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RENORMALIZATIONS OF BRANCHING RANDOM WALKS IN EQUILIBRIUM

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Abstract We study the d-dimensional branching random walk for $d \geq 3$. This process has extremal equilibria for every intensity. We are interested in the large space scale and large space-time scale behavior of the equilibrium state. We show that the fluctuations of space and space-time averages with a non-classical scaling are Gaussian in the limit. For this purpose we use the historical process, which allows a family decomposition. To control the distribution of the families we use the concept of canonical measures and Palm distributions.

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1 Introduction and the main results

We study particle systems which are known to converge to an equilibrium. We are interested in the fluctuation behavior of the equilibrium. In [Zäh01] we studied the invariant measures of the voter model by means of a rescaling limit. We showed that centered sums over a ball with radius r renormalized by $r^{-(d+2)/2}$ are Gaussian in the limit. To this end we partitioned the equilibrium of the voter model where the values of different partition elements are independent. This in turn allows to apply the central limit theorem. We raised the reader's hopes that the techniques used for the voter model can be refined in order to study limiting states of critical branching evolutions in randomly fluctuating media.

In the present paper we investigate the large-space scale structure of non-degenerated equilibria of the critical branching random walk which is an interacting particle systems with the additional structure that a historical process can be defined. This allows a similar family decomposition as in the voter model with the difference that we now deal with unbounded states. We obtain space and space-time renormalization results.

Holley and Stroock established a renormalization result for the branching Brownian motion. The speed up time by α^2 and rescale space by α^{-1}. Then the corresponding process converges to an Ornstein-Uhlenbeck process as $\alpha \to \infty$.

Dawson, Gorostiza and Wakolbinger studied branching systems by rescaling from a different point of view, [DGW01]. They establish results on occupation time fluctuations.

The *critical branching random walk* (BRW) on $\mathbb{Z}^d$ is a particle system whose evolution entails migration and branching:

- During its lifetime each particle moves independently of each other according to a random walk with transition kernel a satisfying $a(x, y) = a(0, y - x)$.
- Each particle has an exponential lifetime with mean $\frac{1}{V}$.
- At the end of its life each particle gives rise to k children with probability p_k, where the mean number of offspring is 1.
- All above random mechanisms are independent.

V is called the branching rate and a the migration kernel. We assume critical binary branching, i.e. $p_0 = p_2 = \frac{1}{2}$.

Such a process has for a transient symmetrized kernel a non-degenerated equilibrium state. We consider large block averages of the configuration and their fluctuations. Our goal is to find the right rescaling to obtain a non-degenerated limit under a renormalization scheme.

If all particles in a ball with radius r were independent one would have to choose the classical rescaling one over the root of the volume of the ball. But the particles are not independent. From corresponding results on super Brownian motion ([DP91]) one is led to conjecture that there are r^{d-2} families each with size of order r^2. So we have to choose the following rescaling term for the spatial block sum

$$\frac{1}{r^2\sqrt{r^{d-2}}} = r^{-\frac{d+2}{2}}. \tag{1.1}$$

Over a time span r^2 most descendents will not leave the ball with radius r. For a space-time block sum with spatial extension r and extension r^2 this yields the rescaling

$$\frac{1}{r^2 r^2 \sqrt{r^{d-2}}} = r^{-\frac{d+6}{2}}. \tag{1.2}$$

These are in fact the normings that lead to non-degenerated limits.

1.1 Formal construction of the process

We introduce the following notations. Let $(E, \mathcal{B}(E))$ be a locally compact Polish space equipped with the Borel σ-algebra.

- By $\mathcal{C}_b(E)$ and $\mathcal{C}_c(E)$ we denote the space of continuous real-valued functions on E that are bounded respectively have compact support. Furthermore let $\mathcal{C}^+(E) = \{f \in \mathcal{C}(E) : f \geq 0\}$.
- Let $\mathcal{M}(E)$ denote the class of all locally finite (or Radon) measures on $\mathcal{B}(E)$. A measure on $\mathcal{B}(E)$ is called locally finite, if it takes finite values on compact sets. On $\mathcal{M}(E)$ we use the o vague topology defined by $\mu_n \to \mu$ iff $\langle \mu_n, f\rangle \to \langle \mu, f\rangle$ for all $f \in \mathcal{C}_c(E)$, where $\langle \mu, f\rangle = \int f d\mu$ for $f : E \to \mathbb{R}$ measurable and μ-integrable. This way $\mathcal{M}(E)$ is a Polish space. Let $\mathcal{M}_f(E) = \{\mu \in \mathcal{M}(E) : \mu(E) < \infty\}$.
- By $\mathcal{P}(\mathcal{M}(E))$ we denote the space of probability measures on $\mathcal{M}(E)$, which is also a Polish space equipped with the weak topology induced by the mappings $\Lambda \to \langle \Lambda, F\rangle$ with $F \in \mathcal{C}_b(\mathcal{M}(E))$.
- Let $\mathcal{N}(E)$ denote the class of all integer-valued measures of $\mathcal{M}(E)$. Furthermore define $\mathcal{N}_f(E) = \{\mu \in \mathcal{N}(E) : \mu(E) < \infty\}$.

Now we construct the BRW. With each particle alive at time t we can associate a unit mass at its position. That means we consider the BRW as a measure-valued process on $\mathbb{Z}^d$. The BRW started with a single initial particle in $x \in \mathbb{Z}^d$ will be denoted by $(\xi_t^x)_{t\geq 0}$. Its state space is $\mathcal{N}_f(\mathbb{Z}^d)$. If the BRW starts with finitely many particles, i.e. in $\mu = \sum_{k=1}^n \delta_{x_k}$ one obtains ξ_t^μ as the independent superposition of BRWs $\{\xi_t^{x_k} : k = 1, \dots, n\}$. The state space is again $\mathcal{N}_f(\mathbb{Z}^d)$. The independent superposition works also for infinite initial configurations. In fact some restriction on the state space has to be made in order to guarantee that the system does not explode.

More precisely, fix a strictly positive function γ on $\mathbb{Z}^d$ with $\sum_{x\in\mathbb{Z}^d} \gamma(x) < \infty$ such that for some constant $M > 0$,

$$\sum_{y\in\mathbb{Z}^d} a(x,y)\gamma(y) \leq M\gamma(x). \tag{1.3}$$

A simple way of obtaining such a γ is to let

$$\gamma(x) = \sum_{n=0}^{\infty} M^{-n} \sum_{y\in\mathbb{Z}^d} a^{(n)}(x,y)\beta(y), \tag{1.4}$$

where $M > 1$, β is strictly positive and bounded with $\sum_{x\in\mathbb{Z}^d}\beta(x) < \infty$ and $a^{(n)}$ are the n-step transition probabilities corresponding to a.

Now let the state space of the BRW be the Liggett-Spitzer space

$$\mathfrak{X} = \{\mu \in \mathcal{N}(\mathbb{Z}^d) : \sum_{x\in\mathbb{Z}^d} \gamma(x)\mu(x) < \infty\}. \tag{1.5}$$

It is called Liggett-Spitzer space since it was introduced by Liggett and Spitzer in [LS81]. The formal construction of the BRW $(\xi_t)_{t\geq 0}$ can be found in [Gre91] Section 1. Indeed it turns out that for $\mu \in \mathfrak{X}$, $\xi_t^\mu \in \mathfrak{X}$ a.s. for all $t > 0$ and hence $\mathfrak{X}$ can be chosen as a state space of the BRW $(\xi_t)_{t\geq 0}$.

Note that any probability measure $\Lambda \in \mathcal{P}(\mathcal{N}(\mathbb{Z}^d))$ with $\sup_{x\in\mathbb{Z}^d}\int \mu(x)\Lambda(d\mu) < \infty$ is concentrated on $\mathfrak{X}$, so that those Λ which are translation invariant and have finite intensity (i.e. $\int \mu(0)\Lambda(d\mu) < \infty$) have their support in $\mathfrak{X}$ (for all choices of γ). A probability measure Λ on $\mathcal{M}(\mathbb{Z}^d)$ is called translation invariant if $\int F(\mu)\Lambda(d\mu) = \int F(\tau_x\mu)\Lambda(d\mu)$ for all $x \in \mathbb{Z}^d$ and all measurable and Λ-integrable $F : \mathcal{M}(\mathbb{Z}^d) \to \mathbb{R}$, where $\tau_x\mu = \mu(\{y \in \mathbb{Z}^d : y + x \in \cdot\})$.

For $\Lambda \in \mathcal{P}(\mathfrak{X})$ we write $\mathcal{L}^\Lambda[\xi_t]$ and $\mathbf{E}^\Lambda[\xi_t]$ for the law and the expectation of ξ_t given that $\mathcal{L}[\xi_0] = \Lambda$. For $\Lambda = \delta_\mu$ with $\mu \in \mathfrak{X}$ we also write $\mathcal{L}^\mu[\xi_t]$ and $\mathbf{E}^\mu[\xi_t]$.

1.2 Basic ergodic theory

Concerning the longtime behavior of the BRW we summarize some known facts. First of all a BRW generated by a finite initial configuration eventually dies out. This is due to the criticality of the branching mechanism. Now the question arises what happens in the situation of an admissible initial measure with infinitely many particles. It turns out that there is a dichotomy depending on whether the symmetrized migration is recurrent or transient. The symmetrized kernel $\hat{a}$ is defined as

$$\hat{a}(x,y) = \frac{a(x,y) + a(y,x)}{2}. \tag{1.6}$$

A recurrent symmetrized particle migration goes along with local extinction while a transient symmetrized migration allows the construction of a non-trivial equilibrium.

To state this result we need the following concept. The probability measure $\Lambda \in \mathcal{P}(\mathcal{M}(\mathbb{Z}^d))$ has an asymptotic density $\varrho : \mathcal{M}(\mathbb{Z}^d) \to [0,\infty]$ if

$$\frac{\mu((-n,n)^d \cap \mathbb{Z}^d)}{|(-n,n)^d \cap \mathbb{Z}^d|} \underset{n\to\infty}{\longrightarrow} \varrho(\mu), \qquad \Lambda\text{-a.s.} \tag{1.7}$$

Let $\Lambda \in \mathcal{P}(\mathcal{M}(\mathbb{Z}^d))$ be translation invariant. By Birkhoff's Ergodic Theorem one knows that Λ has an asymptotic density. If in addition $\int \mu(0)\Lambda(d\mu) < \infty$, then by Birkhoff's Ergodic Theorem the limit of (1.7) is still valid in L^1_Λ.

Now we are ready to state the following fact:

Basic Ergodic Theorem *The longtime behavior of the critical branching random walk depends on whether the symmetrized random walk kernel $\hat{a}$ is recurrent or transient.*

(a) *Assume that $\hat{a}$ is recurrent. If the BRW starts with distribution $\Lambda \in \mathcal{P}(\mathcal{N}(\mathbb{Z}^d))$ with $\sup_{x\in\mathbb{Z}^d} \int \mu(x)\Lambda(d\mu) < \infty$, then*

$$\mathcal{L}^{\Lambda}[\xi_t] \underset{t\to\infty}{\Longrightarrow} \delta_{\underline{0}}, \tag{1.8}$$

where $\underline{0}$ denotes the zero measure.

(b) *Assume that $\hat{a}$ is transient. Then for each $\vartheta \geq 0$ there exists exactly one extremal invariant probability measure $\Lambda_\vartheta \in \mathcal{P}(\mathcal{N}(\mathbb{Z}^d))$ with $\int \mu(0)\Lambda_\vartheta(d\mu) = \vartheta$. This Λ_ϑ is translation invariant. Moreover if $\Lambda \in \mathcal{P}(\mathcal{N}(\mathbb{Z}^d))$ is translation invariant with $\varrho < \infty$ Λ-a.s., where ϱ is the asymptotic density of Λ, which exists by Birkhoff's Ergodic Theorem, then*

$$\mathcal{L}^{\Lambda}[\xi_t] \underset{t\to\infty}{\Longrightarrow} \int \Lambda_{\varrho(\mu)}\Lambda(d\mu). \tag{1.9}$$

Especially if Λ is translation invariant with constant asymptotic density $\vartheta \in [0,\infty)$ (which is equivalent to ergodicity of Λ), then

$$\mathcal{L}^{\Lambda}[\xi_t] \underset{t\to\infty}{\Longrightarrow} \Lambda_\vartheta. \qquad \diamond \tag{1.10}$$

In [Gre91] we find the above result for translation invariant, ergodic initial distributions with finite intensity.

The result in the recurrent case can be extended to the result given here by the comparison argument given in [CG90] in the proof of (5.1).

In the transient case if $\int \mu(0)\Lambda(d\mu) < \infty$ we can use the ergodic decomposition of Λ, which says that Λ can be represented as a mixture of translation invariant, ergodic probability measures, namely $\Lambda = \int \Lambda_\mu \Lambda(d\mu)$, where Λ_μ is translation invariant and ergodic. Moreover Λ_μ has intensity $\rho(\mu)$ for Λ-a.e. μ. By the definition of the BRW we know that the BRW $(\xi_t^\Lambda)_{t\geq 0}$ with initial distribution Λ can be represented as $\xi_t^\Lambda = \int \xi_t^{\Lambda_\mu}\Lambda(d\mu)$. The assertion follows immediately. The result for translation invariant Λ with $\int \mu(0)\Lambda(d\mu) < \infty$ can be extended to the result for translation invariant Λ with a.s. finite asymptotic density by a truncation argument.

1.3 Results: The rescaled fields

We are interested in the large scale properties of the process and hence we are going to study the regime of transient symmetrized migration by means of renormalization of the random field under the equilibrium distribution. Renormalization means forming sums over space and space-time blocks and rescaling their size such that a non-trivial behavior arises.

We begin by introducing some key quantities. Let a_t be the transition probabilities of a continuous time random walk, which jumps after exponential waiting times according to a, i.e.

$$a_t(x,y) = \sum_{n=0}^{\infty} e^{-t}\frac{t^n}{n!}\, a^{(n)}(x,y), \tag{1.11}$$

where $a^{(n)}$ denotes the n-step transition probability of the kernel a. The continuous time transition probabilities $\hat{a}_t$ can be defined analogously.

We will need the Green's function G of the kernel a and the Green's function $\hat{G}$ of the symmetrized kernel $\hat{a}$:

$$G(x) = \int_0^\infty a_t(0,x)\,dt \tag{1.12}$$
$$\hat{G}(x) = \int_0^\infty \hat{a}_t(0,x)\,dt. \tag{1.13}$$

Let $d \geq 3$ and $Z^{(a)} = (Z_1^{(a)}, \dots, Z_d^{(a)})$ denote a random variable with distribution $a(0,\cdot)$. We assume:

$$\text{The group } \mathcal{G} \text{ generated by } \{x : a(0,x) > 0\} \text{ is } d\text{-dimensional.} \tag{1.14}$$

$$Z^{(a)} \text{ has finite second moments; } \mathbf{E}[Z_l^{(a)} Z_k^{(a)}] = \sigma_{l,k};\ l,k = 1,\dots,d. \tag{1.15}$$

Since $d \geq 3$ the first assumption ensures transience of $\hat{a}$. Let $Q = (\sigma_{l,k})_{l,k=1}^d$ denote the matrix of the second moments of a. Let $\bar{Q}(x)$ denote the following quadratic form

$$\bar{Q}(x) = x^{tr} Q^{-1} x, \qquad x \in \mathbb{R}^d. \tag{1.16}$$

(The bar above Q indicates that the quadratic form is defined in terms of Q^{-1}.)

Let ξ be a random variable with the extremal equilibrium law Λ_ϑ given in the Basic Ergodic Theorem (b). For a test function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ (Schwartz space of smooth rapidly decreasing functions) we define

$$F_\vartheta(\varphi) = \sum_{x \in \mathbb{Z}^d} (\xi(x) - \mathbf{E}\xi(x))\varphi(x) = \sum_{x \in \mathbb{Z}^d} (\xi(x) - \vartheta)\varphi(x). \tag{1.17}$$

That means F_ϑ is a generalized random field, cf. [Dob79] and [GV64]. This random field will be renormalized now. Let

$$F_{\vartheta,r}(\varphi) = F_\vartheta(\varphi_r), \tag{1.18}$$

with

$$\varphi_r(x) = h(r)\varphi\left(\frac{x}{r}\right), \tag{1.19}$$

where we have to choose the function $h(r)$ depending on the Green's function of the underlying random walk and decreasing much faster than the classical rescaling

$$\frac{1}{\sqrt{|\{x \in \mathbb{Z}^d : |x| \leq r\}|}} = O(r^{-d/2}). \tag{1.20}$$

Now we formulate the first main result.

Theorem 1 (Space renormalization in equilibrium) *Assume the critical binary branching random walk with translation invariant kernel a on $\mathbb{Z}^d$ with $d \geq 3$. Then under the assumptions (1.14), (1.15) and for $d > 3$ under the additional assumption*

$$\text{for d} = 4: \qquad n^2 \mathbf{P}[|Z^{(a)}| \geq n] = o\left(\frac{1}{\log n}\right) \tag{1.21}$$

$$\text{for d} \geq 5: \qquad \text{finite moments of order } d-1 \tag{1.22}$$

and with the choice

$$h(r) = r^{-\frac{d+2}{2}} \tag{1.23}$$

we obtain

$$F_{\vartheta,r} \xrightarrow[r\to\infty]{d} \sqrt{C_\vartheta}\,\Psi, \tag{1.24}$$

where Ψ is the Gaussian self-similar generalized random field with covariance functional

$$\mathbf{E}[\Psi(\varphi)\Psi(\psi)] = L(\varphi,\psi) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\varphi(x)\psi(y)}{\bar{Q}(y-x)^{(d-2)/2}}\,dx\,dy, \tag{1.25}$$

where $\bar{Q}(x)$ is defined in (1.16). The constant C_ϑ has the form

$$C_\vartheta = \vartheta \frac{V}{4|Q|^{\frac{1}{2}}\pi^{\frac{d}{2}}}\Gamma\left(\frac{d-2}{2}\right), \tag{1.26}$$

where Γ denotes the Gamma function. $\diamond$

The distribution of a generalized random field is a probability measure on the σ-algebra of Borel subsets (with respect to the weak topology) of the dual space $\mathcal{S}'(\mathbb{R}^d)$ of $\mathcal{S}(\mathbb{R}^d)$, cf. [GV64] Chapter III. Since $\mathcal{S}(\mathbb{R}^d)$ is a normed space, the dual $\mathcal{S}'(\mathbb{R}^d)$ is a Banach space. The convergence in (1.24) is weak convergence of probability measures on $\mathcal{S}'(\mathbb{R}^d)$.

The Gaussian self-similar generalized random field Ψ with covariance functional L is defined by its characteristic functional

$$\mathbf{E}[e^{i\Psi(\varphi)}] = e^{-\frac{1}{2}L(\varphi,\varphi)}. \tag{1.27}$$

Obviously the limit in (1.24) has to be self-similar of order $(d-2)/2$. A generalized random field F is called self-similar of order κ if

$$F(\varphi) = F(r^\kappa r^{-d}\varphi\left(\frac{\cdot}{r}\right)). \tag{1.28}$$

Remark 1.1 The properties (1.21) and (1.22) are not only required for technical reasons. If these assumptions are not fulfilled then the Green's function of the underlying random walk has another asymptotics and hence we have to choose a different scaling function.

Since ξ is infinitely divisible, it is easy to see that the constant C_ϑ has to have the form ϑC, where C is a constant, which does not depend on ϑ.

Next we consider the space-time picture in the equilibrium using again renormalization. Let $(\xi_t^{(\infty)})_{t\geq 0}$ be the BRW with initial distribution Λ_ϑ. Particularly $(\xi_t^{(\infty)})_{t\geq 0}$ is stationary. For a test function $\tilde{\varphi} \in \mathcal{S}(\mathbb{R}^d \times [0,\infty))$ we define

$$\tilde{F}_\vartheta(\tilde{\varphi}) = \int_0^\infty ds \sum_{x\in\mathbb{Z}^d} (\xi_s^{(\infty)}(x) - \vartheta)\tilde{\varphi}(x,s). \tag{1.29}$$

The space-time renormalized field is

$$\tilde{F}_{\vartheta,r}(\tilde{\varphi}) = \int_0^\infty ds \sum_{x\in\mathbb{Z}^d} (\xi_s^{(\infty)}(x) - \vartheta)\tilde{\varphi}_r(x,s), \tag{1.30}$$

with

$$\tilde{\varphi}_r(x,t) = \tilde{h}(r)\tilde{\varphi}\left(\frac{x}{r}, \frac{t}{r^2}\right), \tag{1.31}$$

where $\tilde{h}$ will be specified explicitly later on.

We rescale the space coordinate by r and the time coordinate by r^2 in order to get the complete space-time view of the family structure. The reason for this is that a surviving family deriving from one ancestor has at time t a spatial extension of order $t^{1/2}$.

We have to assume mean 0 of a, since otherwise the families would move at speed t and hence are shifted out of the ball with radius $\sqrt{t}$ by time t.

We prove the following space-time renormalization result.

Theorem 2 (Space-time renormalization in equilibrium) *Assume the critical binary branching random walk with translation invariant mean zero kernel a on $\mathbb{Z}^d$ with $d \geq 3$. Then under the assumptions of Theorem 1 and with the choice*

$$\tilde{h}(r) = r^{-\frac{d+6}{2}} \tag{1.32}$$

we obtain

$$\tilde{F}_{\vartheta,r} \underset{r\to\infty}{\overset{d}{\longrightarrow}} \sqrt{\tilde{C}_\vartheta}\tilde{\Psi}, \tag{1.33}$$

where $\tilde{\Psi}$ is the Gaussian generalized random field with covariance functional

$$\tilde{L}(\tilde{\varphi},\tilde{\psi}) = \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz \int_0^\infty ds \int_0^\infty dt\, \tilde{\varphi}(y,s)\tilde{\psi}(z,t)K(s,t;y,z), \tag{1.34}$$

with

$$K(s,t;y,z) = \int_{|t-s|}^\infty du\, b_u(y,z), \tag{1.35}$$

where b_u is given by

$$b_u(y,z) = \frac{1}{(2\pi u)^{\frac{d}{2}}|Q|^{\frac{1}{2}}} \exp\left(-\frac{\bar{Q}(z-y)}{2u}\right), \tag{1.36}$$

with $\bar{Q}(\cdot)$ defined in (1.16). The constant $\tilde{C}_\vartheta$ has the form

$$\tilde{C}_\vartheta = \frac{1}{2}V\vartheta. \qquad \diamond \tag{1.37}$$

The main idea to prove the results is to decompose the equilibrium configuration in independent family clusters. For this purpose we use the historical process, a concept which allows to develop a framework which could be the basis for some future work, namely we would like to establish analogous results in case of state dependent branching systems or catalytic branching systems. The infinite divisibility in the present case provides explicit formulas for some basic objects in the family decomposition. In fact we could use here just these explicit formulas shortening the proof but losing insight.

2 Historical process

The goal of this section is to introduce the historical process associated with the particle system $(\xi_t)_{t\geq 0}$ which records the past of all particles alive at time t. Moreover we prove some properties of the historical process. If we observe the particle system (i.e. the process $(\xi_t)_{t\geq 0}$ with values in $\mathfrak{X}$) at a certain time t we can not say anything about the relationship between the particles. We lost all information about the genealogy of the system. We have to enrich the state space such that the state at time t provides information about the trajectories followed by the particles and its ancestors. The state space of the BRW is $\mathfrak{X}$. The state space of the historical process will be a subset of the space of integer-valued measures on the space of cdlg paths on $\mathbb{Z}^d$.

In our construction we mimic the construction of the historical process for superprocesses in [DP91]. Also Le Gall ([LG89],[LG91]) and Dynkin ([Dyn91]) independently introduced the historical process in the context of superprocesses.

We want to give an intuitive idea of the historical process. Think of the branching random walk as being constructed as a functional of a system where individual particles can be distinguished. With some notational effort such a system can be easily constructed due to the independence between different individuals.

Let $N(t)$ be the number of particles alive at time t and let $y^{(k)}$ be the position of the k-th particle alive at time t ($k = 1, \ldots, N(t)$). Then the BRW has the form $\xi_t = \sum_{k=1}^{N(t)} \delta_{y^{(k)}}$. That means each particle is associated with mass 1 at its position. In the historical process we want to preserve information about the trajectories of those particles still alive at time t and lines of descent but not their individuality. Then it is obviously appropriate to consider the empirical measure, i.e. to assign mass 1 to the trajectories of the particles alive at time t. To get a time independent state space of paths continue the path till time ∞ as a constant. The historical process is then the measure-valued process given by $\sum_{k=1}^{N(t)} \delta_{\tilde{y}^{(k)}(\cdot \wedge t)}$, where $\tilde{y}^{(k)}(\,\cdot \wedge t)$ is shorthand for the stopped trajectory $(\tilde{y}^{(k)}(s \wedge t))_{s\geq 0}$ that had been followed by the k-th particle alive at time t.

We are interested in the equilibrium of the BRW. In order to describe the historical process of the equilibrium process it is useful to view the process with time parameter set $(-\infty, \infty)$, to start the process at time $-s$ and to observe the process at time 0. Then let $s \to \infty$ to obtain the equilibrium historical process. This object will be proved to exist indeed and it will be characterized analytically.

First of all we recall some basic tools and notions before we focus on the historical process in the second part. The main result is Theorem 3, where the equilibrium historical process is described via its Laplace functional and its canonical measure. These ingredients we introduce next.

2.1 Laplace functional, canonical measure and Palm distribution

We consider the Laplace functional of the BRW, which can be used to characterize the law of the BRW completely. This idea is useful on the level of the historical process later on.

Let $f : \mathbb{Z}^d \to [0, \infty)$ be bounded and let $\mu \in \mathfrak{X}$. The Laplace functional of the BRW with initial configuration μ has the form

$$\mathbf{E}^{\mu}[e^{-\langle \xi_t, f\rangle}] = \exp\left(\left\langle \mu, \log(1 - v(t, 1 - e^{-f}; \cdot))\right\rangle\right), \tag{2.1}$$

where $v(t,f;\,\boldsymbol{\cdot}\,)$ is the unique non-negative solution of the following equation

$$\frac{\partial}{\partial t}v(t,f;x) = \Omega v(t,f;\,\boldsymbol{\cdot}\,)(x) - \frac{V}{2}v(t,f;x)^2, \qquad v(0,f;x) = f(x), \tag{2.2}$$

where Ω is the infinitesimal generator of (a_t), i.e.

$$\Omega f(x) = \sum_{y\in\mathbb{Z}^d} [a(x,y) - \delta(x,y)]f(y). \tag{2.3}$$

This can be seen by the following argument. Since all random mechanism are independent, the configuration at time t is the independent superposition of BRWs each with a single ancestor. That means

$$\mathbf{E}^{\mu}[e^{-\langle \xi_t, f\rangle}] = \exp\left(\langle \mu, \log \mathbf{E}^{\delta_\cdot}[e^{-\langle \xi_t, f\rangle}]\rangle\right). \tag{2.4}$$

Hence it suffices to investigate the Laplace functional of a BRW with a single ancestor. A renewal argument shows that $h(t,x) := \mathbf{E}^{\delta_x}[e^{-\langle \xi_t, f\rangle}]$ satisfies the integral equation

$$h(t,x) = e^{-Vt}a_t(e^{-f})(x) + \frac{V}{2}\int_0^t e^{-V(t-s)}a_{t-s}(1 + h^2(s,\,\boldsymbol{\cdot}\,))(x)\, ds. \tag{2.5}$$

Therefore h satisfies the equation

$$\frac{\partial h}{\partial t} = \Omega h + \frac{V}{2}(1-h)^2, \quad h(0,x) = e^{-f(x)}. \tag{2.6}$$

Hence $1 - h(t,x) = v(t, 1 - e^{-f}; x)$.

Equation (2.2) is equivalent to

$$v(t,f;x) = (a_t f)(x) - \frac{V}{2}\int_0^t a_{t-s}\left(v(s,f;\,\boldsymbol{\cdot}\,)^2\right)(x)\, ds. \tag{2.7}$$

Let $\mathcal{H}(\vartheta)$ denote the distribution of the Poisson point process with intensity measure $\vartheta\lambda$, where $\vartheta > 0$ is a parameter and λ denotes the counting measure on $\mathbb{Z}^d$. Note that the Poisson point process is translation invariant with finite intensity, hence the initial state lies a.s. in $\mathfrak{X}$. If there is no other specification then $(\xi_t)_{t\geq 0}$ denotes the BRW started in the Poisson point process with intensity measure $\vartheta\lambda$. One can easily check that the Laplace functional of the BRW ξ_t started with distribution $\mathcal{H}(\vartheta)$ has the form

$$\mathbf{E}^{\mathcal{H}(\vartheta)}[e^{-\langle \xi_t, f\rangle}] = \exp\left(-\langle \vartheta\lambda, v(t, 1-e^{-f};\,\boldsymbol{\cdot}\,)\rangle\right). \tag{2.8}$$

Now we come to the canonical measure of the BRW. Since the BRW starts in the Poisson point process, which is infinitely divisible, the law $\mathcal{L}[\xi_t]$ is also infinitely divisible and hence it has a *canonical measure* denoted with $\mathbf{Q}_t$. More formally, there exists a σ-finite measure $\mathbf{Q}_t$ on $\mathcal{N}(\mathbb{Z}^d)$ such that

$$\mathbf{E}^{\mathcal{H}(\vartheta)}\left[e^{-\langle \xi_t, f\rangle}\right] = \exp\left(-\int_{\mathcal{N}(\mathbb{Z}^d)} (1 - e^{-\langle \mu, f\rangle})\mathbf{Q}_t(d\mu)\right). \tag{2.9}$$

Furthermore we need the concept of Palm distributions. Let $R \in \mathcal{M}(\mathcal{M}(\mathbb{Z}^d))$ with locally finite intensity I, i.e.

$$I(x) = \int_{\mathcal{M}(\mathbb{Z}^d)} \mu(x) R(d\mu) < \infty \quad \text{for all } x \in \mathbb{Z}^d. \tag{2.10}$$

The family $\{(R)_x, x \in \mathbb{Z}^d\}$ is called *Palm distribution* associated with R if

$$\sum_{x \in \mathbb{Z}^d} \int_{\mathcal{M}(\mathbb{Z}^d)} f(x) g(\mu) (R)_x(d\mu) I(x) = \int_{\mathcal{M}(\mathbb{Z}^d)} \langle \mu, f \rangle g(\mu) R(d\mu), \tag{2.11}$$

for all $f : \mathbb{Z}^d \to [0, \infty)$, $g : \mathcal{M}(\mathbb{Z}^d) \to [0, \infty)$ measurable.
Note that for every $x \in \mathbb{Z}^d$ with $I(x) > 0$ the Palm distribution $(R)_x$ arises through a reweighting of the form of a local *size biasing*

$$(R)_x(M) = \frac{\int \mu(\{x\}) 1\!\!1_M(\mu) R(d\mu)}{I(x)}, \quad M \subset \mathcal{M}(\mathbb{Z}^d). \tag{2.12}$$

The Palm distribution of the canonical measure is called canonical Palm distribution.

The canonical Palm distribution of ξ_t has a nice representation as a genealogical tree, see [GRW90]. We need some ingredients.
Let $(\bar{X}_t^x)_{t \geq 0}$ be a continuous time random walk with transition kernel $\bar{a}(y, z) = a(z, y)$ starting at x at time 0. Furthermore let $\{(\xi_t^{s,y})_{t \geq 0}; s \geq 0, y \in \mathbb{Z}^d\}$ be a collection of independent BRWs with transition kernel a starting with one particle at y at time 0. Define

$$\zeta_{t,x} = \int_0^t \xi_s^{s, \bar{X}_s^x} \nu(ds), \tag{2.13}$$

where ν is the law of the Poisson point process on $\mathbb{R}^+$ with intensity V w.r.t. the Lebesgue measure. In [GW91] Theorem 2.3 we find the following result for the branching Brownian motion. The proof can be easily adapted to our case.

Proposition 2.1 *The canonical Palm distribution of ξ_t corresponding to the point x is given by:*

$$(\mathbf{Q}_t)_x = \mathcal{L}[\zeta_{t,x} + \delta_x]. \qquad \diamond \tag{2.14}$$

In other words, $\zeta_{t,x}$ may be thought of as the population of an individual δ_x's ("ego's") relatives. The point x is the starting point of the ancestral line $\bar{X}^x$. At the random branching times in the support of ν an individual is born, whose offspring at time t are relatives of "ego".

2.2 Construction of the historical process

Now we come to the formal construction of the historical process. We introduce the following notations. Let $(E, \mathcal{B}(E))$ be a locally compact Polish space equipped with the Borel σ-algebra.

- Let $D((-\infty, \infty), E) = \{\tilde{y} : (-\infty, \infty) \to E;\ \text{cdlg}\}$ denote the space of right-continuous E-valued paths on $(-\infty, \infty)$ with left limits equipped with the Skorohod topology. Sometimes we write $D(E)$ instead of $D((-\infty, \infty), E)$. We denote $y_t := \tilde{y}(t)$ for $\tilde{y} \in D(E)$.

- Let $D([s,\infty),E) = \{\tilde{y} : [s,\infty) \to E;\ \text{cdlg}\}$. We identify $D([s,\infty),E)$ with a subset of $D((-\infty,\infty),E)$ by setting $y_u = y_s,\ \forall u < s$.

- For $\tilde{y} = (y_s)_{s\in\mathbb{R}} \in D(E)$ we denote by $\tilde{y}^t := (y_{s\wedge t})_{s\in\mathbb{R}}$ the path stopped at time t. Let $D^t(E) = \{\tilde{y} \in D(E) : \tilde{y} = \tilde{y}^t\}$ be the set of all paths stopped at time t.

- Let $\mathcal{M}^t(D(E)) = \{\tilde{\mu} \in \mathcal{M}(D(E)) : \tilde{\mu}(D^t(E)^c) = 0\}$ denote the class of all measures on $D(E)$ supported by $D^t(E)$ and $\mathcal{N}^t(D(E))$ is defined analogously.

- For $\tilde{y}, \tilde{w} \in D(E)$ and $s \in \mathbb{R}$ we define $(\tilde{y}/s/\tilde{w}) \in D(E)$ in the following way

$$(\tilde{y}/s/\tilde{w})_t = \begin{cases} y_t, & t < s \\ w_t, & t \geq s \end{cases}. \tag{2.15}$$

- Let $\pi_t, \bar{\pi}_t$ denote the projection maps

$$\begin{array}{rrcll} \pi_t : & D(E) & \longrightarrow & E; & \tilde{y} \mapsto y_t \\ \bar{\pi}_t : & \mathcal{M}(D(E)) & \longrightarrow & \mathcal{M}(E); & \tilde{\mu} \mapsto \tilde{\mu}(\pi_t^{-1}(\cdot)). \end{array} \tag{2.16}$$

- Let $\mathcal{D}(I)$ be the σ-algebra generated by $\{\pi_t; t \in I\}$ and let $b\mathcal{D}(I) = \{\tilde{f} : D(E) \to \mathbb{R}; \tilde{f} \text{ bounded and } \mathcal{D}(I)\text{-measurable}\}$.

Notice that the ˜-notation is used for objects connected with the path space. In Section 1.3 we used this notation in a similar way. The ˜ always indicates a space-time point of view. The concrete meaning is always clear from the context.

The historical process will be constructed by giving a unique characterization of its Laplace functional. In order to give the historical version of (2.2) we have to replace (a_t) by the transition semigroup of the law of the random walk *path process*. We introduce this object next.

Let $\tilde{\Pi}^t_{s,x}$ denote the law of the path of the basic random walk moving according to kernel a started at time s at x and stopped at time $t > s$.

The path process $\tilde{X}$ associated with the random walk is a time inhomogeneous Markov process with state space $D(\mathbb{Z}^d)$ and time inhomogeneous semigroup $\tilde{a}_{s,t}$ acting on $\mathcal{C}_b(D(\mathbb{Z}^d))$, which is defined by

$$(\tilde{a}_{s,t}\tilde{f})(\tilde{y}) = \int_{D^t([s,\infty),\mathbb{Z}^d)} \tilde{f}(\tilde{y}/s/\tilde{z})\tilde{\Pi}^t_{s,y_s}(d\tilde{z}), \quad \tilde{y} \in D(\mathbb{Z}^d). \tag{2.17}$$

Let $\tilde{\mathfrak{X}}$ be the set of integer-valued measures on the path space such that the time 0 projection gives an element in $\mathfrak{X}$ (the state space of the BRW), i.e.

$$\tilde{\mathfrak{X}} = \{\tilde{\mu} \in \mathcal{N}(D(\mathbb{Z}^d)) : \bar{\pi}_0\tilde{\mu} \in \mathfrak{X}\}. \tag{2.18}$$

The subspace of $\tilde{\mathfrak{X}}$ consisting of measures carried by the set of paths stopped at time t is denoted by $\tilde{\mathfrak{X}}^t$.

Now we can define the historical process as follows:

Definition 2.2 *The historical process $(\tilde{\xi}_t)_{t\in\mathbb{R}}$ is a $\tilde{\mathfrak{X}}$-valued Markov process with $\tilde{\xi}_t \in \tilde{\mathfrak{X}}^t$. The law of $\tilde{\xi}_t$ starting at time s in $\tilde{\mu} \in \tilde{\mathfrak{X}}^s$ is given in terms of its Laplace functional*

$$\mathbf{E}^{s,\tilde{\mu}}[e^{-\langle \tilde{\xi}_t, \tilde{f}\rangle}] = \exp\left(\left\langle \tilde{\mu}, \log\left(1 - \tilde{v}(s,t,1-e^{-\tilde{f}};\,\cdot\,)\right)\right\rangle\right), \quad \tilde{f} \in \mathcal{C}_b^+(D(\mathbb{Z}^d)), \tag{2.19}$$

where $\tilde{v}(s,t,\tilde{f};\,\cdot\,)$ is the unique non-negative solution of

$$\tilde{v}(s,t,\tilde{f};\tilde{y}) = (\tilde{a}_{s,t}\tilde{f})(\tilde{y}) - \frac{V}{2}\int_s^t \tilde{a}_{s,u}\left(\tilde{v}(u,t,\tilde{f};\,\cdot\,)^2\right)(\tilde{y})\,du. \tag{2.20}$$

Note that the historical process started at time s in $\tilde{y} \in D^s(\mathbb{Z}^d)$ puts mass only on paths $\tilde{z}$ with $\tilde{z}^s = \tilde{y}$.

By the Basic Ergodic Theorem (1.10) we have weak convergence of the BRW to the unique invariant measure. Now we want to interpret the equilibrium measure as the configuration we observe at time 0 if we start the process at $-\infty$, furthermore we want to lift this idea to the level of the historical process.

We will see that there exists a limiting law for the historical process by letting $s \to -\infty$ for suitable initial laws. We want to specify these initial laws. (The following steps are taken from [DG96] Section A.1.c.)

A system of independent random walks on $\mathbb{Z}^d$ has a unique extremal equilibrium with intensity $\vartheta\lambda$ (namely a Poisson system) and hence a unique entrance law. That means there exists a unique collection of locally finite measures $\{\tilde{\lambda}_{s,\vartheta}\}_{s\in\mathbb{R}}$ on $D(\mathbb{Z}^d)$ such that:

$$\tilde{\lambda}_{s,\vartheta} \text{ is concentrated on } D^s(\mathbb{Z}^d). \tag{2.21}$$

$$\text{For } A \subset \mathbb{Z}^d\text{: } \tilde{\lambda}_{s,\vartheta}(\{\tilde{y} : y_s \in A\}) = \vartheta\lambda(A). \tag{2.22}$$

$$\text{For } t > s \text{ and } B \in \mathcal{D}([s,\infty))\text{:} \tag{2.23}$$

$$\tilde{\lambda}_{t,\vartheta}(B) = \vartheta \sum_{x\in\mathbb{Z}^d} \tilde{\Pi}^t_{s,x}(B).$$

Note that $\tilde{\lambda}_{s,\vartheta} \in \tilde{\mathfrak{X}}$.

Since we want to construct a decomposition of the configuration into independent families, we need the concept of clan measures. We call a measure $\tilde{\mu} \in \tilde{\mathfrak{X}}$ a *clan measure* if there exists a $\tilde{y} \in D(\mathbb{Z}^d)$ such that

$$\tilde{\mu}(Cl(\tilde{y})^c) = 0, \tag{2.24}$$

where

$$Cl(\tilde{y}) = \{\tilde{z}; \exists\, s : \tilde{z}^s = \tilde{y}^s\}. \tag{2.25}$$

The set of clan measures in $\tilde{\mathfrak{X}}^t$ is denoted by $\tilde{\mathfrak{X}}^{cl,t}$.

We consider the historical process $(\tilde{\xi}_t)_{t\geq s}$ started as the Poisson point process with intensity measure $\tilde{\lambda}_{s,\vartheta}$ at time s. We write $\mathcal{L}^{s,\vartheta}[\tilde{\xi}_t]$ for its law. The corresponding Laplace functional has the form

$$\mathbf{E}^{s,\vartheta}[e^{-\langle \tilde{\xi}_t, \tilde{f}\rangle}] = \exp\left(-\left\langle \tilde{\lambda}_{s,\vartheta}, \tilde{v}(s,t,1-e^{-\tilde{f}};\,\cdot\,)\right\rangle\right), \quad \tilde{f} \in \mathcal{C}_b^+(D(\mathbb{Z}^d)). \tag{2.26}$$

It is infinitely divisible and it has therefore the canonical representation

$$\mathbf{E}^{s,\vartheta}[e^{-\langle\tilde{\xi}_t,\tilde{f}\rangle}] = \exp\left(-\int(1-e^{-\langle\tilde{\mu},\tilde{f}\rangle})\tilde{\mathbf{Q}}_t^s(d\tilde{\mu})\right), \quad \tilde{f}\in\mathcal{C}_b^+(D(\mathbb{Z}^d)), \tag{2.27}$$

where $\tilde{\mathbf{Q}}_t^s$ is the canonical measure. We state the following result, which gives us a nice representation of the canonical Palm distribution.

Proposition 2.3 *Let $\tilde{\mathbf{Q}}_t^s$ denote the canonical measure of $\tilde{\xi}_t$ started as the Poisson point process with intensity measure $\tilde{\lambda}_{s,\vartheta}$ at time s. Then the canonical Palm distribution at $\tilde{y}\in D^t(\mathbb{Z}^d)$ has the following representation:*

$$(\tilde{\mathbf{Q}}_t^s)_{\tilde{y}} = \mathcal{L}[\tilde{\zeta}_{s,t,\tilde{y}}+\delta_{\tilde{y}}] \tag{2.28}$$

with

$$\tilde{\zeta}_{s,t,\tilde{y}} = \int_s^t \tilde{\xi}_t^{u,\tilde{y}^u}\,\nu(du), \tag{2.29}$$

where ν is a random Poisson point measure on $\mathbb{R}$ with intensity V w.r.t. to the Lebesgue measure and $\{(\tilde{\xi}_t^{u,\tilde{z}})_{t\geq u}; u\in\mathbb{R}, \tilde{z}\in D^u(\mathbb{Z}^d)\}$ are independent historical processes started at time u in $\delta_{\tilde{z}}$. $\diamond$

Now we are prepared to state the following result:

Theorem 3 *For $d\geq 3$:*

(i) The law $\mathcal{L}^{s,\vartheta}[\tilde{\xi}_t]$ converges weakly in $\mathcal{P}(\tilde{\mathfrak{X}}^t)$ as $s\to-\infty$ to the law of an infinitely divisible random measure $\tilde{\xi}_t^{-\infty}$ with intensity

$$\mathbf{E}[\langle\tilde{\xi}_t^{-\infty},\tilde{f}\rangle] = \langle\tilde{\lambda}_{t,\vartheta},\tilde{f}\rangle, \qquad \tilde{f}\in\mathcal{C}^+(D(\mathbb{Z}^d))\cap b\mathcal{D}([u,\infty)),\ u\in(-\infty,\infty) \tag{2.30}$$

and Laplace functional

$$\mathbf{E}[e^{-\langle\tilde{\xi}_t^{-\infty},\tilde{f}\rangle}] = e^{-\tilde{v}_t(\tilde{f})}, \qquad \tilde{f}\in\mathcal{C}^+(D(\mathbb{Z}^d))\cap b\mathcal{D}([u,\infty)),\ u\in(-\infty,\infty), \tag{2.31}$$

where

$$\tilde{v}_t(\tilde{f}) = \lim_{s\to-\infty}\left\langle\tilde{\lambda}_{s,\vartheta},\tilde{v}(s,t,1-e^{-\tilde{f}};\cdot)\right\rangle. \tag{2.32}$$

(ii) $\mathcal{L}[\bar{\pi}_t\tilde{\xi}_t^{-\infty}] = \Lambda_\vartheta$, where Λ_ϑ defined in (1.10).

(iii) The canonical measure $\tilde{\mathbf{Q}}_t^{-\infty}$ of $\tilde{\xi}_t^{-\infty}$ and the canonical Palm distribution are supported by the set of clan measures, that means

$$\tilde{\mathbf{Q}}_t^{-\infty}\left((\tilde{\mathfrak{X}}^{cl,t})^c\right) = 0 \tag{2.33}$$

and for $\tilde{\lambda}_{t,\vartheta}$-a.e. $\tilde{y}$

$$(\tilde{\mathbf{Q}}_t^{-\infty})_{\tilde{y}}\left((\tilde{\mathfrak{X}}^{cl,t})^c\right) = 0. \qquad \diamond \tag{2.34}$$

The proofs are deferred to Section 5.

2.3 Family decomposition

Once we have a historical process we can define a corresponding family decomposition of $\tilde{\xi}^{-\infty}$. A particle alive at time s (which corresponds to $\delta_{\tilde{y}}$ for some $\tilde{y} \in D^s(\mathbb{Z}^d)$) and a particle alive at time t (which corresponds to $\delta_{\tilde{z}}$ for some $\tilde{z} \in D^t(\mathbb{Z}^d)$) belong to the same family if $\tilde{z} \in Cl(\tilde{y})$. To identify the law of one family directly is difficult. However in the present situation we have with the infinite divisibility of $\tilde{\xi}_t^{-\infty}$ a powerful tool at hand.

We are interested in two special cases, namely the distribution of the members of one family that are alive at time t and the law of the weighted occupation time of one family.

For the first case note that $\tilde{\xi}_t^{-\infty}$ is infinitely divisible. It is well-known from the theory of infinite divisibility that the canonical measure is the intensity measure for typical elements which the population consists of. Theorem 3 (iii) tells us that the clan measures are typical elements.

To be more precise let $\tilde{\Upsilon}_t$ be a Poisson point process with intensity measure $\tilde{\mathbf{Q}}_t^{-\infty}$. That means $\tilde{\Upsilon}_t$ is a random measure on $\mathcal{N}(D(\mathbb{Z}^d))$. Since $\tilde{\mathbf{Q}}_t^{-\infty}$ is the canonical measure of $\tilde{\xi}_t^{-\infty}$ we get

$$\tilde{\xi}_t^{-\infty} \stackrel{d}{=} \int \tilde{\mu} \tilde{\Upsilon}_t(d\tilde{\mu}). \tag{2.35}$$

Note that the r.h.s. is a countable sum of clan measures.

We define one special numbering of the clans, but any other numbering would do as well. Let $\mathbb{Z}^d = \{x_1, x_2, \dots\}$. Now the historical process induces the following family decomposition, where the families whose members are at time 0 at x_k, are labelled by k and a further randomly chosen index n. Define the following decomposition of $\mathcal{M}(D(\mathbb{Z}^d))$ for $\{x_1, x_2, \dots\}$ given above

$$\begin{aligned} \tilde{\mathcal{M}}_1 &:= \{\tilde{\mu} \in \mathcal{M}(D(\mathbb{Z}^d)) : \bar{\pi}_0\tilde{\mu}(x_1) > 0\} \\ \tilde{\mathcal{M}}_2 &:= \{\tilde{\mu} \in \mathcal{M}(D(\mathbb{Z}^d)) : \bar{\pi}_0\tilde{\mu}(x_2) > 0, \bar{\pi}_0\tilde{\mu}(x_1) = 0\} \\ &\vdots \\ \tilde{\mathcal{M}}_k &:= \{\tilde{\mu} \in \mathcal{M}(D(\mathbb{Z}^d)) : \bar{\pi}_0\tilde{\mu}(x_k) > 0, \forall\, l < k : \bar{\pi}_0\tilde{\mu}(x_l) = 0\} \\ &\vdots \end{aligned} \tag{2.36}$$

Let $\{\tilde{\xi}_t^{k,l}; l \in \mathbb{N}\}$ be independent point processes on $D(\mathbb{Z}^d)$ each with distribution

$$\tilde{\mathbf{P}}_{t,k} = \frac{\tilde{\mathbf{Q}}_t^{-\infty}(\tilde{\mathcal{M}}_k \cap \cdot)}{\tilde{\mathbf{Q}}_t^{-\infty}(\tilde{\mathcal{M}}_k)}. \tag{2.37}$$

Note that $\tilde{\mathbf{Q}}_t^{-\infty}(\tilde{\mathcal{M}}_k) < \infty$. Furthermore let $N_{t,k}$ be Poisson distributed with mean $\theta_{t,k} := \tilde{\mathbf{Q}}_t^{-\infty}(\tilde{\mathcal{M}}_k)$ independent of $\tilde{\xi}_t^{k,l}$. Then we obtain

Corollary 2.4 *The historical equilibrium process can be decomposed as*

$$\tilde{\xi}_t^{-\infty} \stackrel{d}{=} \sum_{k=1}^{\infty} \sum_{l=1}^{N_{t,k}} \tilde{\xi}_t^{k,l}. \tag{2.38}$$

Proof This can be easily seen by the following short calculation. Since $\tilde{\xi}_t^{k,l}$ are independent and $N_{t,k}$ is Poisson distributed with mean $\theta_{t,k} = \tilde{\mathbf{Q}}_t^{-\infty}(\tilde{\mathcal{M}}_k)$ we get

$$\begin{aligned}
\mathbf{E}\left[\exp\Big(-\langle\sum_{k=1}^{\infty}\sum_{l=1}^{N_{t,k}}\tilde{\xi}_t^{k,l},\tilde{f}\rangle\Big)\right] &= \prod_{k=1}^{\infty} e^{-\theta_{t,k}}\sum_{n=0}^{\infty}\frac{\theta_{t,k}^n}{n!}\left(\mathbf{E}\left[\exp(-\langle\tilde{\xi}_t^{k,1},\tilde{f}\rangle)\right]\right)^n \\
&= \prod_{k=1}^{\infty}\exp\left(-\theta_{t,k}\int(1-e^{-\langle\tilde{\mu},\tilde{f}\rangle})\tilde{\mathbf{P}}_{t,k}(d\tilde{\mu})\right). \qquad (2.39)
\end{aligned}$$

By the definition of $\tilde{\mathbf{P}}_{t,k}$ we obtain

$$\mathbf{E}\left[\exp\Big(-\langle\sum_{k=1}^{\infty}\sum_{l=1}^{N_{t,k}}\tilde{\xi}_t^{k,l},\tilde{f}\rangle\Big)\right] = \exp\left(-\int(1-e^{-\langle\tilde{\mu},\tilde{f}\rangle})\tilde{\mathbf{Q}}_t^{-\infty}(d\tilde{\mu})\right), \qquad (2.40)$$

which in turn leads to (2.38), since $\tilde{\mathbf{Q}}_t^{-\infty}$ is the canonical measure of $\tilde{\xi}_t^{-\infty}$. $\square$

As mentioned above the second point we are interested in is the law of the weighted occupation time of one family. Let $g \in \mathcal{C}_c^+(\mathbb{R})$. Recall that $(\xi_t^{(\infty)})$ is the stationary BRW (cf. Section 1.3). The weighted occupation time $\int_0^\infty ds\, g(s)\xi_s^{(\infty)}$ is infinitely divisible, thus we can proceed as above. Let $\mathbf{Q}_{(g)}^{(\infty)}$ be the canonical measure of $\int_0^\infty ds\, g(s)\xi_s^{(\infty)}$, i.e.,

$$\mathbf{E}\left[\exp\left(-\Big\langle\int_0^\infty ds\, g(s)\xi_s^{(\infty)},f\Big\rangle\right)\right] = \exp\left(-\int_{M(\mathbb{Z}^d)}(1-e^{-\langle\mu,f\rangle})\mathbf{Q}_{(g)}^{(\infty)}(d\mu)\right). \qquad (2.41)$$

We group all families in the following way. The first group contains all families whose members occupy x_1. In the second group we collect all families occupying x_2 and not occupying x_1, and so on.

Let $\mathcal{M}_k := \{\bar{\pi}_0\tilde{\mu};\tilde{\mu}\in\tilde{\mathcal{M}}_k\}$. We identify the law of one family in the k-th group with $\mathbf{P}_{g,k}$.

$$\mathbf{P}_{g,k} = \frac{\mathbf{Q}_{(g)}^{(\infty)}(\mathcal{M}_k\cap\cdot)}{\mathbf{Q}_{(g)}^{(\infty)}(\mathcal{M}_k)}. \qquad (2.42)$$

Now we label the families in each group. Let $\{\xi^{g;k,l};k,l\in\mathbb{N}\}$ be a system of independent random variables, where each $\xi^{g;k,l}$ has law $\mathbf{P}_{g,k}$. Furthermore let $N_{g,k}$ be Poisson distributed with mean $\theta_{g,k} := \mathbf{Q}_{(g)}^{(\infty)}(\mathcal{M}_k)$ independent of $\{\xi^{g;k,l}\}$.

Corollary 2.5 *The equilibrium occupation process can be decomposed as*

$$\int_0^\infty ds\, g(s)\xi_s^{(\infty)} \stackrel{d}{=} \sum_{k=1}^{\infty}\sum_{l=1}^{N_{g,k}}\xi^{g;k,l}. \qquad \diamond \qquad (2.43)$$

The proof is similar to the proof of Corollary 2.4.

3 Proof of the spatial renormalization result

We have to show

$$\mathbf{E}[e^{iF_{\vartheta,r}(\varphi)}] \underset{r\to\infty}{\longrightarrow} e^{-\frac{1}{2}C_\vartheta L(\varphi,\varphi)}, \qquad \forall \varphi \in \mathcal{S}(\mathbb{R}^d), \tag{3.1}$$

which is equivalent to

$$\mathcal{L}[F_{\vartheta,r}(\varphi)] \underset{r\to\infty}{\Longrightarrow} \mathcal{N}(0, C_\vartheta L(\varphi,\varphi)), \qquad \forall \varphi \in \mathcal{S}(\mathbb{R}^d). \tag{3.2}$$

The proof of (3.2) is based on the historical process associated with the infinitely old equilibrium distribution and on the characterization of the equilibrium Λ_ϑ as the limit of $\mathcal{L}[\xi_t]$ as $t \to \infty$.

First of all we prove (3.2) for a non-negative smooth test function with compact support, i.e. $\varphi \in \mathcal{C}_c^{\infty,+}(\mathbb{R}^d)$. That means there exists a constant $c > 0$ such that $\mathrm{supp}(\varphi) \subset \mathrm{B}(c)$, where $\mathrm{B}(c)$ is the ball with radius c in $\mathbb{R}^d$.

The proof is split in four parts. In the first part we investigate the first and the second moments of the renormalized field. The second part contains the basic idea that is to avoid the analysis of all moments by giving a decomposition in independent components and checking a form of the Lyapunov condition in the Central Limit Theorem. This works for the following reason. The historical process of the branching random walk allows a cluster decomposition of the equilibrium state. We can view it as an infinitely old system and decompose the configuration into independent clusters of individuals belonging to the same family. The size of the clusters we can control via the third absolute moment.

In part 1 we calculate the first and the second moment of the renormalized field $F_{\vartheta,r}$. The family decomposition is given in part 2. In part 3 we check the Lyapunov condition of the following CLT. The proofs of the lemmas we defer to part 4.

Central Limit Theorem *Let $\{Y_{k,l}^{(r)}; k,l \in \mathbb{N}\}$ be independent real-valued random variables with $\{Y_{k,l}^{(r)}; l \in \mathbb{N}\}$ i.i.d. for each r. Let $\{N_k^{(r)}; k \in \mathbb{N}\}$ be independent random variables, where $N_k^{(r)}$ is Poisson distributed with mean $\theta_k^{(r)}$, which is uniformly bounded in r and k. Assume that $\{Y_{k,l}^{(r)}; k,l \in \mathbb{N}\}$ and $\{N_k^{(r)}; k \in \mathbb{N}\}$ are independent. If*

$$\lim_{r\to\infty} \mathbf{E}\Big[\sum_{k=1}^{\infty}\Big(\sum_{l=1}^{N_k^{(r)}} Y_{k,l}^{(r)} - \mathbf{E}\Big[\sum_{l=1}^{N_k^{(r)}} Y_{k,l}^{(r)}\Big]\Big)\Big]^2 = \sigma^2 \tag{3.3}$$

and

$$\lim_{r\to\infty} \mathbf{E}\Big[\sum_{k=1}^{\infty}\sum_{l=1}^{N_k^{(r)}} |Y_{k,l}^{(r)}|^3\Big] = 0, \tag{3.4}$$

then

$$\mathcal{L}\Big[\sum_{k=1}^{\infty}\sum_{l=1}^{N_k^{(r)}} Y_{k,l}^{(r)} - \mathbf{E}\Big[\sum_{k=1}^{\infty}\sum_{l=1}^{N_k^{(r)}} Y_{k,l}^{(r)}\Big]\Big] \underset{r\to\infty}{\Longrightarrow} \mathcal{N}(0,\sigma^2). \qquad \diamond \tag{3.5}$$

Proof Note that $\{Z_k^{(r)}; k \in \mathbb{N}\}$ defined by

$$Z_k^{(r)} = \sum_{l=1}^{N_k^{(r)}} Y_{k,l}^{(r)} - \mathbf{E}\Big[\sum_{l=1}^{N_k^{(r)}} Y_{k,l}^{(r)}\Big] \tag{3.6}$$

fits in the setting of the usual Central Limit Theorem, since $\mathbf{E}[Z_k^{(r)}] = 0$ and

$$\sum_{k=1}^{\infty} \mathbf{E}[Z_k^{(r)}]^2 \underset{r\to\infty}{\longrightarrow} \sigma^2. \tag{3.7}$$

That the Lyapunov condition is fulfilled for the third moment can be seen by the following observation

$$\sum_{k=1}^{\infty} \mathbf{E}|Z_k^{(r)}|^3 \le O(1) \sum_{k=1}^{\infty} \mathbf{E}\Big[\sum_{l=1}^{N_k^{(r)}} |Y_{k,l}^{(r)}|\Big]^3. \tag{3.8}$$

Since $N_k^{(r)}$ Poisson distributed with mean $\theta_k^{(r)}$ and $\{Y_{k,l}^{(r)}; l \in \mathbb{N}\}$ i.i.d., we obtain

$$\mathbf{E}\Big[\sum_{l=1}^{N_k^{(r)}} |Y_{k,l}^{(r)}|\Big]^3 = (\theta_k^{(r)})^3 \left(\mathbf{E}|Y_{k,1}^{(r)}|\right)^3 + 3(\theta_k^{(r)})^2\, \mathbf{E}|Y_{k,1}^{(r)}|\, \mathbf{E}|Y_{k,1}^{(r)}|^2 + \theta_k^{(r)}\, \mathbf{E}|Y_{k,1}^{(r)}|^3. \tag{3.9}$$

The parameters $\theta_k^{(r)}$ are uniformly bounded, hence

$$\sum_{k=1}^{\infty} \mathbf{E}|Z_k^{(r)}|^3 \le O(1) \sum_{k=1}^{\infty} \theta_k^{(r)}\, \mathbf{E}|Y_{k,1}^{(r)}|^3 = O(1)\mathbf{E}\Big[\sum_{k=1}^{\infty}\sum_{l=1}^{N_k^{(r)}} |Y_{k,l}^{(r)}|^3\Big] \underset{r\to\infty}{\longrightarrow} 0. \tag{3.10}$$

Thus the Lyapunov condition is fulfilled. □

Part 1 (First and second moment of the renormalized field) The first step is to verify that the first and second moments show the needed limiting behavior, which can later be strengthen to give the full Central Limit Theorem.

By Definition (1.18):

$$\mathbf{E}[F_{\vartheta,r}(\varphi)] = 0. \tag{3.11}$$

In part 4 we prove the following lemma.

Lemma 3.1 (Variance Estimate) *Under the assumptions (1.14), (1.15), (1.21) and (1.22)*

$$\mathbf{Var}[F_{\vartheta,r}(\varphi)] = \mathbf{E}\left[F_{\vartheta,r}(\varphi)\right]^2 \underset{r\to\infty}{\longrightarrow} C_\vartheta L(\varphi,\varphi) \tag{3.12}$$

for all $\varphi \in \mathcal{C}_c^{\infty,+}(\mathbb{R}^d)$. ◇

Part 2 (Family decomposition) In Section 2.3 we established the family decomposition of the historical equilibrium process, which we can use to get the family decomposition of ξ, since by Theorem 3 (ii)

$$\xi \stackrel{d}{=} \bar{\pi}_0 \tilde{\xi}_0^{-\infty}. \tag{3.13}$$

By (2.38) we obtain

$$\xi \stackrel{d}{=} \sum_{k=1}^{\infty} \sum_{l=1}^{N_{0,k}} \xi^{k,l}, \tag{3.14}$$

where $\xi^{k,l} = \bar{\pi}_0 \tilde{\xi}_0^{k,l}$. Applying (3.14) to the renormalized field $F_{\vartheta,r}$ leads to

$$F_{\vartheta,r}(\varphi) \stackrel{d}{=} \sum_{k=1}^{\infty} \left(\sum_{l=1}^{N_{0,k}} \left\langle \xi^{k,l}, \varphi_r \right\rangle - \mathbf{E}\Big[\sum_{l=1}^{N_{0,k}} \left\langle \xi^{k,l}, \varphi_r \right\rangle \Big] \right), \tag{3.15}$$

which is a sum of independent random variables. We are done if that sum fulfills the assumptions of the CLT.

Recall that $N_{0,k}$ is Poisson distributed with mean $\theta_{0,k} = \tilde{\mathbf{Q}}_0^{-\infty}(\tilde{\mathcal{M}}_k)$. Hence $\theta_{0,k} \leq \vartheta$, thus the parameters $\theta_{0,k}$ are uniformly bounded.

The convergence of the second moment is given by Lemma 3.1. It remains to check the condition on the third moment.

Part 3 (Lyapunov condition) Recall that the support of φ is contained in $\mathrm{B}(c)$, where $\mathrm{B}(c)$ is the ball with radius c in $\mathbb{R}^d$. Let $B(c)$ be the ball with radius c in $\mathbb{Z}^d$. The two symbols are quite similar, anyway it will be always clear from the context. We observe

$$\mathbf{E}\Big[\sum_{k=1}^{\infty} \sum_{l=1}^{N_{0,k}} |\langle \xi^{k,l}, \varphi_r \rangle|^3 \Big] \leq h(r)^3 \|\varphi\|_\infty^3 \mathbf{E}\Big[\sum_{k=1}^{\infty} \sum_{l=1}^{N_{0,k}} \langle \xi^{k,l}, \mathbb{1}_{B(cr)} \rangle^3 \Big]. \tag{3.16}$$

The sum on the r.h.s. of (3.16) in turn can be written as follows

$$\mathbf{E}\Big[\sum_{k=1}^{\infty} \sum_{l=1}^{N_{0,k}} \langle \xi^{k,l}, \mathbb{1}_{B(cr)} \rangle^3 \Big] = \sum_{k=1}^{\infty} \theta_{0,k} \int \langle \tilde{\mu}, \mathbb{1}_{B(cr)} \circ \pi_0 \rangle^3 \tilde{\mathbf{P}}_{0,k}(d\tilde{\mu}), \tag{3.17}$$

where (recall (2.37))

$$\tilde{\mathbf{P}}_{0,k} = \frac{\tilde{\mathbf{Q}}_0^{-\infty}(\tilde{\mathcal{M}}_k \cap \cdot)}{\tilde{\mathbf{Q}}_0^{-\infty}(\tilde{\mathcal{M}}_k)}, \qquad \theta_{0,k} = \tilde{\mathbf{Q}}_0^{-\infty}(\tilde{\mathcal{M}}_k). \tag{3.18}$$

This yields

$$\mathbf{E}\Big[\sum_{k=1}^{\infty} \sum_{l=1}^{N_{0,k}} \langle \xi^{k,l}, \mathbb{1}_{B(cr)} \rangle^3 \Big] = \int \langle \tilde{\mu}, \mathbb{1}_{B(cr)} \circ \pi_0 \rangle^3 \tilde{\mathbf{Q}}_0^{-\infty}(d\tilde{\mu}) \tag{3.19}$$

and by the definition of the Palm distribution

$$\mathbf{E}\Big[\sum_{k=1}^{\infty} \sum_{l=1}^{N_{0,k}} \langle \xi^{k,l}, \mathbb{1}_{B(cr)} \rangle^3 \Big] = \int \int \mathbb{1}_{B(cr)}(\pi_0(\tilde{y})) \langle \tilde{\mu}, \mathbb{1}_{B(cr)} \circ \pi_0 \rangle^2 (\tilde{\mathbf{Q}}_0^{-\infty})_{\tilde{y}}(d\tilde{\mu}) \tilde{\lambda}_{0,\vartheta}(d\tilde{y}). \tag{3.20}$$

Proposition 2.3 provides

$$\begin{aligned}
&\int\int \mathbb{I}_{B(cr)}(\pi_0(\tilde{y}))\langle\tilde{\mu}, \mathbb{I}_{B(cr)}\circ\pi_0\rangle^2(\tilde{\mathbf{Q}}_0^{-t})_{\tilde{y}}(d\tilde{\mu})\tilde{\lambda}_{0,\vartheta}(d\tilde{y}) \\
&\quad = \int \mathbb{I}_{B(cr)}(\pi_0(\tilde{y}))\mathbf{E}\left[\langle\tilde{\zeta}_{-t,0,\tilde{y}}+\delta_{\tilde{y}}, \mathbb{I}_{B(cr)}\circ\pi_0\rangle^2\right]\tilde{\lambda}_{0,\vartheta}(d\tilde{y}),
\end{aligned} \tag{3.21}$$

where $\tilde{\zeta}_{-t,0,\tilde{y}}$ is defined in (2.29). In part 4 we prove the following lemma

Lemma 3.2 *For $\tilde{\zeta}_{-t,0,\tilde{y}}$ defined in (2.29)*

$$\sup_{t\geq 0}\int \mathbb{I}_{B(r)}(\pi_0(\tilde{y}))\mathbf{E}\left[\langle\tilde{\zeta}_{-t,0,\tilde{y}}, \mathbb{I}_{B(r)}\circ\pi_0\rangle^2\right]\tilde{\lambda}_{0,\vartheta}(d\tilde{y}) = O(r^{d+4}). \qquad \diamond \tag{3.22}$$

On the r.h.s. of (3.21) we can write (with $\tilde{f} = \mathbb{I}_{B(cr)}\circ\pi_0$)

$$\mathbf{E}\left[\langle\tilde{\zeta}_{-t,0,\tilde{y}}+\delta_{\tilde{y}}, \tilde{f}\rangle^2\right] = \mathbf{E}\left[\langle\tilde{\zeta}_{-t,0,\tilde{y}}, \tilde{f}\rangle^2\right] + 2\tilde{f}(\tilde{y})\mathbf{E}\left[\langle\tilde{\zeta}_{-t,0,\tilde{y}}, \tilde{f}\rangle\right] + \tilde{f}(\tilde{y})^2. \tag{3.23}$$

Since the order of the second and the third term is dominated by that of the first one we end up with

$$\int\int \mathbb{I}_{B(cr)}(\pi_0(\tilde{y}))\langle\tilde{\mu}, \mathbb{I}_{B(cr)}\circ\pi_0\rangle^2(\tilde{\mathbf{Q}}_0^{-\infty})_{\tilde{y}}(d\tilde{\mu})\tilde{\lambda}_{0,\vartheta}(d\tilde{y}) = O(r^{d+4}). \tag{3.24}$$

Equations (3.16), (3.20) and (3.24) yield

$$\mathbf{E}\Big[\sum_{k=1}^{\infty}\sum_{l=1}^{N_{0,k}}\Big|\langle\xi^{k,l}, \varphi_r\rangle\Big|^3\Big] = O(r^{-\frac{3}{2}(d+2)}r^{d+4}) = O(r^{-\frac{d-2}{2}}) \underset{r\to\infty}{\longrightarrow} 0. \tag{3.25}$$

To sum up it can be said that (3.15) gives a representation of $F_{\vartheta,r}(\varphi)$ as a sum of independent random variables which fulfills the CLT. So far we only considered non-negative test functions with compact support. In part 4 we prove the following lemma, which ensures the result for test functions of the Schwartz space.

Lemma 3.3 (for general test functions) *If*

$$\mathcal{L}[F_{\vartheta,r}(\varphi)] \underset{r\to\infty}{\Longrightarrow} \mathcal{N}(0, C_\vartheta L(\varphi,\varphi)) \tag{3.26}$$

is valid for all $\varphi\in\mathcal{C}_c^{\infty,+}(\mathbb{R}^d)$, then the statement is true for all $\varphi\in\mathcal{S}(\mathbb{R}^d)$. $\diamond$

This completes the proof of Theorem 1. $\square$

Part 4 (Proofs of the lemmas) We prove the lemmas used in part 1 and 3. First of all we remark that for all $k\in\mathbb{N}$,

$$\sup_{t\geq 0}\mathbf{E}^{\mathcal{H}(\vartheta)}[\xi_t(0)^k] < \infty. \tag{3.27}$$

This well-known fact can be seen by the following argument. Since $\mathbf{E}^{\mathcal{H}(\vartheta)}[\xi_t(0)^k]$ can be written as a sum of terms of the form

$$\sum_{x_1\in\mathbb{Z}^d}\mathbf{E}^{\delta_{x_1}}[\langle\xi_t, f\rangle^{k_1}]\cdots\sum_{x_m\in\mathbb{Z}^d}\mathbf{E}^{\delta_{x_m}}[\langle\xi_t, f\rangle^{k_m}], \tag{3.28}$$

where $k_1 + \dots k_m = k$, we can follow (3.27) from Lemma B.1 in the appendix by an induction. Property (3.27) provides uniform integrability and thus ensures convergence of moments.

Proof of Lemma 3.1 By (1.10) and (3.27):

$$\mathbf{E}[F_{\vartheta,r}(\varphi)]^2 = \lim_{t\to\infty} \mathbf{E}^{\mathcal{H}(\vartheta)}\left[\langle \xi_t - \vartheta\lambda, \varphi_r\rangle^2\right]. \tag{3.29}$$

By (B.6) in the appendix we have

$$\mathbf{E}^{\mathcal{H}(\vartheta)}[\langle \xi_t, \varphi_r\rangle] = \vartheta\langle\lambda, \varphi_r\rangle \tag{3.30}$$

and (B.7) gives us

$$\mathbf{E}^{\mathcal{H}(\vartheta)}[\langle \xi_t, \varphi_r\rangle^2] = \vartheta^2\langle\lambda, \varphi_r\rangle^2 + \vartheta\langle\lambda, \varphi_r^2\rangle + V\vartheta \sum_{x,y\in\mathbb{Z}^d} \varphi_r(x)\varphi_r(y)\int_0^t ds\, \hat{a}_{2s}(x,y). \tag{3.31}$$

This results in

$$\mathbf{E}[F_{\vartheta,r}(\varphi)]^2 = \vartheta\langle\lambda, \varphi_r^2\rangle + \frac{V\vartheta}{2}\sum_{x,y\in\mathbb{Z}^d} \varphi_r(x)\varphi_r(y)\hat{G}(y-x), \tag{3.32}$$

where $\hat{G}$ is defined in (1.13).

Concerning the first term on the r.h.s. of (3.32) we observe that

$$\vartheta\langle\lambda, \varphi_r^2\rangle = \vartheta r^{-(d+2)}\sum_{x\in B(cr)} \varphi\left(\frac{x}{r}\right)^2 = O(r^{-2}), \tag{3.33}$$

since $\mathrm{supp}(\varphi) \subset \mathrm{B}(c)$.

We consider the second term on the r.h.s. of (3.32). In the appendix in Lemma C.1 we establish the following asymptotics

$$\hat{G}(x) \sim \begin{cases} C\cdot\bar{Q}(x)^{-\frac{d-2}{2}} & \text{as } |x|\to\infty;\ \text{if } x\in\mathcal{G}, \\ 0 & \text{if } \ x\in\mathbb{Z}^d\backslash\mathcal{G}, \end{cases} \tag{3.34}$$

with

$$C = \frac{|\mathbb{Z}^d/\mathcal{G}|\Gamma\left(\frac{d-2}{2}\right)}{2\pi^{d/2}|Q|^{1/2}}, \tag{3.35}$$

where $\mathcal{G}$ is defined in (1.14).

Fix $\varepsilon > 0$. In order to employ the asymptotics of $\hat{G}(y-x)$ for $|y-x|$ large in the term $\sum_{x,y}\varphi_r(x)\varphi_r(y)\hat{G}(y-x)$ we split the sum into two parts depending on whether $|y-x| < M$ or $|y-x| \geq M$. Here $M = M(\varepsilon)$ is a constant that depends only on ε and will be chosen below in (3.38).

To show that the sum over $|y-x| < M$ is negligible the crude estimate $\hat{G}(y-x) \leq \hat{G}(0)$ is sufficient to get (recall $\mathrm{supp}(\varphi) \subset \mathrm{B}(c)$)

$$\Big|\sum_{|y-x|<M} \hat{G}(y-x)\, r^{-(d+2)}\varphi\left(\frac{x}{r}\right)\varphi\left(\frac{y}{r}\right)\Big| \leq r^{-2}(2M)^d(2c)^d\|\varphi\|_\infty^2\hat{G}(0). \tag{3.36}$$

It remains to investigate the sum over $|y-x| \geq M$. For $(y-x) \notin \mathcal{G}$ we have $\hat{G}(y-x) = 0$. Hence we get

$$\sum_{|y-x|\geq M} \hat{G}(y-x)\, r^{-(d+2)}\, \varphi\left(\frac{x}{r}\right)\, \varphi\left(\frac{y}{r}\right) = \sum_{\substack{y-x\in\mathcal{G};\\ |y-x|\geq M}} \hat{G}(y-x)\, r^{-(d+2)}\, \varphi\left(\frac{x}{r}\right)\, \varphi\left(\frac{y}{r}\right). \tag{3.37}$$

Let $\tilde{\varepsilon} := \varepsilon|\mathbb{Z}^d/\mathcal{G}|/(C\, L(\varphi,\varphi))$, where C is given in (3.38). From (3.34) we obtain for $(y-x) \in \mathcal{G}; |y-x| > M(\varepsilon)$,

$$\left|\frac{\hat{G}(y-x)}{\frac{C}{\bar{Q}(y-x)^{(d-2)/2}}} - 1\right| < \tilde{\varepsilon}, \qquad \text{where} \qquad C = \frac{|\mathbb{Z}^d/\mathcal{G}|}{2\pi^{\frac{d}{2}}|Q|^{\frac{1}{2}}}\Gamma\left(\frac{d-2}{2}\right). \tag{3.38}$$

We replace $\hat{G}$ by the above expression plus an error term, that means

$$\begin{aligned} &\sum_{\substack{y-x\in\mathcal{G};\\ |y-x|\geq M}} \hat{G}(y-x)\, r^{-(d+2)}\, \varphi\left(\frac{x}{r}\right)\, \varphi\left(\frac{y}{r}\right) \\ &= \sum_{\substack{y-x\in\mathcal{G};\\ |y-x|\geq M}} \left(\hat{G}(y-x) - \frac{C}{\bar{Q}(y-x)^{(d-2)/2}}\right) r^{-(d+2)}\, \varphi\left(\frac{x}{r}\right)\, \varphi\left(\frac{y}{r}\right) \\ &\quad + \sum_{\substack{y-x\in\mathcal{G};\\ |y-x|\geq M}} \left(\frac{C}{\bar{Q}(y-x)^{(d-2)/2}}\right) r^{-(d+2)}\, \varphi\left(\frac{x}{r}\right)\, \varphi\left(\frac{y}{r}\right). \end{aligned} \tag{3.39}$$

We investigate the two sums on the r.h.s. of (3.39) separately. For the second sum we observe that

$$\begin{aligned} \sum_{\substack{y-x\in\mathcal{G};\\ |y-x|\geq M}} \frac{C}{\bar{Q}(y-x)^{(d-2)/2}}\, r^{-(d+2)} \varphi\left(\frac{x}{r}\right)\, \varphi\left(\frac{y}{r}\right) \quad &= \quad C \sum_{\substack{y-x\in\mathcal{G};\\ |y-x|\geq M}} r^{-2d} \frac{\varphi\left(\frac{x}{r}\right)\, \varphi\left(\frac{y}{r}\right)}{\bar{Q}(\frac{y}{r} - \frac{x}{r})^{(d-2)/2}} \\ &\underset{r\to\infty}{\longrightarrow} \quad \frac{C}{|\mathbb{Z}^d/\mathcal{G}|} L(\varphi,\varphi). \end{aligned} \tag{3.40}$$

For the first sum on the r.h.s. of (3.39) we get by (3.38)

$$\begin{aligned} &\left| \sum_{\substack{y-x\in\mathcal{G};\\ |y-x|\geq M}} \left(\hat{G}(y-x) - \frac{C}{\bar{Q}(y-x)^{(d-2)/2}}\right) r^{-(d+2)}\, \varphi\left(\frac{x}{r}\right)\, \varphi\left(\frac{y}{r}\right) \right| \\ &\leq \ \tilde{\varepsilon}\, C \sum_{\substack{y-x\in\mathcal{G};\\ |y-x|\geq M}} \frac{1}{\bar{Q}(y-x)^{(d-2)/2}}\, r^{-(d+2)}\, \varphi\left(\frac{x}{r}\right)\, \varphi\left(\frac{y}{r}\right). \end{aligned} \tag{3.41}$$

The sum on the r.h.s. of (3.41) converges to $L(\varphi,\varphi)/|\mathbb{Z}^d/\mathcal{G}|$ as $r \to \infty$ (analogously to (3.40)). For the given ε we chose $\tilde{\varepsilon}$ in an appropriate way such that

$$\limsup_{r\to\infty} \left| \sum_{\substack{y-x\in\mathcal{G};\\ |y-x|\geq M}} \left(\hat{G}(y-x) - \frac{C}{\bar{Q}(y-x)^{(d-2)/2}}\right) r^{-(d+2)}\, \varphi\left(\frac{x}{r}\right)\, \varphi\left(\frac{y}{r}\right) \right| \leq \varepsilon. \tag{3.42}$$

Combining (3.40) and (3.42) we obtain

$$\sum_{x,y\in\mathbb{Z}^d} \varphi_r(x)\varphi_r(y)\hat{G}(y-x) \underset{r\to\infty}{\longrightarrow} \frac{C}{|\mathbb{Z}^d/\mathcal{G}|} L(\varphi,\varphi). \tag{3.43}$$

By (3.32), (3.33) and (3.43) we get

$$\mathbf{E}[F_{\vartheta,r}(\varphi)]^2 \underset{r\to\infty}{\longrightarrow} \frac{V\vartheta}{4\pi^{\frac{d}{2}}|Q|^{\frac{1}{2}}} \Gamma\left(\frac{d-2}{2}\right) L(\varphi,\varphi). \tag{3.44}$$

This completes the proof. $\square$

Proof of Lemma 3.2 Recall the definition of $\tilde{\zeta}_{-t,T,\tilde{y}}$ from (2.29)

$$\tilde{\zeta}_{-t,T,\tilde{y}} = \int_{-t}^{T} \tilde{\xi}_T^{s,\tilde{y}^s}\,\nu(ds), \tag{3.45}$$

where ν is a random Poisson point measure on $\mathbb{R}$ with intensity V w.r.t to the Lebesgue measure. We proceed in five steps.

Step 0 Clearly we shall need a moment formula. First focus on general moments and condition on the number of Poisson points in the interval $(-t,T]$

$$\begin{aligned}
\mathbf{E}[\langle\tilde{\zeta}_{-t,T,\tilde{y}},\tilde{f}\rangle^n] &= \sum_{m=0}^{\infty} e^{-V(T+t)}\frac{(V(T+t))^m}{m!}\mathbf{E}\left[\langle\tilde{\zeta}_{-t,T,\tilde{y}},\tilde{f}\rangle^n | \nu(-t,T]=m\right] \\
&= \sum_{m=0}^{\infty} e^{-V(T+t)}\frac{(V(T+t))^m}{m!}\int_{(-t,T]^m}\frac{ds_1\cdots ds_m}{(T+t)^m}\mathbf{E}\left[\langle\sum_{k=1}^{m}\tilde{\xi}_T^{s_k,\tilde{y}^{s_k}},\tilde{f}\rangle^n\right],
\end{aligned} \tag{3.46}$$

since the Poisson points conditioned on their number in $(-t,T]$ are uniformly distributed on $(-t,T]$. In particular we obtain the following useful representation of the second moment

$$\begin{aligned}
&\int \tilde{\lambda}_{T,\vartheta}(d\tilde{y})\tilde{f}(\tilde{y})\mathbf{E}[\langle\tilde{\zeta}_{-t,T,\tilde{y}},\tilde{f}\rangle^2] \\
&= \int \tilde{\lambda}_{T,\vartheta}(d\tilde{y})\tilde{f}(\tilde{y})\sum_{m=0}^{\infty} e^{-V(T+t)}\frac{V^m}{m!}\int_{(-t,T]^m} ds_1\cdots ds_m \\
&\quad\times\Big(\sum_k \mathbf{E}[\langle\tilde{\xi}_T^{s_k,\tilde{y}^{s_k}},\tilde{f}\rangle^2] + \sum_{k\neq l}\mathbf{E}[\langle\tilde{\xi}_T^{s_k,\tilde{y}^{s_k}},\tilde{f}\rangle]\mathbf{E}[\langle\tilde{\xi}_T^{s_l,\tilde{y}^{s_l}},\tilde{f}\rangle]\Big),
\end{aligned} \tag{3.47}$$

which can be simplified to

$$\begin{aligned}
&\int \tilde{\lambda}_{T,\vartheta}(d\tilde{y})\tilde{f}(\tilde{y})\mathbf{E}[\langle\tilde{\zeta}_{-t,T,\tilde{y}},\tilde{f}\rangle^2] \\
&= V\int \tilde{\lambda}_{T,\vartheta}(d\tilde{y})\tilde{f}(\tilde{y})\int_{-t}^{T} ds\,\mathbf{E}[\langle\tilde{\xi}_T^{s,\tilde{y}^s},\tilde{f}\rangle^2] \\
&\quad + V^2\int \tilde{\lambda}_{T,\vartheta}(d\tilde{y})\tilde{f}(\tilde{y})\int_{-t}^{T} ds\int_{-t}^{T} du\,\mathbf{E}[\langle\tilde{\xi}_T^{s,\tilde{y}^s},\tilde{f}\rangle]\mathbf{E}[\langle\tilde{\xi}_T^{u,\tilde{y}^u},\tilde{f}\rangle].
\end{aligned} \tag{3.48}$$

We are interested in the case $T = 0$ and in the special test function $\tilde{f} = 1\!\!1_{B(r)} \circ \pi_0$. We can exploit $\mathcal{L}[\bar{\pi}_0(\tilde{\xi}_0^{s,\tilde{y}^s})] = \mathcal{L}^{\delta_{y_s}}[\xi_{-s}]$ to write

$$\int \tilde{\lambda}_{0,\vartheta}(d\tilde{y}) 1\!\!1_{B(r)}(\pi_0(\tilde{y}))\mathbf{E}[\langle \tilde{\zeta}_{-t,0,\tilde{y}}, 1\!\!1_{B(r)} \circ \pi_0 \rangle^2] = I_1(r) + I_2(r), \tag{3.49}$$

where

$$I_1(r) := V \int \tilde{\lambda}_{0,\vartheta}(d\tilde{y}) 1\!\!1_{B(r)}(y_0) \int_{-t}^{0} ds\, \mathbf{E}^{\delta_{y_s}}[\langle \xi_{-s}, 1\!\!1_{B(r)} \rangle^2] \tag{3.50}$$

and

$$I_2(r) := V^2 \int \tilde{\lambda}_{0,\vartheta}(d\tilde{y}) 1\!\!1_{B(r)}(y_0) \int_{-t}^{0} ds \int_{-t}^{0} du\, \mathbf{E}^{\delta_{y_s}}[\langle \xi_{-s}, 1\!\!1_{B(r)} \rangle] \mathbf{E}^{\delta_{y_u}}[\langle \tilde{\xi}_{-u}, 1\!\!1_{B(r)} \rangle]. \tag{3.51}$$

Use (2.23) to get

$$I_1(r) = V\vartheta \sum_{x \in \mathbb{Z}^d} \sum_{i \in B(r)} \int_0^t ds\, a_s(x,i) \mathbf{E}^{\delta_x}[\langle \xi_s, 1\!\!1_{B(r)} \rangle^2]. \tag{3.52}$$

In $I_2(r)$ we can exploit the symmetry in s and u and then we can again apply (2.23) to get

$$I_2(r) = 2V^2\vartheta \sum_{x,y \in \mathbb{Z}^d} \sum_{i \in B(r)} \int_0^t ds \int_s^t du\, a_{u-s}(y,x) a_s(x,i) \mathbf{E}^{\delta_x}[\langle \xi_s, 1\!\!1_{B(r)} \rangle] \mathbf{E}^{\delta_y}[\langle \xi_u, 1\!\!1_{B(r)} \rangle]. \tag{3.53}$$

Now the problem is reduced to estimate (3.52) and (3.53). We proceed in three steps. Namely we consider the terms $I_1(r)$ and $I_2(r)$ separately in steps 2 and 3 below after preparing basic estimates in step 1.

Step 1 First of all we want to establish the basic estimates we will use repeatedly.
Estimate 1

$$\sum_{i \in B(r)} \hat{G}(i) = O(r^2) \tag{3.54}$$

Estimate 2 for sums in (v, h, ρ)

$$\sum_{v \in \mathbb{Z}^d} \sum_{h \in B(r)} \int_\tau^t d\rho\, a_{\rho-\tau}(v,z) a_\rho(v,h) \;\leq\; O(r^2), \tag{3.55}$$

uniformly in z, t and τ.
In the remaining part of step 1 we prove these estimates
Proof of estimate 1 We decompose

$$\sum_{i \in B(r)} \hat{G}(i) = \sum_{k=0}^{\infty} \sum_{i \in B(r,k)} \hat{G}(i), \tag{3.56}$$

where

$$B(r,k) := \{x \in \mathbb{Z}^d : \frac{r}{2^{k+1}} \leq |x| < \frac{r}{2^k}\}. \tag{3.57}$$

By Lemma C.1 we find a constant $C' > 0$ such that for all $x \in \mathbb{Z}^d$

$$\hat{G}(x) \leq \frac{C'}{(|x|+1)^{d-2}}. \tag{3.58}$$

Thus we can estimate

$$\sum_{i \in B(r)} \hat{G}(i) \leq \sum_{k=0}^{\infty} \frac{C'}{(\frac{r}{2^{k+1}}+1)^{d-2}} \left(\frac{r}{2^k}\right)^d \leq O(r^2). \tag{3.59}$$

This completes the proof of (3.54).

Proof of estimate 2 By Lemma C.1 we find a constant $C > 0$ such that for all $y \in \mathbb{Z}^d$

$$\hat{G}(y) \leq \frac{C}{(|y|+1)^{d-2}}. \tag{3.60}$$

We decompose $a_\rho(v,h)$ to get

$$\begin{aligned}
&\sum_{v \in \mathbb{Z}^d} \sum_{h \in B(r)} \int_\tau^t d\rho \, a_{\rho-\tau}(v,z) a_\rho(v,h) \\
&\quad = \sum_{v,v' \in \mathbb{Z}^d} \sum_{h \in B(r)} \int_\tau^t d\rho \, a_{\rho-\tau}(v,z) a_{\rho-\tau}(v,v') a_\tau(v',h) \\
&\quad = \sum_{v' \in \mathbb{Z}^d} \sum_{h \in B(r)} a_\tau(v',h) \int_\tau^t d\rho \, \hat{a}_{2(\rho-\tau)}(z,v'),
\end{aligned} \tag{3.61}$$

where we performed the sum over v in the latter equality. Estimating the integral over ρ by the corresponding Green's function leads to

$$\sum_{v \in \mathbb{Z}^d} \sum_{h \in B(r)} \int_\tau^t d\rho \, a_{\rho-\tau}(v,z) a_\rho(v,h) \ \leq \sum_{v' \in \mathbb{Z}^d} \sum_{h \in B(r)} a_\tau(v',h) \hat{G}(v'-z). \tag{3.62}$$

We want to distinguish between the cases "large" distance between v' and z and "small" distance between v' and z. In the first case we want to exploit that the Green's function is small enough. On the other hand there are not too many pairs (v',z) fulfilling the second condition. Denote for $z \in \mathbb{Z}^d$:

$$B_z(r) = \{y \in \mathbb{Z}^d : |z-y| < r\}. \tag{3.63}$$

We split the sum over v' as follows

$$\begin{aligned}
&\sum_{v' \in \mathbb{Z}^d} \sum_{h \in B(r)} a_\tau(v',h) \hat{G}(v'-z) \\
&\quad = \sum_{v' \notin B_z(r)} \sum_{h \in B(r)} a_\tau(v',h) \hat{G}(v'-z) + \sum_{v' \in B_z(r)} \sum_{h \in B(r)} a_\tau(v',h) \hat{G}(v'-z) \\
&\quad = O(r^{-(d-2)}) \sum_{v' \notin B_z(r)} \sum_{h \in B(r)} a_\tau(v',h) + \sum_{v' \in B(r)} \hat{G}(v') \sum_{h \in B(r)} a_\tau(v'+z,h),
\end{aligned} \tag{3.64}$$

where we used that $\hat{G}(v'-z) = O(r^{-(d-2)})$ for $|v'-z| > r$ by (3.60). In the first term on the r.h.s. of (3.64) we estimate the sum over v' by 1. In the second term we do the same for the sum over h, hence

$$\sum_{v' \in \mathbb{Z}^d} \sum_{h \in B(r)} a_\tau(v', h) \hat{G}(v'-z) \leq O(r^2) + \sum_{v' \in B(r)} \hat{G}(v') \leq O(r^2). \tag{3.65}$$

Combining (3.62) and (3.65) we obtain (3.55).

Step 2 Now we are in the position to establish the order of the term $I_1(r)$ given in (3.52). In the appendix we find formula (B.3) for the second moment of the BRW.

$$\begin{aligned} I_1(r) &= V\vartheta \sum_{x \in \mathbb{Z}^d} \sum_{i \in B(r)} \int_0^t ds\, a_s(x,i) \left[(a_s \mathbb{1}_{B(r)})(x) + V \int_0^s du\, a_{s-u} \Big((a_u \mathbb{1}_{B(r)})^2 \Big)(x) \right] \\ &= I_{1,1}(r) + I_{1,2}(r), \end{aligned} \tag{3.66}$$

where

$$I_{1,1}(r) := V\vartheta \sum_{x \in \mathbb{Z}^d} \sum_{i,j \in B(r)} \int_0^t ds\, a_s(x,i) a_s(x,j) \tag{3.67}$$

and

$$I_{1,2}(r) := V^2\vartheta \sum_{x,y \in \mathbb{Z}^d} \sum_{i,j,g \in B(r)} \int_0^t ds\, a_s(x,i) \int_0^s du\, a_{s-u}(x,y) a_u(y,j) a_u(y,g). \tag{3.68}$$

In case of $I_{1,1}(r)$ we perform the sum over x and then we estimate the integral over s by the corresponding Green's function to get by (3.54)

$$I_{1,1}(r) \leq O(1) \sum_{i,j \in B(r)} \hat{G}(j-i) = O(r^{d+2}). \tag{3.69}$$

In case of $I_{1,2}(r)$ we change the order of integration, such that

$$I_{1,2}(r) = V^2\vartheta \sum_{x,y \in \mathbb{Z}^d} \sum_{i,j,g \in B(r)} \int_0^t du\, a_u(y,j) a_u(y,g) \int_u^t ds\ a_s(x,i) a_{s-u}(x,y). \tag{3.70}$$

Now we use the basic estimate given in (3.55) for the sums in (x,i,s), hence

$$I_{1,2}(r) \leq O(r^2) \sum_{y \in \mathbb{Z}^d} \sum_{j,g \in B(r)} \int_0^t du\, a_u(y,j) a_u(y,g) = O(r^2) I_{1,1}(r) = O(r^{d+4}). \tag{3.71}$$

To sum up it can be said that $I_1(r) = O(r^{d+4})$.

Step 3 Now we investigate $I_2(r)$ given in (3.53). We apply the moment formula given in (B.2) to get

$$I_2(r) = 2V^2\vartheta \sum_{x,y \in \mathbb{Z}^d} \sum_{i,j,g \in B(r)} \int_0^t ds \int_s^t du\, a_{u-s}(y,x) a_s(x,i) a_s(x,j) a_u(y,g). \tag{3.72}$$

We apply the basic estimate in (3.55) for the sums in (y, g, u), which leads to

$$I_2(r) \leq O(r^2) \sum_{x\in\mathbb{Z}^d} \sum_{i,j\in B(r)} \int_0^t ds\, a_s(x,i) a_s(x,j) = O(r^2) I_{1,1}(r) = O(r^{d+4}). \tag{3.73}$$

We end up with $I_2(r) = O(r^{d+4})$. This completes the proof. □

Proof of Lemma 3.3 At first we extend the result to smooth functions which are not necessarily non-negative, then in a second step we drop the bounded support property and consider $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

Let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. Note that the family decomposition in (3.14) does not depend on the test function, thus we get (3.15) also for $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. The same goes for the Lyapunov condition, since (3.16) is still valid. It remains to study the first and the second moment of the rescaled field. The first moment is obviously zero. For calculating the variance we decompose φ in its positive part and its negative part, i.e. $\varphi = \varphi^+ - \varphi^-$ with $\varphi^+, \varphi^- \in \mathcal{C}_c^{\infty,+}(\mathbb{R}^d)$. We get

$$\begin{aligned}\mathbf{E}\left[F_{\vartheta,r}(\varphi)\right]^2 &= \mathbf{E}\left[F_{\vartheta,r}(\varphi^+ - \varphi^-)\right]^2 \\ &= 2\mathbf{E}\left[F_{\vartheta,r}(\varphi^+)\right]^2 + 2\mathbf{E}\left[F_{\vartheta,r}(\varphi^-)\right]^2 - \mathbf{E}\left[F_{\vartheta,r}(\varphi^+ + \varphi^-)\right]^2. \end{aligned}\tag{3.74}$$

For each term we can apply Lemma 3.1, which leads to

$$\mathbf{E}\left[F_{\vartheta,r}(\varphi)\right]^2 \underset{r\to\infty}{\longrightarrow} 2C_\vartheta L(\varphi^+, \varphi^+) + 2C_\vartheta L(\varphi^-, \varphi^-) - C_\vartheta L(\varphi^+ + \varphi^-, \varphi^+ + \varphi^-). \tag{3.75}$$

By the bilinearity of L we obtain the desired result.

Now let $\varphi \in \mathcal{S}(\mathbb{R}^d)$. For $\varepsilon > 0$ there exists a function $\varphi_\varepsilon \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ such that

$$L(|\varphi - \varphi_\varepsilon|, |\varphi - \varphi_\varepsilon|) < \varepsilon. \tag{3.76}$$

Recall from (3.32)

$$\mathbf{E}[F_{\vartheta,r}(\varphi)]^2 = \vartheta\langle\lambda, \varphi_r^2\rangle + \frac{V\vartheta}{2} \sum_{x,y\in\mathbb{Z}^d} \varphi_r(x)\varphi_r(y)\hat{G}(y-x). \tag{3.77}$$

By Lemma C.1 we find a constant $C > 0$ such that

$$\hat{G}(y-x) \leq C \cdot \left(\bar{Q}(y-x)\right)^{-\frac{d-2}{2}} \tag{3.78}$$

for all $x, y \in \mathbb{Z}^d$. Now we can proceed as in (3.40) and estimate

$$\begin{aligned}\limsup_{r\to\infty} \mathbf{E}[F_{\vartheta,r}(\varphi)]^2 &\leq \limsup_{r\to\infty} \left[r^{-2}\langle\vartheta\lambda, \varphi^2\rangle + \sum_{x,y\in\mathbb{Z}^d} \hat{G}(y-x)|\varphi_r(x)||\varphi_r(y)| \right] \\ &\leq C \limsup_{r\to\infty} \sum_{x,y\in\mathbb{Z}^d} r^{-2d}\left(Q\left(\frac{x}{r} - \frac{y}{r}\right)\right)^{-\frac{d-2}{2}} \left|\varphi\left(\frac{x}{r}\right)\right| \left|\varphi\left(\frac{y}{r}\right)\right| \\ &\leq C' L(|\varphi|, |\varphi|), \end{aligned}\tag{3.79}$$

where $C' > 0$ is a constant independent of φ. Applying this estimate to $\varphi - \varphi_\varepsilon$ we obtain

$$\begin{aligned}\limsup_{r\to\infty} \mathbf{E}[F_{\vartheta,r}(\varphi) - F_{\vartheta,r}(\varphi_\varepsilon)]^2 &= \limsup_{r\to\infty} \mathbf{E}[F_{\vartheta,r}(\varphi - \varphi_\varepsilon)]^2 \\ &\leq C'\, L(|\varphi - \varphi_\varepsilon|, |\varphi - \varphi_\varepsilon|) \leq C'\, \varepsilon. \end{aligned} \tag{3.80}$$

From this and the assumption of this lemma that

$$\mathcal{L}[F_{\vartheta,r}(\varphi_\varepsilon)] \underset{r\to\infty}{\Longrightarrow} \mathcal{N}(0, C_\vartheta L(\varphi_\varepsilon, \varphi_\varepsilon)), \tag{3.81}$$

we can conclude

$$\mathcal{L}[F_{\vartheta,r}(\varphi)] \underset{r\to\infty}{\Longrightarrow} \mathcal{N}(0, C_\vartheta L(\varphi, \varphi)). \tag{3.82}$$

This completes the proof. □

4 Proof of the space-time-renormalization result

The proof is organized as the proof of Theorem 1 in Section 3. First of all we consider $\tilde{\varphi}(x,t) = \varphi(x)g(t)$, where φ and g are smooth non-negative functions with compact support, i.e. $\varphi \in \mathcal{C}_c^{\infty,+}(\mathbb{R}^d)$ and $g \in \mathcal{C}_c^{\infty,+}([0,\infty))$. That means there exist constants γ and c such that $\mathrm{supp}(g) \subset [0,\gamma]$ and $\mathrm{supp}(\varphi) \subset \mathrm{B}(c)$, where $\mathrm{B}(c)$ is the ball with radius c in $\mathbb{R}^d$.

Part 1 (Moments of the rescaled field) The first step is to verify that the first and the second moments show the needed limiting behavior, which can be strengthened to give the full Central Limit Theorem.

By Definition (1.30):

$$\mathbf{E}[\tilde{F}_{\vartheta,r}(\tilde{\varphi})] = 0. \tag{4.1}$$

In part 4 we will prove the following lemma.

Lemma 4.1 (Variance Convergence) *Under the assumptions (1.14), (1.15), (1.21) and (1.22)*

$$\mathbf{Var}[\tilde{F}_{\vartheta,r}(\tilde{\varphi})] = \mathbf{E}\left[\tilde{F}_{\vartheta,r}(\tilde{\varphi})\right]^2 \underset{r\to\infty}{\longrightarrow} \tilde{C}_\vartheta \tilde{L}(\tilde{\varphi}, \tilde{\varphi}) \tag{4.2}$$

for $\tilde{\varphi} \in \mathcal{C}_c^+(\mathbb{R}^d \times [0,\infty))$. ◇

Part 2 (Family decomposition) In Section 2.3 we established the family decomposition of the weighted occupation time, which we can use to get a decomposition of $\tilde{F}_{\vartheta,r}(\tilde{\varphi})$ for $\tilde{\varphi}(x,t) = \varphi(x)g(t)$. Note that

$$\tilde{F}_{\vartheta,r}(\tilde{\varphi}) = \Big\langle \int_0^\infty ds\, g_r(s)(\xi_s^{(\infty)} - \mathbf{E}\xi_s^{(\infty)}), \varphi_r \Big\rangle, \tag{4.3}$$

where $g_r(s) = r^{-2}g(s/r^2)$ and $\varphi_r(x) = h(r)\varphi(x/r)$ with $h(r) = r^{-(d+2)/2}$. By (2.43) we obtain

$$\int_0^\infty ds\, g_r(s)\xi_s^{(\infty)} \stackrel{d}{=} \sum_{k=1}^\infty \sum_{l=1}^{N_{g_r,k}} \xi^{g_r;k,l}. \tag{4.4}$$

This leads to

$$\tilde{F}_{\vartheta,r}(\tilde{\varphi}) \stackrel{d}{=} \sum_{k=1}^{\infty} \left(\sum_{l=1}^{N_{g_r,k}} \langle \xi^{g_r;k,l}, \varphi_r \rangle - \mathbf{E}\Big[\sum_{l=1}^{N_{g_r,k}} \langle \xi^{g_r;k,l}, \varphi_r \rangle \Big] \right), \tag{4.5}$$

which is a sum of independent random variables. We are done if that sum fulfills the assumptions of the CLT.

Recall that $N_{g_r,k}$ is Poisson distributed with mean $\theta_{g_r,k} = \mathbf{Q}^{(\infty)}_{(g_r)}(\mathcal{M}_k)$. Hence

$$\theta_{g_r,k} \leq r^{-2} \int_0^{t(r)} ds\, g\left(\frac{s}{r^2}\right) \mathbf{E}[\xi^{(\infty)}_s(x_k)], \tag{4.6}$$

where $t(r) := \gamma r^2$. This yields $\theta_{g_r,k} \leq \gamma\vartheta \|g\|_\infty$, thus the parameters $\theta_{g_r,k}$ are uniformly bounded. The convergence of the second moment is given by Lemma 4.1. It remains to check the condition on the third moment.

Part 3 (Lyapunov) We observe (recall $\text{supp}(\varphi) \subset \mathrm{B}(c)$)

$$\mathbf{E}\Big[\sum_{k=1}^{\infty} \sum_{l=1}^{N_{g_r,k}} |\langle \xi^{g_r;k,l}, \varphi_r \rangle|^3 \Big] \leq h(r)^3 \|\varphi\|^3_\infty \mathbf{E}\Big[\sum_{k=1}^{\infty} \sum_{l=1}^{N_{g_r,k}} \langle \xi^{g_r;k,l}, \mathbb{I}_{B(cr)} \rangle^3 \Big]. \tag{4.7}$$

The sum on the r.h.s. of (4.7) in turn can be written as follows

$$\mathbf{E}\Big[\sum_{k=1}^{\infty} \sum_{l=1}^{N_{g_r,k}} \langle \xi^{g_r;k,l}, \mathbb{I}_{B(cr)} \rangle^3 \Big] = \sum_{k=1}^{\infty} \theta_{g_r,k} \int \langle \mu, \mathbb{I}_{B(cr)} \rangle^3 \mathbf{P}_{g_r,k}(d\mu), \tag{4.8}$$

where

$$\mathbf{P}_{g_r,k} = \frac{\mathbf{Q}^{(\infty)}_{g_r}(\mathcal{M}_k \cap \cdot)}{\mathbf{Q}^{(\infty)}_{g_r}(\mathcal{M}_k)}, \qquad \theta_{g_r,k} = \mathbf{Q}^{(\infty)}_{g_r}(\mathcal{M}_k). \tag{4.9}$$

This yields

$$\mathbf{E}\Big[\sum_{k=1}^{\infty} \sum_{l=1}^{N_{g_r,k}} \langle \xi^{g_r;k,l}, \mathbb{I}_{B(cr)} \rangle^3 \Big] = \int \langle \mu, \mathbb{I}_{B(cr)} \rangle^3 \mathbf{Q}^{(\infty)}_{g_r}(d\mu). \tag{4.10}$$

Note that $\mathbf{Q}^{(\infty)}_{g_r}$ is the canonical measure of $\int_0^{t(r)} ds\, g_r(s) \xi^{(\infty)}_s$, where $t(r) = \gamma r^2$, since $\text{supp}(g) \subset [0, \gamma]$. By (3.27) we know

$$\int \langle \mu, \mathbb{I}_{B(cr)} \rangle^3 \mathbf{Q}^{(\infty)}_{g_r}(d\mu) = \lim_{T\to\infty} \int \langle \mu, \mathbb{I}_{B(cr)} \rangle^3 Q_{g_r;T,t(r)}(d\mu), \tag{4.11}$$

where $\mathbf{Q}_{g_r;T,t(r)}$ is the canonical measure of $\int_T^{T+t(r)} ds\, g_r(s-T)\xi_s$, where $(\xi_s)_{s\geq 0}$ is the BRW started as the Poisson point process with intensity ϑ. That means

$$\mathbf{E}^{\mathcal{H}(\vartheta)}\left[e^{-\langle \int_T^{T+t(r)} ds\, g_r(s-T)\xi_s, f \rangle} \right] = \exp\left(-\int_{\mathcal{N}(\mathbb{Z}^d)} (1 - e^{-\langle \mu, f \rangle}) \mathbf{Q}_{g_r;T,t(r)}(d\mu) \right). \tag{4.12}$$

In part 4 we prove the following lemma.

Lemma 4.2 *Let $Q_{g_r;T,t(r)}$ be the canonical measure of $\int_T^{T+t(r)} ds\, g_r(s-T)\xi_s$, where $g \in \mathcal{C}_c^{\infty,+}([0,\infty))$ and $g_r(s) = r^{-2}g(s/r^2)$. Then*

$$\int \langle \mu, \mathbb{I}_{B(r)} \rangle^3 Q_{g_r;T,t(r)}(d\mu) = O(r^{d+4}) \tag{4.13}$$

uniformly in T. $\diamond$

Combining (4.7), (4.10), (4.11) and (4.13) yields

$$\mathbf{E}\Big[\sum_{k=1}^{\infty} \sum_{l=1}^{N_{g_r,k}} \langle \xi^{g_r;k,l}, \varphi_r \rangle^3\Big] = O(r^{-\frac{d-2}{2}}) \underset{r\to\infty}{\longrightarrow} 0. \tag{4.14}$$

To sum up it can be said that (4.5) gives a representation of $\tilde{F}_{\vartheta,r}(\tilde{\varphi})$ as a sum of independent random variables, which fulfills the assumptions of the CLT. As stated above in part 1 the variance of the limit field has the form

$$\tilde{C}_\theta \tilde{L}(\tilde{\varphi}, \tilde{\varphi}). \tag{4.15}$$

So far we only considered test functions of the form $\tilde{\varphi}(x,s) = \varphi(x)g(s)$ with $\varphi \in \mathcal{C}_c^{\infty,+}(\mathbb{R}^d)$ and $g \in \mathcal{C}_c^{\infty,+}([0,\infty))$. In part 4 we prove the following lemma, which ensures the result for test functions of the Schwartz space.

Lemma 4.3 (for general test functions) *If*

$$\mathcal{L}[\tilde{F}_{\vartheta,r}(\tilde{\varphi})] \underset{r\to\infty}{\Longrightarrow} \mathcal{N}(0, C_\vartheta \tilde{L}(\tilde{\varphi}, \tilde{\varphi})) \tag{4.16}$$

is valid for $\tilde{\varphi}(x,s) = \varphi(x)g(s)$ with $\varphi \in \mathcal{C}_c^{\infty,+}(\mathbb{R}^d)$ and $g \in \mathcal{C}_c^{\infty,+}([0,\infty))$, then the statement is true for $\tilde{\varphi} \in \mathcal{S}(\mathbb{R}^d \times [0,\infty))$. $\diamond$

This completes the proof of Theorem 2. $\square$

Part 4 (Proofs of the lemmas)

Proof of Lemma 4.1 Let $\mathrm{supp}(\tilde{\varphi}) \subset \mathrm{B}(c) \times [0,\gamma]$, hence $\mathrm{supp}(\tilde{\varphi}_r) \subset \mathrm{B}(cr) \times [0, t(r)]$ with $t(r) = \gamma r^2$. Thus we can write

$$\mathbf{E}\left[\tilde{F}_{\vartheta,r}(\tilde{\varphi})\right]^2 = \mathbf{E}^{\Lambda_\vartheta}\left[\int_0^{t(r)} ds\, \langle \xi_s, \tilde{\varphi}_r(\,\cdot\,,s)\rangle\right]^2 - \vartheta^2 \left[\int_0^{t(r)} ds\, \langle \lambda, \tilde{\varphi}_r(\,\cdot\,,s)\rangle\right]^2. \tag{4.17}$$

Hence by the moment formulas (B.37) and (B.38) of the appendix

$$\mathbf{E}\left[\tilde{F}_{\vartheta,r}(\tilde{\varphi})\right]^2 = I_1(r) + I_2(r), \tag{4.18}$$

where

$$I_1(r) := 2\vartheta \sum_{x\in\mathbb{Z}^d} \int_0^{t(r)} ds\, \tilde{\varphi}_r(x,s) \int_s^{t(r)} du\, (a_{u-s}\tilde{\varphi}_r(\,\cdot\,,u))(x) \tag{4.19}$$

and

$$I_2(r) := V\vartheta \sum_{x\in\mathbb{Z}^d} \int_0^{t(r)} ds \left(\int_s^{t(r)} du\, (a_{u-s}\tilde{\varphi}_r(\,\cdot\,,u))(x) \right)^2$$
$$+ V\vartheta \sum_{x\in\mathbb{Z}^d} \int_0^{\infty} ds \left(\int_0^{t(r)} du\, (a_{u+s}\tilde{\varphi}_r(\,\cdot\,,u))(x) \right)^2. \qquad (4.20)$$

We consider $I_1(r)$

$$I_1(r) = 2\vartheta \sum_{x,y\in B(cr)} \int_0^{t(r)} ds\, \tilde{\varphi}_r(x,s) \int_s^{t(r)} du\, a_{u-s}(x,y)\tilde{\varphi}_r(y,u). \qquad (4.21)$$

First of all we estimate $\tilde{\varphi}_r$ by its supremum and then we estimate the sum over y by 1 to get

$$|I_1(r)| \leq 2\vartheta\|\tilde{\varphi}\|_\infty^2 \tilde{h}(r)^2 t(r)^2 |B(cr)| \leq O(r^{-(d+6)}r^4 r^d) = O(r^{-2}) \underset{r\to\infty}{\longrightarrow} 0. \qquad (4.22)$$

The term $I_2(r)$ can be rewritten as

$$I_2(r) = V\vartheta \sum_{x,y,z\in\mathbb{Z}^d} \int_0^{t(r)} ds \int_s^{t(r)} du \int_s^{t(r)} d\rho\, a_{u-s}(x,y)a_{\rho-s}(x,z)\tilde{\varphi}_r(y,u)\tilde{\varphi}_r(z,\rho)$$
$$+ V\vartheta \sum_{x,y,z\in\mathbb{Z}^d} \int_0^{\infty} ds \int_0^{t(r)} du \int_0^{t(r)} d\rho\, a_{u+s}(x,y)a_{\rho+s}(x,z)\tilde{\varphi}_r(y,u)\tilde{\varphi}_r(z,\rho). \qquad (4.23)$$

We change the order of integration to get

$$I_2(r) = V\vartheta \sum_{x,y,z\in\mathbb{Z}^d} \int_0^{t(r)} du \int_0^{t(r)} d\rho\, \tilde{\varphi}_r(y,u)\tilde{\varphi}_r(z,\rho) \int_0^{u\wedge\rho} ds\, a_{u-s}(x,y)a_{\rho-s}(x,z)$$
$$+ V\vartheta \sum_{x,y,z\in\mathbb{Z}^d} \int_0^{t(r)} du \int_0^{t(r)} d\rho\, \tilde{\varphi}_r(y,u)\tilde{\varphi}_r(z,\rho) \int_0^{\infty} ds\, a_{u+s}(x,y)a_{\rho+s}(x,z). \qquad (4.24)$$

Now we substitute $s := u+\rho-2s$ respectively $s := u+\rho+2s$. We end up with

$$I_2(r) = \frac{1}{2}V\vartheta \sum_{x,y,z\in\mathbb{Z}^d} \int_0^{t(r)} du\, \tilde{\varphi}_r(y,u) \int_0^{t(r)} d\rho\, \tilde{\varphi}_r(z,\rho) \int_{|u-\rho|}^{\infty} ds\, a_{\frac{u-\rho+s}{2}}(x,y)a_{\frac{\rho-u+s}{2}}(x,z). \qquad (4.25)$$

Let $\varepsilon > 0$. We will show at the very end (see (4.46) and the sequel) that there exists a constant $C > 0$ such that

$$\sum_{x\in\mathbb{Z}^d} \sum_{y\in B(cr)} \int_0^u d\rho\, a_{s+u-\rho}(x,y)a_s(x,z) \;\leq\; Cr^2, \quad \forall\, z\in\mathbb{Z}^d; u,s\in[0,\infty). \qquad (4.26)$$

We continue the analysis of $I_2(r)$. Let

$$\varepsilon' = \frac{2\varepsilon}{V\vartheta\|\tilde{\varphi}\|_\infty^2 C\gamma(2c)^d}. \tag{4.27}$$

We decompose $I_2(r)$ depending on whether $s \le |u-\rho| + \varepsilon' r^2$ or $s > |u-\rho| + \varepsilon' r^2$

$$I_2(r) = I_{2,1}(r) + I_{2,2}(r), \tag{4.28}$$

where

$$\begin{aligned} I_{2,1}(r) \quad &:= \quad \frac{1}{2}V\vartheta \sum_{x,y,z\in\mathbb{Z}^d} \int_0^{t(r)} du\, \tilde{\varphi}_r(y,u) \int_0^{t(r)} d\rho\, \tilde{\varphi}_r(z,\rho) \\ &\quad \times \int_{|u-\rho|}^{|u-\rho|+\varepsilon' r^2} ds\, a_{\frac{u-\rho+s}{2}}(x,y) a_{\frac{\rho-u+s}{2}}(x,z) \end{aligned} \tag{4.29}$$

and

$$\begin{aligned} I_{2,2}(r) \quad &:= \quad \frac{1}{2}V\vartheta \sum_{x,y,z\in\mathbb{Z}^d} \int_0^{t(r)} du\, \tilde{\varphi}_r(y,u) \int_0^{t(r)} d\rho\, \tilde{\varphi}_r(z,\rho) \\ &\quad \times \int_{|u-\rho|+\varepsilon' r^2}^{\infty} ds\, a_{\frac{u-\rho+s}{2}}(x,y) a_{\frac{\rho-u+s}{2}}(x,z). \end{aligned} \tag{4.30}$$

Consider first $I_{2,1}(r)$. By symmetry in u and ρ it suffices to consider

$$\begin{aligned} I'_{2,1}(r) \quad &:= \quad \frac{1}{2}V\vartheta \sum_{x,y,z\in\mathbb{Z}^d} \int_0^{t(r)} du\, \tilde{\varphi}_r(y,u) \int_0^{u} d\rho\, \tilde{\varphi}_r(z,\rho) \\ &\quad \times \int_{|u-\rho|}^{|u-\rho|+\varepsilon' r^2} ds\, a_{\frac{u-\rho+s}{2}}(x,y) a_{\frac{\rho-u+s}{2}}(x,z), \end{aligned} \tag{4.31}$$

since the remaining term (involving $\int_u^{t(r)} d\rho \ldots$) can be treated in the same way. Recall that $\mathrm{supp}(\tilde{\varphi}) \subset \mathrm{B}(c) \times [0,\gamma]$. In $I'_{2,1}(r)$ we estimate $\tilde{\varphi}_r$ by its supremum. We substitute $s := (s-u+\rho)/2$, afterwards we change the order of integration, hence

$$I'_{2,1}(r) \le V\vartheta r^{-(d+6)}\|\tilde{\varphi}\|_\infty^2 \sum_{x\in\mathbb{Z}^d} \sum_{y,z\in B(cr)} \int_0^{t(r)} du \int_0^{\varepsilon' r^2/2} ds \int_0^u d\rho\, a_{s+u-\rho}(x,y) a_s(x,z). \tag{4.32}$$

Now we apply (4.26) to get (using $|B(cr)| \le (2cr)^d$)

$$I'_{2,1}(r) \le \frac{1}{2}V\vartheta\|\tilde{\varphi}\|_\infty^2 C\gamma(2c)^d \varepsilon' = \varepsilon. \tag{4.33}$$

Return to (4.28). We have to treat now $I_{2,2}(r)$. We need the local CLT given in Proposition D.2 of the appendix, which says

$$a_t(0,x) - b_t(0,x) = t^{-\frac{d}{2}} E(t,x), \tag{4.34}$$

where $E(t,x)\to 0$ as $t\to\infty$ uniformly in x and

$$b_t(0,x)=\frac{1}{(2\pi t)^{d/2}|Q|^{1/2}}e^{-\frac{\bar{Q}(x)}{2t}}. \tag{4.35}$$

We use the decomposition given in (4.34) for $I_{2,2}(r)$, hence

$$I_{2,2}(r)=I_{2,2,1}(r)+I_{2,2,2}(r)+I_{2,2,3}(r), \tag{4.36}$$

where

$$\begin{aligned} I_{2,2,1}(r) &:= \frac{1}{2}V\vartheta\sum_{x,y,z\in\mathbb{Z}^d}\int_0^{t(r)}du\,\tilde{\varphi}_r(y,u)\int_0^{t(r)}d\rho\,\tilde{\varphi}_r(z,\rho)\\ &\qquad\times\int_{|u-\rho|+\varepsilon' r^2}^{\infty}ds\,b_{\frac{u-\rho+s}{2}}(x,y)b_{\frac{\rho-u+s}{2}}(x,z) \end{aligned}\tag{4.37}$$

and

$$\begin{aligned} I_{2,2,2}(r) &:= \frac{1}{2}V\vartheta\sum_{x,y,z\in\mathbb{Z}^d}\int_0^{t(r)}du\,\tilde{\varphi}_r(y,u)\int_0^{t(r)}d\rho\,\tilde{\varphi}_r(z,\rho)\\ &\qquad\times\int_{|u-\rho|+\varepsilon' r^2}^{\infty}ds\,b_{\frac{u-\rho+s}{2}}(x,y)\left(\frac{\rho-u+s}{2}\right)^{-\frac{d}{2}}E\left(\frac{\rho-u+s}{2},z-x\right) \end{aligned}\tag{4.38}$$

and

$$\begin{aligned} I_{2,2,3}(r) &:= \frac{1}{2}V\vartheta\sum_{x,y,z\in\mathbb{Z}^d}\int_0^{t(r)}du\,\tilde{\varphi}_r(y,u)\int_0^{t(r)}d\rho\,\tilde{\varphi}_r(z,\rho)\\ &\qquad\times\int_{|u-\rho|+\varepsilon' r^2}^{\infty}ds\left(\frac{u-\rho+s}{2}\right)^{-\frac{d}{2}}E\left(\frac{u-\rho+s}{2},y-x\right)a_{\frac{\rho-u+s}{2}}(x,z). \end{aligned}\tag{4.39}$$

In $I_{2,2,1}(r)$ we perform the sum over x and then we substitute $u:=u/r^2$, $\rho:=\rho/r^2$ and then $s:=s/r^2$, hence

$$\begin{aligned} I_{2,2,1}(r) &= \frac{1}{2}V\vartheta\sum_{y,z\in\mathbb{Z}^d}r^{-2d}\int_0^{\gamma}du\,\tilde{\varphi}\left(\frac{y}{r},u\right)\int_0^{\gamma}d\rho\,\tilde{\varphi}\left(\frac{z}{r},\rho\right)\\ &\qquad\times\int_{|u-\rho|+\varepsilon'}^{\infty}ds\,\frac{1}{(2\pi s)^{d/2}|Q|^{1/2}}\exp\left(-\frac{1}{2s}\bar{Q}\left(\frac{z}{r}-\frac{y}{r}\right)\right)\\ &\underset{r\to\infty}{\longrightarrow} \frac{1}{2}V\vartheta\int_{\mathbb{R}^d}dy\int_{\mathbb{R}^d}dz\int_0^{\gamma}du\int_0^{\gamma}d\rho\,\tilde{\varphi}(y,u)\tilde{\varphi}(z,\rho)\int_{|u-\rho|+\varepsilon'}^{\infty}ds\,b_s(y,z). \end{aligned}\tag{4.40}$$

Return to (4.36) and consider $I_{2,2,2}(r)$. Let

$$\varepsilon''=\frac{d-2}{2V\vartheta\|\tilde{\varphi}\|_\infty^2(2c)^{2d}\gamma^2}\left(\frac{\varepsilon'}{2}\right)^{\frac{d-2}{2}}\varepsilon. \tag{4.41}$$

There exists a t_0 such that $|E(t,x)| \leq \varepsilon''$ for all $t \geq t_0$ and for all x. Choose $r_0 = \sqrt{2t_0/\varepsilon'}$. In $I_{2,2,2}(r)$ (with $r > r_0$) we can estimate $|E(\frac{\rho-u+s}{2}, z-x)| < \varepsilon''$. Then we perform the sum over x. Furthermore we estimate $\tilde{\varphi}_r$ by its supremum. Afterwards we substitute $u := u/r^2$, $\rho := \rho/r^2$ and then $s := s/r^2$, hence

$$|I_{2,2,2}(r)| \leq \frac{1}{2} V \vartheta \varepsilon'' \|\tilde{\varphi}\|_\infty^2 \sum_{y,z \in B(cr)} r^{-2d} \int_0^\gamma du \int_0^\gamma d\rho \int_{|u-\rho|+\varepsilon'}^\infty ds \left(\frac{\rho-u+s}{2}\right)^{-d/2}. \tag{4.42}$$

Since

$$\int_{|u-\rho|+\varepsilon'}^\infty ds \left(\frac{\rho-u+s}{2}\right)^{-\frac{d}{2}} = \frac{4}{d-2} \left(\frac{\rho-u+|u-\rho|+\varepsilon'}{2}\right)^{-\frac{d-2}{2}} \leq \frac{4}{d-2} \left(\frac{\varepsilon'}{2}\right)^{-\frac{d-2}{2}} \tag{4.43}$$

we obtain

$$|I_{2,2,2}(r)| \leq \frac{2V\vartheta \|\tilde{\varphi}\|_\infty^2 (2c)^{2d} \gamma^2}{d-2} \left(\frac{\varepsilon'}{2}\right)^{-\frac{d-2}{2}} \varepsilon'' = \varepsilon. \tag{4.44}$$

The term $I_{2,2,3}(r)$ can be treated in exactly the same way as $I_{2,2,2}(r)$.
Now let $\varepsilon \to 0$. Note that then also $\varepsilon' \to 0$. By (4.33), (4.40) and (4.44) we obtain

$$I_2(r) \underset{r\to\infty}{\longrightarrow} \frac{1}{2} V\vartheta \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz \int_0^\gamma du \int_0^\gamma d\rho \, \tilde{\varphi}(y,u)\, \tilde{\varphi}(z,\rho) \int_{|u-\rho|}^\infty ds \, b_s(y,z). \tag{4.45}$$

Combining this with (4.22) completes the proof of (4.2).

It remains to show (4.26). First of all we estimate the integral over ρ by the corresponding Green's function to get

$$\sum_{x\in\mathbb{Z}^d} \sum_{y \in B(cr)} \int_0^u d\rho \, a_{s+u-\rho}(x,y) a_s(x,z) \leq \sum_{x\in\mathbb{Z}^d} \sum_{y\in B(cr)} G(y-x) a_s(x,z). \tag{4.46}$$

In the next proof (4.59) we will show the following basic estimate

$$\sum_{x\in\mathbb{Z}^d} \sum_{i \in B(r)} G(i-x) f(x) \leq O(r^2) \sum_{x\in\mathbb{Z}^d} f(x). \tag{4.47}$$

Using this we get

$$\sum_{x\in\mathbb{Z}^d} \sum_{y\in B(cr)} \int_0^u d\rho \, a_{s+u-\rho}(x,y) a_s(x,z) \leq O(r^2) \sum_{x\in\mathbb{Z}^d} a_s(x,z) = O(r^2). \tag{4.48}$$

We proved (4.26), which completes the proof of the variance convergence statement of Lemma 4.1. □

Proof of Lemma 4.2 We proceed in four steps.

Step 0 The third moment can be written in terms of the Laplace functional as

$$\int \langle \mu, f\rangle^3 Q_{g_r;T,t(r)}(d\mu) = \frac{\partial^3}{\partial \alpha^3} \int (1 - e^{-\langle \mu, \alpha f\rangle}) Q_{g_r;T,t(r)}(d\mu) \Big|_{\alpha=0}. \tag{4.49}$$

By Lemma A.2 one can easily check that the Laplace functional of the weighted occupation time of the BRW ξ started at time u with distribution $\mathcal{H}(\vartheta)$ has the form

$$\mathbf{E}^{u,\mathcal{H}(\vartheta)}[e^{-\langle\int_T^{T+t(r)} ds\, g_r(s-T)\xi_s, f\rangle}] = \exp\left(-\langle\vartheta\lambda, 1-h(u,T,t(r),\tilde{\varphi};\,\cdot\,)\rangle\right), \tag{4.50}$$

where h is given in (A.8) and $\tilde{\varphi}(x,s) = f(x)g_r(s-T)$.

By (4.12) and (4.50) we get

$$\int \langle\mu, f\rangle^3 Q_{g_r;T,t(r)}(d\mu) = \langle\vartheta\lambda, -\frac{\partial^3}{\partial\alpha^3}h(0,T,t(r),\alpha\tilde{\varphi};\,\cdot\,)\Big|_{\alpha=0}\rangle. \tag{4.51}$$

Using (A.9) we obtain

$$\begin{aligned}
&-\frac{\partial^3}{\partial\alpha^3}h(0,T,t(r),\alpha\tilde{\varphi};x)\Big|_{\alpha=0}\\
&= a_T\left(-\frac{\partial^3}{\partial\alpha^3}h(T,T,t(r),\alpha\tilde{\varphi};\,\cdot\,)\Big|_{\alpha=0}\right)(x)\\
&\quad + 3V\int_0^T ds\, a_s\Big(-\frac{\partial}{\partial\alpha}h(s,T,t(r),\alpha\tilde{\varphi};\,\cdot\,)\Big|_{\alpha=0}\frac{\partial^2}{\partial\alpha^2}h(s,T,t(r),\alpha\tilde{\varphi};\,\cdot\,)\Big|_{\alpha=0}\Big)(x).
\end{aligned} \tag{4.52}$$

The Laplace functional expands to the k-th moment as follows

$$(-1)^k\frac{\partial^k}{\partial\alpha^k}h(s,T,t(r),\alpha\tilde{\varphi};x)\Big|_{\alpha=0} = \mathbf{E}^{s,\delta_x}\left[\left(\int_T^{T+t(r)} du\,\langle\xi_u, g_r(u-T)f\rangle\right)^k\right]. \tag{4.53}$$

We are interested in the special test function $f = \mathbb{1}_{B(r)}$. Furthermore we estimate g by its supremum. Hence by (4.51)

$$\int \langle\mu, \mathbb{1}_{B(r)}\rangle^3 Q_{g_r;T,t(r)}(d\mu) \le r^{-6}\|g\|_\infty^3(I_1(r)+I_2(r)), \tag{4.54}$$

where

$$I_1(r) := \vartheta\sum_{x\in\mathbb{Z}^d}\mathbf{E}^{\delta_x}\left[\int_0^{t(r)} ds\,\langle\xi_s, \mathbb{1}_{B(r)}\rangle\right]^3 \tag{4.55}$$

and

$$I_2(r) := 3V\vartheta\sum_{x\in\mathbb{Z}^d}\int_0^T ds\,\mathbf{E}^{\delta_x}\left[\int_{T-s}^{T+t(r)-s} du\,\langle\xi_u, \mathbb{1}_{B(r)}\rangle\right]\mathbf{E}^{\delta_x}\left[\int_{T-s}^{T+t(r)-s} du\,\langle\xi_u, \mathbb{1}_{B(r)}\rangle\right]^2. \tag{4.56}$$

Step 1 First of all we want to establish the basic estimates we will use repeatedly.

Estimate 1

$$\sum_{i\in B(r)}\hat{G}(i) = O(r^2) \tag{4.57}$$

Estimate 2 for sums in (v,h,s)

$$\sum_{v\in\mathbb{Z}^d}\sum_{h\in B(r)}\int_0^t ds\, a_{u+s}(v,h)a_{\rho+s}(v,z) \;\le\; O(r^2), \tag{4.58}$$

uniformly in z, t, u and ρ.
Estimate 3 for sums in (x,i)

$$\sum_{x\in\mathbb{Z}^d}\sum_{i\in B(r)} G(i-x)f(x) \le O(r^2)\sum_{x\in\mathbb{Z}^d} f(x), \tag{4.59}$$

where f is a function, which will be specified in the particular situation.
In the remaining part of step 1 we prove these estimates. The first estimate was established in Section 3, recall (3.54). The proof of the second estimate works the same as the proof of (3.55). It remains to prove estimate 3. By Lemma C.1 we find a constant $C>0$ such that for all $y\in\mathbb{Z}^d$,

$$G(y)\le \frac{C}{(|y|+1)^{d-2}}, \tag{4.60}$$

since we assumed mean zero of a.
We decompose

$$\sum_{x\in\mathbb{Z}^d}\sum_{i\in B(r)} G(i-x)f(x) = \sum_{x\notin B(2r)}\sum_{i\in B(r)} G(i-x)f(x) + \sum_{x\in B(2r)}\sum_{i\in B(r)} G(i-x)f(x). \tag{4.61}$$

For the first term on the r.h.s. of (4.61) we can exploit that $|i-x|\ge r$, hence by (4.60)

$$\sum_{x\notin B(2r)}\sum_{i\in B(r)} G(i-x)f(x) \le O(r^2)\sum_{x\notin B(2r)} f(x). \tag{4.62}$$

The second term on the r.h.s. of (4.61) we estimate as follows

$$\sum_{x\in B(2r)}\sum_{i\in B(r)} G(i-x)f(x) \le \sum_{x\in B(2r)}\sum_{j\in B(3r)} G(j)f(x) \le O(r^2)\sum_{x\in B(2r)} f(x) \tag{4.63}$$

by the analogue of (4.57) for G. This completes the proof of (4.59).

Step 2 By (B.12)

$$I_1(r) = I_{1,1}(r)+I_{1,2}(r)+I_{1,3}(r)+I_{1,4}(r), \tag{4.64}$$

where

$$\begin{aligned} I_{1,1}(r) \;&:=\; 3V^2\vartheta\sum_{x\in\mathbb{Z}^d}\int_0^{t(r)} ds\int_s^{t(r)} du\,(a_{u-s}\mathbb{1}_{B(r)})(x) \\ &\qquad\times\int_s^{t(r)} d\rho\, a_{\rho-s}\left(\left(\int_\rho^{t(r)} d\tau\,(a_{\tau-\rho}\mathbb{1}_{B(r)})\right)^2\right)(x) \end{aligned} \tag{4.65}$$

and

$$\begin{aligned} I_{1,2}(r) \quad &:= \quad 6V\vartheta \sum_{x\in\mathbb{Z}^d} \int_0^{t(r)} ds \int_s^{t(r)} du\, (a_{u-s}\mathbb{1}_{B(r)})(x) \\ &\qquad \times \int_s^{t(r)} d\rho\, a_{\rho-s} \left(\mathbb{1}_{B(r)} \int_\rho^{t(r)} d\tau\, (a_{\tau-\rho}\mathbb{1}_{B(r)}) \right)(x) \end{aligned} \tag{4.66}$$

and

$$I_{1,3}(r) := 3V\vartheta \sum_{i\in B(r)} \int_0^{t(r)} ds \int_s^{t(r)} d\rho\, a_{\rho-s} \left(\left(\int_\rho^{t(r)} d\tau\, (a_{\tau-\rho}\mathbb{1}_{B(r)}) \right)^2 \right)(i) \tag{4.67}$$

and

$$I_{1,4}(r) := 6\vartheta \sum_{i\in B(r)} \int_0^{t(r)} ds \int_s^{t(r)} d\rho\, a_{\rho-s} \left(\mathbb{1}_{B(r)} \int_\rho^{t(r)} d\tau\, (a_{\tau-\rho}\mathbb{1}_{B(r)}) \right)(i). \tag{4.68}$$

We treat the four terms separately. The term $I_{1,1}(r)$ can be written as

$$\begin{aligned} I_{1,1}(r) \quad &= \quad 3V^2\vartheta \sum_{x,y\in\mathbb{Z}^d} \sum_{i,j,g\in B(r)} \int_0^{t(r)} ds \int_s^{t(r)} du \int_s^{t(r)} d\rho \int_\rho^{t(r)} d\tau \int_\rho^{t(r)} d\pi \\ &\qquad \times a_{u-s}(x,i) a_{\rho-s}(x,y) a_{\tau-\rho}(y,j) a_{\pi-\rho}(y,g). \end{aligned} \tag{4.69}$$

We estimate the integral over u by the corresponding Green's function in order to apply the basic estimate given in (4.59) for the sums in (x,i) to get

$$\begin{aligned} I_{1,1}(r) \quad &\leq \quad O(r^2) \sum_{x,y\in\mathbb{Z}^d} \sum_{j,g\in B(r)} \int_0^{t(r)} ds \int_s^{t(r)} d\rho \int_\rho^{t(r)} d\tau \int_\rho^{t(r)} d\pi \\ &\qquad \times a_{\rho-s}(x,y) a_{\tau-\rho}(y,j) a_{\pi-\rho}(y,g). \end{aligned} \tag{4.70}$$

Now we perform the sum over x and we estimate the integral over π by the corresponding Green's function. Afterwards we use again (4.59) but now for the sums in (y,g), hence

$$I_{1,1}(r) \leq O(r^4) \sum_{y\in\mathbb{Z}^d} \sum_{j\in B(r)} \int_0^{t(r)} ds \int_s^{t(r)} d\rho \int_\rho^{t(r)} d\tau\, a_{\tau-\rho}(y,j). \tag{4.71}$$

Finally we perform the sum over y. We end up with

$$I_{1,1}(r) \leq O(r^4) t(r)^3 |B(r)| \leq O(r^{d+10}). \tag{4.72}$$

Now we consider $I_{1,2}(r)$.

$$\begin{aligned} I_{1,2}(r) \quad &= \quad 6V\vartheta \sum_{x\in\mathbb{Z}^d} \sum_{i,j,g\in B(r)} \int_0^{t(r)} ds \int_s^{t(r)} du \int_s^{t(r)} d\rho \int_\rho^{t(r)} d\tau \\ &\qquad \times a_{u-s}(x,i) a_{\rho-s}(x,j) a_{\tau-\rho}(j,g). \end{aligned} \tag{4.73}$$

We estimate the integral over u by the corresponding Green's function, then we are in the position to apply (4.59) for the sums in (x,i). Afterwards we perform the sum over x and we estimate the integral over τ by the Green's function, hence

$$I_{1,2}(r) \le O(r^2)t(r)^2 \sum_{j,g\in B(r)} G(g-j) \le O(r^{d+8}) \tag{4.74}$$

by (4.57).

The term $I_{1,3}(r)$ we write as follows

$$\begin{aligned} I_{1,3}(r) \quad &= \quad 3V\vartheta \sum_{x\in\mathbb{Z}^d}\sum_{i,j,g\in B(r)} \int_0^{t(r)} ds \int_s^{t(r)} d\rho \int_\rho^{t(r)} d\tau \int_\rho^{t(r)} d\pi \\ &\quad \times a_{\rho-s}(i,x)a_{\tau-\rho}(x,j)a_{\pi-\rho}(x,g). \end{aligned} \tag{4.75}$$

After estimating the integral over π by the Green's function we can apply the basic estimate given in (4.59) for the sums in (x,g), thus

$$I_{1,3}(r) \le O(r^2) \sum_{x\in\mathbb{Z}^d}\sum_{i,j\in B(r)} \int_0^{t(r)} ds \int_s^{t(r)} d\rho \int_\rho^{t(r)} d\tau \, a_{\rho-s}(i,x)a_{\tau-\rho}(x,j). \tag{4.76}$$

Estimating the integral over τ by the Green's function and applying (4.59) for the sums in (x,j) yield

$$I_{1,3}(r) \le O(r^4) \sum_{x\in\mathbb{Z}^d}\sum_{i\in B(r)} \int_0^{t(r)} ds \int_s^{t(r)} d\rho \, a_{\rho-s}(i,x). \tag{4.77}$$

We perform the sum over x, hence

$$I_{1,3}(r) \le O(r^4)t(r)^2|B(r)| \le O(r^{d+8}). \tag{4.78}$$

Finally $I_{1,4}(r)$ can be written as

$$I_{1,4}(r) = 6\vartheta \sum_{i,j,g\in B(r)} \int_0^{t(r)} ds \int_s^{t(r)} d\rho \int_\rho^{t(r)} d\tau \, a_{\rho-s}(i,j)a_{\tau-\rho}(j,g). \tag{4.79}$$

We estimate the integrals over τ and over ρ by the corresponding Green's functions, hence

$$I_{1,4}(r) \le O(1)t(r) \sum_{i,j,g\in B(r)} G(j-i)G(g-j) \le O(r^{d+6}) \tag{4.80}$$

by (4.57).

Combining (4.72), (4.74), (4.78) and (4.80) leads to

$$I_1(r) \le O(r^{d+10}). \tag{4.81}$$

That means we proved the right order of $I_1(r)$.

Step 3 Return to (4.54). We consider the term $I_2(r)$ given in (4.56). By (B.25)

$$\mathbf{E}^{\delta_x}\left[\int_{T-s}^{T+t(r)-s} du\,\langle\xi_u,\mathbb{1}_{B(r)}\rangle\right]=\int_{T-s}^{T+t(r)-s} du\,(a_u\mathbb{1}_{B(r)})(x) \tag{4.82}$$

and by (B.26)

$$\begin{aligned}
&\mathbf{E}^{\delta_x}\left[\int_{T-s}^{T+t(r)-s} du\,\langle\xi_u,\mathbb{1}_{B(r)}\rangle\right]^2\\
&= V\int_0^{T-s} d\rho\,a_\rho\left(\left(\int_{T-s}^{T+t(r)-s} d\tau\,(a_{\tau-\rho}\mathbb{1}_{B(r)})\right)^2\right)(x)\\
&\quad+V\int_{T-s}^{T+t(r)-s} d\rho\,a_\rho\left(\left(\int_{\rho}^{T+t(r)-s} d\tau\,(a_{\tau-\rho}\mathbb{1}_{B(r)})\right)^2\right)(x)\\
&\quad+2\int_{T-s}^{T+t(r)-s} d\rho\,a_\rho\left(\mathbb{1}_{B(r)}\int_{\rho}^{T+t(r)-s} d\tau\,(a_{\tau-\rho}\mathbb{1}_{B(r)})\right)(x).
\end{aligned} \tag{4.83}$$

Substituting these two equations into (4.56) we get

$$I_2(r)=I_{2,1}(r)+I_{2,2}(r)+I_{2,3}(r), \tag{4.84}$$

where

$$\begin{aligned}
I_{2,1}(r) &:= 3V^2\vartheta\sum_{x\in\mathbb{Z}^d}\int_0^T ds\int_{T-s}^{T+t(r)-s} du\,(a_u\mathbb{1}_{B(r)})(x)\\
&\qquad\times\int_0^{T-s} d\rho\,a_\rho\left(\left(\int_{T-s}^{T+t(r)-s} d\tau\,(a_{\tau-\rho}\mathbb{1}_{B(r)})\right)^2\right)(x)
\end{aligned} \tag{4.85}$$

and

$$\begin{aligned}
I_{2,2}(r) &:= 3V^2\vartheta\sum_{x\in\mathbb{Z}^d}\int_0^T ds\int_{T-s}^{T+t(r)-s} du\,(a_u\mathbb{1}_{B(r)})(x)\\
&\qquad\times\int_{T-s}^{T+t(r)-s} d\rho\,a_\rho\left(\left(\int_{\rho}^{T+t(r)-s} d\tau\,(a_{\tau-\rho}\mathbb{1}_{B(r)})\right)^2\right)(x)
\end{aligned} \tag{4.86}$$

and

$$\begin{aligned}
I_{2,3}(r) &:= 6V\vartheta\sum_{x\in\mathbb{Z}^d}\int_0^T ds\int_{T-s}^{T+t(r)-s} du\,(a_u\mathbb{1}_{B(r)})(x)\\
&\qquad\times\int_{T-s}^{T+t(r)-s} d\rho\,a_\rho\left(\mathbb{1}_{B(r)}\int_{\rho}^{T+t(r)-s} d\tau\,(a_{\tau-\rho}\mathbb{1}_{B(r)})\right)(x).
\end{aligned} \tag{4.87}$$

We treat the three terms separately. The term $I_{2,1}(r)$ can be written as

$$\begin{aligned} I_{2,1}(r) &= 3V^2\vartheta \sum_{x,y\in\mathbb{Z}^d}\sum_{i,j,g\in B(r)} \int_0^T ds \int_{T-s}^{T+t(r)-s} du \int_0^{T-s} d\rho \\ &\quad\times \int_{T-s}^{T+t(r)-s} d\tau \int_{T-s}^{T+t(r)-s} d\pi\, a_u(x,i)a_\rho(x,y)a_{\tau-\rho}(y,j)a_{\pi-\rho}(y,g). \end{aligned} \tag{4.88}$$

We first substitute $s := T-s$, then $u := u-s$, $\rho := s-\rho$, $\tau := \tau - s$ and $\pi := \pi - s$ to get

$$\begin{aligned} I_{2,1}(r) &= 3V^2\vartheta \sum_{x,y\in\mathbb{Z}^d}\sum_{i,j,g\in B(r)} \int_0^T ds \int_0^{t(r)} du \int_0^s d\rho \int_0^{t(r)} d\tau \int_0^{t(r)} d\pi \\ &\quad\times a_{u+s}(x,i)a_{s-\rho}(x,y)a_{\tau+\rho}(y,j)a_{\pi+\rho}(y,g). \end{aligned} \tag{4.89}$$

We exchange the integrals over s and over ρ in order to apply the basic estimate given in (4.58) for the sums in (x,i,s). Hence

$$I_{2,1}(r) \le O(r^2) \sum_{y\in\mathbb{Z}^d}\sum_{j,g\in B(r)} \int_0^T d\rho \int_0^{t(r)} du \int_0^{t(r)} d\tau \int_0^{t(r)} d\pi\, a_{\tau+\rho}(y,j)a_{\pi+\rho}(y,g). \tag{4.90}$$

Now we use the basic estimate given in (4.58) for the sums in (y,g,ρ). We end up with

$$I_{2,1}(r) \le O(r^4)t(r)^3|B(r)| \le O(r^{d+10}). \tag{4.91}$$

Return to (4.84). In $I_{2,2}(r)$ we substitute $s := T-s$, then $u := u-s$ and $\rho := \rho - s$ and then $\tau := \tau - s$ and $\pi := \pi - s$. Afterwards we apply the basic estimate given in (4.58) for the sums in (x,i,s), thus

$$I_{2,2}(r) \le O(r^2) \sum_{y\in\mathbb{Z}^d}\sum_{j,g\in B(r)} \int_0^{t(r)} du \int_0^{t(r)} d\rho \int_\rho^{t(r)} d\tau \int_\rho^{t(r)} d\pi\, a_{\tau-\rho}(y,j)a_{\pi-\rho}(y,g). \tag{4.92}$$

We estimate the integral over π by the corresponding Green's function in order to apply (4.59) for the sums in (y,g), hence

$$I_{2,2}(r) \le O(r^4) \sum_{y\in\mathbb{Z}^d}\sum_{j\in B(r)} \int_0^{t(r)} du \int_0^{t(r)} d\rho \int_\rho^{t(r)} d\tau\, a_{\tau-\rho}(y,j). \tag{4.93}$$

Now we perform the sum over y. We end up with

$$I_{2,2}(r) \le O(r^4)t(r)^3|B(r)| \le O(r^{d+10}). \tag{4.94}$$

The term $I_{2,3}(r)$ can be written as

$$\begin{aligned} I_{2,3}(r) &= 6V\vartheta \sum_{x\in\mathbb{Z}^d}\sum_{i,j,g\in B(r)} \int_0^T ds \int_{T-s}^{T+t(r)-s} du \int_{T-s}^{T+t(r)-s} d\rho \int_\rho^{T+t(r)-s} d\tau \\ &\quad\times a_u(x,i)a_\rho(x,j)a_{\tau-\rho}(j,g). \end{aligned} \tag{4.95}$$

We substitute $s := T - s$, then $u := u - s$ and $\rho := \rho - s$ and then $\tau := \tau - s$ to get

$$\begin{aligned} I_{2,3}(r) &= 6V\vartheta \sum_{x\in\mathbb{Z}^d}\sum_{i,j,g\in B(r)} \int_0^T ds \int_0^{t(r)} du \int_0^{t(r)} d\rho \int_\rho^{t(r)} d\tau \\ & \quad \times a_{u+s}(x,i) a_{\rho+s}(x,j) a_{\tau-\rho}(j,g). \end{aligned} \tag{4.96}$$

We apply (4.58) for the sums in (x, i, s), afterwards we can estimate the integral over τ by the corresponding Green's function, thus

$$I_{2,3}(r) \le O(r^2) t(r)^2 \sum_{j,g\in B(r)} G(g-j) \le O(r^{d+8}) \tag{4.97}$$

by (4.57).

Combining (4.91), (4.94) and (4.97) gives us

$$I_2(r) \le O(r^{d+10}). \tag{4.98}$$

By (4.54), (4.81) and (4.98) we get

$$\int \langle \mu, \mathbb{1}_{B(r)}\rangle^3 Q_{g_r;T,t(r)}(d\mu) = O(r^{d+4}). \tag{4.99}$$

This completes the proof. □

Proof of Lemma 4.3 We get the result by first extending the result to sums of $\phi \cdot g$ in two steps and then by approximation of general functions by sums of product type functions.

At first we prove the result for test functions of the form $\tilde{\varphi}(x,s) = \sum_{n=1}^m \varphi_n(x) g_n(s)$ with $\varphi_n \in \mathcal{C}_c^{\infty,+}(\mathbb{R}^d)$ and $g_n \in \mathcal{C}_c^{\infty,+}([0,\infty))$. By (4.2) we get

$$\mathbf{Var}[\tilde{F}_{\vartheta,r}(\tilde{\varphi})] \underset{r\to\infty}{\longrightarrow} \tilde{C}_\vartheta \tilde{L}(\tilde{\varphi},\tilde{\varphi}). \tag{4.100}$$

It remains to establish the decomposition in a sum of independent random variables and to check the Lyapunov condition. As in (4.4) we obtain (with $g_{n;r}(s) = r^{-2} g_n(s/r^2)$)

$$\int_0^\infty ds\, g_{n;r}(s) \xi_s^{(\infty)} \overset{d}{=} \sum_{k=1}^\infty \sum_{l=1}^{N_{g_{n;r},k}} \xi^{g_{n;r};k,l}, \tag{4.101}$$

where $\{N_{g_{n;r},k}; k \in \mathbb{Z}\}$, $\{\xi^{g_{n;r};k,l} : k,l \in \mathbb{Z}\}$ are independent. Recalling the construction of the family decomposition in Section 2.3 we see that one can choose these random variables to be independent for different k if n varies. Hence (with $\varphi_{n;r}(x) = h(r)\varphi_n(x/r)$)

$$\tilde{F}_{\vartheta,r}(\tilde{\varphi}) \overset{d}{=} \sum_{k=1}^\infty \left(\sum_{n=1}^m \sum_{l=1}^{N_{g_{n;r},k}} \langle \xi^{g_{n;r};k,l}, \varphi_{n;r}\rangle - \mathbf{E}\Big[\sum_{n=1}^m \sum_{l=1}^{N_{g_{n;r},k}} \langle \xi^{g_{n;r};k,l}, \varphi_{n;r}\rangle \Big] \right), \tag{4.102}$$

which is a sum of independent random variables. The Lyapunov condition is fulfilled, since we can estimate

$$\sum_{k=1}^{\infty} \mathbf{E}\left[\left(\sum_{n=1}^{m}\sum_{l=1}^{N_{g_{n;r},k}} \langle \xi^{g_{n;r};k,l}, \varphi_{n;r}\rangle\right)^3\right] \leq 2^{3m} \sum_{n=1}^{m}\sum_{k=1}^{\infty} \mathbf{E}\left[\left(\sum_{l=1}^{N_{g_{n;r},k}} \langle \xi^{g_{n;r};k,l}, \varphi_{n;r}\rangle\right)^3\right]. \qquad (4.103)$$

Now we can argue as in the proof of the CLT in Section 3 and we can apply (4.14) separately for $n = 1, \dots, m$.

Now let $\tilde{\varphi}(x,s) = \sum_{n=1}^{m} \alpha_n \varphi_n(x) g_n(s)$ with $\varphi_n \in \mathcal{C}_c^{\infty,+}(\mathbb{R}^d)$ and $g_n \in \mathcal{C}_c^{\infty,+}([0,\infty))$ and $\alpha_n \in \{-1,1\}$. That means we can decompose $\tilde{\varphi} = \tilde{\varphi}^+ - \tilde{\varphi}^-$, where $\tilde{\varphi}^+$ and $\tilde{\varphi}^-$ are both of the above type. The decomposition (4.102) is still valid with the additional term α_n. The Lyapunov condition is fulfilled. As in (3.74) we obtain the desired variance formula.

Now let $\tilde{\varphi} \in \mathcal{S}(\mathbb{R}^d \times [0,\infty))$. For $\varepsilon > 0$ there exists $\tilde{\varphi}_\varepsilon = \sum_{n=1}^{m} \alpha_n \varphi_n(x) g_n(s)$ with $\varphi_n \in \mathcal{C}_c^{\infty,+}(\mathbb{R}^d)$ and $g_n \in \mathcal{C}_c^{\infty,+}([0,\infty))$ and $\alpha_n \in \{-1,1\}$ (which all depend on ε) such that

$$\tilde{L}(|\tilde{\varphi} - \tilde{\varphi}_\varepsilon|, |\tilde{\varphi} - \tilde{\varphi}_\varepsilon|) < \varepsilon. \qquad (4.104)$$

As in the proof of Lemma 3.3 we can conclude

$$\mathcal{L}[\tilde{F}_{\vartheta,r}(\tilde{\varphi})] \underset{r\to\infty}{\Longrightarrow} \mathcal{N}(0, \tilde{C}_\vartheta \tilde{L}(\tilde{\varphi}, \tilde{\varphi})). \qquad (4.105)$$

This completes the proof. □

We finished the proofs of the lemmas used in part 2 and part 3.

5 Proof of the results on the historical process

In this section we prove Proposition 2.3 on the representation of the canonical Palm distribution of the historical process and we prove Theorem 3 on the properties of the equilibrium historical process.

Proof of Proposition 2.3 We proceed as in the proof of [GW91] Theorem 2.3.

Let $\tilde{f}, \tilde{g} \in \mathcal{C}^+(D(\mathbb{Z}^d)) \cap b\mathcal{D}([s_0,\infty))$ for some $s_0 > -\infty$. We have to show that

$$\int_{D^t(\mathbb{Z}^d)} \tilde{g}(\tilde{y}) \mathbf{E}\left[\exp\left(-\langle \tilde{\zeta}_{s,t,\tilde{y}} + \delta_{\tilde{y}}, \tilde{f}\rangle\right)\right] \tilde{\lambda}_{t,\vartheta}(d\tilde{y}) = \int_{\mathcal{M}(D^t(\mathbb{Z}^d))} \langle \tilde{\mu}, \tilde{g}\rangle e^{-\langle \tilde{\mu}, \tilde{f}\rangle} \tilde{\mathbf{Q}}_t^s(d\tilde{\mu}). \qquad (5.1)$$

Using (2.26) and (2.27) we can write the term on the r.h.s. of (5.1) as

$$\int \langle \tilde{\mu}, \tilde{g}\rangle e^{-\langle \tilde{\mu}, \tilde{f}\rangle} \tilde{\mathbf{Q}}_t^s(d\tilde{\mu}) = \int \frac{\partial}{\partial\alpha} \tilde{v}(s,t,1-e^{-(\tilde{f}+\alpha\tilde{g})};\tilde{y})\Big|_{\alpha=0} \tilde{\lambda}_{s,\vartheta}(d\tilde{y}). \qquad (5.2)$$

Abbreviate

$$\tilde{w}(s,t,\tilde{f},\tilde{g};\tilde{y}) := \frac{\partial}{\partial\alpha} \tilde{v}(s,t,1-e^{-(\tilde{f}+\alpha\tilde{g})};\tilde{y})\Big|_{\alpha=0}. \qquad (5.3)$$

From (2.20) we get

$$\tilde{w}(s,t,\tilde{f},\tilde{g};\tilde{y}) = \tilde{a}_{s,t}(\tilde{g}e^{-\tilde{f}})(\tilde{y}) - V\int_s^t \tilde{a}_{s,u}\left(\tilde{v}(u,t,1-e^{-\tilde{f}};\,\cdot\,)\tilde{w}(u,t,\tilde{f},\tilde{g};\,\cdot\,)\right)(\tilde{y})\,du. \tag{5.4}$$

Applying the Feynman-Kac formula to the corresponding equation for $\partial\tilde{w}/\partial s$ gives

$$\tilde{w}(s,t,\tilde{f},\tilde{g};\tilde{y}) = \mathbf{E}^{s,\tilde{y}}\left[\exp\left(-V\int_s^t \tilde{v}(u,t,1-e^{-\tilde{f}};\tilde{X}_u)\,du\right)\tilde{g}(\tilde{X}_t)e^{-\tilde{f}(\tilde{X}_t)}\right]. \tag{5.5}$$

Now substituting (5.5) into (5.2) yields

$$\begin{aligned}\int \langle\tilde{\mu},\tilde{g}\rangle e^{-\langle\tilde{\mu},\tilde{f}\rangle}\tilde{\mathbf{Q}}_t^s(d\tilde{\mu}) \;&=\; \sum_{x\in\mathbb{Z}^d}\int_{\{\tilde{y}:y_s=x\}}\tilde{\lambda}_{s,\vartheta}(d\tilde{y})\int \Pi_{s,x}^t(d\tilde{z})\,\tilde{g}(\tilde{y}/s/\tilde{z})e^{-\tilde{f}(\tilde{y}/s/\tilde{z})}\\ &\quad\times\exp\left(-V\int_s^t \tilde{v}(u,t,1-e^{-\tilde{f}};(\tilde{y}/s/\tilde{z})^u)\,du\right).\end{aligned} \tag{5.6}$$

Since $\tilde{f},\tilde{g}\in b\mathcal{D}([s_0,\infty))$, we can apply (2.23) to get

$$\int \langle\tilde{\mu},\tilde{g}\rangle e^{-\langle\tilde{\mu},\tilde{f}\rangle}\tilde{\mathbf{Q}}_t^s(d\tilde{\mu}) = \int \tilde{g}(\tilde{y})e^{-\tilde{f}(\tilde{y})}\exp\left(-V\int_s^t \tilde{v}(u,t,1-e^{-\tilde{f}};\tilde{y}^u)\,du\right)\tilde{\lambda}_{t,\vartheta}(d\tilde{y}). \tag{5.7}$$

Comparing (5.7) and (5.1) we see that it suffices to show

$$\mathbf{E}\left[\exp\left(-\langle\tilde{\zeta}_{s,t,\tilde{y}},\tilde{f}\rangle\right)\right] = \exp\left(-V\int_s^t \tilde{v}(u,t,1-e^{-\tilde{f}};\tilde{y}^u)\,du\right). \tag{5.8}$$

In order to do so we observe that

$$\begin{aligned}\mathbf{E}\left[\exp\left(-\langle\tilde{\zeta}_{s,t,\tilde{y}},\tilde{f}\rangle\right)\right] \;&=\; \mathbf{E}\left[\exp\left(-\int_s^t \langle\tilde{\xi}_t^{u,\tilde{y}^u},\tilde{f}\rangle\nu(du)\right)\right]\\ &= \sum_{m=0}^\infty e^{-V(t-s)}\frac{(V(t-s))^m}{m!}\int_{[s,t]^m}\frac{du_1\ldots du_m}{(t-s)^m}\,\mathbf{E}\left[\exp\left(-\sum_{k=1}^m\langle\tilde{\xi}_t^{u_k,\tilde{y}^{u_k}},\tilde{f}\rangle\right)\right]\\ &= \sum_{m=0}^\infty e^{-V(t-s)}\frac{V^m}{m!}\left(\int_s^t du\,\mathbf{E}^{u,\delta_{\tilde{y}^u}}\left[\exp\left(-\langle\tilde{\xi}_t,\tilde{f}\rangle\right)\right]\right)^m\\ &= \sum_{m=0}^\infty e^{-V(t-s)}\frac{V^m}{m!}\left(\int_s^t (1-\tilde{v}(u,t,1-e^{-\tilde{f}};\tilde{y}^u))\,du\right)^m\\ &= \exp\left(-V\int_s^t \tilde{v}(u,t,1-e^{-\tilde{f}};\tilde{y}^u)\,du\right).\end{aligned} \tag{5.9}$$

That means we proved (5.8). This completes the proof of (5.1). □

Proof of Theorem 3 The proof is very similar to the proof of [DP91] Theorem 6.3.
(i) Let $\tilde{f}\in\mathcal{C}^+(D(\mathbb{Z}^d))\cap b\mathcal{D}([s_0,\infty))$ for some $s_0>-\infty$ and $s<s_0$. Using (2.19) one can see

$$\mathbf{E}^{s,\tilde{y}}[\langle\tilde{\xi}_t,\tilde{f}\rangle] = \frac{\partial}{\partial\alpha}\tilde{v}(s,t,\alpha\tilde{f};\tilde{y})\Big|_{\alpha=0} \tag{5.10}$$

and by (2.20)

$$\mathbf{E}^{s,\tilde{y}}[\langle\tilde{\xi}_t,\tilde{f}\rangle] = (\tilde{a}_{s,t}\tilde{f})(\tilde{y}). \tag{5.11}$$

By independence we get

$$\mathbf{E}^{s,\vartheta}[\langle\tilde{\xi}_t,\tilde{f}\rangle] = \langle\tilde{\lambda}_{s,\vartheta},\tilde{a}_{s,t}\tilde{f}\rangle. \tag{5.12}$$

Since $\tilde{f}$ depends only on components $t \geq s_0$ the special structure of the initial distribution $\tilde{\lambda}_{s,\vartheta}$ leads to

$$\langle\tilde{\lambda}_{s,\vartheta},\tilde{a}_{s,t}\tilde{f}\rangle = \langle\tilde{\lambda}_{s_0,\vartheta},\tilde{a}_{s_0,t}\tilde{f}\rangle < \infty \quad \forall s < s_0. \tag{5.13}$$

Since the term on the r.h.s. of (5.13) does not depend on s we get tightness of the family $\{\mathcal{L}^{s,\vartheta}[\langle\tilde{\xi}_t,\tilde{f}\rangle], s < s_0\}$.

Furthermore the Laplace functional $\mathbf{E}^{s,\vartheta}[e^{-\langle\tilde{\xi}_t,\tilde{f}\rangle}]$ is decreasing in s. This is justified by the following observation. By (2.26)

$$\frac{\partial}{\partial s}\mathbf{E}^{s,\vartheta}[e^{-\langle\tilde{\xi}_t,\tilde{f}\rangle}] = -\exp\left(-\langle\tilde{\lambda}_{s,\vartheta},\tilde{v}(s,t,1-e^{-\tilde{f}};\,\cdot\,)\rangle\right)\frac{\partial}{\partial s}\langle\tilde{\lambda}_{s,\vartheta},\tilde{v}(s,t,1-e^{-\tilde{f}};\,\cdot\,)\rangle. \tag{5.14}$$

By (2.20)

$$\langle\tilde{\lambda}_{s,\vartheta},\tilde{v}(s,t,1-e^{-\tilde{f}};\,\cdot\,)\rangle = \langle\tilde{\lambda}_{s,\vartheta},\tilde{a}_{s,t}(1-e^{-\tilde{f}})\rangle - \frac{V}{2}\int_s^t\langle\tilde{\lambda}_{s,\vartheta},\tilde{a}_{s,u}(\tilde{v}(u,t,1-e^{-\tilde{f}};\,\cdot\,)^2)\rangle. \tag{5.15}$$

Since $\tilde{f}$ depends only on components $t \geq s_0$ and since the solution of (2.20) is unique, the functions $\tilde{v}(u,t,1-e^{-\tilde{f}};\tilde{y})$ do not depend on y_s with $s < s_0$. As in (5.13) we get

$$\langle\tilde{\lambda}_{s,\vartheta},\tilde{a}_{s,t}(1-e^{-\tilde{f}})\rangle = \langle\tilde{\lambda}_{s_0,\vartheta},\tilde{a}_{s_0,t}(1-e^{-\tilde{f}})\rangle \quad \forall s < s_0 \tag{5.16}$$

and

$$\langle\tilde{\lambda}_{s,\vartheta},\tilde{a}_{s,u}(\tilde{v}(u,t,1-e^{-\tilde{f}};\,\cdot\,)^2)\rangle = \begin{cases} \langle\tilde{\lambda}_{u,\vartheta},\tilde{v}(u,t,1-e^{-\tilde{f}};\,\cdot\,)^2\rangle; & u < s_0 \\ \langle\tilde{\lambda}_{s_0,\vartheta},\tilde{a}_{s_0,u}(\tilde{v}(u,t,1-e^{-\tilde{f}};\,\cdot\,)^2)\rangle & u \geq s_0 \end{cases}. \tag{5.17}$$

We end up with

$$\frac{\partial}{\partial s}\langle\tilde{\lambda}_{s,\vartheta},\tilde{v}(s,t,1-e^{-\tilde{f}};\,\cdot\,)\rangle = \frac{V}{2}\langle\tilde{\lambda}_{s,\vartheta},\tilde{v}(s,t,1-e^{-\tilde{f}};\,\cdot\,)^2\rangle, \tag{5.18}$$

thus the term on the r.h.s. of (5.14) is non-positive, which in turn yields that the Laplace functional is decreasing in s, i.e., it increases as $s \downarrow -\infty$. Hence the Laplace functional $\mathbf{E}^{s,\vartheta}[e^{-\langle\tilde{\xi}_t,\tilde{f}\rangle}]$ converges as $s \to -\infty$. By tightness we get

$$\mathcal{L}^{s,\vartheta}[\tilde{\xi}_t] \underset{s\to-\infty}{\Longrightarrow} \mathcal{L}[\tilde{\xi}_t^{-\infty}], \tag{5.19}$$

where $\tilde{\xi}_t^{-\infty}$ is infinitely divisible.

It remains to show persistence, i.e.,

$$\mathbf{E}[\langle\tilde{\xi}_t^{-\infty},\tilde{f}\rangle] = \langle\tilde{\lambda}_{t,\vartheta},\tilde{f}\rangle. \tag{5.20}$$

We have to show uniform integrability, which can be written by the definition of the Palm distribution as

$$\lim_{K\to\infty}\sup_{s\le t}\int\tilde\lambda_{t,\vartheta}(d\tilde y)\int(\tilde{\mathbf{Q}}^s_t)_{\tilde y}(d\tilde\mu)\tilde f(\tilde y)\mathbb{1}_{\{\langle\tilde\mu,\tilde f\rangle>K\}}(\tilde\mu)=0. \tag{5.21}$$

To do so we observe the following. Let $K>\|\tilde f\|_\infty$. By (2.28)

$$\int\tilde\lambda_{t,\vartheta}(d\tilde y)\int(\tilde{\mathbf{Q}}^s_t)_{\tilde y}(d\tilde\mu)\tilde f(\tilde y)\mathbb{1}_{\{\langle\tilde\mu,\tilde f\rangle>K\}}(\tilde\mu)=\int\tilde\lambda_{t,\vartheta}(d\tilde y)\tilde f(\tilde y)\mathbf{P}[\langle\tilde\zeta_{s,t,\tilde y}+\delta_{\tilde y},\tilde f\rangle>K]. \tag{5.22}$$

Since $\langle\tilde\zeta_{s,t,\tilde y},\tilde f\rangle$ increases if $s\downarrow-\infty$ we get

$$\begin{aligned}&\int\tilde\lambda_{t,\vartheta}(d\tilde y)\int(\tilde{\mathbf{Q}}^s_t)_{\tilde y}(d\tilde\mu)\tilde f(\tilde y)\mathbb{1}_{\{\langle\tilde\mu,\tilde f\rangle>K\}}(\tilde\mu)\\ &\le\int\tilde\lambda_{t,\vartheta}(d\tilde y)\tilde f(\tilde y)\mathbf{P}\left[\Big(\int_{-\infty}^t\nu(du)\langle\tilde\xi^{u,\tilde y^u}_t,\tilde f\rangle\Big)>K-\|\tilde f\|_\infty\right]\\ &\le\frac{1}{K-\|\tilde f\|_\infty}\int\tilde\lambda_{t,\vartheta}(d\tilde y)\tilde f(\tilde y)\mathbf{E}\left[\int_{-\infty}^t\nu(du)\langle\tilde\xi^{u,\tilde y^u}_t,\tilde f\rangle\right].\end{aligned} \tag{5.23}$$

By (5.11) $\mathbf{E}[\langle\tilde\xi^{u,\tilde y^u}_t,\tilde f\rangle]=(\tilde a_{u,t}\tilde f)(\tilde y^u)$. Moreover we know that ν is independent of $\xi^{u,\tilde y^u}_t$ and has intensity measure V times Lebesgue measure, hence

$$\int\tilde\lambda_{t,\vartheta}(d\tilde y)\tilde f(\tilde y)\mathbf{E}\left[\int_{-\infty}^t\nu(du)\langle\tilde\xi^{u,\tilde y^u}_t,\tilde f\rangle\right]=V\int_{-\infty}^t du\int\tilde\lambda_{t,\vartheta}(d\tilde y)\,\tilde f(\tilde y)(\tilde a_{u,t}\tilde f)(\tilde y^u). \tag{5.24}$$

Since $\tilde f\in\mathcal{C}^+_c(D(\mathbb{Z}^d))$ there exists a constant c such that $\tilde f\le\|\tilde f\|_\infty\,\mathbb{1}_{B(c)}\circ\pi_0$. Hence

$$\begin{aligned}&\int\tilde\lambda_{t,\vartheta}(d\tilde y)\tilde f(\tilde y)\mathbf{E}\left[\int_{-\infty}^t\nu(du)\langle\tilde\xi^{u,\tilde y^u}_t,\tilde f\rangle\right]\\ &\le V\|\tilde f\|^2_\infty\int_{-\infty}^t du\int\tilde\lambda_{t,\vartheta}(d\tilde y)\,\mathbb{1}_{B(c)}(y_0)\,\tilde a_{u,t}(\mathbb{1}_{B(c)}\circ\pi_0)(\tilde y^u).\end{aligned} \tag{5.25}$$

By the definition of $\tilde a_{u,t}$ and $\tilde\lambda_{t,\vartheta}$ we conclude that

$$\int\tilde\lambda_{t,\vartheta}(d\tilde y)\tilde f(\tilde y)\mathbf{E}\left[\int_{-\infty}^t\nu(du)\langle\tilde\xi^{u,\tilde y^u}_t,\tilde f\rangle\right]\le O(1)\sum_{y,z\in B(c)}\int_{-\infty}^0 du\,\hat a_{-2u}(y,z). \tag{5.26}$$

By (5.23) and (5.26) we obtain

$$\int\tilde\lambda_{t,\vartheta}(d\tilde y)\int(\tilde{\mathbf{Q}}^s_t)_{\tilde y}(d\tilde\mu)\tilde f(\tilde y)\mathbb{1}_{\{\langle\tilde\mu,\tilde f\rangle>K\}}(\tilde\mu)\le\frac{O(1)}{K-\|\tilde f\|_\infty}\sum_{y,z\in B(c)}\int_0^\infty du\,\hat a_u(y,z). \tag{5.27}$$

Since $\hat a$ is transient, we get (5.21).

(ii) As in (i) we can prove

$$\mathcal{L}^{\mathcal{H}(\vartheta)}[\xi_t]\underset{t\to\infty}{\Longrightarrow}\mathcal{L}[\xi], \tag{5.28}$$

where ξ is infinitely divisible with intensity

$$\mathbf{E}[\langle\xi,f\rangle]=\langle\vartheta\lambda,f\rangle \tag{5.29}$$

and Laplace functional

$$\mathbf{E}[e^{-\langle \xi, f\rangle}] = e^{-v(f)}, \tag{5.30}$$

where

$$v(f) = \lim_{t\to\infty} \Big\langle \vartheta\lambda, v(t, 1-e^{-f}; \,\cdot\,)\Big\rangle. \tag{5.31}$$

Setting $\tilde{f} = f \circ \pi_t$ in (2.31) we see that $\bar{\pi}_t \tilde{\xi}_t^{-\infty}$ and ξ have the same Laplace functional.

(iii) From (2.28) we obtain the result for the canonical Palm distribution. The desired result for the canonical measure we conclude as in [DP91]. □

Appendix

A Laplace functional

Lemma A.1 *Let $\tilde{f} \in \mathcal{C}_b^+(\mathbb{Z}^d \times [0,\infty))$ and denote $f_s(x) := \tilde{f}(x,s)$. The Laplace functional of the weighted occupation time*

$$h(u,t,\tilde{f};x) := \mathbf{E}^{u,\delta_x}\left[\exp\left(-\int_u^t ds\, \langle \xi_s, f_s\rangle\right)\right] \tag{A.1}$$

is the solution of the following equation

$$h(u,t,\tilde{f};x) = 1 + \int_u^t ds\, a_{s-u}\left(\frac{V}{2}\left(1-h(s,t,\tilde{f};\,\cdot\,)\right)^2 - f_s h(s,t,\tilde{f};\,\cdot\,)\right)(x). \tag{A.2}$$

Proof By a renewal argument

$$h(u,t,\tilde{f};x) = e^{-V(t-u)} v(u,t,\tilde{f};x) + \frac{V}{2}\int_u^t ds\, e^{-V(s-u)} w(u,s,t,\tilde{f};x), \tag{A.3}$$

with

$$\begin{aligned} v(u,t,\tilde{f};x) &= \mathbf{E}^{u,x}\left[e^{-\int_u^t ds\, f_s(X_s)}\right] \\ w(u,s,t,\tilde{f};x) &= \mathbf{E}^{u,x}\left[e^{-\int_u^s d\rho\, f_\rho(X_\rho)}\left(h(s,t,\tilde{f};X_s)^2+1\right)\right], \end{aligned} \tag{A.4}$$

where $(X_s)_{s\geq 0}$ is the continuous time random walk with transition kernel a.

Thus $\partial h/\partial u$ can be expressed in terms of the derivatives of v and w. The form of v and w allows the application of the Feynman-Kac formula to get

$$\frac{\partial}{\partial u} v(u,t,\tilde{f};x) = -\Omega v(u,t,\tilde{f};\,\cdot\,)(x) + f_u(x) v(u,t,\tilde{f};x) \tag{A.5}$$

and

$$\frac{\partial}{\partial u} w(u,s,t,\tilde{f};x) = -\Omega w(u,s,t,\tilde{f};\,\cdot\,)(x) + f_u(x) w(u,s,t,\tilde{f};x). \tag{A.6}$$

Thereby we get

$$\frac{\partial}{\partial u} h(u,t,\tilde{f};x) = -\Omega h(u,t,\tilde{f};\,\cdot\,)(x) - \frac{V}{2}\left(1-h(u,t,\tilde{f};x)\right)^2 + f_u(x) h(u,t,\tilde{f};x). \tag{A.7}$$

This completes the proof. □

Lemma A.2 *Let $\tilde{f} \in \mathcal{C}_b^+(\mathbb{Z}^d \times [0,\infty))$ and denote $f_s(x) := \tilde{f}(x,s)$. The Laplace functional of the weighted occupation time from time T till time $T+t$*

$$h(u,T,t,\tilde{f};x) := \mathbf{E}^{u,\delta_x}\left[\exp\left(-\int_T^{T+t} ds\,\langle \xi_s, f_s\rangle\right)\right] \tag{A.8}$$

solves

$$\begin{aligned} h(u,T,t,\tilde{f};x) &= a_{T-u}\left(h(T,T,t,\tilde{f};\,\cdot\,)\right)(x) \\ &\quad + \frac{V}{2}\int_u^T ds\, a_{s-u}\left((1-h(s,T,t,\tilde{f};\,\cdot\,))^2\right)(x). \end{aligned} \tag{A.9}$$

Proof The proof works analogously to the proof of Lemma A.1 except for the fact that we manage without the Feynman-Kac formula. By a renewal argument

$$\begin{aligned} h(u,T,t,\tilde{f};x) &= e^{-V(T+t-u)}v(u,T,t,\tilde{f};x) + \frac{V}{2}\int_u^T ds\, e^{-V(s-u)}w(u,s,T,t,\tilde{f};x) \\ &\quad + \frac{V}{2}\int_T^{T+t} ds\, e^{-V(s-u)}z(u,T,s,t,\tilde{f};x), \end{aligned} \tag{A.10}$$

with

$$\begin{aligned} v(u,T,t,\tilde{f};x) &= \mathbf{E}^{u,x}[e^{-\int_T^{T+t} ds\, f_s(X_s)}] \\ w(u,s,T,t,\tilde{f};x) &= \mathbf{E}^{u,x}\left[h(s,T,t,\tilde{f};X_s)^2 + 1\right] \\ z(u,T,s,t,\tilde{f};x) &= \mathbf{E}^{u,x}\left[e^{-\int_T^s d\rho\, f_\rho(X_\rho)}\left(h(s,s,T+t-s,\tilde{f};X_s)^2+1\right)\right]. \end{aligned} \tag{A.11}$$

Thus $\partial h/\partial u$ can be expressed in terms of the derivatives of v, w and z. Obviously

$$\frac{\partial}{\partial u}v(u,T,t,\tilde{f};x) = -\Omega v(u,T,t,\tilde{f};\,\cdot\,)(x) \tag{A.12}$$

and the same for w and z. We obtain

$$\frac{\partial}{\partial u}h(u,T,t,\tilde{f};x) = -\Omega h(u,T,t,\tilde{f};\,\cdot\,)(x) - \frac{V}{2}\left(1-h(u,T,t,\tilde{f};x)\right)^2. \tag{A.13}$$

This leads to the assertion. □

B Moment calculations

First of all we establish a recursive formula for the moments of the BRW with a single ancestor.

Lemma B.1 *Let $f:\mathbb{Z}^d \to [0,\infty)$ be bounded. The moments of the BRW with a single initial particle fulfill the following recursive formula*

$$\mathbf{E}^{\delta_x}[\langle \xi_t, f\rangle^n] = a_t(f^n)(x) + \frac{V}{2}\sum_{k=1}^{n-1}\binom{n}{k}\int_0^t ds\, a_{t-s}\left(\mathbf{E}^{\delta_\cdot}[\langle \xi_s, f\rangle^k]\,\mathbf{E}^{\delta_\cdot}[\langle \xi_s, f\rangle^{n-k}]\right)(x). \tag{B.1}$$

Particularly for the first and the second moment we obtain

$$\mathbf{E}^{\delta_x}[\langle \xi_t, f\rangle] = (a_t f)(x) \tag{B.2}$$

$$\mathbf{E}^{\delta_x}[\langle \xi_t, f\rangle^2] = a_t(f^2)(x) + V\int_0^t ds\, a_{t-s}\Big((a_s f)^2\Big)(x). \tag{B.3}$$

Proof Let g be in the domain of the generator of the BRW. Then $g(\xi_t)$ solves the Kolmogorov backward equation

$$\frac{\partial}{\partial t}\mathbf{E}^{\delta_x}[g(\xi_t)] = \Omega\mathbf{E}^{\delta_\cdot}[g(\xi_t)](x) + V\left(\frac{1}{2}\mathbf{E}^{2\delta_x}[g(\xi_t)] + \frac{1}{2}\mathbf{E}^{\underline{0}}[g(\xi_t)] - \mathbf{E}^{\delta_x}[g(\xi_t)]\right). \tag{B.4}$$

In particular by setting $g(\mu) = \langle \mu, f\rangle^n$ and by using the independence of the particles (B.4) becomes

$$\frac{\partial}{\partial t}\mathbf{E}^{\delta_x}[\langle \xi_t, f\rangle^n] = \Omega\mathbf{E}^{\delta_\cdot}[\langle \xi_t, f\rangle^n](x) + \frac{V}{2}\sum_{k=1}^{n-1}\binom{n}{k}\mathbf{E}^{\delta_x}[\langle \xi_t, f\rangle^k]\mathbf{E}^{\delta_x}[\langle \xi_t, f\rangle^{n-k}]. \tag{B.5}$$

This leads to assertion (B.1). □

Now we determine the moments of the BRW started as the Poisson point process.

Proposition B.2 (BRW at time t, Poisson initial distribution) *Let $f : \mathbb{Z}^d \to [0,\infty)$ be bounded. The BRW started as the Poisson point process has the first moment*

$$\mathbf{E}^{\mathcal{H}(\vartheta)}[\langle \xi_t, f\rangle] = \vartheta\langle \lambda, f\rangle \tag{B.6}$$

and the second moment

$$\mathbf{E}^{\mathcal{H}(\vartheta)}[\langle \xi_t, f\rangle^2] = \vartheta^2\langle \lambda, f\rangle^2 + \vartheta\langle \lambda, f^2\rangle + V\vartheta\sum_{x,y\in\mathbb{Z}^d} f(x)f(y)\int_0^t ds\,\hat{a}_{2s}(x,y). \tag{B.7}$$

Proof Lemma B.1 provides the first and the second moments of the BRW with a single initial particle. Since we constructed the BRW with a general initial state as superposition of BRWs with single ancestors, we obtain for the first moment

$$\mathbf{E}^{\mathcal{H}(\vartheta)}[\langle \xi_t, f\rangle] = \mathbf{E}^{\mathcal{H}(\vartheta)}\Big[\langle\sum_{x\in\mathbb{Z}^d}\sum_{k=1}^{\xi_0(x)}\xi_t^{x,k}, f\rangle\Big] = \sum_{x\in\mathbb{Z}^d}\sum_{n=0}^{\infty} e^{-\vartheta}\frac{\vartheta^n}{n!}\sum_{k=1}^{n}\mathbf{E}[\langle \xi_t^{x,k}, f\rangle], \tag{B.8}$$

where $\{(\xi_t^{x,k})_{t\geq 0}; k\in\mathbb{N}\}$ are independent BRWs started with a single initial particle in x. This leads to

$$\mathbf{E}^{\mathcal{H}(\vartheta)}[\langle \xi_t, f\rangle] = \sum_{x\in\mathbb{Z}^d}\vartheta\,\mathbf{E}^{\delta_x}[\langle \xi_t, f\rangle]. \tag{B.9}$$

This completes the proof of (B.6). Assertion (B.7) can be proven analogously. □

Now we come to the moments of the weighted occupation times.

Proposition B.3 (Weighted occupation time, single initial particle) *Let $\tilde f \in \mathcal{C}_b^+(\mathbb{Z}^d \times [0,\infty))$ and denote $f_s(x) := \tilde f(x,s)$. The weighted occupation time of the BRW with a single initial particle has the first moment*

$$\mathbf{E}^{\delta_x}\left[\int_0^t ds\,\langle \xi_s, f_s\rangle\right] = \int_0^t ds\,(a_s f_s)(x) \tag{B.10}$$

and the second moment

$$\mathbf{E}^{\delta_x}\left[\int_0^t ds\,\langle \xi_s, f_s\rangle\right]^2 = \int_0^t ds\,a_s\left(V\left(\int_s^t d\rho\,(a_{\rho-s}f_\rho)\right)^2 + 2f_s\int_s^t d\rho\,(a_{\rho-s}f_\rho)\right)(x) \tag{B.11}$$

and the third moment

$$\begin{aligned}
&\mathbf{E}^{\delta_x}\left[\int_0^t ds\,\langle \xi_s, f_s\rangle\right]^3 \\
&= 3V^2\int_0^t ds\,a_s\left(\int_s^t d\rho\,(a_{\rho-s}f_\rho)\int_s^t d\tau\,a_{\tau-s}\left(\left(\int_\tau^t d\pi\,(a_{\pi-\tau}f_\pi)\right)^2\right)\right)(x) \\
&\quad + 6V\int_0^t ds\,a_s\left(\int_s^t d\rho\,(a_{\rho-s}f_\rho)\int_s^t d\tau\,a_{\tau-s}\left(f_\tau\int_\tau^t d\pi\,(a_{\pi-\tau}f_\pi)\right)\right)(x) \\
&\quad + 3V\int_0^t ds\,a_s\left(f_s\int_s^t d\tau\,a_{\tau-s}\left(\left(\int_\tau^t d\pi\,(a_{\pi-\tau}f_\pi)\right)^2\right)\right)(x) \\
&\quad + 6\int_0^t ds\,a_s\left(f_s\int_s^t d\tau\,a_{\tau-s}\left(f_\tau\int_\tau^t d\pi\,(a_{\pi-\tau}f_\pi)\right)\right)(x).
\end{aligned} \tag{B.12}$$

Proof We consider the first moment. Let

$$h_1(u,t,\tilde f;x) := \mathbf{E}^{u,\delta_x}\left[\int_u^t ds\,\langle \xi_s, f_s\rangle\right], \tag{B.13}$$

which can be expressed in terms of the Laplace functional

$$h_1(u,t,\tilde f;x) = -\frac{\partial}{\partial\alpha}h(u,t,\alpha\tilde f;x)\Big|_{\alpha=0}, \tag{B.14}$$

where h is given in (A.1). We know from Lemma A.1 that $h(u,t,\tilde f;x)$ solves

$$h(u,t,\tilde f;x) = 1 + \int_u^t ds\,a_{s-u}\left(\frac{V}{2}\left(1-h(s,t,\tilde f;\,\cdot\,)\right)^2 - f_s h(s,t,\tilde f;\,\cdot\,)\right)(x). \tag{B.15}$$

Substituting equation (B.15) into (B.14) leads to

$$h_1(u,t,\tilde f;x) = \int_u^t ds\,(a_{s-u}f_s)(x). \tag{B.16}$$

Now we turn our attention to the second moment. Let

$$h_2(u,t,\tilde f;x) := \mathbf{E}^{u,\delta_x}\left[\int_u^t ds\,\langle \xi_s, f_s\rangle\right]^2, \tag{B.17}$$

which is in terms of the Laplace functional

$$h_2(u,t,\tilde{f};x) = \frac{\partial^2}{\partial\alpha^2} h(u,t,\alpha\tilde{f};x)\Big|_{\alpha=0}. \tag{B.18}$$

Hence by (B.15) again

$$h_2(u,t,\tilde{f};x) = \int_u^t ds\, a_{s-u}\left(V h_1(s,t,\tilde{f};\,\cdot\,)^2 + 2f_s h_1(s,t,\tilde{f};\,\cdot\,)\right)(x). \tag{B.19}$$

Applying (B.16) we get

$$h_2(u,t,\tilde{f};x) = \int_u^t ds\, a_{s-u}\left(V\left(\int_s^t d\rho\,(a_{\rho-s}f_\rho)\right)^2 + 2f_s\int_s^t d\rho\,(a_{\rho-s}f_\rho)\right)(x). \tag{B.20}$$

Now we come to the third moment. Let

$$h_3(u,t,\tilde{f};x) := \mathbf{E}^{u,\delta_x}\left[\int_u^t ds\,\langle\xi_s, f_s\rangle\right]^3, \tag{B.21}$$

which is in terms of the Laplace functional

$$h_3(u,t,\tilde{f};x) = -\frac{\partial^3}{\partial\alpha^3} h(u,t,\alpha\tilde{f};x)\Big|_{\alpha=0}. \tag{B.22}$$

Hence by (B.15) again

$$h_3(u,t,\tilde{f};x) = \int_u^t ds\, a_{s-u}\Big(3V h_1(s,t,\tilde{f};\,\cdot\,)h_2(s,t,\tilde{f};\,\cdot\,) + 3f_s h_2(s,t,\tilde{f};\,\cdot\,)\Big)(x). \tag{B.23}$$

Applying (B.16) and (B.20) we get

$$\begin{aligned} &h_3(u,t,\tilde{f};x) \\ &= 3V^2\int_u^t ds\, a_{s-u}\left(\int_s^t d\rho\,(a_{\rho-s}f_\rho)\int_s^t d\tau\, a_{\tau-s}\left(\left(\int_\tau^t d\pi\,(a_{\pi-\tau}f_\pi)\right)^2\right)\right)(x) \\ &\quad + 6V\int_u^t ds\, a_{s-u}\left(\int_s^t d\rho\,(a_{\rho-s}f_\rho)\int_s^t d\tau\, a_{\tau-s}\left(f_\tau\int_\tau^t d\pi\,(a_{\pi-\tau}f_\pi)\right)\right)(x) \\ &\quad + 3V\int_u^t ds\, a_{s-u}\left(f_s\int_s^t d\tau\, a_{\tau-s}\left(\left(\int_\tau^t d\pi\,(a_{\pi-\tau}f_\pi)\right)^2\right)\right)(x) \\ &\quad + 6\int_u^t ds\, a_{s-u}\left(f_s\int_s^t d\tau\, a_{\tau-s}\left(f_\tau\int_\tau^t d\pi\,(a_{\pi-\tau}f_\pi)\right)\right)(x). \end{aligned} \tag{B.24}$$

This completes the proof. □

Proposition B.4 (Late weighted occupation time, single initial particle) *Let $\tilde{f} \in \mathcal{C}_b^+(\mathbb{Z}^d \times [0,\infty))$ and denote $f_s(x) := \tilde{f}(x,s)$. The first moment of the weighted occupation time from time T till time $T+t$ of the BRW with a single initial particle is*

$$\mathbf{E}^{\delta_x}\left[\int_T^{T+t} ds\,\langle\xi_s, f_s\rangle\right] = \int_T^{T+t} ds\,(a_s f_s)(x). \tag{B.25}$$

The second moment is

$$\mathbf{E}^{\delta_x}\left[\int_T^{T+t} ds\,\langle \xi_s, f_s\rangle\right]^2 = V\int_0^T ds\, a_s\left(\left(\int_T^{T+t} d\rho\,(a_{\rho-s}f_\rho)\right)^2\right)(x)$$
$$+\int_T^{T+t} ds\, a_s\left(V\left(\int_s^{T+t} d\rho\,(a_{\rho-s}f_\rho)\right)^2 + 2f_s\int_s^{T+t} d\rho\,(a_{\rho-s}f_\rho)\right)(x). \tag{B.26}$$

Proof The proof works analogously as the proof of Proposition B.3 except for using Lemma A.2 instead of Lemma A.1. We conclude that the first moment

$$h_1(u,T,t,\tilde f;x) := \mathbf{E}^{u,\delta_x}\left[\int_T^{T+t} ds\,\langle \xi_s, f_s\rangle\right] \tag{B.27}$$

solves

$$h_1(u,T,t,\tilde f;x) = a_{T-u}\left(h_1(T,T,t,\tilde f;\,\cdot\,)\right)(x), \tag{B.28}$$

which in turn means by (B.16)

$$h_1(u,T,t,\tilde f;x) = \int_T^{T+t} ds\,(a_{s-u}f_s)(x). \tag{B.29}$$

For the second moment we get that

$$h_2(u,T,t,\tilde f;x) := \mathbf{E}^{u,\delta_x}\left[\int_T^{T+t} ds\,\langle \xi_s, f_s\rangle\right]^2 \tag{B.30}$$

solves

$$h_2(u,T,t,\tilde f;x) = a_{T-u}\left(h_2(T,T,t,\tilde f;\,\cdot\,)\right)(x) + V\int_u^T ds\, a_{s-u}\left(h_1(s,T,t,\tilde f;\,\cdot\,)^2\right)(x). \tag{B.31}$$

By (B.20) and (B.29) we end up with assertion (B.26). This completes the proof. $\square$

Proposition B.5 (Late weighted occupation time, Poisson initial distribution) *Let $\tilde f \in \mathcal{C}_b^+(\mathbb{Z}^d\times[0,\infty))$ and denote $f_s(x) := \tilde f(x,s)$. The weighted occupation time from time T till time $T+t$ of the BRW started as the Poisson point process has the first moment*

$$\mathbf{E}^{\mathcal{H}(\vartheta)}\left[\int_T^{T+t} ds\,\langle \xi_s, f_s\rangle\right] = \vartheta\int_T^{T+t} ds\,\langle \lambda, f_s\rangle \tag{B.32}$$

and the second moment

$$\mathbf{E}^{\mathcal{H}(\vartheta)}\left[\int_T^{T+t} ds\,\langle \xi_s, f_s\rangle\right]^2$$
$$= \vartheta^2\left(\int_T^{T+t} ds\,\langle \lambda, f_s\rangle\right)^2 + 2\vartheta\sum_{x\in\mathbb{Z}^d}\int_T^{T+t} ds\, f_s(x)\int_s^{T+t} d\rho\,(a_{\rho-s}f_\rho)(x)$$
$$+V\vartheta\sum_{x\in\mathbb{Z}^d}\int_0^T ds\left(\int_T^{T+t} d\rho\,(a_{\rho-s}f_\rho)(x)\right)^2 + V\vartheta\sum_{x\in\mathbb{Z}^d}\int_T^{T+t} ds\left(\int_s^{T+t} d\rho\,(a_{\rho-s}f_\rho)(x)\right)^2. \tag{B.33}$$

Proof In order to apply the moment formulas of the BRW with a single initial particle we decompose

$$\xi_s = \sum_{x\in\mathbb{Z}^d} \sum_{k=1}^{\xi_0(x)} \xi_s^{x,k}, \tag{B.34}$$

where $\{(\xi_s^{x,k})_{s\geq 0}; k \in \mathbb{N}\}$ are independent BRWs started with a single initial particle in x. For the first moment we obtain

$$\mathbf{E}^{\mathcal{H}(\vartheta)}\left[\int_T^{T+t} ds\,\langle \xi_s, f_s\rangle\right] = \vartheta \sum_{x\in\mathbb{Z}^d} \mathbf{E}^{\delta_x}\left[\int_T^{T+t} ds\,\langle \xi_s, f_s\rangle\right]. \tag{B.35}$$

Using (B.25) leads to assertion (B.32). For the second moment we obtain

$$\begin{aligned}\mathbf{E}^{\mathcal{H}(\vartheta)}\left[\int_T^{T+t} ds\,\langle \xi_s, f_s\rangle\right]^2 &= \vartheta^2 \left(\sum_{x\in\mathbb{Z}^d} \mathbf{E}^{\delta_x}\left[\int_T^{T+t} ds\,\langle \xi_s, f_s\rangle\right]\right)^2 \\ &\quad + \vartheta \sum_{x\in\mathbb{Z}^d} \mathbf{E}^{\delta_x}\left[\int_T^{T+t} ds\,\langle \xi_s, f_s\rangle\right]^2 .\end{aligned} \tag{B.36}$$

Using (B.25) and (B.26) leads to assertion (B.33), thus we are done. □

Proposition B.6 (Equilibrium weighted occupation time) *Let $\tilde{f} \in \mathcal{C}_b^+(\mathbb{Z}^d \times [0,\infty))$ and denote $f_s(x) := \tilde{f}(x,s)$. The first moment of the weighted occupation time in equilibrium is*

$$\mathbf{E}^{\Lambda_\vartheta}\left[\int_0^t ds\,\langle \xi_s, f_s\rangle\right] = \vartheta \int_0^t ds\,\langle \lambda, f_s\rangle. \tag{B.37}$$

The second moment has the form

$$\begin{aligned}\mathbf{E}^{\Lambda_\vartheta}\left[\int_0^t ds\,\langle \xi_s, f_s\rangle\right]^2 &= \vartheta^2\left(\int_0^t ds\,\langle \lambda, f_s\rangle\right)^2 + 2\vartheta \sum_{x\in\mathbb{Z}^d}\int_0^t ds\, f_s(x)\int_s^t d\rho\,(a_{\rho-s}f_\rho)(x) \\ + V\vartheta \sum_{x\in\mathbb{Z}^d}\int_0^\infty ds\left(\int_0^t d\rho\,(a_{s+\rho}f_\rho)(x)\right)^2 &+ V\vartheta\sum_{x\in\mathbb{Z}^d}\int_0^t ds\left(\int_s^t d\rho\,(a_{\rho-s}f_\rho)(x)\right)^2 .\end{aligned} \tag{B.38}$$

Proof Recall the Basic Ergodic Theorem of Section 1.2. By (1.10) and (3.27)

$$\mathbf{E}^{\Lambda_\vartheta}\left[\int_0^t ds\,\langle \xi_s, f_s\rangle\right] = \lim_{T\to\infty} \mathbf{E}^{\mathcal{H}(\vartheta)}\left[\int_T^{T+t} ds\,\langle \xi_s, f_{s-T}\rangle\right] \tag{B.39}$$

and

$$\mathbf{E}^{\Lambda_\vartheta}\left[\int_0^t ds\,\langle \xi_s, f_s\rangle\right]^2 = \lim_{T\to\infty} \mathbf{E}^{\mathcal{H}(\vartheta)}\left[\int_T^{T+t} ds\,\langle \xi_s, f_{s-T}\rangle\right]^2 . \tag{B.40}$$

Now we apply (B.32) respectively (B.33). This leads to assertions (B.37) and (B.38). □

C The asymptotic behavior of the Green's function on $\mathbb{Z}^d$

Lemma C.1 *Assume $\hat{G}$ defined by (1.13). Then*

$$\hat{G}(x) \sim \begin{cases} C \cdot \bar{Q}(x)^{-\frac{d-2}{2}} & \text{as } \quad |x| \to \infty;\ x \in \mathcal{G}, \\ 0 & \text{if } \quad x \in \mathbb{Z}^d \backslash \mathcal{G}, \end{cases} \tag{C.1}$$

where

$$C = \frac{|\mathbb{Z}^d/\mathcal{G}|\Gamma(\frac{d-2}{2})}{2\pi^{d/2}|Q|^{1/2}}, \tag{C.2}$$

where $\bar{Q}$ is given by (1.16).

Proof The little calculation

$$\hat{G}(x) = \int_0^\infty \hat{a}_s(0,x)\, ds = \int_0^\infty \sum_{n=0}^\infty e^{-s} \frac{s^n}{n!} \hat{a}^{(n)}(0,x)\, ds = \sum_{n=0}^\infty \hat{a}^{(n)}(0,x) = \hat{G}^{(dis)}(x) \tag{C.3}$$

shows that the Green's function of the continuous time kernel is equal to the discrete time one. The kernel $\hat{a}$ is symmetric and therefore mean zero. Furthermore we observe that Q given in (1.15) is the covariance matrix of $\hat{a}$.

Hence we need the asymptotics of the discrete time Green's function $\hat{G}^{(dis)}$. In [Zäh01] Corollary 2 we find

$$\hat{G}(x) \sim \begin{cases} C \cdot \bar{Q}(x)^{-\frac{d-2}{2}} & \text{as } \quad |x| \to \infty;\ x \in \mathcal{G}, \\ 0 & \text{if } \quad x \in \mathbb{Z}^d \backslash \mathcal{G}, \end{cases} \tag{C.4}$$

where

$$C = \frac{|\mathbb{Z}^d/\mathcal{G}|\Gamma(\frac{d-2}{2})}{2\pi^{d/2}|Q|^{1/2}}. \tag{C.5}$$

This completes the proof. □

D Local Central Limit Theorems

Proposition D.1 (Continuous time version of 7.P9 of [Spi64]) *For a strongly aperiodic random walk $(\xi_t)_{t\geq 0}$ with kernel a started in the origin, which has mean 0 and a finite second moment,*

$$(2\pi t)^{d/2}\mathbf{P}[\xi_t = x] \underset{t\to\infty}{\longrightarrow} |Q|^{-1/2}, \qquad \forall\, x \in \mathbb{Z}^d, \tag{D.1}$$

where $|Q|$ is the determinant of the covariance matrix Q of a.

Proof The following proof is the adaption of the proof of Theorem 7.P9 of [Spi64] to the continuous time random walk. By the definition of a continuous time random walk we know

$$\mathbf{P}[\xi_t = x] = \mathbf{P}[S_{\tau(t)} = x], \tag{D.2}$$

where $\tau(t)$ is a Poisson random variable with mean t and $S_n = X_1 + \cdots + X_n$ with $\{X_k; k \in \mathbb{N}\}$ i.i.d. with distribution a.

Let φ denote the characteristic function of X_1. Obviously

$$\sum_{y \in \mathbb{Z}^d} \mathbf{P}[S_{\tau(t)} = y] e^{iz\cdot y} = e^{-t} e^{t\varphi(z)}. \tag{D.3}$$

Multiplying $e^{-iz\cdot x}$ and integrating over $\mathbb{T} = \{z \in \mathbb{R}^d : |z_k| \leq \pi, k = 1, \ldots, d\}$ on both sides leads to

$$(2\pi t)^{d/2} \mathbf{P}[\xi_t = x] = (2\pi)^{-d/2} t^{d/2} \int_{\mathbb{T}} e^{-t} e^{t\varphi(z)} e^{-ix\cdot z} dz, \tag{D.4}$$

and after substituting $z := \sqrt{t} z$

$$(2\pi t)^{d/2} \mathbf{P}[\xi_t = x] = (2\pi)^{-d/2} \int_{\sqrt{t}\mathbb{T}} e^{-t} e^{t\varphi(z/\sqrt{t})} e^{-ix\cdot z/\sqrt{t}} dz. \tag{D.5}$$

We decompose the term on the r.h.s. of the latter equation in a principle term plus error terms as follows

$$\begin{aligned}(2\pi t)^{d/2} \mathbf{P}[\xi_t = x] \quad &= \quad (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-\frac{1}{2} z^{tr} Q z} e^{-ix\cdot z/\sqrt{t}} dz \\ &\quad + I_1(t, c_1) - I_2(t, c_1) + I_3(t, c_1, c_2) + I_r(t, c_2), \end{aligned} \tag{D.6}$$

with

$$\begin{aligned} I_1(t, c_1) \quad &:= \quad (2\pi)^{-d/2} \int_{z; |z| \leq c_1} \left[e^{-t} e^{t\varphi(z/\sqrt{t})} - e^{-\frac{1}{2} z^{tr} Q z} \right] e^{-ix\cdot z/\sqrt{t}} dz \\ I_2(t, c_1) \quad &:= \quad (2\pi)^{-d/2} \int_{z; |z| > c_1} e^{-\frac{1}{2} z^{tr} Q z} e^{-ix\cdot z/\sqrt{t}} dz \\ I_3(t, c_1, c_2) \quad &:= \quad (2\pi)^{-d/2} \int_{z; c_1 < |z| \leq c_2\sqrt{t}} e^{-t} e^{t\varphi(z/\sqrt{t})} e^{-ix\cdot z/\sqrt{t}} dz \\ I_4(t, c_2) \quad &:= \quad (2\pi)^{-d/2} \int_{z \in \sqrt{t}\mathbb{T}; |z| > c_2\sqrt{t}} e^{-t} e^{t\varphi(z/\sqrt{t})} e^{-ix\cdot z/\sqrt{t}} dz, \end{aligned} \tag{D.7}$$

with c_2 small enough such that if $|z| \leq c_2\sqrt{t}$ then $z \in \sqrt{t}\mathbb{T}$. First of all we show that the principle term converges as required, namely by the majorized convergence

$$I_0(t) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-\frac{1}{2} z^{tr} Q z} e^{-ix\cdot z/\sqrt{t}} dz \underset{t\to\infty}{\longrightarrow} (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-\frac{1}{2} z^{tr} Q z} dz \tag{D.8}$$

and the term on the r.h.s. in turn is just $|Q|^{1/2}$ by the following argument. Since Q is symmetric there exists an orthogonal matrix S such that

$$S^{tr} Q S = \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_d \end{pmatrix}, \tag{D.9}$$

where $\alpha_1, \ldots, \alpha_k > 0$ are the eigenvalues of Q. We substitute $z := S^{tr} z$ to get

$$(2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-\frac{1}{2} z^{tr} Q z} dz = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-\frac{1}{2} \sum_{k=1}^d \alpha_k z_k^2} dz = (2\pi)^{-d/2} \prod_{k=1}^{d} \frac{\sqrt{2\pi}}{\sqrt{\alpha_k}} = |Q|^{-1/2}. \quad \text{(D.10)}$$

This completes the proof of $I_0(t) \underset{t\to\infty}{\longrightarrow} |Q|^{-1/2}$. We are done if we show that all error terms converge to zero.

First of all we mention that

$$\lim_{z\to 0} \frac{1-\varphi(z)}{z^{tr} Q z} = \frac{1}{2}. \quad \text{(D.11)}$$

This is proven in [Spi64] 7.P7.

Using (D.11) we get

$$e^{-t} e^{t\varphi(z/\sqrt{t})} = \exp\left(- \frac{1-\varphi(z/\sqrt{t})}{(z/\sqrt{t})^{tr} Q (z/\sqrt{t})} z^{tr} Q z \right) \underset{t\to\infty}{\longrightarrow} e^{-\frac{1}{2} z^{tr} Q z}. \quad \text{(D.12)}$$

By majorized convergence we obtain $I_1(t, c_1) \underset{t\to\infty}{\longrightarrow} 0$ for fixed c_1.

The term I_2 can be estimated by

$$|I_2(t, c_1)| \le (2\pi)^{-d/2} \int_{z; |z| > c_1} e^{-\frac{1}{2} z^{tr} Q z} dz. \quad \text{(D.13)}$$

For I_3 we get an analogous estimate if we use (D.11) and (D.12) to show that for c_2 small enough

$$e^{-t} e^{t\varphi(z/\sqrt{t})} \le e^{-\frac{1}{4} z^{tr} Q z}, \quad \forall z, |z| \le c_2 \sqrt{t}. \quad \text{(D.14)}$$

This justifies

$$|I_3(t, c_1, c_2)| \le (2\pi)^{-d/2} \int_{z; |z| > c_1} e^{-\frac{1}{4} z^{tr} Q z} dz. \quad \text{(D.15)}$$

Due to (D.13) and (D.15) I_2 and I_3 can be made arbitrarily small by choosing c_2 small enough and c_1 large enough.

By [Spi64] 7.P8 $|\varphi(z)| = 1$ iff each coordinate of z is a multiple of 2π. That means there exists a constant $\delta = \delta(c_2)$ such that

$$|\varphi(z)| < 1 - \delta, \quad \forall z \in \mathbb{T}, |z| > c_2. \quad \text{(D.16)}$$

We get

$$e^{-t} e^{t\varphi(z)} \le e^{-\delta t} \underset{t\to\infty}{\longrightarrow} 0. \quad \text{(D.17)}$$

We conclude $I_4(t, c_2) \underset{t\to\infty}{\longrightarrow} 0$. This completes the proof. □

Proposition D.2 *For a strongly aperiodic random walk (ξ_t) with kernel a started in the origin, which has mean 0 and has finite second moments,*

$$\lim_{t\to\infty} \left[(2\pi t)^{d/2} \mathbf{P}[\xi_t = x] - |Q|^{-\frac{1}{2}} e^{-\frac{1}{2t} x^{tr} Q^{-1} x} \right] = 0 \quad \text{(D.18)}$$

uniformly in $x \in \mathbb{Z}^d$.

Proof Analogous to the proof of Proposition D.1 we adapt the proof of the remark of [Spi64], which is to be found right after Theorem 7.P9.

As in (D.5) we write

$$(2\pi t)^{d/2}\mathbf{P}[\xi_t = x] = (2\pi)^{-d/2}\int_{\sqrt{t}\mathbb{T}} e^{-t}e^{t\varphi(z/\sqrt{t})}e^{-ix\cdot z/\sqrt{t}}dz \tag{D.19}$$

and as in (D.6) we decompose the term on the r.h.s. of the latter equation in a principle term plus error terms as follows

$$\begin{aligned}(2\pi t)^{d/2}\mathbf{P}[\xi_t = x] &= (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{-\frac{1}{2}z^{tr}Qz}e^{-ix\cdot z/\sqrt{t}}dz \\ &\quad + I_1(t,c_1) + I_2(t,c_1) + I_3(t,c_1,c_2) + I_r(t,c_2).\end{aligned} \tag{D.20}$$

We are done if we show

$$(2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{-\frac{1}{2}z^{tr}Qz}e^{-ix\cdot z/\sqrt{t}}dz = |Q|^{-\frac{1}{2}}e^{-\frac{1}{2t}x^{tr}Q^{-1}x}, \tag{D.21}$$

since the estimates of the error terms are uniformly in x.

We substitute $z := S^{tr}z$, where S is given by (D.9) to get

$$\begin{aligned}(2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{-\frac{1}{2}z^{tr}Qz}e^{-ix\cdot z/\sqrt{t}}dz &= (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{-\frac{1}{2}\sum_{k=1}^d \alpha_k z_k^2}e^{-i(S^{tr}x)\cdot z/\sqrt{t}}dz \\ &= (2\pi)^{-d/2}\prod_{k=1}^d\int_{\mathbb{R}} e^{-\frac{1}{2}\alpha_k z_k^2}e^{-i(S^{tr}x)_k z_k/\sqrt{t}}dz_k.\end{aligned} \tag{D.22}$$

We observe

$$\begin{aligned}\int_{\mathbb{R}} e^{-cx^2-ibx}dx &= e^{-b^2/(4c)}\int_{\mathbb{R}} e^{-c(x+ib/(2c))^2}dx = e^{-b^2/(4c)}\int_{\mathbb{R}+ib/(2c)} e^{-cz^2}dz \\ &= e^{-b^2/(4c)}\int_{\mathbb{R}} e^{-cz^2}dz = e^{-b^2/(4c)}\sqrt{\pi/c}.\end{aligned} \tag{D.23}$$

We apply (D.23) to (D.22), hence

$$\begin{aligned}(2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{-\frac{1}{2}z^{tr}Qz}e^{-ix\cdot z/\sqrt{t}}dz &= (2\pi)^{-d/2}\prod_{k=1}^d \exp\left(-\frac{(S^{tr}x)_k^2}{2\alpha_k t}\right)\sqrt{\frac{2\pi}{\alpha_k}} \\ &= \exp\left(-\sum_{k=1}^d \frac{(S^{tr}x)_k^2}{2\alpha_k t}\right)\prod_{k=1}^d \alpha_k^{-1/2} \\ &= |Q|^{-\frac{1}{2}}e^{-\frac{1}{2t}x^{tr}Q^{-1}x}.\end{aligned} \tag{D.24}$$

This completes the proof. □

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