} \widehat{\psi}\big(2^n B_0(x+t^j) - 2^n B_0(y+t^j)\big)\, d\mathbf{t}\right] \\ &\quad + \mathbb{E}\left[\int_{[0,1]^{2Np}\setminus S_n} \prod_{j=1}^{2p} \widehat{\psi}\big(2^n B_0(x+t^j) - 2^n B_0(y+t^j)\big)\, d\mathbf{t}\right], \end{aligned} \tag{3.71}$$

where $\mathbf{t} := (t^1, \ldots, t^{2p})$ and

$$\begin{aligned} S_n := \bigcup_{k=1}^{2p} \bigcup_{\ell=1}^{N} \Big\{ \mathbf{t} \in [0,1]^{2Np} :\ & \left|t_\ell^k - t_\ell^j\right| > r_n \ \text{ and} \\ & \left|x_\ell + t_\ell^k - t_\ell^j - y_\ell\right| > r_n\ {}^\forall j \ne k \Big\}, \end{aligned} \tag{3.72}$$

and $r_n := c_{3,21} 4^{-n}(n+1)^2$, where $c_{3,21} > 0$ is a constant whose value will be determined later. We consider the integral over S_n first; it can be rewritten as

$$\begin{aligned} &\mathbb{E}\left[\int_{S_n}\int_{\mathbb{R}^{2p}} \prod_{j=1}^{2p} e^{i\xi^j[2^n B_0(x+t^j) - 2^n B_0(y+t^j)]} \psi(\xi^j)\, d\xi\, d\mathbf{t}\right] \\ &= \int_{S_n}\int_{\mathbb{R}^{2p}} e^{-\frac{1}{2}\mathrm{Var}\left(\sum_{j=1}^{2p} \xi^j[2^n B_0(x+t^j) - 2^n B_0(y+t^j)]\right)} \prod_{j=1}^{2p} \psi(\xi^j)\, d\xi\, d\mathbf{t}, \end{aligned} \tag{3.73}$$

where $\xi := (\xi^1, \ldots, \xi^{2p})$.

Now ψ is supported on $[-\frac{5}{2}, -\frac{1}{2}] \cup [\frac{1}{2}, \frac{5}{2}]$. The sectorial LND of B_0 and (3.72) imply that for all $\mathbf{t} \in S_n$ and $\xi \in \mathbb{R}^{2p}$ with $|\xi^j| \in [\frac{1}{2}, \frac{5}{2}]$ for every $1 \le j \le 2p$, we have

$$\begin{aligned} &\mathrm{Var}\left(\sum_{j=1}^{2p} \xi^j \left[2^n B_0(x+t^j) - 2^n B_0(y+t^j)\right]\right) \\ &\ge \mathrm{Var}\Big(\ \xi^k \left[2^n B_0(x+t^k) - 2^n B_0(y+t^k)\right] \Big|\ B_0(x+t^j), B_0(y+t^j),\ j \ne k \Big) \\ &\ge \frac{1}{4}\, 2^{2n} \mathrm{Var}\Big(\ B_0(x+t^k)\ \Big|\ B_0(x+t^j),\ j \ne k;\ B_0(y+t^i),\ {}^\forall 1 \le i \le 2p \Big) \\ &\ge c_{3,22}\, 2^{2n} \sum_{\ell=1}^{N} \min_{1 \le j \ne k \le 2p} \left\{ |t_\ell^k - t_\ell^j|\ ,\ \ |x_\ell + t_\ell^k - y_\ell - t_\ell^j| \right\} \\ &\ge c_{3,22} 2^{2n} r_n = c_{3,22} c_{3,21} (n+1)^2. \end{aligned} \tag{3.74}$$

In the above $c_{3,22} > 0$ is a constant depending on ε and again we have used the fact that $x_\ell + t_\ell^k \geq \varepsilon$ for every $1 \leq \ell \leq N$.

By combining (3.73) and (3.74), we obtain

$$\mathbb{E}\left[\left|\int_{S_n}\int_{\mathbb{R}^{2p}}\prod_{j=1}^{2p} e^{i\xi^j\left[2^n B_0(x+t^j)-2^n B_0(y+t^j)\right]}\,\psi(\xi^j)\,d\xi\,d\mathbf{t}\right|\right] \leq e^{-c_{3,23}n^2}. \tag{3.75}$$

Later we will choose the constant $c_{3,21}$ such that $c_{3,23}$ is sufficiently large. Thus the integral in (3.71) over S_n can be neglected.

Now, we consider the integral in (3.71) over $T_n := [0,1]^{2Np}\backslash S_n$, which can be written as

$$\begin{aligned}
T_n = \Big\{\mathbf{t} \in [0,1]^{2Np} : {}^{\forall}k \in \{1,\ldots,2p\},\ {}^{\forall}\ell \in \{1,\ldots,N\},\\
{}^{\exists}j_{\ell,1} \neq k \ \text{ s.t. } \left|t_\ell^k - t_\ell^{j_{\ell,1}}\right| \leq r_n \\
\text{or } {}^{\exists}j_{\ell,2} \neq k \text{ s.t. } \left|x_\ell + t_\ell^k - y_\ell - t_\ell^{j_{\ell,2}}\right| \leq r_n\Big\} \\
= \bigcap_{k=1}^{2p}\bigcap_{\ell=1}^{N}\Bigg(\Big\{\mathbf{t} \in [0,1]^{2Np} :\ \min_{j_{\ell,1}\neq k}\left|t_\ell^k - t_\ell^{j_{\ell,1}}\right| \leq r_n\Big\} \\
\cup\Big\{\mathbf{t} \in [0,1]^{2Np} :\ \min_{j_{\ell,2}\neq k}\left|x_\ell + t_\ell^k - y_\ell - t_\ell^{j_{\ell,2}}\right| \leq r_n\Big\}\Bigg).
\end{aligned} \tag{3.76}$$

From (3.76), we can see that T_n is a union of at most $(4p)^{2Np}$ sets of the form:

$$A_{\mathbf{j}} = \left\{\mathbf{t} \in [0,1]^{2Np} :\ \max_{\substack{1\leq k\leq 2p\\ 1\leq \ell\leq N}}\left|z_\ell + t_\ell^k - t_\ell^{j_{\ell,k}}\right| \leq r_n\right\}, \tag{3.77}$$

where $\mathbf{j} := (j_{\ell,k} : 1 \leq k \leq 2p,\ 1 \leq \ell \leq N)$ has the property that $j_{\ell,k} \neq k$ and where $z_\ell = 0$ or $x_\ell - y_\ell$.

The following lemma estimates the Lebesgue measure of T_n. Here and throughout, λ_ℓ denotes the Lebesgue measure on $\mathbb{R}^\ell$ for all integers $\ell \geq 1$.

Lemma 3.8 *For any positive even number m, all $z_1, \ldots, z_m \in \mathbb{R}$, every sequence $\{\ell_1, \ldots, \ell_m\} \subseteq \{1, \ldots, m\}$ satisfying $\ell_j \neq j$, and for each $r \in (0,1)$, we have*

$$\lambda_m\left\{s \in [0,1]^m :\ \max_{k\in\{1,\ldots,m\}}|z_k + s_k - s_{\ell_k}| \leq r\right\} \leq (2r)^{m/2}. \tag{3.78}$$

We now continue with the proof of Lemma 3.7 and defer the proof of Lemma 3.8 to the end of this section.

It follows from (3.76), (3.77) and Lemma 3.8 that

$$\lambda_{2Np}(T_n) \leq (4p)^{2Np}\,(2r_n)^{Np}. \tag{3.79}$$

We proceed to estimate the integral in (3.71) over T_n. It is bounded above by

$$\begin{aligned}
&\int_{T_n} \mathbb{E}\left[\prod_{j=1}^{2p} \left|\widehat{\psi}(2^n B_0(x+t^j) - 2^n B_0(y+t^j))\right|\right] d\mathbf{t} \\
&= \int_{T_n} \mathbb{E}\left[\cdots;\ D_n\right] d\mathbf{t} + \int_{T_n} \mathbb{E}\left[\cdots;\ D_n^c\right] d\mathbf{t} =: I_1 + I_2,
\end{aligned} \tag{3.80}$$

where

$$D_n := \left\{ \max_{1\le j\le 2p} \left|B_0(x+t^j) - B_0(y+t^j)\right| > 2^{-7n/8} \right\}. \tag{3.81}$$

Since $\widehat{\psi}$ is a rapidly decreasing function, we derive from (3.79) that

$$\begin{aligned}
I_1 &\le \lambda_{2Np}(T_n)\, \widehat{\psi}(2^{n/8})\, \mathbb{P}(D_n) \\
&\le (4p)^{2Np}\, (2r_n)^{Np} \exp\{-c_{3,24} n\} \\
&= c_{3,25}^p\, (pn)^{2Np}\, 2^{-2Nnp} \exp\{-c_{3,24} n\}.
\end{aligned} \tag{3.82}$$

Note that $c_{3,24} > 0$ can be chosen arbitrarily large, I_1 can be made very small [In particular, I_1 is smaller than the last term in (3.83) below]. On the other hand, by the Cauchy-Schwarz inequality, I_2 is at most

$$\begin{aligned}
&\int_{T_n} \mathbb{P}\left\{ \max_{1\le j\le 2p} |B_0(x+t^j) - B_0(y+t^j)| \le 2^{-7n/8} \right\} d\mathbf{t} \\
&\le (\lambda_{2Np}(T_n))^{1/2} \\
&\quad \times \left[\int_{[0,1]^{2Np}} \mathbb{P}^2 \left\{ \max_{1\le j\le 2p} |B_0(x+t^j) - B_0(y+t^j)| \le 2^{-7n/8} \right\} d\mathbf{t} \right]^{1/2} \\
&\le c_{3,26}^p\, (pn)^{Np}\, 2^{-Nnp}\, [(2p)!]^{N/2} \left(2^{-7np/2} n^{2p} + 2^{-7np/2} |y-x|^{-2p} \right)^{1/2} \\
&\le c_{3,27}^p\, (pn)^{2Np}\, 2^{-Nnp-(7np/4)} |y-x|^{-p}.
\end{aligned} \tag{3.83}$$

In the above we have used (3.79) and (3.51) in Remark 3.5.

Combining (3.71), (3.75) with $c_{3,23}$ large, (3.80), (3.82) and (3.83), we obtain

$$\left\| J(x,y,n) \right\|_{2p}^{2p} \le c_{3,28}^p\, (pn)^{2Np}\, 2^{-(N+(7/4))np}\, |y-x|^{-p} \tag{3.84}$$

for all integers $1 \le p \le c_{3,29}\, n$.

By using (3.84), the Borel-Cantelli lemma and the modulus of continuity of B, we can derive (3.70) in the same way as in the proof of Theorem 3.3. □

Now we conclude the proof of Theorem 3.6. Thanks to Lemma 3.7, we have

$$\begin{aligned}
2^n \iint \left| J(x,y,n) \right| \mu(dx)\, \mu(dy) &\le c_{3,28} 2^n n^{2N} (2+\alpha)^{-n} \iint \frac{\mu(dx)\, \mu(dy)}{|y-x|^{1/2}} \\
&\le c_{3,30}\, n^{2N} \left(\frac{2}{2+\alpha} \right)^n.
\end{aligned} \tag{3.85}$$

This implies (3.68), and finishes our proof of Theorem 3.6. □

Proof [Proof of Lemma 3.8] Given a sequence $\{\ell_1, \dots, \ell_m\}$ satisfying $\ell_j \neq j$, we introduce a partition of the points $\{s_1, \dots, s_m\}$ as follows:

We call (s_i, s_{ℓ_i}) a *pair* if $|z_i + s_i - s_{\ell_i}| \le r$ for some $z_i \in \{z_1, \dots, z_m\}$. We say that a sequence of pairs $C = \{(s_{i_1}, s_{\ell_{i_1}}), \dots, (s_{i_\tau}, s_{\ell_{i_\tau}})\}$ forms a *chain* if for every $(s_{i_j}, s_{\ell_{i_j}}) \in C$, there exists $(s_{i'_j}, s_{\ell_{i'_j}}) \in C$ such that $\{s_{i_j}, s_{\ell_{i_j}}\} \cap \{s_{i'_j}, s_{\ell_{i'_j}}\} \neq \emptyset$. First, we start with the pair (s_1, s_{ℓ_1}) and form the maximum chain C_1. Let Γ_1 be the collection of all s_i's contained in some pairs of C_1. Secondly we pick a pair (s_j, s_{ℓ_j}) from the remaining pairs, and let Γ_2 be the collection of all s_i's in the maximal chain containing (s_j, s_{ℓ_j}). Clearly, $\Gamma_1 \cap \Gamma_2 = \emptyset$. Continue this procedure, we get a partition $\Gamma_1, \dots, \Gamma_L$ of $\{s_1, \dots, s_m\}$, where L is determined by the sequence $\{\ell_1, \dots, \ell_m\}$. Denote $\tau_i = \#\Gamma_i$, so that $\tau_i \ge 2$, $\sum_{i=1}^L \tau_i = m$, and $L \le m/2$.

With the above partition, we can write

$$\begin{aligned} &\lambda_m \left\{ s \in [0,1]^m : \max_{1 \le k \le m} |z_k + s_k - s_{\ell_k}| \le r \right\} \\ &= \prod_{i=1}^L \int_{[0,1]^{\tau_i}} \mathbf{1}_{\{|z_k + s_k - s_{\ell_k}| \le r,\ s_k,\ s_{\ell_k} \in \Gamma_i\}}\, d\bar{s}^i, \end{aligned} \tag{3.86}$$

where $\bar{s}^i \in [0,1]^{\tau_i}$ denotes the vector formed by the elements in Γ_i.

In order to estimate the integral on $[0,1]^{\tau_i}$ $(i = 1, \dots, L)$, we note that there is a natural tree structure in each Γ_i. We pick any element $s_{i_0} \in \Gamma_i$ as the root of the tree; its first generation offspring are the elements $s_{i_{01}}, \dots s_{i_{0k_1}} \in \Gamma_i$ such that for every $j = 1, \dots, k_1$ either $(s_{i_0}, s_{i_{0j}})$ or $(s_{i_{0j}}, s_{i_0})$ is a pair. Similarly, we can find the offspring of each $s_{i_{0j}}$, which are the second generation offspring of s_{i_0}. Continuing this procedure, we obtain a tree associated to Γ_i. If this tree has q generations, then we can rewrite the multiple integral over $[0,1]^{\tau_i}$ as a τ_i-layer iterative integral by integrating the qth generation offspring at first, then the $(q-1)$th generation and so on. With the exception of the outside-most layer, each layer of integration contributes a factor $2r$. Hence, we have

$$\int_{[0,1]^{\tau_i}} \mathbf{1}_{\{|z_k + s_k - s_{\ell_k}| \le r,\ s_k,\ s_{\ell_k} \in \Gamma_i\}}\, d\bar{s}^i \le (2r)^{\tau_i - 1}. \tag{3.87}$$

Thus, it follows from (3.86) and (3.87) that

$$\lambda_m \left\{ s \in [0,1]^m : \max_{1 \le k \le m} |z_k + s_k - s_{\ell_k}| \le r \right\} \le (2r)^{\sum_{i=1}^L (\tau_i - 1)} \le (2r)^{m/2}. \tag{3.88}$$

This finishes the proof of Lemma 3.8. □

Remark 3.9 When $d > 1$ and $N > d/2$, our proof of Lemma 3.4 breaks down. See (3.49), where the integral on $I_{\mathbf{j}}^{\mathcal{B}}$ can not be neglected anymore if $d > 1$. In this general case, we do not know whether Theorems 3.3 and 3.6 remain valid.

The following question was raised by Kaufman (1989) for Brownian motion in $\mathbb{R}$. It is still open, and we reformulate it for the Brownian sheet.

Question 3.10 Suppose $N > d/2$. Is it true that, with probability 1, $B(F+t)$ has interior points for some $t \in [0,1]^N$ for every Borel set $F \subset (0,\infty)^N$ with $\dim_{\mathrm{H}} F > d/2$?

Xiao (1997) has proved that for the (N,d)-Brownian sheet B with $N > d/2$ and for any Borel set $F \subseteq (0,\infty)^N$,

$$\dim_{\mathrm{P}} B(F) = 2\,\mathrm{Dim}_{d/2} F \qquad \text{a.s.,} \tag{3.89}$$

where $\mathrm{Dim}_{d/2} F$ denotes the $(d/2)$-dimensional "packing-dimension profile" of F defined by Falconer and Howroyd (1997).

In light of (3.89) and Theorem 3.3, we may ask the following natural question:

Question 3.11 Suppose $N > d/2$. Is it true that, with probability 1, for every Borel set $F \subset (0,\infty)^N$,

$$\dim_{\mathrm{P}} B(F+t) = 2\,\mathrm{Dim}_{d/2} F \quad \text{for almost all } \ t \in [0,1]^N? \tag{3.90}$$

4 Hausdorff and packing dimensions of the multiple points

Let $k \geq 2$ be an integer. We say that $x \in \mathbb{R}^d$ is a *k-multiple point* of B if there exist k distinct points $t^1, \ldots, t^k \in (0,\infty)^N$ such that $x = B(t^1) = \cdots = B(t^k)$. Let M_k be the set of k-multiple points of B. Define L_k to be the random set,

$$\left\{ (t^1, \ldots, t^k) \in (0,\infty)^{Nk} : \ t^i\text{'s are distinct and } B(t^1) = \cdots = B(t^k) \right\}. \tag{4.1}$$

This is the set of *k-multiple times* of B.

Rosen (1984) has proved that if $Nk > (k-1)d/2$, then B has k-multiple points and

$$\dim_{\mathrm{H}} L_k = Nk - (k-1)d/2 \qquad \text{a.s.} \tag{4.2}$$

It is possible to show that this formula holds also for $\dim_{\mathrm{P}} L_k$. Rosen's proof of (4.2) proceeds by studying the regularity of the sample functions of the intersection local times of the sheet.

In the following, we determine the Hausdorff and packing dimensions of M_k. An earlier attempt has been made in Chen (1994). But there are gaps in the proof of his Lemma 2.2 [see lines 2–4 and line -6 on page 57].

In Section 3 we have fixed the gaps in Chen's proof. In particular, see Lemma 3.1 above. This is a correct version of Chen's Lemma 2.2. It is proved by appealing to the sectorial LND of the sheet. Thus, we can prove the following.

Theorem 4.1 *Let $k \geq 2$ be an integer, and suppose that $N \leq d/2$ and $Nk > (k-1)d/2$. Then,*

$$\dim_{\mathrm{H}} M_k = \dim_{\mathrm{P}} M_k = 2Nk - (k-1)d \qquad a.s. \tag{4.3}$$

Proof L et $\widetilde{L}_k$ denote the projection of L_k onto $\mathbb{R}_+^N$; in particular, note that $M_k = B(\widetilde{L}_k)$. It follows from Theorem 1.3 and (4.2) that

$$\dim_{\mathrm{P}} M_k \leq 2Nk - (k-1)d \quad \text{a.s.} \tag{4.4}$$

Since $\dim_{\mathrm{H}} M_k \leq \dim_{\mathrm{P}} M_k$, it remains to show that

$$\dim_{\mathrm{H}} M_k \geq 2Nk - (k-1)d \quad \text{a.s.} \tag{4.5}$$

For this purpose, let $I_1, \ldots, I_k$ be k pairwise disjoint subcubes of $[\varepsilon, 1]^N$ and let $I = \prod_{j=1}^k I_j$. Define $M_k(I)$ to be the collection of all $x \in \mathbb{R}^d$ for which there exists distinct times $t^1, \ldots, t^k \in I$ such that $x = B(t^1) = \cdots = B(t^k)$. Thus, $M_k(I)$ are the k-multiple points of B which correspond to time-points that are in I. The argument in Chen (1994), coupled with our Lemma 3.1 in place of Chen's Lemma 2.2, correctly implies that $\mathbb{P}\{\dim_{\mathrm{H}} M_k(I) \geq 2Nk - (k-1)d\} > 0$. On the other hand, the independent increment property of the Brownian sheet implies a zero-one law for $\dim_{\mathrm{H}} M_k$; see pages 110 and 115 of Rosen (1984). Whence, (4.5) follows. $\square$

Question 4.2 Are there nice, exact Hausdorff and packing measure functions for M_k and L_k? This question is closely related to the regularity, in time, of the intersection local times of the sheet. Such regularity theorems are likely to be interesting in their own right. For such results on Brownian motion, see LeGall (1987, 1989).

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