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Solutions of Stochastic Differential Equations obeying the Law of the Iterated Logarithm, with Applications to Financial Markets*

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Abstract

By using a change of scale and space, we study a class of stochastic differential equations (SDEs) whose solutions are drift–perturbed and exhibit asymptotic behaviour similar to standard Brownian motion. In particular sufficient conditions ensuring that these processes obey the Law of the Iterated Logarithm (LIL) are given. Ergodic–type theorems on the average growth of these non-stationary processes, which also depend on the asymptotic behaviour of the drift coefficient, are investigated. We apply these results to inefficient financial market models. The techniques extend to certain classes of finite–dimensional equation.

Key words: stochastic differential equations, Brownian motion, Law of the Iterated Logarithm, Motoo's theorem, stochastic comparison principle, stationary processes, inefficient market.

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1 Introduction

The following Law of the Iterated Logarithm is one of the most important results on the asymptotic behaviour of finite-dimensional standard Brownian motion:

$$\limsup_{t\to\infty} \frac{|B(t)|}{\sqrt{2t\log\log t}} = 1, \quad \text{a.s.} \tag{1.1}$$

Classical work on iterated logarithm–type results, as well as associated lower bounds on the growth of transient processes, date back to Dvoretzky and Erdős [5]. There is an interesting literature on iterated logarithm results and the growth of lower envelopes for self-similar Markov processes (cf. e.g., Rivero [16], Chaumont and Pardo [4]) which exploit a Lamperti representation [13], processes conditioned to remain positive (cf. Hambly et al. [10]), and diffusion processes with special structure (cf. e.g. Bass and Kumagi [3]).

In contrast to these papers, the analysis here is inspired by work of Motoo [15] on iterated logarithm results for Brownian motions in finite dimensions, in which the asymptotic behaviour is determined by means of time change arguments which reduce the process under study to a stationary one. Our paper concentrates mainly on iterated logarithm upper bounds of solutions of stochastic differential equations, as well as obtaining lower envelopes for the growth rate. Our goal is to establish these results under the minimum continuity and asymptotic conditions on the drift and diffusion coefficients. An advantage of this approach is that it enables us to analyse a class of equations of the form

$$dX(t) = f(X(t))\,dt + g(X(t))\,dB(t)$$

for which $xf(x)/g^2(x)$ tends to a finite limit as $x \to \infty$ in the case when f and g are regularly varying at infinity. Ergodic–type theorems are also presented. We also show how results can be extended to certain classes of non-autonomous and finite-dimensional equations. We employ extensively comparison arguments of various kinds throughout.

In [1], Appleby et al. studied general conditions which ensure a scalar stochastic differential equation with Markov switching obeys the Law of the Iterated Logarithm. In our work here, we are concerned with similar problems for SDEs without switching. In particular, for a parameterised family of SDEs, we observe that solutions can change from being recurrent to transient when a critical value of the bifurcation parameter is exceeded. Despite this, the solutions still obey the Law of the Iterated Logarithm in the sense of (1.1). Between this paper and [1], we examine the extent to which the drift can be perturbed so that in the long-run the size of the large deviations remains the same as those of standard Brownian motion.

In [14], Mao shows that if X is the solution of the d–dimensional equation

$$dX(t) = f(X(t),t)\,dt + g(X(t),t)\,dB(t), \quad t \geq 0$$

and if there exist positive real numbers ρ, K such that for all $x \in \mathbb{R}^d$ and $t \geq 0$, $x^T f(x,t) \leq \rho$, and $||g(x,t)||_{op} \leq K$ (where $||\cdot||_{op}$ denotes the operator norm), then

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t\log\log t}} \leq K\sqrt{e}, \quad \text{a.s.} \tag{1.2}$$

The main steps of the Mao's proof are as follows: first, make a suitable Itô transformation; then estimate the size of the Itô integral term by a Riemann integral using the *exponential martingale* *inequality* (EMI); and finally apply *Gronwall's inequality* (GI) to determine the asymptotic rate of growth.

In contrast, the results in this paper are established through a combination of *comparison principles* and *Motoo's theorem*. Motoo's theorem (cf. [15]) determines the *exact* asymptotic growth rate of the partial maxima of a *stationary* or *asymptotically stationary* process governed by an autonomous SDE. Motoo [15] also gives a proof of the Law of the Iterated Logarithm for a finite-dimensional Brownian motion. This proof is crucially reliant on applying a change in both space and scale. He considers an autonomous non-stationary δ–dimensional Bessel process R_δ, which is governed by the scalar equation

$$dR_\delta(t) = \frac{\delta - 1}{2R_\delta(t)}\,dt + dB(t) \tag{1.3}$$

with $R_\delta(0) = r_0 \geq 0$. The Bessel process R_δ is transformed into an autonomous process with finite speed measure (i.e., a process that possesses a limiting distribution) to which the Motoo's theorem can be applied. More precisely, if we let

$$S_\delta(t) = e^{-t} R_\delta^2(e^t - 1), \tag{1.4}$$

then

$$dS_\delta(t) = (\delta - S_\delta(t))\,dt + 2\sqrt{S_\delta(t)}\,d\tilde{B}(t). \tag{1.5}$$

It is reasonable to ask whether a combination of space and scale transformations of this classic type could reduce a general non-stationary autonomous SDE to one with finite speed measure to which Motoo's theorem could then be applied. If we consider general transformations of the form

$$Y(t) = \lambda(t)P(X(\gamma(t)))$$

where $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is increasing, $P \in C^2(\mathbb{R};\mathbb{R})$ and $\lambda \in C^1(\mathbb{R}^+;\mathbb{R}^+)$ (and is related to γ), the resulting SDE for Y will be non–autonomous, and in particular, will have non-autonomous diffusion coefficient. Adapting the proof of Motoo's theorem to cope with SDEs with non-autonomous diffusion coefficients introduces formidable difficulties, because the independence of excursions, on which the proof relies, can no longer be assured.

However, in this paper, with the well–known stochastic comparison principle (which assumes an order on the drift coefficients), we are able to investigate a much wider class of SDEs which are related to (1.3) through (1.4) and which give rise to equations of the type (1.5). In addition, with ordinary Itô transformations, we could map an even wider class of nonlinear equations onto a family of SDEs whose asymptotic behaviour is understood. This is shown in [2]. A detailed discussion on the relative advantages and disadvantages of this comparison-Motoo technique with the existing EMI–Gronwall approach can also be found in [2].

Also in [1], Appleby et al. applied processes obeying the Law of the Iterated Logarithm to financial market models which are inefficient in the sense of Fama. In this paper, we further investigate some ergodic–like properties of these processes. Under some reasonable assumptions of regarding the market, we establish two main results. First, we show that the largest fluctuations from the trend growth rate of the price are of the same size as in a related efficient market model. Second, we show that these fluctuations are "on average" greater than those in the efficient model, in a sense later made precise.

This paper considers a number of closely related equations, and proves a number of diverse asymptotic results. In order to understand the relationships between these results and to ease the readers' path through the paper, we give a synopsis and discussion about the main results, as well as their applications in Section 3. Full statements of the theorems and detailed proofs are found in succeeding sections.

2 Preliminaries

Throughout the paper, the set of non-negative real numbers is denoted by $\mathbb{R}^+$. The space of $d \times m$ matrices with real entries is denoted by $\mathbb{R}^{d\times m}$; in the case when $m = 1$, we write $\mathbb{R}^{d\times 1} = \mathbb{R}^d$. Let $L^1([a, b]; \mathbb{R}^d)$ be the family of Borel measurable functions $h : [a, b] \to \mathbb{R}^d$ such that $\int_a^b |h(x)|dx < \infty$. If x and y are two real numbers, then the maximum and minimum of x and y are denoted by $x \vee y$ and $x \wedge y$ respectively. Let $|x|$ be the Euclidean norm of a column or a row vector $x \in \mathbb{R}^d$; $||A||$ and $||A||_{op}$ denote the Frobenius norm and operator norm respectively for any $A \in \mathbb{R}^{d\times m}$. Note that

$$||A||_{op} \le ||A|| \quad \text{and} \quad ||A|| \le \sqrt{m}||A||_{op}.$$

Moreover, we use the Landau symbol for functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$:

$$f = \mathcal{O}(g^{-1}) \iff \limsup_{t\to\infty} |g(t)||f(t)| < \infty.$$

We use $(\Omega, \mathscr{F}, \{\mathscr{F}(t)\}_{t\ge 0}, \mathbb{P})$ to denote a complete filtered probability space. The abbreviation *a.s.* stands for *almost surely*. We always assume that both the drift and the diffusion coefficients of SDEs being studied satisfy the local Lipschitz condition even if this is not explicitly stated. If an autonomous scalar SDE has drift coefficient $f(\cdot)$ and non-degenerate diffusion coefficient $g(\cdot)$, then a scale function and speed measure of the solution of this SDE are defined by

$$s_c(x) = \int_c^x e^{-2\int_c^y \frac{f(z)}{g^2(z)}dz}dy, \quad m(dx) = \frac{2dx}{s'(x)g^2(x)}, \quad c,\, x \in I := (l, r) \tag{2.1}$$

respectively, where I is the state space of the process. These functions help us to determine the recurrence and stationarity of a process on I (cf. e.g. [12]). Moreover, Feller's test for explosions (cf. e.g. [12]) allows us to examine whether a process will never escape from its state space in finite time. This in turn relies on whether

$$v(l+) = v(r-) = \infty$$

or not, where v is defined as

$$v_c(x) = \int_c^x s_c'(y) \int_c^y \frac{2dz}{s_c'(z)g^2(z)}dy, \; c, x \in I. \tag{2.2}$$

As mentioned in the introduction, Motoo's Theorem is an important tool in determining the pathwise largest deviations for stationary or asymptotically stationary processes. We state Motoo's theorem in this section for future use.

Theorem 2.1. *(Motoo) Let X be the unique continuous real-valued process satisfying the following equation*

$$dX(t) = f(X(t))dt + g(X(t))dB(t), \quad t \ge 0,$$

with $X(0) = x_0$. Let s and m be the scale function and speed measure of X as defined in (2.1)*, and let $h : (0,\infty) \to (0,\infty)$ be an increasing function with $h(t) \to \infty$ as $t \to \infty$. If X is recurrent on (l,∞) (or $[l,\infty)$ in the case when l is an instantaneous reflecting point) and $m(l,\infty) < \infty$, then*

$$\mathbb{P}\left[\limsup_{t\to\infty} \frac{X(t)}{h(t)} \ge 1\right] = 1 \text{ or } 0$$

according to whether

$$\int_{t_0}^{\infty} \frac{1}{s(h(t))}\,dt = \infty \quad \text{or} \quad \int_{t_0}^{\infty} \frac{1}{s(h(t))}\,dt < \infty, \quad \text{for some } t_0 > 0.$$

The following lemma may be proven by applying Motoo's theorem directly to it. We will use it frequently.

Lemma 2.2. *Let U be the unique continuous adapted solution of the following equation*

$$dU(t) = (-aU(t) + b)\,dt + c\sqrt{|U(t)|}\,dB(t), \quad t \ge 0,$$

with $U(0) = u_0 > 0$, where a, b and c are positive real numbers. Then $U(t) \ge 0$ for all $t \ge 0$ a.s. Moreover U is recurrent, has finite speed measure, and obeys

$$\limsup_{t\to\infty} \frac{U(t)}{\log t} = \frac{c^2}{2a}, \quad \text{a.s.}$$

Throughout the paper, we repeatedly use Doob's theorem for the representation of a continuous martingale in terms of standard one-dimensional Brownian motion. We state the theorem in this section for notational convenience and future reference.

Theorem 2.3. *(Doob) Suppose M is a continuous local martingale defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, and the square variation $\langle M\rangle$ is an absolutely continuous function of t for $\mathbb{P}$-almost every ω. Then there is an extended space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$ of $(\Omega, \mathscr{F}, \mathbb{P})$ on which is defined a one-dimensional Brownian motion $W = \{W(t), \tilde{\mathscr{F}}(t); 0 \le t < \infty\}$ and a $\tilde{\mathscr{F}}(t)$-adapted process X with $\tilde{\mathbb{P}}$-a.s.*

$$\int_0^t X^2(s)ds < \infty, \quad 0 \le t < \infty,$$

such that we have the representations $\tilde{\mathbb{P}}$-a.s.

$$M(t) = \int_0^t X(s)\,dW(s), \quad \langle M\rangle(t) = \int_0^t X^2(s)\,ds, \quad 0 \le t < \infty.$$

In the proof of the above theorem, the new Brownian motion W is constructed by a continuous local martingale with respect to the original probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and another Brownian motion, say $\widehat{B}$, which is defined on the extended part of $(\Omega, \mathscr{F}, \mathbb{P})$ in $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$. Moreover, $\widehat{B}$ is independent of M. Therefore in this paper, any conclusion made with respect to the extended measure $\tilde{\mathbb{P}}$ about the underlying semimartingale (with martingale component M) defined on $(\Omega, \mathscr{F}, \mathbb{P})$ coincides with that for the measure $\mathbb{P}$. Therefore we do not make explicit reference to the probability spaces when stating results.

3 Synopsis and Discussion of Main Results

In this section, we give a brief discussion of the results proven in this paper. First, we state the Law of the Iterated Logarithm and other results on asymptotic growth bounds for transient solutions of autonomous SDEs. Second, we discuss general non-autonomous equations for which the LIL holds, under some uniform estimates on the drift. Third, we give comprehensive results for a parameterised family of autonomous SDEs with constant diffusion coefficient which do not require uniform estimates on the drift. Finally, we discuss some extensions of these results to multi–dimensional SDEs, as well as applications of the results to weakly inefficient financial markets.

3.1 Transient processes

Our first main result, Theorem 4.3, concerns transient solutions of the scalar autonomous stochastic differential equation

$$dX(t) = f(X(t))\,dt + g(X(t))\,dB(t) \tag{3.1}$$

where $f:\mathbb{R}\to\mathbb{R}$, $g(x)=\sigma$ for $x\in\mathbb{R}$, and

$$\lim_{x\to\infty} xf(x) = L_\infty > \frac{\sigma^2}{2}. \tag{3.2}$$

If we define $A := \{\omega : \lim_{t\to\infty} X(t,\omega) = \infty\}$, then $\mathbb{P}[A] > 0$, and we can show that the solution X obeys

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} = |\sigma|, \quad \text{a.s. on } A \tag{3.3}$$

and

$$\liminf_{t\to\infty} \frac{\log\frac{X(t)}{\sqrt{t}}}{\log\log t} = -\frac{1}{\frac{2L_\infty}{\sigma^2}-1}, \quad \text{a.s. on } A.$$

X exhibits similar transient behaviour at minus infinity if

$$\lim_{x\to-\infty} xf(x) = L_{-\infty} > \frac{\sigma^2}{2}. \tag{3.4}$$

These results are established by comparing X with a general Bessel process which has similar behaviour to X. The asymptotic behaviour of the Bessel process is given in Lemma 4.1. The modulus of a finite-dimensional Brownian motion (i.e., a Bessel process) with dimension greater than two is known to be transient, and when the dimension is less than or equal to two, the process is recurrent on the positive real line. However, for general Bessel processes, the critical dimension altering its behaviour does not have to be an integer. This fact is eventually captured in Theorem 4.3 by condition (3.2) (or (3.4)). More precisely, if exactly one of the parameters L_∞ and $L_{-\infty}$ is greater than the critical value $\sigma^2/2$, then the process tends to infinity or minus infinity almost surely while still obeying the Law of the Iterated Logarithm. If on the other hand L_∞ and $L_{-\infty}$ are both greater than $\sigma^2/2$, and we denote the event $\{\omega : \lim_{t\to\infty} X(t,\omega) = -\infty\}$ by $\tilde{A}$, we have that $\mathbb{P}[\tilde{A}] = 1 - \mathbb{P}[A]$. Furthermore both probabilities are positive and can be computed explicitly in terms of the scale function and the deterministic initial value of the process (cf. [12, Proposition 5.5.22]). Motoo's theorem also helps us to find an exact pathwise lower bound on the growth rate of the process.

This result could also be very useful in determining the pathwise decay rates of asymptotically stable SDEs. In Theorem 4.5, the constant diffusion coefficient σ is replaced by a state–dependent coefficient $g(\cdot)$ tending to σ as x tends to infinity, and similar results are obtained by means of a random time–change argument. Theorem 4.3 lays the foundation for further results concerning transient solutions of more general equations with unbounded diffusion coefficients. For example, suppose X obeys (3.1), where g is strictly positive and regularly varying at infinity with index β $(0 < \beta < 1)$, and f and g are related via

$$\lim_{x\to\infty} \frac{xf(x)}{g^2(x)} = L_\infty > \frac{1}{2}.$$

Then by Itô's rule, if A is as previously defined, it is easy to show that

$$\limsup_{t\to\infty} \frac{X(t)}{G^{-1}(\sqrt{2t\log\log t})} = 1, \quad \text{a.s. on } A$$

and

$$\liminf_{t\to\infty} \frac{\log \frac{G(X(t))}{\sqrt{t}}}{\log\log t} = -\frac{1-\beta}{2L_\infty - 1}, \quad \text{a.s. on } A,$$

where G is defined as

$$G(x) = \int_c^x \frac{1}{g(y)}\,dy, \quad c \in \mathbb{R}.$$

We leave the details of this result to the interested reader. Another application of these results is given in the next section: we make use of the upper envelope of the growth rate (3.3) to determine upper bounds for a more general type of equation whose solutions obey the Law of the Iterated Logarithm.

3.2 General conditions and ergodicity

In Section 5, we state and prove three theorems which give sufficient conditions ensuring Law of the Iterated Logarithm–type asymptotic behaviour, and which enable us to prove further results later in the paper. We will study the one-dimensional non-autonomous equation

$$dX(t) = f(X(t),t)\,dt + \sigma\,dB(t), \quad t \geq 0, \tag{3.5}$$

with $X(0) = x_0$. From the results in Section 4, in Theorem 5.1 it can be shown that

$$\sup_{(x,t)\in\mathbb{R}\times\mathbb{R}^+} xf(x,t) = \rho > 0 \tag{3.6}$$

implies

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t\log\log t}} \leq |\sigma|, \quad \text{a.s.} \tag{3.7}$$

Furthermore, in Theorem 5.3, we prove that

$$\inf_{(x,t)\in\mathbb{R}\times\mathbb{R}^+} xf(x,t) = \mu > -\frac{\sigma^2}{2} \tag{3.8}$$

implies

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t\log\log t}} \geq |\sigma|, \quad \text{a.s.} \tag{3.9}$$

Hence if both (3.6) and (3.8) are satisfied, we can determine the exact growth rate of the partial maxima. Moreover, we establish an ergodic–type theorem on a suitably scaled the average value of the process, as described by the following inequalities:

$$\limsup_{t\to\infty} \frac{\int_0^t \frac{X^2(s)}{(1+s)^2}\,ds}{\log t} \leq 2\rho + \sigma^2, \quad \text{a.s.} \tag{3.10}$$

$$\liminf_{t\to\infty} \frac{\int_0^t \frac{X^2(s)}{(1+s)^2}\,ds}{\log t} \geq 2\mu + \sigma^2 > 0, \quad \text{a.s.} \tag{3.11}$$

These results can be deduced from [17, Exercise XI.1.32]. (3.7) is obtained by the construction of two transient processes as described in Section 4. This gives an alternative proof to Theorem 3.1 in [1].

It appears that a condition of the form (3.6) is necessary to ensure that the solution obeys the LIL. Suppose for instance in equation (3.1) that there is $\alpha \in (0,1)$ such that $x^\alpha f(x) \to C > 0$ as $x \to \infty$. Then $X(t) \to \infty$ on some event Ω' with positive probability and

$$\lim_{t\to\infty} \frac{X(t)}{t^{\frac{1}{1+\alpha}}} = [C(1+\alpha)]^{\frac{1}{1+\alpha}}, \quad \text{a.s. on } \Omega',$$

which obviously violates the Law of the Iterated Logarithm (cf. [9, Theorem 4.17.5]).

It is worth noticing that ρ does not appear in the estimate in (3.7). This fact is used in Theorem 7.3 which deals with multi-dimensional systems. However ρ does affect the average value of X in the long-run, as seen in (3.10). As mentioned in the introduction, by the Motoo–comparison approach, the estimate on the constant on the righthand side of (3.7) has been reduced by a factor of $\sqrt{e}$ compared to that obtained by the EMI-Gronwall method. In addition, this approach enables us to find the lower estimate (3.9), which is the same size as the upper estimate. This has not been achieved to date by the exponential martingale inequality approach. Condition (3.8) is sufficient but not necessary for securing an LIL–type of lower bound, as will be seen in Theorem 5.6.

We noted already that the parameters ρ and μ in the drift do not affect the growth of the partial maxima as given by (3.7) and (3.9). However, (3.10) and (3.11) show that these parameters are important in determining the "average" size of the process, with larger contributions from the drift leading to larger average values. To cast further light on this we consider the related deterministic differential equation

$$x'(t) = f(x(t)), \quad t \geq 0, \tag{3.12}$$

where $xf(x) \to C > 0$ as $x \to \infty$. Then it is easy to verify that $x^2(t)/2t \to C$ as $t \to \infty$, which implies

$$\lim_{t\to\infty} \frac{\int_0^t \frac{x^2(s)}{(1+s)^2}ds}{\log t} = 2C. \tag{3.13}$$

This simple example is interesting for a number of reasons. Firstly, it can be seen as motivating the stochastic results (3.10) and (3.11), or as an easily and independently verified corollary of

(3.10) and (3.11). Secondly, it gives insight into the "average" long-run value of X: the fact that $x(t)/\sqrt{t} \to \sqrt{2C}$ as $t \to \infty$ obeys (3.13) suggests, in the sense of (3.10) and (3.11), that $|X(t)|^2$ is "on average" $C_1^2 t$ as $t \to \infty$ for some constant C_1. (3.10) and (3.11) may also be seen as a generalisation of a known result for Brownian motions without drift. Indeed, using (3.10) and (3.11) with $\rho = \mu = 0$, the Brownian motion $X(t) := \sigma B(t)$ must also obey

$$\lim_{t\to\infty} \frac{\int_0^t \frac{X^2(s)}{(1+s)^2}\,ds}{\log t} = \sigma^2, \quad \text{a.s.} \tag{3.14}$$

(3.14) indicates that the large excursions of Brownian motion excursions in the solution of (3.5) contributes the σ^2 term in (3.10) and (3.11). In this case, the statement that $|X(t)|^2$ is "on average" $\sigma^2 t$ is justified not only in the sense of (3.14), but because $\mathbb{E}[X^2(t)] = \sigma^2 t$. These two extreme cases (where there is no diffusion in the first, and no drift in the second) indicate that the contributions of drift and diffusion are of similar magnitude, and this is reflected in (3.10) and (3.11). Finally, it is an easy consequence of Theorem 5.1 and 5.3 that $|X(t)|^2$ grows "on average" like t as $t \to \infty$, because

$$x_0^2 + (2\mu + \sigma^2)t \le \mathbb{E}[X^2(t)] \le x_0^2 + (2\rho + \sigma^2)t, \quad t \ge 0.$$

Theorem 5.6 deals with processes with integrable drift coefficients. For an autonomous equation with drift coefficient $f \in L^1(\mathbb{R};\mathbb{R})$ and constant diffusion coefficient, there exist positive constants $\{C_i\}_{i=1,2,3,4}$ such that

$$C_1 \le \limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} \le C_2, \quad \text{a.s.}$$
$$-C_3 \le \liminf_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} \le -C_4, \quad \text{a.s.}$$

Formulae for the constants in these estimates can be found in Section 5. These processes are recurrent and can be transformed to other processes which are drift-free, have bounded diffusion coefficient, and which preserve the largest fluctuation size. This result is consistent with those in [9, Chapter 4], which roughly say that if the drift coefficient is zero on average along the real line and the diffusion coefficient $g(x)$ has a positive limit σ as $|x| \to \infty$, the process has a limiting normal distribution with mean zero and variance $\sigma^2 t$. This is precisely the distribution of the Brownian motion $\sigma B(t)$ at time t.

3.3 Recurrent processes

In Section 6, we investigate the scalar autonomous equation

$$dX(t) = f(X(t))\,dt + \sigma\,dB(t) \tag{3.15}$$

where the drift coefficient satisfies

$$\lim_{x\to\infty} xf(x) = L_\infty \le \sigma^2/2 \quad \text{and} \quad \lim_{x\to-\infty} xf(x) = L_{-\infty} \le \sigma^2/2. \tag{3.16}$$

These hypotheses are complementary to those discussed in Section 3.1. Using Feller's test (cf.[12]), it can be shown that under condition (3.16), X is no longer transient but in fact recurrent on the real

$L_{-\infty}$ \ L_∞	$(-\infty, -\frac{1}{2})$	$[-\frac{1}{2}, 0)$	$(0, \frac{1}{2}]$	$(\frac{1}{2}, \infty)$
$(-\infty, -\frac{1}{2})$	asymptotically stationary violates LIL Theorem 6.1	recurrent **C**, **D** Theorem 6.4	recurrent **B** Theorem 6.7 Part (i)	$\lim_{t\to\infty} X(t) = \infty$ **A** Theorem 4.3
$[-\frac{1}{2}, 0)$	recurrent **C**, **D** Theorem 6.4	recurrent **C**, **D** Theorem 6.3	recurrent **B** Theorem 6.7 Part (i)	$\lim_{t\to\infty} X(t) = \infty$ **A** Theorem 4.3
$(0, \frac{1}{2}]$	recurrent **B** Theorem 6.7 Part (ii)	recurrent **B** Theorem 6.7 Part (ii)	recurrent **C**, **D** Theorem 6.3	$\lim_{t\to\infty} X(t) = \infty$ **A** Theorem 4.3
$(\frac{1}{2}, \infty)$	$\lim_{t\to\infty} X(t) = -\infty$ **A** Corollary 4.4	$\lim_{t\to\infty} X(t) = -\infty$ **A** Corollary 4.4	$\lim_{t\to\infty} X(t) = -\infty$ **A** Corollary 4.4	$\lim_{t\to\infty} X(t) = \pm\infty$ **A** Theorem 4.3, Corollary 4.4

Figure 1: Asymptotic behaviour of X obeying (3.1) where $\lim_{x\to\infty} xf(x) = L_\infty$ and $\lim_{x\to-\infty} xf(x) = L_{-\infty}$ and $g(x) = 1$. **A** signifies that X obeys the Law of the Iterated Logarithm exactly; **B** that $|X(t)|$ is bounded above and below by $\sqrt{2t\log_2 t}$ as $t \to \infty$; **C** that X has a polynomial upper Liapunov exponent equal to $1/2$; and **D** that the asymptotic behaviour is consistent with the Law of the Iterated Logarithm.

line. However results in Section 4 together with Theorem 5.6 (which deals with integrable drift) suggest that solutions should still have asymptotic behaviour similar to the LIL. This idea motivates us to prove similar results in the recurrent case to those already obtained for transient processes. The upper bound (3.3) given by Theorem 5.1 automatically applies, while difficulties arise in finding the lower bound on the limsup without condition (3.8), particularly when L_∞ and $L_{-\infty}$ are of the same sign. The subdivision of the main result into various theorems is necessitated by slight distinctions in the proofs, which in turn depends on the value of both L_∞ and $L_{-\infty}$. The results are summarised in the case $\sigma = 1$ in Figure 1.

Theorem 6.1 is a direct result of Motoo's theorem: it shows that $-\sigma^2/2$ is another critical value at which the behaviour of the process changes from being stationary (or asymptotically stationary) to non-stationary. The LIL is no longer valid when $L_{\pm\infty} < -\sigma^2/2$. By constructing another asymptotically stationary process as a lower bound for X^2 and X in Theorem 6.3 and 6.4 respectively, we obtain the following exact estimate on the polynomial Liapunov exponent of $|X|$:

$$\limsup_{t\to\infty} \frac{\log|X(t)|}{\log t} = \frac{1}{2}, \quad \text{a.s.} \tag{3.17}$$

(3.17) is of course a less precise result than the LIL. It shows that the partial maxima of the solution grows at least as fast as $K_\varepsilon t^{(1-\varepsilon)/2}$ for $\varepsilon \in (0, 1)$ and some positive K_ε. However, (3.17) is still consistent with the LIL. Using the same construction (see Lemma 6.6) and comparison techniques, together with Theorem 5.6, we obtain Theorem 6.7, which gives upper and lower estimates on the growth rate of the partial maxima.

Note that we have excluded the cases $L_{\pm\infty} = 0$ from Figure 1 for the purpose of stating consistent results on pairs of intervals for L_∞ and $L_{-\infty}$. Nonetheless Theorem 6.7 covers the case when at least one of $L_{\pm\infty} = 0$ and the drift coefficient f changes sign an even number of times. In particular, if f remains non–negative or non–positive on the real line, X can be pathwise compared with the Brownian motion $\{\sigma B(t)\}_{t\geq 0}$ directly, so an exact estimate can be obtained (Corollary 6.8). Otherwise, Theorem 6.3 and 6.4 are sufficient to cover the rest of the cases (Remark 6.5).

3.4 Multi-dimensional processes

In Section 7, we generalise results from Section 5 to the following d-dimensional equation driven by an m–dimensional Brownian motion

$$dX(t) = f(X(t), t)\,dt + g(X(t), t)\,dB(t). \tag{3.18}$$

Theorem 7.1 extends the result of Theorem 5.1 to SDEs with bounded diffusion coefficients under a condition similar to (3.6). Using a random time-change, we prove that

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t\log\log t}} \leq C_a, \quad \text{a.s.}$$

where $C_a := \sup_{(x,t)\in\mathbb{R}^d\times\mathbb{R}^+} ||g(x,t)||_{op}$. In a similar manner, Theorem 7.2 extends Theorem 5.3 in $\mathbb{R}^d$. The generalisation of these results to unbounded diffusion coefficients can be found in [2]. Finally, Theorem 7.3 shows under multi-dimensional analogues of conditions (3.6) and (3.8), the asymptotic large deviations of Euclidean norm of a multi-dimensional process are $\mathcal{O}(\sqrt{2t\log\log t})$. Moreover under some additional assumptions, the largest fluctuations of the norm is given by the co-ordinate process with the largest fluctuations. This result is an extension of the LIL for a d-dimensional Brownian motion (1.1). Mao (cf. [14]) pointed out the fact that the independent individual components of the multi-dimensional Brownian motion are not simultaneously of the order $\sqrt{2t\log\log t}$, for otherwise we would have $\sqrt{d}$ rather than unity on the right-hand side of (1.1). We establish these facts for drift–perturbed finite–dimensional Brownian motions. To simplify the analysis, we look at the following equation in $\mathbb{R}^d$:

$$dX(t) = f(X(t), t)\,dt + \Gamma\,dB(t), \quad t \geq 0 \tag{3.19}$$

where Γ is a $d \times d$ diagonal invertible matrix with diagonal entries $\{\gamma_i\}_{1\leq i\leq d}$. If $\langle x, f(x,t)\rangle \leq \rho$, then

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t\log\log t}} \leq \max_{1\leq i\leq d} |\gamma_i|, \quad \text{a.s.}$$

Furthermore if there exists one coordinate process X_i with drift coefficient f_i satisfying (3.8), then we have

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t\log\log t}} \geq |\gamma_i|, \quad \text{a.s.}$$

In the more general case that Γ is any invertible matrix, with the same conditions as above, the proof of this result can be easily adapted to show that with respect to the norm $|x|_\Gamma := |\Gamma^{-1}x|$, the solution of (3.19) satisfies

$$\limsup_{t\to\infty} \frac{|X(t)|_\Gamma}{\sqrt{2t\log\log t}} = 1, \quad \text{a.s.}$$

3.5 Applications to inefficient financial markets

According to Fama [6; 7], when efficiency refers only to historical information which is contained in every private trading agent's information set, the market is said to be *weakly efficient* (cf.[8, Definition 10.17]. Weak efficiency implies that successive price changes (or returns) are independently distributed. Formally, let the market model be described by a probability triple $(\Omega, \mathscr{F}, \mathbb{P})$. Suppose that trading takes place in continuous time, and that there is one risky security. Let $h > 0$, $t \geq 0$ and let $r_h(t+h)$ denote the return of the security from t to $t+h$, and let $S(t)$ be the price of the risky security at time t. Also let $\mathscr{F}(t)$ be the collection of historical information available to every market participant at time t. Then the market is weakly efficient if

$$\mathbb{P}[r_h(t+h) \leq x | \mathscr{F}(t)] = \mathbb{P}[r_h(t+h) \leq x], \quad \forall\, x \in \mathbb{R},\ h > 0,\ t \geq 0.$$

Here the information $\mathscr{F}(t)$ which is publicly available at time t is nothing other than the generated σ-algebra of the price $\mathscr{F}^S(t) = \sigma\{S(u) : 0 \leq u \leq t\}$. An equivalent definition of weak efficiency in this setting is that

$$r_h(t+h) \text{ is } \mathscr{F}^S(t)\text{-independent, for all } h > 0 \text{ and } t \geq 0. \tag{3.20}$$

Geometric Brownian Motion is the classical stochastic process that is used to describe stock price dynamics in a weakly efficient market. More concretely, it obeys the linear SDE

$$dS(t) = \mu S(t)\,dt + \sigma S(t)\,dB(t), \quad t \geq 0 \tag{3.21}$$

with $S(0) > 0$. Here $S(t)$ is the price of the risky security at time t, μ is the appreciation rate of the price, and σ is the volatility. It is well-known that the logarithm of S grows linearly in the long-run. The increments of $\log S$ are stationary and Gaussian, which is a consequence of the driving Brownian motion. That is, for a fixed time lag h,

$$r_h(t+h) := \log\frac{S(t+h)}{S(t)} = (\mu - \frac{1}{2}\sigma^2)h + \sigma(B(t+h) - B(t))$$

is Gaussian distributed. Clearly $r_h(t+h)$ is $\mathscr{F}^B(t)$-independent, because B has independent increments. Therefore if $\mathscr{F}^B(t) = \mathscr{F}^S(t)$, it follows that the market is weakly efficient. To see this, note that S being a strong solution of (3.21) implies that $\mathscr{F}^S(t) \subseteq \mathscr{F}^B(t)$. On the other hand, since

$$\log S(t) = \log S(0) + (\mu - \frac{1}{2}\sigma^2)t + \sigma B(t), \quad t \geq 0,$$

we can rearrange for B in terms of S to get that $\mathscr{F}^B(t) \subseteq \mathscr{F}^S(t)$, and hence $\mathscr{F}^B(t) = \mathscr{F}^S(t)$. Due to this reason, equation (3.21) has been used to model stock price evolution under the classic Efficient Market Hypothesis.

In order to reflect the phenomenon of occasional weak inefficiency resulting from feedback strategies widely applied by investors, in [1] SDEs whose solutions obey the Law of the Iterated Logarithm are applied to inefficient financial market models. More precisely, a semi-martingale X, which is slightly drift-perturbed and obeys the Law of the Iterated Logarithm, is introduced into equation (3.21) as the driving semimartingale instead of Brownian motion. It is shown that if a process S_* satisfies

$$dS_*(t) = \mu S_*(t)\,dt + S_*(t)\,dX(t), \quad t \geq 0, \quad S_*(0) > 0, \tag{3.22}$$

then S_* preserves some of the main characteristics of the standard Geometric Brownian Motion S. More precisely, the size of the long-run large deviations from the linear trend of the cumulative returns is preserved, along with the exponential growth of S. This is despite the fact that the increments of $\log S_*$ are now correlated and non-Gaussian.

In this paper, we further investigate the effect of this drift perturbation on the cumulative returns in (3.22) with the process X satisfying (3.5) or (3.15), say. We do not wish to provide a complicated and empirically precise model, but rather a simple and tractable model, and to interpret the mathematical results.

With a modest bias in the trend (e.g. captured by condition (3.6) and (3.8)), the excursions in prices from the linear trend are no longer independent. The largest possible sizes of these excursions coincide with those under no bias (as seen in (3.7) and (3.9)). However, by ergodic–type results (e.g. (3.10) and (3.11)), the stronger the positive bias that the investors have, the larger the average values of price excursions, and consequently the smaller the volatility that arises around the average values. This causes the price to persist on average further from the long-run growth trend that the GBM model would allow. This is made precisely in (3.24) below. This persistence could make investors believe that the cumulative returns are close to their true values and are unbiased, which might cause a more dramatic fall in cumulative returns later on. Moreover, if the market is even more pessimistic after a relatively large drop in returns, the bias tends to have a longer negative impact on the market.

In the model presented below, we presume that the returns evolve according to the strength of the various agents trading in the market. At a given time, each agent determines a threshold which signals whether the market is overbought or oversold. The agents become more risk averse in their trading strategies when these overbought or oversold thresholds are breached. If we make the simplifying assumption that one agent is representative of all, then the threshold level is simply the weighted average of the threshold for all the individuals.

Using these ideas, we are led to study the equation

$$dX(t) = f(X(t))[1 - \alpha I_{\{|X(t)| > k\sigma\sqrt{t}\}}]\,dt + \sigma\,dB(t). \tag{3.23}$$

Here f is assumed continuous and odd on $\mathbb{R}$ so that the positive and negative returns are treated symmetrically. Moreover, in order that the bias be modest, we require $\lim_{|x|\to\infty} xf(x) = L \in (0, \sigma^2/2]$. In (3.23), I is the indicator function, and $\alpha \in (0,1]$ measures the extent of short-selling or "going long" in the market. Here an increased α is associated with an increased tendency to sell short or go long. We presume that investors believe that the de–trended security returns are given by Brownian motion without drift, and the returns obey the Law of the Iterated Logarithm. Moreover, we assume that the investors can estimate the value of σ by tracking the size of the largest deviations.

We briefly indicate how the threshold level is arrived. The standard Brownian motion (which the investors believe models the security return) is scaled by σ, and therefore, at time t, has standard deviation $\sigma\sqrt{t}$. If each agent i chooses a multiple k_i of this standard deviation as his/her threshold level, and assuming that all agents are representative, there exists a weighted coefficient k, such that $k\sigma\sqrt{t}$ measures the overall market threshold level. In practice, the value of k might be different for price increases and falls. We treat two situations with one fixed k here for simplicity.

Given these assumptions, we prove the following. First, X is recurrent on $\mathbb{R}$ and obeys the Law of the Iterated Logarithm by the results in Section 5 and 6. Second, we determine the long-run average value of the de–trended cumulative returns by proving the following ergodic–type theorem:

$$\lim_{t\to\infty}\frac{\int_0^t \frac{X^2(s)}{(1+s)^2}\,ds}{\log t}=\Lambda_{L,\sigma,\alpha,k}>\sigma^2, \quad \text{a.s.} \tag{3.24}$$

Here, $\Lambda_{L,\sigma,\alpha,k}$ measures the market bias from the unbiased value of σ^2. It can be computed and is given in Section 8. Our assumptions on parameters ensure that $\Lambda_{L,\sigma,\alpha,k}>\sigma^2$. This means that the presence of bias increases the "average size" of the departures of the returns from the trend growth rate.

To establish (3.24), we first transform the solution X of (3.23) into a process Y by a change in both time and scale; second, we construct two equations with continuous and time-homogenous drift coefficients and with finite speed measures, such that Y is trapped between the solutions of these equations; third, by adjusting certain auxiliary parameters, we obtain an ergodic–type theorem for Y, which in turn implies (3.24). From a mathematical point of view, we have proved an ergodic–type theorem for a non–autonomous equation using the stochastic comparison principle.

Finally, we confirm that equation (3.22) with X satisfying (3.23) does represent an inefficient market in the *weak* sense, i.e., we want to show that

$$r_{*,h}(t+h) \text{ is } \mathscr{F}^{S_*}(t)\text{-dependent, for all } h>0 \text{ and } t\geq 0, \tag{3.25}$$

where r_* is the return. It is easy to verify that

$$S_*(t)=S_*(0)e^{(\mu-\frac{1}{2}\sigma^2)t+X(t)}, \quad X(t)=\log\frac{S_*(t)}{S_*(0)}-(\mu-\frac{1}{2}\sigma^2)t, \quad t\geq 0.$$

Therefore $\mathscr{F}^{S_*}(t)=\mathscr{F}^X(t)$. In the proof of the main result of this section, we establish the strong existence and uniqueness of the solution of equation (3.23) (this requires a little care because of the discontinuity of the drift coefficient). Since $X(0)=0$ is deterministic, and X is a strong solution, we have $\mathscr{F}^X(t)\subseteq\mathscr{F}^B(t)$ for $t\geq 0$. On the other hand, by writing $F(t,x):=f(x)[1-\alpha I_{\{|x|>k\sigma\sqrt{t}\}}]$, we get

$$B(t)=\frac{1}{\sigma}\left(X(t)-\int_0^t F(s,X(s))ds\right), \quad t\geq 0.$$

Hence $\mathscr{F}^B(t)\subseteq\mathscr{F}^X(t)$ for $t\geq 0$. Consequently $\mathscr{F}^{S_*}(t)=\mathscr{F}^B(t)=\mathscr{F}^X(t)$ for $t\geq 0$. So we may replace $\mathscr{F}^{S_*}(t)$ by $\mathscr{F}^B(t)$ in (3.25). Next, the increments $r_{*,h}$ of $\log S_*$ obey

$$\begin{aligned} r_{*,h}(t+h) &:= \log \frac{S_*(t+h)}{S_*(t)} \\ &= (\mu - \frac{1}{2}\sigma^2)h + \sigma(B(t+h) - B(t)) + \int_t^{t+h} F(s,X(s))ds \\ &= (\mu - \frac{1}{2}\sigma^2)h + (X(t+h) - X(t)). \end{aligned}$$

Now suppose for some $t \geq 0$, that $r_{*,h}(t+h)$ is $\mathscr{F}^B(t)$-independent. Since $[(\mu - \frac{1}{2}\sigma^2)h + \sigma(B(t+h) - B(t))]$ is $\mathscr{F}^B(t)$-independent, $\int_t^{t+h} F(s,X(s))ds$ must also be $\mathscr{F}^B(t)$-independent. However, by the Markov property of X, $\int_t^{t+h} F(s,X(s))ds$ is a functional of $X(t)$ and the increments of B. Hence, $\int_t^{t+h} F(s,X(s))ds$ is $\mathscr{F}^X(t)$-dependent, and since $\mathscr{F}^X(t) = \mathscr{F}^B(t)$, this gives a contradiction. Therefore (3.25) is proved.

4 Asymptotic Behaviour of Transient Processes

In this section, we study processes which obey (3.1) and are transient, obeying $|X(t)| \to \infty$ as $t \to \infty$. To do this, we introduce an auxiliary process: let $\delta > 2$ and consider

$$dY(t) = \sigma^2 \frac{\delta - 1}{2Y(t)} dt + \sigma\, dB(t) \quad \text{for } t \geq 0, \tag{4.1a}$$

$$Y(0) = y_0 > 0, \tag{4.1b}$$

where y_0 is deterministic. The solution of the above equation is a generalised Bessel process of dimension higher than 2. $\delta > 2$ does not have to be an integer. If $\delta > 2$ is an integer, then $Y(t) = \sigma|W(t)|$ where W is a δ–dimensional Brownian motion. Therefore, in the general case, we expect Y to grow to infinity like e.g. a three-dimensional Bessel process. This can be confirmed by [12, Chapter 3.3 Section C]. In fact, as proven in the following lemma, Y should also obey the Law of the Iterated Logarithm. The proof is the same in spirit as that in Motoo [15], but is briefly given here in the language of stochastic differential equations in order to be consistent with the techniques of this paper. We moreover employ Motoo's techniques to establish a lower bound on the growth rate.

Lemma 4.1. *Let $\delta > 2$ and Y be the unique continuous adapted process which obeys* (4.1). *Then Y is a positive process a.s., and satisfies*

$$\limsup_{t\to\infty} \frac{Y(t)}{\sqrt{2t \log\log t}} = |\sigma| \quad a.s. \tag{4.2}$$

and

$$\liminf_{t\to\infty} \frac{\log \frac{Y(t)}{\sqrt{t}}}{\log\log t} = -\frac{1}{\delta - 2}, \quad a.s. \tag{4.3}$$

Proof. Let $Z(t) = Y(t)^2$. By Itô's rule, we get

$$dZ(t) = \sigma^2\,\delta\,dt + 2\sqrt{Z(t)}\sigma\,d\widehat{B}(t), \quad t \geq 0$$

with $Z(0) = y_0^2$, where by Doob's martingale representation theorem, we have replaced the original Brownian motion B by $\widehat{B}$ in an extended probability space. Therefore

$$\begin{aligned}
Z(e^t - 1) &= y_0^2 + \int_0^{e^t-1} \sigma^2\delta\,ds + \int_0^{e^t-1} 2\sqrt{Z(s)}\sigma\,d\widehat{B}(s) \\
&= y_0^2 + \int_0^t \sigma^2\delta e^s\,ds + \int_0^t 2\sigma\sqrt{Z(e^s-1)}e^{\frac{s}{2}}\,dW(s),
\end{aligned}$$

where W is again another Brownian motion. If $\widetilde{Z}(t) = Z(e^t - 1)$, then

$$d\widetilde{Z}(t) = \sigma^2\delta e^t\,dt + 2\sigma\sqrt{\widetilde{Z}(t)}e^{\frac{t}{2}}\,dW(t), \quad t \geq 0.$$

If $H(t) := e^{-t}\widetilde{Z}(t)$, then $H(0) > 0$ and H obeys

$$dH(t) = (\sigma^2\delta - H(t))\,dt + 2\sigma\sqrt{H(t)}\,dW(t), \quad t \geq 0. \tag{4.4}$$

Therefore by Lemma 2.2, we have

$$\limsup_{t\to\infty} \frac{H(t)}{2\log t} = \sigma^2, \quad \text{a.s.} \tag{4.5}$$

Using the definition of Y in terms of H and Z we obtain (4.2).

To prove (4.3), consider the transformation $H_*(t) := 1/H(t)$. H_* is well-defined, a.s. positive, and by Itô's rule obeys

$$dH_*(t) = [(4\sigma^2 - \sigma^2\delta)H_*^2(t) + H_*(t)]\,dt - 2\sigma\frac{H_*^2(t)}{\sqrt{H_*(t)}}\,dW(t), \quad t \geq 0.$$

It is easy to show that the scale function of H_* satisfies

$$s_{H_*}(x) = K_1 \int_1^x y^{\frac{\delta-4}{2}} e^{\frac{1}{2\sigma^2 y}}\,dy, \quad x \in \mathbb{R},$$

for some positive constant K_1, and H_* obeys all the conditions of Motoo's theorem. By L'Hôpital's rule, for some positive constant K_2, we have

$$\lim_{x\to\infty} \frac{s_{H_*}(x)}{x^{\frac{\delta-2}{2}}} = K_2.$$

Let $h_1(t) = t^{2/(\delta-2)}$. Then for some $t_1 > 0$,

$$\int_{t_1}^\infty \frac{1}{s_{H_*}(h_1(t))}\,dt \geq \int_{t_1}^\infty \frac{2}{K_2 t}\,dt = \infty.$$

Hence

$$\limsup_{t\to\infty} \frac{H_*(t)}{t^{\frac{2}{\delta-2}}} \geq 1, \quad \text{a.s.}$$

On the other hand, for $\epsilon \in (0, \delta-2)$,

$$\lim_{x\to\infty} \frac{s_{H_*}(x)}{x^{\frac{\delta-2-\epsilon}{2}}} = \infty.$$

Let $h_2(t) = t^{2/(\delta-2-\epsilon-\theta)}$, where $\theta \in (0, \delta-2-\epsilon)$. Then for some $t_2 > 0$, we get

$$\int_{t_2}^{\infty} \frac{1}{s_{H_*}(h_2(t))}\, dt \leq \int_{t_2}^{\infty} \frac{1}{t^{\frac{\delta-2-\epsilon}{\delta-2-\epsilon-\theta}}}\, dt < \infty,$$

a.s. on an a.s. event $\Omega_{\epsilon,\theta} := \Omega_\epsilon \cap \Omega_\theta$, where Ω_ϵ and Ω_θ are both a.s. events. From this by letting $\epsilon \downarrow 0$ and $\theta \downarrow 0$ through rational numbers, it can be deduced that

$$\limsup_{t\to\infty} \frac{\log H_*(t)}{\log t} = \frac{2}{\delta-2}, \quad \text{a.s. on } \cap_{\epsilon,\theta\in\mathbb{Q}} \Omega_{\epsilon,\theta}.$$

Using the relation between H_* and Y, we get the desired result (4.3). □

Corollary 4.2. *Let $\delta > 2$ and Y be the unique continuous adapted process which obeys* (4.1a)*, but with $Y(0) = y_0 < 0$. Then Y obeys*

$$\liminf_{t\to\infty} \frac{Y(t)}{\sqrt{2t\log\log t}} = -|\sigma|, \quad \textit{a.s.} \tag{4.6}$$

and

$$\liminf_{t\to\infty} \frac{\log \frac{|Y(t)|}{\sqrt{t}}}{\log\log t} = -\frac{1}{\delta-2}, \quad \textit{a.s.} \tag{4.7}$$

Proof. Letting $Y_*(t) = -Y(t)$ and applying the same analysis as Lemma 4.1 to Y_*, the results can be easily shown. The details are omitted. □

We are now in a position to determine the asymptotic behaviour of (3.1) when the diffusion coefficient is constant.

Theorem 4.3. *Let X be the unique continuous adapted process which obeys* (3.1)*. Let $A := \{\omega : \lim_{t\to\infty} X(t,\omega) = \infty\}$. If*

$$\lim_{x\to\infty} x f(x) = L_\infty; \tag{4.8}$$

$$g(x) = \sigma, \;\; x \in \mathbb{R},$$

where $\sigma \neq 0$ and $L_\infty > \sigma^2/2$, then $\mathbb{P}[A] > 0$ and X satisfies

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} = |\sigma| \quad \textit{a.s. on } A, \tag{4.9}$$

and

$$\liminf_{t\to\infty} \frac{\log \frac{X(t)}{\sqrt{t}}}{\log\log t} = -\frac{1}{\frac{2L_\infty}{\sigma^2} - 1}, \quad \textit{a.s. on } A. \tag{4.10}$$

Proof. First note that given $L_\infty > \sigma^2/2$, the existence of such a non-null event A in the sample space is guaranteed by Feller's test [12, Proposition 5.5.22]. From now on, we assume that we are working in A, and will frequently suppress ω-dependence and A a.s. qualifications accordingly. We compare X with $Y_{+\epsilon}$, where $Y_{+\epsilon}$ is given by

$$dY_{+\epsilon}(t) = \frac{L_\infty + \epsilon}{Y_{+\epsilon}(t)}\,dt + \sigma\,dB(t), \quad t \geq 0$$

with $Y_{+\epsilon}(0) > 0$ and $(L_\infty + \epsilon) > (L_\infty - \epsilon) > \sigma^2/2$, so that L_∞ takes the same role as δ in (4.1) as we let $\epsilon \downarrow 0$. Since $\lim_{x\to\infty} xf(x) = L_\infty$ and $\lim_{t\to\infty} X(t) = \infty$, there exists $T_1(\epsilon,\omega) > 0$, such that for all $t \geq T_1(\epsilon,\omega)$, $L_\infty - \epsilon < X(t)f(X(t)) < L_\infty + \epsilon$ and $X(t) > 0$. Hence $(L_\infty - \epsilon)/X(t) < f(X(t)) < (L_\infty + \epsilon)/X(t)$, $t \geq T_1(\epsilon,\omega)$. Let $\Delta(t) = Y_{+\epsilon}(t) - X(t)$. We now consider three cases:
Case 1: if $X(T_1) < Y_{+\epsilon}(T_1)$, i.e., $\Delta(T_1) > 0$, we claim that

$$\text{for all} \quad t > T_1(\epsilon,\omega), \quad X(t) < Y_{+\epsilon}(t).$$

Suppose to the contrary there exists a minimal $t^* > T_1(\epsilon,\omega)$ such that $X(t^*) = Y_{+\epsilon}(t^*)$. Then $\Delta(t^*) = 0$ and $\Delta'(t^*) \leq 0$. But

$$\Delta'(t) = \frac{L_\infty + \epsilon}{Y_{+\epsilon}(t)} - f(X(t)) > \frac{L_\infty + \epsilon}{Y_{+\epsilon}(t)} - \frac{L_\infty + \epsilon}{X(t)}, \quad \text{for all } t \geq T_1(\epsilon,\omega),$$

so

$$\Delta'(t^*) > \frac{L_\infty + \epsilon}{Y_{+\epsilon}(t^*)} - \frac{L_\infty + \epsilon}{X(t^*)} = 0,$$

which gives a contradiction.
Case 2: if $X(T_1) > Y_{+\epsilon}(T_1) > 0$, i.e., $\Delta(T_1) < 0$, we show that

$$\text{for all} \quad t \geq T_1(\epsilon,\omega), \quad X(t) \leq Y_{+\epsilon}(t) - \Delta(T_1).$$

Now for all $t \geq T_1(\epsilon,\omega)$,

$$\Delta'(t) = \frac{L_\infty + \epsilon}{Y_{+\epsilon}(t)} - f(X(t)) > \frac{L_\infty + \epsilon}{Y_{+\epsilon}(t)} - \frac{L_\infty + \epsilon}{X(t)} = \frac{-\Delta(t)(L_\infty + \epsilon)}{Y_{+\epsilon}(t)X(t)}. \tag{4.11}$$

In particular

$$\Delta'(T_1) > \frac{-\Delta(T_1)(L_\infty + \epsilon)}{Y_{+\epsilon}(T_1)X(T_1)} > 0. \tag{4.12}$$

There are now two possibilities: either $X(t) > Y(t)$ for all $t > T_1(\epsilon,\omega)$ or there is $T_2(\omega) > T_1(\epsilon,\omega)$, such that $X(T_2) = Y_{+\epsilon}(T_2)$. If $X(t) > Y_{+\epsilon}(t)$, $\forall\, t > T_1(\epsilon,\omega)$, then $\Delta'(t) > 0$, so Δ is increasing on $[T_1(\epsilon,\omega),\infty)$. Therefore $Y_{+\epsilon}(t) - X(t) = \Delta(t) > \Delta(T_1)$, we are done. The analysis of the situation where there exists $T_2(\omega) > T_1(\epsilon,\omega)$ such that $X(T_2) = Y_{+\epsilon}(T_2)$ is dealt with by case 3.
Case 3: if $X(T_1) = Y_{+\epsilon}(T_1)$, i.e., $\Delta(T_1) = 0$, we claim that

$$\text{for all} \quad t > T_1(\epsilon,\omega), \quad X(t) < Y_{+\epsilon}(t).$$

We note first from (4.12) that $\Delta'(T_1) > 0$. Hence, there exists $T_3(\omega) > T_1(\epsilon,\omega)$ such that $\Delta(t) > 0$ for $t \in (T_1, T_3)$. Suppose, in contradiction to the claim, that $T_3(\omega)$ is such that $\Delta(T_3) = 0$. Then $\Delta'(T_3) \leq 0$, which is impossible by (4.11).

Combining the above results, for almost all ω in A, there exists a random variable C_+ such that

$$X(t,\omega) \le Y_{+\epsilon}(t,\omega) + C_+(T_1(\epsilon,\omega)), \quad t \ge T_1(\epsilon,\omega). \tag{4.13}$$

Therefore

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} \le \limsup_{t\to\infty} \frac{Y_{+\epsilon}(t)}{\sqrt{2t\log\log t}}. \tag{4.14}$$

A lower estimate on X can now be deduced by a similar argument. For the same ϵ, define $Y_{-\epsilon}$ by

$$dY_{-\epsilon}(t) = \frac{L_\infty - \epsilon}{Y_{-\epsilon}(t)}\,dt + \sigma\,dB(t), \quad t \ge 0$$

with $Y_{-\epsilon}(0) > 0$. Note that $L_\infty - \epsilon > \sigma^2/2$, so $Y_{-\epsilon}$ is guaranteed to be positive. Then, by arguing as above, we obtain an analogous result to (4.13), namely that

$$X(t,\omega) \ge Y_{-\epsilon}(t,\omega) - C_-(T_4(\epsilon,\omega)), \quad t \ge T_4(\epsilon,\omega), \tag{4.15}$$

for some $T_4(\epsilon,\omega) > 0$ and random variable C_-. This implies

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} \ge \limsup_{t\to\infty} \frac{Y_{-\epsilon}(t)}{\sqrt{2t\log\log t}}. \tag{4.16}$$

We are now in a position to prove (4.9). Using (4.14), and letting Ω^*_ϵ be the a.s. event on which

$$\limsup_{t\to\infty} \frac{Y_{+\epsilon}(t)}{\sqrt{2t\log\log t}} = \sigma,$$

we have

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} \le |\sigma|, \quad \text{a.s. on } \Omega^*_\epsilon \cap A.$$

Letting $\Omega_* = \cap_{\epsilon\in\mathbb{Q}^+\cap(0,1)}\Omega^*_\epsilon$, it follows that

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} \le |\sigma|, \quad \text{a.s. on } \Omega_* \cap A, \tag{4.17}$$

as required. Similarly using (4.16), and letting $\Omega^*_{-\epsilon}$ be the a.s. event on which

$$\limsup_{t\to\infty} \frac{Y_{-\epsilon}(t)}{\sqrt{2t\log\log t}} = \sigma,$$

we have

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} \ge |\sigma|, \quad \text{a.s. on } A \cap \Omega^*_\epsilon.$$

With $\Omega_{**} = \cap_{\epsilon\in\mathbb{Q}\cap(0,1)}\Omega^*_{-\epsilon}$, it follows that

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} \ge |\sigma|, \quad \text{a.s. on } A \cap \Omega_{**} \tag{4.18}$$

as required. Combining (4.17) and (4.18) gives (4.9).

To prove (4.10), notice that $Y_{+\epsilon}$ obeys (4.1) with $\delta = \delta_\epsilon = 1 + 2(L_\infty + \epsilon)/\sigma^2$. Then, by (4.3) we have

$$\liminf_{t\to\infty} \frac{\log \frac{Y_{+\epsilon}(t)}{\sqrt{t}}}{\log\log t} = -\frac{1}{\delta_\epsilon - 2} = -\frac{1}{2(L_\infty + \epsilon)/\sigma^2 - 1}, \quad \text{a.s. on } \Omega_\epsilon^+ \tag{4.19}$$

where Ω_ϵ^+ is an almost sure event. Therefore by (4.13), a.s. on $A \cap \Omega_\epsilon^+$ we have

$$\liminf_{t\to\infty} \frac{\log \frac{X(t)}{\sqrt{t}}}{\log\log t} \le -\frac{1}{2(L_\infty + \epsilon)/\sigma^2 - 1}.$$

If $A^* = A \cap \{\cap_{\epsilon\in\mathbb{Q}\cap(0,1)} \Omega_\epsilon^+\}$, then A^* is an a.s. subset of A and

$$\liminf_{t\to\infty} \frac{\log \frac{X(t)}{\sqrt{t}}}{\log\log t} \le -\frac{1}{2L_\infty/\sigma^2 - 1}, \quad \text{a.s. on } A^*. \tag{4.20}$$

Proceeding similarly with $Y_{-\epsilon}$ and using (4.15) we can prove that

$$\liminf_{t\to\infty} \frac{\log \frac{X(t)}{\sqrt{t}}}{\log\log t} \ge -\frac{1}{2L_\infty/\sigma^2 - 1}, \quad \text{a.s. on } A^{**}, \tag{4.21}$$

where A^{**} is an a.s. subset of A. Combining (4.20) and (4.21) now yields (4.10). $\square$

Depending on the value of $L_{-\infty}$, by Feller's test, we can compute the probability of the event A defined in the previous theorem. Suppose that $L_\infty > \sigma^2/2$. If $L_{-\infty} \le \sigma^2/2$, then $\mathbb{P}[A] = 1$. If $L_{-\infty} > \sigma^2/2$, and we define $\tilde{A} := \{\omega : \lim_{t\to\infty} X(t,\omega) = -\infty\}$, then $A \cup \tilde{A}$ is an a.s. event, and $\mathbb{P}[A]$, $\mathbb{P}[\tilde{A}] \in (0,1)$. The exact values of $\mathbb{P}[A]$ and $\mathbb{P}[\tilde{A}]$ depend on the deterministic initial value of X. In a similar manner, we can prove similar results when the roles of L_∞ and $L_{-\infty}$ are interchanged. By Corollary 4.2, it is not difficult to show the following result. The details of the proof are omitted.

Corollary 4.4. *Let X be the unique continuous adapted process which obeys* (3.1). *Let $\tilde{A} := \{\omega : \lim_{t\to\infty} X(t,\omega) = -\infty\}$. If*

$$\lim_{x\to\infty} xf(x) = L_{-\infty}, \quad g(x) = \sigma, \quad x \in \mathbb{R}$$

where $\sigma \ne 0$ and $L_{-\infty} > \sigma^2/2$, then $\mathbb{P}[\tilde{A}] > 0$ and X satisfies

$$\liminf_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} = -|\sigma| \quad \textit{a.s. on } \tilde{A},$$

and

$$\liminf_{t\to\infty} \frac{\log \frac{|X(t)|}{\sqrt{t}}}{\log\log t} = -\frac{1}{\frac{2L_{-\infty}}{\sigma^2} - 1}, \quad \textit{a.s. on } \tilde{A}.$$

Theorem 4.3 can now be used to prove a more general result for (3.1), where instead of being constant, g now obeys

$$\forall\, x \in \mathbb{R}, \quad g(x) \ne 0, \quad \lim_{x\to\infty} g(x) = \sigma \in \mathbb{R}/\{0\}. \tag{4.22}$$

Theorem 4.5. *Let X be the unique continuous adapted process which obeys* (3.1). *Let $A := \{\omega : \lim_{t\to\infty} X(t,\omega) = \infty\}$. If there exist positive real numbers L_∞ and σ such that $L_\infty > \sigma^2/2$, f obeys* (4.8)*, and g obeys* (4.22)*, then X satisfies* (4.9) *and* (4.10).

Proof. Define the local martingale

$$M(t) = \int_0^t g(X(s))\,dB(s), \quad t \geq 0.$$

Therefore, by (4.22) we have

$$\lim_{t\to\infty} \frac{1}{t}\langle M\rangle(t) = \lim_{t\to\infty} \frac{1}{t}\int_0^t g^2(X(s))\,ds = \sigma^2, \quad \text{a.s. on } A. \tag{4.23}$$

For each $0 \leq s < \infty$, define the stopping time $\nu(s) := \inf\{t \geq 0 : \langle M\rangle(t) > s\}$. By the time-change theorem for martingales [12, Theorem 3.4.6], the process defined as $W(t) := M(\nu(t))$ is a standard Brownian motion with respect to the filtration $\mathcal{Q}(t) := \mathcal{F}(\nu(t))$. If $\widetilde{X}(t) := X(\nu(t))$, then

$$d\widetilde{X}(t) = \frac{f(\widetilde{X}(t))}{g^2(\widetilde{X}(t))}\,dt + dW(t), \quad t \geq 0.$$

Now, since $\lim_{t\to\infty} xf(x)/g^2(x) = L_\infty/\sigma^2 > 1/2$, by Theorem 4.3, for almost all $\omega \in A$,

$$\limsup_{t\to\infty} \frac{\widetilde{X}(t)}{\sqrt{2t\log\log t}} = 1, \quad \liminf_{t\to\infty} \frac{\log \frac{\widetilde{X}(t)}{\sqrt{t}}}{\log\log t} = -\frac{1}{\frac{2L_\infty}{\sigma^2} - 1}.$$

That is for almost all $\omega \in A$,

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2\langle M\rangle(t)\log\log\langle M\rangle(t)}} = 1, \quad \liminf_{t\to\infty} \frac{\log \frac{X(t)}{\sqrt{\langle M\rangle(t)}}}{\log\log\langle M\rangle(t)} = -\frac{1}{\frac{2L_\infty}{\sigma^2} - 1}. \tag{4.24}$$

Combining (4.23) with these limits, the desired assertion can be obtained. □

A similar result can be developed in the case when $X(t) \to -\infty$ as $t \to \infty$ under the assumptions that $xf(x) \to L_{-\infty} > \sigma^2/2$ and $g(x) \to \sigma$ as $x \to -\infty$. The proof is essentially the same as that of Theorem 4.5, and hence omitted.

5 General Conditions Ensuring the Law of the Iterated Logarithm and Ergodicity

Theorem 5.1. *Let X be the unique continuous adapted process satisfying* (3.5). *If there exists a positive real number ρ such that*

$$\forall\,(x,t) \in \mathbb{R}\times\mathbb{R}^+, \quad xf(x,t) \leq \rho, \tag{5.1}$$

then

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t\log\log t}} \leq |\sigma|, \quad \textit{a.s.} \tag{5.2}$$

and

$$\limsup_{t\to\infty} \frac{\int_0^t \frac{X^2(s)}{(1+s)^2}\,ds}{\log t} \le 2\rho + \sigma^2, \quad a.s. \tag{5.3}$$

Proof. Without loss of generality, we can choose $\rho > \sigma^2/2$. Then by Itô's rule,

$$dX^2(t) = (2X(t)f(X(t),t) + \sigma^2)\,dt + 2X(t)\sigma\,dB(t).$$

Let $Z(t) = X^2(t)$, $t \ge 0$. Define $\gamma(x) = x/|x|$ for $x \neq 0$ and $\gamma(0) = 1$. Then

$$W(t) := \int_0^t \gamma(X(s))dB(s)$$

is a standard Brownian motion with respect to $\mathscr{F}^B$, and we have

$$dZ(t) = (2X(t)f(X(t),t) + \sigma^2)dt + 2\sigma\sqrt{Z(t)}dW(t).$$

Now consider the process X_u defined by

$$dX_u(t) = (2\rho + \sigma^2)dt + 2\sigma\sqrt{|X_u(t)|}dW(t) \tag{5.4}$$

with $X_u(0) > X^2(0)$. Arguing as in the forthcoming Theorem 5.3, it can be shown that $X_u(t) \ge 0$ for all $t \ge 0$ a.s. This means that the absolute values in the diffusion coefficient in (5.4) can be omitted. Hence by the Ikeda and Watanabe comparison theorem (cf. [11]), $X_u(t) \ge X^2(t)$ for all $t \ge 0$ a.s. From the proof of Lemma 4.1, we know that $\mathbb{P}[\lim_{t\to\infty} X_u(t) = \infty] = 1$. Moreover, X_u obeys

$$\limsup_{t\to\infty} \frac{X_u(t)}{2t\log\log t} \le \sigma^2 \quad \text{a.s.}$$

Hence the assertion (5.2) is obtained.

The second part of the theorem can be easily deduced from the fact $X_u(t) \ge X^2(t)$ for all $t \ge 0$ a.s., and (5.4) by Exercise XI.1.32 in [17], which is stated below as Lemma 5.2. □

Lemma 5.2. *Suppose that Q is the unique continuous adapted process satisfying*

$$dQ(t) = \delta\,dt + 2\sqrt{Q(t)}\,dB(t), \quad t \ge 0$$

with $Q(0) \ge 0$ and $\delta > 0$. Then Q obeys

$$\lim_{t\to\infty} \frac{\int_1^t \frac{Q(s)}{s^2}ds}{\log t} = \delta, \quad a.s.$$

We now establish lower bounds corresponding to the upper bounds given in the previous theorem.

Theorem 5.3. *Let X be the unique continuous adapted process satisfying* (3.5). *If there exists a real number μ such that*

$$\inf_{(x,t)\in\mathbb{R}\times\mathbb{R}^+} xf(x,t) = \mu > -\frac{\sigma^2}{2}, \tag{5.5}$$

then

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t\log\log t}} \geq |\sigma|, \quad a.s. \tag{5.6}$$

Moreover,

$$\liminf_{t\to\infty} \frac{\int_0^t \frac{X^2(s)}{(1+s)^2}\,ds}{\log t} \geq 2\mu+\sigma^2, \quad a.s. \tag{5.7}$$

Proof. We begin with a change in both time and scale on X to transform it to a process which can be compared with a stationary process. Let $Y(t) = e^{-t}X(\frac{1}{2}(e^{2t}-1))$. By Itô's rule, it can be shown that for $t \geq 0$

$$dY^2(t) = \left[-2Y^2(t) + 2Y(t)e^t f\left(Y(t)e^t, \frac{1}{2}(e^{2t}-1)\right) + \sigma^2\right]dt + 2\sigma\sqrt{Y^2(t)}\,dW(t)$$

with $Y^2(0) = x_0^2$ where W is the $\mathscr{F}^B$–adapted standard Brownian motion introduced in the proof of Theorem 5.1. Consider the processes governed by the following two equations,

$$dY_1(t) = (-2Y_1(t) + 2\mu + \sigma^2)\,dt + 2\sigma\sqrt{|Y_1(t)|}\,dW(t), \tag{5.8}$$

$$dY_2(t) = (-2Y_2(t))\,dt + 2\sigma\sqrt{|Y_2(t)|}\,dW(t) \tag{5.9}$$

with $x_0^2 \geq Y_1(0) \geq Y_2(0) = 0$. Instead of applying Lemma 2.2 directly, we give more details on estimating the asymptotic growth rate of Y_1 using Motoo's theorem. By Yamada and Watanabe's uniqueness theorem (cf.[12, Proposition 5.2.13]), $Y_2(t) = 0$ for all $t \geq 0$ a.s. for all $t \geq 0$. Applying the Ikeda-Watanabe comparison theorem twice, we have $Y^2(t) \geq Y_1(t) \geq Y_2(t) = 0$ for all $t \geq 0$ a.s. Hence the absolute values in (5.8) can be removed. Now it is easy to check that a scale function and the speed measure of Y_1 are

$$s_{Y_1}(x) = e^{-\frac{1}{\sigma^2}} \int_1^x e^{\frac{y}{\sigma^2}} y^{-\frac{2\mu+\sigma^2}{2\sigma^2}}\,dy, \quad m_{Y_1}(dx) = \frac{1}{2}\sigma^2 e^{-\frac{1}{\sigma^2}} e^{\frac{-x}{\sigma^2}} x^{\frac{2\mu+\sigma^2}{2\sigma^2}-1}\,dx$$

respectively. Without loss of generality, we can choose $\mu \in (-\sigma^2/2, \sigma^2/2]$. Then $s_{Y_1}(\infty) = \infty$, $s_{Y_1}(0) > -\infty$ and $m_{Y_1}(0,\infty) < \infty$. In addition, the function defined by (2.2) and associated with Y_1 satisfies $v(0) < \infty$. So by Feller's test for explosions, Y_1 reaches zero within finite time on some event. A direct calculation confirms that $m_{Y_1}(\{0\}) = 0$. By the definition of an instantaneously reflecting point (cf. e.g.[17, Chapter VII, Definition 3.11]), we conclude that zero is a reflecting barrier for Y_1, and hence Y_1 is an a.s. recurrent process with finite speed measure. Thus Motoo's theorem in Section 2 can be applied. Let $h(t) = \sigma^2\log t$. Since $\mu \in (-\sigma^2/2, \sigma^2/2]$, by L'Hôpital's rule

$$\lim_{x\to\infty} \frac{s_{Y_1}(x)}{e^{\frac{x}{\sigma^2}}} = \lim_{x\to\infty} x^{-\frac{2\mu+\sigma^2}{2\sigma^2}} = 0.$$

This implies that there exists $x_* > 0$ such that for all $x > x_*$, $s_{Y_1}(x) < e^{x/\sigma^2}$. Since h is an increasing function, there exists $t_0 > 0$ such that for all $t > t_0$, $h(t) > x_*$, so $s_{Y_1}(h(t)) < t$. Hence

$$\int_{t_0}^\infty \frac{1}{s_{Y_1}(h(t))}\,dt \geq \int_{t_0}^\infty \frac{1}{t}\,dt = \infty.$$

Therefore, by Motoo's theorem

$$\limsup_{t\to\infty} \frac{Y^2(t)}{\log t} \geq \limsup_{t\to\infty} \frac{Y_1(t)}{\log t} \geq \sigma^2, \quad \text{a.s.}$$

Using the relation between X and Y, we get the desired result (5.6).

For the second part of the conclusion, consider the following equation

$$dZ(t) = (2\mu + \sigma^2)\, dt + 2\sigma \sqrt{|Z(t)|}\, dW(t), \quad t \geq 0,$$

with $Z(0) \leq x_0^2$. Then $X^2(t) \geq Z(t) \geq 0$ for $t \geq 0$ a.s. Again, by applying Lemma 5.2 to Z, (5.7) is proved. □

The following corollary combines Theorem 4.5 with Theorem 5.1, and shows that the condition that the diffusion coefficient be constant can be relaxed.

Corollary 5.4. *Let X be the unique continuous adapted process satisfying the equation*

$$dX(t) = f(X(t), t)\, dt + g(X(t))\, dB(t), \quad t \geq 0,$$

with $X(0) = x_0$. Suppose $g : \mathbb{R} \to \mathbb{R}$ is even and satisfies

$$\forall\, x \in \mathbb{R}, \quad g(x) \neq 0, \quad \lim_{|x|\to\infty} g(x) = \sigma \in \mathbb{R}/\{0\}. \tag{5.10}$$

Let g also satisfy

$$\forall\, x, y \in \mathbb{R}, \quad |g(x) - g(y)| \leq h(|x - y|),$$

where $h : \mathbb{R}^+ \to \mathbb{R}^+$ is increasing and $\int_{0^+} h^{-2}(u)du = \infty$. If there exists a positive constant ρ such that f satisfies (5.1)*, then X obeys* (5.2).

Proof. Since f satisfies (5.1), then $\forall\,(x, t) \in \mathbb{R}/\{0\} \times \mathbb{R}^+$ and $\epsilon > 0$, $-(\rho + \epsilon)/|x| < f(x, t) < (\rho + \epsilon)/|x|$. Without loss of generality, we can choose $\rho > \sigma^2/2 \vee g^2(0)/2$. Consider the equation

$$dX_u(t) = \frac{\rho + \epsilon}{X_u(t)}\, dt + g(X_u(t))\, dB(t), \quad t \geq 0 \tag{5.11}$$

with $X_u(0) > x_0 \vee 0$. It is easy to check that the scale function of X_u satisfies $s_{X_u}(\infty) < \infty$ and $s_{X_u}(0) = -\infty$. Thus $\mathbb{P}[\lim_{t\to\infty} X_u(t) = \infty] = 1$. Moreover $v_{X_u}(\infty) = v_{X_u}(0) = \infty$, which implies that $\mathbb{P}[X_u(t) > 0;\ \forall\, 0 < t < \infty] = 1$. Therefore by the Ikeda–Watanabe comparison theorem [11, Chapter VI, Theorem 1.1], $X(t) \leq X_u(t)$ for $t \geq 0$ a.s. Similarly, we can construct another process X_l which also satisfies (5.11), but with $X_l(0) < x_0 \wedge 0$. Then $\mathbb{P}[\lim_{t\to\infty} X_l(t) = -\infty] = 1$ and $\mathbb{P}[X_l(t) < 0;\ \forall\, 0 < t < \infty] = 1$. Thus $X(t) \geq X_l(t)$ for $t \geq 0$ a.s. Now by Theorem 4.5,

$$\limsup_{t\to\infty} \frac{X_u(t)}{\sqrt{2t \log\log t}} = -\liminf_{t\to\infty} \frac{X_l(t)}{\sqrt{2t \log\log t}} = |\sigma|, \quad \text{a.s.}$$

Therefore X obeys (5.2). □

The next corollary applies the ergodic theorem conclusions of Theorem 5.1, 5.3 and Lemma 5.2 to the process with non-constant diffusion coefficient dealt with in Theorem 4.5. The proof, which we supply here, is similar to that of Lemma 5.2.

Corollary 5.5. *Let X be the unique continuous adapted process which obeys* (3.1). *Let $A := \{\omega : \lim_{t\to\infty} X(t,\omega) = \infty\}$. If there exist positive real numbers L_∞ and σ such that $L_\infty > \sigma^2/2$, f obeys* (4.8)*, and g obeys* (4.22)*, then X satisfies*

$$\lim_{t\to\infty} \frac{\int_0^t \frac{X^2(s)}{(1+s)^2} ds}{\log t} = 2L_\infty + \sigma^2, \quad \textit{a.s. on } A. \tag{5.12}$$

Proof. Applying the transformation $Y(t) := (e^{-t/2}X(e^t-1))^2$ for $t \geq 0$, we get

$$Y(t) = x_0^2 - \int_0^t Y(s)\,ds + \int_0^t 2\tilde{X}(s)f(\tilde{X}(s))\,ds + \int_0^t g^2(\tilde{X}(s))\,ds \\ + \int_0^t 2\tilde{X}(s)e^{\frac{-s}{2}} g(\tilde{X}(s))\,d\tilde{B}(s), \tag{5.13}$$

where $\tilde{X}(t) := X(e^t-1)$, and as before, $\tilde{B}$ is another standard Brownian motion in an extended probability space. It can be verified that for almost all $\omega \in A$,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \tilde{X}(s)f(\tilde{X}(s))\,ds = L_\infty, \quad \lim_{t\to\infty} \frac{1}{t}\int_0^t g^2(\tilde{X}(s))\,ds = \sigma^2. \tag{5.14}$$

Let

$$M(t) := \int_0^t 2\tilde{X}(s)e^{\frac{-s}{2}} g(\tilde{X}(s))\,d\tilde{B}(s),$$

so that M has the quadratic variation

$$\langle M\rangle(t) := \int_0^t 4\tilde{X}^2(s)e^{-s}g^2(\tilde{X}(s))\,ds.$$

We have

$$\lim_{t\to\infty} \frac{\langle M\rangle(t)}{\int_0^t Y(s)\,ds} = 4\sigma^2, \quad \text{a.s. on } A. \tag{5.15}$$

Suppose $D := \{\omega : \lim_{t\to\infty}\langle M\rangle(t) < \infty\}$ with $\mathbb{P}[D] > 0$. Then $\int_0^\infty Y(s)\,ds < \infty$, a.s. on $A\cap D$. Thus

$$\lim_{t\to\infty} \frac{Y(t)}{t} = 2L_\infty + \sigma^2, \quad \text{a.s. on } A\cap D,$$

which contradicts

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} = |\sigma|, \quad \text{a.s. on } A. \tag{5.16}$$

Therefore $\mathbb{P}[\lim_{t\to\infty}\langle M\rangle(t) = \infty] = 1$. Note that (5.16) implies $\lim_{t\to\infty} Y(t)/t = 0$ a.s. on A. Also,

$$\lim_{t\to\infty} \frac{M(t)}{\int_0^t Y(s)\,ds} = \lim_{t\to\infty} \frac{M(t)}{\langle M\rangle(t)} \cdot \frac{\langle M\rangle(t)}{\int_0^t Y(s)\,ds} = 0, \quad \text{a.s. on } A.$$

Now since for all $t \geq 0$, $Y(t) \geq 0$ a.s., we have

$$\int_0^t Y(s)\,ds \leq x_0^2 + \int_0^t 2\tilde{X}(s)f(\tilde{X}(s))\,ds + \int_0^t g^2(\tilde{X}(s))\,ds + M(t).$$

By first dividing both sides by $\int_0^t Y(s)\,ds$, then taking limits as $t \to \infty$ and using (5.14), and finally by rearranging the resulting inequality, we get

$$\liminf_{t\to\infty} \frac{t}{\int_0^t Y(s)\,ds} \geq \frac{1}{2L_\infty + \sigma^2}, \quad \text{a.s. on } A.$$

Hence

$$\limsup_{t\to\infty} \frac{\int_0^t Y(s)\,ds}{t} \leq 2L_\infty + \sigma^2, \quad \text{a.s. on } A.$$

Finally, since

$$\lim_{t\to\infty} \frac{M(t)}{t} = \lim_{t\to\infty} \frac{M(t)}{\int_0^t Y(s)\,ds} \cdot \frac{\int_0^t Y(s)\,ds}{t} = 0, \quad \text{a.s. on } A,$$

by (5.13) we get

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t Y(s)\,ds = 2L_\infty + \sigma^2, \quad \text{a.s. on } A,$$

from which the desired result (5.12) can be obtained. ☐

Besides being of independent interest, the following result will be used extensively in Section 6 to prove comparison results. It is a special case of a result in [1]. The result in [1] covers equations with Markovian switching.

Theorem 5.6. *Let X be the unique continuous adapted process satisfying* (3.15) *with $X(0) = x_0$. If $f \in L^1(\mathbb{R};\mathbb{R})$, then there exist positive real numbers $\{C_i\}_{i=1,2,3,4}$ such that*

$$C_1 \leq \limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} \leq C_2, \quad a.s. \tag{5.17}$$

$$-C_3 \leq \liminf_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} \leq -C_4, \quad a.s. \tag{5.18}$$

where

$$C_1 = \frac{|\sigma| e^{\frac{-2}{\sigma^2}\sup_{x\in\mathbb{R}}\int_0^x f(z)dz}}{e^{\frac{-2}{\sigma^2}\int_0^\infty f(z)dz}}, \quad C_2 = \frac{|\sigma| e^{\frac{-2}{\sigma^2}\inf_{x\in\mathbb{R}}\int_0^x f(z)dz}}{e^{\frac{-2}{\sigma^2}\int_0^\infty f(z)dz}},$$
$$C_3 = \frac{|\sigma| e^{\frac{-2}{\sigma^2}\inf_{x\in\mathbb{R}}\int_0^x f(z)dz}}{e^{\frac{2}{\sigma^2}\int_{-\infty}^0 f(z)dz}}, \quad C_4 = \frac{|\sigma| e^{\frac{-2}{\sigma^2}\sup_{x\in\mathbb{R}}\int_0^x f(z)dz}}{e^{\frac{2}{\sigma^2}\int_{-\infty}^0 f(z)dz}}.$$

6 Recurrent Processes with Asymptotic Behaviour Close to the Law of the Iterated Logarithm

In this section, we again study solutions of (3.15), where the drift coefficient satisfies

$$\lim_{x\to\infty} xf(x) = L_\infty \le \frac{\sigma^2}{2} \quad \text{and} \quad \lim_{x\to-\infty} xf(x) = L_{-\infty} \le \frac{\sigma^2}{2}. \tag{6.1}$$

As mentioned previously, the solutions are no longer transient but are now recurrent on the real line. The results obtained vary according to the values of L_∞ and $L_{-\infty}$. We classify these results into four main cases. The first result is a direct and easy application of Motoo's theorem. However, we state it as a theorem here for two reasons: first, it shows that $-\sigma^2/2$ is another critical value for the process; second, it provides a way to construct a process with known behaviour to which we can compare processes in the other three cases.

Theorem 6.1. *Let X be the unique continuous adapted process satisfying* (3.15). *If f satisfies* (6.1) *and $L_\infty \in (-\infty, -\sigma^2/2)$, $L_{-\infty} \in (-\infty, -\sigma^2/2)$, then X is recurrent and has finite speed measure. Moreover X obeys*

$$\limsup_{t\to\infty} \frac{\log X(t)}{\log t} = \frac{1}{1-2L_\infty/\sigma^2}, \quad \limsup_{t\to\infty} \frac{\log(-X(t))}{\log t} = \frac{1}{1-2L_{-\infty}/\sigma^2}, \quad a.s.$$

Hence

$$\limsup_{t\to\infty} \frac{\log|X(t)|}{\log t} = \frac{1}{1-2(L_\infty \vee L_{-\infty})/\sigma^2}, \quad a.s.$$

Proof. Condition (6.1) implies that for any $\epsilon > 0$, there exists $x_\epsilon > 0$ such that

$$L_\infty - \epsilon < xf(x) < L_\infty + \epsilon < -\frac{\sigma^2}{2}, \quad x > x_\epsilon;$$
$$L_{-\infty} - \epsilon < xf(x) < L_{-\infty} + \epsilon < -\frac{\sigma^2}{2}, \quad x < -x_\epsilon.$$

It can be shown that setting $c = x_\epsilon$ in (2.1), for any $x > x_\epsilon$, a scale function of X satisfies

$$\int_{x_\epsilon}^{x} \left(\frac{y}{x_\epsilon}\right)^{\frac{-2(L_\infty+\epsilon)}{\sigma^2}} dy \le s(x) \le \int_{x_\epsilon}^{x} \left(\frac{y}{x_\epsilon}\right)^{\frac{-2(L_\infty-\epsilon)}{\sigma^2}} dy. \tag{6.2}$$

Since $L_\infty \in (-\infty, -\sigma^2/2)$, we have $s(\infty) = \infty$. A similar estimate can be used to get $s(-\infty) = -\infty$. For some constants $K_{1,\epsilon}$ and $K_{2,\epsilon}$, the speed measure can be estimated by

$$m(0,\infty) \le K_{1,\epsilon} + K_{2,\epsilon} \int_{x_\epsilon}^{\infty} x^{\frac{2(L_\infty+\epsilon)}{\sigma^2}} dx < \infty.$$

Similarly $m(-\infty, 0) < \infty$, so $m(-\infty,\infty) < \infty$. Hence X is recurrent on $\mathbb{R}$ and has finite speed measure. We can therefore apply Motoo's theorem to X. By L'Hôpital's rule, we have

$$\begin{aligned} 0 \le \limsup_{x\to\infty} \frac{s(x)}{x^{1-\frac{2(L_\infty-\epsilon)}{\sigma^2}}} &\le \lim_{x\to\infty} \frac{e^{-\frac{2}{\sigma^2}\int_0^{x_\epsilon} f(z)\,dz - \frac{2}{\sigma^2}\int_{x_\epsilon}^{x} \frac{L_\infty-\epsilon}{z}\,dz}}{(1-\frac{2(L_\infty-\epsilon)}{\sigma^2})x^{-2(L_\infty-\epsilon)/\sigma^2}} \\ &= \frac{K_{3,x_\epsilon}}{1-\frac{2(L_\infty-\epsilon)}{\sigma^2}} \end{aligned}$$

for some positive real number K_{3,x_ϵ}. So if $h_1(t)=t^{1/[1-2(L_\infty-\epsilon)/\sigma^2]}$, we get

$$\int_1^\infty \frac{1}{s(h_1(t))}\,dt \geq \int_1^\infty \frac{1}{K_{4,\epsilon}t}\,dt=\infty,$$

for some positive real number $K_{4,\epsilon}$. Hence

$$\limsup_{t\to\infty}\frac{X(t)}{t^{\frac{1}{1-2(L_\infty-\epsilon)/\sigma^2}}}\geq 1,\quad \text{a.s. on an a.s. event } \Omega_\epsilon,$$

which implies

$$\limsup_{t\to\infty}\frac{\log X(t)}{\log t}\geq \frac{1}{1-2(L_\infty-\epsilon)/\sigma^2},\quad \text{a.s. on } \Omega_\epsilon.$$

By considering the a.s. event $\Omega^*=\cap_{\epsilon\in\mathbb{Q}}\Omega_\epsilon$, we have

$$\limsup_{t\to\infty}\frac{\log X(t)}{\log t}\geq \frac{1}{1-2L_\infty/\sigma^2},\quad \text{a.s. on } \Omega^*. \tag{6.3}$$

Similarly using (6.2) for some positive constant $K_{5,\epsilon}$, we obtain

$$\liminf_{x\to\infty}\frac{s(x)}{x^{1-2(L_\infty+\epsilon)/\sigma^2}}\geq \frac{K_{5,\epsilon}}{1-\frac{2(L_\infty+\epsilon)}{\sigma^2}}>0.$$

If we choose $h_2(t)=t^{\frac{1+\epsilon}{1-2(L_\infty+\epsilon)/\sigma^2}}$, then for some positive constant $K_{6,\epsilon}$,

$$\int_1^\infty \frac{1}{s(h_2(t))}\,dt \leq \int_1^\infty \frac{1}{K_{6,\epsilon}t^{1+\epsilon}}\,dt<\infty.$$

Hence

$$\limsup_{t\to\infty}\frac{X(t)}{t^{\frac{1+\epsilon}{1-2(L_\infty+\epsilon)/\sigma^2}}}\leq 1,\quad \text{a.s. on } \Omega_\epsilon. \tag{6.4}$$

Letting $\epsilon\downarrow 0$ through rational numbers, and combining with (6.3) we get

$$\limsup_{t\to\infty}\frac{\log X(t)}{\log t}= \frac{1}{1-2L_\infty/\sigma^2},\quad \text{a.s. on } \Omega^*. \tag{6.5}$$

Now let $Y(t)=-X(t)$, $g(x)=-f(-x)$ and $\tilde{B}(t)=-B(t)$. Then

$$\lim_{x\to\infty}xg(x)=\lim_{x\to\infty}-xf(-x)=\lim_{y\to-\infty}yf(y)=L_{-\infty}$$

and

$$dY(t)=g(Y(t))\,dt+\sigma\,d\tilde{B}(t).$$

Hence by applying the line of argument above we obtain

$$\limsup_{t\to\infty}\frac{Y(t)}{t^{\frac{1}{1-2(L_{-\infty}-\epsilon)/\sigma^2}}}\geq 1,\quad \text{a.s. on some a.s. event } \tilde{\Omega}_\epsilon,$$

$$\limsup_{t\to\infty}\frac{Y(t)}{t^{\frac{1+\epsilon}{1-2(L_{-\infty}+\epsilon)/\sigma^2}}}\leq 1,\quad \text{a.s. on } \tilde{\Omega}_\epsilon,$$

and so as before, we have

$$\limsup_{t\to\infty}\frac{\log Y(t)}{\log t}=\frac{1}{1-2L_{-\infty}/\sigma^2},\quad \text{a.s. on some a.s. event } \tilde{\Omega}^*.$$

Finally combining the above limit with (6.5), we get

$$\limsup_{t\to\infty}\frac{\log|X(t)|}{\log t}=\frac{1}{1-2(L_\infty\vee L_{-\infty})/\sigma^2},\quad \text{a.s.}$$

□

The previous theorem is not the main focus of this paper, as it applies to stationary or asymptotic stationary processes. It shows that such processes do not behave asymptotically in a manner close to the LIL. However, taking the results of Theorem 6.1, Theorem 4.3 and Theorem 4.5 together, we can exclude the necessity to study these regions of $(L_\infty, L_{-\infty}, \sigma^2)$ parameter space further.

The rest of our analysis focusses on the regions of $(L_\infty, L_{-\infty}, \sigma^2)$ parameter space not covered by these results. Before moving on to the next theorem, we give a lemma which allows us to construct appropriate comparison processes.

Lemma 6.2. *Suppose $f:\mathbb{R}\to\mathbb{R}$ is locally Lipschitz continuous and satisfies* (6.1). *If $L_\infty\in[-\sigma^2/2,\infty)$ and $L_{-\infty}\in[-\sigma^2/2,\infty)$ and $f(0)=0$, then for every $\epsilon>0$ there exists an odd function $q_\epsilon:\mathbb{R}\to\mathbb{R}$ such that*

$$q_\epsilon \text{ is locally Lipschitz continuous on } \mathbb{R}; \tag{6.6a}$$

$$\lim_{x\to\pm\infty} xq_\epsilon(x)=-\frac{\sigma^2}{2}-\epsilon; \tag{6.6b}$$

$$f(x)\ge q_\epsilon(x),\quad x\ge 0; \tag{6.6c}$$

$$f(x)\le q_\epsilon(x),\quad x\le 0. \tag{6.6d}$$

Moreover, the function $G_\epsilon:(-\infty,\infty)\to\mathbb{R}$ defined by $G_\epsilon(x)=\sqrt{|x|}q_\epsilon(\sqrt{|x|})$ is globally Lipschitz continuous on $(-\infty,\infty)$.

Proof. For every $\epsilon>0$ there exists $x_\epsilon>1$ such that

$$L_\infty-\frac{\epsilon}{2}<xf(x)<L_\infty+\frac{\epsilon}{2},\quad x>x_\epsilon, \tag{6.7}$$

$$L_{-\infty}-\frac{\epsilon}{2}<xf(x)<L_{-\infty}+\frac{\epsilon}{2},\quad x<-x_\epsilon. \tag{6.8}$$

Since f is locally Lipschitz continuous, there is a constant $K>0$ such that

$$|f(x)-f(y)|\le K|x-y|,\quad |x|\vee|y|\le 1. \tag{6.9}$$

Now define $f_\epsilon:[x_\epsilon,\infty)\to\mathbb{R}$ by $f_\epsilon(x)=(L_\infty\wedge L_{-\infty}-\varepsilon/2)x^{-1}$ and

$$C_\epsilon=1+K+\left\{(-\min_{x\in[1,x_\epsilon]}f(x))\vee\max_{x\in[-x_\epsilon,-1]}f(x)\vee 0\right\}+[-f_\epsilon(x_\epsilon)]^+,$$

where

$$[x]^+ := \begin{cases} x, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Then

$$C_\epsilon \geq 1 + K; \quad C_\epsilon + f_\epsilon(x_\epsilon) \geq 1. \tag{6.10}$$

Also

$$-C_\epsilon < f(x), \quad x \in [1, x_\epsilon] \tag{6.11}$$

and

$$C_\epsilon > f(x), \quad x \in [-x_\epsilon, -1]. \tag{6.12}$$

By the second inequality in (6.10), and the fact that $L_\infty \wedge L_{-\infty} \geq -\sigma^2/2$, we may define $\delta_\epsilon : [x_\epsilon, \infty) \to [0, \infty)$ by

$$\delta_\epsilon(x) = \frac{\frac{\sigma^2}{2} + L_\infty \wedge L_{-\infty} + \frac{\epsilon}{2}}{\frac{\frac{\sigma^2}{2} + L_\infty \wedge L_{-\infty} + \frac{\epsilon}{2}}{f_\epsilon(x_\epsilon) + C_\epsilon} + x - x_\epsilon}, \quad x \geq x_\epsilon.$$

Now we define the candidate function q_ϵ. It is given for $x \geq 0$ by

$$q_\epsilon(x) = \begin{cases} -C_\epsilon x, & x \in [0, 1], \\ -C_\epsilon, & x \in (1, x_\epsilon], \\ f_\epsilon(x) - \delta_\epsilon(x), & x > x_\epsilon, \end{cases}$$

and extended for $x \leq 0$ according to $q_\epsilon(x) = -q_\epsilon(-x)$. Clearly q_ϵ is odd by definition, and is obviously Lipschitz continuous on $(-x_\epsilon, x_\epsilon)$. Since

$$\lim_{x \to x_\epsilon^+} q_\epsilon(x) = f_\epsilon(x_\epsilon) - \delta_\epsilon(x_\epsilon) = f_\epsilon(x_\epsilon) - f_\epsilon(x_\epsilon) - C_\epsilon = -C_\epsilon = q_\epsilon(x_\epsilon),$$

we have that q_ϵ is locally Lipschitz continuous on $\mathbb{R}$. Noting that

$$\lim_{x \to \infty} x f_\epsilon(x) = L_\infty \wedge L_{-\infty} - \frac{\epsilon}{2}, \quad \lim_{x \to \infty} x \delta_\epsilon(x) = \frac{\sigma^2}{2} + L_\infty \wedge L_{-\infty} + \frac{\epsilon}{2},$$

we get

$$\lim_{x \to \infty} x q_\epsilon(x) = L_\infty \wedge L_{-\infty} - \frac{\epsilon}{2} - \left(\frac{\sigma^2}{2} + L_\infty \wedge L_{-\infty} + \frac{\epsilon}{2} \right) = -\frac{\sigma^2}{2} - \epsilon.$$

Since q_ϵ is odd, the same limit pertains as $x \to -\infty$.

Finally, we show that $x f(x) \geq x q_\epsilon(x)$, $x \in \mathbb{R}$. For $x \in [0, 1]$, because $f(0) = 0$, and (6.9) holds, we have $|f(x)| \leq K|x| = Kx$. Hence

$$f(x) \geq -Kx \geq -Kx - x \geq -C_\epsilon x = q_\epsilon(x).$$

For $x \in [-1, 0]$ we have $|f(x)| \leq K|x| = -Kx$. Hence

$$f(x) \leq -Kx \leq -Kx - x \leq -C_\epsilon x = q_\epsilon(x),$$

where we have used the first inequality of (6.10) to deduce the third inequality in each case, and the definition of q_ϵ and the fact that it is an odd function at the last steps.

By (6.11), for $x \in [1, x_\epsilon]$ we have $q_\epsilon(x) = -C_\epsilon < f(x)$, and as q_ϵ is odd, for $x \in [-x_\epsilon, -1]$ using (6.12) we get $q_\epsilon(x) = C_\epsilon > f(x)$. It remains to establish inequalities on (x_ϵ, ∞) and $(-\infty, -x_\epsilon)$. We noted earlier that $\delta_\epsilon(x) > 0$ for $x > x_\epsilon$. Hence, by the definition of q_ϵ, this fact and (6.7) yield

$$q_\epsilon(x) = f_\epsilon(x) - \delta_\epsilon(x) < f_\epsilon(x) = \frac{L_\infty \wedge L_{-\infty} - \epsilon/2}{x} \le \frac{L_\infty - \epsilon/2}{x} < f(x),$$

for $x > x_\epsilon$, as required. We now consider the case when $x < -x_\epsilon$. Since q_ϵ is odd, we get

$$q_\epsilon(x) = -q_\epsilon(-x) = -f_\epsilon(-x) + \delta_\epsilon(-x) > -f_\epsilon(-x),$$

the last step coming from the fact that $\delta_\epsilon(-x) > 0$ for $-x > x_\epsilon$. By the definition of f_ϵ, we have

$$q_\epsilon(x) > \frac{L_\infty \wedge L_{-\infty} - \epsilon/2}{x}, \quad x < -x_\epsilon.$$

Thus, as $x < 0$, we get

$$x q_\epsilon(x) < L_\infty \wedge L_{-\infty} - \frac{\epsilon}{2} \le L_{-\infty} - \frac{\epsilon}{2} < x f(x),$$

using (6.8) at the last step. Hence $x q_\epsilon(x) < x f(x)$ for $x < -x_\epsilon$.

We conclude by dealing with the continuity of G_ϵ. For $x \in [0,1]$ we have $G_\epsilon(x) = -C_\epsilon x$, so G_ϵ is Lipschitz continuous on $[0,1)$. Since for any $M > 1$ the functions $x \mapsto \sqrt{x}$ and $x \mapsto q_\epsilon(x)$ are Lipschitz continuous from $[1,M] \to [1,\sqrt{M}]$ and $[1,\sqrt{M}] \to \mathbb{R}$ respectively, the composition $[1,M] \to \mathbb{R} : x \mapsto q_\varepsilon(\sqrt{x})$ is Lipschitz continuous. Thus the product $G_\epsilon : [1,M] \to \mathbb{R} : x \mapsto G_\epsilon(x) = \sqrt{x} q_\epsilon(\sqrt{x})$ is Lipschitz continuous. Since $M > 1$ is arbitrary, recalling that G_ϵ is Lipschitz continuous on $[0,1)$ and continuous at $x = 1$, we have that G_ϵ is locally Lipschitz continuous on $[0,\infty)$. Moreover, as $\sqrt{\cdot}$ and $q_\epsilon(\cdot)$ are actually globally Lipschitz continuous on $[1,\infty)$, and G_ϵ is Lipschitz continuous on $[0,1]$, it follows that G_ϵ is globally Lipschitz continuous on $[0,\infty)$. Finally since G_ϵ is an even function, it is also globally Lipschitz continuous on $\mathbb{R}$. □

Armed with this result, we are now in a position to determine the asymptotic behaviour for X when $L_\infty \in [-\sigma^2/2, \sigma^2/2]$, $L_{-\infty} \in [-\sigma^2/2, \sigma^2/2]$.

Theorem 6.3. *Let X be the unique continuous adapted process satisfying* (3.15). *Suppose f satisfies* (6.1) *and there exists at least one $x_* \in \mathbb{R}$ such that $f(x_*) = 0$. If $L_\infty \in [-\sigma^2/2, \sigma^2/2]$ and $L_{-\infty} \in [-\sigma^2/2, \sigma^2/2]$, then X is recurrent and satisfies*

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t \log\log t}} \le |\sigma|, \quad \textit{a.s.}$$

Moreover

$$\limsup_{t\to\infty} \frac{\log|X(t)|}{\log t} = \frac{1}{2}, \quad \textit{a.s.} \tag{6.13}$$

Proof. Again, the first part of the conclusion can be obtained immediately by Theorem 5.1. Therefore we also have the following upper estimate

$$\limsup_{t\to\infty} \frac{\log|X(t)|}{\log t} \le \frac{1}{2}, \quad \text{a.s.}$$

For the rest of the proof, the main idea is to compare X^2 with a squared stationary process described in Theorem 6.1. In what follows we fix $\epsilon \in (0,1)$. By hypothesis, there exists at least one $x_* \in \mathbb{R}$ such that $f(x_*) = 0$. Consider the process $\widetilde{X}$ governed by the following equation,

$$d\widetilde{X}(t) = \tilde{f}(\widetilde{X}(t))\,dt + \sigma\,dB(t), \quad t \geq 0,$$

where $\widetilde{X}(t) = X(t) - x_*$ and $\tilde{f}(x) = f(x + x_*)$. Thus $\tilde{f}(0) = 0$. By Itô's rule, we have

$$\begin{aligned} d\widetilde{X}^2(t) &= \left(2\widetilde{X}(t)\tilde{f}(\widetilde{X}(t)) + \sigma^2\right)dt + 2\widetilde{X}(t)\sigma\,dB(t) \\ &= \left[2\left(\widetilde{X}(t)\tilde{f}(\widetilde{X}(t)) - \widetilde{X}(t)q_\epsilon(\widetilde{X}(t))\right) + 2\widetilde{X}(t)q_\epsilon(\widetilde{X}(t)) + \sigma^2\right]dt + 2\widetilde{X}(t)\sigma\,dB(t). \end{aligned}$$

If q_ϵ is defined as in the previous lemma, then for all $t \geq 0$, $\phi(t) := \tilde{X}(t)\tilde{f}(\tilde{X}(t)) - \tilde{X}(t)q_\epsilon(\tilde{X}(t)) \geq 0$. Since q_ϵ is odd, we can rewrite the above equation governing $\widetilde{X}^2(t) =: Y(t)$ as

$$dY(t) = (2\phi(t) + 2\sqrt{|Y(t)|}q_\epsilon(\sqrt{|Y(t)|}) + \sigma^2)\,dt + 2\sqrt{|Y(t)|}\sigma\,dW(t)$$

where $Y(0) = (x_0 - x_*)^2$ and W is another Brownian motion in an extended space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$. Consider now the processes governed by the following two equations

$$\begin{aligned} dY_\epsilon(t) &= (2\sqrt{|Y_\epsilon(t)|}q_\epsilon(\sqrt{|Y_\epsilon(t)|}) + \sigma^2)\,dt + 2\sqrt{|Y_\epsilon(t)|}\sigma\,dW(t) \\ dY_0(t) &= (2\sqrt{|Y_0(t)|}q_\epsilon(\sqrt{|Y_0(t)|}))\,dt + 2\sqrt{|Y_0(t)|}\sigma\,dW(t) \end{aligned}$$

with $Y(0) \geq Y_\epsilon(0) \geq Y_0(0) = 0$. Since the drift coefficient of Y_0 is globally Lipschitz continuous by the previous lemma, we can use Yamada and Watanabe's uniqueness theorem, as well as the Ikeda-Watanabe comparison theorem to show that for every $\epsilon \in (0,1)$, there exists an a.s. event Ω_ϵ, such that $Y(t) \geq Y_\epsilon(t) \geq Y_0(t) = 0$ for all $t \geq 0$ a.s. on Ω_ϵ. Therefore all the absolute values can be removed. Now by the definition and properties of q_ϵ, it is easy to check that the scale function and the speed measure of Y_ϵ satisfy

$$s(\infty) = \infty, \quad s(0) > -\infty, \quad \text{and } m(0,\infty) < \infty$$

respectively. A similar argument to that used in Theorem 5.3 shows that zero is a reflecting barrier for Y_ϵ. Therefore Y_ϵ is a recurrent process on $\mathbb{R}^+$ with finite speed measure to which we can apply Motoo's theorem in order to determine the growth rate of its largest deviations. Now since $\lim_{x\to\infty} \sqrt{x}q(\sqrt{x}) = -\sigma^2/2 - \epsilon$, for the same ϵ, there exists x_ϵ such that for all $x > x_\epsilon$,

$$-\frac{\sigma^2}{2} - \epsilon(1+\epsilon) < \sqrt{x}q_\epsilon(\sqrt{x}) < -\frac{\sigma^2}{2} - \epsilon(1-\epsilon).$$

Let s be the scale function of Y_ϵ, then for some real positive constants $K_{1,\epsilon}$,

$$0 \leq \limsup_{x\to\infty} \frac{s(x)}{x^{1+\epsilon(1+\epsilon)/\sigma^2}} \leq \lim_{x\to\infty} \frac{\int_{x_\epsilon}^x \left(\frac{y}{x_\epsilon}\right)^{\frac{\epsilon(1+\epsilon)}{\sigma^2}} dy}{x^{1+\epsilon(1+\epsilon)/\sigma^2}} = \frac{K_{1,\epsilon}}{1+\epsilon(1+\epsilon)/\sigma^2}.$$

If we choose $h(t) = t^{\frac{1}{1+\epsilon(1+\epsilon)/\sigma^2}}$, then

$$\int_1^\infty \frac{1}{s(h(t))}\,dt \geq \int_1^\infty \frac{1}{t}\,dt = \infty.$$

Again by Motoo's theorem we have

$$\limsup_{t\to\infty} \frac{Y_\epsilon(t)}{t^{1/(1+\epsilon(1+\epsilon)/\sigma^2)}} \geq 1, \quad \text{a.s. on an a.s. event } \Omega_\epsilon^*,$$

which implies

$$\limsup_{t\to\infty} \frac{\log Y_\epsilon(t)}{\log t} \geq \frac{1}{1+\epsilon(1+\epsilon)/\sigma^2}, \quad \text{a.s. on } \Omega_\epsilon^*.$$

Hence on the a.s. event $\Omega_\epsilon^{**} = \Omega_\epsilon \cap \Omega_\epsilon^*$,

$$\limsup_{t\to\infty} \frac{\log Y(t)}{\log t} \geq \frac{1}{1+\epsilon(1+\epsilon)/\sigma^2} \quad \text{a.s.}$$

Considering the a.s. event $\Omega^* = \cap_{\epsilon\in\mathbb{Q}} \Omega_\epsilon^{**}$, we have

$$\limsup_{t\to\infty} \frac{\log Y(t)}{\log t} \geq 1, \quad \text{a.s.}$$

which implies

$$\limsup_{t\to\infty} \frac{\log |\widetilde{X}(t)|}{\log t} \geq \frac{1}{2}, \quad \text{a.s.,}$$

and hence the result. □

Using the same technique employed to prove Theorem 6.3, we may construct a locally Lipschitz continuous function q_ϵ such that for all $x \in \mathbb{R}$, $f(x) \geq q_\epsilon(x)$, and $\lim_{|x|\to\infty} xq_\epsilon(x) = -\sigma^2/2 - \epsilon$. Instead of comparing pathwise with X^2, we construct a solution with drift coefficient q_ϵ and directly compare it with X. The proof is left to the reader.

Theorem 6.4. *Let X be the unique continuous adapted process satisfying* (3.15). *Suppose f satisfies* (6.1) *and there exists at least one $x_* \in \mathbb{R}$ such that $f(x_*) = 0$. If $L_{-\infty} \in (-\infty, -\sigma^2/2)$ and $L_\infty \in [-\sigma^2/2, 0]$, or $L_\infty \in (-\infty, -\sigma^2/2)$ and $L_{-\infty} \in [-\sigma^2/2, 0]$, then X is recurrent and obeys*

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t\log\log t}} \leq |\sigma|, \quad \textit{a.s.}$$

Moreover,

$$\limsup_{t\to\infty} \frac{\log |X(t)|}{\log t} = \frac{1}{2}, \quad \textit{a.s.}$$

Remark 6.5.

Even though zeros are not included on the intervals for $L_{\pm\infty}$ in Figure 1 in Section 3, the construction of q_ϵ in either Theorem 6.3 or Theorem 6.4 covers the case when at least one of L_∞ and $L_{-\infty}$ is zero. Therefore (6.13) always holds provided the drift coefficient f reaches zero at least once. However, if f changes its sign an even number of times, more precise estimates on the growth rate can be obtained, even when at least one of L_∞ and $L_{-\infty}$ is zero. Lemma 6.6 and Theorem 6.7 deal with this case. In particular, if f remains non-negative (or non-positive) on the real line, we can compare X with the Brownian motion $\{\sigma B(t)\}_{t\geq 0}$ directly. This fact is stated in Corollary 6.8 without proof.

In order to apply a comparison argument to the next category of parameter values, we need to construct an appropriate drift coefficient, just as was done in Lemma 6.2 and Theorem 6.3.

Lemma 6.6. *Suppose $f:\mathbb{R}\to\mathbb{R}$ is locally Lipschitz continuous and satisfies* (6.1).

(i) If $L_{-\infty}\in(-\infty,0]$ and $L_{\infty}\in[0,\infty)$, and there exists $x_*>0$ such that for all $|x|>x_*$, $f(x)\geq 0$, then there exists an even function $q_{x_*}:\mathbb{R}\to\mathbb{R}$ such that for all $x\in\mathbb{R}$, $f(x)\geq q_{x_*}(x)$.

(ii) If $L_{\infty}\in(-\infty,0]$ and $L_{-\infty}\in[0,\infty)$, and there exists $x_*>0$ such that for all $|x|>x_*$, $f(x)\leq 0$, then there exists an even function $q_{x_*}:\mathbb{R}\to\mathbb{R}$ such that for all $x\in\mathbb{R}$, $f(x)\leq q_{x_*}(x)$.

Moreover, q_{x_*} in either case is globally Lipschitz continuous.

Proof. Under the conditions in Part (i), define $C:=\min_{x\in[-x_*,x_*]}f(x)\wedge 0$ and construct q_{x_*} according to:

$$q_{x_*}(x)=\begin{cases} C, & |x|<x_*,\\ -Cx+C+Cx_*, & x_*\leq x\leq x_*+1,\\ Cx+C+Cx_*, & -x_*-1\leq x\leq -x_*,\\ 0, & |x|>x_*+1.\end{cases}$$

It is obvious that q_{x_*} is even, globally Lipschitz continuous, and $f(x)\geq q_{x_*}(x)$ for all $x\in\mathbb{R}$. By a similar argument, we get the second part of the assertion. □

Theorem 6.7. *Let X be the unique continuous adapted process satisfying* (3.15)*, and suppose f satisfies* (6.1).

(i) If $L_{-\infty}\in(-\infty,0]$ and $L_{\infty}\in[0,\sigma^2/2]$, and there exists $x_*>0$ such that for all $|x|>x_*$, $f(x)\geq 0$, then X is recurrent and there exists a deterministic $\varsigma>0$ such that

$$\varsigma\leq\limsup_{t\to\infty}\frac{X(t)}{\sqrt{2t\log\log t}}\leq|\sigma|,\quad a.s.$$

(ii) If $L_{\infty}\in(-\infty,0]$ and $L_{-\infty}\in[0,\sigma^2/2]$, and there exists $x_*>0$ such that for all $|x|>x_*$, $f(x)\leq 0$, then X is recurrent and there exists a deterministic $\varsigma>0$ such that

$$-|\sigma|\leq\liminf_{t\to\infty}\frac{X(t)}{\sqrt{2t\log\log t}}\leq-\varsigma,\quad a.s.$$

Proof. We show assertion (i) first. Consider another process Y governed by the equation

$$dY(t)=q_{x_*}(Y(t))\,dt+\sigma\,dB(t),\quad t\geq 0,$$

with $Y(0)\leq X(0)$, where q_{x_*} is the function defined in Lemma 6.6. Note that $q_{x_*}\in L^1(\mathbb{R};\mathbb{R})$, so by Theorem 5.6, we have

$$\varsigma\leq\limsup_{t\to\infty}\frac{Y(t)}{\sqrt{2t\log\log t}},\quad \text{a.s.}$$

where

$$\varsigma=\frac{|\sigma|e^{\frac{-2}{\sigma^2}\sup_{x\in\mathbb{R}}\int_0^x q_{x_*}(z)\,dz}}{e^{\frac{-2}{\sigma^2}\int_0^\infty q_{x_*}(z)\,dz}}.$$

By Lemma 6.6 part (i), $f(x) \geq q_{x_*}(x)$ for all $x \in \mathbb{R}$, so a comparison argument gives

$$\varsigma \leq \limsup_{t\to\infty} \frac{Y(t)}{\sqrt{2t\log\log t}} \leq \limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}}, \quad \text{a.s.}$$

Combining this with the result of Theorem 5.1, we get the first part of the theorem. For part (ii), let $\bar{X}(t) = -X(t)$, $\bar{f}(x) = -f(-x)$ and $\bar{B}(t) = B(t)$. Then $\bar{X}$ obeys

$$d\bar{X}(t) = \bar{f}(\bar{X}(t))\,dt + \sigma\, d\bar{B}(t).$$

Now

$$\lim_{x\to\infty} x\bar{f}(x) = \lim_{y\to-\infty} (-y)(-f(y)) = \lim_{y\to-\infty} yf(y) = L_{-\infty} > 0.$$

Similarly $\lim_{y\to-\infty} y\bar{f}(y) = L_{\infty} < 0$. Therefore by the first part of the proof we get

$$\varsigma \leq \limsup_{t\to\infty} \frac{\bar{X}(t)}{\sqrt{2t\log\log t}}, \quad \text{a.s.}$$

which implies

$$\liminf_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} \leq -\varsigma, \quad \text{a.s.}$$

Combining this limit with the result of Theorem 5.1, the second assertion is proved. □

Corollary 6.8. *Let X be the unique continuous adapted process satisfying* (3.15).

(i) Suppose f is non-negative on the real line. If $L_{-\infty} \in (-\infty, 0]$ and $L_{\infty} \in [0, \sigma^2/2]$, then X is recurrent and satisfies

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} = \limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t\log\log t}} = |\sigma|, \quad \textit{a.s.}$$

(ii) Suppose f is non-positive on the real line. If $L_{\infty} \in (-\infty, 0]$ and $L_{-\infty} \in [0, \sigma^2/2]$, then X is recurrent and satisfies

$$\liminf_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} = \limsup_{t\to\infty} \frac{-|X(t)|}{\sqrt{2t\log\log t}} = -|\sigma|, \quad \textit{a.s.}$$

The lower estimate on the asymptotic growth rate of partial maxima of $|X|$ in this section can also be obtained when the limit in condition (6.1) is replaced by a limit superior or limit inferior in the appropriate way. For example, in Theorem 6.3, we can amend the condition (6.1) to $\liminf_{x\to-\infty} xf(x) = L_{-\infty}$ and $\limsup_{x\to\infty} xf(x) = L_{\infty}$. Hence we are able to estimate the growth rate of the partial maxima (or minima) of solutions in this section in terms of either the Law of the Iterated Logarithm or the polynomial Liapunov exponent for all real values of L_{∞} and $L_{-\infty}$.

7 Generalisation to multi-dimensional Systems

In this section, we generalize some of the main results in the scalar case to finite-dimensional processes. We show that analogous results can be obtained by using the same technique under adjusted conditions.

Theorem 7.1. *Let X be the unique continuous adapted process satisfying the d-dimensional equation* (3.18)*, where $X(0)=x_0\in\mathbb{R}^d$, $f:\mathbb{R}^d\times\mathbb{R}^+\to\mathbb{R}^d$, $g:\mathbb{R}^d\times\mathbb{R}^+\to\mathbb{R}^{d\times m}$ and B is a m-dimensional Brownian motion. If there exist positive real numbers ρ, C_a and C_b such that*

$$\forall\,(x,t)\in\mathbb{R}^d\times\mathbb{R}^+,\quad x^T f(x,t)\le\rho; \tag{7.1a}$$

$$\forall\,(x,t)\in\mathbb{R}^d\times\mathbb{R}^+,\ ||g(x,t)||_{op}\le C_a,\ \inf_{|x|\in\mathbb{R}^d/\{0\}}\frac{\sqrt{\sum_{j=1}^m\left(\sum_{i=1}^d x_i g_{ij}(x,t)\right)^2}}{|x|}\ge C_b. \tag{7.1b}$$

then

$$\limsup_{t\to\infty}\frac{|X(t)|}{\sqrt{2t\log\log t}}\le C_a,\quad a.s. \tag{7.2}$$

Proof. Define

$$\Phi(x,t)=\begin{cases}\dfrac{\sqrt{\sum_{j=1}^m(\sum_{i=1}^d x_i g_{ij}(x))^2}}{||x||}, & x\ne 0,\\ \sigma\in[C_b,C_a], & x=0.\end{cases} \tag{7.3}$$

Note that by (7.1b) and the Cauchy-Schwarz inequality,

$$\begin{aligned}C_b\le\Phi(x,t)&=\frac{\sqrt{\sum_{j=1}^m(\sum_{i=1}^d x_i g_{ij}(x,t))^2}}{|x|}\le\frac{\sqrt{\sum_{i=1}^d x_i^2\sum_{j=1}^m\sum_{i=1}^d g_{ij}^2(x,t)}}{|x|}\\&=\frac{|x|\cdot\|g(x,t)\|}{|x|}=\|g(x,t)\|\le C_a\sqrt{m},\quad x\ne 0,\quad t\ge 0.\end{aligned} \tag{7.4}$$

Now define θ by

$$\theta(t)=\int_0^t\Phi^2(X(s),s)\,ds,\quad t\ge 0.$$

Then $\lim_{t\to\infty}\theta(t)=\infty$. Also, define the stopping time $\eta(t)=\inf\{s>0:\theta(s)>t\}$: thus $\eta(t)=\theta^{-1}(t)$. Define $\tilde{X}(t):=X(\eta(t))$ and $\mathscr{K}(t)=\mathscr{F}(\eta(t))$ for all $t\ge 0$ (where $(\mathscr{F}(t))_{t\ge 0}$ is the original filtration). Then $\tilde{X}$ is $\mathscr{K}(t)$–adapted. Furthermore, for $1\le i\le d$, we have

$$\tilde{X}_i(t)=X_i(\eta(t))=X_i(0)+\int_0^{\eta(t)}f_i(X(s))\,ds+N_i(t) \tag{7.5}$$

where

$$N_i(t)=\int_0^{\eta(t)}\sum_{j=1}^m g_{ij}(X(s),s)\,dB_j(s). \tag{7.6}$$

$N = (N_1, N_2, \ldots, N_d)$ is a d–dimensional local martingale with respect to the filtration $\mathscr{K}(t)$. By Problem 3.4.5 in [12], it can be verified that the cross variation of N is given by

$$\begin{aligned}\langle N_i, N_k\rangle(t) &= \int_0^{\eta(t)} \sum_{j=1}^m g_{ij}(X(s),s) g_{kj}(X(s),s)\, ds \\ &= \int_0^t \sum_{j=1}^m g_{ij}(\tilde{X}(s),\eta(s)) g_{kj}(\tilde{X}(s),\eta(s))/\Phi^2(\tilde{X}(s),\eta(s))\, ds,\end{aligned}$$

Then there is an extension of $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$ of $(\Omega, \mathscr{F}, \mathbb{P})$ on which is defined a m–dimensional Brownian motion $\tilde{B} = \{(\tilde{B}_1(t), \tilde{B}_1(t), \ldots, \tilde{B}_m(t)); \tilde{\mathscr{K}}(t); 0 \le t < \infty\}$ such that

$$N_i(t) = \int_0^t \sum_{j=1}^m g_{ij}(\tilde{X}(s),\eta(s))/\Phi(\tilde{X}(s),\eta(s))\, d\tilde{B}_j(s), \quad \widetilde{\mathbb{P}}\text{-a.s.}$$

The filtration $\tilde{\mathscr{K}}(t)$ in the extended space is such that $\tilde{X}$ is $\tilde{\mathscr{K}}(t)$–adapted. Similarly, we obtain

$$\int_0^{\eta(t)} f_i(X(s),s)\, ds = \int_0^t f_i(\tilde{X}(s),\eta(s))/\Phi^2(\tilde{X}(s),\eta(s))\, ds.$$

Therefore

$$d\tilde{X}_i(t) = \frac{f_i(\tilde{X}(t),\eta(t))}{\Phi^2(\tilde{X}(t),\eta(t))}\, dt + \frac{1}{\Phi(\tilde{X}(t),\eta(t))} \sum_{j=1}^m g_{ij}(\tilde{X}(t),\eta(t))\, d\tilde{B}_j(t).$$

Next define $a : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^+$

$$a(x,t) = \sum_{j=1}^r \left(\sum_{i=1}^d x_i g_{ij}(x,t) \right)^2. \tag{7.7}$$

By (7.1b), $a(x,t) > 0$ for all $t \ge 0$ and $x \ne 0$. Define for $j = 1, \ldots, m$ the functions $\Phi_j : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}$

$$\Phi_j(x,t) = \frac{1}{\sqrt{a(x,t)}} \sum_{i=1}^d x_i g_{ij}(x,t), \quad x \ne 0$$

and $\Phi_j(x,t) = 1/\sqrt{m}$ for $x = 0$ and $t \ge 0$. Then

$$\frac{1}{\Phi(x,t)} \sum_{i=1}^d x_i g_{ij}(x,t) = |x| \Phi_j(x,t), \quad x \in \mathbb{R}^d, \tag{7.8}$$

$$\sum_{j=1}^m \Phi_j^2(x,t) = 1, \quad x \in \mathbb{R}^d. \tag{7.9}$$

Now, applying Itô's rule to $Z(t) = |\tilde{X}(t)|^2$, we get

$$\begin{aligned}dZ(t) = &\left[\frac{2\tilde{X}^T(t) f(\tilde{X}(t),\eta(t)) + \|g(\tilde{X}(t),\eta(t))\|^2}{\Phi^2(\tilde{X}(t),\eta(t))} \right] dt \\ &+ 2 \sum_{j=1}^m \left(\frac{1}{\Phi(\tilde{X}(t),\eta(t))} \sum_{i=1}^d \tilde{X}_i(t) g_{ij}(\tilde{X}(t),\eta(t)) \right) d\tilde{B}_j(t)\end{aligned}$$

so by (7.8) and $|\tilde{X}(t)| = \sqrt{Z(t)}$ we have

$$dZ(t) = \left[\frac{2\tilde{X}^T(t) f(\tilde{X}(t), \eta(t)) + \|g(\tilde{X}(t), \eta(t))\|^2}{\Phi^2(\tilde{X}(t), \eta(t))} \right] dt + 2\sqrt{Z(t)} \sum_{j=1}^{m} \Phi_j(\tilde{X}(t), \eta(t))\, d\tilde{B}_j(t).$$

Finally define

$$W(t) = \int_0^t \sum_{j=1}^{m} \Phi_j(\tilde{X}(s), \eta(s))\, d\tilde{B}_j(s), \quad t \ge 0.$$

By (7.9) and e.g. [12, Theorem 3.3.16], W is a standard Brownian motion adapted to $(\tilde{\mathscr{K}}(t))_{t\ge 0}$ such that

$$dZ(t) = \left[\frac{2\tilde{X}^T(t) f(\tilde{X}(t), \eta(t)) + \|g(\tilde{X}(t), \eta(t))\|^2}{\Phi^2(\tilde{X}(t), \eta(t))} \right] dt + 2\sqrt{Z(t)} dW(t). \tag{7.10}$$

Now it is easy to see that the drift coefficient of (7.10) is bounded above by $K_u := (2\rho + mC_a^2)/C_b^2$ due to (7.1). Consider the process governed by the equation

$$dZ_u(t) = K_u\, dt + 2\sqrt{|Z_u(t)|}\, dW(t), \quad t \ge 0,$$

with $Z_u(0) \ge x_0^2$. A similar argument as given in the proof of Theorem 5.3 shows that Z_u is non-negative. Applying the comparison theorem again, we have, for almost all $\omega \in \Omega$, $0 \le Z(t) \le Z_u(t)$ for all $t \ge 0$. Let $V_u(t) := e^{-t} Z_u(e^t - 1)$. By Itô's rule, it can be shown that

$$dV_u(t) = (-V_u(t) + K_u)\, dt + 2\sqrt{|V_u(t)|}\, d\widetilde{W}(t), \quad t \ge 0,$$

where $\widetilde{W}$ is another one-dimensional Brownian motion. Applying Lemma 2.2, we obtain

$$\limsup_{t\to\infty} \frac{V_u(t)}{2\log t} = 1, \quad \text{a.s.}$$

Using the relation between V_u and Z_u, and then comparing Z_u with Z, we get

$$\limsup_{t\to\infty} \frac{Z(t)}{2t \log\log t} \le \limsup_{t\to\infty} \frac{Z_u(t)}{2t \log\log t} \le 1, \quad \text{a.s.}$$

Since $\eta^{-1}(t) = \theta(t)$ and $Z(t) = |X(\eta(t))|^2$ for $t \ge 0$, we have

$$\limsup_{t\to\infty} \frac{|X(t)|^2}{2\theta(t) \log\log \theta(t)} \le 1, \quad \text{a.s.}$$

By (7.4), $C_b^2\, t \le \theta(t) \le C_a^2\, t$ for all $t \ge 0$ a.s. Thus

$$\limsup_{t\to\infty} \frac{|X(t)|^2}{2t \log\log t} \le C_a^2, \quad \text{a.s.}$$

The assertion (7.2) is therefore proven. □

We now establish the corresponding lower bound.

Theorem 7.2. *Let X be the unique continuous adapted process satisfying the d-dimensional equation* (3.18)*, where B is a m-dimensional Brownian motion. If* (7.1b) *holds and there exists a positive real number μ such that*

$$\inf_{(x,t)\in\mathbb{R}^d\times\mathbb{R}^+}\left(2x^T f(x,t)+||g(x,t)||^2\right)=\mu, \tag{7.11}$$

then

$$\limsup_{t\to\infty}\frac{|X(t)|}{\sqrt{2t\log\log t}}\geq C_b, \quad a.s. \tag{7.12}$$

Proof. Proceeding in the same way as in the previous theorem, we arrive at the process Z governed by

$$dZ(t) \quad = \quad \frac{2X^T(\eta(t))f(X(\eta(t)),\eta(t))+||g(X(\eta(t)),\eta(t))||^2}{\Phi^2(X(\eta(t)),\eta(t))}dt \quad + \quad 2\sqrt{Z(t)}\,dW(t),$$

where Φ is as defined in (7.3). By condition (7.11), it is obvious that the drift coefficient is bounded below by $K_l := \mu/(mC_a^2)$. Let Z_l be the non-negative process with $Z(0)\geq Z_l(0)\geq 0$ which satisfies the SDE

$$dZ_l(t)=K_l\,dt+2\sqrt{Z_l(t)}\,dW(t), \quad t\geq 0.$$

Then $Z(t)\geq Z_l(t)$, for all $t\geq 0$ a.s. Applying the same change in time and scale to Z_l as in the previous proof, and defining $V_l(t):=e^{-t}Z_l(e^t-1)$, we get

$$dV_l(t)=(-V_l(t)+K_l)\,dt+2\sqrt{|V_l(t)|}\,d\widetilde{W}(t), \quad t\geq 0.$$

Applying Lemma 2.2 again yields

$$\limsup_{t\to\infty}\frac{V_l(t)}{2\log t}=1, \quad \text{a.s.}$$

Following a similar argument as in Theorem 7.1, we get the desired result (7.12). $\square$

Our last theorem covers the special case where the diffusion coefficient is constant, diagonal and invertible. In this result, we use the notation $\langle x,y\rangle$ to denote the standard inner product of x and y in $\mathbb{R}^d$, and e_i as the i–th standard basis vector.

Theorem 7.3. *Let B be a d-dimensional Brownian motion and X be the unique continuous adapted process satisfying the d-dimensional equation*

$$dX(t)=f(X(t),t)\,dt+\Gamma\,dB(t), \quad t\geq 0 \tag{7.13}$$

with $X(0)=x_0\in\mathbb{R}^d$, where $f:\mathbb{R}^d\times\mathbb{R}^+\to\mathbb{R}^d$ and Γ is a $d\times d$ diagonal and invertible matrix with diagonal entries $\{\gamma_i\}_{1\leq i\leq d}$.

(i) If there exists a positive real number ρ such that

$$\forall\,(x,t)\in\mathbb{R}^d\times\mathbb{R}^+, \quad x^T f(x,t)\leq\rho, \tag{7.14}$$

then

$$\limsup_{t\to\infty}\frac{|X(t)|}{\sqrt{2t\log\log t}}\leq\max_{1\leq i\leq d}|\gamma_i|, \quad a.s. \tag{7.15}$$

(ii) If there exists $i \in \{1, 2...d\}$ such that

$$\inf_{(x,t)\in\mathbb{R}^d\times\mathbb{R}^+} \langle x, e_i\rangle\langle f(x,t), e_i\rangle = \mu > -\frac{\gamma_i^2}{2}, \tag{7.16}$$

then

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t\log\log t}} \geq |\gamma_i|, \quad a.s. \tag{7.17}$$

(iii) Moreover, if (7.14) holds, and there exists $i \in \{1, 2...d\}$ such that (7.16) holds and $|\gamma_i| = \max_{1\leq j\leq d}|\gamma_j|$, then

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t\log\log t}} = |\gamma_i|, \quad a.s.$$

Proof. It is obvious that part (iii) of the conclusion is a consequence of part (i) and (ii). To prove part (i), let $Y(t) := \Gamma^{-1}X(t)$, $\tilde{f}(x,t) = \Gamma^{-1}f(\Gamma x, t)$, so that

$$dY(t) = \tilde{f}(Y(t), t)\,dt + I_d\,dB(t), \quad t \geq 0.$$

Therefore

$$d|Y(t)|^2 = (2Y^T(t)\tilde{f}(Y(t), t) + d)\,dt + 2Y^T(t)\,dB(t), \quad t \geq 0.$$

Define $Z(t) := |Y(t)|^2$. Then the above equation can be written as

$$dZ(t) = (2Y^T(t)\tilde{f}(Y(t), t) + d)\,dt + 2\sqrt{Z(t)}\,dW(t), \quad t \geq 0,$$

where W is another one–dimensional Brownian motion. If we can show that

$$\forall (y,t) \in \mathbb{R}^d \times \mathbb{R}^+, \quad y^T\tilde{f}(y,t) \leq K, \tag{7.18}$$

for some positive K, then the non-negative process governed by

$$dZ_u(t) = (2K + d)\,dt + 2\sqrt{Z_u(t)}\,dW(t), \quad t \geq 0,$$

with $Z_u(0) \geq x_0^2$ satisfies $Z_u(t) \geq Z(t)$ for all $t \geq 0$ almost surely. As in the proof of the previous theorem, we have

$$\limsup_{t\to\infty} \frac{Z(t)}{2t\log\log t} \leq \limsup_{t\to\infty} \frac{Z_u(t)}{2t\log\log t} \leq 1, \quad \text{a.s.}$$

Thus

$$\limsup_{t\to\infty} \frac{\sqrt{\frac{X_1^2(t)}{\gamma_1^2} + \frac{X_2^2(t)}{\gamma_2^2} + \ldots + \frac{X_d^2(t)}{\gamma_d^2}}}{\sqrt{2t\log\log t}} \leq 1, \quad \text{a.s.}$$

Since

$$\frac{1}{\max_{1\leq i\leq d}|\gamma_i|}\sqrt{X_1^2(t) + \cdots + X_d^2(t)} \leq \sqrt{\frac{X_1^2(t)}{\gamma_1^2} + \frac{X_2^2(t)}{\gamma_2^2} + \ldots + \frac{X_d^2(t)}{\gamma_d^2}},$$

assertion (7.15) is proved. Now it is left to show (7.18). Let $y := \Gamma^{-1}x$, so that for $1 \le i \le d$, the i-th components are related by $y_i = x_i/\gamma_i$. Hence condition (7.14) gives

$$\begin{aligned} y^T \tilde{f}(y,t) &= y^T \Gamma^{-1} f(\Gamma y, t) = \Sigma_{i=1}^d \frac{y_i}{\gamma_i} f_i(\Gamma y, t) \\ &= \Sigma_{i=1}^d \frac{x_i}{\gamma_i^2} f_i(x,t) \le \frac{1}{\min_{1\le i\le d} \gamma_i^2} \Sigma_{i=1}^d x_i f_i(x,t) \le \frac{\rho}{\min_{1\le i \le d}\gamma_i^2}. \end{aligned}$$

The proof of part (i) is complete. For part (ii), note for each $1 \le i \le d$ and all $t \ge 0$, that $|X(t)| \ge |X_i(t)|$. Consider a particular X_i which is governed by

$$dX_i(t) = f_i(X(t),t)\,dt + \gamma_i\,dB_i(t), \quad t \ge 0.$$

Here by (7.16) and Theorem 5.3, we have

$$\limsup_{t\to\infty} \frac{|X_i(t)|}{\sqrt{2t\log\log t}} \ge |\gamma_i|, \quad \text{a.s.}$$

and so the inequality (7.17) is obvious. □

8 Application to a Financial Market Model

In this section, for the purposes mentioned in Section 3.5, we present an ergodic–type theorem for the solution of the equation

$$dX(t) = f(X(t))[1 - \alpha I_{\{|X(t)|>k\sigma\sqrt{t}\}}]\,dt + \sigma\,dB(t). \tag{8.1}$$

Theorem 8.1. *Suppose f is locally Lipschitz continuous and odd on $\mathbb{R}$, and satisfies,*

$$\lim_{|x|\to\infty} xf(x) = L \in (0, \sigma^2/2], \quad f(x) \ge 0 \quad \textit{for all } x \ge 0. \tag{8.2}$$

Let x_0 be deterministic, $0 < \alpha \le 1$, $\sigma > 0$, $k > 0$ and I be the indicator function. Then there is a unique strong continuous solution X of (8.1) with $X(0) = x_0$. Moreover, X obeys

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t\log\log t}} = \sigma, \quad \textit{a.s.}$$

and

$$\lim_{t\to\infty} \frac{\int_0^t \frac{X^2(s)}{(1+s)^2}\,ds}{\log t} = \Lambda_{L,\sigma,\alpha,k} \quad \textit{a.s.,} \tag{8.3}$$

where

$$\Lambda_{L,\sigma,\alpha,k} := \frac{\int_0^{k^2\sigma^2} e^{\frac{-x}{2\sigma^2}} x^{\frac{\sigma^2+2L}{2\sigma^2}}\,dx + (k^2\sigma^2)^{\frac{L\alpha}{\sigma^2}} \int_{k^2\sigma^2}^\infty e^{\frac{-x}{2\sigma^2}} x^{\frac{\sigma^2+2L(1-\alpha)}{2\sigma^2}}\,dx}{\int_0^{k^2\sigma^2} e^{\frac{-x}{2\sigma^2}} x^{\frac{2L-\sigma^2}{2\sigma^2}}\,dx + (k^2\sigma^2)^{\frac{L\alpha}{\sigma^2}} \int_{k^2\sigma^2}^\infty e^{\frac{-x}{2\sigma^2}} x^{\frac{2L(1-\alpha)-\sigma^2}{2\sigma^2}}\,dx} > \sigma^2. \tag{8.4}$$

Remark 8.2.

In the case when $f(x) = 0$, then $L = 0$, and we can independently prove (3.14), which is consistent with (8.3) ($\Lambda_{L,\sigma,\alpha,k} = \sigma^2$). On the other hand, letting $L \to 0$ in (8.4) yields $\lim_{L\to 0^+} \Lambda_{L,\sigma,\alpha,k} = \sigma^2$.

Remark 8.3.

As claimed earlier, we have $\Lambda_{L,\sigma,\alpha,k} > \sigma^2$ under the hypotheses of Theorem 8.1. To see this, for $L \in (0, \sigma^2/2]$, let

$$I := \int_0^{k^2\sigma^2} e^{\frac{-x}{2\sigma^2}} x^{\frac{2L-\sigma^2}{2\sigma^2}}\, dx$$

and

$$J := (k^2\sigma^2)^{\frac{L\alpha}{\sigma^2}} \int_{k^2\sigma^2}^{\infty} e^{\frac{-x}{2\sigma^2}} x^{\frac{2L(1-\alpha)-\sigma^2}{2\sigma^2}}\, dx.$$

Integration by parts gives

$$\int_0^{k^2\sigma^2} e^{\frac{-x}{2\sigma^2}} x^{\frac{\sigma^2+2L}{2\sigma^2}}\, dx = -2e^{\frac{-k^2}{2}} k^{1+\frac{2L}{\sigma^2}} \sigma^{3+\frac{2L}{\sigma^2}} + (\sigma^2 + 2L)I$$

and

$$(k^2\sigma^2)^{\frac{L\alpha}{\sigma^2}} \int_{k^2\sigma^2}^{\infty} e^{\frac{-x}{2\sigma^2}} x^{\frac{\sigma^2+2L(1-\alpha)}{2\sigma^2}}\, dx = 2e^{\frac{-k^2}{2}} k^{1+\frac{2L}{\sigma^2}} \sigma^{3+\frac{2L}{\sigma^2}} + (\sigma^2 + 2L(1-\alpha))J.$$

Then by (8.4)

$$\Lambda_{L,\sigma,\alpha,k} = \sigma^2 + \frac{2LI + 2L(1-\alpha)J}{I+J} > \sigma^2,$$

as claimed.

Proof of Theorem 8.1. We first discuss the existence of a strong solution of (8.1), which is not immediately obvious because the drift coefficient of (8.1) is discontinuous. However, by condition (8.2) and the continuity of f, the drift coefficient of X is uniformly bounded on $[0,\infty) \times \mathbb{R}$. Therefore, we may apply Proposition 5.3.6 and Remark 5.3.7 in [12] (which are based on Girsanov's theorem) to obtain a weak solution. Moreover, by Corollary 5.3.11 in [12], the weak solution of (8.1) is unique in the sense of probability law. On the other hand, Theorem V.41.1 in [18] by Nakao and Le Gall gives us the pathwise uniqueness of the solution. This, together with the weak existence implies the existence of a strong solution by Corollary 5.3.23 in [12]. For a given initial value x_0, and a fixed Brownian motion B, this strong solution is unique.

By the Ikeda–Watanabe comparison theorem which only requires the continuity of one of the drift coefficients in the two equations being compared, the first part of the theorem can easily be obtained by Theorem 5.1 and 5.3.

Now consider the transformation $Y(t) := e^{-t}X^2(e^t - 1)$. By Itô's rule, and the fact that f is odd, there exists a standard Brownian motion W such that

$$dY(t) = \big(-Y(t) + \sigma^2 + 2\sqrt{Y(t)}e^{\frac{t}{2}} f(\sqrt{Y(t)}e^{\frac{t}{2}})[1 - \alpha I_{\{Y(t)>k^2\sigma^2(1-e^{-t})\}}]\big)\, dt + 2\sigma\sqrt{Y(t)}\, dW(t). \quad (8.5)$$

For any $0 < \varepsilon < 1/2$, there exists a deterministic $T_{1,\varepsilon} > 0$ such that for all $t > T_{1,\varepsilon}$, $e^{-t} < \varepsilon$, so $k^2\sigma^2(1-\varepsilon) < k^2\sigma^2(1-e^{-t}) < k^2\sigma^2$. Due to (8.2) and continuity of f, there exists a $K > L(1+\varepsilon)$ such that for all $x \in \mathbb{R}$, $xf(x) < K$, and there exists a deterministic $x_\varepsilon > 0$ such that for all $x > x_\varepsilon$, $L(1-\varepsilon) < xf(x) < L(1+\varepsilon)$. For any $0 < \eta < 1 \wedge k^2\sigma^2(1-\varepsilon)$, there exists a deterministic $T_{2,\varepsilon,\eta} > T_{1,\varepsilon}$ such that $e^{T_{2,\varepsilon,\eta}/2}\sqrt{\eta} = x_\varepsilon$. Thus for all $t > T_{2,\varepsilon,\eta}$ and $Y(t) > \eta$, $L(1-\varepsilon) < \sqrt{Y(t)}e^{t/2}f(\sqrt{Y(t)}e^{t/2}) < L(1+\varepsilon)$. Choose θ_1, $\theta_2 > 0$ so small that $\theta_1 < 2L$, $\theta_1 \vee \theta_2 \vee \eta < k^2\sigma^2/6$, which implies $\eta+\theta_1 < k^2\sigma^2(1-\varepsilon)-\theta_2$. Now consider $Y_u := Y_{u,\varepsilon,\eta,\theta_1,\theta_2}$ and $Y_l := Y_{l,\varepsilon,\eta,\theta_1,\theta_2}$ governed by the following two equations respectively: for $t \geq T_{2,\varepsilon,\eta}$,

$$dY_u(t) = [-Y_u(t) + \sigma^2 + 2G_u(Y_u(t))]\,dt + 2\sigma\sqrt{Y_u(t)}\,dW(t), \tag{8.6}$$

$$dY_l(t) = [-Y_l(t) + \sigma^2 + 2G_l(Y_l(t))]\,dt + 2\sigma\sqrt{Y_l(t)}\,dW(t) \tag{8.7}$$

with Y_l and Y_u chosen so that $0 \leq Y_l(T_{2,\varepsilon,\eta}) < Y(T_{2,\varepsilon,\eta}) < Y_u(T_{2,\varepsilon,\eta})$ a.s., where $G_u : \mathbb{R}^+ \to \mathbb{R}^+/\{0\}$ is defined by

$$G_u(x) = \begin{cases} K, & 0 \leq x < \eta, \\ -\frac{K-L(1+\varepsilon)}{\theta_1}x + (K + \frac{K-L(1+\varepsilon)}{\theta_1}\eta), & \eta \leq x < \eta+\theta_1, \\ L(1+\varepsilon), & \eta+\theta_1 \leq x < k^2\sigma^2, \\ -\frac{L\alpha(1+\varepsilon)}{\theta_2}x + L(1+\varepsilon)(1+\frac{\alpha k^2\sigma^2}{\theta_2}), & k^2\sigma^2 \leq x < k^2\sigma^2+\theta, \\ L(1-\alpha)(1+\varepsilon), & k^2\sigma^2+\theta_2 \leq x. \end{cases}$$

$G_l : \mathbb{R}^+ \to \mathbb{R}^+$ is defined by

$$G_l(x) = \begin{cases} 0, & 0 \leq x < \eta, \\ \frac{L(1-\varepsilon)}{\theta_1}x - \frac{L(1-\varepsilon)\eta}{\theta_1}, & \eta \leq x < \eta+\theta_1, \\ L(1-\varepsilon), & \eta+\theta_1 \leq x < k_\varepsilon - \theta_2, \\ -\frac{L\alpha(1-\varepsilon)}{\theta_2}x + L(1-\alpha)(1-\varepsilon) + \frac{L\alpha k_\varepsilon(1-\varepsilon)}{\theta_2}, & k_\varepsilon - \theta_2 \leq x < k_\varepsilon, \\ L(1-\alpha)(1-\varepsilon), & k_\varepsilon \leq x, \end{cases}$$

where $k_\varepsilon := k^2\sigma^2(1-\varepsilon)$. Note that G_u and G_l are globally Lipschitz continuous on $\mathbb{R}^+$. Again by Ikeda–Watanabe's comparison theorem, it can be verified that $Y_l(t) \leq Y(t) \leq Y_u(t)$ for all $t \geq T_{2,\varepsilon,\eta}$ a.s. on an a.s. event $\Omega_* := \Omega_{\varepsilon,\eta,\theta_1,\theta_2}$. Choose $c \in (\eta+\theta_1, k^2\sigma^2(1-\varepsilon)-\theta_2)$ in definition (2.1). Then direct calculations on a scale function and speed measure of Y_l give that

$$
\begin{aligned}
\zeta_{1,\varepsilon,\eta,\theta_1,\theta_2} &:= \int_0^\infty x\, m_{Y_l}(dx) \\
&= \frac{1}{2\sigma^2}\Bigg[\int_0^\eta e^{\frac{c-2L(1-\varepsilon)}{2\sigma^2}}\left(\frac{\eta+\theta_1}{c}\right)^{\frac{\sigma^2+2L(1-\varepsilon)}{2\sigma^2}}\left(\frac{\eta}{\eta+\theta_1}\right)^{\frac{-2L(1-\varepsilon)\eta/\theta_1+\sigma^2}{2\sigma^2}} e^{\frac{-x}{2\sigma^2}}\left(\frac{x}{\eta}\right)^{\frac{1}{2}} dx \\
&\qquad + \int_\eta^{\eta+\theta_1} e^{\frac{c-2L(1-\varepsilon)(\eta+\theta_1)/\theta_1}{2\sigma^2}}\left(\frac{\eta+\theta_1}{c}\right)^{\frac{\sigma^2+2L(1-\varepsilon)}{2\sigma^2}} e^{\frac{2L(1-\varepsilon)/\theta_1-1}{2\sigma^2}x} \\
&\qquad\qquad \left(\frac{x}{\eta+\theta_1}\right)^{\frac{\sigma^2-2L(1-\varepsilon)\eta/\theta_1}{2\sigma^2}} dx \\
&\qquad + \int_{\eta+\theta_1}^{k^2\sigma^2(1-\varepsilon)-\theta_2} e^{\frac{c-x}{2\sigma^2}}\left(\frac{x}{c}\right)^{\frac{\sigma^2+2L(1-\varepsilon)}{2\sigma^2}} dx \\
&\qquad + \int_{k^2\sigma^2(1-\varepsilon)-\theta_2}^{k^2\sigma^2(1-\varepsilon)} \left(\frac{k^2\sigma^2(1-\varepsilon)-\theta_2}{c}\right)^{\frac{\sigma^2+2L(1-\varepsilon)}{2\sigma^2}} e^{\frac{c-x-2L\alpha(1-\varepsilon)(x-k^2\sigma^2(1-\varepsilon)+\theta_2)/\theta_2}{2\sigma^2}} \\
&\qquad\qquad \left(\frac{x}{k^2\sigma^2(1-\varepsilon)-\theta_2}\right)^{\frac{\sigma^2+2L(1-\alpha)(1-\varepsilon)+2L\alpha k^2\sigma^2(1-\varepsilon)^2/\theta_2}{2\sigma^2}} dx \\
&\qquad + \int_{k^2\sigma^2(1-\varepsilon)}^\infty c^{\frac{-\sigma^2-2L(1-\varepsilon)}{2\sigma^2}} e^{\frac{c-2L\alpha(1-\varepsilon)}{2\sigma^2}} (k^2\sigma^2(1-\varepsilon)-\theta_2)^{\frac{2L\alpha(1-\varepsilon)-2L\alpha k^2\sigma^2(1-\varepsilon)^2/\theta_2}{2\sigma^2}} \\
&\qquad\qquad (k^2\sigma^2(1-\varepsilon))^{\frac{2L\alpha k^2(1-\varepsilon)^2}{2\theta_2}} e^{\frac{-x}{2\sigma^2}} x^{\frac{\sigma^2+2L(1-\alpha)(1-\varepsilon)}{2\sigma^2}} dx\Bigg] < \infty. \qquad (8.8)
\end{aligned}
$$

Similar calculations give $\int_0^\infty m_{Y_l}(dx) =: \zeta_{2,\varepsilon,\eta,\theta_1,\theta_2} < \infty$. Hence by the ergodic theorem [18, Theorem V.53.1], for almost all $\omega \in \Omega_*$,

$$
\liminf_{t\to\infty}\frac{1}{t}\int_0^t Y(s)\,ds = \liminf_{t\to\infty}\frac{1}{t}\int_{T_{2,\varepsilon,\eta}}^t Y(s)\,ds \geq \lim_{t\to\infty}\frac{1}{t}\int_{T_{2,\varepsilon,\eta}}^t Y_l(s)\,ds = \frac{\zeta_{1,\varepsilon,\eta,\theta_1,\theta_2}}{\zeta_{2,\varepsilon,\eta,\theta_1,\theta_2}}. \qquad (8.9)
$$

Now we let the parameters tend to zero through rational numbers in the order ε, θ_1, θ_2 and η. We consider each term in the square brackets in (8.8) in turn. As $\varepsilon \downarrow 0$, the first integral on the interval $(0,\eta)$ becomes

$$
J_1 := e^{\frac{c-2L}{2\sigma^2}} c^{\frac{-\sigma^2-2L}{2\sigma^2}} (\eta+\theta_1)^{\frac{L}{\sigma^2}} \left(\frac{\eta+\theta_1}{\eta}\right)^{\frac{L\eta}{\sigma^2\theta_1}} \int_0^\eta e^{\frac{-x}{2\sigma^2}} x^{\frac{1}{2}}\,dx.
$$

Hence

$$
\lim_{\eta\to 0}(\lim_{\theta_1\to 0} J_1) = \lim_{\eta\to 0} e^{\frac{c-2L}{2\sigma^2}} c^{\frac{-\sigma^2-2L}{2\sigma^2}} \eta^{\frac{L}{\sigma^2}} e^{\frac{L}{\sigma^2}} \int_0^\eta e^{\frac{-x}{2\sigma^2}} x^{\frac{1}{2}}\,dx = 0.
$$

Similarly, as $\varepsilon \downarrow 0$, the second integral becomes

$$
J_2 := e^{\frac{c-2L(\eta+\theta_1)/\theta_1}{2\sigma^2}} c^{\frac{-\sigma^2-2L}{2\sigma^2}} (\eta+\theta_1)^{\frac{L+L\eta/\theta_1}{\sigma^2}} \int_\eta^{\eta+\theta_1} e^{\frac{2L/\theta_1-1}{2\sigma^2}x} x^{\frac{\sigma^2-2L\eta/\theta_1}{2\sigma^2}}\,dx.
$$

Since $\theta_1 < 2L$, we have

$$J_2 \le e^{\frac{c}{2\sigma^2} - \frac{L\eta}{\sigma^2\theta_1} - \frac{L}{\sigma^2}} c^{\frac{-\sigma^2-2L}{2\sigma^2}} (\eta+\theta_1)^{\frac{L+L\eta/\theta_1}{\sigma^2}} e^{\frac{L(\eta+\theta_1)}{\theta_1\sigma^2}} (\eta+\theta_1)^{\frac{1}{2}} \theta_1.$$

Hence $\lim_{\theta_1\to 0} J_2 = 0$. For the third integral, as $\varepsilon, \theta_1, \theta_2$ and η tend to zero, it tends to

$$\int_0^{k^2\sigma^2} e^{\frac{c-x}{2\sigma^2}} \left(\frac{x}{c}\right)^{\frac{\sigma^2+2L}{2\sigma^2}} dx.$$

Also as $\varepsilon \downarrow 0$, the fourth integral becomes

$$J_4 := e^{\frac{c}{2\sigma^2} + \frac{L\alpha k^2}{\theta_2} - \frac{L\alpha}{\sigma^2}} c^{\frac{-\sigma^2-2L}{2\sigma^2}} (k^2\sigma^2 - \theta_2)^{\frac{L\alpha}{\sigma^2} - \frac{L\alpha k^2}{\theta_2}} \int_{k^2\sigma^2-\theta_2}^{k^2\sigma^2} e^{\frac{-(1+2L\alpha/\theta_2)x}{2\sigma^2}} x^{\frac{\sigma^2+2L(1-\alpha)+2L\alpha k^2\sigma^2/\theta_2}{2\sigma^2}} dx.$$

It can be verified that

$$J_4 \le c^{\frac{-\sigma^2-2L}{2\sigma^2}} e^{\frac{c-k^2\sigma^2+\theta_2}{2\sigma^2}} (k^2\sigma^2-\theta_2)^{\frac{L\alpha}{\sigma^2}} (k^2\sigma^2)^{\frac{1}{2}+\frac{L(1-\alpha)}{\sigma^2}} \left(\frac{k^2\sigma^2}{k^2\sigma^2-\theta_2}\right)^{\frac{L\alpha k^2}{\theta_2}} \theta_2.$$

Letting $\theta_2 \downarrow 0$, since $\lim_{\theta_2\to 0}\left(\frac{k^2\sigma^2}{k^2\sigma^2-\theta_2}\right)^{\frac{L\alpha k^2}{\theta_2}} = e^{\frac{L\alpha}{\sigma^2}}$, we have $\lim_{\theta_2\to 0} J_4 = 0$. Finally, as $\varepsilon \downarrow 0$, the last integral becomes

$$J_5 := c^{\frac{-\sigma^2-2L}{2\sigma^2}} e^{\frac{c}{2\sigma^2}} e^{\frac{-L\alpha}{\sigma^2}} (k^2\sigma^2-\theta_2)^{\frac{L\alpha}{\sigma^2}} \left(\frac{k^2\sigma^2}{k^2\sigma^2-\theta_2}\right)^{\frac{L\alpha k^2}{\theta_2}} \int_{k^2\sigma^2}^{\infty} e^{\frac{-x}{2\sigma^2}} x^{\frac{\sigma^2+2L(1-\alpha)}{2\sigma^2}} dx.$$

Letting $\theta_2 \downarrow 0$, we have

$$\lim_{\theta_2\to 0} J_5 = c^{\frac{-\sigma^2-2L}{2\sigma^2}} e^{\frac{c}{2\sigma^2}} (k^2\sigma^2)^{\frac{L\alpha}{\sigma^2}} \int_{k^2\sigma^2}^{\infty} e^{\frac{-x}{2\sigma^2}} x^{\frac{\sigma^2+2L(1-\alpha)}{2\sigma^2}} dx.$$

Hence

$$\lim_{\varepsilon,\theta_1,\theta_2,\eta\to 0} \zeta_{1,\varepsilon,\eta,\theta_1,\theta_2} = \frac{1}{2\sigma^2} c^{\frac{-\sigma^2-2L}{2\sigma^2}} e^{\frac{c}{2\sigma^2}} \Bigg(\int_0^{k^2\sigma^2} e^{\frac{-x}{2\sigma^2}} x^{\frac{\sigma^2+2L}{2\sigma^2}} dx + (k^2\sigma^2)^{\frac{L\alpha}{\sigma^2}} \int_{k^2\sigma^2}^{\infty} e^{\frac{-x}{2\sigma^2}} x^{\frac{\sigma^2+2L(1-\alpha)}{2\sigma^2}} dx \Bigg).$$

In a similar fashion, we can verify that as $\varepsilon \downarrow 0$, $\theta_1 \downarrow 0$, $\theta_2 \downarrow 0$ and $\eta \downarrow 0$, $\zeta_{2,\varepsilon,\eta,\theta_1,\theta_2}$ also tends to a finite limit. Indeed,

$$\lim_{\varepsilon,\theta_1,\theta_2,\eta\to 0} \zeta_{2,\varepsilon,\eta,\theta_1,\theta_2} = \frac{1}{2\sigma^2} c^{\frac{-\sigma^2-2L}{2\sigma^2}} e^{\frac{c}{2\sigma^2}} \Bigg(\int_0^{k^2\sigma^2} e^{\frac{-x}{2\sigma^2}} x^{\frac{2L-\sigma^2}{2\sigma^2}} dx + (k^2\sigma^2)^{\frac{L\alpha}{\sigma^2}} \int_{k^2\sigma^2}^{\infty} e^{\frac{-x}{2\sigma^2}} x^{\frac{2L(1-\alpha)-\sigma^2}{2\sigma^2}} dx \Bigg).$$

This implies that

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t Y(s)\,ds \ge \Lambda_{L,\sigma,\alpha,k}, \quad \text{a.s. on } \Omega_{**} := \cap_{\{\varepsilon,\eta,\theta_1,\theta_2\in\mathbb{Q}\}}\Omega_*. \tag{8.10}$$

where $\Lambda_{L,\sigma,\alpha,k}$ is given by (8.4) and Ω_{**} is an a.s. event. In an analogous manner, by the definition of G_u, we have

$$\begin{aligned}
\kappa_{1,\varepsilon,\eta,\theta_1,\theta_2} &:= \int_0^\infty x\, m_{Y_u}(dx)\\
&= \frac{1}{2\sigma^2}\Bigg[\int_0^\eta e^{\frac{c+2K-2L(1+\varepsilon)}{2\sigma^2}}\left(\frac{\eta+\theta_1}{c}\right)^{\frac{\sigma^2+2L(1+\varepsilon)}{2\sigma^2}}\left(\frac{\eta}{\eta+\theta_1}\right)^{\frac{\sigma^2+2K+2(K-L(1+\varepsilon))\eta/\theta_1}{2\sigma^2}}\\
&\qquad e^{\frac{-x}{2\sigma^2}}\left(\frac{x}{\eta}\right)^{\frac{\sigma^2+2K}{2\sigma^2}}dx\\
&\quad+\int_\eta^{\eta+\theta_1} e^{\frac{c+2(K-L(1+\varepsilon))(\eta+\theta_1)/\theta_1}{2\sigma^2}}\left(\frac{\eta+\theta_1}{c}\right)^{\frac{\sigma^2+2L(1+\varepsilon)}{2\sigma^2}} e^{-\frac{1+2(K-L(1+\varepsilon))/\theta_1}{2\sigma^2}x}\\
&\qquad\left(\frac{x}{\eta+\theta_1}\right)^{\frac{\sigma^2+2K+2(K-L(1+\varepsilon))\eta/\theta_1}{2\sigma^2}}dx\\
&\quad+\int_{\eta+\theta_1}^{k^2\sigma^2} e^{\frac{c-x}{2\sigma^2}}\left(\frac{x}{c}\right)^{\frac{\sigma^2+2L(1+\varepsilon)}{2\sigma^2}}dx\\
&\quad+\int_{k^2\sigma^2}^{k^2\sigma^2+\theta_2} e^{\frac{c+2L(1+\varepsilon)\alpha k^2\sigma^2/\theta_2}{2\sigma^2}}\left(\frac{k^2\sigma^2}{c}\right)^{\frac{\sigma^2+2L(1+\varepsilon)}{2\sigma^2}} e^{-\frac{1+2L(1+\varepsilon)\alpha/\theta_2}{2\sigma^2}x}\\
&\qquad\left(\frac{x}{k^2\sigma^2}\right)^{\frac{\sigma^2+2L(1+\varepsilon)(1+\alpha k^2\sigma^2/\theta_2)}{2\sigma^2}}dx\\
&\quad+\int_{k^2\sigma^2+\theta_2}^\infty e^{\frac{c-2L(1+\varepsilon)\alpha}{2\sigma^2}}\left(\frac{k^2\sigma^2}{c}\right)^{\frac{\sigma^2+2L(1+\varepsilon)}{2\sigma^2}}\left(\frac{k^2\sigma^2+\theta_2}{k^2\sigma^2}\right)^{\frac{\sigma^2+2L(1+\varepsilon)(1+\alpha k^2\sigma^2/\theta_2)}{2\sigma^2}} e^{\frac{-x}{2\sigma^2}}\\
&\qquad\left(\frac{x}{k^2\sigma^2+\theta_2}\right)^{\frac{\sigma^2+2L(1-\alpha)(1+\varepsilon)}{2\sigma^2}}dx\Bigg] < \infty.
\end{aligned}$$

Similar calculations give $\int_0^\infty m_{Y_u}(dx) =: \kappa_{2,\varepsilon,\eta,\theta_1,\theta_2} < \infty$. Also by the ergodic theorem,

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t Y(s)\,ds \le \lim_{t\to\infty}\frac{1}{t}\int_0^t Y_u(s)\,ds = \frac{\kappa_{1,\varepsilon,\eta,\theta_1,\theta_2}}{\kappa_{2,\varepsilon,\eta,\theta_1,\theta_2}}, \quad \text{a.s. on } \Omega_*. \tag{8.11}$$

Again, let $\varepsilon \downarrow 0$, $\theta_1 \downarrow 0$, $\theta_2 \downarrow 0$ and $\eta \downarrow 0$ through rational numbers and proceeding as for Y_l, we get the same limit $\Lambda_{L,\sigma,\alpha,k}$ as obtained the lower bound. Combining this with (8.11) and (8.10), we have

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t Y(s)\,ds = \Lambda_{L,\sigma,\alpha,k}, \quad \text{a.s. on } \Omega_{**}.$$

Using the relation $Y(t) = e^{-t}X^2(e^t-1)$, the desired result (8.3) is obtained. □

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