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Large–range constant threshold growth model in one dimension

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Abstract

We study a one dimensional constant threshold model in continuous time. Its dynamics have two parameters, the range n and the threshold ϑ. An unoccupied site x becomes occupied at rate 1 as soon as there are at least ϑ occupied sites in $[x-n, x+n]$. As n goes to infinity and ϑ is kept fixed, the dynamics can be approximated by a continuous space version, which has an explicit invariant measure at the front. This allows us to prove that the speed of propagation is asymptoticaly $\frac{n^2}{2\vartheta}$.

Key words: growth model; asymptotic propagation velocity; invariant distribution.

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1 Introduction

Identification of limiting shape in non–trivial random growth models is known to be a hard problem and there are but a few results. These are mostly limited to exactly solvable models, see [12], [7], [8]. Approximation results are also scarce [4], [6], [2].

Here we consider growth models with large range. These are particularly interesting from the point of view of excitable media [5] and are also popular in mean–field approximations.

Perhaps the simplest generalization of additive growth models is the threshold growth model [1]. These conceptually simple dynamics have two parameters: the *neighbourhood* $\mathscr{N} \subset \mathbb{Z}^d$ and the integer *threshold* $\vartheta > 0$. In this paper we will only consider growth in continuous time, therefore the dynamics are described by the evolution of the set of occupied points $A_t \subset \mathbb{Z}^d$, $t \geq 0$. The initial set A_0 is assumed deterministic and most usually finite. The sets A_t are increasing in time, obeying the rule that $x \notin A_t$ becomes occupied at rate 1 as soon as $|(x+\mathscr{N}) \cap A_t| \geq \vartheta$. Whenever convenient we will refer to occupied points as 1's and unoccupied ones as 0's. The transition of a point from state 0 to state 1 will be sometimes refered to as a jump.

We will now formalize the notion of large range. We assume that the elementary neighbourhood $\mathscr{N}_E \subset \mathbb{R}^d$ is convex, nondegenerate (i.e., $\mathrm{Vol}_d(\mathscr{N}_E) > 0$), and symmetric (i.e., $-\mathscr{N}_E = \mathscr{N}_E$). Then, we let $\mathscr{N} = n\mathscr{N}_E \cap \mathbb{Z}^d$ for some large n.

Different assumptions on the behaviour of ϑ lead to several interesting regimes. For example, the case where ϑ is proportional to n^d leads to conservation law PDE. However, the regime in which ϑ grows to infinity much more slowly than n^d seems to have the same characteristics as the one with constant ϑ. These regimes will be the focus of our subsequent work. From here on we concentrate on the case of constant ϑ, i.e., when ϑ does not change with n.

Our main aim is an explicit identification of the leading term in asymptotic behaviour of A_t. It seems like that for constant ϑ this is only possible in one dimension. For these reasons and because there are other interesting features, not present in higher dimensions, we assume $d = 1$ henceforth. In this setting there is only one possible neighbourhood (up to a scaling constant), namely $\mathscr{N}_E = [-1, 1]$ hence the two parameters are n and ϑ.

To understand what happens when n grows to infinity we will define a continuous process which will resemble our original discrete process. In Section 2 we will study this continuous process, and in Section 3 we will show that the resemblance of the continuous process to the discrete one, no matter how weak it may seem at the moment, is more than coincidential. The proof will be made using a standard coupling argument, and the result, which is the main result of this paper, will in a way justify the interchange of limits of the time and space component. Before stating this main theorem we need to introduce the limiting process and some notation.

We rescale the lattice by factor $\frac{1}{n}$ so that the size of the neighbourhood is fixed. Simultaneously we slow down the time by replacing the time variable t by $\frac{t}{n}$. We get a process on discrete set with the distance between points equal to $\frac{1}{n}$ where particles are added (in unit interval) with rate 1. This process then resembles the following growth process on point locations (finite or countable sets with no limit points) in $\mathbb{R}$. Consider a space–time Poisson point location $\mathscr{P}$ on $\mathbb{R} \times [0, \infty)$, with constant intensity 1. Beginning with a deterministic point location $\tilde{A}_0$ we declare, for every $(x, t) \in \mathscr{P}$, $\tilde{A}_t = \tilde{A}_{t-} \cup \{x\}$ if and only if $|\tilde{A}_{t-} \cap [x-1, x+1]| \geq \vartheta$. (Note that there will only be finitely many additions to $\tilde{A}_t$ in every compact space–time region.) Other constructions of similar type have appeared in the literature [10], [11].

We define r_t to be the rightmost point x in $\mathbb{Z}$ for which $|A_t \cap [x-n, x+n]| \geq \vartheta$. We will assume $r_0 = 0$ and $A_0 \subset (-\infty, -1]$. Then $A_t \subset (-\infty, r_t - 1]$. Note that there are exactly $\vartheta - 1$ particles in $A_t \cap [r_t - n + 1, r_t - 1]$, that there is a particle at $r_t - n$ and there are no particles in A_t that are to the right of r_t. For this reason we call the interval $[r_t - n, r_t]$ the *front neighbourhood*. Since there is at time t a 1 at $r_t - n$ and there is a 0 at r_t we sometimes omit these two extreme points. We also analogously define $\tilde{r}_t$ to be the rightmost point x in $\mathbb{R}$ for which $|\tilde{A}_t \cap [x-1, x+1]| \geq \vartheta$, and assume $\tilde{r}_0 = 0$ and $\tilde{A}_0 \subset (-\infty, 0)$. These assumptions seem to restrict initial sets but it turns out that these restrictions are not crucial. There are a few classes of sets that do not obey these restrictions. For example, if for some initial set A_0 there is no point x in $\mathbb{Z}$ for which $|A_0 \cap [x-n, x+n]| \geq \vartheta$, the set A_t equals A_0 for all t. On the other hand, if there are infinitely many points in A_0 from where the growth originates, we can examine every such point by itself. If these points are close together, the occupied sets will rapidly join, and if the points are far away, we can locally look at them as in our case with the assumptions above. There is also the case where there are some points in state 1 to the right of the initial front neighbourhood. It can be seen that if there are only finitely many of them, it will eventually happen that the front neighbourhood will surpass them – our restrictions are then met. On the other hand, if there are infinitely many points in A_0 to the right of the front neighbourhood this can mean a serious increase in the propagation speed. This case will not be considered in this article.

The propagation velocities are then characterized by

$$w_n = \lim_{t \to \infty} \frac{r_t}{t} \qquad \text{and} \qquad w = \lim_{t \to \infty} \frac{\tilde{r}_t}{t}.$$

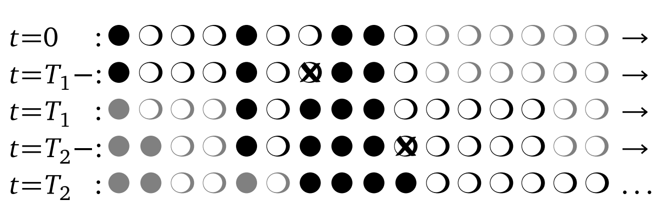
We will prove that these limits exist in Lemma 2 and Lemma 3. Then, Theorem 3 will tell us that $w = \frac{1}{2\vartheta}$. Finally, the coupling argument will help us prove the main result of our paper, namely

Theorem 1. *Let w_n denote the propagation velocities in the discrete model with neighbourhood $[-n, n]$. Then*

$$\lim_{n \to \infty} \frac{w_n}{n^2} = \frac{1}{2\vartheta}.$$

This theorem basically means that the interchange of limits ($n \to \infty$ and $t \to \infty$) works.

It may be helpful to present the model with parameters $n = 9$ and $\vartheta = 4$. In the following schemas the white circle denotes an unoccupied site, and the black one an occupied one. The front neighbourhood is emphasised and the other points are depicted in light grey. Every white circle in the front neighbourhood has equal probability of being the first one to jump from state 0 to state 1. The circle that does jump is denoted with a black cross. The jumps occur at times T_k. Here we see a few steps at the front.



Note that there was a transition of the second site between the times T_1 and T_2 but this transition has no effect on the front neighbourhood. It is clear that the dynamics restricted to the interval $[r_t - n + 1, r_t]$, translated to $[1, n]$ and looked at times $T_1, T_2, \dots$ defines a discrete time Markov chain on all size $\vartheta - 1$ subsets of $\{1, 2, \dots, n-1\}$.

This chain is irreducible and aperiodic, hence there exists a unique invariant measure, which in turn determines w_n. The details are covered in Section 3, specifically in Lemma 3. The same lemma also deals with the effect of random times when these transitions occur.

By the same argument, the limiting dynamics $\tilde{A}_t$ gives rise to a discrete time Markov chain on subsets of $[0, 1]$ of size $\vartheta - 1$. The state of this chain at time t is given by

$$(X_1(t), \dots, X_{\vartheta-1}(t)), \quad 0 \le X_1(t) \le \dots \le X_{\vartheta-1}(t) \le 1\,.$$

The transition of random vector X (from time T_k) at time T_{k+1} is defined by a uniformly distributed random variable U_k which simulates the position of the first particle in the neighbourhood jumping from 0 to 1. The random variable U_k is independent of X for any t, independent of t and also independent of all the random variables $U_l, l < k$. The transition can be described in the following way: if U_k is smaller than $X_1(T_k)$, then

$$\begin{aligned}
&(X_1(T_{k+1}), \dots, X_{\vartheta-1}(T_{k+1})) = \\
&\quad = (X_1(T_k) - U_k, \dots, X_{\vartheta-1}(T_k) - U_k)\,,
\end{aligned}$$

and if there are exactly m components of $X(T_k)$ smaller than U_k

$$\begin{aligned}
&(X_1(T_{k+1}), \dots, X_{\vartheta-1}(T_{k+1})) = \\
&\quad = (X_2(T_k) - X_1(T_k), \dots, X_m(T_k) - X_1(T_k), \\
&\qquad U_k - X_1(T_k), X_{m+1}(T_k) - X_1(T_k), \dots, X_{\vartheta-1}(T_k) - X_1(T_k))\,.
\end{aligned}$$

To simplify matters we will observe the processes only at jump times T_n. With a slight change of notation we conveniently label jump times by $1, 2, \dots$ and write $X(n) = X(T_n)$. In this interpretation, a new particle appears, and the neighbourhood moves so far to the right that the new neighbourhood still has ϑ particles in state 1, with one of them at the left edge of the neighbourhood. In a way we eliminated the component of a random time, at which these jumps occur. As already said, we will deal with this part of dynamics in Section 3. This defines a discrete time Markov chain on an infinite state space. Relevant results in the field (like Aperiodic Ergodic Theorem 13.0.1 in [9]) show that the distribution of this Markov chain converges to a unique invariant measure. The exact way how this invariant measure determines the limiting speed w will be given in Lemma 2.

The most trivial case is when ϑ equals 1. Then the Markov chain is constant, i.e. $X(t) = ()$ and the speed of spreading is $w = E[U] = \frac{1}{2}$.

The rest of the article is organized as follows: in Section 2 we consider the smallest nontrivial case $\vartheta = 2$, and then procede to determine the invariant measure in the general case (Theorem 2). Section 3 shows that the propagation velocity in discrete state space converges to propagation velocity in continuous state space. Together with the results from Section 2 Theorem 1 tells us that the speed of propagation is asymptoticaly $\frac{n^2}{2\vartheta}$. The final section adds some further insights into described models and identifies additional open problems.

2 Continuous state space Markov chain

Our goal in this section is to determine the invariant measure of the continuous state space Markov chain $X(t)$ which we defined in the previous section. The general theory of Markov chains on general state spaces is thoroughly described in [9]. Since each of our chains is recurrent and irreducible all the theorems known from discrete state space Markov chains are also valid, like the existence and uniqueness of invariant measure (Theorem 10.0.1 in [9]), the ergodic theorem (Theorem 13.0.1 in [9]) and the law of large numbers (Theorem 17.1.7 in [9]).

We will use the mentioned theorems but first we take a look at a few illustrative examples and then continue from them. Finally, we will identify the invariant measure and deduce from it the speed of propagation.

Example 1. *Assume that $\vartheta = 2$. Then it is easy to see that the Markov chain is described by single component $X(k)$, and $X(k+1) = |X(k) - U_k|$. For the sake of simplicity we write U for U_k, X for $X_1(k)$ and X' for $X_1(k+1)$. It follows that $X' = |X - U|$.*

The distribution of X' can be derived from the distribution of X in the following way. First we note that

$$\begin{aligned} P(X' \in (x, x + \mathrm{d}x)) &= P(X \in (U + x, U + x + \mathrm{d}x)) \\ &+ \quad P(X \in (U - x - \mathrm{d}x, U - x)) \end{aligned} \tag{1}$$

From here on we will assume that X is continuously distributed with a density function $f(x)$. This assumption simplifies the analysis, and since we know from the general theory there exists a unique invariant measure, independent of the initial distribution, this does not change the final result. Using conditional expectation and first order approximation we get

$$\begin{aligned} P(X \in (U + x, U + x + \mathrm{d}x)) &= E(E[1(X \in (U + x, U + x + \mathrm{d}x))|U]) \\ &\approx E(f(U + x)\mathrm{d}x) = \left(\int_0^1 f(u + x)\mathrm{d}u\right)\mathrm{d}x . \end{aligned} \tag{2}$$

Like usually $f(x, \mathrm{d}x) \approx g(x, \mathrm{d}x)$ means that $\lim_{\mathrm{d}x\to 0} \frac{f(x,\mathrm{d}x)}{g(x,\mathrm{d}x)} = 1$. Similar equality holds for the other term in (1). Summing these two terms and denoting the density of X' by g we get

$$g(x) = \int_0^1 f(u + x)\mathrm{d}u + \int_0^1 f(u - x)\mathrm{d}u$$

We use the fact that f is a density on $[0, 1]$ to see

$$g(x) = \int_0^1 f(u)\mathrm{d}u + \int_x^{1-x} f(u)\mathrm{d}u = 1 + \int_x^{1-x} f(u)\mathrm{d}u .$$

If X is uniformly distributed ($f(x) \equiv 1$ on $[0, 1]$), X' has density $g(x) = 2(1 - x)$.

It follows that the stationary distribution solves the integral equation

$$f(x) = 1 + \int_x^{1-x} f(u)\mathrm{d}u . \tag{3}$$

First we use a standard argument. The right hand side is a continuous function of x, so f is continuous too. However, the right hand side of this equation is then a differentiable function, so we know that f has to be differentiable too. Taking the derivative of f gives

$$f'(x) = -f(1-x) - f(x)$$

and using (3) for the righthand side yields

$$f'(x) = -2\,.$$

We get $f(x) = 2(1-x)$. We have shown that this is the unique stationary density function.

It may be worth mentioning that for every initial distribution this particular Markov chain reaches its equilibrium in at most two steps, as we show in Proposition 1. Also, the distribution after the first step is always continuous, regardless of the initial distribution. Proposition 1 redetermines the invariant measure of Example 1 with a simpler argument, however, the methods used in Example 1 can be extended to the case $\vartheta > 2$ so it is meant as an introduction to Theorem 3.

Proposition 1. *If X is a random variable with values in $[0,1]$, U_1 and U_2 are uniformly distributed on $[0,1]$ and X, U_1 and U_2 are independent, then the random variable*

$$||X - U_1| - U_2|$$

is continuously distributed with density $2(1-x)$ on $[0,1]$.

Proof. Start by

$$P(|X - U_1| > t) = \int_0^1 \mathrm{d}u \int_0^1 1(|x-u| > t)\mu_X(\mathrm{d}x)$$

and since the condition $|x-u| > t$ holds for $u - t > x$ or $x > t + u$ we get

$$\begin{aligned} P(|X - U_1| > t) &= \int_0^1 \mathrm{d}u \int_{[0,u-t)} \mu_X(\mathrm{d}x) + \int_0^1 \mathrm{d}u \int_{(u+t,1]} \mu_X(\mathrm{d}x) \\ &= \int_0^{1-t} P(X < s)\mathrm{d}s + \int_t^1 P(X > s)\mathrm{d}s \end{aligned}$$

So, we know

$$P(|X - U_1| \le t) = 1 - \int_0^{1-t} P(X < s)\mathrm{d}s - \int_t^1 P(X > s)\mathrm{d}s$$

which is a.e. differentiable function of t. So we can write

$$\frac{\mathrm{d}}{\mathrm{d}t} P(|X - U_1| \le t) = P(X < 1-t) + P(X > t)\,.$$

This proves the well known fact that $|X - U_1|$ is continuously distributed.

Let us denote the density of $|X - U_1|$ by f_1 and its distribution function by F. It follows by the argument above and the fact that F is continuous that $||X - U_1| - U_2|$ has a density equal to

$$f_2(x) = F(1-x) + 1 - F(x)\,.$$

Since

$$
\begin{aligned}
F(1-x) &= 1-\int_0^{1-(1-x)} P(X<s)\mathrm{d}s - \int_{1-x}^1 P(X>s)\mathrm{d}s \\
&= 1-x-\int_0^{x} P(X<s)\mathrm{d}s + \int_{1-x}^1 P(X<s)\mathrm{d}s
\end{aligned}
$$

and

$$
\begin{aligned}
F(x) &= 1-\int_0^{1-x} P(X<s)\mathrm{d}s - \int_{x}^1 P(X>s)\mathrm{d}s \\
&= x-\int_0^{1-x} P(X<s)\mathrm{d}s + \int_{x}^1 P(X<s)\mathrm{d}s
\end{aligned}
$$

it follows that

$$
f_2(x) = 2(1-x)\,.
$$

The density f_2 is independent of the initial distribution of X. □

Remark 1. *Proposition 1 could also be proved by a coupling argument.*

Now we take a look at arbitrary ϑ. Since one occupied point is fixed at the left border of front neighbourhood there are only $\vartheta-1$ free parameters; hence we introduce $\eta = \vartheta - 1$.

In the proof of the next theorem we will use a technical Lemma to determine the form of the stationary distribution. We will also use the Lemma to show how we could prove the uniqueness of the stationary distribution. The Lemma seems interesting by itself:

Lemma 1. *Let n be arbitrary positive integer and $h: \mathbb{R}^n \to \mathbb{R}^+$ be arbitrary integrable function on D where $D = \{\mathbf{x}; \mathbf{x} = (x_1, x_2, \ldots, x_n), 0 \le x_1 \le \ldots \le x_n \le 1\}$. With $x_0 = 0$ we have*

$$
\int_D \mathrm{d}\mathbf{x} \int_0^{1-x_n} \left(\sum_{j=0}^{n} h(u+x_0, u+x_1, \ldots, \widehat{u+x_j}, \ldots, u+x_n) \right) \mathrm{d}u = \int_D h(x_1, \ldots, x_n)\mathrm{d}\mathbf{x}\,.
$$

We use the standard notation $(x_1, x_2, \ldots, \widehat{x_j}, \ldots, x_\eta) = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_\eta)$.

Proof. With the simultaneus introduction of a new function and variables

$$
\begin{aligned}
y_1 = x_1, y_2 = x_2 - x_1, \ldots, y_n = x_n - x_{n-1} \\
h^*(y_1, y_2, \ldots, y_n) = h(x_1, x_2, \ldots, x_n)
\end{aligned}
$$

the right hand side of equality equals

$$
\int_\Delta h^*(y_1, \ldots, y_n)\mathrm{d}\mathbf{y}\,,
$$

where $\Delta = \{(y_1, y_2, \dots, y_n); \forall i{:}y_i \geq 0, \sum_{i=1}^{n} y_1 \leq 1\}$ is the standard simplex (Jacobian of the transformation equals 1). For the left hand side of the equality we first observe

$$h(u+x_0, u+x_1, \dots, \widehat{u+x_j}, \dots, u+x_n) = h^*(u, y_1, y_2, \dots, y_{j-1}, y_j + y_{j+1}, y_{j+2}, \dots, y_n)$$

(with the last term $h(u+x_0, u+x_1, \dots, u+x_{n-1}) = h^*(u, y_1, y_2, \dots, y_{n-1})$). So we have

$$\int_D \mathrm{d}\mathbf{x} \int_0^{1-x_n} \left(\sum_{j=0}^{n} h(u+x_0, u+x_1, \dots, \widehat{u+x_j}, \dots, u+x_n) \right) \mathrm{d}u$$
$$= \sum_{j=0}^{n} \int_\Delta \mathrm{d}\mathbf{y} \int_0^{1-\sum_{i=1}^{n} y_i} h^*(u, y_1, y_2, \dots, y_{j-1}, y_j + y_{j+1}, y_{j+2}, \dots, y_n) \mathrm{d}u\,.$$

If we denote $y_0 = u$ this actually equals

$$\sum_{j=0}^{n} \int_{\Delta'} h^*(y_0, y_1, y_2, \dots, y_{j-1}, y_j + y_{j+1}, y_{j+2}, \dots, y_n) \mathrm{d}\mathbf{y}'$$

where $\Delta' = \{(y_0, y_1, y_2, \dots, y_n); \forall i{:}y_i \geq 0, \sum_{i=0}^{n} y_1 \leq 1\}$ is again a simplex. Introducing the variables

$$z_1 = y_0, z_2 = y_1, \dots, z_j = y_{j-1}, z_{j+1} = y_j + y_{j+1}, z_{j+2} = y_{j+2}, \dots, z_n = y_n, t = y_j$$

for $j = 0, \dots, n-1$, and

$$z_1 = y_0, z_2 = y_1, \dots, z_n = y_{n-1}, t = y_n$$

for $j = n$, we have

$$\int_{\Delta'} h^*(y_0, y_1, y_2, \dots, y_{j-1}, y_j + y_{j+1}, y_{j+2}, \dots, y_n) \mathrm{d}\mathbf{y}' = \int_{\substack{\mathbf{z} \in \Delta \\ 0 \leq t \leq z_{j+1}}} h^*(z_1, z_2, \dots, z_n) \mathrm{d}\mathbf{z} \mathrm{d}t$$

for $j = 0, \dots, n-1$, and

$$\int_\Delta^{\prime} h^*(y_0, y_1, \dots, y_{n-1}) \mathrm{d}\mathbf{y}' = \int_{\substack{\mathbf{z} \in \Delta \\ 0 \leq t, t + \sum_{i=1}^{n} z_i \leq 1}} h^*(z_1, z_2, \dots, z_n) \mathrm{d}\mathbf{z} \mathrm{d}t\,.$$

Integrating over t and summing up all these terms gives us

$$\int_\Delta h^*(z_1, z_2, \dots, z_n)(z_1 + \ldots + z_n + 1 - \sum_{i=1}^{n} z_i) \mathrm{d}\mathbf{z}\,,$$

which is the right hand side of equality in lemma and so we are done. □

Theorem 2. *For the observed Markov chain there exists a unique stationary distribution with density function*

$$f_{\mathbf{X}}(x_1, x_2, \dots, x_\eta) = 2^\eta \eta! \prod_{i=1}^{\eta} (1 - x_i) 1(0 \leq x_1 \leq x_2 \leq \cdots \leq x_\eta \leq 1)\,.$$

Proof. First let us look at the transformation of X at a given step. This actually is a transition rule of our Markov chain. Define a uniformly distributed random variable U, independent of X. As in the Example 1 we assume that X has a continuous density. Then we can write

$$P(\cap_{i=1}^{\eta}\{X_i' \in (x_i, x_i + \mathrm{d}x_i)\}) \approx f_{\mathbf{X}'}(x_1, x_2, \ldots, x_\eta)\mathrm{d}x_1\mathrm{d}x_2 \ldots \mathrm{d}x_\eta\,.$$

and, using U and X to describe X',

$$\begin{aligned}P\left(\bigcap_{i=1}^{\eta}\{X_i' \in (x_i, x_i + \mathrm{d}x_i)\}\right) &= P\left(\bigcap_{i=1}^{\eta}\{X_i - U \in (x_i, x_i + \mathrm{d}x_i)\}\right) + \\ &+ \sum_{j=1}^{\eta} P\left(\bigcap_{i=1}^{j-1}\{X_{i+1} - X_1 \in (x_i, x_i + \mathrm{d}x_i)\} \cap \{U - X_1 \in (x_j, x_j + \mathrm{d}x_j)\}\cap\right. \\ &\qquad \left. \cap \bigcap_{i=j+1}^{\eta}\{X_i - X_1 \in (x_i, x_i + \mathrm{d}x_i)\}\right)\end{aligned}$$

Integrating we derive

$$\begin{aligned}f_{\mathbf{X}'}(x_1, x_2, \ldots, x_\eta) = &\int_0^{1-x_\eta} f_{\mathbf{X}}(u + x_1, u + x_2, \ldots, u + x_\eta)\mathrm{d}u + \\ &+ \sum_{j=1}^{\eta}\int_0^{1-x_\eta} f_{\mathbf{X}}(u, u + x_1, u + x_2, \ldots, \widehat{u + x_j}, \ldots, u + x_\eta)\mathrm{d}u.\end{aligned}$$

Defining $x_0 = 0$, the new density $f_{\mathbf{X}'}$ can be written as

$$f_{\mathbf{X}'}(x_1, x_2, \ldots, x_\eta) = \int_0^{1-x_\eta}\left(\sum_{j=0}^{\eta} f_{\mathbf{X}}(u + x_0, u + x_1, \ldots, \widehat{u + x_j}, \ldots, u + x_\eta)\right)\mathrm{d}u \tag{4}$$

We define the operator L with

$$(Lf)(x_1, x_2, \ldots, x_\eta) = \int_0^{1-x_\eta}\left(\sum_{j=0}^{\eta} f(u + x_0, u + x_1, \ldots, \widehat{u + x_j}, \ldots, u + x_\eta)\right)\mathrm{d}u$$

and then try to find its fixed point, function g:

$$Lg = g\,. \tag{5}$$

For $\vartheta = 2$ $(\eta = 1)$ we already know the equation and also its solution. For $\vartheta = 3$ $(\eta = 2)$ we get

$$g(x, y) = \int_0^{1-y}\big(g(x + u, y + u) + g(u, x + u) + g(u, y + u)\big)\,\mathrm{d}u$$

which can not be transformed into a differential equation. This is also the case for $\vartheta > 3$.

Now we show how the uniqueness of the invariant measure could be proved without the Markov Chain Uniqueness Theorem. Let f and g be the functions with the property

$$f = Lf \qquad \text{and} \qquad g = Lg\,.$$

We compute the norm $||f - g||_1$ on D:

$$||f - g||_1 = \int_D |f(\mathbf{x}) - g(\mathbf{x})| \mathrm{d}\mathbf{x}.$$

Using the stationary property of f and g we get

$$\begin{aligned} ||f - g||_1 \quad &= \quad \int_D \left| \int_0^{1-x_\eta} \left(\sum_{j=0}^{\eta} (f(u+x_0, u+x_1, \dots, \widehat{u+x_j}, \dots, u+x_\eta) - \right.\right. \\ &\left.\left. -g(u+x_0, u+x_1, \dots, \widehat{u+x_j}, \dots, u+x_\eta)) \right) \mathrm{d}u \right| \mathrm{d}\mathbf{x}. \end{aligned}$$

Using triangle inequality we get

$$\begin{aligned} ||f - g||_1 \quad &\le \quad \int_D \int_0^{1-x_\eta} \left(\sum_{j=0}^{\eta} |f(u+x_0, u+x_1, \dots, \widehat{u+x_j}, \dots, u+x_\eta) - \right. \\ &\left. -g(u+x_0, u+x_1, \dots, \widehat{u+x_j}, \dots, u+x_{\eta-1})| \right) \mathrm{d}u \mathrm{d}\mathbf{x}, \end{aligned}$$

with equality only when all the terms are of the same sign. Finally, using Lemma 1 for the function $h = |f - g|$ gives us that

$$\begin{aligned} &\int_D \int_0^{1-x_\eta} \left(\sum_{j=0}^{\eta} |f(u+x_0, u+x_1, \dots, \widehat{u+x_j}, \dots, u+x_\eta) - \right. \\ &\left. -g(u+x_0, u+x_1, \dots, \widehat{u+x_j}, \dots, u+x_\eta)| \right) \mathrm{d}u \mathrm{d}\mathbf{x} = ||f - g||_1. \end{aligned}$$

So the terms $f(u+x_0, u+x_1, \dots, \widehat{u+x_j}, \dots, u+x_\eta) - g(u+x_0, u+x_1, \dots, \widehat{u+x_j}, \dots, u+x_\eta)$ are of the same sign for all j and u, for all possible values of x_i. It is not hard to see that this is only possible when $f - g$ is of the same sign. Since f and g are densities, their integrals are equal, so they must be almost surely equal.

We can now finish the proof of Theorem 2. It remains to check that the function suggested in theorem really satisfies the transformation condition (5). We integrate once more.

If $g(x_1, \dots, x_\eta) = c_\eta \prod_{i=1}^{\eta} (1 - x_i)$, then from

$$\frac{\mathrm{d}}{\mathrm{d}u} \left(-c_\eta \prod_{i=0}^{\eta} (1 - u - x_i) \right) = \sum_{j=0}^{\eta} \left(c_\eta \prod_{i=0, i \ne j}^{\eta} (1 - u - x_i) \right)$$

it follows that

$$-\frac{d}{du}(1-u)g(u+x_1,\dots,u+x_\eta)=\sum_{j=0}^{n}g(u+x_0,u+x_1,\dots,\widehat{u+x_j},\dots,u+x_\eta)\,.$$

Integrating from 0 to $1-x_\eta$ we get on the right-hand side

$$\int_0^{1-x_\eta}\sum_{j=0}^{\eta}g(u+x_0,u+x_1,\dots,\widehat{u+x_j},\dots,u+x_\eta)\mathrm{d}u\,,$$

which actually is the right-hand side of (5). On the other hand we get

$$\begin{aligned}-\int_0^{1-x_\eta}\frac{\mathrm{d}}{\mathrm{d}u}(1-u)g(u+x_1,\ \dots\ ,u+x_\eta)\mathrm{d}u &= -\,(1-u)g(u+x_1,\dots,u+x_\eta)\Big|_0^{1-x_\eta}\\ &= -x_\eta g(1-x_\eta+x_1,\dots,1-x_\eta+x_\eta)+g(x_1,\dots,x_\eta)\,.\end{aligned}$$

The first term equals 0 so g is stationarity. Recognizing that the density in question equals the density of an ordered sample of equally distributed random variables with density $2(1-x)1_{[0,1]}(x)$ implies $c_\eta=2^\eta\eta!$. This finishes the proof of Theorem. $\square$

The final part of this section is the computation of the propagation velocity from the invariant measure. From the theory of order statistics we know

$$f_{X_k}(x)=2k\binom{\vartheta-1}{k}(1-x)^{2\vartheta-2k-1}(1-(1-x)^2)^{k-1}\,,$$

see for example [3].

It follows from the result above

$$f_{X_1}(x_1)=2(\vartheta-1)(1-x_1)^{2\vartheta-3}\,,$$

which is a density of a Beta $(1,2\vartheta-2)$ distribution, and so we know $E(X_1)=\frac{1}{2\vartheta-1}$ and $E({X_1}^2)=\frac{1}{\vartheta(2\vartheta-1)}$.

Besides X_1 (the first nontrivial point in state 1 in the front neighbourhood) we are really interested only in $X_{\vartheta-1}$ (the rightmost point in state 1):

$$f_{X_{\vartheta-1}}(x)=2(\vartheta-1)(1-x)(1-(1-x)^2)^{\vartheta-2}$$

As with X_1 we need $E(X_{\vartheta-1})$ and $\mathrm{var}(X_{\vartheta-1})$. Simple computations gives us $E(X_{\vartheta-1})=1-2^{2\vartheta-1}\frac{\vartheta!(\vartheta-1)!}{(2\vartheta)!}$ and $E(X^2_{\vartheta-1})=1-2^{2\vartheta}\frac{\vartheta!(\vartheta-1)!}{(2\vartheta)!}+\frac{1}{\vartheta}$.

The speed of growth can be identified with the shift of the left border of the front neighbourhood.

Lemma 2. *The speed of spreading w exists and equals*

$$w=E[\min\{X_1,U\}]\,.$$

Proof. We stated in the introduction that $w = \lim_{t\to\infty} \frac{\tilde{r}_t}{t}$. We can write w in the form

$$w = \lim_{t\to\infty} \frac{\sum_{s=1}^{t}(\tilde{r}_s - \tilde{r}_{s-1})}{t}. \tag{6}$$

Here, $\tilde{r}_s - \tilde{r}_{s-1}$ is the width of a jump of the front neighbourhood at time s which is in fact the function of $X(s-1)$, the state of Markov chain at time $s-1$, and of U_s, the position of a particle changing its state to 1. A straightforward application of transtion rule of Markov chain $X(t)$ shows that $\tilde{r}_s - \tilde{r}_{s-1} = \min\{X_1(s-1), U_s\}$. So, the right hand side of (6) becomes

$$\lim_{t\to\infty} \frac{\sum_{s=1}^{t} \min\{X_1(s-1), U_s\}}{t}.$$

Since the Law of large numbers for Markov chains in general state–space applies (see Theorem 17.1.7 in [9]), the above limit exists and the speed equals

$$w = E[\min\{X_1, U\}].$$

□

The speed could also be identified as the shift of the rightmost point in state 1,

$$E[(U - X_{\vartheta-1})_+].$$

It comes with no surprise that these two values are the same. Another characterisation of the propagating velocity will be mentioned in the last section.

Proposition 2. *Let U be a random variable, uniformly distributed on $[0,1]$, and let X be an arbitrary random variable with values on $[0,1]$. Let U and X be independent. Then*

$$E(\min(U,X)) = E(X) - \frac{1}{2}E(X^2).$$

Proof. Using conditional expectation we compute

$$\begin{aligned} E(\min(U,X)) &= E(E[\min(U,X)|X]) = E\left(\int_0^1 \min(u,X)\mathrm{d}u\right) \\ &= E\left(\frac{X^2}{2} + X - X^2\right) = E(X) - \frac{1}{2}E(X^2). \end{aligned}$$

□

Theorem 3. *The speed of propagation in the continuous model equals*

$$w = \frac{1}{2\vartheta}.$$

Proof. We use Lemma 2, Proposition 2 and known moments of X_1 to find

$$w = E(\min(U, X_1)) = E(X_1) - \frac{1}{2}E({X_1}^2) = \frac{1}{2\vartheta - 1} - \frac{1}{2\vartheta(2\vartheta - 1)}$$
$$= \frac{1}{2\vartheta}\,.$$

□

Remark 2. *Since* $\min(u, x) + (u - x)_+ = u$ *it follows that*

$$E((U - X)_+) = \frac{1}{2} - E(X) + \frac{1}{2}E(X^2)\,.$$

So if we take the alternative definiton of w *and use moments of* $X_{\vartheta-1}$ *we get*

$$E((U - X_{\vartheta-1})) = \frac{1}{2} - E(X_{\vartheta-1}) + \frac{1}{2}E({X_{\vartheta-1}}^2) = \frac{1}{2\vartheta}\,.$$

The result is not unexpected. There are ϑ *points in* $[0, 1]$*, so every point (on average) is in this neighbourhood for* ϑ *steps. And because every entry is uniformly distributed, it enters the neighbourhood (again on average) at* $\frac{1}{2}$*. So in* ϑ *steps it has to traverse the distance of* $\frac{1}{2}$*.*

3 Coupling discrete and continuous model

Now we connect the speed in finite n (discrete states) with the speed in the limit process (continuous states). We need to set up an appropriate coupling argument between the discrete process and the continuous one. However, before doing that we will eliminate the effect of random time of the jump. Remember that r_t is the rightmost point x in $\mathbb{Z}$ for which $|A_t \cap [x - n, x + n]| \geq \vartheta$. Let $T_1, T_2, \ldots$ be the (random) times when new points in the front neighbourhood change state from 0 to 1. So r_t changes exactly at times $T_1, T_2, \ldots$. We say that at these times the front jumps. The average speed of propagation over m such jumps then equals

$$\frac{r_{T_m}}{T_m} = \frac{\sum_{i=1}^{m}(r_{T_i} - r_{T_{i-1}})}{\sum_{i=1}^{m}(T_i - T_{i-1})}.$$

Since $T_i - T_{i-1}$ is the time between $(i-1)$–th and i–th jump of r_t, the $T_i - T_{i-1}$ are i.i.d. random variables, exponentially distributed with parameter $n - \vartheta$. By the strong law of large numbers $\frac{1}{m}\sum_{i=1}^{m}(T_i - T_{i-1})$ converges to $\frac{1}{n-\vartheta}$. As in the proof of Lemma 2 the variables $r_{T_i} - r_{T_{i-1}}$ can be written as a function of a Markov chain of the wavefront as described in the introduction. Since the chain is aperiodic and irreducible

$$\frac{1}{m}\sum_{i=1}^{m}(r_{T_i} - r_{T_{i-1}}) \tag{7}$$

converges a.e. to a limit $\hat{w}_n$ by the Law of large numbers. We just proved the existence of w_n and $\hat{w}_n$ and their mutual connection. We state these facts in the following Lemma:

Lemma 3. *The average speed of the continuous process converges to $\hat{w}_n \cdot (n - \vartheta)$ so we have*

$$w_n = \hat{w}_n(n - \vartheta)\ .$$

Now we prove the main theorem, already stated in the Introduction.

Theorem 1. *Let w_n denote the propagation velocities in the discrete model with neighbourhood $[-n, n]$. Then*

$$\lim_{n\to\infty} \frac{w_n}{n^2} = \frac{1}{2\vartheta}.$$

Proof. Using the preceding Lemma 3 we need to show that $\hat{w}_n \sim n \cdot w$. In the proof of this we use the standard coupling argument.

First we list some facts about the one dimensional process. If A_0 is a valid starting set (i.e., such that there exists a point in state 0 with at least ϑ state–1–points in its neighbourhood and all the points right to it are in state 0) and A_0' is the set A_0 without one occupied point, then $r_t \geq r_t'$, i.e. the front growing from A_0 is farther than the front growing from A_0' (with the same underlying Poisson location process). This is clear by monotonicity. The same conclusion holds if A_0' is obtained from A_0 by moving one occupied point to the left. This fact follows from a simple coupling argument. The final and trivial fact is that r_t and r_t' are the same for A_0 and $A_0' = A_0 \cap [r_0 - n, r_0]$, i.e. the points left of the front do not affect its propagation to the right.

This also means that the convergence of (7) is uniform over all initial configurations of the front neighbourhood: if we move all the points in the initial neighbourhood to the left, we get the dynamics below the one starting from the initial one, and if we move all the points to the right, we get the dynamics above. But both these dynamics can be coupled in a natural way (every point in the lower dynamics generates a point in the upper dynamics, with their distance being exactly the size of the neighbourhood).

Finally, we describe the natural coupling between the discrete and continuous processes. Namely, we project the continuous Poisson point location $\mathscr{P}$ to a discrete space grid, to get the discrete Poisson point location $\lfloor \mathscr{P} \rfloor = \{(\lfloor nx \rfloor, t); (x, t) \in \mathscr{P}\}$, which uniquely defines the evolution of the discrete space process. Of course we already eliminated the time component so we will only be interested in location–component of $\mathscr{P}$ (and $\lfloor \mathscr{P} \rfloor$).

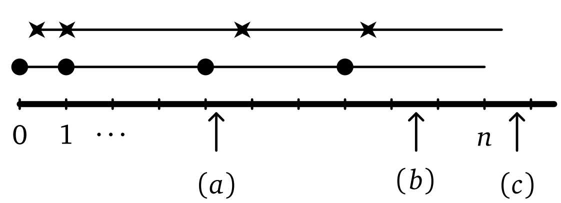


Figure 1: Three basically different posibilities in coupling continuous and discrete processes: in case (a) the continuous one moves while the discrete one does not change, in case (b) they stay approximately aligned, and in case (c) the discrete one overtakes the continuous one.

One of the complications which we will need to address below is that a point in $\mathscr{P}$ in the front always results in a new occupied point whereas this is not true for the point in $\lfloor \mathscr{P} \rfloor$ in the discrete model. This fact enables the continuous model to ocassionaly overtake the discrete one. It is also possible for the discrete model to "overtake" the continuous one. This happens whenever there is a point in continuous process just outside the front neighbourhood so that the clipped point in discrete model still is in the front neighbourhood. The point is both this occasions are relatively rare. Figure 1 shows the front neighbourhoods in discrete and continuous settings. The size of the neighbourhood is n, the points in the discrete front are denoted by a circle and the points in the continuous front by a cross. There are three possibilities where a new particle can appear. If the $\mathscr{P}$ process generates the point at location (a), it means that the discrete front does not change but the continuous one does. If the point appears at position (b), both fronts move, and if the point appears at position (c), the continuous front does not change (since the point is outside the front) but the discrete one changes (the corresponding point in $\lfloor \mathscr{P} \rfloor$ is in the front). The probability that the case (a) takes place is $\frac{\vartheta}{n+1}$ and the probability that the case (c) occurs is less than $\frac{1}{n+1}$.

We start by proving that the discrete process is in the limit above the continuous one,

$$\liminf_{n\to\infty} \frac{\hat{w}_n}{n} \geq w\,. \tag{8}$$

Let $b_k = r_{T_k} - r_{T_{k-1}}$ be the size of the k–th shift of the front in the continuous dynamics, and let a_k be the size of the k–th shift of the front in (coupled) discrete dynamics. The starting configuration for both dynamics should be aligned, meaning that the points in discrete configuration are the rounded down points of the continuous dynamics (as in the picture above). With probability at least $\frac{n-\vartheta}{n+1}$ the underlying Poisson point location $\mathscr{P}$ is such that the first step will keep both configurations (at the front) aligned. So both configuration of the front stay aligned over m consecutive steps with probability at least $\left(\frac{n-\vartheta}{n+1}\right)^m$. In this case the sum of shifts is nearly equal (the difference can be at most 1, depending on the initial and last configurations). So we have

$$a_1 + a_2 + \ldots + a_m \geq nb_1 + nb_2 + \ldots + nb_m - 1\,. \tag{9}$$

If something goes wrong, we still have

$$a_1 + a_2 + \ldots + a_m \geq 0\,.$$

We can choose large enough n and m so that the probability of both processes being aligned for m steps is arbitrarily close to 1. Then we have (using law of large numbers for the continuous process) for every $\varepsilon > 0$ and large enough n and m (take for example $m = \sqrt{n}$ to have just one parameter) that in the case (9)

$$a_1 + a_2 + \ldots a_m \geq nm(w - \varepsilon) - 1$$

and so

$$E[a_1 + a_2 + \ldots a_m] \geq (nm(w-\varepsilon) - 1)\left(\frac{n-\vartheta}{n+1}\right)^m\,.$$

After observing

$$\frac{\sum_{k=1}^{j} a_k}{j} \geq \frac{\sum_{l=1}^{\lfloor j/m \rfloor} \frac{\sum_{k=(l-1)m+1}^{lm} a_k}{m}}{j/m}$$

and applying law of large numbers for the discrete process we finaly get

$$\frac{\hat{w}_n}{n} \geq \left(w - \varepsilon - \frac{1}{nm}\right)\left(\frac{n-\vartheta}{n+1}\right)^m\,.$$

We have proved (8) from above.

This approach is not valid for the upper bound since we cannot establish the upper bound for the sum of shifts in the discrete dynamics if the discrete dynamics overtakes the continuous one (if the possibility (c) from the Figure 1 takes place, the fronts are not aligned any more and there is a possibility that in the next few steps the discrete front moves without the continuous one, and then they are too far away from each other to make coupling work again).

Another remark is that the lower bound we estabished for $\frac{\hat{w}_n}{n}$ is not a very good one, which we know from the case $\vartheta = 2$.

For the conclusion of the proof we show that the continuous dynamics is in the limit faster than the discrete one:

$$\limsup_{n\to\infty} \frac{\hat{w}_n}{n} \leq w\,. \tag{10}$$

What we will do is take the original continuous dynamics and use it to generate a new, slower continuous dynamics. We will do it in such a way that the new and the original continuous dynamics will generate exactly the same discrete dynamics. But as we already saw in the picture above we need to take a look at the interval $[0, n+1]$ of continuous dynamics since this interval affects the discrete dynamics. So now the scaling factor will be $n+1$.

Let us take the initial discrete configuration and make a continuous configuration such that an occupied point x in the discrete setting corresponds to a point $\frac{x}{n+1}$ in the continuous setting. The continuous dynamics is propelled by the Poisson point location $\mathscr{P}$. We slow it down by the following mechanism: every point that jumps in the front is immediately rounded down to a grid $\{\frac{k}{n+1}; k \in \mathbb{Z}\}$ and moreover, if there is already a point at this location, it is omitted. The shift of this process in k–th try of a step is denoted by c_k and its limiting speed is w_n^*. Trivially it is true that $w_n^* \leq w$ since the rounded down process cannot overtake the original one. But if we take the original continuous process or the slowed–down one and use it to generate the discrete one, we get the same discrete process. So it is true that

$$\sum_{k=1}^{m} a_k^* = \sum_{k=1}^{m} (n+1)c_k\,,$$

from where it follows

$$\frac{\sum_{k=1}^{m} a_k^*}{m} = (n+1)\frac{\sum_{k=1}^{m} c^k}{m} \to (n+1)w_n^* \leq (n+1)w\,.$$

Note that some of the c_k could be equal to 0 (and therefore also a_k^*) , but the jumps of discrete process, denoted by a_k, are all strictly positive. Since the probability of a_k^* being 0 equals $\frac{\vartheta}{n+1}$ we have

$$\frac{\sum_{k=1}^{m} a_k^*}{m} = \frac{\sum_{k=1}^{M(m)} a_k}{M(m)} \frac{M(m)}{m} \to \hat{w}_n \cdot \frac{n+1-\vartheta}{n+1}\,,$$

where $M(m)$ is the number of indices $i \leq m$ for which $a_i^* = 0$ and the convergence is a.e. by the law of large numbers. Combining last two results yields

$$\hat{w}_n \cdot \frac{n+1-\vartheta}{n+1} \leq (n+1)w$$

or

$$\frac{\hat{w}_n}{n} \le w\frac{(n+1)^2}{n(n+1-\vartheta)}\,.$$

which proves (10). It follows

$$\lim_{n\to\infty}\frac{\hat{w}_n}{n} = w$$

or, since ϑ is constant,

$$\lim_{n\to\infty}\frac{w_n}{n^2} = w.$$

This finishes the proof of Theorem 1. □

4 Final remarks

Let us briefly return to the original process on $\mathbb{Z}$. The neighbourhood of a single particle consists of n particles on both sides of it, and the threshold is ϑ. Since there are finitely many different states of the particles in the rightmost neighbourhood and the transitions from one to another state is clear, the transition matrix of this Markov chain can be explicitly determined. With the standard tools (finding the left-hand side eigenvector of the eigenvalue 1) we can determine the stationary distribution of this Markov chain, and from that we can compute the speed of spreading. For $\vartheta = 2$ this is a simple task and the computed rescaled speed equals $\frac{(n+1)(3n-2)}{12n(n-1)}$. For $\vartheta = 3$ these computations are tedious but feasible. After rescaling we get the speed of spreading equal to $\frac{(n-1)(n+1)}{2n(3n-5)}$. The case $\vartheta = 4$ is much more challenging and the speed equals $\frac{(n+1)\left(15n^3-105n^2+230n-152\right)}{120n(n-2)(n-3)^2}$. Note that the values of upper expressions decrease to $\frac{1}{4}$, $\frac{1}{6}$ and $\frac{1}{8}$ respectively as n grows to infinity.

The computed values of w and $\hat{w}_n$ for small n indicate that the value of $\frac{\hat{w}_n}{n}$ is a decreasing function of both n and ϑ. The values for $n \le 10$ are presented in Table 1 and their numerical values are shown in Table 2. The state–space quickly becomes too large to compute the invariant measure. For small n ($n \le 6$) it can be done by hand and Mathematica is able to compute the values of the speed for $n \le 15$. Table 2 poses an interesting question if the value of $\frac{\hat{w}_n}{n}$ is decreasing for all values n, ϑ.

We proved that

$$\lim_{n\to\infty}\frac{w_n\vartheta}{n^2} = \frac{1}{2}.$$

We suspect that this result may be also valid for slowly growing ϑ, for instance $\vartheta = O(\log n)$. This could be proved by coupling two continuous processes in a natural way (the particle that jumps from 0 to 1 at the same position in both front neighbourhoods). The graph in Figure 2 shows the L_1 distance between the two coupled processes with parameter $\vartheta = 5$ over the first 1000 steps. We can clearly see that the convergence is not monotone but there are two monotone piecewise constant functions which sandwich the L_1 distance. The upper bound decreases rarely but when it does it decreases substantially. On the other hand the lower bound decreases more often but in smaller steps. The quotient of the upper bound and the lower bound is at most $\vartheta - 1$. The ratio of the number of jumps of the upper bound and the number of jumps of the lower bound also seems to converge to $\vartheta - 1$. The proof that these two functions converge to 0 rapidly enough remains an open problem and will be addressed in future work.

Table 1: Values of $\frac{\hat{w}_n}{n}$ depending on n and ϑ.

$\vartheta \setminus n$	2	3	4	5	6	7	8	9	10
2	$\frac{1}{2}$	$\frac{7}{18}$	$\frac{25}{72}$	$\frac{13}{40}$	$\frac{14}{45}$	$\frac{19}{63}$	$\frac{33}{112}$	$\frac{125}{432}$	$\frac{77}{270}$
3		$\frac{1}{3}$	$\frac{15}{56}$	$\frac{6}{25}$	$\frac{35}{156}$	$\frac{3}{14}$	$\frac{63}{304}$	$\frac{20}{99}$	$\frac{99}{500}$
4			$\frac{1}{4}$	$\frac{31}{150}$	$\frac{301}{1620}$	$\frac{243}{1400}$	$\frac{331}{2000}$	$\frac{1087}{6804}$	$\frac{3047}{19600}$
5				$\frac{1}{5}$	$\frac{21}{124}$	$\frac{46}{301}$	$\frac{185}{1296}$	$\frac{45}{331}$	$\frac{2849}{21740}$
6					$\frac{1}{6}$	$\frac{127}{882}$	$\frac{3025}{23184}$	$\frac{6821}{55944}$	$\frac{24673}{212625}$
7						$\frac{1}{7}$	$\frac{255}{2032}$	$\frac{622}{5445}$	$\frac{2915}{27284}$
8							$\frac{1}{8}$	$\frac{511}{4590}$	$\frac{28501}{279900}$
9								$\frac{1}{9}$	$\frac{1023}{10220}$
10									$\frac{1}{10}$

Table 2: Numerical values of $\frac{\hat{w}_n}{n}$ depending on n and ϑ.

$\vartheta \setminus n$	2	3	4	5	6	7	8	9	10
2	0.5	0.3889	0.3472	0.325	0.3111	0.3016	0.2946	0.2894	0.2852
3		0.3333	0.2679	0.24	0.2244	0.2143	0.2072	0.202	0.198
4			0.25	0.2067	0.1858	0.1736	0.1655	0.1598	0.1555
5				0.2	0.1694	0.1528	0.1427	0.136	0.131
6					0.1667	0.144	0.1305	0.1219	0.116
7						0.1429	0.1255	0.1142	0.1068
8							0.125	0.1113	0.1018
9								0.1111	0.1001
10									0.1

Another word could be said about the set $B_t = \{x \in A_t; \{0, 1, \ldots, x\} \subseteq A_t\}$ which is the largest fully occupied set of points (again just looking to the right). It is quite obvious that the distance of the farthest point in B_t and the farthest point in A_t is approximately constant (by the Law of large numbers) so the set of fully occupied points propagates (in limit) with the same speed as the first occupied point does. The third definition of the propagating velocity would therefore give the same value.

The final remark we present is about the same regime in discrete time. Every point in state 0 at time t with enough points in state 1 in its neighbourhood jumps to state 1 in time $t+1$ with probability $p > 0$. It is fairly obvious that the normalized limiting propagation velocity equals 1 (independent of p). This is a consequence of the law of large numbers. If we look at the front neighbourhood for large n at time t there are ϑ points in state 1. In time $t+1$ there are arbitrary many sites in state 1 with probability arbitrary close to 1 in the vicinity of the border of the neighbourhood so the front neighbourhood moves for nearly n.

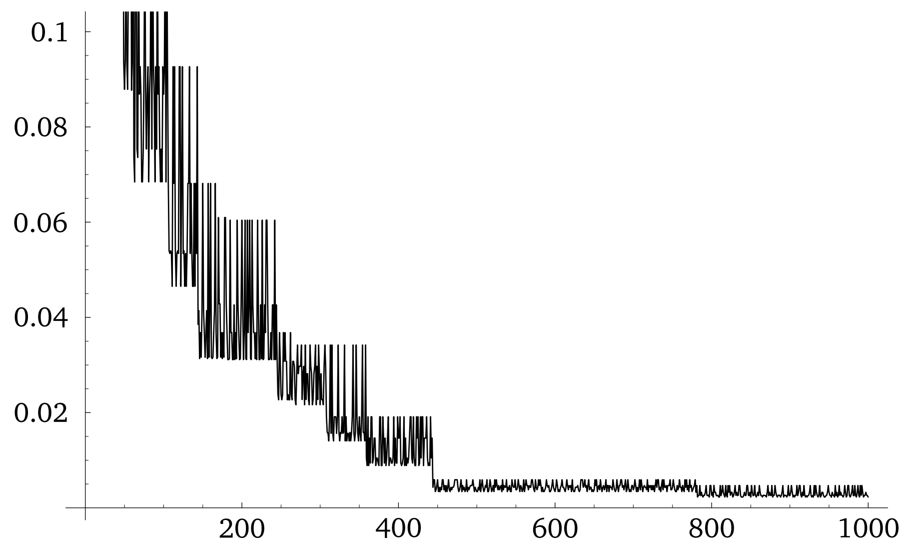


Figure 2: The evolution of the distance between two coupled processes

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