Electronic Journal of Probability

Vol. 7 (2002) Paper no. 20, pages 1–22.

Journal URL
http://www.math.washington.edu/~ejpecp/
Paper URL
http://www.math.washington.edu/~ejpecp/EjpVol7/paper20.abs.html

DIFFUSION IN LONG-RANGE CORRELATED ORNSTEIN-UHLENBECK FLOWS

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Abstract: We study a diffusion $d\mathbf{x}(t) = \mathbf{V}(t, \mathbf{x}(t))\, dt + \sqrt{2D_0} d\mathbf{w}(t)$ with a random Markovian and Gaussian drift for which the usual (spatial) Péclet number is infinite. We introduce a temporal Péclet number and we prove that, under the finiteness of the temporal Péclet number, the laws of scaled diffusions $\epsilon\mathbf{x}(t/\epsilon^2)$, $t \geq 0$ converge weakly, as $\epsilon \to 0$, to the law of a Brownian motion. We also show that the effective diffusivity has a finite, nonzero limit as $D_0 \to 0$.

Keywords and phrases: Ornstein-Uhlenbeck fields, martingale central limit theorem, homogenization, power-law spectrum.

AMS subject classification (2000): Primary 60F17, 35B27; secondary 60G44.

Submitted to EJP on Nov. 20, 2001. Final version accepted on May 31, 2002.

[1] Research supported by NSF grant DMS-9971322
[2] Research supported by the State Committee for Scientific Research of Poland grant Nr 2 PO3A 017 17

1 Introduction

The motion of a passive scalar in a random velocity is described by Itô's stochastic differential equation

$$d\mathbf{x}(t) = \mathbf{V}(t, \mathbf{x})dt + \sqrt{2D_0}d\mathbf{w}(t), \quad \mathbf{x}(0) = \mathbf{0} \tag{1.1}$$

where $\mathbf{V} = (V_1, \cdots, V_d) : \mathbb{R} \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ is a d-dimensional random vector field with incompressible ($\nabla_{\mathbf{x}} \cdot \mathbf{V}(t, \mathbf{x}) \equiv 0$) realizations and $\mathbf{w}(t)$, $t \geq 0$ is a d-dimensional standard Brownian motion, independent of $\mathbf{V}$. The coefficient $D_0 > 0$ is called the molecular diffusivity.

We are particularly interested in the following class of velocity fields: $\mathbf{V}$ is a time-space stationary, centered ($\mathbf{E}\mathbf{V}(0, \mathbf{0}) = \mathbf{0}$) Gaussian, Markovian field with the co-variance matrix given by

$$\begin{aligned} \mathbf{R}(\mathbf{x}, t) &:= \mathbf{E}[\mathbf{V}(t, \mathbf{x}) \otimes \mathbf{V}(0, \mathbf{0})] \\ &= \int_{R^d} \mathrm{e}^{-|\mathbf{k}|^{2\beta} t} \cos(\mathbf{k} \cdot \mathbf{x})\, \Gamma(\mathbf{k})\, \mathcal{E}(|\mathbf{k}|) |\mathbf{k}|^{1-d}\, d\mathbf{k}, \quad \beta > 0 \end{aligned} \tag{1.2}$$

with $\Gamma(\mathbf{k}) := \mathbf{I} - \mathbf{k} \otimes \mathbf{k}|\mathbf{k}|^{-2}$ and the power-law energy spectrum

$$\mathcal{E}(k) = a(k)k^{1-2\alpha}, \quad k = |\mathbf{k}| > 0, \tag{1.3}$$

where $a(k)$ is a (ultraviolet or infrared) cut-off function to ensure the finiteness of the integral (1.2). This class of velocity fields plays an important role in statistical hydrodynamics because the particular member with $\beta = 1/3, \alpha = 4/3$ satisfies Kolmogorov-Obukhov's self-similarity hypothesis for the developed turbulence.

The main object of interest here is the large-scale *diffusive scaling*

$$\mathbf{x}^{\varepsilon}(t) := \varepsilon \mathbf{x}(t/\varepsilon^2), \quad \varepsilon \downarrow 0. \tag{1.4}$$

In the case of $\alpha \geq 1$ the *infrared* cutoff is necessary. Once the infrared cutoff is in place, the velocity field is spatially homogeneous and temporally strongly mixing. Then, with an additional arbitrary ultraviolet cutoff to ensure regularity, the motion ($D_0 \geq 0$) on the large (integral) scale is diffusive by the results of [5].

For $\alpha < 1$ the infrared cutoff is optional (thus long-range correlation is possible) but an ultraviolet cutoff is necessary. However, we will only assume that

$$\sup_{k \geq 1} k^n\, a(k) < +\infty, \forall n \geq 1. \tag{1.5}$$

Our main objective is to prove a sharp convergence theorem for the diffusive limit in flows with *long-range* correlation.

It is well known (see [1], Corollary of Theorem 3.4.1) that under these assumptions almost all realizations of the field are jointly continuous in $(t, \mathbf{x})$ and of C^{∞}-class in $\mathbf{x}$ for any t fixed. One can further prove the global existence and uniqueness of solutions of (1.1). We denote by Q_{ε} the laws of the scaled trajectories $\mathbf{x}_{\varepsilon}(t) = \varepsilon \mathbf{x}(t/\varepsilon^2)$, $t \geq 0$ in $C([0, +\infty); \mathbb{R}^d)$. The main theorem of this paper is formulated as follows.

Theorem 1 *Let $D_0 > 0$ and let $\mathbf{V}(t, \mathbf{x})$, $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$ be a stationary, centered Gaussian velocity field with the co-variance matrix given by* (1.2) *with $\alpha < 1$, $\beta \geq 0$ and $\alpha + \beta < 1$ and the cut-off function $a(\cdot)$ satisfying* (1.5)*. Then the laws Q_ε converge weakly over $C([0, +\infty); \mathbb{R}^d)$, as $\varepsilon \downarrow 0$, to a Wiener measure with a non-trivial co-variance matrix $2\mathbf{D} \geq 2D_0\mathbf{I}$.*

The following questions arise naturally: Does the diffusion limit hold when $D_0 = 0$? Do the diffusion coefficients $\mathbf{D}$ (called the effective diffusivity) established in Theorem 1 have a non-zero limit as D_0 tends to zero? We don't know the answers. However, we can prove the following.

Theorem 2 *Let $\mathbf{D}(D_0)$ be half the covariance matrix of the limiting Brownian motion as a function of the molecular diffusivity D_0. We have*

$$\limsup_{D_0 \to 0} D_{i,j}(D_0) < \infty, \quad i, j = 1, ..., d, \tag{1.6}$$

$$\liminf_{D_0 \to 0} D_{i,i}(D_0) > 0, \quad i = 1, ..., d. \tag{1.7}$$

Beside the framework and techniques developed in the paper, the main interest of the theorems is that they establish a new regime for the diffusive limit. Previous diffusive limit theorems have been proved either for random flows that have finite Péclet number

$$\mathrm{Pe} := D_0^{-1} \sqrt{\int_{\mathbb{R}^d} \sum_{i=1}^{d} \hat{R}_{ii}(0, \mathbf{k}) |\mathbf{k}|^{-2} \, d\mathbf{k}} < \infty$$

[2, 8] or for Markovian flows that are strongly mixing in time [5, 11]. For the flows considered here, finite Péclet number means $\alpha < 0$ while temporal mixing means $\beta = 0$. In the regime $\alpha + \beta < 1, \alpha > 0, \beta > 0$ the velocity neither has finite Péclet number nor is temporally mixing. Since $\alpha + \beta < 1$ if and only if

$$\int_0^\infty \sum_{i=1}^{d} R_{ii}(t, 0) \, dt < \infty$$

our results suggest the introduction of the *temporal* Péclet number defined as

$$D_0^{-1} \int_0^\infty \sum_{i=1}^{d} R_{ii}(t, 0) \, dt$$

whose convergence may be an alternative general condition for long-time diffusive behavior. The condition $\alpha + \beta < 1$ is believed and partially shown to be sharp (see [6] for more discussion).

Without loss of generality we will set $D_0 = 1$ till we turn to the proof of Theorem 2.

2 Function spaces and random fields

In the sequel, we shall denote by $\mathcal{T}_0 := (\Omega, \mathcal{V}, \mathbb{P})$ the probability space of random vector fields and by $\mathcal{T}_1 := (\Sigma, \mathcal{W}, Q)$ the probability space of the (molecular) Brownian motion. Their respective expectations are denoted by $\mathbf{E}$ and $\mathbf{M}$. The trajectory $\mathbf{x}(t)$, $t \geq 0$ is then a stochastic process over the probability space $\mathcal{T}_0 \otimes \mathcal{T}_1$. The space $\mathcal{T}_0$ is detailed in the rest of this section.

2.1 Spatially homogeneous Gaussian measures

Let $\mathbb{H}^m_\rho$ be the Hilbert space of d-dimensional incompressible vector fields that is the completion of $\mathcal{S}_{\rm div} := \{f \in \mathcal{S}(\mathbb{R}^d;\mathbb{R}^d) : \nabla_{\mathbf{x}} \cdot f = 0\}$ with respect to the norm

$$||f||^2_{\mathbb{H}^m_\rho} := \int\limits_{\mathbb{R}^d} (|f(\mathbf{x})|^2 + |\nabla_{\mathbf{x}} f(\mathbf{x})|^2 + \cdots + |\nabla^m_{\mathbf{x}} f(\mathbf{x})|^2)\vartheta_\rho(\mathbf{x})\, d\mathbf{x}$$

for any positive integer m and the weight function $\vartheta_\rho(\mathbf{x}) := (1+|\mathbf{x}|^2)^{-\rho}$, where $\rho > d/2$. When $m, \rho = 0$ we shall write $\mathbb{L}^2{}_{\rm div} := \mathbb{H}^0_0$. Let the Gaussian measure μ be the probability law of $\mathbf{V}(0,\cdot) \in \mathbb{H}^m_\rho$ as given by the correlation functions (1.2) with (1.3) and (1.5). Let

$$\mathcal{T}_2 := (\mathbb{H}^m_\rho, \mathcal{B}(\mathbb{H}^m_\rho), \mu),$$

$L^p(\mu) := L^p(\mathcal{T}_2)$, $1 \le p \le +\infty$ and $L^p_0(\mu) := \{F \in L^p(\mu) : \int F\, d\mu = 0\}$. We will suppress writing the measure μ when there is no danger of confusion. Both here and in the sequel $\mathcal{B}(\mathcal{M})$ is the Borel σ-algebra of subsets of a metric space $\mathcal{M}$. On $\mathbb{H}^m_\rho$ we define a group of μ-preserving transformations $\tau_{\mathbf{x}} : \mathbb{H}^m_\rho \to \mathbb{H}^m_\rho$, $\tau_{\mathbf{x}} f(\cdot) = f(\cdot + \mathbf{x})$, $\mathbf{x} \in \mathbb{R}^d$. This group is ergodic and stochastically continuous, hence $U^{\mathbf{x}} F(f) := F(\tau_{\mathbf{x}}(f))$, $\mathbf{x} \in \mathbb{R}^d$, $F \in L^p$ defines a C_0-group of isometries on any L^p, $1 \le p < +\infty$. We denote $D_p F := \partial_{x_p} U^{\mathbf{x}} F\Big|_{\mathbf{x}=\mathbf{0}}$, $p = 1, \cdots, d$ its L^2-generators. Let $\mathcal{C}^\infty_b$ be the space consisting of such elements $F \in L^\infty$, for which $F(\mathbf{x}, f) := F(\tau_{\mathbf{x}} f)$ are C^∞ in $\mathbf{x}$, μ a.s., with derivatives of all orders bounded by deterministic constants (i.e. constants independent of f). For any $1 \le p < +\infty$ and a positive integer m let $W^{p,m}$ be the Sobolev space as the closure of $\mathcal{C}^\infty_b$ in the norm

$$\|F\|^p_{p,m} := \sum_{m_1+\cdots+m_d \le m} \|D_1^{m_1} \cdots D_d^{m_d} F\|^p_{L^p}.$$

This definition can be extended in an obvious way to include the case of $p = +\infty$.

Let $(\cdot,\cdot)_{L^2}$ denote the generalized pairing between tempered distributions and the Schwartz test functions. Let

$$F_\varphi(f) := (f, \varphi)_{\mathbb{L}^2}, \quad f \in \mathbb{H}^m_\rho, \quad \varphi \in \mathcal{S}_{\rm div}. \tag{2.1}$$

Obviously $F_\varphi \in (\mathbb{H}^m_\rho)^*$ and $\mathcal{B}(\mathbb{H}^m_\rho)$ is the smallest σ-algebra with respect to which all F_φ, $\varphi \in \mathcal{S}_{\rm div}$ are measurable.

Let $B(\mathbf{k}, l) \subseteq \mathbb{R}^d$ be the Euclidean ball of radius l centered at $\mathbf{k}$. Set

$$\mathcal{S}_{\rm div}(l) := \{\varphi \in \mathcal{S}_{\rm div} : \ \text{supp}\ \widehat{\varphi} \cap B(\mathbf{0}, l) = \phi\}.$$

We denote by $\mathcal{P}_0$ the space of all polynomials over $\mathbb{H}^m_\rho$, i.e.

$$\mathcal{P}_0 := \text{span}\ \big[F : F(f) := (\varphi_1, f)_{\mathbb{L}^2{}_{\rm div}} \cdots (\varphi_N, f)_{\mathbb{L}^2{}_{\rm div}}, \tag{2.2}$$

$$f \in \mathbb{H}^m_\rho,\ N \ge 1 \text{ a positive integer}, \varphi_j \in \mathcal{S}_{\rm div}, 0 \notin \text{supp}\, \hat{\varphi}_j,\ \forall j\big].$$

$\mathcal{P}_0$ is a dense subset of L^2 ([9], Theorem 2.11, p. 21). Notice that

$$\mathcal{P}_0 = \bigcup_{l>0} \mathcal{P}_0(l),$$

where

$$\mathcal{P}_0(l) := \text{span } \Big[F : F(f) := (\varphi_1, f)_{\mathbb{L}^2{}_{\text{div}}} \cdots (\varphi_N, f)_{\mathbb{L}^2{}_{\text{div}}}, \\ f \in \mathbb{H}^m_\rho,\ N \geq 1 \text{ a positive integer}, \varphi_1, \cdots, \varphi_N \in \mathcal{S}_{\text{div}}(l)\Big]. \tag{2.3}$$

Also,

$$U^{\mathbf{x}}(\mathcal{P}_0) \subseteq \mathcal{P}_0, \quad \forall \mathbf{x} \in \mathbb{R}^d. \tag{2.4}$$

By $\mathcal{H}(l)$ we denote the L^2-closure of $\mathcal{P}_0(l)$ and by $\mathcal{Q}(l)$ the orthogonal projection onto $\mathcal{H}(l)$. Obviously $\mathcal{Q}(l)F = E_\mu[F|\Sigma(l)]$, $F \in L^2$, where $E_\mu[\,\cdot\,|\Sigma(l)]$ denotes the conditional expectation operator with respect to the the σ-algebra $\Sigma(l)$ generated by all polynomials from $\mathcal{P}_0(l)$ in the probability space $\mathcal{T}_2$. Hence we can extend $\mathcal{Q}(l)$ to a positivity preserving contraction operator $\mathcal{Q}(l) : L^p \to L^p$ for any $1 \leq p \leq +\infty$. Notice that

$$U^{\mathbf{x}} F_\varphi = F_{\tau_{-\mathbf{x}}\varphi} \in \mathcal{H}(l), \quad \forall \mathbf{x} \in \mathbb{R}^d,\, \varphi \in \mathcal{S}_{\text{div}}(l), \tag{2.5}$$

hence the following proposition holds.

Proposition 1 *$U^{\mathbf{x}}\mathcal{Q}(l)F = \mathcal{Q}(l)U^{\mathbf{x}}F$, for any $F \in L^1$, $\mathbf{x} \in \mathbb{R}^d$, $l > 0$.*

Proof. Observe that for any integer $n \geq 1$, a bounded, Borel measurable function $G : \mathbb{R}^{2n} \to \mathbb{R}$ and $\varphi_1, \cdots, \varphi_n \in \mathcal{S}_d$ we have

$$U^{\mathbf{x}} G(F_{\varphi_1}, \cdots, F_{\varphi_n}) = G(U^{\mathbf{x}} F_{\varphi_1}, \cdots, U^{\mathbf{x}} F_{\varphi_n}),\ \mathbf{x} \in \mathbb{R}^d. \tag{2.6}$$

¿From (2.5) we infer that $U^{\mathbf{x}}(\mathcal{H}(l)) = \mathcal{H}(l)$ and the conclusion of the proposition follows. ∎

2.2 Ornstein-Uhlenbeck Velocity Field

Let $W(t)$, $t \geq 0$ be a cylindrical Wiener process over the probability space $\mathcal{T}_0 = (\Omega, \mathcal{V}, \mathbb{P})$ so that $dW(t)$ is a divergence-free, space-time-white-noise Gaussian field. Let $B : \mathbb{L}^2{}_{\text{div}} \to \mathbb{H}^m_\rho$ be the continuous extension of the operator defined on $\mathcal{S}_{\text{div}}$ as

$$\widehat{B\psi}(\mathbf{k}) = \sqrt{2\mathcal{E}\left(|\mathbf{k}|\right)}|\mathbf{k}|^{(1+2\beta-d)/2}\hat{\psi}(\mathbf{k}), \quad \psi \in \mathcal{S}_{\text{div}} \tag{2.7}$$

where $\mathcal{E}(k)$ is given by (1.3)-(1.5). Here and below $\hat{\psi}$ denotes the Fourier transform of ψ. It can be shown (see [7], Proposition 2) that B is a Hilbert-Schmidt operator.

Let

$$\widehat{S(t)\psi}(\mathbf{k}) := \mathrm{e}^{-|\mathbf{k}|^{2\beta}t}\hat{\psi}(\mathbf{k}),\ \psi \in \mathcal{S}_{\text{div}}. \tag{2.8}$$

It can be shown (see [7], Proposition 2) that $S(t)$, $t \geq 0$ extends to a C_0-semigroup of operators on $\mathbb{H}^m_\rho$, provided that β is an integer. For a non-integral β the above is still true provided that $d/2 < \rho < d/2 + \beta$. S_{div} is a core of the generator $-A$ of the semigroup and

$$\widehat{A\psi}(\mathbf{k}) = |\mathbf{k}|^{2\beta}\hat{\psi}(\mathbf{k}),\ \psi \in \mathcal{S}_{\text{div}}.$$

We also introduce the operator $C : \mathbb{L}^2{}_{\text{div}} \to \mathbb{H}^m_\rho$ defined as the continuous extension of

$$\widehat{C\psi}(\mathbf{k}) = \sqrt{2\mathcal{E}\left(|\mathbf{k}|\right)}|\mathbf{k}|^{(1-d)/2}\hat{\psi}(\mathbf{k}), \quad \psi \in \mathcal{S}_{\text{div}}. \tag{2.9}$$

The same argument as in the proof of Part 1) of Proposition 2 of [7] yields that C is Hilbert-Schmidt.

Let

$$V_f(t) = S(t)f + \int_0^t S(t-s)\, BdW(s). \tag{2.10}$$

By $V_\mu(t)$, $t \geq 0$ we denote the process $V_f(t)$, $t \geq 0$ over $\mathcal{T}_0 \otimes \mathcal{T}_2$ with the random initial condition f, distributed according to μ and independent of the cylindrical Wiener process $W(t)$, $t \geq 0$. Let $V(t) := \mathbf{V}(t, \cdot)$, $t \geq 0$ be $\mathbb{H}^m_\rho$-valued, continuous trajectory stochastic process. Its law in $C([0, +\infty); \mathbb{H}^m_\rho)$ coincides with that of $V_\mu(t)$, $t \geq 0$ and in what follows we shall identify those two processes. The measure μ is stationary (see Section 2.3 of [7]). Its ergodicity follows from Lemma 1 below (Corollary 1). A direct calculation shows that for any bounded and measurable $G, H : \mathbb{R}^N \to \mathbb{R}$, $\varphi_1, \cdots, \varphi_N \in \mathcal{S}_{\text{div}}$ one has

$$\mathbf{E}\left[G((V(t), \varphi_1)_{\mathbb{L}^2}, \cdots, (V(t), \varphi_N)_{\mathbb{L}^2}) H((V(0), \varphi_1)_{\mathbb{L}^2}, \cdots, (V(0), \varphi_N)_{\mathbb{L}^2})\right]$$

$$= \mathbf{E}\left[H((V(t), \varphi_1)_{\mathbb{L}^2}, \cdots, (V(t), \varphi_N)_{\mathbb{L}^2}) G((V(0), \varphi_1)_{\mathbb{L}^2}, \cdots, (V(0), \varphi_N)_{\mathbb{L}^2})\right],$$

therefore μ is reversible. We denote by R^t, $t \geq 0$, $\mathcal{L}$ and $\mathcal{E}_{\mathcal{L}}(\cdot, \cdot)$ respectively the L^2-semigroup, generator and Dirichlet form corresponding to the process $V(t)$, $t \geq 0$.

A useful formula for the Dirichlet form of linear functionals is given by the following.

Proposition 2 *For any $\varphi \in \mathcal{S}_{div}$, the linear functional F_φ as defined by* (2.1) *is in the domain $D(\mathcal{L})$ of $\mathcal{L}$ and*

$$\mathcal{E}_{\mathcal{L}}(F_\varphi, F_\varphi) = \int_{\mathbb{R}^d} |\mathbf{k}|^{2\beta} |\widehat{\varphi}(\mathbf{k})|^2 \frac{\mathcal{E}(|\mathbf{k}|)}{|\mathbf{k}|^{d-1}}\, d\mathbf{k}. \tag{2.11}$$

Proof. Suppose first that

$$\widehat{\varphi} \in C_0^\infty(\mathbb{R}^d \setminus \{\mathbf{0}\}; \mathbb{R}^d). \tag{2.12}$$

Then, a direct calculation shows that $F_\varphi \in D(\mathcal{L})$ and $\mathcal{L}F_\varphi(f) = (A\varphi, f)_{\mathbb{L}^2}$ and (2.11) follows. On the other hand for an arbitrary $\varphi \in \mathcal{S}_{\text{div}}$ one can choose a sequence of φ_n, $n \geq 1$ such that their respective Fourier transforms $\widehat{\varphi}_n$ satisfy (2.12) and

$$\lim_{n \uparrow +\infty} \int_{\mathbb{R}^d} |\mathbf{k}|^{2\beta} \, |\widehat{\varphi}_n(\mathbf{k}) - \widehat{\varphi}(\mathbf{k})|^2 \frac{\mathcal{E}(|\mathbf{k}|)}{|\mathbf{k}|^{d-1}}\, d\mathbf{k} = 0. \tag{2.13}$$

¿From (2.13) we conclude both that $F_{\varphi_n} \to F_\varphi$, $n \uparrow +\infty$ and that $\mathcal{L}F_{\varphi_n} \to \mathcal{L}F_\varphi$ in the L^2-sense. This implies that $F_\varphi \in D(\mathcal{L})$. We obtain (2.11) by passing to the limit in the expression for $\mathcal{E}_{\mathcal{L}}(F_{\varphi_n}, F_{\varphi_n})$. ∎

The following proposition holds.

Proposition 3 *Operators R^t and $\mathcal{Q}(l)$ commute i.e.*

$$[R^t, \mathcal{Q}(l)] = \mathbf{0} \tag{2.14}$$

for any $t \geq 0$, $l > 0$.

Proof. Suppose that $n \geq 1$, $\varphi_1, \cdots, \varphi_n \in \mathcal{S}_{\rm div}(l)$ and $G(f) := F_{\varphi_1}(f) \cdots F_{\varphi_n}(f)$, $f \in \mathbb{H}^m_\rho$. We have

$$R^t G(f) = \mathbf{E} G(V_f(t)), \tag{2.15}$$

with V_f given by (2.10). By Theorem 1.36 p. 16 of [9] there exists a polynomial $P(x_1, \cdots, x_n)$, $(x_1, \cdots, x_n) \in \mathbb{R}^n$ such that the right hand side of (2.15) is given by $P(F_{S(t)\varphi_1}(f), \cdots, F_{S(t)\varphi_n}(f)) \in \mathcal{H}(l)$. Hence for any polynomial $G \in \mathcal{H}(l)$ we have $R^t G \in \mathcal{H}(l)$, $t \geq 0$. The L^2-density of polynomials in $\mathcal{H}(l)$ implies that $R^t(\mathcal{H}(l)) \subseteq \mathcal{H}(l)$ or equivalently

$$\mathcal{Q}(l) R^t \mathcal{Q}(l) = R^t \mathcal{Q}(l).$$

Since R^t and $\mathcal{Q}(l)$ are self-adjoint, by taking the adjoint of both sides of this equality we arrive at $R^t \mathcal{Q}(l) = \mathcal{Q}(l) R^t$. Hence, (2.16) follows. ∎

As a consequence of the above proposition, we have

$$\mathcal{E}_{\mathcal{L}}(F, F) \geq \mathcal{E}_{\mathcal{L}}(\mathcal{Q}(l)F, \mathcal{Q}(l)F) \quad \forall\, F \in L^2_0. \tag{2.16}$$

3 The main lemma

The purpose of this section is to prove the following estimate.

Lemma 1 *There exists an absolute constant $C_0 > 0$ such that*

$$\mathcal{E}_{\mathcal{L}}(F, F) \geq C_0\, \beta \int_0^{+\infty} l^{2\beta - 1} \|\mathcal{Q}(l) F\|^2_{L^2}\, dl \quad \forall\, F \in L^2_0,$$

where $\mathcal{Q}(l)$ is the projection onto $\mathcal{H}(l)$ – the L^2-closure of $\mathcal{P}_0(l)$ as defined in (2.3).

Since, by Lemma 1, $\mathcal{E}_{\mathcal{L}}(F, F) = 0$ implies that $\mathcal{Q}(l)(F) = 0, \forall l$, and hence F is a constant, we have the following.

Corollary 1 *Measure μ is ergodic.*

The proof of Lemma 1 (Section 3.3) uses the general scheme of periodization and periodic approximation (Section 3.1 and 3.2) which is valid for general stationary Markov fields.

3.1 Periodization of the Ornstein-Uhlenbeck flow

For an arbitrary integer $n \geq 1$ let $\Lambda_n := \{\mathbf{j} \in \mathbf{Z}^d : 0 < |\mathbf{j}| \leq n 2^n\}$. Suppose that $0 \leq \phi^{(n)}_{\mathbf{0}} \leq 1$ is a C^∞ smooth function such that

$$\text{supp } (\phi^{(n)}_{\mathbf{0}}) \subseteq \Delta^{(n)}_{\mathbf{0}} := \{\mathbf{k} = (k_1, \cdots, k_d) \in \mathbb{R}^d : -2^{-n-1} \leq k_i < 2^{-n-1}\}$$

and $\phi^{(n)}_{\mathbf{0}}(\mathbf{k}) \equiv 1$ when $-2^{-n-1}(1 - 2^{-n}) \leq k_i \leq 2^{-n-1}(1 - 2^{-n})$, $i = 1, \cdots, d$. Let $\phi^{(n)}_{\mathbf{j}}(\mathbf{k}) := \phi^{(n)}_{\mathbf{0}}(\mathbf{k} - \mathbf{k}_{\mathbf{j}})$, where $\mathbf{k}_{\mathbf{j}} := \mathbf{j} 2^{-n}$ for $\mathbf{j} = (j_1, \cdots, j_d) \in \Lambda_n$.

We define

$$H_n^m := \{f : f(\mathbf{x}) = \sum_{\mathbf{j}\in\Lambda_n} \Gamma(\mathbf{k_j}) \left[a_\mathbf{j} \cos(\mathbf{k_j}\cdot\mathbf{x}) + b_\mathbf{j} \sin(\mathbf{k_j}\cdot\mathbf{x})\right], \quad \text{for some } a_\mathbf{j},\, b_\mathbf{j} \in R^d\}$$

considered as the subspace of the Sobolev space of all divergence free vector fields $f : R^d \to R^d$ that are $2^{n+1}\pi$ periodic, possessing m generalized derivatives. Let $j_n : H_n^m \to \mathbb{H}_\rho^m$ be the inclusion map. Since $\mathbb{H}_\rho^m \subseteq \mathcal{S}'(\mathbb{R}^d;\mathbb{R}^d)$ we can define for any $f \in \mathbb{H}_\rho^m$ its Fourier transform $\hat{f} = (\hat{f}_1, \cdots \hat{f}_d)$, via the relation $\hat{f}_k(\phi) := f_k(\hat{\phi})$, for any $\phi \in \mathcal{S}(\mathbb{R}^d)$ (see [12] p. 5). We set $\pi_n : \mathbb{H}_\rho^m \to H_n^m$ by the formula

$$\pi_n f(\mathbf{x}) := \sum_{\mathbf{j}\in\Lambda_n} \left[X_\mathbf{j}^{(n)}(f) \cos(\mathbf{k_j}\cdot\mathbf{x}) + Y_\mathbf{j}^{(n)}(f) \sin(\mathbf{k_j}\cdot\mathbf{x})\right],$$

where $X_\mathbf{j}^{(n)}(f) := \Gamma(\mathbf{k_j})\mathrm{Re}\, \hat{f}(\phi_\mathbf{j}^{(n)})$, $Y_\mathbf{j}^{(n)}(f) := -\Gamma(\mathbf{k_j})\mathrm{Im}\, \hat{f}(\phi_\mathbf{j}^{(n)})$. Notice that $\pi_n j_n = id_n$ - the identity map on H_n^m. $X_\mathbf{j}^{(n)}(f)$, $Y_\mathbf{j}^{(n)}(f)$, $\mathbf{j} \in \Lambda_n$ are independent, centered real Gaussian vectors over the probability space $\mathcal{T}_2$. Their co-variance matrix is

$$S_\mathbf{j}^{(n)} := \Gamma(\mathbf{k_j}) \left(\int_{R^d} \left|\phi_\mathbf{j}^{(n)}(\mathbf{k})\right|^2 \mathcal{E}(|\mathbf{k}|)|\mathbf{k}|^{1-d}\, d\mathbf{k} \right) \Gamma(\mathbf{k_j}).$$

The images of Wiener process $C\mathbf{W}(t)$, $t \geq 0$ under π_n are finite dimensional Brownian motions given by

$$\pi_n C\mathbf{W}(t) = \sum_{\mathbf{j}\in\Lambda_n} \sqrt{S_\mathbf{j}^{(n)}} \left[\mathbf{w}_\mathbf{j}^{(n)}(t) \cos(\mathbf{k_j}\cdot\mathbf{x}) + \tilde{\mathbf{w}}_\mathbf{j}^{(n)}(t) \sin(\mathbf{k_j}\cdot\mathbf{x})\right], \; t \geq 0, \tag{3.1}$$

where $\mathbf{w}_\mathbf{j}^{(n)}(t)$, $\tilde{\mathbf{w}}_\mathbf{j}^{(n)}(t)$, $t \geq 0$ are independent standard d dimensional Brownian motions, $\mathbf{j} \in \Lambda_n$. Let us set $a_\mathbf{j}(0; f) = a_\mathbf{j}$, $b_\mathbf{j}(0; f) = b_\mathbf{j}$, $\mathbf{j} \in \Lambda_n$ for any $f = \sum_{\mathbf{j}\in\Lambda_n} [a_\mathbf{j} \cos(\mathbf{k_j}\cdot\mathbf{x}) + b_\mathbf{j} \sin(\mathbf{k_j}\cdot\mathbf{x})] \in H_n^m$ and consider the d-dimensional Ornstein-Uhlenbeck processes $a_\mathbf{j}(t; f)$, $b_\mathbf{j}(t; f)$, $t \geq 0$ given by

$$\begin{cases} da_\mathbf{j}(t; f) = -|\mathbf{k_j}|^{2\beta} a_\mathbf{j}(t; f)\, dt + |\mathbf{k_j}|^\beta \sqrt{S_\mathbf{j}^{(n)}} d\mathbf{w}_\mathbf{j}^{(n)}(t), \\ a_\mathbf{j}(0; f) = a_\mathbf{j}, \end{cases} \tag{3.2}$$

$$\begin{cases} db_\mathbf{j}(t; f) = -|\mathbf{k_j}|^{2\beta} b_\mathbf{j}(t; f) dt + |\mathbf{k_j}|^\beta \sqrt{S_\mathbf{j}^{(n)}} d\tilde{\mathbf{w}}_\mathbf{j}^{(n)}(t), \\ b_\mathbf{j}(0; f) = b_\mathbf{j}. \end{cases} \tag{3.3}$$

We define an Ornstein-Uhlenbeck process in H_n^m as

$$\mathbf{V}_f^{(n)}(t)(\mathbf{x}) := \sum_{\mathbf{j}\in\Lambda_n} \left[a_\mathbf{j}(t; f) \cos(\mathbf{k_j}\cdot\mathbf{x}) + b_\mathbf{j}(t; f) \sin(\mathbf{k_j}\cdot\mathbf{x})\right].$$

Let $\mu_n := \mu\pi_n^{-1}$ and $\Pi_n : L^2(\mu_n) \to L^2$, $J_n : \mathcal{P}_0 \to L^2(\mu_n)$ be linear maps given by

$$\Pi_n F(f) = F(\pi_n(f)) \tag{3.4}$$
$$J_n F(f) = F(j_n(f)). \tag{3.5}$$

The process $\mathbf{V}_{\mu_n}^{(n)}(t)$, $t \geq 0$, is a stationary process defined over the probability space $(\Omega \times H_n^m, \mathcal{V} \otimes \mathcal{B}(H_n^m), P \otimes \mu_n)$ as $\mathbf{V}_f^{(n)}(t)$, $t \geq 0$ with the initial condition f distributed according to μ_n. This process gives rise to a random, time-space stationary and spatially $2^{n+1}\pi$-periodic vector field

$$\mathbf{V}_f^{(n)}(t, \mathbf{x}) := \mathbf{V}_f^{(n)}(t)(\mathbf{x}) \quad \text{and} \quad \mathbf{V}^{(n)}(t, \mathbf{x}) := \mathbf{V}_{\mu_n}^{(n)}(t)(\mathbf{x}). \tag{3.6}$$

We denote by R_n^t, $\mathcal{L}_n$, $\mathcal{E}_{\mathcal{L}_n}(\cdot,\cdot)$ the $L^2(\mu_n)$-semigroup, generator and Dirichlet form corresponding to the process $\mathbf{V}_{\mu_n}^{(n)}(t)$, $t \geq 0$.

Let M_n be the cardinality of Λ_n and c_l the cardinality of those $\mathbf{j}$-s for which $|\mathbf{k_j}| \leq l$. We denote by ν, ν_l the Gaussian measures on $(\mathbb{R}^d)^{2M_n}$, $(\mathbb{R}^d)^{2c_l}$ with the corresponding characteristic functions

$$\varphi(\xi_\mathbf{j}, \eta_\mathbf{j}; \mathbf{j} \in \Lambda_n) = \prod_{\mathbf{j} \in \Lambda_n} \exp\left\{-\frac{1}{2}\left(S_\mathbf{j}^{(n)}\xi_\mathbf{j} \cdot \xi_\mathbf{j} + S_\mathbf{j}^{(n)}\eta_\mathbf{j} \cdot \eta_\mathbf{j}\right)\right\} \tag{3.7}$$

and

$$\varphi_l(\xi_\mathbf{j}, \eta_\mathbf{j}; |\mathbf{k_j}| \leq l) = \prod_{|\mathbf{k_j}| \leq l} \exp\left\{-\frac{1}{2}\left(S_\mathbf{j}^{(n)}\xi_\mathbf{j} \cdot \xi_\mathbf{j} + S_\mathbf{j}^{(n)}\eta_\mathbf{j} \cdot \eta_\mathbf{j}\right)\right\}. \tag{3.8}$$

For any monomial

$$G(a_\mathbf{j}, b_\mathbf{j}; \mathbf{j} \in \Lambda_n) := \prod_{\mathbf{j} \in \Lambda_n} a_\mathbf{j}^{\alpha_\mathbf{j}} b_\mathbf{j}^{\beta_\mathbf{j}},$$

with $\alpha_\mathbf{j}$, $\beta_\mathbf{j} \geq 0$ nonnegative integers we define

$$UG(f) := G(a_\mathbf{j}(0; f), b_\mathbf{j}(0; f); \mathbf{j} \in \Lambda_n),\ f \in H_n^m. \tag{3.9}$$

Set

$$\phi_\mathbf{j}^a(\mathbf{x}) = \cos(\mathbf{k_j} \cdot \mathbf{x}), \quad \phi_\mathbf{j}^b(\mathbf{x}) = \sin(\mathbf{k_j} \cdot \mathbf{x}),\ \mathbf{j} \in \Lambda_n.$$

We have

$$\begin{aligned} F(\cdot) &:= \prod_{\mathbf{j} \in \Lambda_n} (\phi_\mathbf{j}^a, \cdot)_{\mathbb{L}^2{}_{\mathrm{div}}(\mathbf{T}_n^d)}^{\alpha_\mathbf{j}} (\phi_\mathbf{j}^b, \cdot)_{\mathbb{L}^2{}_{\mathrm{div}}(\mathbf{T}_n^d)}^{\beta_\mathbf{j}} \\ &= UG \in J_n(\mathcal{P}_0). \end{aligned} \tag{3.10}$$

The operator U extends to a unitary map between $L^2(\nu)$ and $L^2(\mu_n)$. As in Section 2.2 let $\mathcal{H}_n(l)$ be the L^2-closure of the polynomials $J_n(\mathcal{P}_0(l))$ and let $\mathcal{Q}_n(l)$ be the corresponding orthogonal projection. Notice that

$$U(\mathcal{K}_n(l)) = \mathcal{H}_n(l), \tag{3.11}$$

where $\mathcal{K}_n(l)$ is the L^2-closure of polynomials depending only on variables $a_\mathbf{j}, b_\mathbf{j}$ corresponding to those $\mathbf{j}$-s for which $|\mathbf{k_j}| > l$.

3.2 Periodic approximation

In the next proposition we show that $\mathbf{V}_{\mu_n}^{(n)}(t)$, $t \geq 0$ is an approximation of $\mathbf{V}_\mu(t)$, $t \geq 0$.

Proposition 4 *i) For any $F \in \mathcal{P}_0$ we have*

$$\lim_{n\uparrow+\infty} \Pi_n R_n^t J_n F = R^t F \quad \text{in } L^2. \tag{3.12}$$

ii) For any $F \in \mathcal{P}_0$ we have $J_n F \in D(\mathcal{E}_{\mathcal{L}_n})$ and

$$\lim_{n\uparrow+\infty} \mathcal{E}_{\mathcal{L}_n}(J_n F, J_n F) = \mathcal{E}_{\mathcal{L}}(F, F). \tag{3.13}$$

The class of polynomials $\mathcal{P}_0$ is a core of $\mathcal{E}_{\mathcal{L}}$.

iii) Let $F = UG$ with $G \in L^2((\mathbb{R}^d)^{2M_n})$ then

$$\mathcal{Q}_n(l)F(f) = UG(l)(f), \tag{3.14}$$

where

$$G(l)(a_{\mathbf{j}}, b_{\mathbf{j}}; |\mathbf{k}_{\mathbf{j}}| > l) := \underbrace{\int \cdots \int}_{a_{\mathbf{j}},\, b_{\mathbf{j}}:|\mathbf{k}_{\mathbf{j}}|\le l} G(a_{\mathbf{j}}, b_{\mathbf{j}}; \mathbf{j} \in \Lambda_n) d\nu_l \tag{3.15}$$

and for any $F \in \mathcal{P}_0$

$$\lim_{n\uparrow+\infty} \|\mathcal{Q}_n(l) J_n F\|_{L^2(\mu_n)} = \|\mathcal{Q}(l)F\|_{L^2(\mu)}. \tag{3.16}$$

Proof. *Part i).* It suffices to verify that (3.12) holds for the polynomials. Let $\varphi_1, \cdots, \varphi_N \in \mathcal{S}_{\rm div}$ and

$$F(\cdot) = (\varphi_1, \cdot)_{\mathbb{L}^2{}_{\rm div}} \cdots (\varphi_N, \cdot)_{\mathbb{L}^2{}_{\rm div}}, \tag{3.17}$$

then

$$\Pi_n R_n^t J_n F(f) = \mathbf{E}\left[(\varphi_1, j_n(\mathbf{V}^{(n)}_{\pi_n(f)}(t)))_{\mathbb{L}^2{}_{\rm div}} \cdots (\varphi_N, j_n(\mathbf{V}^{(n)}_{\pi_n(f)}(t)))_{\mathbb{L}^2{}_{\rm div}}\right] \tag{3.18}$$

and the right hand side of (3.18) can be expressed as a finite sum of certain products made of expressions of the form

$$\sum_{\mathbf{j}\in\Lambda_n} \int_{R^d} \left[X^{(n)}_{\mathbf{j}} \cos(\mathbf{k}_{\mathbf{j}} \cdot \mathbf{x}) + Y^{(n)}_{\mathbf{j}} \sin(\mathbf{k}_{\mathbf{j}} \cdot \mathbf{x})\right] \varphi_k(\mathbf{x})\, d\mathbf{x}$$

and

$$\sum_{\mathbf{j}\in\Lambda_n} \int_{R^d}\int_{R^d} \left(1 - \mathrm{e}^{-2|\mathbf{k}_{\mathbf{j}}|^{2\beta} t}\right) S^{(n)}_{\mathbf{j}} \cos(\mathbf{k}_{\mathbf{j}} \cdot (\mathbf{x} - \mathbf{x}'))\, \varphi_k(\mathbf{x}) \cdot \varphi_l(\mathbf{x}')\, d\mathbf{x}\, d\mathbf{x}'.$$

Taking into account the definitions of $X^{(n)}_{\mathbf{j}}$, $Y^{(n)}_{\mathbf{j}}$ we conclude that as $n \uparrow +\infty$ the right hand side of (3.18) tends to

$$\mathbf{E}\left[(\varphi_1, \mathbf{V}_f(t))_{\mathbb{H}^m_\rho} \cdots (\varphi_N, \mathbf{V}_f(t))_{\mathbb{H}^m_\rho}\right] = R^t F(f),$$

in the L^2 sense.

Part ii). Note that $R^t(\mathcal{P}_0) \subseteq \mathcal{P}_0$ so $\mathcal{P}_0$ is a core of $\mathcal{L}$ and for F as in (3.17)

$$\mathcal{L}F(f) = \sum_{k=1}^{N} (\varphi_1, f)_{\mathbb{L}^2{}_{\rm div}} \cdots (A\varphi_k, f)_{\mathbb{L}^2{}_{\rm div}} \cdots (\varphi_N, f)_{\mathbb{L}^2{}_{\rm div}}.$$

Likewise, (3.18) implies that $R_n^t J_n(\mathcal{P}_0) \subseteq J_n(\mathcal{P}_0)$, $\forall\, t \geq 0$, so $J_n(\mathcal{P}_0)$ is a core of $\mathcal{L}_n$ and

$$\mathcal{L}_n J_n F(f) = \sum_{k=1}^{N} (\varphi_1, j_n(f))_{\mathbb{L}^2_{\rm div}} \cdots (A\varphi_k, j_n(f))_{\mathbb{L}^2_{\rm div}} \cdots (\varphi_N, j_n(f))_{\mathbb{L}^2_{\rm div}},$$

$$\Pi_n J_n F(f) = (\varphi_1, \pi_n j_n(f))_{\mathbb{L}^2_{\rm div}} \cdots (\varphi_N, \pi_n j_n(f))_{\mathbb{L}^2_{\rm div}},$$

for $F(\cdot) = (\varphi_1, \cdot)_{\mathbb{L}^2_{\rm div}} \cdots (\varphi_N, \cdot)_{\mathbb{L}^2_{\rm div}} \in \mathcal{P}_0$. Thus,

$$\lim_{n\uparrow+\infty} [\|\Pi_n J_n F - F\|_{L^2} + \|\Pi_n \mathcal{L}_n J_n F - \mathcal{L} F\|_{L^2}] = 0$$

and (3.13) follows for all $F \in \mathcal{P}_0$.

Part iii) Notice that the orthogonal projection $\widetilde{\mathcal{Q}}_n(l)$ onto $\mathcal{K}_n(l)$ in $L^2((\mathbb{R}^d)^{2M_n})$ is the conditional expectation with respect to the σ-algebra generated by the functions in variables $a_{\mathbf{j}}, b_{\mathbf{j}}$, $|\mathbf{k}_{\mathbf{j}}| > l$ only. Thus, for $G := G(a_{\mathbf{j}}, b_{\mathbf{j}}; \mathbf{j} \in \Lambda_n)$ we have

$$\left(\widetilde{\mathcal{Q}}_n(l) G\right)(a_{\mathbf{j}}, b_{\mathbf{j}}; |\mathbf{k}_{\mathbf{j}}| > l) = \underbrace{\int \cdots \int}_{a_{\mathbf{j}},\, b_{\mathbf{j}}: |\mathbf{k}_{\mathbf{j}}| \leq l} G(a_{\mathbf{j}}, b_{\mathbf{j}}; \mathbf{j} \in \Lambda_n) d\nu_l,$$

which in turn implies (3.14), thanks to (3.11). (3.16) follows from the fact that the co-variance matrices of the fields $\mathbf{V}^{(n)}$ approximate, as $n \uparrow +\infty$, the co-variance matrix of the field $\mathbf{V}$. ∎

3.3 Proof of Lemma 1

The conclusion of Lemma 1 follows from Proposition 4 and the following.

Lemma 2 *There exists an absolute constant $C > 0$, independent of n, such that*

$$\mathcal{E}_{\mathcal{L}_n}(J_n F, J_n F) \geq C\,\beta \int_0^{+\infty} l^{2\beta-1} \|\mathcal{Q}_n(l) J_n F\|^2_{L^2(\mu_n)}\, dl$$

for all $F \in \mathcal{P}_0$ such that $\int F d\mu = 0$.

Proof. Let F be the polynomial given by (3.17). We have $J_n F = UG$, for a certain polynomial $G(a_{\mathbf{j}}, b_{\mathbf{j}}; \mathbf{j} \in \Lambda_n)$ (cf. (3.5), (3.9)), i.e.

$$J_n F(V_f^{(n)}(t)) = G(a_{\mathbf{j}}(t; f), b_{\mathbf{j}}(t; f); \mathbf{j} \in \Lambda_n).$$

Consequently,

$$\mathcal{E}_{\mathcal{L}_n}(J_n F, J_n F) = \sum_{\mathbf{j} \in \Lambda_n} |\mathbf{k}_{\mathbf{j}}|^{2\beta} \mathcal{E}_{\mathbf{j}}(G, G) \tag{3.19}$$

with

$$\mathcal{E}_{\mathbf{j}}(G, G) := \frac{1}{2} \underbrace{\int \cdots \int}_{(\mathbb{R}^d)^{2M_n}} \left(S_{\mathbf{j}}^{(n)} \nabla_{a_{\mathbf{j}}} G \cdot \nabla_{a_{\mathbf{j}}} G + S_{\mathbf{j}}^{(n)} \nabla_{b_{\mathbf{j}}} G \cdot \nabla_{b_{\mathbf{j}}} G \right) d\nu.$$

The above argument generalizes to any polynomial $F \in \mathcal{P}_0$. We shall denote by G the corresponding polynomial in $a_{\mathbf{j}}, b_{\mathbf{j}}$, $\mathbf{j} \in \Lambda_n$. For any integer $m \ge 1$ we can write that

$$
\begin{aligned}
\mathcal{E}_{\mathcal{L}_n}(J_nF, J_nF) &= \sum_{k=0}^{+\infty} \sum_{|\mathbf{k_j}| \in [k/m,(k+1)/m)} |\mathbf{k_j}|^{2\beta} \mathcal{E}_{\mathbf{j}}(G,G) \\
&\ge \sum_{k=0}^{+\infty} \left(\frac{k}{m}\right)^{2\beta} \sum_{|\mathbf{k_j}| \in [k/m,(k+1)/m)} \mathcal{E}_{\mathbf{j}}(G,G) \\
&= \sum_{k=0}^{+\infty} \left(\frac{k}{m}\right)^{2\beta} \left[\mathcal{F}\left(\frac{k}{m}\right) - \mathcal{F}\left(\frac{k+1}{m}\right)\right] \qquad (3.20)
\end{aligned}
$$

where

$$
\mathcal{F}(l) := \sum_{|\mathbf{k_j}| \ge l} \mathcal{E}_{\mathbf{j}}(G,G).
$$

A simple calculation shows that the right side of (3.20) is greater than or equal to

$$
\frac{2\,\beta}{m} \sum_{k=0}^{+\infty} \left(\frac{k}{m}\right)^{2\beta-1} \mathcal{F}\left(\frac{k}{m}\right).
$$

Moreover, by Jensen's inequality,

$$
\mathcal{F}(l) \ge \mathcal{F}'(l) := \sum_{|\mathbf{k_j}| \ge l} \mathcal{E}_{\mathbf{j}}(G(l), G(l)),
$$

where

$$
G(l)(a_{\mathbf{j}},\, b_{\mathbf{j}} : |\mathbf{k_j}| > l) := \int \cdots \int_{a_{\mathbf{j}},\, b_{\mathbf{j}} : |\mathbf{k_j}| \le l} G(a_{\mathbf{j}},\, b_{\mathbf{j}} : \mathbf{j} \in \Lambda_n)\, d\nu_l.
$$

Because J_nF has zero mean, so does $G(l)$ and the coercivity of $\mathcal{E}_{\mathbf{j}}$ implies

$$
\mathcal{F}(l) \ge \mathcal{F}'(l) \ge C\|G(l)\|^2_{L^2(\nu)} = C\|\mathcal{Q}_n(l)F\|^2_{L^2(\mu_n)}
$$

with an absolute constant C independent of n. Here we have used the fact that

$$
\mathcal{Q}_n(l)F(f) = UG(l)(f).
$$

The conclusion of the lemma follows upon the passage to the limit with $m \uparrow +\infty$. ∎

4 Lagrangian velocity process

In what follows we introduce the so-called *Lagrangian canonical process* over the probability space $\mathcal{T}_0 \otimes \mathcal{T}_1$, with the state space $\mathbb{H}^m_\rho$ that, informally speaking, describes the random environment viewed from the moving particle. Let $\mathbf{x}_f(t)$, $t \ge 0$ be the trajectory of (1.1) with the drift replaced by $\mathbf{V}_f(t,\mathbf{x}) := V_f(t)(\mathbf{x})$, $(t,\mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$. Let $\eta(t) := \tau_{\mathbf{x}(t)}V(t)$, $t \ge 0$ and $\eta_f(t) := \tau_{\mathbf{x}_f(t)}V_f(t)$,

$t \geq 0$. $\eta(t)$, $t \geq 0$ is a continuous, Markov process (see e.g. [10] Theorem 1 p. 424) i.e. there exists Q^t, $t \geq 0$ a C_0-semigroup Q^t, $t \geq 0$ of Markovian operators on L^2 satisfying

$$Q^t F(f) = \mathbf{EM} F(\eta_f(t)) \tag{4.1}$$

and

$$\mathbf{E}[F(\eta(t+h))|\mathcal{V}_t] = Q^h F(\eta(t)), \quad t, h \geq 0 \tag{4.2}$$

where $\mathcal{V}_t$, $t \geq 0$ is the natural filtration corresponding to the Lagrangian process. Moreover thanks to incompressibility of $\mathbf{V}$ the measure μ is stationary, i.e. $\int Q^t F \, d\mu = \int F \, d\mu$, $t \geq 0$. Ergodicity of the measure for the semigroup Q^t, $t \geq 0$ follows from the ergodicity for R^t, $t \geq 0$ (see Theorem 1 p. 424 of [10]). The generator of the process is given by

$$\mathcal{M}F(f) = \Delta F(f) + \mathcal{L}F(f) + V \cdot \nabla F(f), \forall F \in \mathcal{C}_{\mathcal{L}} := \mathcal{C}_b^\infty \cap D(\mathcal{L}), \; f \in \mathbb{H}^m_\rho$$

where $V = (V_1, \cdots, V_d)$ is a random vector over $\mathcal{T}_2$ given by

$$V(f) := f(\mathbf{0}), \;\; f \in \mathbb{H}^m_\rho, \tag{4.3}$$

$\nabla := (D_1, \cdots, D_d)$, $\Delta := D_1^2 + \cdots + D_d^2$. In order to make sense of (4.3) we need to assume that $m > [d/2] + 1$. The set $\mathcal{C}_{\mathcal{L}}$ is dense in L^2. In what follows we shall also consider a family of approximate Lagrangian processes obtained as follows. Let $V^{(n)} = (V_1^{(n)}, \cdots, V_d^{(n)})$, $n \geq 1$ be random vectors over $\mathcal{T}_2$ with components belonging to $\mathcal{C}_b^\infty$ such that $\nabla \cdot V^{(n)} = 0$ and $\lim_{n\uparrow+\infty} \|V - V^{(n)}\|_{L^p_d} = 0$ for all $1 \leq p < +\infty$. We define a random field $\mathbf{V}^{(n)}(t, \mathbf{x}) := V^{(n)}(\tau_{\mathbf{x}}(V(t)))$, $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$ and set $\eta^{(n)}(t) := \tau_{\mathbf{x}^{(n)}(t)}(V(t))$, $t \geq 0$ where $\mathbf{x}^{(n)}(t; \omega, \sigma)$, $t \geq 0$ is a solution of (1.1) with $\mathbf{V}^{(n)}$ as the drift. One can choose $V^{(n)}$, $n \geq 1$, (for details on this point see the remark before formula (11) in [10]) in such a way that

$$\lim_{n\uparrow+\infty} \mathbf{ME} \sup_{0 \leq t \leq T} |\mathbf{x}(t; \omega, \sigma) - \mathbf{x}_n(t; \omega, \sigma)|^2 = 0. \tag{4.4}$$

As before we introduce also the process $\eta_f^{(n)}(t)$, $t \geq 0$ corresponding to the trajectories that are the solutions of (1.1) with the drift $\mathbf{V}_f^{(n)}(t, \mathbf{x}) := V^{(n)}(\tau_{\mathbf{x}}(V_f(t)))$, $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$. All the facts stated for $\eta(t)$, $t \geq 0$ hold also for $\eta^{(n)}(t)$, $t \geq 0$. In particular these processes are Markovian with the respective semigroups Q_n^t, $t \geq 0$. These semigroups satisfy $\lim_{n\uparrow+\infty} \|Q^t f - Q_n^t f\|_{L^p_d} = 0$ for any $t \geq 0$ and $f \in L^p$. The generator of the approximate process is given by

$$\mathcal{M}_n F = \Delta F + \mathcal{L}_n F + V^{(n)} \cdot \nabla F \quad \text{ for } \; F \in \mathcal{C}_{\mathcal{L}_n}. \tag{4.5}$$

$\mathcal{C}_{\mathcal{L}_n}$ is a core of $\mathcal{M}_n$ (see [10] Theorem 2 p. 424).
On $\mathcal{C}_{\mathcal{L}} \times \mathcal{C}_{\mathcal{L}}$ we define a non-negative definite bilinear form

$$(f, g)_+ := \mathcal{E}_{\mathcal{L}}(f, g) + \int \nabla f \cdot \nabla g \, d\mu. \tag{4.6}$$

The form is closable and we denote by $\mathcal{H}_+$ the completion of $\mathcal{C}_{\mathcal{L},0} = \mathcal{C}_{\mathcal{L}} \cap L^2_0$ under the norm $\| \cdot \|_+ := (\cdot, \cdot)_+^{1/2}$. It is easy to observe that $\mathcal{H}_+ = W^{2,1} \cap \mathcal{D}(\mathcal{E}_L)$. The scalar product $(\cdot, \cdot)_+$ over $\mathcal{H}_+$ is the Dirichlet form associated with the Markovian process $\xi_t := \tau_{\mathbf{w}(t)} \mathbf{V}(t)$, $t \geq 0$ with

$\mathbf{w}(t)$, $t \geq 0$ a standard d-dimensional Brownian motion independent of $\mathbf{V}(t)$, $t \geq 0$. We denote also by $\mathcal{H}_-^0$ the space of all $F \in L_0^2$ for which

$$\|F\|_- := \sup_{\|G\|_+=1} \int F\, G\, d\mu < +\infty.$$

The completion of $\mathcal{H}_-^0$ in the $\|\cdot\|_-$ norm shall be denoted by $\mathcal{H}_-$.

5 Proof of Theorem 1

5.1 Corrector field and energy identity

Proposition 5 *Under the assumptions of Theorem* 1*, we have $V_p \in \mathcal{H}_-$, for any $p = 1, \cdots, d$.*

Proof. Denote by H_1 the L^2-closure of the space $\mathcal{C}_0 := [F_\varphi : \varphi \in \mathcal{S}_{\rm div},\, \mathbf{0} \notin \text{ supp } \varphi]$ and by Π the corresponding orthogonal projection. Notice that $R^t(\mathcal{C}_0) \subseteq \mathcal{C}_0$ and in consequence

$$R^t \Pi - \Pi R^t = \mathbf{0}, \quad t > 0. \tag{5.1}$$

Let $G_p(F) := (V_p, F)_{L^2}$, $F \in \mathcal{H}_+$. For any $F_\varphi \in \mathcal{C}_0$ we have

$$G_p(F_\varphi) = (V_p, F_\varphi)_{L^2} = \int_{\mathbb{R}^d} \mathcal{E}(|\mathbf{k}|) \left(e_p - \frac{k_p \mathbf{k}}{|\mathbf{k}|^2} \right) \cdot \widehat{\varphi}(\mathbf{k}) |\mathbf{k}|^{1-d}\, d\mathbf{k} \tag{5.2}$$

with $e_p = \underbrace{(0, \cdots, 1, \cdots, 0)}_{\text{p-th position}}$, $p = 1, \cdots, d$. Hence, by the Cauchy inequality

$$\begin{aligned} \left| \int V_p\, F_\varphi\, d\mu \right| &\leq \left\{ \int_{\mathbb{R}^d} \frac{\mathcal{E}(|\mathbf{k}|)}{|\mathbf{k}|^{2\beta}} \frac{d\mathbf{k}}{|\mathbf{k}|^{d-1}} \right\}^{1/2} \left\{ \int_{\mathbb{R}^d} |\mathbf{k}|^{2\beta} \mathcal{E}(|\mathbf{k}|) |\widehat{\varphi}(\mathbf{k})|^2 \frac{d\mathbf{k}}{|\mathbf{k}|^{d-1}} \right\}^{1/2} \\ &\leq c \mathcal{E}_{\mathcal{L}}(F_\varphi, F_\varphi)^{1/2} c \|F_\varphi\|_+. \end{aligned}$$

We have obtained therefore that $G_p(F) \leq c\|F\|_+$, $F \in H_1$. Since $V_p \in H_1$ we have, for an arbitrary $F \in \mathcal{H}_+$,

$$G_p(F) = G_p(\Pi F) \leq c \mathcal{E}_{\mathcal{L}}(\Pi F, \Pi F)^{1/2} \stackrel{(5.1)}{\leq} c \mathcal{E}_{\mathcal{L}}(F, F)^{1/2} \leq c\|F\|_+ . \blacksquare$$

For any $\lambda > 0$ we define the so-called λ-correctors $\chi_\lambda^{(p)} \in D_0(\mathcal{M}) := D(\mathcal{M}) \cap L_0^2$, $p = 1, \cdots, d$ as the unique solutions of the resolvent equations

$$-\mathcal{M}\chi_\lambda^{(p)} + \lambda \chi_\lambda^{(p)} = V_p, \quad p = 1, \cdots, d. \tag{5.3}$$

Proposition 6 $\chi_\lambda^{(p)} \in \mathcal{H}_+$ *for any $p = 1, \cdots, d$ and*

$$\begin{aligned} \mathcal{E}_{\mathcal{L}}(\chi_\lambda^{(p)}, \Phi) + \int \nabla \chi_\lambda^{(p)} \cdot \nabla \Phi\, d\mu + \int V \cdot \nabla \chi_\lambda^{(p)} \Phi\, d\mu + \lambda \int \chi_\lambda^{(p)} \Phi\, d\mu \\ = G_p(\Phi) \quad \forall\, \Phi \in L^\infty \cap \mathcal{H}_+. \end{aligned} \tag{5.4}$$

Proof. Let $V^{(n)}$ be as in Section 4 and let $\chi^{(p)}_{n,\lambda}$ be the λ-corrector for the generator $\mathcal{M}_n$, i.e. $-\mathcal{M}_n\chi^{(p)}_{n,\lambda}+\lambda\chi^{(p)}_{n,\lambda}=V_p$. By virtue of Theorem 2 of [10] we infer that $\chi^{(p)}_{n,\lambda}\in\mathcal{H}_+\cap L^2_0$ and

$$\mathcal{E}(\chi^{(p)}_{n,\lambda},\Phi)+\int V^{(n)}\cdot\nabla\chi^{(p)}_{n,\lambda}\,\Phi\,d\mu+\lambda\int\chi^{(p)}_{n,\lambda}\,\Phi\,d\mu=\int V_p\,\Phi\,d\mu\quad\forall\Phi\in\mathcal{H}_+\cap L^2 \tag{5.5}$$

hence, substituting $\Phi:=\chi^{(p)}_{n,\lambda}$ we deduce that

$$\|\chi^{(p)}_{n,\lambda}\|^2_+ +\lambda\|\chi^{(p)}_{n,\lambda}\|^2_{L^2}=\int\chi^{(p)}_{n,\lambda}\,V_p\,d\mu\le\|G_p\|_-\|\chi^{(p)}_{n,\lambda}\|_+. \tag{5.6}$$

Thus, there exists $C>0$ independent of λ,n such that $\|\chi^{(p)}_{n,\lambda}\|_+\le C$ and the set $\mathcal{A}_p:=\{\chi^{(p)}_{n,\lambda}: 1\ge\lambda>0,\ n\ge 1\}$ is $\mathcal{H}_+$-weakly pre-compact. ¿From the resolvent representation

$$\chi^{(p)}_{\lambda}(f)\ =\ \int_0^{+\infty}\mathrm{e}^{-\lambda t}\mathbf{EM}V_p(\eta_f(t)) \tag{5.7}$$

$$\chi^{(p)}_{n,\lambda}(f)\ =\ \int_0^{+\infty}\mathrm{e}^{-\lambda t}\mathbf{EM}V_{n,p}(\eta^{(n)})_f(t) \tag{5.8}$$

and (4.4) we deduce $\lim_{n\uparrow+\infty}\|\chi^{(p)}_{n,\lambda}-\chi^{(p)}_{\lambda}\|_{L^2}=0$. Therefore $\chi^{(p)}_{\lambda}\in\mathcal{H}_+\cap L^2_0$ and $\chi^{(p)}_{n,\lambda}\rightharpoonup\chi^{(p)}_{\lambda}$, as $n\uparrow+\infty$ $\mathcal{H}_+$-weakly. In addition

$$\|\chi^{(p)}_{\lambda}\|^2_+ +\lambda\|\chi^{(p)}_{\lambda}\|^2_{L^2}\le C \tag{5.9}$$

for some constant $C>0$ independent of $\lambda>0$. Letting $n\uparrow+\infty$ in (5.5) we obtain (5.4). ∎

From (5.9) we know that $\chi^{(p)}_{\lambda}$, $1\ge\lambda>0$ is weakly compact in $\mathcal{H}_+$. Let $\chi^{(p)}_*$ be an $\mathcal{H}_+$-weak limit point of $\chi^{(p)}_{\lambda}$, as $\lambda\downarrow 0$. It satisfies the following equation

$$\mathcal{E}_{\mathcal{L}}(\chi^{(p)}_*,\Phi)+\int\nabla\chi^{(p)}_*\cdot\nabla\Phi\,d\mu+\int V\cdot\nabla\chi^{(p)}_*\Phi\,d\mu \tag{5.10}$$
$$=G_p(\Phi),\quad\forall\,\Phi\in L^\infty\cap\mathcal{H}_+.$$

¿From (5.6) the *energy estimate* follows:

$$\|\chi^{(p)}_*\|^2_+\le G_p(\chi^{(p)}_*).$$

The crucial observation is the following.

Lemma 3 *Let $\chi^{(p)}_*$ be an $\mathcal{H}_+$-weak limit of $\{\chi^{(p)}_{\lambda}\}$. Then*

$$\|\chi^{(p)}_*\|^2_+=G_p(\chi^{(p)}_*)\quad\textit{(energy identity.)} \tag{5.11}$$

Additionally, if $\chi^{(p)}_{1,*}$, $\chi^{(p)}_{2,*}$ are two $\mathcal{H}_+$-weak limiting points of $\chi^{(p)}_{\lambda}$, as $\lambda\downarrow 0$ then

$$\left\|\frac{\chi^{(p)}_{1,*}+\chi^{(p)}_{2,*}}{2}\right\|^2_+=\frac{1}{2}\,G_p(\chi^{(p)}_{1,*}+\chi^{(p)}_{2,*}). \tag{5.12}$$

Proof. Set

$$V = V_{\le l} + V_{>l},$$

with $V_{<l} = (V_{1,\le l}, \cdots, V_{d,\le l})$, $V_{>l} = (V_{1,>l}, \cdots, V_{d,>l})$ where

$$V_{p,\le l} := (\mathbf{I} - \mathcal{Q}(l))V_p \quad \text{and} \quad V_{p,>l} := \mathcal{Q}(l)V_p, \quad p = 1, \cdots d.$$

Let $\chi_{*,>l}^{(p)} := \mathcal{Q}(l)\chi_*^{(p)} \in L_0^2 \cap \mathcal{H}_+$. Thanks to Proposition 1 we have

$$\mathcal{Q}(l)(V \cdot \nabla\chi_*^{(p)}) = \mathcal{Q}(l)(V_{\le l} \cdot \nabla\chi_*^{(p)}) + V_{>l} \cdot \nabla\chi_{*,>l}^{(p)}.$$

Let

$$g_a := f_a(\chi_{*,>l}^{(p)}), \tag{5.13}$$

$$H_a := F_a(\chi_{*,>l}^{(p)}), \tag{5.14}$$

where $f_a(r) := -a^\gamma \vee (r \wedge a^\gamma)$, $a > 0$, $\gamma > 0$ and F_a is the integral of f_a satisfying $F_a(0) = 0$. Substituting into (5.10) g_a for the test function we conclude that

$$\mathcal{E}_{\mathcal{L}}(\chi_*^{(p)}, g_a) + \int \nabla\chi_*^{(p)} \cdot \nabla g_a \, d\mu + \int V_{\le l} \cdot \nabla\chi_*^{(p)} g_a \, d\mu \tag{5.15}$$

$$+ \int V_{>l} \cdot \nabla\chi_*^{(p)} g_a \, d\mu = G_p(g_a) \quad \forall \, a > 0.$$

From the contraction property for Dirichlet forms we have $\|g_a\|_+ \le \|\chi_{*,>l}^{(p)}\|_+$, so the set g_a, $a > 0$ is $\mathcal{H}_+$-weakly pre-compact. To prove that g_a $\mathcal{H}_+$-weakly converges as $a \uparrow +\infty$ we show that for any $F \in W^{1,\infty}$

$$\lim_{a\uparrow+\infty} \int \nabla g_a \cdot \nabla F \, d\mu = \int \nabla\chi_{*,>l}^{(p)} \cdot \nabla F \, d\mu. \tag{5.16}$$

Observe that

$$\nabla g_a = f_a'(\chi_{*,>l}^{(p)})\nabla\chi_{*,>l}^{(p)}$$

and $|f_a'(\chi_{*,>l}^{(p)}) - 1| \le \min\{a^{-\gamma}|\chi_{*,>l}^{(p)}|, \mathbf{1}\}$. We infer therefore that

$$\int \left|\nabla g_a - \nabla\chi_{*,>l}^{(p)}\right| \, d\mu \le 2\|f_a'(\chi_{*,>l}^{(p)}) - 1\|_{L^2}\|\nabla\chi_{*,>l}^{(p)}\|_{L_d^2} \tag{5.17}$$

where $L_d^2 := [F = (F_1, \cdots, F_d) : F_i \in L^2, \, i = 1, \cdots, d]$ with the norm $\|F\|_{L_d^2}^2 := \sum_{i=1}^d \|F_i\|_{L^2}^2$. The right hand side of (5.17) tends to 0, as $a \uparrow +\infty$. Hence we have (5.16) and $g_a \rightharpoonup \chi_{*,>l}^{(p)}$, as $a \uparrow +\infty$, $\mathcal{H}_+$-weakly. As a consequence, we have that

$$\lim_{a\uparrow+\infty} (\chi_{*,>l}^{(p)}, g_a)_+ = \|\chi_{*,>l}^{(p)}\|_+^2. \tag{5.18}$$

The same argument shows also that $\lim_{a\uparrow+\infty} G_p(g_a) = G_p(\chi_{*,>l}^{(p)})$. Notice that $V_{\le l}$ is L^2-orthogonal to any F_φ, with $\varphi \in \mathcal{S}_{\rm div}(l)$ Hence by virtue of the classical Kolmogorov-Rozanov Theorem (see e.g. Theorem 10.1 p. 181 in [13]) $V_{\le l}$ is independent of $\Sigma(l)$. Since $\|g_a\|_{L^2} \le \|\chi^{(p)}_{*,>l}\|_{L^2}$, $a > 0$ we obtain, using independence of $V_{\le l}$ and any g_a, $a > 0$, that

$$\int |V_{\le l}\, g_a|^2\, d\mu = \|V_{\le l}\|^2_{L^2_d}\, \|g_a\|^2_{L^2} \le \|V\|^2_{L^2_d} \|\chi^{(p)}_{*,>l}\|^2_{L^2}.$$

Thus $\{V_{\le l}\, g_a,\, a > 0\}$ is L^2_d-weakly convergent to $V_{\le l}\, \chi^{(p)}_{*,>l}$ and

$$\lim_{a\uparrow+\infty} \int V_{\le l}\cdot\nabla\chi^{(p)}_* g_a\, d\mu = \int V_{\le l}\cdot\nabla\chi^{(p)}_*\chi^{(p)}_{*,>l}\, d\mu.$$

The last term on the left hand side of (5.15) equals

$$\int V_{>l}\cdot\nabla\chi^{(p)}_{*,>l}\, g_a\, d\mu = \int V_{>l}\cdot\nabla H_a\, d\mu = \int \nabla\cdot(V_{>l} H_a)\, d\mu = 0,$$

with H_a given by (5.14). Here we used the fact that $\nabla\cdot V_{>l} = 0$. Thus, we have proved that

$$(\chi^{(p)}_*, \chi^{(p)}_{*,>l})_+ + \int \nabla\chi^{(p)}_*\cdot\nabla\chi^{(p)}_{*,>l}\, d\mu + \int V_{\le l}\cdot\nabla\chi^{(p)}_*\chi^{(p)}_{*,>l}\, d\mu \tag{5.19}$$

$$= G_p(\chi^{(p)}_{*,>l}).$$

Observe that the integrand appearing in the last term on the left hand side of (5.19) belongs to L^1 and

$$\begin{aligned}\int \left|\mathbf{V}_{\le l}\cdot\nabla\chi^{(p)}_*\, \mathcal{Q}(l)\chi^{(p)}_*\right| d\mu \quad &\le \quad \left(\int |\mathbf{V}_{\le l}|^2 \left|\mathcal{Q}(l)\chi^{(p)}_*\right|^2 d\mu\right)^{1/2} \|\nabla\chi^{(p)}_*\|_{L^2_d} \\ &= \quad \|\mathbf{V}_{\le l}\|_{L^2_d}\, \|\mathcal{Q}(l)\chi^{(p)}_*\|_{L^2}\, \|\nabla\chi^{(p)}_*\|_{L^2_d}\end{aligned} \tag{5.20}$$

Here we have used the fact that $V_{\le l}$ and $\mathcal{Q}(l)\chi^{(p)}_*$ are independent. A elementary calculation shows that $\|V_{\le l}\|_{L^2_d} \sim l^{1-\alpha}$, $l \ll 1$. On the other hand, Lemma 1 implies that $\|\mathcal{Q}(l)\chi^{(p)}_*\|_{L^2} \sim o(1) l^{-\beta}$, $l \ll 1$ so the right hand side of (5.20) is of the order of magnitude $\sim o(1) l^{1-\alpha-\beta}$ as $l \ll 1$. By Proposition 3,

$$\|\chi^{(p)}_{*,>l}\|_+ \le \|\chi^{(p)}_*\|_+ \quad \forall\, l > 0 \tag{5.21}$$

so the set $\{\chi^{(p)}_{*,>l}\, l > 0\}$ is $\mathcal{H}_+$-weakly pre-compact. Since

$$\lim_{l\downarrow 0} (\chi^{(p)}_{*,>l}, F)_{L^2} = (\chi^{(p)}_{*,>l_0}, F)_{L^2} \quad \forall\, F \in \mathcal{H}(l_0) \text{ and any } l_0 > 0$$

$\chi^{(p)}_{*,>l} \rightharpoonup \chi^{(p)}_*$ $\mathcal{H}_+$-weakly as $l \downarrow 0$. Letting $l \downarrow 0$ in (5.19) we deduce (5.11). To show (5.12) we note that $1/2(\chi^{(p)}_{1,*} + \chi^{(p)}_{2,*})$ also satisfies (5.10) and (5.12) follows after a repetition of the preceding argument. ∎

From Lemma 3 and (5.6) we conclude immediately the following.

Corollary 2 *For any $p = 1, \cdots, d$ we have*

$$\lim_{\lambda\downarrow 0} \lambda \|\chi^{(p)}_\lambda\|^2_{L^2} = 0. \tag{5.22}$$

In addition $\chi^{(p)}_\lambda$ converges $\mathcal{H}_+$-strongly, as $\lambda \downarrow 0$.

5.2 Convergence of finite dimensional distributions

According to (5.3)

$$\varepsilon x_p\left(\frac{t}{\varepsilon^2}\right)=\sqrt{2}\varepsilon w_p\left(\frac{t}{\varepsilon^2}\right)+\varepsilon\int\limits_0^{t/\varepsilon^2}V_p(\eta(s))ds=R_\varepsilon(t)+\varepsilon N_\varepsilon^{(p)}\left(\frac{t}{\varepsilon^2}\right), \tag{5.23}$$

where

$$N_\varepsilon^{(p)}(t):=\sqrt{2}w_p(t)+\chi_{\varepsilon^2}^{(p)}(\eta(t))-\chi_{\varepsilon^2}^{(p)}(\eta(0))-\int\limits_0^t\mathcal{M}\chi_{\varepsilon^2}^{(p)}(\eta(s))ds \tag{5.24}$$

is a square integrable martingale with respect to the filtration $\mathcal{Z}_t$, $t\geq 0$ relative to $\eta(t)$, $t\geq 0$. By $N_\varepsilon(t):=(N_\varepsilon^{(1)}(t),\cdots,N_\varepsilon^{(d)}(t))$, $t\geq 0$ we denote the respective $\mathbb{R}^d$-valued martingale. The remainder

$$R_\varepsilon^{(p)}(t):=\varepsilon^3\int\limits_0^{t/\varepsilon^2}\chi_{\varepsilon^2}^{(p)}(\eta(s))\,ds+\varepsilon\chi_{\varepsilon^2}^{(p)}(\eta(0))-\varepsilon\chi_{\varepsilon^2}^{(p)}\left(\eta\left(\frac{t}{\varepsilon^2}\right)\right) \tag{5.25}$$

satisfies

$$\mathbf{ME}|R_\varepsilon^{(p)}(t)|\leq\varepsilon^3\int\limits_0^{t/\varepsilon^2}\mathbf{ME}|\chi_{\varepsilon^2}^{(p)}(\eta(s))|\,ds+2\varepsilon\|\chi_{\varepsilon^2}^{(p)}\|_{L^2}\leq\varepsilon\|\chi_{\varepsilon^2}^{(p)}\|_{L^2}(t+2). \tag{5.26}$$

The right member of (5.26) tends to 0 as $\varepsilon\downarrow 0$ by virtue of Lemma 2. A standard calculation shows that, for any $\varepsilon_1,\varepsilon_2>0$,

$$\mathbf{ME}|N_{\varepsilon_1}^{(p)}(t)-N_{\varepsilon_2}^{(p)}(t)|^2=-2\,t\,(\mathcal{M}(\chi_{\varepsilon_1^2}^{(p)}-\chi_{\varepsilon_2^2}^{(p)}),\chi_{\varepsilon_1^2}^{(p)}-\chi_{\varepsilon_2^2}^{(p)})_{L^2}.$$

We claim that, in fact

$$-(\mathcal{M}(\chi_{\varepsilon_1^2}^{(p)}-\chi_{\varepsilon_2^2}^{(p)}),\chi_{\varepsilon_1^2}^{(p)}-\chi_{\varepsilon_2^2}^{(p)})_{L^2}=\|\chi_{\varepsilon_1^2}^{(p)}-\chi_{\varepsilon_2^2}^{(p)}\|_+^2. \tag{5.27}$$

Accepting this claim, its proof shall be presented momentarily, we conclude that

$$\mathbf{ME}|N_{\varepsilon_1}^{(p)}(t)-N_{\varepsilon_2}^{(p)}(t)|^2=2\,t\|\chi_{\varepsilon_1^2}^{(p)}-\chi_{\varepsilon_2^2}^{(p)}\|_+^2. \tag{5.28}$$

By virtue of the second part of Corollary 2, (5.28) and Kolmogorov's inequality for martingales we deduce immediately that for an arbitrary $\varrho>0$ we can find $\varepsilon_0>0$ such that for $0<\varepsilon<\varepsilon_0$

$$\mathbf{ME}\sup_{0\leq t\leq T}\left[\varepsilon N_\varepsilon^{(p)}\left(\frac{t}{\varepsilon^2}\right)-\varepsilon N_{\varepsilon_0}^{(p)}\left(\frac{t}{\varepsilon^2}\right)\right]^2\leq C\|\chi_{\varepsilon^2}^{(p)}-\chi_{\varepsilon_0^2}^{(p)}\|_+^2T<\varrho T,\,\forall\,T\geq 0 \tag{5.29}$$

for some constant $C>0$.

Stationarity and ergodicity of $\eta(t)$, $t\geq 0$ implies that $N_{\varepsilon_0}(t)$, $t\geq 0$ is a martingale with stationary and ergodic increments The classical martingale central limit theorem of Billingsley (see [3], Theorem 23.1, p. 206) implies that finite dimensional distributions of $\varepsilon N_{\varepsilon_0}(t/\varepsilon^2)$, $t\geq 0$ tend weakly, as $\varepsilon \downarrow 0$, to those of a Brownian Motion whose co-variance equals $2\mathbf{D}(\varepsilon_0) := 2[D_{p,q}(\varepsilon_0)]$ with

$$D_{p,q}(\varepsilon_0) = (\chi^{(p)}_{\varepsilon_0^2}, \chi^{(q)}_{\varepsilon_0^2})_+ + \delta_{p,q}.$$

Passing to the limit $\varepsilon_0 \downarrow 0$ we conclude that the f.d.d. of $\mathbf{x}_\varepsilon(t)$, $t \ge 0$, as $\varepsilon \downarrow 0$, converges to the Wiener measure with the co-variance matrix given by $2\mathbf{D} = 2[D_{p,q}]$, where

$$D_{p,q} := (\chi^{(p)}_*, \chi^{(q)}_*)_+ + \delta_{p,q}. \tag{5.30}$$

Proof of (5.27). Using (4.5) we conclude that

$$-(\mathcal{M}_n(\chi^{(p)}_{n,\varepsilon_1^2} - \chi^{(p)}_{n,\varepsilon_2^2}), \chi^{(p)}_{n,\varepsilon_1^2} - \chi^{(p)}_{n,\varepsilon_2^2})_{L^2} = \|\chi^{(p)}_{n,\varepsilon_1^2} - \chi^{(p)}_{n,\varepsilon_2^2}\|_+^2. \tag{5.31}$$

For an arbitrary $\lambda > 0$ the correctors $\chi^{(p)}_{n,\lambda}$ converge both strongly in L^2 and weakly in $\mathcal{H}_+$, as $n \uparrow +\infty$. In consequence $\chi^{(p)}_\lambda$ satisfies (5.5) with $\mathbf{V}$ in place of $\mathbf{V}_n$ and test functions $\Phi \in \mathcal{H}_+ \cap L^\infty$. Choosing $f_a(\chi^{(p)}_\lambda)$ as the test functions and letting $a \uparrow +\infty$ we conclude $\|\chi^{(p)}_\lambda\|_+^2 + \lambda\|\chi^{(p)}_\lambda\|_{L^2}^2 = (V_p, \chi^{(p)}_\lambda)_{L^2}$, which in turn implies that

$$\lim_{n\uparrow+\infty} [\|\chi^{(p)}_{n,\lambda} - \chi^{(p)}_\lambda\|_{L^2} + \|\chi^{(p)}_{n,\lambda} - \chi^{(p)}_\lambda\|_+] = 0. \tag{5.32}$$

Since

$$(\mathcal{M}(\chi^{(p)}_{\varepsilon_1^2} - \chi^{(p)}_{\varepsilon_2^2}), \chi^{(p)}_{\varepsilon_1^2} - \chi^{(p)}_{\varepsilon_2^2})_{L^2} = (\varepsilon_1^2\chi^{(p)}_{\varepsilon_1^2} - \varepsilon_2^2\chi^{(p)}_{\varepsilon_2^2}, \chi^{(p)}_{\varepsilon_1^2} - \chi^{(p)}_{\varepsilon_2^2})_{L^2} \tag{5.33}$$

we conclude, thanks to (5.32) that the right hand side of (5.33) equals

$$\lim_{n\uparrow+\infty} (\varepsilon_1^2\chi^{(p)}_{n,\varepsilon_1^2} - \varepsilon_2^2\chi^{(p)}_{n,\varepsilon_2^2}, \chi^{(p)}_{n,\varepsilon_1^2} - \chi^{(p)}_{n,\varepsilon_2^2})_{L^2}$$

$$= \lim_{n\uparrow+\infty} (\mathcal{M}_n(\chi^{(p)}_{n,\varepsilon_1^2} - \chi^{(p)}_{n,\varepsilon_2^2}), \chi^{(p)}_{n,\varepsilon_1^2} - \chi^{(p)}_{n,\varepsilon_2^2})_{L^2} \stackrel{(5.31)}{=}$$

$$\lim_{n\uparrow+\infty} \|\chi^{(p)}_{n,\varepsilon_1^2} - \chi^{(p)}_{n,\varepsilon_2^2}\|_+^2 = \|\chi^{(p)}_{\varepsilon_1^2} - \chi^{(p)}_{\varepsilon_2^2}\|_+^2.$$

That, in consequence, validates our claim (5.27)

5.3 Tightness

The proof of tightness uses the forward-backward martingale decomposition of additive functionals of Markov processes (see e.g. [14]). Since $\mathcal{C} := \bigcap_{m\ge1} W^{\infty,m} \cap D(\mathcal{L})$ is a core of the generator $\Delta + \mathcal{L}$ (see [10], Theorem 2, p. 424), for any $\sigma > 0$ one can find $u_p \in \mathcal{C}$ such that $\|(\Delta + \mathcal{L})u_p - V_p\|_- < \sigma$. Exactly the same argument as used in the proof of formula (3.11) in [14] leads to the estimate

$$\mathbf{ME}\left\{\varepsilon^2 \sup_{0\le t\le T} \left| \int_0^{t/\varepsilon^2} [V_p(\eta(s)) - (\Delta + \mathcal{L})u_p(\eta(s))]\, ds \right|^2 \right\} \tag{5.34}$$

$$\leq 14T\|V_p - (\Delta + \mathcal{L})u_p\|_-^2 < 14T\sigma^2.$$

According to [10], Theorem 2, $u_p \in D(\mathcal{M})$ and $\mathcal{M}u_p = \Delta u_p + \mathcal{L}u_p + V \cdot \nabla u_p$. It is easy to verify that also $u_p \in D(\mathcal{M}^*)$ and $\mathcal{M}^* u_p = \mathcal{L}u_p + \Delta u_p - V \cdot \nabla u_p$. We conclude therefore that

$$\varepsilon \int_0^{t/\varepsilon^2} (\Delta + \mathcal{L})u_p(\eta(s))\, ds = \varepsilon N_{u_p}^{\leftarrow}(t/\varepsilon^2) + \varepsilon N_{u_p}^{\rightarrow}(t/\varepsilon^2)$$

where

$$N_{u_p}^{\rightarrow}(t) := u_p(\eta(t)) - u_p(\eta(0)) - \int_0^t \mathcal{M}u_p(\eta(s))\, ds, \quad t \geq 0$$

is a (continuous trajectory) martingale with respect to the standard filtration $\mathcal{Z}_t$, $t \geq 0$ and

$$N_{u_p}^{\leftarrow}(t) := u_p(\eta(0)) - u_p(\eta(t)) - \int_0^t \mathcal{M}^* u_p(\eta(s))\, ds, \quad t \geq 0$$

is a backward (continuous trajectory) martingale. By this we mean that $N_{u_p}^{\leftarrow}(T) - N_{u_p}^{\leftarrow}(T-t)$ is a $\mathcal{G}_{T-t}$, $0 \leq t \leq T$ martingale for any $T > 0$, where $\mathcal{G}_t$ the σ-algebra generated by $\eta(s)$, $s \geq t$. By virtue of [3], Theorem 23.1, p. 206, both $\varepsilon N_{u_p}^{\rightarrow}([t/\varepsilon^2])$, $t \geq 0$ and $\varepsilon N_{u_p}^{\leftarrow}([t/\varepsilon^2])$, $t \geq 0$, $1 > \varepsilon > 0$ are tight in $D[0,T]$, for any $T > 0$, $p = 1, \cdots, d$. Let $X_n := \sup_{n-1 \leq t \leq n} |N_{u_p}^{\rightarrow}(t) - N_{u_p}^{\rightarrow}(n-1)|^2$. The sequence X_n, $n \geq 1$ is stationary with $\mathbf{ME}X_1 = \|u_p\|_-^2 < +\infty$. By virtue of the Mean Ergodic Theorem we have that $1/N \max\{X_1, \cdots, X_N\} \to 0$ as $N \to +\infty$ in the L^1 sense, thus,

$$\lim_{\varepsilon \downarrow 0} \mathbf{ME} \sup_{0 \leq t \leq T} \varepsilon^2 \left| N_{u_p}^{\rightarrow}(t/\varepsilon^2) - N_{u_p}^{\rightarrow}([t/\varepsilon^2]) \right|^2 = 0. \tag{5.35}$$

Therefore $\varepsilon N_{u_p}^{\rightarrow}(t/\varepsilon^2)$, $t \geq 0$,$1 > \varepsilon > 0$ is tight in $D[0,T]$ for any $T > 0$, $p = 1, \cdots, d$. The same holds for $\varepsilon N_{u_p}^{\leftarrow}(t/\varepsilon^2)$, $t \geq 0$,$1 > \varepsilon > 0$. This together with (5.34) yield tightness of $\mathbf{x}_\varepsilon(t)$, $t \geq 0$, $1 > \varepsilon > 0$ in $D([0,T]; \mathbb{R}^d)$. Continuity of the trajectories implies tightness in $C([0,+\infty); \mathbb{R}^d)$. ∎

6 Proof of Theorem 2

Now we turn to the proof of Theorem 2 for the limit of vanishing molecular diffusivity $D_0 \to 0$. With the molecular diffusivity D_0, the expression (5.30) for the effective diffusivity $\mathbf{D}$ becomes

$$D_{p,q} := (\chi_*^{(p)}(D_0), \chi_*^{(q)}(D_0))_+ + D_0 \delta_{p,q}, \tag{6.1}$$

where the bilinear form becomes

$$(f,g)_+ := \mathcal{E}_{\mathcal{L}}(f,g) + D_0 \int \nabla f \cdot \nabla g \, d\mu \tag{6.2}$$

and $\chi_*^{(p)}(D_0)$ is the corrector field corresponding to D_0. Following exactly the same argument the uniform estimate (5.9) is valid for some constant C independent of λ and D_0.

Passing to the limit $\lambda \to 0$, we have

$$\|\chi_*^{(p)}(D_0)\|_+ \le C \tag{6.3}$$

which, together with (6.1), implies the upper bound stated in Theorem 2.

For the lower bound, we turn to Eq. (5.10), which after some elementary approximation argument can be written in the form

$$\mathcal{E}_{\mathcal{L}}(\chi_*^{(p)}(D_0), V_p) + D_0 \int \nabla \chi_*^{(p)}(D_0) \cdot \nabla V_p \, d\mu + \int V \cdot \nabla \chi_*^{(p)}(D_0) V_p \, d\mu = G_p(V_p). \tag{6.4}$$

We want to show that, if the infimum of $\mathbf{D}$ is zero as D_0 tends to zero, then the entire left side of (6.4) drops out in the limit while the right side equals $\|V_p\|_{L^2}^2 > 0$, thus leading to contradiction.

Let us assume the infimum of $\mathbf{D}$ as $D_0 \to 0$ is zero and take an infimum-achieving subsequence of $\chi_*^{(p)}(D_0), p = 1, ..., d$. For that sequence we shall have

$$\sum_{p=1}^{d} \mathcal{E}_{\mathcal{L}}(\chi_*^{(p)}(D_0), \chi_*^{(p)}(D_0)) \to 0. \tag{6.5}$$

By (6.3), the Cauchy inequality and the assumption, both the first and second term on the left side of equation (6.4) vanishes.

Let us denote $W_p := V \cdot \nabla V_p$ and

$$\|W_p\|_{H_-(\mathcal{L})} := \sup_{\mathcal{E}_{\mathcal{L}}(F,F)=1} \int W_p \, F \, d\mu.$$

Then

$$\begin{aligned} \|W_p\|_{H_-(\mathcal{L})}^2 &= \int_0^{+\infty} (R^t \, W_p, W_p)_{L^2} \, dt \\ &= \int_0^{+\infty} \mathbf{E} \left[V(t) \cdot \nabla V_p(t) \, V(0) \cdot \nabla V_p(0) \right]. \end{aligned} \tag{6.6}$$

The right side of (6.6) can be explicitly calculated using Feynman diagrams and the result is

$$\sum_{p=1}^{d} \|W_p\|_{H_-(\mathcal{L})}^2$$

$$= d \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{|\mathbf{k}|^{2\beta} + |\mathbf{k}'|^{2\beta}} \times \frac{\mathcal{E}(|\mathbf{k}|)\mathcal{E}(|\mathbf{k}'|)}{|\mathbf{k}|^{d-1}|\mathbf{k}'|^{d-1}} \left\{ |\mathbf{k}'|^2 - \frac{(\mathbf{k} \cdot \mathbf{k}')^2}{|\mathbf{k}|^2} \right\} d\mathbf{k} \, d\mathbf{k}' < +\infty$$

for $\alpha + \beta < 1$. The third term on the left hand side of (6.4) can be therefore bounded by

$$\|W_p\|_{-} \mathcal{E}_{\mathcal{L}}^{1/2}(\chi_*^{(p)}(D_0), \chi_*^{(p)}(D_0)), \; p = 1, \cdots, d,$$

which vanishes by (6.5). Theorem 2 is proved.

References

[1] R. J. Adler, *Geometry of Random Fields*, Wiley, New York (1981)

[2] M. Avellaneda and A.J. Majda, *Mathematical models with exact renormalization for turbulent transport. Commun. Math. Phys.*, 131, 381-429 (1990).

[3] P. Billingsley, *Convergence of Probability Measures*, Wiley, New York (1968).

[4] Da Prato, G. and Zabczyk, J., *Ergodicity for Infinite Dimensional Systems*, Cambridge Univ. Press, Cambridge (1996).

[5] A. Fannjiang and T. Komorowski, *Turbulent diffusion in Markovian flows*, Ann. Appl. Prob. 9 (1999) 591-610.

[6] A. Fannjiang and T. Komorowski, *Diffusive and non-diffusive limits of transport in non-mixing flows with power-law spectra.* SIAM J. Appl. Math. 62:3(2002), 909-923.

[7] A. Fannjiang, T. Komorowski and S.Peszat, *Lagrangian dynamics for a passive tracer in a class of Gaussian Markovian flows.* To appear, Stoch. Proc. Appl.

[8] A. Fannjiang and G. Papanicolaou, *Diffusion in turbulence. Probab. Th. Rel. Fields* 105 (1996), 279-334.

[9] S. Janson, *Gaussian Hilbert Spaces*, Cambridge Tracts in Math. 129, Cambridge Univ. Press, 1997.

[10] T. Komorowski, *An abstract Lagrangian process related to convection-diffusion of a passive tracer in a Markovian flow*, *Bull. Pol. Ac. Sci.* 48 (2000), 413-427.

[11] L. Koralov, *Transport by time dependent stationary flow. Comm. Math. Phys.* 199:3 (1999), 649–681.

[12] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. II*, Academic Press, New York, London, 1978.

[13] Yu. A. Rozanov, *Stationary Random Processes*, Holden-Day, San Fransisco, Cambridge, London, Amsterdam 1967.

[14] L. Wu, *Forward-backward martingale decomposition and compactness results for additive functionals of stationary ergodic Markov processes. Ann. Inst. Henri Poincare* 35 (1999), 121-141.