Electronic Journal of Probability

Vol. 14 (2009), Paper no. 42, pages 1198–1221.

Journal URL
http://www.math.washington.edu/~ejpecp/

# Expansions For Gaussian Processes And Parseval Frames

Harald Luschgy
Universität Trier, FB IV-Mathematik, D-54286 Trier, Germany.
E-mail: luschgy@uni-trier.de

Gilles Pagès
Laboratoire de Probabilités et Modèles aléatoires
UMR 7599, Université Paris 6, case 188
4, pl. Jussieu, F-75252 Paris Cedex 5.
E-mail:gilles.pages@upmc.fr

**Abstract**

We derive a precise link between series expansions of Gaussian random vectors in a Banach space and Parseval frames in their reproducing kernel Hilbert space. The results are applied to pathwise continuous Gaussian processes and a new optimal expansion for fractional Ornstein-Uhlenbeck processes is derived.

**Key words:** Gaussian process, series expansion, Parseval frame, optimal expansion, fractional Ornstein-Uhlenbeck process.

**AMS 2000 Subject Classification:** Primary 60G15, 42C15.

Submitted to EJP on May 28, 2008, final version accepted February 23, 2009.

# 1 Introduction

Series expansions is a classical issue in the theory of Gaussian measures (see [2], [9], [17]). Our motivation for a new look on this issue finds its origin in recent new expansions for fractional Brownian motions (see [14], [1], [5], [6], [7]).

Let $(E, ||\cdot||)$ be a real separable Banach space and let $X : (\Omega, A, P) \to E$ be a centered Gaussian random vector with distribution $P_X$. In this article we are interested in series expansions of $X$ of the following type. Let $\xi_1, \xi_2, \dots$ be i.i.d. $N(0,1)$-distributed real random variables. A sequence $(f_j)_{j\geq 1} \in E^{\mathbb{N}}$ is called *admissible* for $X$ if

$$\sum_{j=1}^{\infty} \xi_j f_j \text{ converges a.s. in } E \tag{1.1}$$

and

$$X \stackrel{d}{=} \sum_{j=1}^{\infty} \xi_j f_j. \tag{1.2}$$

By adding zeros finite sequences in $E$ may be turned into infinite sequences and thus also serve as admissible sequences.

We observe a precise link to frames in Hilbert spaces. A sequence $(f_j)_{j\geq 1}$ in a real separable Hilbert space $(H, (\cdot,\cdot))$ is called *Parseval frame* for $H$ if $\sum_{j=1}^{\infty} (f_j, h) f_j$ converges in $H$ and

$$\sum_{j=1}^{\infty} (f_j, h) f_j = h \tag{1.3}$$

for every $h \in H$. Again by adding zeros, finite sequences in $H$ may also serve as frames. For the background on frames the reader is referred to [4]. (Parseval frames correspond to tight frames with frame bounds equal to 1 in [4].)

**Theorem 1.** *Let $(f_j)_{j\geq 1} \in E^{\mathbb{N}}$. Then $(f_j)$ is admissible for $X$ if and only if $(f_j)$ is a Parseval frame for the reproducing kernel Hilbert space of $X$.*

We thus demonstrate that the right notion of a "basis" in connection with expansions of $X$ is a Parseval frame and not an orthonormal basis for the reproducing kernel Hilbert space of $X$. The first notion provides the possibility of redundancy and is more flexible as can be seen $e.g.$ from wavelet frames. It also reflects the fact that "sums" of two (or more) suitable scaled expansions of $X$ yield an expansion of $X$.

The paper is organized as follows. In Section 2 we investigate the general Banach space setting in the light of frame theory and provide the proof of Theorem 1. Section 3 contains applications to pathwise continuous processes $X = (X_t)_{t\in I}$ viewed as $C(I)$-valued random vectors where $I$ is a compact metric space. >Furthermore, we comment on optimal expansions. Fractional Brownian motions serve as illustration. Section 4 contains a new optimal expansion for fractional Ornstein-Uhlenbeck processes.

It is convenient to use the symbols $\sim$ and $\approx$ where $a_n \sim b_n$ means $a_n/b_n \to 1$ and $a_n \approx b_n$ means $0 < \liminf a_n/b_n \leq \limsup a_n/b_n < \infty$.

## 2 The Banach space setting

Let $(E, \|\cdot\|)$ be a real separable Banach space. For $u \in E^*$ and $x \in E$, it is convenient to write

$$\langle u, x \rangle$$

in place of $u(x)$. Let $X : (\Omega, \mathscr{A}, P) \to E$ be a centered Gaussian random vector with distribution $P_X$. The covariance operator $C = C_X$ of $X$ is defined by

$$C : E^* \to E,\ Cu := E\langle u, X\rangle X. \tag{2.1}$$

This operator is linear and (norm-)continuous. Let $H = H_X$ denote the reproducing kernel Hilbert space (Cameron Martin space) of the symmetric nonnegative definite kernel

$$E^* \times E^* \to R, (u, v) \mapsto \langle u, Cv \rangle$$

(see [17], Propositions III.1.6. and III.1.7). Then $H$ is a Hilbert subspace of $E$, that is $H \subset E$ and the inclusion map is continuous. The reproducing property reads

$$(h, Cu)_H = \langle u, h \rangle, u \in E^*, h \in H \tag{2.2}$$

where $(\cdot,\cdot)_H$ denotes the scalar product on $H$ and the corresponding norm is given by

$$\|h\|_H = \sup\{|\ \langle u, h\rangle\ | : u \in E^*, \langle u, Cu\rangle \le 1\}. \tag{2.3}$$

In particular, for $h \in H$,

$$\|h\| \le \sup_{\|u\|\le 1} \langle u, Cu\rangle^{1/2} \|h\|_H = \|C\|^{1/2}\|h\|_H. \tag{2.4}$$

The $\|\cdot\|_H$-closure of $A \subset H$ is denoted by $\overline{A}^{(H)}$. Furthermore, $H$ is separable, $C(E^*)$ is dense in $(H, \|\cdot\|_H)$, the unit ball

$$U_H := \{h \in H : \|h\|_H \le 1\}$$

of $H$ is a compact subset of $E$,

$$\text{supp}(P_X) = (\ker C)^\perp := \{x \in E :\ \langle u, x\rangle = 0 \text{ for every } u \in \ \ker C\} = \overline{H} \text{ in } E$$

and

$$H = \{x \in E :\ \|x\|_H < \infty\} \tag{2.5}$$

where $\|x\|_H$ is formally defined by (2.3) for every $x \in E$. As for the latter fact, it is clear that $\|h\|_H < \infty$ for $h \in H$. Conversely, let $x \in E$ with $\|x\|_H < \infty$. Observe first that $x \in \overline{H}$. Otherwise, by the Hahn-Banach theorem, there exists $u \in E^*$ such that $u|\overline{H} = 0$ and $\langle u, x\rangle > 0$. Since, $\langle u, Cu\rangle = 0$ this yields

$$\|x\|_H \ge \sup_{a>0} a\langle u, x\rangle = \infty,$$

a contradiction. Now consider $C(E^*)$ as a subspace of $(H, \|\cdot\|_H)$ and define $\varphi : C(E^*) \to R$ by $\varphi(Cu) := \langle u, x\rangle$. If $Cu_1 = Cu_2$, then using $(\ker C)^\perp = \overline{H}, \langle u_1 - u_2, x\rangle = 0$. Therefore, $\varphi$ is well defined. The map $\varphi$ is obviously linear and it is bounded since

$$\|\varphi\| = \sup\{|\varphi(Cu)| : u \in E^*, \|Cu\|_H \le 1\} = \|x\|_H < \infty$$

by (2.2). By the Hahn-Banach theorem there exists a linear bounded extension $\tilde{\varphi}:\overline{C(E^*)}^{(H)}\to R$ of $\varphi$. Then, since $\overline{C(E^*)}^{(H)}=H$, by the Riesz theorem there exists $g\in H$ such that $\tilde{\varphi}(h)=(h,g)_H$ for every $h\in H$. Consequently, using (2.2),

$$\langle u,x\rangle=\varphi(Cu)=(Cu,g)_H=\langle u,g\rangle$$

for every $u\in E^*$ which gives $x=g\in H$.

The key is the following characterization of admissibility. It relies on the Ito-Nisio theorem. Condition $(v)$ is an abstract version of Mercer's theorem (cf. [15]. p. 43). Recall that a subset $G\subset E^*$ is said to be separating if for every $x,y\in E, x\neq y$ there exists $u\in G$ such that $\langle u,x\rangle\neq\langle u,y\rangle$.

**Lemma 1.** *Let $(f_j)_{j\geq 1}\in E^{\mathbb{N}}$. The following assertions are equivalent.*

$(i)$ *The sequence $(f_j)_{j\geq 1}$ is admissible for $X$.*

$(ii)$ *There is a separating linear subspace $G$ of $E^*$ such that for every $u\in G$,*

$$(\langle u,f_j\rangle)_{j\geq 1} \text{ is admissible for } \langle u,X\rangle.$$

$(iii)$ *There is a separating linear subspace $G$ of $E^*$ such that for every $u\in G$,*

$$\sum_{j=1}^{\infty}\langle u,f_j\rangle^2=\langle u,Cu\rangle.$$

$(iv)$ *For every $u\in E^*$,*

$$\sum_{j=1}^{\infty}\langle u,f_j\rangle f_j=Cu.$$

$(v)$ *For every $a>0$,*

$$\sum_{j=1}^{\infty}\langle u,f_j\rangle\langle v,f_j\rangle=\langle u,Cv\rangle$$

*uniformly in $u,v\in\{y\in E^*:\|y\|\leq a\}$.*

**Proof.** Set $X_n:=\sum_{j=1}^{n}\xi_j f_j$. (i) $\Rightarrow$ (v). $X_n$ converges a.s. in $E$ to some $E$-valued random vector $Y$, say, with $X\stackrel{d}{=}Y$. It is well known that this implies $X_n\to Y$ in $L_E^2$. Therefore,

$$|\sum_{j=1}^{n}\langle u,f_j\rangle\langle v,f_j\rangle-\langle u,Cv\rangle|=|E\langle u,X_n\rangle\langle v,X_n\rangle-E\langle u,Y\rangle\langle v,Y\rangle|$$

$$=|E\langle u,Y-X_n\rangle\langle v,Y-X_n\rangle|\leq a^2E\|Y-X_n\|^2\to 0 \quad \text{as} \quad n\to\infty$$

uniformly in $u,v\in\{y\in E^*:\|y\|\leq a\}$. (v) $\Rightarrow$ (iv) $\Rightarrow$ (iii) is obvious. (iii) $\Rightarrow$ (i). For every $u\in G$,

$$E\exp(i\langle u,X_n\rangle)=\exp(-\sum_{j=1}^{n}\langle u,f_j\rangle^2/2)\to\exp(-\langle u,Cu\rangle/2)=E\exp(i\langle u,X\rangle).$$

The assertion (i) follows from the Ito-Nisio theorem (cf. [17], p. 271). (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is obvious. $\square$

Note that the preceding lemma shows in particular that $(f_j)_{j\geq 1}$ is admissible for $X$ if and only if $(f_{\sigma(j)})_{j\geq 1}$ is admissible for $X$ for (some) every permutation $\sigma$ of $N$ so that $\sum \xi_j f_j$ converges unconditionally a.s. in $E$ for such sequences and all the $a.s.$ limits under permuations of $N$ have distribution $P_X$.

It is also an immediate consequence of Lemma 1(v) that admissible sequences $(f_j)$ satisfy $\|f_j\| \to 0$ since by the Cauchy criterion, $\lim_{j\to\infty} \sup_{||u||\leq 1} \langle u, f_j\rangle^2 = 0$.

The corresponding lemma for Parseval frames reads as follows.

**Lemma 2.** *Let $(f_j)_{j\geq 1}$ be a sequence in a real separable Hilbert space $(K, (\cdot,\cdot)_K)$. The following assertions are equivalent.*

*(i) The sequence $(f_j)$ is a Parseval frame for $K$.*

*(ii) For every $k \in K$,*

$$\lim_{n\to\infty} || \sum_{j=1}^{n} (k, f_j)_K f_j ||_K = ||k||_K.$$

*(iii) There is a dense subset $G$ of $K$ such that for every $k \in G$,*

$$\sum_{j=1}^{\infty} (k, f_j)_K^2 = ||k||_K^2.$$

*(iv) For every $k \in K$,*

$$\sum_{j=1}^{\infty} (k, f_j)_K^2 = ||k||_K^2.$$

**Proof.** (i) $\Rightarrow$ (ii) is obvious. (ii) $\Rightarrow$ (iv). For every $k \in K, n \in N$,

$$\begin{aligned} 0 &\leq || \sum_{j=1}^{n} (k, f_j)_K f_j - k||_K^2 \\ &= || \sum_{j=1}^{n} (k, f_j)_K f_j||_K^2 - 2 \sum_{j=1}^{n} (k, f_j)_K^2 + ||k||_K^2 \end{aligned}$$

so that

$$2 \sum_{j=1}^{n} (k, f_j)_K^2 \leq || \sum_{j=1}^{n} (k, f_j)_K f_j||_K^2 + ||k||_K^2.$$

Hence

$$\sum_{j=1}^{\infty} (k, f_j)_K^2 \leq ||k||_K^2.$$

Using this inequality we obtain conversely for $k \in K, n \in N$

$$\begin{aligned}
||\sum_{j=1}^{n}(k,f_j)_K f_j||_K^2 &= \sup_{||g||_K \le 1}(g, \sum_{j=1}^{n}(k,f_j)_K f_j)_K^2 \\
&= \sup_{||g||_K \le 1}(\sum_{j=1}^{n}(k,f_j)_K (g,f_j)_K)^2 \\
&\le \sum_{j=1}^{n}(k,f_j)_K^2 \sup_{||g||_K \le 1}\sum_{j=1}^{n}(g,f_j)_K^2 \\
&\le \sum_{j=1}^{n}(k,f_j)_K^2.
\end{aligned}$$

Hence

$$||k||_K^2 \le \sum_{j=1}^{\infty}(k,f_j)_K^2.$$

(iv) $\Rightarrow$ (iii) is obvious. (iii) $\Rightarrow$ (i). Since $G$ is dense in $K$, for $k \in K$ there exist $k_n \in G$ satisfying $k_n \to k$ so that $\lim_{n\to\infty}(k_n, f_j)_K^2 = (k, f_j)_K^2$ for every $j$. Fatou's lemma for the counting measure in $N$ implies

$$\begin{aligned}
\sum_{j=1}^{\infty}(k,f_j)_K^2 &\le \liminf_{n\to\infty}\sum_{j=1}^{\infty}(k_n,f_j)_K^2 \\
&= \lim_{n\to\infty}||k_n||_K^2 = ||k||_K^2.
\end{aligned}$$

Therefore, one easily checks that $\sum_{j=1}^{\infty} c_j f_j$ converges in $K$ for every $c = (c_j) \in l_2(N)$ and

$$T : l_2(N) \to K, T(c) := \sum_{j=1}^{\infty} c_j f_j$$

is linear and continuous (see [4], Theorem 3.2.3). Consequently, the frame operator

$$TT^* : K \to K, TT^*k = \sum_{j=1}^{\infty}(k,f_j)_K f_j$$

is linear and continuous. By (ii),

$$(TT^*k, k)_K = \sum_{j=1}^{\infty}(k,f_j)_K^2 = ||k||_K^2$$

for every $k \in G$ and thus $(TT^*k,k)_K = ||k||_K^2$ for every $k \in K$. This implies $TT^*k = k$ for every $k \in K$. $\square$

The preceeding lemma shows that the series (1.3) converges unconditionally. Note further that a Parseval frame $(f_j)$ for $K$ satisfies $\{f_j : j \ge 1\} \subset U_K$, since

$$||f_m||_K^4 + \sum_{j \ne m}(f_m, f_j)_K^2 = \sum_{j=1}^{\infty}(f_m,f_j)_K^2 = ||f_m||_K^2,$$

$\overline{span}\{f_j : j \geq 1\} = K$ and it is an orthonormal basis for $K$ if and only if $||f_j||_K = 1$ for every $j$.

**Proof of Theorem 1.** The "if" part is an immediate consequence of the reproducing property (2.2) and Lemmas 1 and 2 since for $u \in E^*$,

$$\sum_{j=1}^{\infty} \langle u, f_j \rangle^2 = \sum_{j=1}^{\infty} (Cu, f_j)_H^2 = ||Cu||_H^2 = \langle u, Cu \rangle.$$

The "only if" part. By Lemma 1,

$$||f_j||_H = \sup\{|\langle u, f_j \rangle| : \langle u, Cu \rangle \leq 1\} \leq 1$$

so that by (2.5), $\{f_j : j \geq 1\} \subset H$. Again the assertion follows immediately from (2.2) and Lemmas 1 and 2 since $C(E^*)$ is dense in $H$. $\square$

The covariance operator admits factorizations $C = SS^*$, where $S : K \to E$ is a linear continuous operator and $(K, (\cdot,\cdot)_K)$ a real separable Hilbert space, which provide a useful tool for expansions. It is convenient to allow that $S$ is not injective. One gets

$$\begin{aligned} S(K) &= H, \qquad (2.6)\\ *[.4em](Sk_1, Sk_2)_H &= (k_1, k_2)_K, k_1 \in K, k_2 \in (\mathrm{ker} S)^{\perp},\\ *[.4em]\|S\| &= \|S^*\| = \|C\|^{1/2},\\ *[.4em]\overline{S^*(E^*)} &= (\mathrm{ker} S)^{\perp} \text{ in } K,\\ *[.4em](\mathrm{ker} S^*)^{\perp} &:= \{x \in E : \langle u, x \rangle = 0 \ \forall u \in \mathrm{ker} S^*\} = \overline{H} \text{ in } E. \end{aligned}$$

Notice that factorizations of $C$ correspond to linear continuous operators $T : K \to H$ satisfying $TT^* = I$ via $S = JT$, where $J : H \to E$ denotes the inclusion map.

A sequence $(e_j)$ in $K$ is called *Parseval frame sequence* if it is a Parseval frame for $\overline{span}\{e_j : j \geq 1\}$.

**Proposition 1.** *Let $C = SS^*, S : K \to E$ be a factorization of $C$ and let $(e_j)$ be a Parseval frame sequence in $K$ satisfying $(kerS)^{\perp} \subset \overline{span}\{e_j : j = 1, 2, \ldots\}$. Then $(S(e_j))$ is admissible for $X$. Conversely, if $(f_j)$ is admissible for $X$ then there exists a Parseval frame sequence $(e_j)$ in $K$ satisfying $(kerS)^{\perp} = \overline{span}\{e_j : j = 1, 2, \ldots\}$ such that $S(e_j) = f_j$ for every $j$.*

**Proof.** Let $K_0 := \overline{\mathrm{span}}\{e_j : j = 1, 2, \ldots\}$. Since by (2.6)

$$S^*(E^*) \subset (\mathrm{ker} S)^{\perp} \subset K_0,$$

one obtains for every $u \in E^*$, by Lemma 2,

$$\sum_j \langle u, Se_j \rangle^2 = \sum_j (S^* u, e_j)_K^2 = \|S^* u\|_K^2 = \langle u, Cu \rangle.$$

The assertion follows from Lemma 1. Conversely, if $(f_j)$ is admissible for $X$ then $(f_j)$ is a Parseval frame for $H$ by Theorem 1. Set $e_j := (S|(\mathrm{ker} S)^{\perp})^{-1}(f_j) \in (\mathrm{ker} S)^{\perp}$. Then by (2.6) and Lemma 2, for every $k \in (\mathrm{ker} S)^{\perp}$,

$$\sum_j (k, e_j)_K^2 = \sum_j (Sk, f_j)_H^2 = ||Sk||_H^2 = ||k||_K^2$$

so that again by Lemma 2, $(e_j)$ is a Parseval frame for $(\ker S)^\perp$. □

EXAMPLES • Let $S : H \to E$ be the inclusion map. Then $C = SS^*$.
• Let $K$ be the closure of $E^*$ in $L^2(P_X)$ and $S : K \to E, Sk = Ek(X)X$. Then $S^* : E^* \to K$ is the natural embedding. Thus $C = SS^*$ and $S$ is injective (see (2.6)). ($K$ is sometimes called the energy space of $X$.) One obtains

$$H = S(K) = \{Ek(X)X : k \in K\}$$

and

$$(Ek_1(X)X, Ek_2(X)X)_H = \int k_1 k_2 dP_X.$$

• Let $E$ be a Hilbert space, $K = E$ and $S = C^{1/2}$. Then $C = SS^* = S2$ and $(\ker S)^\perp = \overline{H}$. Consequently, if $(e_j)$ is an orthonormal basis of the Hilbert subspace $\overline{H}$ of $E$ consisting of eigenvectors of $C$ and $(\lambda_j)$ the corresponding nonzero eigenvalues, then $(\sqrt{\lambda_j} e_j)$ is admissible for $X$ and an orthonormal basis of $(H, (\cdot,\cdot)_H)$ (Karhunen-Loève basis).

Admissible sequences for $X$ can be charaterized as the sequences $(Se_j)_{j\geq 1}$ where $(e_j)$ is a fixed orthonormal basis of $K$ and $S$ provides a factorization of $C$. That every sequence $(Se_j)$ of this type is admissible follows from Proposition 1.

**Theorem 2.** *Assume that $(f_j)_{j\geq 1}$ is admissible for $X$. Let $K$ be an infinite dimensional real separable Hilbert space and $(e_j)_{j\geq 1}$ an orthonormal basis of $K$. Then there is a factorization $C = SS^*, S : K \to E$ such that $S(e_j) = f_j$ for every $j$.*

**Proof.** First, observe that $\sum_{j=1}^{\infty} c_j f_j$ converges in $E$ for every $(c_j)_j \in l_2(N)$. In fact, using Lemma 1,

$$\begin{aligned} \|\sum_{j=n}^{n+m} c_j f_j\|^2 &= \sup_{\|u\|\leq 1} \langle u, \sum_{j=n}^{n+m} c_j f_j\rangle^2 \\ &\leq \sum_{j=n}^{n+m} c_j^2 \sup_{\|u\|\leq 1} \sum_{j=1}^{\infty} \langle u, f_j\rangle^2 \\ &= \sum_{j=n}^{n+m} c_j^2 \sup_{\|u\|\leq 1} \langle u, Cu\rangle \\ &= \sum_{j=n}^{n+m} c_j^2 \|C\| \to 0,\ n, m \to \infty \end{aligned}$$

and thus the sequence is Cauchy in $E$. Now define $S : K \to E$ by

$$S(k) := \sum_{j=1}^{\infty} (k, e_j)_K f_j$$

where $\sum (k, e_j)_K f_j$ converges in $E$ since $((k, e_j)_K)_j \in l_2(N)$. $S$ is obviously linear. Moreover, for $k \in K$, using again Lemma 1,

$$\begin{aligned} \|Sk\|^2 &= \sup_{\|u\|\leq 1} \langle u, Sk\rangle^2 \\ &= \sup_{\|u\|\leq 1} (\sum_{j=1}^{\infty} (k, e_j)_K \langle u, f_j\rangle)^2 \\ &\leq \|k\|_K^2 \|C\|. \end{aligned}$$

Consequently, $S$ is continuous and $S(e_j) = f_j$ for every $j$. (At this place one needs orthonormality of $(e_j)$.) Finally, $S^*(u) = \sum_{j=1}^{\infty} \langle u, f_j \rangle e_j$ and hence

$$SS^*u = \sum_{j=1}^{\infty} \langle u, f_j \rangle f_j = Cu$$

for every $u \in E^*$ by Lemma 1. □

It is an immediate consequence of the preceding theorem that an admissible sequence $(f_j)$ for $X$ is an orthonormal basis for $H$ if and only if $(f_j)$ is $l_2$-independent, that is $\sum_{j=1}^{\infty} c_j f_j = 0$ for some $(c_j) \in l_2(N)$ implies $c_j = 0$ for every $j$. In fact, $l_2$-independence of $(f_j)$ implies that the operator $S$ in Theorem 2 is injective.

Let $F$ be a further separable Banach space and $V : E \to F$ a $P_X$*-measurable linear transfromation*, that is, $V$ is Borel measurable and linear on a Borel measurable subspace $D_V$ of $E$ with $P_X(D_V) = 1$. Then $H_X \subset D_V$, the operator $VJ_X : H_X \to F$ is linear and continuous, where $J_X : H_X \to E$ denotes the inclusion map and $V(X)$ is centered Gaussian with covariance operator

$$C_{V(X)} = VJ_X(VJ_X)^* \tag{2.7}$$

(see [11], [2], Chapter 3.7). Consequently, by (2.6)

$$\begin{aligned} H_{V(X)} &= V(H_X), \\ *[.4em](Vh_1, Vh_2)_{H_{V(X)}} &= (h_1, h_2)_{H_X}, h_1 \in H_X, h_2 \in (\ker(V \mid H_X))^{\perp}. \end{aligned} \tag{2.8}$$

Note that the space of $P_X$-measurable linear transfromation $E \to F$ is equal to the $L^p_F(P_X)$-closure of the space of linear continuous operators $E \to F, p \in [1, \infty)$ (see [11]).

>From Theorem 1 and Proposition 1 one may deduce the following proposition.

**Proposition 2.** *Assume that $V : E \to F$ is a $P_X$-measurable linear transformation. If $(f_j)_{j\geq 1}$ is admissible for $X$ in $E$, then $(V(f_j))_{j\geq 1}$ is admissible for $V(X)$ in $F$. Conversely, if $V|H_X$ is injective and $(g_j)_{j\geq 1}$ an admissible sequence for $V(X)$ in $F$, then there exists a sequence $(f_j)_{j\geq 1}$ in $E$ which is admissible for $X$ such that $V(f_j) = g_j$ for every $j$.*

Example Let $X$ and $Y$ be jointly centered Gaussian random vectors in $E$ and $F$, respectively. Then $E(Y|X) = V(X)$ for some $P_X$-measurable linear transformation $V : E \to F$. The cross covariance operator $C_{YX} : E^* \to F$, $C_{YX}u = E\langle u, x \rangle Y$ can be factorized as $C_{YX} = U_{YX}S_X^*$, where $C_X = S_X S_X^*$ is the energy factorization of $C_X$ with $K_X$ the closure of $E^*$ in $L^2(P_X)$ and $U_{YX} : K_X \to F, U_{YX}k = Ek(X)Y$. Then

$$V = U_{YX}S_X^{-1} \text{ on } H_X$$

(see [11]). Consequently, if $(f_j)_{j\geq 1}$ is admissible for $X$ in $E$ then $(U_{YX}S_X^{-1}f_j)_{j\geq 1}$ is admissible for $E(Y|X)$ in $F$.

# 3 Continuous Gaussian processes

Now let $I$ be a compact metric space and $X = (X_t)_{t\in I}$ be a real pathwise continuous centered Gaussian process. Let $E := \mathscr{C}(I)$ be equipped with the sup-norm $\|x\| = \sup_{t\in I} |x(t)|$ so that the norm dual $\mathscr{C}(I)^*$ coincides with the space of finite signed Borel measures on $I$ by the Riesz theorem. Then $X$ can be seen as a $\mathscr{C}(I)$-valued Gaussian random vector and the covariance operator $C : \mathscr{C}(I)^* \to \mathscr{C}(I)$ takes the form

$$\begin{aligned} Cu(t) &= \langle \delta_t, Cu\rangle = \langle C\delta_t, u\rangle \\ &= \langle EX_t X, u\rangle = \int_I EX_t X_s du(s). \end{aligned} \tag{3.1}$$

**Corollary 1.** *Let $(f_j)_{j\geq 1} \in \mathscr{C}(I)^{\mathbb{N}}$.*
*(a) If*

$$EX_s X_t = \sum_{j=1}^{\infty} f_j(s) f_j(t) \text{ for every } s,t \in I$$

*then $(f_j)$ is admissible for $X$.*

*(b) If*

$$\sum_{j=1}^{\infty} f_j(t)^2 < \infty \text{ for every } t \in I$$

*and if the process $Y$ with $Y_t = \sum_{j=1}^{\infty} \xi_j f_j(t)$ has a pathwise continuous modification $X$, then $(f_j)$ is admissible for $X$ and $X = \sum_{j=1}^{\infty} \xi_j f_j$ a.s.*

**Proof.** (a) For $u \in G := \text{span}\,\{\delta_t : t \in I\}, u = \sum_{i=1}^{m} \alpha_i \delta_{t_i}$ we have

$$\langle u, Cu\rangle = \sum_{i=1}^{m}\sum_{k=1}^{m} \alpha_i \alpha_k EX_{t_i} X_{t_k}$$

and

$$\sum_{j=1}^{n} \langle u, f_j\rangle^2 = \sum_{i=1}^{m}\sum_{k=1}^{m} \alpha_i \alpha_k \sum_{j=1}^{n} f_j(t_i) f_j(t_k)$$

so that

$$\sum_{j=1}^{\infty} \langle u, f_j\rangle^2 = \langle u, Cu\rangle.$$

Since $G$ is a separating subspace of $\mathscr{C}(I)^*$ the assertion follows from Lemma 1.
(b) Notice that $\sum \xi_j f_j(t)$ converges a.s. in $R$ and $Y$ is a centered Gaussian process. Hence $X$ is centered Gaussian. Since

$$EX_s X_t = EY_s Y_t = \sum_{j=1}^{\infty} f_j(s) f_j(t) \text{ for every } s,t \in I,$$

the assertion follows from (a). $\square$

>Factorizations of $C$ can be obtained as follows. For Hilbert spaces $K_i$, let $\oplus_{i=1}^m K_i$ denote the Hilbertian (or $l_2-$)direct sum.

**Lemma 3.** *For $i \in \{1, \dots, m\}$, let $K_i$ be a real separable Hilbert space. Assume the representation*

$$EX_sX_t = \sum_{i=1}^{m} (g_s^i, g_t^i)_{K_i}, s, t \in I$$

*for vectors $g_t^i \in K_i$. Then*

$$S : \oplus_{i=1}^m K_i \to \mathscr{C}(I), Sk(t) := \sum_{i=1}^{m} (g_t^i, k_i)_{K_i}$$

*is a linear continuous operator, $(kerS)^\perp = \overline{span}\{(g_t^1, \dots, g_t^m) : t \in I\}$ and $C = SS^*$.*

**Proof.** Let $K := \oplus_{i=1}^m K_i$ and $g_t := (g_t^1, \dots, g_t^m)$. Then $EX_sX_t = (g_s, g_t)_K$ and $Sk(t) = (g_t, k)_K$. First, observe that

$$\sup_{t \in I} \|g_t\|_K \leq \|C\|^{1/2} < \infty.$$

Indeed, for every $t \in I$, by (3.1),

$$\|g_t\|_K^2 = EX_t^2 = \langle \delta_t, C\delta_t \rangle \leq \|C\|.$$

The function $Sk$ is continuous for $k \in$ span $\{g_s : s \in I\}$. This easily implies that $Sk$ is continuous for every $k \in \overline{\text{span}}\{g_s : s \in I\}$ and thus for every $k \in K$. $S$ is obviously linear and

$$\|Sk\| = \sup_{t \in I} | (g_t, k)_K | \leq \|C\|^{1/2} \|k\|_K.$$

Finally, $S^*(\delta_t) = g_t$ so that

$$SS^*\delta_t(s) = Sg_t(s) = EX_sX_t = C\delta_t(s)$$

for every $s, t \in I$. Consequently, for every $u \in \mathscr{C}(I)^*, t \in I$,

$$\begin{aligned} SS^*u(t) &= \langle SS^*u, \delta_t \rangle = \langle u, SS^*\delta_t \rangle \\ &= \langle u, C\delta_t \rangle = \langle Cu, \delta_t \rangle = Cu(t) \end{aligned}$$

and hence $C = SS^*$. $\square$

Example Let $K$ be the first Wiener chaos, that is $K = \overline{\text{span}}\{X_t : t \in I\}$ in $L^2(P)$ and $g_t = X_t$. Then $Sk = EkX$ and $S$ is injective. If for instance $X = W$ (Brownian motion) and $I = [0, T]$, then

$$K = \left\{ \int_0^T f(s) dW_s : f \in L^2([0, T], dt) \right\}.$$

We derive from the preceeding lemma and Proposition 1 the following corollary.

**Corollary 2.** *Assume the situation of Lemma 3. Let $(e^i_j)_j$ be a Parseval frame sequence in $K_i$ satisfying $\{g^i_t : t \in I\} \subset \overline{span}\ \{e^i_j : j = 1,2,\dots\}$. Then, $(S_i(e^i_j))_{1\le i\le m,j}$ is admissible for $X$, where $S_i k(t) = (g^i_t, k)_{K_i}$.*

The next corollary implies the well known fact that the Karhunen-Loève expansion of $X$ in some Hilbert space $L^2(I,\mu)$ already converges uniformly in $t \in I$. It appears as special case of Proposition 2.

**Corollary 3.** *Let $\mu$ be a finite Borel measure on $I$ with $supp(\mu) = I$ and let $V : \mathscr{C}(I) \to L^2(I,\mu)$ denote the natural (injective) embedding. Let $(g_j)_{j\ge1}$ be admissible for $V(X)$ in $L^2(I,\mu)$. Then there exists a sequence $(f_j)_{j\ge1}$ in $\mathscr{C}(I)$ which is admissible for $X$ such that $V(f_j) = g_j$ for every $j$.*

The admissibility feature is stable under tensor products. For $i \in \{1,\dots,d\}$, let $I_i$ be a compact metric space and $X^i = (X^i_t)_{t\in I_i}$ a continuous centered Gaussian process. Set $I := \Pi^d_{i=1} I_i$ and let $X = (X_t)_{t\in I}$ be a continuous centered Gaussian process with covariance function

$$EX_s X_t = \Pi^d_{i=1} EX^i_{s_i} X^i_{t_i}, s,t \in I. \tag{3.2}$$

For instance, $X := \otimes^d_{i=1} X^i$ satisfies (3.2) provided $X1,\dots,X^d$ are independent. For real separable Hilbert spaces $K_i$, let $\widehat{\otimes}^d_{i=1} K_i$ denote the $d$-fold Hilbertian tensor product.

**Proposition 3.** *For $i \in \{1,\dots,d\}$, let $(f^i_j)_{j\ge1}$ be an admissible sequence for $X^i$ in $\mathscr{C}(I_i)$. Then*

$$(\otimes^d_{i=1} f^i_{j_i})_{\underline{j}=(j_1,\dots,j_d)\in\mathbb{N}^d}$$

*is admissible for $X$ with covariance (3.2) in $\mathscr{C}(I)$. Furthermore, if $C_{X^i} = S_i S^*_i, S_i : K_i \to \mathscr{C}(I_i)$ is a factorization of $C_{X^i}$, then $\otimes^d_{i=1} S_i : \widehat{\otimes}^d_{i=1} K_i \to \mathscr{C}(I)$ provides a factorization of $C_X$.*

**Proof.** For $i \in \{1,\dots,d\}$, let $K_i$ be a real separable Hilbert space and $(e^i_j)_j$ an orthonormal basis of $K_i$. Then $(\otimes^d_{i=1} e^i_{j_i})_{\underline{j}}$ is an orthonormal basis of $K := \widehat{\otimes}^d_{i=1} K_i$.

If $C_{X^i} = S_i S^*_i, S_i : K_i \to \mathscr{C}(I_i)$ is a factorization of $C_{X^i}$, set $g^i_t := S^*_i \delta_t, t \in I_i$. Then $EX^i_s X^i_t = (g^i_s, g^i_t)_{K_i}$ and hence, by (3.2)

$$EX_s X_t = \Pi^d_{i=1}(g^i_{s_i}, g^i_{t_i})_{K_i} = (\otimes^d_{i=1} g^i_{s_i}, \otimes^d_{i=1} g^i_{t_i})_K, s,t \in I.$$

Consequently, by Lemma 3

$$U : K \to \mathscr{C}(I), Uk(t) = (\otimes^d_{i=1} g^i_{t_i}, k)_K$$

provides a factorization of $C_X$. Since

$$\begin{aligned} U(\otimes^d_{i=1} e^i_{j_i})(t) &= \Pi^d_{i=1}(g^i_{t_i}, e^i_{j_i})_{K_i} \\ &= \Pi^d_{i=1} S_i e^i_{j_i}(t_i) = \otimes^d_{i=1}(S_i e^i_{j_i})(t) \\ &= (\otimes^d_{i=1} S_i)(\otimes^d_{i=1} e^i_{j_i})(t), t \in I, \end{aligned}$$

we obtain $U = \otimes^d_{i=1} S_i$ and thus $\otimes^d_{i=1} S_i$ provides a factorization of $C_X$.

If $(f^i_j)_{j\ge1}$ is admissible for $X^i$, then by Theorem 2 assuming now that $K_i$ is infinite dimensional, there is a factorization $C_{X^i} = T_i T^*_i, T_i : K_i \to \mathscr{C}(I_i)$ such that $T_i(e^i_j) = f^i_j$ for every $j$. Since

$\otimes_{i=1}^{d} T_i : K \to \mathscr{C}(I)$ provides a factorization of $C_X$ as shown above and $(\otimes_{i=1}^{d} T_i)(\otimes_{i=1}^{d} e_{j_i}^i) = \otimes_{i=1}^{d} f_{j_i}^i$, it follows from Proposition 1 that $(\otimes_{i=1}^{d} f_{j_i}^i)_{\underline{j}\in N^d}$ is admissible for $X$. □

**Comments on optimal expansions.** For $n \in N$, let

$$l_n(X) := \inf\{E||\sum_{j=n}^{\infty} \xi_j f_j|| : (f_j)_{j\geq 1} \in \mathscr{C}(I)^{N} \text{ admissible for } X\}. \tag{3.3}$$

Rate optimal solutions of the $l_n(X)$-problem are admissible sequences $(f_j)$ for $X$ in $\mathscr{C}(I)$ such that

$$E||\sum_{j=n}^{\infty} \xi_j f_j|| \approx l_n(X) \quad \text{as} \quad n \to \infty.$$

For $I = [0, T]^d \subset R^d$, consider the covariance operator $R = R_X$ of $X$ on $L^2(I, dt)$ given by

$$R : L^2(I, dt) \to L^2(I, dt), Rk(t) = \int_I EX_s X_t k(s) ds. \tag{3.4}$$

Using (3.1) we have $R_X = VC_X V^*$, where $V : C(I) \to L^2(I, dt)$ denotes the natural (injective) embedding. The choice of Lebesgue measure on I is the best choice for our purposes (see (A1)). Let $\lambda_1 \geq \lambda_2 \geq \ldots > 0$ be the ordered nonzero eigenvalues of $R$ (each written as many times as its multiplicity).

**Proposition 4.** *Let $I = [0, T]^d$. Assume that the eigenvalues of $R$ satisfy*

*(A1) $\lambda_j \geq c_1 j^{-2\vartheta} \log(1 + j)^{2\gamma}$ for every $j \geq 1$ with $\vartheta > 1/2, \gamma \geq 0$ and $c_1 > 0$*

*and that $X$ admits an admissible sequence $(f_j)$ in $\mathscr{C}(I)$ satisfying*

*(A2) $||f_j|| \leq c_2 j^{-\vartheta} \log(1 + j)^{\gamma}$ for every $j \geq 1$with $c_2 < \infty$,*

*(A3) $f_j$ is a-Hölder-continuous and $[f_j]_a \leq c_3 j^b$ for every $j \geq 1$ with $a \in (0, 1], b \in R$ and $c_3 < \infty$, where*

$$[f]_a = \sup_{s\neq t} \frac{|f(s) - f(t)|}{|s - t|^a}$$

*(and $|t|$ denotes the $l_2$-norm of $t \in R^d$).*
*Then*

$$l_n(X) \approx n^{-(\vartheta-\frac{1}{2})}(\log n)^{\gamma+\frac{1}{2}} \quad as \quad n \to \infty \tag{3.5}$$

*and $(f_j)$ is rate optimal.*

**Proof.** The lower estimate in (3.5) follows from (A1) (see [8], Proposition 4.1) and from (A2) and (A3) follows

$$E||\sum_{j=n}^{\infty} \xi_j f_j|| \leq c_4 n^{-(\vartheta-\frac{1}{2})}(\log(1 + n))^{\gamma+\frac{1}{2}}$$

for every $n \geq 1$, (see [13], Theorem 1). $\square$

Concerning assumption (A3) observe that we have by (2.2) and (3.1) for $h \in H = H_X, s, t \in I$,

$$h(t) =< \delta_t, h >= (h, C\delta_t)_H$$

and

$$||C(\delta_s - \delta_t)||_H^2 =< \delta_s - \delta_t, C(\delta_s - \delta_t) >= E|X_s - X_t|^2$$

so that

$$\begin{aligned} |h(s) - h(t)| &= |(h, C(\delta_s - \delta_t))_H| \\ &\leq ||h||_H ||C(\delta_s - \delta_t)||_H \\ &= ||h||_H (E|X_s - X_t|^2)^{1/2}. \end{aligned} \tag{3.6}$$

Consequently, since admissible sequences are contained in the unit ball of $H$, (A3) is satisfied with $b = 0$ provided $I \to L^2(P), t \mapsto X_t$ is $a$-Hölder-continuous.

The situation is particularly simple for Gaussian sheets.

**Corollary 4.** *Assume that for $i \in \{1, \ldots, d\}$, the continuous centered Gaussian process $X^i = (X_t^i)_{t \in [0,T]}$ satisfies (A1) - (A3) for some admissible sequence $(f_j^i)_{j \geq 1}$ in $\mathscr{C}([0,T])$ with parameters $\vartheta_i, \gamma_i, a_i, b_i$ such that $\gamma_i = 0$ and let $X = (X_t)_{t \in I}, I = [0,T]^d$ be the continuous centered Gaussian sheet with covariance (3.2). Then*

$$l_n(X) \approx n^{-(\vartheta - \frac{1}{2})} (\log n)^{\vartheta(m-1) + \frac{1}{2}} \tag{3.7}$$

*with $\vartheta = \min_{1 \leq i \leq d} \vartheta_i$ and $m = card\{i \in \{1, \ldots, d\} : \vartheta_i = \vartheta\}$ and a decreasing arrangement of $(\otimes_{i=1}^d f_{j_i}^i)_{\underline{j} \in \mathbb{N}^d}$ is rate optimal for $X$.*

**Proof.** In view of Lemma 1 in [13] and Proposition 3, the assertions follow from Proposition 4. $\square$

Examples The subsequent examples may serve as illustrations.

• Let $W = (W_t)_{t \in [0,T]}$ be a *standard Brownian motion*. Since $EW_sW_t = s \wedge t = \int_0^T 1_{[0,s]}(u) 1_{[0,t]}(u) du$, the (injective) operator

$$S : L^2([0,T], dt) \to \mathscr{C}([0,T]), \quad Sk(t) = \int_0^t k(s) ds$$

provides a factorization of $C_W$ so that we can apply Corollary 2. The orthonormal basis $e_j(t) = \sqrt{2/T} \cos(\pi(j - 1/2)t/T), j \geq 1$ of $L^2([0,T], dt)$ yields the admissible sequence

$$f_j(t) = Se_j(t) = \frac{\sqrt{2T}}{\pi(j - 1/2)} \sin(\frac{\pi(j - 1/2)t}{T}), \ j \geq 1 \tag{3.8}$$

for $W$ (Karhunen-Loève basis of $H_W$) and $e_j(t) = \sqrt{2/T} \sin(\pi jt/T)$ yields the admissible sequence

$$g_j(t) = \frac{\sqrt{2T}}{\pi j} (1 - \cos(\frac{\pi jt}{T})), \ j \geq 1.$$

Then

$$f_j^1(t) = \frac{1}{\sqrt{2}} f_j(t) \quad = \quad \frac{\sqrt{T}}{\pi(j-1/2)} \sin(\frac{\pi(j-1/2)t}{T}),\ j \geq 1 \tag{3.9}$$
$$f_j^2(t) = \frac{1}{\sqrt{2}} g_j(t) \quad = \quad \frac{\sqrt{T}}{\pi j}(1-\cos(\frac{\pi j t}{T})),\ j \geq 1$$

is a Parseval frame for $H_W$ and hence admissible for $W$. The trigonometric basis $e_0(t) = 1/\sqrt{T}$, $e_{2j}(t) = \sqrt{2/T}\cos(2\pi jt/T), e_{2j-1}(t) = \sqrt{2/T}\sin(2\pi jt/T)$ of $L^2([0,T],dt)$ yields the admissible sequence

$$f_0(t) = \frac{t}{\sqrt{T}}, f_{2j}(t) = \frac{\sqrt{T}}{\sqrt{2}\pi j}\sin(\frac{2\pi jt}{T}), \tag{3.10}$$
$$f_{2j-1}(t) = \frac{\sqrt{T}}{\sqrt{2}\pi j}(1-\cos(\frac{2\pi jt}{T})), j \geq 1$$

(Paley-Wiener basis of $H_W$). By Proposition 4, all these admissible sequences for $W$ (with $f_{2j} := f_j^1, f_{2j-1} := f_j^2$, say in (3.9)) are rate optimal.

Assume that the wavelet system $2^{j/2}\psi(2^j\cdot -k), j,k \in \mathbb{Z}$ is an orthonormal basis (or only a Parseval frame) for $L^2(R,dt)$. Then the restrictions of these functions to [0,T] clearly provide a Parseval frame for $L^2([0,T],dt)$ so that the sequence

$$f_{j,k}(t) = S(2^{j/2}\psi(2^j\cdot -k))(t) = 2^{-j/2}\int_{-k}^{2^j t-k} \psi(u)du, j,k \in \mathbb{Z}$$

is admissible for $W$. If $\psi \in L^1(R,dt)$ and $\Psi(x) := \int_{-\infty}^x \psi(u)du$, then this admissible sequence takes the form

$$f_{j,k}(t) = 2^{-j/2}(\Psi(2^j t-k) - \Psi(-k)), j,k \in \mathbb{Z}. \tag{3.11}$$

• We consider the Dzaparidze-van Zanten expansion of the *fractional Brownian motion* $X = (X_t)_{t\in[0,T]}$ with Hurst index $\rho \in (0,1)$ and covariance function

$$EX_sX_t = \frac{1}{2}(s^{2\rho} + t^{2\rho} - |s-t|^{2\rho}).$$

These authors discovered in [5] for $T = 1$ a time domain representation

$$EX_sX_t = (g_s^1, g_t^1)_K + (g_s^2, g_t^2)_K$$

with $K = L^2([0,1],dt)$ and kernels $g_t^i \in L^2([0,1],dt)$. Hence by Lemma 3, the operator

$$S : L^2([0,1],dt) \oplus L^2([0,1],dt) \to \mathscr{C}([0,1]),\ S(k_1,k_2)(t) = \int_0^1 g_t^1(s)k_1(s)ds + \int_0^1 g_t^2(s)k_2(s)ds$$

provides a factorization of $C_X$ so that for every pair of orthonormal bases $(e_j^1)_{j\geq 1}$ and $(e_j^2)_{j\geq 1}$ of $L^2([0,1],dt)$,

$$f_j^i(t) = \int_0^1 g_t^i(s)e_j^i(s)ds,\ j \geq 1,\ i = 1,2$$

is admissible in $\mathscr{C}([0,1])$ for $X$. By Corollary 2, this is a consequence of the above representation of the covariance function (and needs no extra work). Then Dzaparidze and van Zanten [5] could calculate $f_j^i$ explicitely for the Fourier-Bessel basis of order $-\rho$ and $1-\rho$, respectively and arrived at the admissible family in $\mathscr{C}([0,1])$

$$
\begin{aligned}
f_j^1(t) &= \frac{c_\rho \sqrt{2}}{\mid J_{1-\rho}(x_j) \mid x_j^{\rho+1}} \sin(x_j t), j \geq 1 \\
f_j^2(t) &= \frac{c_\rho \sqrt{2}}{\mid J_{-\rho}(y_j) \mid y_j^{\rho+1}} (1-\cos(y_j t)), j \geq 1
\end{aligned}
$$

where $J_\nu$ denotes the Bessel function of the first kind of order $\nu$, $0 < x_1 < x_2 < \ldots$ are the positive zeros of $J_{-\rho}, 0 < y_1 < y_2 < \ldots$ the positive zeros of $J_{1-\rho}$ and $c_\rho^2 = \Gamma(1+2\rho)\sin(\pi\rho)/\pi$. Consequently, by self-similarity of $X$, the sequence

$$
\begin{aligned}
f_j^1(t) &= \frac{T^\rho c_\rho \sqrt{2}}{\mid J_{1-\rho}(x_j) \mid x_j^{\rho+1}} \sin(\frac{x_j t}{T}), j \geq 1 \\
f_j^2(t) &= \frac{T^\rho c_\rho \sqrt{2}}{\mid J_{-\rho}(y_j) \mid y_j^{\rho+1}} (1-\cos(\frac{y_j t}{T})), j \geq 1
\end{aligned}
\tag{3.12}
$$

in $\mathscr{C}([0,T])$ is admissible for $X$. Using Lemma 1, one can deduce (also without extra work)

$$
EX_sX_t = \sum_{j=1}^{\infty} f_j^1(s) f_j^1(t) + \sum_{j=1}^{\infty} f_j^2(s) f_j^2(t)
$$

uniformly in $(s,t) \in [0,T]^2$. Rate optimality of (3.12) (using an arrangement like $f_{2j} := f_j^1, f_{2j-1} := f_j^2$) is shown in [6] based on the work [8] and is also an immediate consequence of Proposition 4 since

$$
x_j \sim y_j \sim \pi j, J_{1-\rho}(x_j) \sim J_{-\rho}(y_j) \sim \frac{\sqrt{2}}{\pi} j^{-1/2}
$$

(see [5]), and the eigenvalues satisfy $\lambda_j \sim c j^{-(1+2\rho)}$ as $j \to \infty$ (see [3], [12]).

In the ordinary Brownian motion case $\rho = 1/2$, (3.12) coincides with (3.9). The interesting extension of (3.10) to fractional Brownian motions is discussed in [7] and extensions of the wavelet expansion (3.11) can be found in [1], [14].

• Let $X = (X_t)_{t \in [0,T]}$ be *Brownian bridge* with covariance

$$
EX_sX_t = s \wedge t - \frac{st}{T} = \int_0^T (1_{[0,s]}(u) - \frac{s}{T})(1_{[0,t]}(u) - \frac{t}{T}) du.
$$

By Lemma 3, the operator

$$
S : L^2([0,T], dt) \to \mathscr{C}([0,T]), Sk(t) = \int_0^t k(s) ds - \frac{t}{T} \int_0^T k(s) ds
$$

provides a factorization of $C_X$ and $\ker S = \mathrm{span}\{1_{[0,T]}\}$. The choice $e_j(t) = \sqrt{2/T}\cos(\pi jt/T), j \geq 1$ of an orthonormal basis of $(\ker S)^{\perp}$ yields admissibility of

$$f_j(t) = Se_j(t) = \frac{\sqrt{2T}}{\pi j}\sin(\frac{\pi jt}{T}), j \geq 1 \tag{3.13}$$

for $X$ (Karhunen-Loève basis of $H_X$). By Proposition 4, this sequence is rate optimal.

• One considers the *stationary Ornstein-Uhlenbeck process* as the solution of the Langevin equation

$$dX_t = -\alpha X_t dt + \sigma dW_t, t \in [0,T]$$

with $X_0$ independent of $W$and $N(0, \frac{\sigma 2}{2\alpha})$-distributed, $\sigma > 0, \alpha > 0$. It admits the explicit representation

$$X_t = e^{-\alpha t}X_0 + \sigma e^{-\alpha t}\int_0^t e^{\alpha s}dW_s$$

and

$$EX_sX_t = \frac{\sigma^2}{2\alpha}e^{-\alpha|s-t|} = \frac{\sigma^2}{2\alpha}e^{-\alpha(s+t)} + \sigma 2e^{-\alpha(s+t)}\int_0^{s\wedge t} e^{2\alpha u}du.$$

Thus the (injective) operator

$$S : R \oplus L^2([0,T], dt) \to \mathscr{C}([0,T]), S(c,k)(t) = \frac{c\sigma}{\sqrt{2\alpha}}e^{-\alpha t} + \sigma\int_0^t e^{-\alpha(t-s)}k(s)ds$$

provides a factorization of $C_X$ so that for every Parseval frame $(e_j)_{j\geq 1}$ for $L^2([0,T], dt)$, the functions

$$f_0(t) = \frac{\sigma}{\sqrt{2\alpha}}e^{-\alpha t},\ f_j(t) = \sigma\int_0^t e^{-\alpha(t-s)}e_j(s)ds,\ j \geq 1 \tag{3.14}$$

provide an admissible sequence for $X$. For instance the choice of the orthonormal basis $\sqrt{2/T}\cos(\pi(j-1/2)t/T), j \geq 1$ implies that (3.14) is rate optimal. This follows from Lemma 1 in [13] and Proposition 4.

Another representation is given by the Lamperti transformation $X = V(W)$ for the linear continuous operator

$$V : \mathscr{C}([0, e^{2\alpha T}]) \to \mathscr{C}([0,T],\ Vx(t) = \frac{\sigma}{\sqrt{2\alpha}}e^{-\alpha t}x(e^{2\alpha t}).$$

The admissible sequence $(f_j)$ in $\mathscr{C}([0, e^{2\alpha T}])$ for $(W_t)_{t\in[0,e^{2\alpha T}]}$ from (4.1) yields the admissible sequence

$$\tilde{f}_j(t) = Vf_j(t) = \frac{\sigma}{\sqrt{\alpha}\pi(j-1/2)}e^{\alpha(T-t)}\sin(\pi(j-1/2)e^{-2\alpha(T-t)}), j \geq 1 \tag{3.15}$$

for $X$. By Proposition 4, the sequence (3.15) is rate optimal.

• Sheet versions can be deduced from Proposition 3 and Corollary 4 (and need no extra work).

# 4 Optimal expansion of a class of stationary Gaussian processes

## 4.1 Expansion of fractional Ornstein-Uhlenbeck processes

The fractional Ornstein-Uhlenbeck process $X^\rho = (X_t^\rho)_{t\in\mathbb{R}}$ of index $\rho \in (0,2)$ is a continuous stationary centered Gaussian process having the covariance function

$$EX_s^\rho X_t^\rho = e^{-\alpha|s-t|^\rho}, \alpha > 0. \tag{4.1}$$

We derive explicit optimal expansions of $X^\rho$ for $\rho \le 1$. Let

$$\gamma^\rho : R \to R, \gamma^\rho(t) = e^{-\alpha|t|^\rho}$$

and for a given $T > 0$, set

$$\beta_0(\rho) := \frac{1}{2T}\int_{-T}^{T}\gamma^\rho(t)dt, \beta_j(\rho) := \frac{1}{T}\int_{-T}^{T}\gamma^\rho(t)\cos(\pi jt/T)dt, j \ge 1. \tag{4.2}$$

**Theorem 3.** *Let $\rho \in (0,1]$. Then $\beta_j(\rho) > 0$ for every $j \ge 0$ and the sequence*

$$f_0 = \sqrt{\beta_0(\rho)}, \quad f_{2j} = \sqrt{\beta_j(\rho)}\cos(\pi jt/T), \quad f_{2j-1}(t) = \sqrt{\beta_j(\rho)}\sin(\pi jt/T), j \ge 1 \tag{4.3}$$

*is admissible for $X^\rho$ in $\mathscr{C}([0,T])$. Furthermore,*

$$l_n(X^\rho) \approx n^{-\rho/2}(\log n)^{1/2} \quad as \quad n \to \infty$$

*and the sequence (4.3) is rate optimal.*

**Proof.** Since $\gamma^\rho$ is of bounded variation (and continuous) on [-T,T], it follows from the Dirichlet criterion that its (classical) Fourier series converges pointwise to $\gamma^\rho$ on [-T,T], that is using symmetry of $\gamma^\rho$,

$$\gamma^\rho(t) = \beta_0(\rho) + \sum_{j=1}^{\infty}\beta_j(\rho)\cos(\pi jt/T), t \in [-T,T].$$

Thus one obtains the representation

$$EX_sX_t = \gamma^\rho(s-t) = \beta_0(\rho) + \sum_{j=1}^{\infty}\beta_j(\rho)[\cos(\pi js/T)\cos(\pi jt/T) + \sin(\pi js/T)\sin(\pi jt/T)], s,t \in [0,T]. \tag{4.4}$$

This is true for every $\rho \in (0,2)$. If $\rho = 1$, then integration by parts yields

$$\beta_0(1) = \frac{1-e^{-\alpha T}}{\alpha T}, \quad \beta_j(1) = \frac{2\alpha T(1-e^{-\alpha T}(-1)^j)}{\alpha^2T^2 + \pi^2 j2}, \ j \ge 1. \tag{4.5}$$

In particular, we obtain $\beta_j(1) > 0$ for every $j \ge 0$. If $\rho \in (0,1)$, then $\gamma^\rho_{|[0,\infty)}$ is the Laplace transform of a suitable one-sided strictly $\rho$-stable distribution with Lebesgue-density $q_\rho$. Consequently, for

$j \geq 1$,

$$\begin{aligned}
\beta_j(\rho) &= \frac{2}{T}\int_0^T e^{-\alpha t^\rho}\cos(\pi jt/T)dt \qquad (4.6)\\
&= \int_0^\infty \frac{2}{T}\int_0^T e^{-tx}\cos(\pi jt/T)dt q_\rho(x)dx\\
&= \int_0^\infty \frac{2xT(1-e^{-xT}(-1)^j)}{x^2T^2+\pi^2j^2} q_\rho(x)dx.
\end{aligned}$$

Again, $\beta_j(\rho) > 0$ for every $j \geq 0$. It follows from (4.4) and Corollary 1(a) that the sequence $(f_j)_{j\geq 0}$ defined in (4.3) is admissible for $X^\rho$ in $\mathscr{C}([0,T])$.

Next we investigate the asymptotic behaviour of $\beta_j(\rho)$ as $j \to \infty$ for $\rho \in (0,1)$. The spectral measure of $X^\rho$ still for $\rho \in (0,2)$ is a symmetric $\rho$-stable distribution with continuous density $p_\rho$ so that

$$\begin{aligned}
\gamma^\rho(t) &= \int_{\mathbb{R}} e^{itx}p_\rho(x)dx\\
&= 2\int_0^\infty \cos(tx)p_\rho(x)dx,\ t \in R
\end{aligned}$$

and the spectral density satisfies the high-frequency condition

$$p_\rho(x) \sim c(\rho)x^{-(1+\rho)} \quad \text{as} \quad x \to \infty \qquad (4.7)$$

where

$$c(\rho) = \frac{\alpha\Gamma(1+\rho)\sin(\pi\rho/2)}{\pi}.$$

Since by the Fourier inversion formula

$$p_\rho(x) = \frac{1}{\pi}\int_0^\infty \gamma^\rho(t)\cos(tx)dt,\ x \in R,$$

we obtain for $j \geq 1$,

$$\begin{aligned}
\beta_j(\rho) &= \frac{2}{T}\int_0^T \gamma^\rho(t)\cos(\pi jt/T)dt\\
&= \frac{2}{T}\left(\int_0^\infty \gamma^\rho(t))\cos(\pi jt/T)dt - \int_T^\infty \gamma^\rho(t)\cos(\pi jt/T)dt\right)\\
&= \frac{2\pi}{T}p_\rho(\pi j/T) - \frac{2}{T}\int_T^\infty \gamma^\rho(t))\cos(\pi jt/T)dt
\end{aligned}$$

Integrating twice by parts yields

$$\int_T^\infty \gamma^\rho(t)\cos(\pi jt/T)dt = O(j^{-2})$$

for any $\rho \in (0,2)$ so that for $\rho \in (0,1)$

$$\begin{aligned} \beta_j(\rho) &\sim \frac{2\pi}{T} p_\rho(\pi j/T) \\ &\sim \frac{2\pi T^\rho c(\rho)}{(\pi j)^{1+\rho}} \\ &= \frac{2\alpha T^\rho \Gamma(1+\rho)\sin(\pi\rho/2)}{(\pi j)^{1+\rho}} \quad \text{as} \quad j \to \infty. \end{aligned} \tag{4.8}$$

We deduce from (4.5) and (4.8) that the admissible sequence (4.3) satisfies the conditions (A2) and (A3) from Proposition 4 with parameters $\vartheta = (1+\rho)/2$, $\gamma = 0$, $a = 1$ and $b = (1-\rho)/2$. Furthermore, by Theorem 3 in Rosenblatt [16], the asymptotic behaviour of the eigenvalues of the covariance operator of $X^\rho$ on $L^2([0,T],dt)$ (see (3.4)) for $\rho \in (0,2)$ is as follows:

$$\lambda_j \sim \frac{2T^{1+\rho}\pi c(\rho)}{(\pi j)^{1+\rho}} \quad \text{as} \quad j \to \infty. \tag{4.9}$$

Therefore, the remaining assertions follow from Proposition 4. □

Here are some comments on the above theorem.

First, note that the admissible sequence (4.3) is not an orthonormal basis for $H = H_{X^\rho}$ but only a Parseval frame at least in case $\rho = 1$. In fact, it is well known that for $\rho = 1$,

$$||h||_H^2 = \frac{1}{2}(h(0)^2 + h(T)^2) + \frac{1}{2\alpha}\int_0^T (h'(t)^2 + \alpha^2 h(t)^2)dt$$

so that *e.g.*

$$||f_{2j-1}||_H^2 = \frac{1 - e^{-\alpha T}(-1)^j}{2} < 1.$$

A result corresponding to Theorem 3 for fractional Ornstein-Uhlenbeck sheets on $[0,T]^d$ with covariance structure

$$EX_sX_t = \prod_{i=1}^d e^{-\alpha_i |s_i - t_i|^{\rho_i}}, \ \alpha_i > 0, \ \rho_i \in (0,1] \tag{4.10}$$

follows from Corollary 4.

As a second comment, we wish to emphasize that, unfortunately, in the nonconvex case $\rho \in (1,2)$ it is not true that $\beta_j(\rho) \geq 0$ for every $j \geq 0$ so that the approach of Theorem 3 no longer works. In fact, starting again from

$$\beta_j(\rho) = \frac{2\pi}{T} p_\rho(\pi_j/T) - \frac{2}{T}\int_T^\infty \gamma^\rho(t)\cos(\pi j t/T)dt,$$

three integrations by parts show that

$$\begin{aligned} \beta_j(\rho) &= \tfrac{2\pi}{T} p_\rho(\pi j/T) + \tfrac{(-1)^{j+1} 2T\alpha\rho e^{-\alpha T^\rho} T^{\rho-1}}{(\pi j)^2} + O(j^{-3}) \\ &= \tfrac{(-1)^{j+1} 2T\alpha\rho e^{-\alpha T^\rho} T^{\rho-1}}{(\pi j)^2} + O(j^{-(1+\rho)}), \ j \to \infty. \end{aligned}$$

This means that for any $T > 0$, the $2T$-periodic extension of $\gamma^{\rho}_{|[-T,T]}$ is not nonnegative definite for $\rho \in (1,2)$ in contrast to the case $\rho \in (0,1]$.

## 4.2 Expansion of stationary Gaussian processes with convex covariance function

As concerns admissibility, it is interesting to observe that the convex function $\gamma(t) = e^{-\alpha t^{\rho}}$ , $\rho \in (0,1]$ in Theorem 3 can be replaced by any convex on $(0,\infty)$ function $\gamma$ going to zero at infinity. Some additional natural assumptions related on its regularity at 0 or the rate of decay of its spectral density at infinity then provide the optimality.

Namely, let $X = (X)_{t\in\mathbb{R}}$ be a continuous stationary centered Gaussian process with $EX_sX_t = \gamma(s-t)$, $\gamma : R \to R$. Then $\gamma$ is continuous, symmetric and nonnegative definite. Set

$$\beta_0 := \frac{1}{2T}\int_{-T}^{T} \gamma(t)dt, \quad \beta_j := \frac{1}{T}\int_{-T}^{T} \gamma(t)\cos(\pi jt/T)dt,\ j \geq 1.$$

Then the extension of Theorem 3 reads as follows.

**Theorem 4.** $(a)$ ADMISSIBLITY. *Assume that the function $\gamma$ is convex, positive on $(0,\infty)$ and $\gamma(\infty) := \lim_{t\to\infty}\gamma(t) = 0$. Then $X$ admits a spectral density $p$, $\beta_j \geq 0$ for every $j \geq 0$ and the sequence defined by*

$$f_0 \;=\; \sqrt{\beta_0}, \quad f_{2j}(t) = \sqrt{\beta_j}\cos(\pi jt/T), \quad f_{2j-1}(t) = \sqrt{\beta_j}\sin(\pi jt/T), \quad j \geq 1 \qquad (4.11)$$

*is admissible for the continuous process $X$ in $\mathscr{C}([0,T])$.*

OPTIMALITY. *Assume furthermore that one of the following two conditions is satisfied by $\gamma$ for some $\delta \in (1,2]$:*

$(A_\delta)$ *$\gamma \in L^1([T,\infty))$, the spectral density $p \in L^2(R, dx)$ and the high-frequency condition*

$$p(x) \sim cx^{-\delta} \;\; as \;\; x \to \infty,$$

*or*

$(B_\delta)$ *$\gamma(t) = \gamma(0) - a|t|^{\delta-1} + b(t)$, $|t| \leq T$ with $a > 0$ and $b : [-T,T] \to R$ is a $(\delta+\eta)$-Hölder function for some $\eta > 0$, null at zero.*

*Then,*

$$l_n(X) \approx n^{-(\delta-1)/2}(\log n)^{1/2} \quad as \quad n \to \infty$$

*and the sequence (4.11) is rate optimal.*

**Remarks.** • Assumption $(A_\delta)$ is related to the spectral density whereas $(B_\delta)$ is related to the (right) regularity of $\gamma$ at 0. In fact, these assumptions are somewhat similar since the rate of decay of $p$ is closely related to the right regularity of $\gamma$ at 0. So, in practice it mainly depends on which quantity is straightforwardly available for a given process.

• The notion of $\beta$-Hölder regularity of a function $g$ means that the $[\beta]$ first derivatives of $g$ do exist on $[-T,T]$ and that $g^{([\beta])}$ is $\beta-[\beta]$-Hölder. In fact, in [16], th enotion is still a bit more general ($g^{([\beta])}$ is assumed to $\beta-[\beta]$-Hölder in a $L^1(dt)$-sense).

**Proof.** $(a)$ The function $\gamma$ is positive, convex over $(0,\infty)$ and $\gamma(\infty)=0$ so that $\gamma$ is in fact a Polya-type function. Hence its right derivative $\gamma'$ is non-decreasing with $\gamma'(\infty)=0$, the spectral measure of $X$ admits a Lebesgue-density $p$ and

$$\gamma(t)=\int_{(0,\infty)}\left(1-\frac{|t|}{s}\right)^{+}d\nu(s)$$

for all $t\in R$, where $\nu$ is a finite Borel measure on $(0,\infty)$ with mass $\gamma(0)$ (see [10], Theorems 4.3.1 and 4.3.3 for details). Therefore, using Fubini's theorem, it is enough to show the positivity of the numbers $\beta_j$ for functions of the type $\gamma(t)=(1-\frac{|t|}{s})^{+}$, $s\in(0,\infty)$. But in this case an integration by parts yields

$$\beta_0=\frac{T\wedge s}{T}(1-\frac{T\wedge s}{2s})\geq 0 \text{ and } \beta_j=\frac{2T}{s(\pi j)^2}(1-\cos(\pi j(T\wedge s)/T))\geq 0,\quad j\geq 1.$$

Now one proceeds along the lines of the proof of Theorem 3. Since $\gamma$ is of bounded variation on $[-T,T]$, the representation (4.4) of $EX_sX_t$ is true with $\beta_j(\rho)$ replaced by $\beta_j$ so that the sequence $(f_j)_{j\geq 0}$ is admissible for $X$ in $\mathscr{C}([0,T])$.

$(b)$ Assume $(A_\delta)$. Using $\gamma\in L^1(R,dt)$ and the Fourier inversion formula, one gets for $j\geq 1$

$$\beta_j=\frac{2\pi}{T}p(\pi j/T)-\frac{2}{T}\int_T^\infty \gamma(t)\cos(\pi jt/T)dt.$$

Since $\gamma(\infty)=\gamma'(\infty)=0$, integrating twice by parts yields

$$\int_T^\infty \gamma(t)\cos(\pi jt/T)dt=O(j^{-2}),$$

hence

$$\beta_j=O(j^{-\delta})\quad \text{as}\quad j\to\infty$$

in view of $\delta\leq 2$. Furthermore, the assumption $p\in L^2(R,dx)$ and the high-frequency condition yield

$$\lambda_j\sim c_1 j^{-\delta}\quad \text{as}\quad j\to\infty$$

for an appropriate constant $c_1\in(0,\infty)$ (see [16]). Now, one derives from Proposition 4 the remaining assertions.

Assume $(B_\delta)$. An integration by parts (using that $\gamma$ is even) yields

$$\beta_j=-\frac{2}{\pi j}\int_0^T \sin\left(\frac{\pi jt}{T}\right)\gamma'(t)\,dt$$

where $\gamma'$ denotes the right derivative of $\gamma$. Consequently, for $\varepsilon\in(0,T]$, using that $\gamma'\leq 0$ and is non-decreasing,

$$\beta_j=|\beta_j|\leq\frac{2}{\pi j}(\gamma(0)-\gamma(\varepsilon))+\frac{\gamma(0)-\gamma(\varepsilon)}{\varepsilon}\frac{4T}{(\pi j)^2}$$

At this stage, Assumption $(B_\delta)$ yields

$$\beta_j \le C\left(\frac{\varepsilon^{\delta-1}}{j} + \frac{\varepsilon^{\delta-2}}{j^2}\right)$$

where $C$ is positive real constant. Setting $\varepsilon = T/j$, implies as expected $\beta_j = O(j^{-\delta})$.
On the other hand, calling upon Theorem 2 in [16], shows that the eigenvalues $\lambda_j$ of the covariance operator on $L^2([0,T],dt)$ (indexed in decreasing order) satisfy

$$\lambda_j \sim 2\,\kappa\,\Gamma(\delta)\left(\frac{T}{\pi j}\right)^{\delta} \qquad \text{as } j \to \infty$$

where $\kappa = -a\cos\left(\frac{\pi}{2}\delta\right) > 0$ and $\Gamma$ denotes the Gamma function. One concludes as above. $\square$

**Acknowlegdement.** It is a pleasure to thank Wolfgang Gawronski for helpful discussions on Section 4.

## References

[1] AYACHE, A., TAQQU, M.S., Rate optimality of Wavelet series approximations of fractional Brownian motion, *J. Fourier Anal. and Appl.*, **9** (2003), 451-471.

[2] BOGACHEV, V.I., *Gaussian Measures*, AMS, 1998.

[3] BRONSKI, J.C., Small ball constants and tight eigenvalue asymptotics for fractional Brownsian motions, *J. Theoret. Probab.* 16 (2003), 87-100.

[4] CHRISTENSEN, O., *An Introduction to Frames and Riesz Bases*, Birkhäuser, Boston, 2003.

[5] DZHAPARIDZE, K., VAN ZANTEN, H., A series expansion of fractional Brownian motion, *Probab. Theory Relat. Fields*, **130** (2004), 39-55.

[6] DZHAPARIDZE, K., VAN ZANTEN, H., Optimality of an explicit series expansion of the fractional Brownian sheet, *Statist. Probab. Letters* **71** (2005), 295 - 301.

[7] DZHAPARIDZE, K., VAN ZANTEN, H., Krein's spectral theory and the Paley-Wiener expansion of fractional Brownian motion, *Ann. Probab.* **33** (2005), 620-644.

[8] KÜHN, T., LINDE, W., Optimal series representation of fractional Brownian sheets, *Bernoulli* 8 (2002), 669 - 696.

[9] LEDOUX, M., TALAGRAND, M., *Probability in Banach Spaces*, Springer, Berlin, 1991.

[10] LUKACS, E., *Characteristic Functions*, second edition, Griffin, London, 1970.

[11] LUSCHGY, H., Linear estimators and radonifying operators, *Theory Probab. Appl.* **40** (1995), 205-213.

[12] LUSCHGY, H., PAGÈS, G., Sharp asymptotics of the functional quantization problem for Gaussian processes, *Ann. Probab.* 32 (2004), 1574- 1599.

[13] LUSCHGY, H., PAGÈS, G., High-resolution product quantization for Gaussian processes under sup-norm distortion, *Bernoulli* 13 (2007), 653-671.

[14] MEYER, Y., SELLAN, F. TAQQU, M.S., Wavelets, generalized white noise and fractional integration: the synthesis of fractional Brownian motion, *J. Fourier Anal. and Appl.* **5** (1999), 465-494.

[15] NEVEU, J., *Processus Aléatoires Gaussiens*, Les Presses de l'Université de Montréal, 1968.

[16] ROSENBLATT, M., Some results on the asymptotic behaviour of eigenvalues for a class of integral equations with translation kernel, *J. Math. Mech.* 12 (1963), 619 - 628.

[17] VAKHANIA, N.N., TARIELADZE, V.I., CHOBANYAN, S.A., *Probability Distributions on Banach Spaces*, Kluwer, Boston, 1987.