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Some families of increasing planar maps

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Abstract

Stack-triangulations appear as natural objects when one wants to define some families of increasing triangulations by successive additions of faces. We investigate the asymptotic behavior of rooted stack-triangulations with $2n$ faces under two different distributions. We show that the uniform distribution on this set of maps converges, for a topology of local convergence, to a distribution on the set of infinite maps. In the other hand, we show that rescaled by $n^{1/2}$, they converge for the Gromov-Hausdorff topology on metric spaces to the continuum random tree introduced by Aldous. Under a distribution induced by a natural random construction, the distance between random points rescaled by $(6/11)\log n$ converge to 1 in probability.
We obtain similar asymptotic results for a family of increasing quadrangulations.

Key words: stackmaps, triangulations, Gromov-Hausdorff convergence, continuum random tree.

AMS 2000 Subject Classification: Primary 60C05, 60F17.

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1 Introduction

Consider a rooted triangulation of the plane. Choose a finite triangular face ABC and add inside a new vertex O and the three edges AO, BO and CO. Starting at time 1 from a single rooted triangle, after k such evolutions, a triangulation with $2k+2$ faces is obtained. The set of triangulations $\triangle_{2k}$ with $2k$ faces that can be reached by this growing procedure is not the set of all rooted triangulations with $2k$ faces. The set $\triangle_{2k}$ – called the set of stack-triangulations with $2k$ faces – can be naturally endowed with two very different probability distributions:

- the first one, very natural for the combinatorial point of view, is the uniform distribution $\mathbb{U}^{\triangle}_{2k}$,
- the second probability $\mathbb{Q}^{\triangle}_{2k}$ maybe more realistic following the description given above, is the probability induced by the construction when the faces where the insertion of edges are done are chosen uniformly among the existing finite faces.

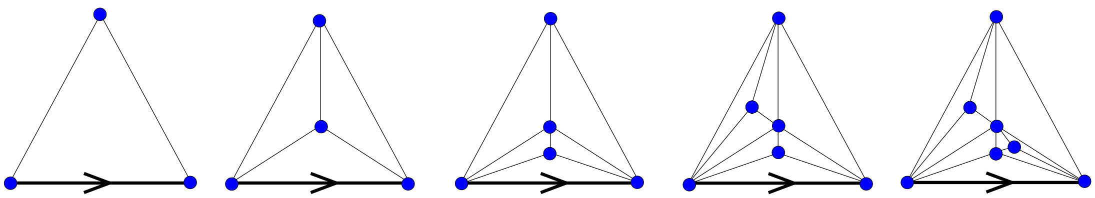

Figure 1: Iterative construction of a stack-triangulation. Note that three different histories lead to the final triangulation.

The aim of this paper is to study these models of random maps. Particularly, we are interested in large maps when the number of faces tends to $+\infty$. It turns out that this model of triangulations is combinatorially simpler that the set of all triangulations. Under the two probabilities $\mathbb{Q}^{\triangle}_{2k}$ and $\mathbb{U}^{\triangle}_{2k}$ we exhibit a *global limit behavior* of these maps.

A model of increasing quadrangulations is also treated at the end of the paper. In few words this model is as follows. Begin with the rooted square and successively choose a finite face $ABCD$, add inside a node O and two new edges: AO and OC (or BO and OD). When these two choices of pair of edges are allowed we get a model of quadrangulations that we were unable to treat as wanted (see Section 8.1). When only a suitable choice is possible, we get a model very similar to that of stack-triangulations that may be endowed also with two different natural probabilities. The results obtained are, up to the normalizing constants, the same as those obtained for stack-triangulations. For sake of briefness, only the case of stack-triangulations is treated in details.

We present below the content of the paper and a rough description of the results, the formal statements being given all along the paper.

1.1 Contents

In Section 2 we define formally the set of triangulations $\triangle_{2n}$ and the two probabilities $\mathbb{U}^{\triangle}_{2n}$ and $\mathbb{Q}^{\triangle}_{2n}$. This section contains also a bijection between $\triangle_{2n}$ and the set $\mathcal{T}^{\mathsf{ter}}_{3n-2}$ of ternary trees with

$3n-2$ nodes deeply used in the paper. In Section 3 are presented the two topologies considered in the paper:

- the first one is an ultra-metric topology called *topology of local convergence*. It aims to describe the limiting local behavior of a sequence of maps (or trees) around their roots,

- the second topology considered is the *Gromov-Hausdorff topology* on the set of compact metric spaces. It aims to describe the limiting behavior of maps (or trees) seen as metric spaces where the distance is the graph distance. The idea here is to normalize the distance in maps by the order of the diameter in order to observe a limiting behavior.

In Section 4.1 are recalled some facts concerning Galton-Watson trees conditioned by the size n, when the offspring distribution is $\nu_{\mathsf{ter}} = \frac{1}{3}\delta_3 + \frac{2}{3}\delta_0$ (the tree is ternary in this case). It is recalled (Section 4.2) that they converge under the topology of local convergence to an infinite branch, (the spine or infinite line of descent) on which are grafted some critical ternary Galton-Watson trees; rescaled by $n^{1/2}$ they converge for the Gromov-Hausdorff topology to the continuum random tree (CRT), introduced by Aldous [1] (Section 5.1).

Sections 4 and 5 are devoted to the statements and the proofs of the main results of the paper concerning random triangulations under $\mathbb{U}_{2n}^{\triangle}$, when $n \to +\infty$. The strongest theorems of these parts, that may also be considered as the strongest results of the entire paper, are:

- the weak convergence of $\mathbb{U}_{2n}^{\triangle}$ for the topology of local convergence to a measure on infinite triangulations (Theorem 12, Section 4),

- the convergence in distribution of the metric of stack-triangulations for the Gromov-Hausdorff topology (the distance being the graph distance divided by $\sqrt{6n}/11$) to the CRT (Theorem 15, Section 5). It is up to our knowledge, the only case where the convergence of the metric of a model of random maps is proved (apart from trees).

Section 7 is devoted to the study of $\triangle_{2n}$ under $\mathbb{Q}_{2n}^{\triangle}$. Under this distribution, there is no local convergence around the root, its degree going a.s. to $+\infty$. Theorem 21 says that seen as metric spaces they converge normalized by $(6/11)\log n$, in the sense of the finite dimensional distributions, to the discrete distance on $[0,1]$ (the distance between different points is 1). Hence, there is no weak convergence for the Gromov-Hausdorff topology, the space $[0,1]$ under the discrete distance being not compact. Section 7.2 contains some elements stating the speed of growing of the maps (the evolution of the node degrees, or the size of a sub-map).

Section 8 is devoted to the study of a model of quadrangulations very similar to that of stack-triangulations, and to some questions related to another family of growing quadrangulations.

Last, the Appendix, Section 9, contains the proofs that have been extracted from the text for sake of clarity.

1.2 Literature about stack-triangulations

The fact that stack-triangulations are in bijection with ternary trees, is well known, and will be proved in Section 1, using the idea of Darrasse and Soria [16].

Stack-triangulations appear in the literature for very various reasons. In Bernardi and Bonichon [7], stack-triangulations are shown to be in bijection with intervals in the Kreweras lattice (and realizers being both minimal and maximal). The set of stack triangulations coincides also with the set of plane triangulations having a unique Schnyder wood (see Felsner and Zickfeld [21]).

These triangulations appear also around the problem of graph uniquely 4-colorable. A graph G is uniquely 4-colorable if it can be colored with 4 colors, and if every 4-coloring of G produces the same partition of the vertex set into 4 color classes. There is an old conjecture saying that the family of maps having this property is the set of stack-triangulations. We send the interested reader to Böhme & al. [10] and references therein for more information on the question.

As illustrated on Figure 2, these triangulations appear also in relation with Apollonian circles. We refer to Graham & al. [25], and to several other works of the same authors, for remarkable properties of these circles.

The so-called Apollonian networks, are obtained from Apollonian space-filling circles packing. First, we consider the Apollonian space-filling circles packing. Start with three adjacent circles as on Figure 2. The hole between them is filled by the unique circle that touches all three, forming then three new smaller holes. The associated triangulations is obtained by adding an edge between the center of the new circle C and the three centers of the circles tangent to C. If each time a unique hole receives a circle, the set of triangulation that may be obtained is the set of stack-triangulations. If each hole received a circle altogether at the same time, we get the model of Apollonian networks. We refer to Andrade & al. [3] and references therein for some properties of this model of networks.

The random Apollonian model of network studied by Zhou & al. [47], Zhang & al. [45], and Zhang & al. [46] (when their parameters d is 2) coincides with our model of stack-triangulations under $\mathbb{Q}^{\triangle}$. Using physicist methodology and simulations they study among others the degree distribution (which is seen to respect a power-law) and the distance between two points taken at random (that is seen to be around $\log n$).

Darrasse and Soria [16] obtained the degree distribution on a model of "Boltzmann" stacked triangulations, where this time, the size of the quadrangulations is random, and uniformly distributed conditionally to its size. Bodini, Darrasse and Soria [9], computed the limiting distribution (and the moment convergence) of the distance of a random node to the root, and between two random nodes under $\mathbb{U}_{2n}^{\triangle}$ (these results are obtained with a method absolutely different to those involved to prove Theorem 15). Their results is in accordance with Theorem 15.

We end the introduction by reviewing the known asymptotic behaviors of quadrangulations and triangulations with n faces under the uniform distribution (or close distributions in some sense).

1.3 Literature about convergence of maps

We refer to Angel & Schramm [5], Chassaing & Schaeffer [14] Bouttier & al. [11] for an overview of the history of the study of maps from the combinatorial point of view, and to the references therein for the link with the 2-dimensional quantum gravity of physicists. We here focus on the main results concerning the convergence of maps. We exclude the results concerning trees (which are indeed also planar maps).

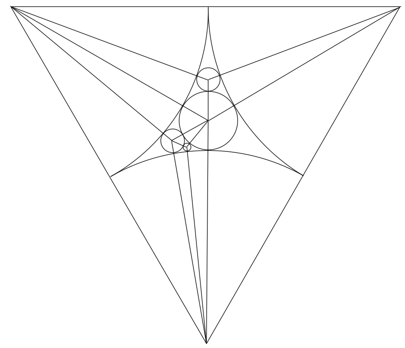

Figure 2: Construction of Apollonian's circles by successive insertions of circles (the starting point is three tangent circles). To get the triangulation associated, add an edge between the center of the new circle C and the three centers of the circles tangent to C.

In the very last years, many studies concerning the behavior of large maps have been published. The aim in these works was mainly to define or to approach a notion of limiting map. Appeared then two different points of view, two different topologies to measure this convergence.

Angel & Schramm [5] showed that the uniform distribution on the set of rooted triangulations with n faces (in fact several models of triangulations are investigated) converges weakly for a topology of local convergence (see Section 3.1) to a distribution on the set of infinite but locally finite triangulations. In other words, for any r, the sub-map $S_r(n)$ obtained by keeping only the nodes and edges at distance smaller or equal to r from the root vertex, converges in distribution toward a limiting random map S_r. By a theorem of Kolmogorov this allows to show the convergence of the uniform measure on triangulations with n faces to a measure on the set of infinite but locally finite rooted triangulations (see also Krikun [28] for a simple description of this measure). Chassaing & Durhuus [13] obtained then a similar result, with a totally different approach, on uniform rooted quadrangulations with n faces.

The second family of results concerns the convergence of rescaled maps: the first one in this direction has been obtained by Chassaing & Schaeffer [14] who studied the limiting profile of quadrangulations. The (cumulative) profile $(\mathrm{Prof}(k), k \geq 0)$ of a rooted graph, defined in Section 5, gives the successive number of nodes at distance smaller than k from the root. Chassaing & Schaeffer [14, Corollary 4] showed that

$$\left(\frac{\mathrm{Prof}((8n/9)^{1/4}x)}{n}\right)_{x\geq 0} \to (\mathcal{J}[m, m+x])_{x\geq 0}$$

where the convergence holds weakly in $D([0,+\infty),\mathbb{R})$. The random probability measure $\mathcal{J}$ is ISE the Integrated super Brownian excursion. ISE is the (random) occupation measure of the Brownian snake with lifetime process the normalized Brownian excursion, and m is the minimum of the support of $\mathcal{J}$. The radius, i.e. the largest distance to the root, is also shown to converge, divided by $(8n/9)^{1/4}$, to the range of ISE. Then,
– Marckert & Mokkadem [39] showed the same result for pointed quadrangulations with n faces,
– Marckert & Miermont [37] showed that up to a normalizing constant, the same asymptotic holds for pointed rooted bipartite maps under Boltzmann distribution with n faces, (the weight

of a bipartite map is $\prod_{f \text{ face of m}} w_{\deg(f)}$ where the $(w_{2i})_{i\geq 0}$ is a "critical sequence of weight"),
– Weill [44] obtained the same results as those of [37] in the rooted case,
– Miermont [40] provided the same asymptotics for rooted pointed Boltzmann maps with n faces with no restriction on the degree,
– Weill and Miermont [41] obtained the same result as [40] in the rooted case.

All these results imply that if one wants to find a (finite and non trivial) limiting object for rescaled maps, the edge-length in maps with n faces has to be fixed to $n^{-1/4}$ instead of 1. In Marckert & Mokkadem [39], quadrangulations are shown to be obtained as the gluing of two trees, thanks to the Schaeffer's bijection (see [43; 14; 39]) between quadrangulations and well labeled trees. They introduce also a notion of random compact continuous map, "the Brownian map", a random metric space candidate to be the limit of rescaled quadrangulations. In [39] the convergence of rescaled quadrangulations to the Brownian map is shown but not for a "nice topology". As a matter of fact, the convergence in [39] is a convergence of the pair of trees that encodes the quadrangulations to a pair of random continuous trees, that also encodes, in a sense similar to the discrete case, a continuous object that they name the Brownian map. "Unfortunately" this convergence does not imply – at least not in an evident way – the convergence of the rescaled quadrangulations viewed as metric spaces to the Brownian map for the Gromov-Hausdorff topology.

Some authors think that the Brownian map is indeed the limit, after rescaling, of classical families of maps (those studied in [14; 39; 37; 44; 40; 41]) for the Gromov-Hausdorff topology. Evidences in this direction have been obtained by Le Gall [31] who proved the following result. He considers M_n a $2p$-angulations with n faces under the uniform law. Then, he shows that at least along a suitable subsequence, the metric space consisting of the set of vertices of M_n, equipped with the graph distance rescaled by the factor $n^{1/4}$, converges in distribution as $n \to \infty$ towards a limiting random compact metric space, in the sense of the Gromov-Hausdorff distance. He proved that the topology of the limiting space is uniquely determined independently of p and of the subsequence, and that this space can be obtained as the quotient of the CRT for an equivalence relation which is defined from Brownian labels attached to the vertices. Then Le Gall & Paulin [32] show that this limiting space is topologically a sphere. The description of the limiting space is a little bit different from the Brownian map but one may conjecture that these two spaces are identical.

Before coming back to our models and results we would like to stress on two points.
• The topology of local convergence (on non rescaled maps) and the Gromov-Hausdorff topology (on rescaled map) are somehow orthogonal topologies. The Gromov-Hausdorff topology considers only what is at the scaling size (the diameter, the distance between random points, but not the degree of the nodes for example). The topology of local convergence considers only what is at a finite distance from the root. In particular, it does not measure at all the phenomenons that are at the right scaling factor, if this scaling goes to $+\infty$. This entails that in principle one may not deduce any non-trivial limiting behavior for the Gromov-Hausdorff topology from the topology of local convergence, and vice versa.
• There is a conjecture saying that properly rescaled random planar maps conditioned to be large should converge to a limiting continuous random surface, whose law should not depend up to scaling constant from the family of reasonable maps that are sample. This conjecture still holds even if the family of stack-maps studied here converges to some objects that can not be the limit of uniform quadrangulations. The reason is that stack-maps are in some sense not

reasonable maps.

2 Stack-triangulations

2.1 Planar maps

A planar map m is a proper embedding without edge crossing of a connected graph in the sphere. Two planar maps are identical if one of them can be mapped to the other by a homeomorphism that preserves the orientation of the sphere. A planar map is a quadrangulation if all its faces have degree four, and a triangulation if all its faces have degree three. There is a difference between the notions of planar maps and planar graphs, a planar graph having possibly several non-homeomorphic embeddings on the sphere.

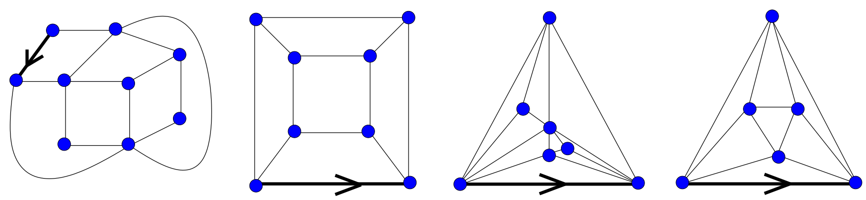

Figure 3: Two rooted quadrangulations and two rooted triangulations.

In this paper we deal with rooted planar maps (m, E): an oriented edge $E = (E_0, E_1)$ of m is distinguished. The point E_0 is called the root vertex of m. Two rooted maps are identical if the homeomorphism preserves also the distinguished oriented edge. Rooting maps like this allows to avoid non-trivial automorphisms. By a simple projection, rooted planar maps on the sphere are in one to one correspondence with rooted planar maps on the plane, where the root of the latter is adjacent to the infinite face (the unbounded face) and is oriented in such a way that the infinite face lies on its right, as on Figure 3. From now on, we work on the plane.

For any map m, we denote by $V(m), E(m), F(m), F^\circ(m)$ the sets of vertices, edges, faces and finite faces of m; for any v in $V(m)$, we denote by $\deg(v)$ the degree of v. The graph distance d_G between two vertices of a graph G is the number of edges in a shortest path connecting them. The set of nodes of a map m equipped with the graph distance denoted by d_m is naturally a metric space. The study of the asymptotic behavior of (m, d_m) under various distributions is the main aim

2.2 The stack-triangulations

We build here $\triangle_{2k}$ the set of *stack-triangulations* with $2k$ faces, for any $k \geq 1$.

Set first $\triangle_2 = \{\Theta\}$ where Θ denotes the unique rooted triangle (the first map in Figure 1). Assume that $\triangle_{2k}$ is defined for some $k \geq 1$ and is a set of rooted triangulations with $2k$ faces. We now define $\triangle_{2(k+1)}$. Let

$$\triangle^\bullet_{2k} = \{(m, f) \mid m \in \triangle_{2k}, f \in F^\circ(m)\}$$

be the set of rooted triangulations from $\triangle_{2k}$ with a distinguished finite face. We now introduce an application Φ from $\triangle^{\bullet}_{2k}$ onto the set of all rooted triangulations with $2(k+1)$ faces (we should write Φ_k). For any $(m,f) \in \triangle^{\bullet}_{2k}$, $\Phi(m,f)$ is the following rooted triangulation: draw m in the plane, add a point x inside the face f and three non-crossing edges inside f between x and the three vertices of f adjacent to x (see Figure 4). The obtained map has $2k+2$ faces.

We call $\triangle_{2(k+1)} = \Phi(\triangle^{\bullet}_{2k})$ the image of this application.

On Figure 3, the first triangulation is in $\triangle_{10}$ (see also Figure 1). The second one is not in $\triangle_8$ since it has no internal node having degree 3.

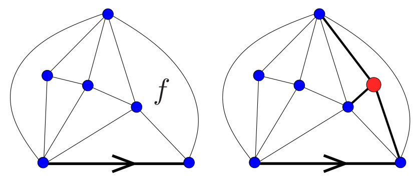

Figure 4: A triangulation (m,f) with a distinguished face and its image by Φ.

Definition 1. *We call internal vertex of a stack-triangulation m every vertex of m that is not adjacent to the infinite face (all the nodes but three).*
We call history of a stack-triangulation $m_k \in \triangle_{2k}$ any sequence $\big((m_i, f_i), i = 1, \dots, k-1\big)$ such that $m_i \in \triangle_{2i}$, $f_i \in F^{\circ}(m_i)$ and $m_{i+1} = \Phi(m_i, f_i)$. We let $\mathcal{H}(m)$ be the set of histories of m, and $H_{\triangle}(k) = \{\mathcal{H}(m) \mid m \in \triangle_{2k}\}$.

We define here a special drawing $\mathcal{G}(m)$ of a stack-triangulation m. The embedding $\mathcal{G}(\Theta)$ of the unique rooted triangle Θ is fixed at position $E_0 = (0,0)$, $E_1 = (1,0)$, $E_2 = e^{i\pi/3}$ (where E_0, E_1, E_2 are the three vertices of Θ, and (E_0, E_1) its root). The drawing of its edges are straight lines drawn in the plane. To draw $\mathcal{G}(m)$ from $\mathcal{G}(m')$ when $m = \Phi(m', f')$, add a point x in the center of mass of f', and three straight lines between x and the three vertices of f' adjacent to x. The faces of $\mathcal{G}(m)$ hence obtained are geometrical triangles. Presented like this, $\mathcal{G}(m)$ seems to depend on the history of m used in its construction, and thus we should have written $\mathcal{G}_h(m)$ instead of $\mathcal{G}(m)$, where the index h would have stood for the history h used. But it is easy to check (see Proposition 2) that if h, h' are both in $\mathcal{H}(m)$ then $\mathcal{G}_{h'}(m) = \mathcal{G}_h(m)$.

Definition 2. *The drawing $\mathcal{G}(m)$ is called the canonical drawing of m.*

2.2.1 Two distributions on $\triangle_{2k}$

For any $k \geq 1$, we denote by $\mathbb{U}^{\triangle}_{2k}$ the uniform distribution on $\triangle_{2k}$.
We now define a second probability $\mathbb{Q}^{\triangle}_{2k}$. For this, we construct on a probability space $(\Omega, \mathbb{P})$ a process $(M_n)_{n\geq 1}$ such that M_n takes its values in $\triangle_{2n}$ as follows: first M_1 is the rooted triangle Θ. At time $k+1$, choose a finite face F_k of M_k uniformly among the finite faces of M_k and this independently from the previous choices and set

$$M_{k+1} = \Phi(M_k, F_k).$$

We denote by $\mathbb{Q}^{\triangle}_{2k}$ the distribution of M_k. Its support is exactly $\triangle_{2k}$.

The weight of a map under $\mathbb{Q}_{2k}^{\triangle}$ being proportional to its number of histories, it is easy to check that $\mathbb{Q}_{2k}^{\triangle} \neq \mathbb{U}_{2k}^{\triangle}$ for $k \geq 4$.

2.3 Combinatorial facts

We begin this section where is presented the bijection between ternary trees and stack-triangulations with some considerations about trees.

2.3.1 Definition of trees

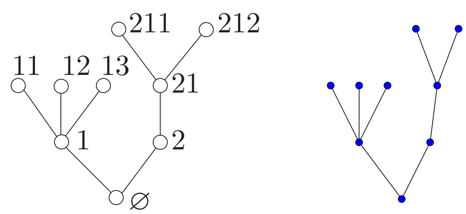

Figure 5: A rooted tree and its usual representation on the plane.

Consider the set $W = \bigcup_{n\geq 0} \mathbb{N}^n$ of finite words on the alphabet $\mathbb{N} = \{1, 2, 3, \dots\}$ where by convention $\mathbb{N}^0 = \{\varnothing\}$. For $u = u_1 \dots u_n, v = v_1 \dots v_m \in W$, we let $uv = u_1 \dots u_n v_1 \dots v_m$ be the concatenation of the words u and v.

For $m_1, \dots, m_p \in \mathbb{N}$, we let $\{m_1, \dots, m_p\}^\star = \cup_{n\geq 0} \{m_1, \dots, m_p\}^n$ be the set of finite words with letters $m_1, \dots, m_p$.

Definition 3. *A planar tree* $\mathbf{t}$ *is a subset of* W
- *containing the root-vertex* $\varnothing$*,*
- *such that if* $ui \in \mathbf{t}$ *for some* $u \in W$ *and* $i \in \mathbb{N}$*, then* $u \in \mathbf{t}$*,*
- *and such that if* $ui \in \mathbf{t}$ *for some* $u \in W$ *and* $i \in \mathbb{N}$*, then* $uj \in \mathbf{t}$ *for all* $j \in \{1, \dots, i\}$*.*

We denote by $\mathcal{T}$ the set of planar trees. For any $u \in t$, let $c_u(t) = \max\{i \mid ui \in t\}$ be the number of *children* of u. The elements of a tree t are called *nodes*, a node having no child a *leaf*, the other nodes the *internal nodes*. The set of leaves of t will be denoted by ∂t, and its set of internal nodes by t°. The number of nodes of a tree t is denoted by $|t|$.

A binary (resp. ternary) tree t is a planar tree such that $c_u(t) \in \{0, 2\}$ (resp. $c_u(t) \in \{0, 3\}$) for any $u \in t$. We denote by $\mathcal{T}^{\mathsf{bin}}$ and $\mathcal{T}^{\mathsf{ter}}$ the set of finite or infinite binary and ternary trees, and by $\mathcal{T}_n^{\mathsf{bin}}$ and $\mathcal{T}_n^{\mathsf{ter}}$ the corresponding set of trees with n nodes.

If u and v are two nodes in t, we denote by $u \wedge v$ the *deepest common ancestor* of u and v, i.e. the largest word w prefix to both u and v (the node $u \wedge v$ is in t). The length $|u|$ of a word $u \in W$ is called the height or depth of u, or graph distance of u to the root, if considered as a vertex of some tree. For $u = u_1 \dots u_n \in t$, we let $u[j] = u_1 \dots u_j$ and $[[\varnothing, u]] = \{\varnothing, u[1], \dots, u[n]\}$ be the ancestral line of u back to the root. For any tree t and u in t, the *fringe subtree* $t_u := \{w \mid uw \in t\}$ is in some sense, the subtree of t rooted in u. Finally recall that the lexicographical order (LO) on W induces a total ordering of the nodes of any tree.

We now give a formalism to describe the growth of trees. We denote by $\mathcal{T}^{\mathsf{ter}\bullet}_{3n+1} := \{(t,u) \mid t \in \mathcal{T}^{\mathsf{ter}}_{3n+1}, u \in \partial t\}$ the set of ternary trees with $3n+1$ nodes with a distinguished leaf. Very similarly with the function Φ defined in Section 2.2, we define the application ϕ from $\mathcal{T}^{\mathsf{ter}\bullet}_{3k+1}$ into $\mathcal{T}^{\mathsf{ter}}_{3k+4}$ as follows; for any $(t,u) \in \mathcal{T}^{\mathsf{ter}\bullet}_{3k+1}$, let $t' := \phi(t,u)$ be the tree $t \cup \{u1, u2, u3\}$ obtained from t by the replacement of the leaf u by an internal node having 3 children.

Definition 4. *As for maps (see Definition 1), for any tree $t \in \mathcal{T}^{\mathsf{ter}}_{3k-2}$, a history of a tree t is a sequence $h' = \big((t_i,u_i), i = 1,\dots,k-1\big)$ such that $(t_i,u_i) \in \mathcal{T}^{\mathsf{ter}\bullet}_{3i-2}$ and $t_{i+1} = \phi(t_i,u_i)$. The set of histories of t is denoted by $\mathcal{H}(t)$, and we denote $H_{\mathcal{T}}(k) = \{\mathcal{H}(t) \mid t \in \mathcal{T}^{\mathsf{ter}}_{3k-2}\}$.*

2.3.2 The fundamental bijection between stack-triangulations and ternary trees

Before explaining the bijection we use between $\triangle_{2K}$ and $\mathcal{T}^{\mathsf{ter}}_{3K-2}$ we define a function Λ which will play an eminent role in our asymptotic results concerning the metrics in maps. Let $W_{1,2,3}$ be the set of words containing at least one occurrence of each element of $\Sigma_3 = \{1,2,3\}$ as for example 321, 123, 113211213123. Let $u = u_1 \dots u_k$ be a word on the alphabet Σ_3. Define $\tau_1(u) := 0$ and $\tau_2(u) := \inf\{i \mid i > 0, u_i = 1\}$, the rank of the first apparition of 1 in u. For $j \geq 3$, define

$$\tau_j(u) := \inf\{i \mid i > \tau_{j-1}(u) \text{ such that } u_{1+\tau_{j-1}(u)} \dots u_i \in W_{1,2,3}\}.$$

This amounts to decomposing u into subwords, the first one ending when the first 1 appears, the subsequent ones ending each time that each of the three letters 1, 2 and 3 has appeared again. For example if $u = 22123122131$ then $\tau_1(u) = 0, \tau_2(u) = 3, \tau_3(u) = 6, \tau_4(u) = 10$. Denote by

$$\Lambda(u) = \max\{i \mid \tau_i(u) \leq |u|\} \tag{1}$$

the number of these non-overlapping subwords. Further for two words (or nodes) $u = wa_1 \dots a_k$ and $v = wb_1 \dots b_l$ with $a_1 \neq b_1$, (in this case $w = u \wedge v$,) set

$$\Lambda(u,v) = \Lambda(a_1 \dots a_k) + \Lambda(b_1 \dots b_l). \tag{2}$$

We call the one or two parameters function Λ the *passage function*.

We now describe a bijection $\Psi^{\triangle}_K$ between $\triangle_{2K}$ and $\mathcal{T}^{\mathsf{ter}}_{3K-2}$ having a lot of important properties. This bijection is inspired from Darrasse & Soria [16].

Proposition 1. *For any $K \geq 1$ there exists a bijection*

$$\begin{array}{rccl} \Psi^{\triangle}_K : & \triangle_{2K} & \longrightarrow & \mathcal{T}^{\mathsf{ter}}_{3K-2} \\ & m & \longmapsto & t := \Psi^{\triangle}_K(m) \end{array}$$

such that:
(a) Each internal node u of m corresponds bijectively to an internal node v of t. We denote for sake of simplicity by u' the image of u.
(b) Each leaf of t corresponds bijectively to a finite triangular face of m.
(c) For any u internal node of m, $\Lambda(u') = d_m(E_0,u)$.
(d) For any u and v internal nodes of m

$$\big|d_m(u,v) - \Lambda(u',v')\big| \leq 4. \tag{3}$$

(e) Let u be an internal node of m. We have

$$\deg_m(u) = 3 + \#\{v' \in t^\circ \mid v' = u'w', w' \in 1\,\{2,3\}^\star \cup 3\,\{1,2\}^\star \cup 2\,\{1,3\}^\star\}, \qquad (4)$$

where the set in (4) is the union of the subtrees of t° rooted in $u'1$, $u'2$ and $u'3$ formed by the "binary trees" having no nodes containing a 1, resp. a 2, resp a 3.

We will write $\Psi^\triangle$ instead of $\Psi_K^\triangle$ when no confusion on K is possible.

Property (e) in Proposition 1 is given in Darrasse & Soria [16], where it is used to derive the asymptotic degree distribution of a random node under a Boltzmann distribution (see Section 6). We give below a complete proof of Proposition 1. The quotes around binary trees signal that by construction these branching structures do not satisfy the requirements of Definition 3.

The existence of a bijection between $\triangle_{2K}$ and $\mathcal{T}^{\mathsf{ter}}_{3K-2}$ is well known and is a simple consequence of the ternary decomposition of the maps in $\triangle_{2K}$, as illustrated on Figure 6: in the first step of the construction of m, the insertion of the three first edges incident to the node x in the triangle Θ splits it into three parts that behave clearly as stack-triangulations. The node x may be recovered at any time since it is the unique vertex incident to the three vertices incident to the infinite face. The bijection induced by this decomposition (this is illustrated on Figure 6)

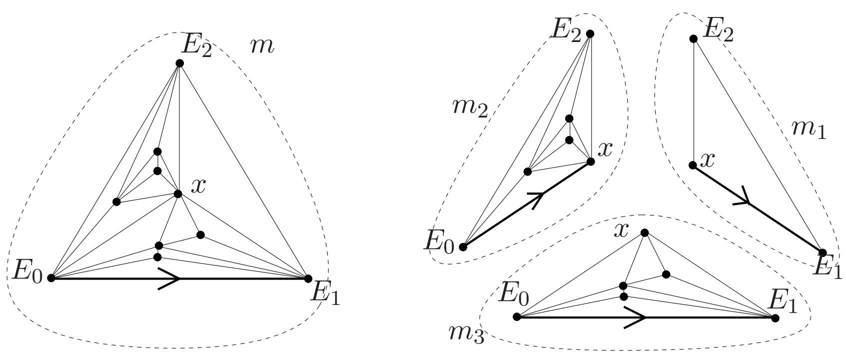

Figure 6: Decomposition of a stack-triangulation using the recovering of the first inserted node.

can be defined in order to encode the distance between the nodes in the maps, and then to get the properties announced in Proposition 1. This construction, presented below, is inspired by Darrasse & Soria [16].

The proof of Proposition 1 we propose raises on an iterative argument, and then will raise on the notion of histories. Since a stack-triangulation generally owns several histories, we need to show some consistence properties of the construction, more or less intuitively clear. The consistence needed relies on an association between the triangular faces of the canonical drawing introduced in Definition 2 and the words on Σ_3: thanks to the canonical drawing, there is a sense to talk of a face f without referring to a map, and thanks to our construction of trees, there is a sense to talk of a node u – which is a word – without referring to a tree. We will call *canonical face* a geometrical face corresponding to a canonical drawing, given together with an oriented edge: the notation $f = (A, B, C)$ will refer to the canonical face f admitting (A, B) as oriented edge.

Let us now design a bijection $\psi^{\triangle}$ which associates a word in Σ_3 with each canonical face. The image by $\psi^{\triangle}$ of the unique canonical face (E_0, E_1, E_2) of the unique rooted triangle Θ on $\triangle_2$ is $\varnothing$, the empty word on Σ_3. We now proceed by induction and consider $\mathcal{F}_K$ the set of canonical faces belonging to at least one of the canonical drawings of a map of $\triangle_{2K}$. Assume by induction that for any face f in $\mathcal{F}_K$, $\psi^{\triangle}(f)$ is well defined and is a word of Σ_3. Assume also that there is only one canonical face associated with a geometrical face: if (x_1, x_2, x_3) and (y_1, y_2, y_3) are elements of $\mathcal{F}_K$ associated with the same geometrical face, then $(x_1, x_2, x_3) = (y_1, y_2, y_3)$.

Let $f = (A, B, C)$ be a canonical face belonging to $\mathcal{F}_K$. The growing of a map having f as a face, in the face f, is as explained above, obtained by inserting a node x in f and three edges between x and the nodes A, B and C. The three "new" canonical faces are set to be (B, C, x), (A, x, C), (A, B, x) (this fixes the respected oriented edges, that are chosen in such a way that the infinite face lies on the right of each of these new faces seen as maps, and then allow a successive decomposition, see Figure 7). If the image of f by $\psi^{\triangle}$ is u, we associate respectively with the three "new" faces the nodes $u1$, $u2$ and $u3$. The quotes around "new" signal that a face in $\mathcal{F}_K$ may belong also in some $\mathcal{F}_j$ for $j < K$, and then these new faces may "already" belong to $\mathcal{F}_K$. Since the procedure of construction of the faces does not depend on the time K, the association of $u1, u2$ and $u3$ with the new faces is consistent in time. One now can check easily that $\psi^{\triangle}$ is now defined for any face of $\mathcal{F}_{K+1}$, and that the properties assumed on $\mathcal{F}_K$ are inherited in $\mathcal{F}_{K+1}$.

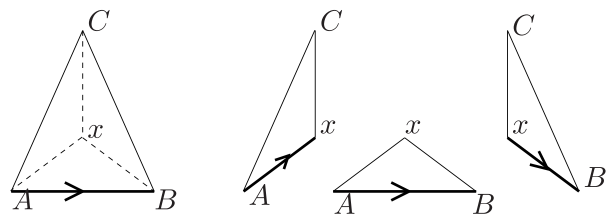

Figure 7: Heritage of the canonical orientation of the faces. If the first face is sent on u, then the other ones, from left to right are sent on $u2, u3$ and $u1$

Now, the bijection $\psi^{\triangle}$ induces a bijection $\psi^{\triangle}_K$ between the set $H_{\triangle}(K)$ of histories of the maps of $\triangle_{2K}$ and $H_{\mathcal{T}}(K)$ the set of histories of the trees of $\mathcal{T}^{\text{ter}}_{3K-2}$ (for any $K \geq 1$). More precisely, the application $\psi^{\triangle}_K$ is defined as follows. The history of the unique stack-triangulation with 4 faces is $(\Theta, (E_0, E_1, E_2))$, and we fix its image to be $(\{\varnothing\}, \varnothing)$ the tree reduced to the root vertex, marked on this node (which is a leaf). Let now $K \geq 3$ be fixed and let $h_K = \big((m_i, f_i), i = 1, \dots, K-1\big)$ be a history of a triangulation m_K of $\triangle_{2K}$. Recall the content of Section 2.2. In particular we have $m_K = \Phi(m_{K-1}, f_{K-1})$.

By induction assume that a tree-history $h'_{K-1} := \big((t_i, u_i), i = 1, \dots, K-2\big)$ is associated with $h_{K-1} := \big((m_i, f_i), i = 1, \dots, K-2\big)$ by $\psi^{\triangle}_{K-1}$. Particularly, we assume by induction that for any $i \leq K-2$, u_i is given by $\psi^{\triangle}(f_i)$ (that is the node marked in t_i corresponds to the face marked in m_i). More globally, thinking to the construction induced by the history, this implies

$$t^{\circ}_{K-1} = \{\psi^{\triangle}(f_i) \mid 1 \leq i \leq K-2\}.$$

To define h'_K, we let

$$\begin{cases} u_{K-1} = \psi^{\triangle}(f_{K-1}) \\ t_{K-1} = t_{K-2} \cup \{\psi^{\triangle}(f_{K-2})1, \psi^{\triangle}(f_{K-2})2, \psi^{\triangle}(f_{K-2})3\} \end{cases}.$$

Since $u_{K-1} = \psi^{\triangle}(f_{K-1})$ is a leaf of t_{K-2}, t_{K-1} is indeed a tree, and also u_{K-1} is a leaf of t_{K-1}; hence $h'_K := \big((t_i, u_i), i = 1, \dots, K-1\big)$ is a history of a tree, say t_K. We then set:

$$\psi_K^{\triangle}(h_K) = h'_K. \tag{5}$$

This ends the induction. It turns out that $\psi_K^{\triangle}$ is a bijection, as stated in the next Lemma. Before stating it, we introduce a notation: if $h_K = \big((m_i, f_i), i = 1, \dots, K-1\big)$ is a history of m_K then for any $j < K$, we let h_j be $\big((m_i, f_i), i = 1, \dots, j-1\big)$ the history restricted to the $j-1$ first steps: h_j is the history of a map denoted by m_j; accordingly, we do the same for tree-histories.

Lemma 2. *For any $K \geq 1$, $\psi_K^{\triangle}$ is a bijection between $H_{\triangle}(K)$ and $H_{\mathcal{T}}(K)$ such that:*
(i) The family $(\psi_K^{\triangle}, K \geq 1)$ is consistent: if $\psi_K^{\triangle}(h_K) = h'_K$ then for any $j < K$,

$$\psi_j^{\triangle}(h_j) = h'_j.$$

(ii) Robustness: $h(1)$ and $h(2)$ are two histories of m iff $\psi_K^{\triangle}(h(1)) = \psi_K^{\triangle}(h(2))$ are histories of the same tree t.

This Lemma follows easily the construction of $\psi^{\triangle}$. Now, the point (ii) of this Lemma allows to build an application $\Psi_K^{\triangle,\star} : \triangle_{2K} \to \mathcal{T}_{3K-2}^{\mathsf{ter}}$ by associating m_K with t_K (this bijection $\Psi_K^{\triangle,\star}$ has all the nice properties announced in Proposition 1).

Note 1. *We may rephrase what we have done: take any history of a given map m, and construct iteratively the corresponding ternary tree using $\psi^{\triangle}$. The last tree obtained does not depend on the history chosen, but only on m.*

Lemma 3. *Let $\Psi_K^{\triangle} : \triangle_{2K} \to \mathcal{T}_{3K-2}^{\mathsf{ter}}$ the application defined above.*
i) $\Psi_K^{\triangle,\star}$ is a bijection;
ii) Let $t = \Psi_K^{\triangle,\star}(m)$. The properties assertion (a), (b), (c), (d), (e) of Proposition (1) holds true.

Proof. The application $\Psi_K^{\triangle,\star}$ is a bijection thanks to the previous Lemma (ii). To prove (ii) of the present Lemma we introduce the notion of type of a face, and of a node (of a word on Σ_3). For any face (u, v, w) in m, define

$$\text{type}(u, v, w) := \left(d_m(E_0, u), d_m(E_0, v), d_m(E_0, w)\right), \tag{6}$$

the distance of u, v, w to the root-vertex of m. Since u, v, and w are neighbors, the type of any triangle is (i, i, i), $(i, i, i+1)$, $(i, i+1, i+1)$ for some i, or a permutation of this.

We then prolong the construction of Φ given above, and mark the nodes of t with the types of the corresponding faces. For any internal node $u' \in t$ with $\text{type}(u') = (i, j, k)$,

$$\begin{cases} \text{type}(u'1) = (& 1 + i \wedge j \wedge k, & j, & k &), \\ \text{type}(u'2) = (& i, & 1 + i \wedge j \wedge k, & k &), \\ \text{type}(u'3) = (& i, & j, & 1 + i \wedge j \wedge k &) \end{cases} \tag{7}$$

as one can easily check with a simple figure: this corresponds as said above to the fact that if the leaf u is associated with the “empty” triangle (A, B, C), then the insertion of a node x in

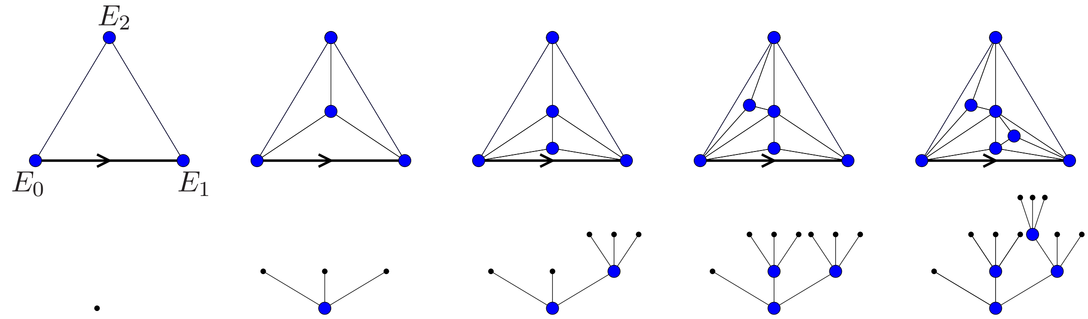

Figure 8: Construction of the ternary tree associated with an history of a stack-triangulation

(A,B,C) is translated by the insertion in the tree of the nodes $u1$ (resp. $u2$, $u3$) associated with (x,B,C) (resp. (A,x,B), (A,B,x)). Formula (7) gives then the types of these three faces. Using that type$(\varnothing) = (0,1,1)$, giving t the types of all nodes are known and are obtained via the deterministic evolution rules (7). The distance of any internal node u to the root of m is computed as follows: assume that u has been inserted at a certain date in a face $f = (A,B,C)$. Then clearly its distance to the root vertex is

$$d_m(E_0, u) = g(\mathrm{type}(f)),$$

where $g(i,j,k) = 1 + (i \wedge j \wedge k)$. Moreover, since an internal node in m corresponds to the insertion of three children in the tree, each internal node u of m corresponds to an internal node u' of t and

$$d_m(E_0, u) = g(\mathrm{type}(u')).$$

It remains to check that for any $u' \in t$,

$$g(\mathrm{type}(u')) = \Lambda(u') \tag{8}$$

as defined above. This is a simple exercise: the initial type (that of $\varnothing$) varies along a branch of t only when a 1 occurs in the nodes. Then the type passes from (i,i,i) to $(i+1,i+1,i+1)$ when the three letters 1, 2 and 3 has appeared: this corresponds to the incrementation of the distance to the root in the triangulation. This leads to (c).

Note 2. *The notion of type of a face f is canonical as we saw, when we proved that it is a function of the ancestors of $u = \psi^{\triangle}(f)$. Showing this property directly seems a bit ugly.*

(d) Consider u and v two internal nodes of m. The node $w' = u' \wedge v'$ corresponds to the smallest canonical face $f = (A,B,C)$ containing u and v. Assume $u' = w'1\ldots$ and $v' = w'2\ldots$, then u and v belong respectively to the canonical faces (w,B,C) and (A,w,C). Therefore there exists $x \in \{w,A,B,C\}$ such that $d_m(u,v) = d_m(u,x) + d_m(v,x)$ which leads directly to $|d_m(u,v) - (d_m(u,w) + d_m(v,w))| \le 2$. The remaining cases are treated similarly. Let us investigate now the relation between w and u and $\Lambda(a_1 \ldots a_j)$ in the case where $u = wa_1 \ldots a_j$. Each triangle appearing in the construction of m behaves as a copy of m except that its type is not necessarily $(i,i{+}1,i{+}1)$ (as was the type of $\varnothing$). Then the distance of the node $u = wa_1 \ldots a_j$ to w may be not exactly $\Lambda(a_1 \ldots a_j)$. We now show that

$$|d_m(w,u) - \Lambda(a_1 \ldots a_j)| \le 1.$$

This difference comes from the initialization of the counting of the non-overlapping subwords from $W_{1,2,3}$ in $a_1 \dots a_j$. This counting has to begin when a face of type (i, i, i) has been reached. Since we no longer consider the distance between u and the root but between u and w, the definition of the type has to be slightly modified. Let $u = (A, B, C)$ be the canonical face associated to $w'a_1 \dots a_j$, we define $\mathrm{type}_w(u)$ as $(d_m(A, w), d_m(B, w), d_m(C, w))$.

Let $a = a_1 \dots a_k$ be a word on Σ_3. Define $\tau_1'(a) := 1$, $\tau_2'(a) := \inf\{i \,|\, i > 1, a_i = a_1$, the rank of the second apparition of a_1 in a and $\tau_j'(a) = \tau_j(a)$ for $j \geq 3$ (the definition of τ_j is given in Equation (39)). Lastly we set

$$\Lambda_w(a) = \max\{i \,|\, \tau_i(a) \leq |a|\}.$$

Let $u = wa_1 \dots a_j$, it is clear that $d_m(u, w) = \Lambda_w(a_1 \dots a_j)$ (see the proof of Property (c) of Proposition 1). Furthermore for any word a on the alphabet Σ_3,

$$|\Lambda(z) - \Lambda_w(z)| \leq 1,$$

which concludes the proof.

(e) Let u be an internal vertex of m and let $f = (A, B, C)$ be the canonical face containing u. Let v be a vertex of m. Then $d_m(u, v) = 1$ if $v = A, B, C$ or if $v' = u'a_1 \dots a_j$ for a certain $a_1 \dots a_j \in \mathcal{W}_3$. Assume $a_1 = 1$. Then $(\psi^{\triangle})^{-1}(u'a_1) = (u, B, C)$. Now $d_m(u, v) = 1$ if and only if u is an adjacent vertex to $(\psi^{\triangle})^{-1}(u'a_1 \dots a_j)$. Furthermore, such a face is of the form (u, B, y), (u, x, y) or (u, x, C) meaning that $a_2 \dots a_j \in \{2, 3\}^\star$ (which can be done by induction). The two remaining cases $a_1 = 2$ and $a_1 = 3$ are done in the same way. $\square$

2.4 Induced distribution on the set of ternary trees

The bijection $\Psi_K^{\triangle}$ transports the distributions $\mathbb{U}_{2K}^{\triangle}$ and $\mathbb{Q}_{2K}^{\triangle}$ on the set of ternary trees $\mathcal{T}_{3K-2}^{\mathsf{ter}}$.
1) First, the distribution

$$\mathbb{U}_{3K-2}^{\mathsf{ter}} := \mathbb{U}_{2K}^{\triangle} \circ (\Psi_K^{\triangle})^{-1} \tag{9}$$

is simply the uniform distribution on $\mathcal{T}_{3K-2}^{\mathsf{ter}}$ since $\Psi_K^{\triangle}$ is a bijection.
2) The distribution

$$\mathbb{Q}_{3K-2}^{\mathsf{ter}} := \mathbb{Q}_{2K}^{\triangle} \circ (\Psi_K^{\triangle})^{-1} \tag{10}$$

is the distribution giving a weight to a tree proportional to its number of histories, that is the number of histories of the corresponding triangulation.

We want to give here another representation of the distribution $\mathbb{Q}_{3K-2}^{\mathsf{ter}}$.

Definition 5. *We call increasing ternary tree* $\mathbf{t} = (T, l)$ *a pair such that:*
• T *is the set of internal nodes of a ternary tree,*
• l *is a bijective application between* T *(viewed as a set of nodes) onto* $\{1, \dots, |T|\}$ *such that* l *is increasing along the branches (thus* $l(\varnothing) = 1$*).*

Notice that T is not necessarily a tree as defined in Section 2.3.1: for example T may be $\{\varnothing, 2\}$.

Let $\mathcal{I}_K^{\mathsf{ter}}$ denotes the set of increasing ternary trees (T, l) such that $|T| = K$ (i.e. T is the set of internal nodes of a tree in $\mathcal{T}_{3K+1}^{\mathsf{ter}}$).

The number of histories of a ternary tree $t \in \mathcal{T}^{\mathsf{ter}}_{3K-2}$ is given by the

$$w_{K-1}(t^\circ) = \#\{(t^\circ, l) \in \mathcal{I}^{\mathsf{ter}}_{K-1}\}$$

the number of increasing trees having t° as first coordinate, in other words, with shape t°. Indeed in order to record the number of histories of t an idea is to mark the internal nodes of t by their apparition time, the root being marked 1. Hence the marks are increasing along the branches, and there is a bijection between $\{1, \dots, K-1\}$ and the set of internal nodes of t. Conversely, any labeling of t° with marks having these properties corresponds indeed to a history of m. Thus

Lemma 4. *For any $K \geq 1$, the distribution $\mathbb{Q}^{\mathsf{ter}}_{3K-2}$ has the following representation*

$$\mathbb{Q}^{\mathsf{ter}}_{3K-2}(t) = C_{K-1} \cdot w_{K-1}(t^\circ), \qquad \textit{for any } t \in \mathcal{T}^{\mathsf{ter}}_{3K-2}$$

where C_{K-1} is the constant $C_{K-1} := \left(\sum_{t' \in \mathcal{T}^{\mathsf{ter}}_{3K-2}} w_{K-1}(t'^\circ)\right)^{-1}$.

3 Topologies

3.1 Topology of local convergence

The topology induced by the distance d_L defined below will be called "topology of local convergence". Its aim is to describe an asymptotic behavior of maps (or more generally graphs) around their root. We stress on the fact that no rescaling is involved in this part.

We borrow some considerations from Angel & Schramm [5]. Let $\mathcal{M}$ be the set of rooted maps (m, e) where $e = (E_0, E_1)$ is the distinguished edge of m. The maps from $\mathcal{M}$ are not assumed to be finite, but only locally finite, i.e. the degree of the vertices are finite. For any $r \geq 0$, denote by $B_m(r)$ the map having as set of vertices

$$V(B_m(r)) = \{u \in V(m) \mid d_m(u, E_0) \leq r\},$$

the vertices in m with graph distance to E_0 non greater than r, and having as set of edges, the edges in $E(m)$ between the vertices of $V(B_m(r))$.

For any $m = (\mathsf{m}_1, e)$ and $m' = (\mathsf{m}', e')$ in $\mathcal{M}$ set

$$d_L(m, m') = 1/(1+k) \tag{11}$$

where k is the supremum of the radius r such that $B_m(r)$ and $B_{m'}(r)$ are equals as rooted maps. The application d_L is a metric on the space $\mathcal{M}$. A sequence of rooted maps converges to a given rooted map m (for the metric d_L) if eventually they are equivalent with m on arbitrarily large combinatorial balls around their root. In this topology, all finite maps are isolated points, and infinite maps are their accumulation points. The space $\mathcal{M}$ is complete for the distance d_L since given a Cauchy sequence of locally finite embedded rooted maps it is easy to see that it is possible to choose for them embeddings that eventually agree on balls of any fixed radius around the root. Thus, the limit of the sequence exists (as a locally finite embedded maps). In other words, the space $\mathcal{T}$ of (locally finite embedded rooted) maps is complete.

The space of triangulations (or of quadrangulations) endowed with this metric is not compact since one may find a sequence of triangulations being pairwise at distance 1. The topology on the space of triangulations induces a weak topology on the linear space of measures supported on planar triangulations.

3.2 Gromov-Hausdorff topology

The other topology we are interested in will be the suitable tool to describe the convergence of rescaled maps to a limiting object. The point of view here is to consider maps endowed with the graph distance as metric spaces. The topology considered – called the Gromov-Hausdorff topology – is the topology of the convergence of compact (rooted) metric spaces. We borrow some considerations from Le Gall & Paulin [32] and from Le Gall [30, Section 2]. We send the interested reader to these works and references therein.

First, recall that the Hausdorff distance in a metric space (E, d_E) is a distance between the compact sets of E; for K_1 and K_2 compacts in E,

$$d_{Haus(E)}(K_1, K_2) = \inf\{r \mid K_1 \subset K_2^r, K_2 \subset K_1^r\}$$

where $K^r = \cup_{x\in K} B_E(x, r)$ is the union of open balls of radius r centered on the points of K. Now, given two pointed (i.e. with a distinguished node) compact metric spaces $((E_1, v_1), d_1)$ and $((E_2, v_2), d_2)$, the Gromov-Hausdorff distance between them is

$$d_{GH}(E_1, E_2) = \inf\{d_{Haus(E)}(\phi_1(E_1), \phi_2(E_2)) \vee d_E(\phi_1(v_1), \phi_2(v_2))\},$$

where the infimum is taken on all metric spaces E and all isometric embeddings ϕ_1 and ϕ_2 from (E_1, d_1) and (E_2, d_2) in (E, d_E). Let $\mathbb{K}$ be the set of all isometric classes of compact metric spaces, endowed with the Gromov-Hausdorff distance d_{GH}. It turns out that $(\mathbb{K}, d_{GH})$ is a complete metric space, which makes it appropriate to study the convergence in distribution of $\mathbb{K}$-valued random variables.

The Gromov-Hausdorff convergence is then a consequence of any convergence of E'_n to E'_∞, when E'_n and E'_∞ are some isomorphic embeddings of E_n and E_∞ in a common metric space (E, d_E). In the proofs, we exhibit a space (E, d_E) where this convergence holds; hence, the results of convergence we get are stronger than the only convergence for the Gromov-Hausdorff topology. In fact, it holds for a sequence of parametrized spaces.

4 Local convergence of stack-triangulations under $\mathbb{U}_{2n}^{\triangle}$

We first begin by giving some information about Galton-Watson trees conditioned by the size. These facts will be used also in Section 5.

4.1 Galton-Watson trees conditioned by the size

Consider $\nu_{\mathsf{ter}} := \frac{2}{3}\delta_0 + \frac{1}{3}\delta_3$ as a (critical) offspring distribution of a Galton-Watson (GW) process starting from one individual. Denote by P^{ter} the law of the corresponding GW family tree; we will also write P_n^{ter} instead of $P^{\mathsf{ter}}\big(\,.\, \big| |\mathbf{t}| = n\big)$.

Lemma 5. *P_{3n+1}^{ter} is the uniform distribution on $\mathcal{T}_{3n+1}^{\mathsf{ter}}$.*

Proof. A ternary tree t with $3n+1$ nodes has n internal nodes having 3 children and $2n+1$ leaves with degree 0. Hence $P_{3n+1}^{\mathsf{ter}}(\{t\}) = 3^{-n}(2/3)^{2n+1}/P^{\mathsf{ter}}(\mathcal{T}_{3n+1}^{\mathsf{ter}})$. This is constant on $\mathcal{T}_{3n+1}^{\mathsf{ter}}$ and has support $\mathcal{T}_{3n+1}^{\mathsf{ter}}$. □

The conclusion is that for any $K \geq 1$

$$P^{\mathsf{ter}}_{3K-2} = \mathbb{U}^{\mathsf{ter}}_{3K-2}. \tag{12}$$

Following (9), this gives us a representation of the uniform distribution on $\mathcal{T}^{\mathsf{ter}}_{3K-2}$ in terms of conditioned GW trees. This will be our point of view in the sequel of the paper.

We would like to point out that the number of ternary trees with a given number of nodes, as well as the number of forests $\#\mathcal{F}^r_{\mathsf{ter}}(k)$ of ternary trees with r roots and a total number of nodes K are well known:

$$\#\mathcal{T}^{\mathsf{ter}}_{3K+1} = \frac{1}{3K+1}\binom{3K+1}{K} \text{ and } \#\mathcal{F}^r_{\mathsf{ter}}(K) = \frac{r}{K}\binom{K}{(K-r)/3}. \tag{13}$$

These formulas are consequence of the so-called rotation/conjugation principle due to Raney, or Dvoretzky-Motzkin (see Pitman [42], Section 5.1 for more information on this principle).

We now state some results concerning the local convergence of uniform ternary trees. The limiting random tree will be used to build the limiting random maps, local limit of stacked-triangulations.

4.2 Local convergence of uniform ternary trees

We endow $\mathcal{T}^{\mathsf{ter}}$ with the local distance d_L defined in (11): instead of redefining an ad hoc metric similar to d_L on the set of planar trees, we identify the set of trees with the set of rooted planar maps with one face (this is classical, and corresponds to the embedding of planar trees in the plane respecting the cyclical order around the vertices). Under this metric, the accumulation points of sequences of trees (t_K) such that $|t_K| = 3K-2$ are infinite trees. It is known that the sequence $(P^{\mathsf{ter}}_{3K-2})$ converges weakly for the topology of local convergence. Let us describe a random tree $\mathbf{t}^{\mathsf{ter}}_\infty$ under the limit distribution, denoted by P^{ter}_∞.

Let $\mathcal{W}_3$ be the infinite complete ternary tree and let (X_i) be a sequence of i.i.d. r.v. uniformly distributed on Σ_3. Define

$$L^{\mathsf{ter}}_\infty = (X(j), j \geq 0) \tag{14}$$

the infinite path in $\mathcal{W}_3$ starting from the root $(\varnothing)$ and containing the words $X(j) := X_1 \dots X_j$ for any $j \geq 1$. Take a sequence $(t(i))$ of GW trees under P^{ter} and graft them on the neighbors of L^{ter}_∞, that is on the nodes of $\mathcal{W}_3$ at distance 1 of L^{ter}_∞ (sorted according to the LO). The tree obtained is $\mathbf{t}^{\mathsf{ter}}_\infty$. In the literature the branch L^{ter}_∞ is called the *spine* or the *infinite line of descent* in $\mathbf{t}^{\mathsf{ter}}_\infty$.

Proposition 6. *(Gillet [23]) When $n \to +\infty$, P^{ter}_{3n+1} converges weakly to P^{ter}_∞ for the topology of local convergence.*

This result is due to Gillet [23, Section III] (see Theorems III.3.1, III.4.2, III.4.3, III.4.4).

Note 3. The distribution P^{ter}_∞ is usually called "size biased GW trees". We send the interested reader to Section 2 in Lyons & al. [34] to have an overview of this object. In particular, this distribution is known to be the limit for local convergence of critical GW trees conditioned by the non extinction.

4.3 Local convergence of stacked-triangulations

The first aim of this part is to define a map m_∞ built thanks to $\mathbf{t}_\infty^{\mathsf{ter}}$ with the help of a limiting "bijection" analogous to the functions $\Psi_K^{\triangle}$'s. Some problems arise when one wants to draw or define an infinite map on the plane since we have to deal with accumulation points and possible infinite degree of vertices. We come back on this point in Section 4.3.1. We now describe a special class of infinite trees – we call them *thin ternary trees* – that will play an important role further.

Definition 6. *An infinite line of descent in a tree is a sequence $(u_i, i \geq 0)$ such that: u_0 is the root $\varnothing$, and u_i is a child of u_{i-1} for any $i \geq 1$. We call thin ternary tree a ternary tree having a unique infinite line of descent $L = (u_i, i \geq 0)$, satisfying moreover $\Lambda(u_n) \underset{n}{\longrightarrow} \infty$ (which will be written $\Lambda(L) = \infty$). The set of thin ternary trees is denoted by $\mathcal{T}_{\mathsf{thin}}^{\mathsf{ter}}$.*

Lemma 7. *The support of P_∞^{ter} is included in $\mathcal{T}_{\mathsf{thin}}^{\mathsf{ter}}$.*

Proof. By construction L_∞^{ter} is an infinite line of descent in $\mathbf{t}_\infty^{\mathsf{ter}}$ that satisfies clearly a.s. $\Lambda(L_\infty^{\mathsf{ter}}) = +\infty$. This line is a.s. unique because the sequence $(t(i))$ of grafted trees are critical GW trees and then have a.s. all a finite size. $\square$

For any tree t, finite or not, denote the $\Lambda-$ball of t of radius r by

$$B_r^\Lambda(t) := \{u \mid u \in t, \Lambda(u) \leq r\}.$$

Lemma 8. *For any tree $t \in \mathcal{T}_{\mathsf{thin}}^{\mathsf{ter}}$ and any $r \geq 0$, $\#B_r^\Lambda(t)$ is finite.*

Proof. Let L be the unique infinite line of descent of t. Since $\Lambda(L) = +\infty$, $B_r^\Lambda(t)$ contains only a finite part say $[\![\varnothing, u]\!]$ of L. This part is connected since Λ is non decreasing: if $w = uv$ for two words u and v then $\Lambda(w) \geq \Lambda(u)$. Using again that Λ is non decreasing, $B_r^\Lambda(t)$ is contained in $[\![\varnothing, u]\!]$ union the finite set of finite trees rooted on the neighbors of $[\![\varnothing, u]\!]$. $\square$

Proposition 9. *If a sequence of trees (t_n) converges for the local topology to a thin tree t, then for any $r \geq 0$ there exists N_r such that for any $n \geq N_r$, $B_r^\Lambda(t_n) = B_r^\Lambda(t)$.*

Proof. Suppose that this is not true. Then take the smallest r for which there does not exists such a N_r (then $r \geq 1$ since the property is clearly true for $r = 0$). Let l_r be the length of the longest word in $B_r^\Lambda(t)$. Since $d_L(t_n, t) \leq 1/(l_r + 1)$ for n say larger than N'_r, for those n the words in t_n and t with at most l_r letters coincide. This implies that $B_r^\Lambda(t) \subset B_r^\Lambda(t_n)$ and that this inclusion is strict for a sub-sequence (t_{n_k}) of (t_n). Hence one may find a sequence of words w_{n_k} such that: $\Lambda(w_{n_k}) = r$, $w_{n_k} \in t_{n_k}$, $w_{n_k} \notin t$. Let w'_{n_k} be the smallest (for the LO) elements of (t_{n_k}) with this property. In particular, the father $w_{n_k}^f$ of w'_{n_k} satisfies either:
(a) $\Lambda(w_{n_k}^f) = r - 1$ or,
(b) $\Lambda(w_{n_k}^f) = r$ and then $w_{n_k}^f$ belongs to $B_r^\Lambda(t)$.
For n large enough, say larger than N_{r-1}, $B_{r-1}^\Lambda(t_n)$ coincides with $B_{r-1}^\Lambda(t)$ (since r is the first number for which this property does not hold). Hence, the set $S_f = \{w_{n_k}^f \mid n_k \geq N_{r-1} \wedge N'_r\} \subset B_r^\Lambda(t)$ is finite by the previous Lemma. Then the sequence (w'_{n_k}) takes its values in the set of children of the nodes of S_f, the finite set say S_r. Consider an accumulation point p of (w'_{n_k}). The point p is in the finite set $\{w'_{n_k}, k \geq 0\}$ and then not in t. But p is in t since t contains all (finite) accumulations points of all sequences (x_n), where $x_n \in t_n$. This is a contradiction. $\square$

4.3.1 A notion of infinite map

This section is much inspired by Angel & Schramm [5] and Chassaing & Durhuus [13, Section 6].

We call infinite map m, the embedding of a graph in the plane having the following properties:

(α) it is locally-finite, that is the degree of all nodes is finite,

(β) if $(\rho_n, n \geq 1)$ is a sequence of points belonging to distinct edges of m, then accumulation points of (ρ_n) are not on m (neither on the edges or on the vertices of m).

This last condition ensures that no face is created artificially. For example, we want to avoid a drawing of an infinite graph where each node has degree 2 (an infinite graph line, in some sense) that would create two faces or more, as would result by a drawing of this graph where the two extremities accumulate on the same point. Avoiding the creation of artificial faces allows to ensure that homeomorphisms of the plane are still the right tools to discriminate similar objects. In the following we define an application $\Psi_\infty^\triangle$ that associates with a tree t of $\mathcal{T}_{\mathsf{thin}}^{\mathsf{ter}}$ an infinite map $\Psi_\infty^\triangle(t)$ of the plane. Before this, let us make some remarks. Let $t \in \mathcal{T}_{\mathsf{thin}}^{\mathsf{ter}}$, for any r, set $t(r)$ the tree having as set of internal nodes $B_r^\Lambda = B_r^\Lambda(t)$. We have clearly $d_L(t(r), t) \underset{r}{\to} 0$. Moreover, since $t(r)$ is included in $t(r+1)$, the map $m_r = (\Psi^\triangle)^{-1}(t(r))$ is "included" in m_{r+1}. The quotes are there to recall that we are working on equivalence classes modulo homeomorphisms and that the inclusion is not really defined stricto sensu. In order to have indeed an inclusion, an idea is to use the canonical drawing (see Definition 2) : the inclusion $\mathcal{G}(m_r) \subset \mathcal{G}(m_{r+1})$ is clear if one uses a history leading to m_{r+1} that passes from m_r, which is possible thanks to Property (i) of Proposition 2 and the fact that $t(r) \subset t(r+1)$. Now $(\mathcal{G}(m_r))$ is a sequence of increasing graphs. Let $\mathcal{G}_t$ be defined as the map $\cup_r \mathcal{G}(m_r)$ and having as set of nodes and edges those belonging to at least one of the $\mathcal{G}(m_r)$.

Proposition 10. *For any thin tree t, the map $\mathcal{G}_t$ satisfies (α) and (β).*

Proof. The first assertion comes from the construction and the finiteness of the balls B_r^Λ (by Lemma 8). For the second assertion, just notice that for any r, only a unique face of m_r contains an infinite number of faces of $\mathcal{G}_t$. Indeed, $t(r)$ is included in t and t owns only one infinite line of descent L. Hence among the set of fringe subtrees $\{t_u \mid u \in t(r)\}$ of t (each of them corresponding to the nodes that will be inserted in one of the triangular faces of m_r) only one has an infinite cardinality. It remains to check that the edges do not accumulate, and for this, we have only to follow the sequence of triangles (F_k) that contains an infinite number of faces, those corresponding with the nodes of L. Moreover, by uniqueness of the infinite line of descent in t, the family of triangles (F_k) forms a decreasing sequence for the inclusion. Consider now the subsequence F_{n_k} where $g(\mathrm{type}(F_{n_k})) = g(\mathrm{type}(F_{n_{k-1}})) + 1$. The triangle F_{n_k} has then all its sides different from $F_{n_{k-1}}$. Hence any accumulation points ρ of (ρ_n) (as defined in (β)) must belong to $\cap F_k$. By the previous argument, ρ does not belong to any side of those triangles, which amounts to saying that ρ lies outside m. $\square$

Proposition 11. *Let (t_n) be a sequence of trees, $t_n \in \mathcal{T}_{3n-2}^{\mathsf{ter}}$, converging for the local topology to a thin tree t. Then the sequence of maps $(\Psi_n^\triangle)^{-1}(t_n)$ converges to $\mathcal{G}_t$ for the local topology.*

Proof. If (t_n) converges to t then for any r, there exists n_r such that for any $n \geq n_r$, $B_r^\Lambda(t_n) = B_r^\Lambda(t)$. Hence, if n is large enough, $d_L((\Psi_n^\triangle)^{-1}(t_n), \mathcal{G}_t) \leq 1/(r+1)$. $\square$

We have till now, work on topological facts, separated in some sense from the probabilistic considerations. It remains to deduce the probabilistic properties of interest.

4.3.2 A law on the set of infinite stack-maps

The set $\mathcal{T}^{\mathsf{ter}}$ is a Polish space for the topology d_L. In such a space, Skorohod's representation theorem (see e.g. [27, Theorem 4.30]) applies. Since P^{ter}_{3n-2} converges to P^{ter}_∞, there exists a space Ω on which are defined altogether $\tilde{\mathbf{t}}_\infty, \tilde{\mathbf{t}}_1, \tilde{\mathbf{t}}_2, \ldots$, such that $\tilde{\mathbf{t}}_n \sim P^{\mathsf{ter}}_{3n-2}$ for any n, $\tilde{\mathbf{t}}_\infty \sim P^{\mathsf{ter}}_\infty$, and such that $\tilde{\mathbf{t}}_n \xrightarrow[n]{(a.s.)} \tilde{\mathbf{t}}_\infty$. Moreover, thanks to Lemma 7, $\tilde{\mathbf{t}}_\infty$ is a.s. a thin tree.

We then work on this space Ω and use the almost sure properties of $\tilde{\mathbf{t}}_\infty$. The convergence in distribution of our theorem will be a consequence of the a.s. convergence on Ω.

Definition 7. *We denote by $P^\triangle_\infty$ the distribution of $m_\infty := \mathcal{G}_{\mathbf{t}_\infty}$.*

A simple consequence of Proposition 11 is the following assertion. Since $d_L(\tilde{\mathbf{t}}_n, \tilde{\mathbf{t}}_\infty) \xrightarrow[n]{(a.s.)} 0$ then

$$d_L\left((\Psi_n^\triangle)^{-1}(\tilde{\mathbf{t}}_n), \mathcal{G}_{\tilde{\mathbf{t}}_\infty}\right) \xrightarrow[n]{(a.s.)} 0. \tag{15}$$

This obviously implies the following result.

Theorem 12. $(\mathbb{U}^\triangle_{2n})$ *converges weakly to $P^\triangle_\infty$ for the topology of local convergence.*

5 Asymptotic under the Gromov-Hausdorff topology

The asymptotic behavior of GW trees under P^{ter}_n is very well studied. We focus in this section on the limiting behavior under the Gromov-Hausdorff topology. The facts described here will be used later in the proof of the theorems stating the convergence of stack-triangulations. In addition we stress on the fact that the limit of rescaled stack-maps under the uniform distribution is the same limit as the one of GW trees: the continuum random tree.

5.1 Gromov-Hausdorff convergence of rescaled GW trees

We present here the limit of rescaled GW trees conditioned by the size for the Gromov-Hausdorff topology. We borrow some considerations from Le Gall & Weill [33] and Le Gall [30].

We adopt the same normalizations as Aldous [1; 2]: the Continuum Random Tree (CRT) $\mathcal{T}_{2\mathsf{e}}$ can be defined as the real tree coded by twice a normalized Brownian excursion $\mathsf{e} = (\mathsf{e}_t)_{t\in[0,1]}$. Indeed, any function f with duration 1 and satisfying moreover $f(0) = f(1) = 0$, and $f(x) \geq 0, x \in [0,1]$ may be viewed as coding a continuous tree as follows (illustration can be found on Figure 9). For every $s, s' \in [0,1]$, we set

$$m_f(s,s') := \inf_{s\wedge s' \leq r \leq s\vee s'} f(r).$$

We then define an equivalence relation on $[0,1]$ by setting $s \underset{f}{\sim} s'$ if and only if $f(s) = f(s') = m_f(s, s')$. Finally we put

$$d_f(s, s') = f(s) + f(s') - 2\, m_f(s, s') \tag{16}$$

and note that $d_f(s, s')$ only depends on the equivalence classes of s and s'.

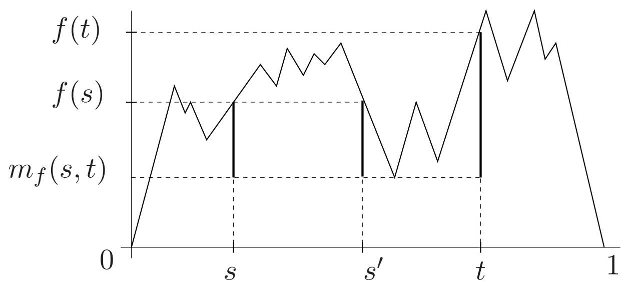

Figure 9: Graph of a continuous function f satisfying $f(0) = f(1) = 0$ and $f(x) \geq 0$ on [0,1]. In this example $s \underset{f}{\sim} s'$ and the distance $d_f(s,t) = d_f(s',t) = f(s) + f(t) - 2\, m_f(s,t)$ is the sum of the lengths of the vertical segments.

Then the quotient space $\mathcal{T}_f := [0,1]/ \underset{f}{\sim}$ equipped with the metric d_f is a compact $\mathbb{R}$-tree (see e.g. Section 2 of [18]). In other words, it is a compact metric space such that for any two points σ and σ' there is a unique arc with endpoints σ and σ' and furthermore this arc is isometric to a compact interval of the real line. We view $\mathcal{T}_f$ as a rooted $\mathbb{R}$-tree, whose root ρ is the equivalence class of 0.

The CRT is the metric space $(\mathcal{T}_{2\mathsf{e}}, d_{2\mathsf{e}})$. In addition to the usual genealogical order of the tree, the CRT $\mathcal{T}_{2\mathsf{e}}$ inherits a LO from the coding by $2\mathsf{e}$, in a way analogous to the ordering of (discrete) plane trees from the left to the right.

Discrete trees T are now equipped with their graph distances d_T.

Proposition 13. *The following convergence holds for the GH topology. Under P^{ter}_{3n+1},*

$$\left(T, \frac{d_T}{\sqrt{3n/2}}\right) \xrightarrow[n]{(d)} (\mathcal{T}_{2\mathsf{e}}, d_{2\mathsf{e}}).$$

Proof. The convergence for the GH topology is a consequence of the convergence for any suitable encoding of trees. The offspring distribution $\nu_{\mathsf{ter}} := \frac{2}{3}\delta_0 + \frac{1}{3}\delta_3$ is critical (in other words has mean 1) and variance 2. The convergence of rescaled GW trees conditioned by their size is proved by Aldous [1; 2]. (See also Le Gall [30] or Marckert & Mokkadem [38], Section 6 of Pitman [42]). ☐

5.2 Gromov-Hausdorff convergence of stack-triangulations

We begin with a simple asymptotic result concerning the function Λ defined in Section 2.3.2.

Lemma 14. *Let $(X_i)_{i\geq 1}$ be a sequence of random variables uniform in $\Sigma_3 = \{1,2,3\}$, and independent. Let W_n be the word $X_1 \dots X_n$.*
(i) $n^{-1}\Lambda(W_n) \xrightarrow[n]{(a.s.)} \Lambda_\triangle$ *where*

$$\Lambda_\triangle := 2/11. \tag{17}$$

(ii) $\mathbb{P}(|\Lambda(W_n) - n\Lambda_\triangle| \geq n^{1/2+u}) \underset{n}{\to} 0$ *for any* $u > 0$.

Proof. If W is the infinite sequence (X_i), clearly $\tau_2(W) \sim \mathsf{Geometric}(1/3)$ and for $i \geq 3$, the $(\tau_i(W) - \tau_{i-1}(W))'s$ are i.i.d., independent also from τ_2, and are distributed as $1+G_1+G_2$ where $G_1 \sim \mathsf{Geometric}(1/3)$ and $G_2 \sim \mathsf{Geometric}(2/3)$ [the distribution $\mathsf{Geometric}(p)$ is $\sum_{k\geq 1} p(1-p)^{k-1}\delta_k$]. It follows that $\mathbb{E}(\tau_i(W) - \tau_{i-1}(W)) = 11/2$ for $i \geq 3$ and $\mathbb{E}(\tau_2(W)) = 3 < +\infty$. By the renewal theorem assertion (i) holds true. For the second assertion, write

$$\{|\Lambda(W_n) - n\Lambda_\triangle| \geq n^{1/2+u}\} = \{\tau_{n\Lambda_\triangle + n^{1/2+u}} \leq n\} \cup \{\tau_{n\Lambda_\triangle - n^{1/2+u}} \geq n\}.$$

By the Bienaymé-Tchebichev inequality the probability of the events in the right hand side goes to 0. □

For every integer $n \geq 2$, let M_n be a random rooted map under $\mathbb{U}^\triangle_{2n}$. Denote by m_n the set of vertices of M_n and by d_{m_n} the graph distance on m_n. We view (m_n, d_{m_n}) as a random variable taking its values in the space of isometric classes of compact metric spaces.

Theorem 15. *Under* $\mathbb{U}^\triangle_{2n}$,

$$\left(m_n, \frac{d_{m_n}}{\Lambda_\triangle\sqrt{3n/2}}\right) \xrightarrow[n]{(d)} (\mathcal{T}_{2\mathsf{e}}, d_{2\mathsf{e}}),$$

for the Gromov-Hausdorff topology on compact metric spaces.

This theorem is a corollary of the following stronger Theorem stating the convergence of maps seen as parametrized metric spaces. In order to state this theorem, we need to parametrize the map M_n. The set of internal nodes of m_n inherits of an order, the LO on trees, thanks to the function $\Psi^\triangle_n$. Let $u(r)$ be the rth internal node of m_n for $r \in \{0, \dots, n-1\}$. Denote by $d_{m_n}(k,j)$ the distance between $u(k)$ and $u(j)$ in m_n. We need in the following theorem to interpolate d_{m_n} between the integer points to obtain a continuous function. Any smooth enough interpolation is suitable. [For example, define d_{m_n} as the plane interpolation on the triangles with integer coordinates of the form $(a,b), (a+1,b), (a,b+1)$ and $(a,b+1), (a+1,b+1), (a+1,b)$].

Theorem 16. *Under* $\mathbb{U}^\triangle_{2n}$,

$$\left(\frac{d_{m_n}(ns,nt)}{\Lambda_\triangle\sqrt{3n/2}}\right)_{(s,t)\in[0,1]^2} \xrightarrow[n]{(d)} (d_{2\mathsf{e}}(s,t))_{(s,t)\in[0,1]^2}\,, \tag{18}$$

where the convergence holds in $C[0,1]^2$ (even if not indicated, the space $C[0,1]$ and $C[0,1]^2$ are equipped with the topology of uniform convergence).

The proof of this Theorem is postponed to Section 9.1.

Proof of Theorem 15. To explain why Theorem 15 is a consequence of Theorem 16, we introduce the notion of correspondence between two pointed compact metric spaces. Let $((E,v),d)$ and $((E',v'),d')$ be two pointed compact metric spaces. We say that $\mathcal{R}\subset E\times E'$ is a *correspondence* between $((E,v),d)$ and $((E',v')d')$ if $(v,v')\in\mathcal{R}$ and for every $x\in E$ (resp. $x'\in E'$) there exists $y'\in E'$ (resp. $y\in E$) such that $(x,y')\in\mathcal{R}$ (resp. $(x',y)\in\mathcal{R}$). The distorsion of $\mathcal{R}$ is defined by

$$\mathrm{dis}(\mathcal{R})=\sup\big\{|d(x,y)-d'(x',y')|\,:\,(x,x'),(y,y')\in\mathcal{R}\big\}.$$

It has been proved in [19] that

$$d_{GH}(E,E')=\frac{1}{2}\inf\{dis(\mathcal{R})\,:\,\mathcal{R}\in\mathcal{C}\},\tag{19}$$

where $\mathcal{C}$ denotes the set of all correspondences between $((E,v),d)$ and $((E',v'),d')$.

Let m_n be a uniform stack-triangulation with $2n$ faces. Then one can construct a correspondence $\mathcal{R}_n$ between $((m_n,E_0),d_{m_n}/\Lambda_\triangle\sqrt{3n/2})$ and a continuous tree $\mathcal{T}_{2\mathsf{e}}$ thanks to the parametrization of m_n, that is

$$\mathcal{R}_n=\{(u(ns),s)\in m_n\times\mathcal{T}_{2\mathsf{e}}\,:\,s\in[0,1]\}.$$

Now using Theorem 16, by Skorohod's representation theorem there exists a space Ω where a copy $\widetilde{d_{m_n}}$ of d_{m_n}, and a copy $\tilde{d_{2\mathsf{e}}}$ of $d_{2\mathsf{e}}$ satisfies $\tilde{d_{m_n}}\xrightarrow[n]{(a.s.)}\widetilde{d_{2\mathsf{e}}}$ in $C[0,1]^2$. On this space the correspondence $\mathrm{dis}(\widetilde{\mathcal{R}_n})=\sup\{|\widetilde{d_{m_n}}(ns,nt)/\Lambda_\triangle\sqrt{3n/2}-\widetilde{d_{2\mathsf{e}}}(s,t)|\,:\,(s,t)\in[0,1]^2\}$. On the space Ω, a.s. $\mathrm{dis}(\widetilde{\mathcal{R}_n})\to 0$, which implies Theorem 15 (together with (19)). $\square$

The profile $\mathrm{Prof}_m:=(\mathrm{Prof}_m(t),t\geq 0)$ of a map m with root vertex E_0 is the càdlàg-process

$$\mathrm{Prof}_m(t)=\#\{u\in V(m)\mid d_m(E_0,u)\leq t\},\ \text{for any } t\geq 0.$$

The radius $R(m)=\max\{d_m(u,E_0)\mid u\in V(m)\}$ is the largest distance to the root vertex in m.

As a corollary of Theorem 15 or Theorem 16, we have:

Corollary 17. *Under $\mathbb{U}^{\triangle}_{2n}$, the process*

$$\Big(n^{-1}\,\mathrm{Prof}_{m_n}(\Lambda_\triangle\sqrt{3n/2}\,v)\Big)_{v\geq 0}\xrightarrow[n]{(d)}\left(\int_0^v l^x_{2\mathsf{e}}dx\right)_{v\geq 0}\tag{20}$$

where $l^x_{2\mathsf{e}}$ stands for the local time of twice the Brownian excursion $2\mathsf{e}$ at position x at time 1, and where the convergence holds in distribution in the set $D[0,+\infty)$ of càdlàg functions endowed with the Skorohod topology. Moreover

$$\frac{R(m_n)}{\Lambda_\triangle\sqrt{3n/2}}\xrightarrow[n]{(d)}2\max\mathsf{e}$$

Proof. Let $D_n(s)=\frac{d_{m_n}(ns,0)}{\Lambda_\triangle\sqrt{3n/2}}$ be the interpolated distance to E_0. By (18), $(D_n(s))_{s\in[0,1]}\xrightarrow[n]{(d)}(2\mathsf{e}(s))_{s\in[0,1]}$ in $C[0,1]$. By Skorohod's representation theorem there exists a space Ω where a copy $\tilde{D}_n$ of D_n, and a copy $\tilde{\mathsf{e}}$ of e satisfies $\tilde{D}_n \xrightarrow[n]{(a.s.)} 2\tilde{\mathsf{e}}$ in $C[0,1]$. We work from now on on this space, and write $\widetilde{\mathrm{Prof}}_n$ the profile corresponding to $\tilde{D}_n$. For any v such that $\Lambda_\triangle\sqrt{3n/2}\,v$ is an integer,

$$n^{-1}\widetilde{\mathrm{Prof}}_n(\Lambda_\triangle\sqrt{3n/2}\,v) = \int_0^1 \mathbb{1}_{\tilde{D}_n(s)\le v}\,ds.$$

For every v, a.s., $\int_0^1 \mathbb{1}_{\tilde{D}_n(s)\le v}\,ds \to \int_0^v l^x_{2\tilde{\mathsf{e}}}\,dx$. To see this, take any $\varepsilon > 0$ and check that $\|\tilde{D}_n - 2\tilde{\mathsf{e}}\|_\infty \to 0$ yields

$$\int_0^1 \mathbb{1}_{2\tilde{\mathsf{e}}(s)\le v-\varepsilon}\,ds \le \int_0^1 \mathbb{1}_{\tilde{D}_n(s)\le v}\,ds \le \int_0^1 \mathbb{1}_{2\tilde{\mathsf{e}}(s)\le v+\varepsilon}\,ds. \tag{21}$$

Since the Borelian measure $\mu_{2\mathsf{e}}(B) = \int_0^1 \mathbb{1}_{2e(s)\in B}\,ds$ has no atom a.s., $v \to \int_0^v l^x_{2\mathsf{e}}dx$ is continuous and non-decreasing. Hence since $v \to \int_0^1 \mathbb{1}_{D_n(s)\le v}\,ds$ is non decreasing and by (21) we have $\int_0^1 \mathbb{1}_{D_n(s)\le v}\,ds \to \int_0^v l^x_{2\mathsf{e}}\,dx$ a.s. for any $v \ge 0$. Thus, $(v \to \int_0^1 \mathbb{1}_{D_{ns}\le v}\,ds) \to (v \to \int_0^v l^x_{2\mathsf{e}}\,dx)$ in $C[0,1]$. This yields the convergence of Prof_{m_n} as asserted in (20).

For the second assertion, note that $f \to \max f$ is continuous on $C[0,1]$. Since $\tilde{D}_n \xrightarrow[n]{(a.s.)} 2\tilde{\mathsf{e}}$ then $\max \tilde{D}_n \xrightarrow[n]{(a.s.)} \max 2\mathsf{e}$, and then also in distribution. □

6 Asymptotic behavior of the typical degree

The results obtained in this part are summed up in the following Proposition.

Proposition 18. *Let m_n be a map $\mathbb{U}^\triangle_{2n}$ distributed, $u(1)$ the first node inserted in m_n, and $\mathbf{u}$ be a random node chosen uniformly among the internal nodes of m_n.*
(i) $\deg_{m_n}(u(1)) \xrightarrow[n]{(d)} X$ *where for any $k \ge 0$,* $\mathbb{P}(X = k+3) = \frac{k}{k+3}\binom{2k+2}{k}\frac{2^{k+3}}{3^{2k+3}}$,
(ii) $\deg_{m_n}(\mathbf{u}) \xrightarrow[n]{(d)} Y$ *where for any $k \ge 0$,* $\mathbb{P}(Y = k+3) = \frac{1}{k+3}\binom{2k+2}{k}\frac{2^{k+3}}{3^{2k+2}}$.

The *(ii)* point has been shown by Darrasse & Soria [16], in the case of a model of stack-triangulations under a Boltzmann model. Assertion *(ii)* follows nevertheless their work, and a technical lemma saying that in a Galton-Watson tree $\mathbf{t}$ conditioned to have a total size n with offspring distribution ν_{ter}, the fringe subtree $\mathbf{t_u}$ taken at a random node $\mathbf{u}$, converges in distribution to a Galton-Watson tree under ν_{ter} (with no conditioning) when $n \to +\infty$. This argument should be detailed in a forthcoming work of Darrasse & Soria. We give below an elementary proof of this Lemma avoiding the Boltzmann distribution, and the generating function.

Lemma 19. *Let T be a random tree under $\mathbb{U}^{\mathsf{ter}}_{3n+1}$ and $\mathbf{u}$ be chosen uniformly in T°. We have* $|T_\mathbf{u}| \xrightarrow[n]{(d)} \mathbf{K}$ *where* $\mathbb{P}(\mathbf{K} = 3k+1) = \frac{2^{2k+1}}{3^{3k}(3k+1)}\binom{3k+1}{k}$, *for $k \ge 1$. Moreover, conditionally on $|T_\mathbf{u}| = m$, $T_\mathbf{u}$ has the uniform distribution in $\mathcal{T}^{\mathsf{ter}}_m$.*

Note 4. *There are several ways to check that the limiting sequences in Proposition 18 and Lemma 19 define indeed some probability distributions. One may use an approach using generating functions. Alternatively, one may use probabilistic arguments.*

For the sum in the Lemma, proceed as follows: Consider the family tree of a Galton-Watson process with the critical offspring distribution ν_{ter} *starting from one individual. Each of the* $\frac{1}{3k+1}\binom{3k+1}{k}$ *ternary trees having* $3k+1$ *nodes, has weight* $\frac{2^{2k+1}}{3^{3k+1}}$ *under this law. The a.s. extinction of this Galton-Watson process rewrites* $\sum_{k\geq 1}\frac{2^{2k+1}}{3^{3k}(3k+1)}\binom{3k+1}{k}=1$.

Using the same idea, check that $\sum_{k\geq 0}\frac{3}{2k+3}\binom{2k+3}{k}\frac{2^{k+3}}{3^{2k+3}}$ *is the extinction probability of a binary Galton-Watson process with offspring distribution* $\mu(0)=2/3$ *and* $\mu(2)=1/3$*, starting with 3 individuals: since it is sub-critical, this probability is 1.*

A probabilistic proof of $\sum_{k\geq 1}\frac{k}{k+3}\binom{2k+2}{k}\frac{2^{k+3}}{3^{2k+3}}=1$ *runs as follows: one checks that this sum equals* $\frac{1}{3}\sum_{k\geq 0}\mathbb{P}(S_{2k+4}=-4)$ *where* S_j *is a sum of* j *i.i.d. random variables taking values* -1 *or +1 with probability* $2/3$ *and* $1/3$*. Then* $\sum_{k\geq 0}\mathbb{P}(S_{2k+4}=-4)=\mathbb{E}(\sum_j \mathbb{1}_{S_j=-4})$ *is the mean time passed by the random walk* S*, at position* -4*: the identity is exact since* -4 *can be reached only at even dates. But the mean sojourn time in a given position with negative ordinate is 3, since the drift of the random walk is* $-1\times(2/3)+1\times(1/3)=-1/3$.

Proof of Lemma 19. Consider

$$\mathcal{T}^{\mathsf{ter}\star}_{3n+1}:=\{(t,u)\mid t\in\mathcal{T}^{\mathsf{ter}}_{3n+1},u\in t^\circ\},\quad \mathcal{T}^{\mathsf{ter}\bullet}_{3n+1}:=\{(t,u)\mid t\in\mathcal{T}^{\mathsf{ter}}_{3n+1},u\in\partial t\}$$

the set of ternary trees with a distinguished internal node, resp. leaf. For any tree t and $u\in t$ set $t[u]=\{v\in t\mid v \text{ is not a descendant of } u\}$. Each element (t,u) of $\mathcal{T}^{\mathsf{ter}\star}_{3n+1}$ can be decomposed bijectively as a pair $[(t[u],u),t_u]$ where $(t[u],u)$ is a tree with a marked leaf, and t_u is a ternary tree having at least one internal node. Hence, for any n, the function ρ defined by $\rho(t,u):=[(t[u],u),t_u]$ is a bijection from $\mathcal{T}^{\mathsf{ter}\star}_{3n+1}$ onto $\bigcup_{k=1}^n\left(\mathcal{T}^{\mathsf{ter}\bullet}_{3(n-k)+1}\times\mathcal{T}^{\mathsf{ter}}_{3k+1}\right)$.

Since the trees in $\mathcal{T}^{\mathsf{ter}}_{3n+1}$ have the same number of internal nodes, choosing a tree T uniformly in $\mathcal{T}^{\mathsf{ter}}_{3n+1}$ and then a node u uniformly in T°, amounts to choosing a marked tree (T,u) uniformly in $\mathcal{T}^{\mathsf{ter}\star}_{3n+1}$. We then have, for any fixed k,

$$\mathbb{U}^{\mathsf{ter}}_{3n+1}(|T_{\mathbf{u}}|=3k+1)\quad=\quad\#\mathcal{T}^{\mathsf{ter}\bullet}_{3(n-k)+1}\#\mathcal{T}^{\mathsf{ter}}_{3k+1}\left(\#\mathcal{T}^{\mathsf{ter}\star}_{3n+1}\right)^{-1}. \tag{22}$$

When $n\to+\infty$, this tends to the result announced in the Lemma, using (13), $\#\mathcal{T}^{\mathsf{ter}\bullet}_{3m+1}=(2m+1)\#\mathcal{T}^{\mathsf{ter}}_{3m+1}$ and $\#\mathcal{T}^{\mathsf{ter}\star}_{3n+1}=n\#\mathcal{T}^{\mathsf{ter}}_{3n+1}$ and

$$\#\mathcal{T}^{\mathsf{ter}}_{3n+1}=\frac{1}{3n+1}\binom{3n+1}{n}\sim\sqrt{\frac{3}{\pi}}\frac{3^{3n}}{2^{2n+2}n^{3/2}}. \tag{23}$$

To conclude that we have indeed a convergence in distribution of $\deg_T(\mathbf{u})$ under $\mathbb{U}^{\mathsf{ter}}_{3n+1}$ to $\mathbf{K}$, we use that the limit sums to 1 (see Note 4). The second assertion of the Lemma is clear. □

Proof of Proposition 18. As illustrated on Figure 10, for any $t\in\mathcal{T}^{\mathsf{ter}}$, we let

$$t^{deg}:=\{v\mid v\in t, v\in 1\{2,3\}^\star\cup 2\{1,3\}^\star\cup 3\{1,2\}^\star\}.$$

In general t^{deg} is a forest of three pseudo-trees: pseudo here means that the connected components of t^{deg} have a tree structure but do not satisfies the first and third points in Definition

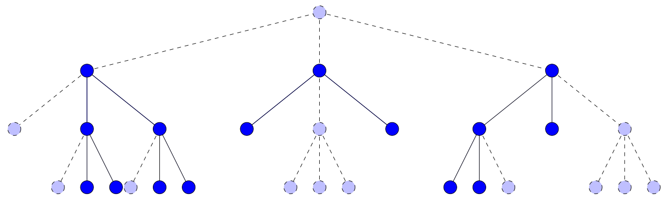

Figure 10: A ternary tree t and t^{deg}. Plain vertices belong to t^{deg}.

3. For sake of compactness, we will however up to a slight abuse of language call these three pseudo-trees, binary trees (combinatorially their are binary trees).

(i) Let T be a tree $\mathbb{U}^{\mathsf{ter}}_{3n-2}$ distributed and $m = (\Psi^{\triangle}_n)^{-1}(T)$. By Property (iii) of Proposition 1,

$$\deg_m(u(1)) = 3 + \#(T^{deg} \cap T^{\circ}), \tag{24}$$

or in other words $\mathbb{U}^{\triangle}_{2n}(\deg(u(1) = k+3) = \mathbb{U}^{\mathsf{ter}}_{3n-2}(|T^{deg}| = 2k+3)$. Each ternary tree t not reduced to the root vertex can be decomposed in a unique way as a pair (t^{deg}, f) where $f := (t(1), \dots, t(k)) \in (\mathcal{T}^{\mathsf{ter}})^k$ is a forest of ternary trees, and $k = \#(t^{deg} \cap t^{\circ})$. Let $\mathcal{F}^n_{\mathsf{bin}}(k)$ (resp $\mathcal{F}^n_{\mathsf{ter}}(k)$) be the set of forests composed with n binary (resp. ternary) trees and total number of nodes k. For $0 \le k < n-1$, we get:

$$\mathbb{U}^{\mathsf{ter}}_{3n-2}(|T^{deg}| = 2k+3) = \frac{\#\mathcal{F}^3_{\mathsf{bin}}(2k+3)\#\mathcal{F}^k_{\mathsf{ter}}(3n-2k-6)}{\#\mathcal{T}^{\mathsf{ter}}_{3n-2}}. \tag{25}$$

A well known consequence of the rotation/conjugation principle (see Pitman [42] Section 5.1) is that

$$\#\mathcal{F}^m_{\mathsf{bin}}(n) = \frac{m}{n}\binom{n}{(n-m)/2}, \quad \text{and } \#\mathcal{F}^m_{\mathsf{ter}}(n) = \frac{m}{n}\binom{n}{(n-m)/3} \tag{26}$$

with the convention that $\binom{a}{b}$ is 0 if b is negative or non integer. We then have

$$\mathbb{U}^{\mathsf{ter}}_{3n-2}(|T^{deg}| = 2k+3) = \frac{\frac{3}{2k+3}\binom{2k+3}{k}\frac{k}{3n-2k-6}\binom{3n-2k-6}{n-k-2}}{\frac{1}{3n-2}\binom{3n-2}{n-1}}. \tag{27}$$

We get $\mathbb{U}^{\triangle}_{2n}(\deg_{m_n}(u(1)) = k+3) \underset{n}{\to} \frac{k}{k+3}\binom{2k+2}{k}\frac{2^{k+3}}{3^{2k+3}}$, limit which is indeed a probability distribution (see Note 4).

(ii) Now let m_n be $\mathbb{U}^{\triangle}_{2n}$ distributed and $\mathbf{u}$ be a uniform internal node of m_n. Let $T = \Psi^{\triangle}(m_n)$ and $\mathbf{u}'$ be the internal node of T corresponding to $\mathbf{u}$. We have this time $\mathbb{U}^{\triangle}_{2n}(\deg_m(\mathbf{u}) = k+3) = \mathbb{U}^{\mathsf{ter}}_{3n-2}(|T^{deg}_{\mathbf{u}'}| = 2k+3)$. First by a simple counting argument,

$$\mathbb{U}^{\mathsf{ter}}_{3n-2}\left(|T^{deg}_{\mathbf{u}'}| = 2k+3 \;\middle|\; |T_{\mathbf{u}'}| = 3j-2\right) = \mathbb{U}^{\mathsf{ter}}_{3j-2}\left(|T^{deg}| = 2k+3\right).$$

Conditioning on $|T_{\mathbf{u}'}|$, using Formulas (22) and (27) we get after simplification

$$q_{n,k} := \mathbb{U}^{\mathsf{ter}}_{3n-2}\left(|T^{deg}_{\mathbf{u}'}| = 2k+3\right) = \sum_{j \ge k+2} q_{n,k,j} \tag{28}$$

where

$$q_{n,k,j} = (\mathbb{1}_{j\le n})\ \frac{3}{2k+3}\binom{2k+3}{k}\frac{\frac{k}{3j-2k-6}\binom{3j-2k-6}{j-k-2}\binom{3(n-j)}{n-j}}{\binom{3n-3}{2n-1}}.$$

We have $\lim_n \binom{3(n-j)}{n-j}/\binom{3n-3}{2n-1} = 2^{2j-1}/3^{3j-3}$, and thus

$$\sum_{j\ge k+2}^{\infty} \lim_n q_{n,k,j} = \tilde{q}_k := \frac{3}{2k+3}\binom{2k+3}{k}\frac{2^{k+3}}{3^{2k+3}}, \tag{29}$$

which is the probability distribution announced to be the limit of $q_{n,k}$. To end the proof we have to explain why the exchange $\lim_n$ and $\sum_{j\ge k+2}$ is legal.

By Fatou's lemma, for any k one has

$$\limsup_n q_{n,k} = \overline{q_k} \ge \underline{q_k} = \liminf_n q_{n,k} \ge \tilde{q}_k.$$

Since $(\tilde{q}_k)$ is a probability distribution, $\sum \tilde{q}_k = 1$. And clearly for any K, $\overline{q_1} + \cdots + \overline{q_K} = \limsup_n q_{n,1} + \cdots + q_{n,K} \le 1$. Hence, we have that $\sum \overline{q_k} = 1$ (it is greater than $\sum \tilde{q}_k = 1$ and smaller than 1), this implies that $\sum \underline{q_k} = \sum \overline{q_k} = \sum \tilde{q}_k = 1$ and then for any k, $\overline{q_k} = \underline{q_k} = \tilde{q}_k$ and then $\lim_k q_k$ exists and is $\tilde{q}_k$. $\square$

7 Asymptotic behavior of stack-triangulations under $\mathbb{Q}^{\triangle}_{2n}$

We first present a result concerning ternary trees under $\mathbb{Q}^{\mathsf{ter}}_{3n-2}$.

Proposition 20. *Let* $\mathbf{t}$ *be a random tree under* $\mathbb{Q}^{\mathsf{ter}}_{3n-2}$*, and* $\mathbf{u}$ *and* $\mathbf{v}$ *be two nodes chosen uniformly and independently in* $\mathbf{t}^\circ$*; let* $\mathbf{w} = \mathbf{u} \wedge \mathbf{v}$ *be their deepest common ancestor.*
1) We have $\left(\frac{3}{2}\log n\right)^{-1/2}\left(|\mathbf{u}| - \frac{3}{2}\log n, |\mathbf{v}| - \frac{3}{2}\log n\right) \xrightarrow[n]{(d)} (N_1, N_2)$ *where* N_1 *and* N_2 *are independent centered Gaussian r.v. with variance 1.*
2) Let $\mathbf{a_0}\ldots\mathbf{a_k}$ *and* $\mathbf{b_0}\ldots\mathbf{b_l}$ *be the unique words such that* $\mathbf{u} = \mathbf{w}\mathbf{a_0}\ldots\mathbf{a_k}$ *and* $\mathbf{v} = \mathbf{w}\mathbf{b_0}\ldots\mathbf{b_l}$*. Set*

$$\mathbf{u}^\star = \mathbf{a_1}\ldots\mathbf{a_k} \text{ and } \mathbf{v}^\star = \mathbf{b_1}\ldots\mathbf{b_l}.$$

Conditionally to $(|\mathbf{u}^\star|, |\mathbf{v}^\star|)$ *(their lengths)* $\mathbf{u}^\star$ *and* $\mathbf{v}^\star$ *are independent random words composed with* $|\mathbf{u}^\star|$ *and* $|\mathbf{v}^\star|$ *independent letters uniformly distributed in* $\Sigma_3 = \{1,2,3\}$*.*
3) For any $a_n \to +\infty$*, we have* $|\mathbf{w}|/a_n \xrightarrow[n]{proba.} 0$*.*

[Notice that in the second assertion, the words $\mathbf{a_0}\ldots\mathbf{a_k}$ and $\mathbf{b_0}\ldots\mathbf{b_l}$ may be empty.]

The following theorem may be considered as the strongest result of this section. As explained in its proof, it is an immediate consequence of Proposition 20.

Theorem 21. *Let* M_n *be a stack-triangulation under* $\mathbb{Q}^{\triangle}_{2n}$*. Let* $k \in \mathbb{N}$ *and* $\mathbf{v}_1, \ldots, \mathbf{v}_k$ *be* k *nodes of* M_n *chosen independently and uniformly among the internal nodes of* M_n*. We have*

$$\left(\frac{D_{M_n}(\mathbf{v}_i, \mathbf{v}_j)}{3\Lambda_\triangle \log n}\right)_{(i,j)\in\{1,\ldots,k\}^2} \xrightarrow[n]{proba.} (1_{i\ne j})_{(i,j)\in\{1,\ldots,k\}^2}$$

The matrix $(1_{i\ne j})_{(i,j)\in\{1,\ldots,k\}^2}$ *is the matrix of the discrete distance on a set of* k *points.*

This is consistent with the computations of Zhou [47] and Zhang & al [45].

Proof. The convergence stated in the theorem is equivalent to the (in appearance weaker) following statement: if $\mathbf{v_1}$ and $\mathbf{v_2}$ are two i.i.d. uniform random internal nodes in M_n, then

$$\frac{D_{M_n}(\mathbf{v}_1, \mathbf{v}_2)}{3\Lambda_{\triangle} \log n} \xrightarrow[n]{proba.} 1.$$

Now, consider $\mathbf{t} := \Psi^{\triangle}(M_n)$ and write

$$|D_{M_n}(\mathbf{v}_1, \mathbf{v}_2) - \Lambda(\mathbf{v}'_1, \mathbf{v'}_2)| \leq 4$$

(by (3)) where $\mathbf{v}'_1$ and $\mathbf{v'}_2$ are the internal nodes corresponding to $\mathbf{v}_1$ and $\mathbf{v}_2$ by the bijection $\Psi^{\triangle}$. Notice that $\mathbf{v}'_1$ and $\mathbf{v'}_2$ are i.i.d. uniform in $\mathbf{t}^{\circ}$. Now write,

$$|\Lambda(\mathbf{v}'_1, \mathbf{v'}_2) - \Lambda(\mathbf{v}'^{\star}_1) - \Lambda(\mathbf{v}'^{\star}_2)| \leq 2$$

and use Proposition 20 : asymptotically $\mathbf{v_1}'^{\star}$ and $\mathbf{v_2}'^{\star}$ have each of them a length around $3 \log n/2$ (more precisely $\mathbb{P}(|\mathbf{v}'^{\star}_i - 3/2 \log n| \geq \varepsilon \log n) \to 0$ for $i \in \{1, 2\}$ and any $\varepsilon > 0$) and the letters of $\mathbf{v_1}'^{\star}$ and $\mathbf{v_2}'^{\star}$ are independent. A simple application of Lemma 14 gives the result. $\square$

We now focus on Proposition 20 and on its proof. This Proposition is more or less part of the folklore. Bergeron & al [6], in particular in Theorem 8 and Example 1 p.7, proved that

$$\left(\frac{3}{2} \log n\right)^{-1/2} \left(|\mathbf{u}| - \frac{3}{2} \log n\right) \xrightarrow[n]{(d)} N_1. \tag{30}$$

Below we present a formal proof of this proposition using a "Poisson-Dirichlet fragmentation" point of view, very close to that used in Broutin & al. [12, Section 7] where the height of increasing trees is investigated. They show that in increasing trees the asymptotic proportion $n^{-1}(|t_1|, \dots, |t_d|)$ of nodes in the subtrees of the root are given by a Poisson-Dirichlet distribution. The point of view developed below is different in nature, since we first take a Poisson-Dirichlet fragmentation and then show that the fragmentation tree is distributed as an increasing tree, leading then at once to the convergence of $n^{-1}(|t_1|, \dots, |t_d|)$. The following Subsection is mostly contained in the more general work of Dong & al. [17] (particularly Section 5). We give a straight exposition below for the reader convenience, in a quite different vocabulary.

7.1 Poisson-Dirichlet fragmentation

We construct here a representation of the distribution $\mathbb{Q}^{\triangle}_{3K-2}$ as the distribution of the underlying tree of a fragmentation tree. Let begin with the description of the deterministic fragmentation tree associated with a sequence of choices $\mathbf{b} = (b_i)_{i \geq 1}$, $b_i \in [0, 1]$ and a sequence $\mathbf{y} = (y^u)_{u \in \mathcal{W}_3} = (y^u_1, y^u_2, y^u_3)_{u \in \mathcal{W}_3}$ indexed by the infinite complete ternary tree $\mathcal{W}_3 = \bigcup_{n \geq 0} \{1, 2, 3\}^n$, where for any $i \in \{1, 2, 3\}$ and $u \in \mathcal{W}_3$, $y^u_i > 0$ and $\sum_{i=1}^{3} y^u_i = 1$. The sequence (y^u) may be thought as the fragmentation structure associated with the tree.

With these two sequences we associate a sequence $F_n = F(n, \mathbf{b}, \mathbf{y})$ of ternary trees with $3n + 1$ leaves, where each node is marked with an interval as follows.

– At time 0, F_0 is the tree $\{\varnothing\}$ (reduced to the root) marked by $I_{\varnothing} = [0, 1)$.

– Assume now that F_i is built, and is a ternary tree with $3i+1$ nodes each marked with an interval included in $[0,1)$, and such that the leaves-intervals $(I_u, u \in \partial T_i)$ form a partition of $[0,1)$. Then the tree F_{i+1} is obtained from F_i as follows. Consider $u^\star$ the leaf whose associated interval $I_{u^\star}$ contains b_{i+1}. Give to $u^\star$ the 3 children $u^\star 1, u^\star 2, u^\star 3$. Now split the interval $I_{u^\star}$ into $(I_{u^\star 1}, I_{u^\star 2}, I_{u^\star 3})$ with respective size proportions given by $y^{u^\star}$: if $I_{u^\star} = [a,b)$ then set $I_{u^\star i} = [a+(b-a)\sum_{j=1}^{i-1} y_j^{u^\star}, a+(b-a)\sum_{j=1}^{i} y_j^{u^\star})$ for every $i \in \{1,2,3\}$. Let $\Omega_{\mathcal{F}}$ be the set of fragmentation trees (a tree where each node is marked by an interval). We define the projection π from $\Omega_{\mathcal{F}}$ to $\mathcal{T}^{\mathsf{ter}}$ as the application sending a fragmentation tree F to its underlying tree $\pi(F)$, that is the tree F without marks.

We now let $\mathbf{b}$ and $\mathbf{y}$ be random. For $d \geq 2$ consider the simplex

$$\Delta_{d-1} = \left\{ x = (x_1, \dots, x_d) \mid x_i \geq 0 \text{ for every } i \in \{1, \dots, d\} \text{ and } \sum_{i=1}^{d} x_i = 1 \right\}.$$

The $d-1$-dimensional Dirichlet distribution with parameter $\alpha \in (0,+\infty)$, denoted $\mathrm{Dir}_{d-1}(\alpha)$, is the probability measure (on Δ_{d-1}) with density

$$\mu_{d,\alpha}(x_1, \dots, x_d) := \frac{\Gamma(d\alpha)}{\Gamma(\alpha)^d \Gamma(d)}\, x_1^{\alpha-1} \dots x_d^{\alpha-1} \tag{31}$$

with respect to dS_d the uniform measure on Δ_{d-1}.

Consider the following discrete time process $(\mathbf{F}_n)$ where $\mathbf{F}_n = F(n, \mathbf{B}, \mathbf{Y})$, $\mathbf{B}$ is a sequence of i.i.d. random variables uniform on $[0,1]$, and $\mathbf{Y} = (Y^u)_{u \in \mathcal{W}_d}$ is a sequence of i.i.d. r.v. with $\mathrm{Dir}_{d-1}(\alpha)$ distribution (independent from $\mathbf{B}$), with $d=3$, and $\alpha = \frac{1}{d-1} = 1/2$. When like here, the choice of the interval that will be fragmented is equal to the size of the fragment, the fragmentation is said to be biased by the size.

Proposition 22. *If $d=3$ and $\alpha = \frac{1}{d-1}$ for any $K \geq 1$ the distribution of $\pi(\mathbf{F}_K)$ is $\mathbb{Q}^{\mathsf{ter}}_{3K+1}$.*

Note 5. *The construction done in the ternary case, can be extended in the d-ary case without any problem (with $d \geq 2$). The distribution of the underlying fragmentation tree corresponding to the Dirichlet distribution $\mu_{d,\alpha}$ with $\alpha = \frac{1}{d-1}$ is a distribution on d-ary tree similar to $\mathbb{Q}^{\mathsf{ter}}_{3K+1}$: it corresponds to d-ary increasing trees, and can also be constructed thanks to successive insertions of internal nodes uniformly on the existing leaves.*

Proof. Denote by $\mathbb{P}^F_{3K+1}$ the distribution of $\pi(\mathbf{F}_K)$ and $\mathbf{t}^{(K)}$ a r.v. having this distribution. Knowing $Y^\varnothing = (Y_1^\varnothing, Y_2^\varnothing, Y_3^\varnothing)$, the distribution of the vector $(|\mathbf{t}_1^{(K)\circ}|, |\mathbf{t}_2^{(K)\circ}|, |\mathbf{t}_3^{(K)\circ}|)$ giving the size of the subtrees of $\mathbf{t}^{(K)}$ is multinomial $(K-1, Y_1^\varnothing, Y_2^\varnothing, Y_3^\varnothing)$; indeed for $i \in \{1,2,3\}$, $|\mathbf{t}_i^{(K)\circ}| = \#\{l \in \{1, \dots, K-1\} \,|\, B_l \in I_i\}$ where $I_i = [\sum_{j=1}^{i-1} Y_j^\varnothing, \sum_{j=1}^{i} Y_j^\varnothing)$. Let us integrate this. We have

$$\mathbb{P}^F_{3K+1}(\{t \mid |t_i^\circ| = k_i, i \in \{1,2,3\}\}) = \int_{\Delta_2} \binom{K-1}{k_1, k_2, k_3} x_1^{k_1} x_2^{k_2} x_3^{k_3} \mu_{3,\frac{1}{2}}(x_1,x_2,x_3) dS_3(x_1,x_2,x_3) \tag{32}$$

for any non negative integers k_1, k_2, k_3 summing to $K-1$. This leads to

$$\mathbb{P}^F_{3K+1}(\{t \mid |t_i^\circ| = k_i, i \in \{1,2,3\}\}) = \binom{K-1}{k_1, k_2, k_3} \frac{\Gamma(3/2)}{\Gamma(1/2)^3} \frac{\prod_{i=1}^{3} \Gamma(k_i + 1/2)}{\Gamma(k_1+k_2+k_3+3/2)}. \tag{33}$$

The comparison with $\mathbb{Q}^{\mathsf{ter}}$ is done as follows. Let count the number of constructions leading to a tree t such that $|t_i^\circ| = k_i, i \in \{1,2,3\}$. The sum of the number of histories of the trees with m internal nodes is $N_m := \prod_{i=0}^{m-1}(2i+1)$ since at each step of the history, we choose one leaf of the tree and at the k-th step, the tree has $2k+1$ leaves. Hence

$$\mathbb{Q}^{\mathsf{ter}}_{3K+1}(\{t \mid |t_i^\circ| = k_i, i \in \{1,2,3\}\}) = \binom{K-1}{k_1, k_2, k_3} \frac{\prod_{i=1}^3 N_{k_i}}{N_K}.$$

Since

$$\Gamma(k_i + 1/2) = 2^{-k_i} N_{k_i} \Gamma(1/2)$$

and

$$\Gamma(k_1 + k_2 + k_3 + 3/2) = 2^{-k_1-k_2-k_3} N_{k_1+k_2+k_3+1} \Gamma(3/2),$$

and after simplification, we check that $\mathbb{Q}^{\mathsf{ter}}_{3K+1}(\{t \mid |t_i^\circ| = k_i, i \in \{1,2,3\}\}) = \mathbb{P}^F_{3K+1}(\{t \mid |t_i^\circ| = k_i, i \in \{1,2,3\}\})$, that is the size of the subtrees of the root have the same distribution under $\mathbb{Q}^{\mathsf{ter}}_{3K+1}$ or under $\mathbb{P}^F_{3K+1}$.

To conclude the proof of the proposition, we use a recursive argument, by conditioning on the subtree sizes. Fix k_1, k_2, k_3 summing to $K-1$, and consider the law of $(t_1^\circ, t_2^\circ, t_3^\circ)$ conditionally on $\{|t_i^\circ| = k_i, i \in \{1,2,3\}\}$ under $\mathbb{Q}^{\mathsf{ter}}_{3K+1}$, and next under $\mathbb{P}^F_{3K+1}$. One then checks easily that under this conditioning and under each law, the subtrees $(t_1^\circ, t_2^\circ, t_3^\circ)$ is a vector of independent random variables, and have respectively the law $\mathbb{Q}^{\mathsf{ter}}_{3k_1+1} \times \mathbb{Q}^{\mathsf{ter}}_{3k_2+1} \times \mathbb{Q}^{\mathsf{ter}}_{3k_3+1}$ and $\mathbb{P}^F_{3k_1+1} \times \mathbb{P}^F_{3k_2+1} \times \mathbb{P}^F_{3k_3+1}$ (the fundamental reason for the fragmentation case is that independent random variables $U_1, \dots, U_k$ conditioned to belong to an interval I are again independent and uniform in I). $\square$

Proof of Proposition 20. Let r fixed, and let $n > r$. Let us examine the probability that $\mathbf{w}$ is not in $\mathbf{t}^{(r)} = \pi(\mathbf{F}_r)$. Conditionally on $I(r) := (I_u, u \in \mathbf{t}^{(r)})$ the vector size $(|\mathbf{t}_u^{(n)\circ}|, u \in \partial\mathbf{t}^{(r)})$, giving the number of internal nodes in the fringe subtrees of $\mathbf{t}^{(n)}$ at the leaves of $\mathbf{t}^{(r)}$, has the multinomial $(n-r, (|I_u|, u \in \partial\mathbf{t}^{(r)}))$ distribution. The event $\{\mathbf{w} \notin \mathbf{t}^{(r)\circ}\}$ is equal to the event

$$E_r := \{\mathbf{u} \text{ and } \mathbf{v} \text{ belong to the same subtree of the family } (\mathbf{t}_u^{(n)\circ}, u \in \partial\mathbf{t}^{(r)})\}.$$

Knowing $I(r)$ the probability of E_r is $\sum_{u\in\mathbf{t}^{(r)}} |I_u|^2 \leq \max|I_u| \sum_{u\in\mathbf{t}^{(r)})} |I_u| = \max|I_u|$, since this event occurs only if the two variables uniform b_i corresponding to $\mathbf{u}$ and $\mathbf{v}$ belong to the same interval I_u. Hence

$$\mathbb{P}(|\mathbf{w}| \geq r) \leq \mathbb{P}(\mathbf{w} \notin \mathbf{t}^{(r)\circ}) \leq \mathbb{E}\left(\max_{u\in\partial\mathbf{t}^{(r)}} |I_u|\right) \xrightarrow[r\to+\infty]{} 0 \tag{34}$$

since in a size biased fragmentation process where the fragmentation measure does not charge 0, the maximal size of the fragments goes a.s. to 0 when the time $\to +\infty$. Property (3) follows.

We now prove (1). By the strong long of large number, knowing $I(r)$,

$$\forall u \in \partial\mathbf{t}^{(r)}, \quad \frac{|\mathbf{t}_u^{(n)\circ}|}{n} \xrightarrow[n]{(a.s.)} |I_u| \tag{35}$$

and then one will assume now that N is chosen such that for any $n \geq N$, $\frac{|\mathbf{t}_u^{(n)\circ}|}{n} \in [|I_u|/2, 2|I_u|]$ for any $u \in \partial\mathbf{t}^{(r)}$ (the lengths $|I_u|$ are a.s. all non zero). Again, conditionally to their sizes

$(|\mathbf{t}_u^{(n)\circ}|, u \in \partial \mathbf{t}^{(r)}) = (s_1, \dots, s_{2r+1})$, the trees $(\mathbf{t}_u^{(n)\circ}, u \in \partial \mathbf{t}^{(r)})$ are independent trees, the ith having the law $\mathbb{Q}_{s_i}^{\mathsf{ter}}$.

Consider the event $E_{r,\varepsilon} := \{\max I(r) < \varepsilon\}$. As said above, for any $\varepsilon' > 0$, any $\varepsilon > 0$, if r is large enough, $\mathbb{P}(E_{r,\varepsilon}) \geq 1 - \varepsilon'$. Let us condition by $E_{r,\varepsilon}$, and let us choose $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{t}^{(n)\circ}$, with $n > N$. With probability close to 1 (under $E_{r,\varepsilon}$), the random nodes $\mathbf{u}$ and $\mathbf{v}$ belong to two different subtrees $|\mathbf{t}_w^{(n)\circ}|$ and $|\mathbf{t}_{w'}^{(n)\circ}|$ for $\{w, w'\} \subset \partial \mathbf{t}^{(r)}$; in this case, their distance in $\mathbf{t}^{(n)}$ is, up to 2, the sum of their respective height in $|\mathbf{t}_w^{(n)\circ}|$ and $|\mathbf{t}_{w'}^{(n)\circ}|$. Let us condition additionally on w and w'. Knowing that $\mathbf{u}$ is in $|\mathbf{t}_w^{(n)\circ}|$, $\mathbf{u}$ is uniform in this tree. By the characterization of the law of $\mathbf{t}_w^{(n)\circ}$ conditionally to its size $a_n := |\mathbf{t}_w^{(n)\circ}|$, and (30), we get

$$\left(\frac{3}{2}\log a_n\right)^{-1/2}\left(\mathbf{u} - \frac{3}{2}\log a_n\right) \xrightarrow[n]{(d)} N_1; \tag{36}$$

Since conditionally on $I(r)$, for any n large enough $a_n \in [|I_w|n/2, 2|I_w|n]$, then

$$\left(\frac{3}{2}\log n\right)^{-1/2}\left(\mathbf{w} - \frac{3}{2}\log n\right) \xrightarrow[n]{(d)} N_1, \tag{37}$$

and the same hold for $\mathbf{v}$, the limiting variable being noted N_2. Since conditionally to their size, $\mathbf{t}_w^{(n)\circ}$ and $\mathbf{t}_{w'}^{(n)\circ}$ are independent, the distance between $\mathbf{u}$ and $\mathbf{v}$ is conditionally to these sizes, independent. Since this independence holds whatever are w and w', and since the limit of $\left(\left(\frac{3}{2}\log n\right)^{-1/2}(\mathbf{u} - \frac{3}{2}\log n), \left(\frac{3}{2}\log n\right)^{-1/2}(\mathbf{v} - \frac{3}{2}\log n)\right)$ is (N_1, N_2) with N_1, N_2 i.i.d. Gaussian $N(0,1)$, whatever are $w \neq w'$ fixed in $\mathbf{t}^{(r)}$, (i) holds true.

Property (ii) follows a simple symmetry argument. □

We give now some indications about the limiting behavior of triangulations under the law $\mathbb{Q}_{2n}^{\triangle}$.

7.2 Some features of large maps under $\mathbb{Q}^{\triangle}$

Some asymptotic results allowing to understand the behavior of large maps under $\mathbb{Q}^{\triangle}$ can also be proved using the fragmentations processes. In particular using that the size of a subtree rooted on a given node u evolves (asymptotically) linearly in time (this is due, as said before, to the rate of insertions of nodes in T_u which is constant and given by $|I_u|$), the same results holds true for a fixed face in the triangulation. Moreover, the length $|I_u|$ is the product of $|u|$ marginals of Poisson-Dirichlet random variables. Hence $N_n(f)$ the number of internal nodes present in the canonical face f at time n behaves as follows: $n^{-1}N_n(f)$ converges a.s. toward a random variable N_f almost surely in $(0,1)$. This fragmentation point of view allows to prove much more : the a.s. joint convergence of $n^{-1}(N_{f_1}, \dots, N_{f_k})$ for the (disjoint or not) faces f_i of m_j toward a limiting random variable taking its value in $\mathbb{R}^k$, and whose limiting distribution may be described in terms of product of Poisson-Dirichlet random variables.

The degree of a node may also be followed when n goes to $+\infty$. If $v(j)$ denotes the jth node inserted in m_n, one may prove that $\deg(v(j))$ goes to infinity with n. The degree of a node follows indeed a simple Markov chain since it increases if and only if a node is inserted in a face adjacent to $v(j)$ and this occurs with a probability equals to $\deg(v(j))$ divided by the current number of internal faces. Denoting by D_j^n the degree of $\deg(v(j))$ at time n (recall that $D_j^j = 3$), under $\mathbb{Q}_{2n}^{\triangle}$, we have that for $n > j$ and $k \geq 3$, conditionally on D_j^n

$$D_j^{n+1} = D_j^n + B\left(D_j^n/(2n-1)\right) \tag{38}$$

where we have denoted by $B(p)$ a Bernoulli random variable with parameter p (in other words $\mathbb{Q}_{2(n+1)}^{\triangle}(\deg(v(j)) = k+1) = \frac{k}{2n-1}\mathbb{Q}_{2n}^{\triangle}(deg(v(j)) = k) + \frac{2n-k-2}{2n-1}\mathbb{Q}_{2n}^{\triangle}(deg(v(j)) = k+1))$.
This chain has the same dynamics as the following simple model of urn. Consider an urn with 3 white balls and $2j-2$ black balls at time 0. At each step pick a ball and replace it in the urn. If the picked ball is white then add one white ball and one black ball, and if it is black, add two black balls. The number N_j^t of white balls at time t has the same law as D_j^{j+t} (the number of black balls behaves as the number of finite faces of m_{j+t} not incident to $v(j)$). This model of urn has been studied in Flajolet & al. [22, p.94] (to use their results, take $a_0 = 3$, $b_0 = 2j-2$, $\sigma = 2$, $\alpha = 1$ and replace n by $n-j$). For example, we derive easily from their results the following proposition.

Proposition 23. *Let m_n be a map $\mathbb{Q}_{2n}^{\triangle}$ distributed and $v(j)$ the j-th node inserted, for $n > j$ and $1 \leq k \leq n-j$, we get*

$$\mathbb{Q}_{2n}^{\triangle}(deg_{m_n}(v(j)) = k+3) = \frac{\Gamma(n-j+1)\Gamma(j+\frac{1}{2})}{\Gamma(n+\frac{1}{2})}\binom{k+2}{k}\sum_{i=0}^{k}(-1)^i\binom{k}{i}\binom{n-\frac{i}{2}-2}{n-j}$$

where $\binom{a}{b} = a(a-1)\dots(a-b+1)/b!$.

This model of urns has also been studied by Janson [26]; Theorem 1.3 in [26] gives the asymptotic behavior of urns under these dynamics, depending on the initial conditions. The discussion given in Section 3.1 of [26] shows that the asymptotic behavior of $D_j(n)$ is quite difficult to describe. One may use (38) to see that $\mathbb{E}(D_j^{n+1} \mid D_j^n) = D_j^n(1 + \frac{1}{2n-1})$ to show that $(M_j^n)_{n\geq j}$ defined by

$$M_j^n = D_j^n/u_n$$

is a $\mathcal{F}_n$ martingale, where $\mathcal{F}_n = \sigma(D_j^k, j \leq k \leq n)$ for any sequence u_n such that $u_{n+1} = u_n(2n)/(2n-1)$. This allows to see that

$$\mathbb{E}(D_j^n) = \mathbb{E}(D_j^j)\prod_{k=j}^{n-1}(2k)/(2k-1) = 3\prod_{k=j}^{n-1}(2k)/(2k-1).$$

This indicates that for a fixed j the expectation $\mathbb{E}(D_j^n)$ grows as $\sqrt{n}$. Some other regimes may be obtained: for $t \in (0,1)$, $\mathbb{E}(D_{nt}^n) \to 3(1-t)^{-1}$ when n goes to $+\infty$. (We recall that any triangulation with $2n$ faces has $3n$ edges and $n+2$ nodes; hence the mean degree of a node in $6n/(n+2)$ in any triangulation).

8 Two families of increasing quadrangulations

We present here two families of quadrangulations. The first one, quite natural, resists to our investigations. The second one, that may appear to be quite unnatural, is in fact very analogous to stack-triangulations, and is studied with the same tools.

8.1 A first model of growing quadrangulations

This is the simplest model, and we present it rapidly: the construction starts from a rooted square. Assume that a quadrangulation has yet be constructed. Choose a finite face $f = (A, B, C, D)$ and a diagonal AC or BD. Then add inside f, a node x and the two edges Ax and xC or the two edges Bx and xD. Each finite face inherits a root from the construction in the following way : assume that f is rooted in (A, B), if we add the edges Ax and xC, then the new faces (A, B, C, x) and (A, x, C, B) are rooted respectively in (B, C) and (x, C) otherwise the new faces (A, B, x, D) and (x, B, C, D) are rooted respectively in (A, B) and (x, B). It is easy to check that the root of each face is well defined and does not depend of the order of insertions of the edges. The set $\square'_k$ is then the set of quadrangulations with k bounded faces reached by this procedure starting with the rooted square (formally define a growing procedure Φ_4, similar to Φ of Section 2.2, using $\square'^{\bullet}_k = \{(m, f, \alpha) \mid m \in \square'_k, f \in F^\circ(m), \alpha \in \{0, 1\}\}$ the rooted quadrangulations from $\square'_k$ with a distinguished finite face marked with 0 or 1, and add in $f = (A, B, C, D)$ the pair of edges $\{Ax, xC\}$ if $\alpha = 0$ and $\{Bx, xD\}$ otherwise).

There is again some bijections between $\square'_k$ and some set of trees, but we were unable to define on the corresponding trees a device allowing to study the distance in the maps (under the uniform distribution, as well as under the distribution induced by the construction when both f and α are iteratively uniformly chosen). We conjecture that they behave asymptotically in terms of metric spaces as triangulations under $\mathbb{Q}^{\triangle}_{2k}$ and $\mathbb{U}^{\triangle}_{2k}$ up to some normalizing constant.

We describe below a bijection between $\square'_k$ and the set of trees having no nodes having only one child. There exists also a bijection with Schröder trees (trees where the nodes have 0,1 or 2 children) with k internal nodes.

Proposition 24. *For any $k \geq 2$, there exists a bijection Ψ_k between $\square'_k$ and the set of trees having k leaves, no nodes of outdegree 1 and with a root marked 0 or 1.*

For $k = 1$, $\square'_k = \{s\}$ the rooted square and in this case we may set $\Psi(s) = \{\varnothing\}$, the tree reduced to a (non marked) leaf.

Proof. Assume that $k \geq 2$. Let (A, B, C, D) denotes the exterior face of every map $m \in \square'_k$. Split $\square'_k$ into two subsets $\square'_{k,0}$ and $\square'_{k,1}$, where $\square'_{k,0}$ (resp. $\square'_{k,1}$) is the set of maps $m \in \square'_k$ which contains an internal node x and the two edges Ax and xC (resp. Bx and xD). Notice that m cannot contain at the same time an internal node x and Ax and xB. It is easy to see that the rotation of $\pi/2$ is a bijection between $\square'_{k,0}$ and $\square'_{k,1}$. We then focus on $\square'_{k,0}$ and explain the bijection between $\square'_{k,0}$ and the set of trees having no nodes of outdegree 1 and k leaves. Let $x_1, \dots, x_j$ be the $j \geq 1$ internal points of m, adjacent to A and C. These points (if properly labeled) define $j+1$ submaps $m_1, \dots, m_{i+j}$ of m with border (A, x_i, C, x_{i+1}) for $i = 0$ to j where $B = x_0$ and $D = x_{j+1}$. We then build $t = \Psi_k(m)$ by sending m onto the root of t, and m_i to the ith child of the root of t. Each of the submaps m_i can also be decomposed in the same way (the root of m_i induces an ordering on its internal vertices which permits to repeat the construction) except that by maximality of the set $\{x_1, \dots, x_j\}$, the face (A, x_i, C, x_{i+1}) is either empty or contains an internal node y adjacent to x_i and x_{i+1}. The coloring of the nodes (except the root) is then useless. $\square$

8.2 A family of stack-quadrangulations

The construction presented here is very similar to that of stack-triangulations; some details will be skipped when the analogy will be clear enough. The difference with the model of quadrangulations of Section 8.1 is that given a face $f = (A, B, C, D)$, only a suitable choice of pair of edges (either $\{Ax, xC\}$ or $\{Bx, xD\}$) will be allowed. This choice amounts to forbidding "parallel" pair of edges of the type (Ax, xC) and $(Ax', x'C)$.

Formally, set first $\square_1 = \{s\}$ where s is the unique rooted square. The unique element of $\square_2$ (say s_2) is obtained as follows. Label by (E_0, E_1, E_2, E_3) the vertices of s, such that (E_0, E_1) is the root of s. To get s_2, draw s in the plane, add in the bounded face of s a node x and the two edges E_0x and xE_2 in this face. We define now $\square_k$ recursively asking to the maps m with border (A, B, C, D) and rooted in (A, B) to have the following property of decomposability. If $k \geq 2$ there exists a unique node x in the map m, such that Ax and xC are edges of m. Moreover the submaps m_1 and m_2 of m with respective borders (A, x, C, D) (rerooted in (x, C)) and (A, B, C, x) (rerooted in (B, C)) belong both to the sets $\cup_{j<k}\square_j$, more precisely $(m_1, m_2) \in \cup_{j=1}^{k-1}\square_j \times \square_{k-j}$ (see an illustration on Figure 11).

This rerooting operation corresponds to distinguish a diagonal in each face (once for all) on which the following insertion inside this face, if any, will take place.

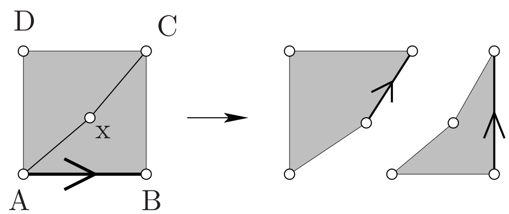

Figure 11: The decomposition is well defined thanks to the uniqueness of a node x adjacent to both A and C.

Any map belonging to $\square_k$ is a rooted quadrangulation having k internal faces. There exists again a canonical drawing of these maps, where the border (E_0, E_1, E_2, E_3) (rooted in (E_0E_1)) of the quadrangulations is sent on a fixed square of the plane, and where, when it exists, the unique node x adjacent to both E_0 and E_2 is sent of the center of mass of (E_0, E_1, E_2, E_3), the construction being continued recursively in the submaps m_1 and m_2 (the edges are straight lines).

There exists also a sequential construction of this model, more suitable to define the distribution of interest.

8.2.1 Sequential construction of $\square_k$

We introduce a labeling of the nodes of $\square_k$ by some integers. The idea here is double. This labeling will distinguish the right diagonal where the (only allowed) pair of edges will be inserted, and also will be used to count the number of histories leading to a given map. A labeled map may be viewed as a pair (m, l) where m is an unlabeled map and l an application from $V(m)$ onto the set of integers.

We then consider $\square_k^\ell$ the set of label quadrangulations having k internal faces, built as follows. First $\square_1^\ell$ contains the unique labeled rooted map (s,l) with vertices (E_0,E_1,E_2,E_3) rooted in (E_0,E_1) and labeled by

$$l(E_0)=4, l(E_1)=3, l(E_2)=2, l(E_3)=1.$$

Assume now that $\square_k^\ell$ has been defined for some $k\geq 1$, and is a set of quadrangulations with k internal faces (and thus $k+3$ vertices), where the vertices are labeled by different integers from $\{1,\dots,k+3\}$. To construct $\square_{k+1}^\ell$ from $\square_k^\ell$ we consider an application Φ_4^ℓ from $\square_k^{\ell,\bullet} = \{((m,l),f) \mid m\in\square_k^\ell, f\in F^\circ(m)\}$ taking its values on the set of labeled quadrangulations with $k+1$ finite faces. The map $\Phi_4^l((m,l),f)$ is the map obtained by the following procedure:
- draw m in the plane;
- denote by (A,B,C,D) the vertices of f, such that A has the largest label (and thus C is at the opposite diagonal of A in f),
- add a point x labeled $k+4$ in f and the two edges Ax and xC in f. The obtained labeled map is $\Phi_4^l((m,l),f)$.

We denote by $\square_{k+1}^\ell$ the set $\Phi_4^\ell(\square_k^{\ell,\bullet})$.

For $(m,l)\in\square_k^\ell$, we call π_k (or more simply π) the function defined by $\pi_k\big((m,l)\big)=m$; this is simply the application that erases the labels of a labeled map. We now show that $\pi(\square_k^\ell)\subset\square_k$, in other words, taking into account that we have no doubt on the size of the maps, or on the fact that it is a quadrangulation, we just have to check that the no parallel pairs of edges have been constructed. Since the vertex of f whose label is the greatest (among the four vertices of f) is the most recent vertex inserted in the map (among the four vertices of f), it is clear that this cannot produce parallel pairs of edges.

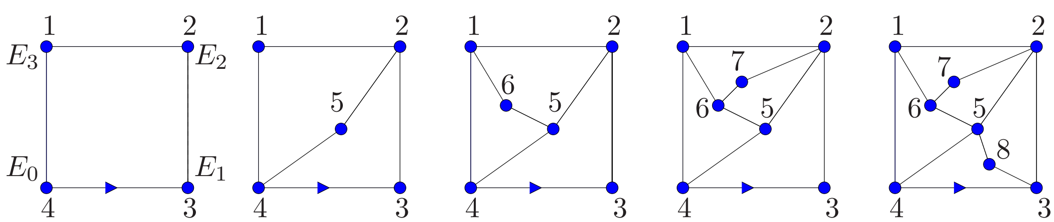

Figure 12: A sequence of quadrangulations obtained by successive insertions of pair of edges.

Consider a labeled map $(m_k,l_k)\in\square_k^\ell$ for some $k\geq 2$. There exists a unique map (m_{k-1},l_{k-1}) in $\square_{k-1}^\ell$ such that $(m_k,l_k)=\Phi_4^\ell(m_{k-1},l_{k-1})$. It is obtained from (m_k,l_k) by the suppression of the node with largest label together with the two edges that are incident to this node. Hence, each map (m_k,l_k) characterizes uniquely a "legal" history of $m_k=\pi(m_k,l_k)$, meaning that for i, m_{i+1} is obtained from m_i by the insertion of two edges, and for any i, m_i is in $\square_i$. From now on, we will make a misuse of language and confound the histories of a stack-quadrangulation $m_k\in\square_k$ and $\pi^{-1}(m_k)$.

We denote by $\mathbb{U}_k^\square$ the uniform distribution on $\square_k$ and as done for triangulations in Section 2.2.1, $\mathbb{Q}_k^\square$ denotes the distribution of $\pi(M_k,l_k)$ when $M_{i+1}=\Phi_4^\ell((M_i,l_i),F_i)$, where M_1 is the unique element of $\square_1$ and where F_i is chosen uniformly among the internal faces of M_i (all the F_i are independent). The support of $\mathbb{Q}_k^\square$ is the set $\square_k$, and one may check that $\mathbb{Q}_k^\square\neq\mathbb{U}_k^\square$ for $k\geq 4$.

8.3 The function Λ'

As in Section 2.3.2, we define a function Λ' to express the distance between any pair of nodes u and v in a stack-quadrangulation m in terms of a tree associated bijectively. Let

$$W_{1,2} = \{12, 21\}^\star \cdot \{11, 22\} \cdot \{1, 2\},$$

be the set of words on $\Sigma_2 = \{1, 2\}$, beginning with any number of occurrences of 12 or 21, followed by 11 or 22, then by a 1 or a 2. Notice that all the words of $W_{1,2}$ have an odd length. For example $u = 12\,21\,21\,11\,2 \in W_{1,2}$.

Let $u = u_1 \dots u_k$ be a word on the alphabet Σ_2. Define $\tau_1(u) := 0$ and for $j \geq 2$,

$$\tau_j(u) := \inf\{i \mid i \geq \tau_{j-1}(u) \text{ such that } u_{1+\tau_{j-1}(u)} \dots u_i \in W_{1,2}\}. \tag{39}$$

This amounts to decomposing u into subwords belonging to $W_{1,2}$. We denote by $\tilde{\Lambda}'(u) = \max\{i \mid \tau_i(u) \leq |u|\}$, then $u = u_1 \dots u_{\tau_{\tilde{\Lambda}'(u)}(u)}\tilde{u}$, where $\tilde{u} \notin W_{1,2}$. Lastly we define $\Lambda'(u)$ as

$$\Lambda'(u) = \tilde{\Lambda}'(u) + \begin{cases} 0 & \text{if } |\tilde{u}| \text{ is even and } \tilde{u} \text{ does not end with 11 or 22} \\ 1 & \text{otherwise} \end{cases}$$

Further, for two words $u = wa_1 \dots a_k$ and $v = wb_1 \dots b_l$ (with $a_1 \neq b_1$), set as in the triangulation case $\Lambda'(u, v) = \Lambda'(a_1 \dots a_k) + \Lambda'(b_1 \dots b_l)$.

We now give a proposition for stack-quadrangulations similar to Proposition 1.

Proposition 25. *For any $K \geq 1$, there exists a bijection*

$$\begin{array}{rccc} \Psi_K^{\square}: & \square_K & \longrightarrow & \mathcal{T}_{2K-1}^{\mathsf{bin}} \\ & m & \longmapsto & t := \Psi_K^{\square}(m) \end{array}$$

such that :
(a) each internal node u of m corresponds bijectively to an internal node u' of t.
(b) Each leaf of t corresponds bijectively to a finite quadrangular face of m.
(c) For any u internal node of m, $\Lambda'(u') = d_m(E_0, u)$.
(d) For any u and v internal nodes of m

$$\left|d_m(u, v) - \Lambda'(u', v')\right| \leq 4. \tag{40}$$

(e) Let u be an internal node of m. We have

$$\deg_m(u) = 2 + \#\{v' \in t^\circ \mid v' = u'w', |w'| \geq 2, w' \in \{12, 21\}^\star\}.$$

The existence of a bijection between $\square_K$ and $\mathcal{T}_{2K-1}^{\mathsf{bin}}$ comes from the recursive decomposition of a stack-quadrangulation along the first pair of edges inserted (which can be determined at any time since there is a unique node x adjacent to both A and C in any $m \in \square_K$, for $K \geq 2$).

Proof. The proof of this Proposition is very similar to that of Proposition 2. We only sketch the main lines. We propose a bijection that does not follow the decomposition provided in Figure 11, but which is illustrated in Figure 13.

First, the maps in $\square_K$ own also a canonical drawing as said above, hence we can consider the set $\mathcal{F}_K^{\square}$ of canonical faces belonging to at least one of the canonical drawings of a map of $\square_K$ (which is the analogue of the set $\mathcal{F}_K$ introduced in Section 2.3.2). We keep the convention that $f = (A, B, C, D)$ refers to the canonical face f admitting (A, B) as distinguished edge.

We construct a bijection $\psi^{\square}$ which associates a word of $\Sigma_2^{\star}$ with each canonical face of a stack-quadrangulation. There is a unique canonical face (B, C, D, A) in $\mathcal{F}_2^{\square}$, its image by $\psi^{\square}$ is set to be the empty word on Σ_2. We proceed then by induction: assume that $\psi^{\square}$ is well defined on $\mathcal{F}_j^{\square}$ for $j \leq K$ and let $f = (A, B, C, D)$ be a canonical face of $\mathcal{F}_K^{\square}$. The insertion of a node x and of the two corresponding edges in the face f (in a quadrangulation having f as a face) gives birth to two "new" canonical faces which are set to be (B, x, D, A) and (B, x, D, C). If the image of f by $\psi^{\square}$ is u, we associate respectively to these two new faces the nodes $u1$ and $u2$. Notice that $u1$ (resp. $u2$) corresponds to the face situated on the left (resp. on the right) of the distinguished edge (B, x) (see Figure 13).

With slight modifications in the proof of Proposition 1, we see that the bijection $\psi^{\square}$ induces a bijection $\psi_K^{\square}$ between the set $H_{\square}(K)$ of histories of maps of $\square_K$ and $H_{\mathcal{T}}(K)$ the set of histories of trees of $\mathcal{T}_{2K-1}^{\mathsf{bin}}$ (for any $K \geq 1$).

We associate now with any stack-quadrangulation a binary tree as represented on Figure 13. Formally let m_K be a stack-quadrangulation and h_K be one of its history, we define t_K *the* tree of $\mathcal{T}_{2K-1}^{\mathsf{bin}}$ whose history is $\psi_K^{\square}$. The tree t_K is well defined thanks to the properties of consistence and robustness of $\psi_K^{\square}$, details are given for the case of stack-triangulations in Lemma 2. We finally set $\Psi_K^{\square}(m_K) := t_K$.

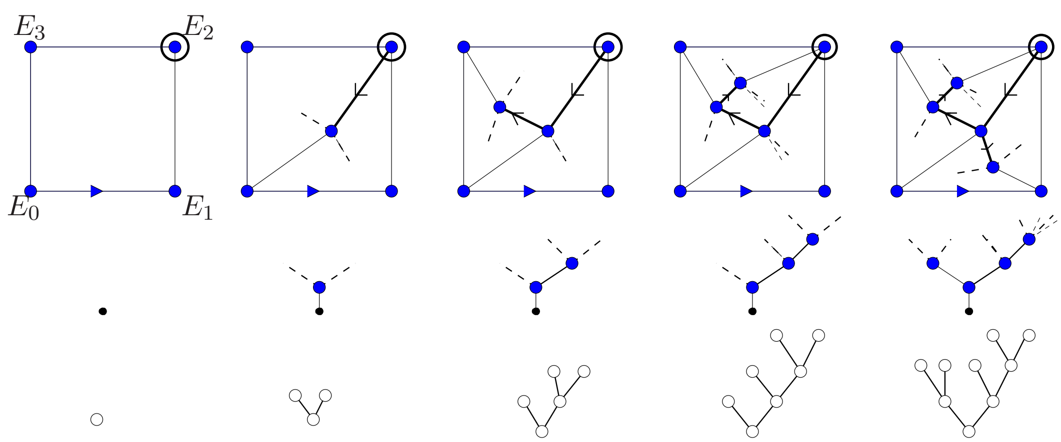

Figure 13: A sequence of quadrangulations obtained by successive insertions of pair of edges.

To prove properties (c), d and (e) we introduce a notion of type of faces in a stack-quadrangulation (or type of a node in the corresponding tree) as in the proof of Proposition 2. For any face $f = (A, B, C, D)$ in m such that $O(f) = (A, B)$, we set:

$$\text{type}(A, B, C, D) := (d_m(E_0, A), d_m(E_0, B), d_m(E_0, C), d_m(E_0, D))$$

the 4-tuple of the distance of A, B, C and D to the root vertex of m. It is well known that in a quadrangulation, the type of any face is $(i, i+1, i, i+1)$ or $(i, i+1, i+2, i+1)$, for some i, or a circular permutation of this.

As the types of the faces arising in the construction are not modified by the insertions of new edges, we mark any node of $t = \Psi^{\square}(m)$ with the type of the corresponding face. It is then easy to check that for u' an internal node of t with $\mathrm{type}(u') = (a, b, c, d)$, we have $d_m(u, E_0) = 1 + b \wedge d$ and

$$\begin{cases} \mathrm{type}(u'1) = (\quad b, \quad 1 + b \wedge d, \quad d, \quad a \quad), \\ \mathrm{type}(u'2) = (\quad b, \quad 1 + b \wedge d, \quad d, \quad c \quad), \end{cases} \tag{41}$$

Property (c) follows directly from (41) using the fact that $\mathrm{type}(\varnothing) = (1, 2, 1, 0)$. Properties (d) and (e) are deduced directly by the same arguments as for triangulations. $\square$

8.3.1 Asymptotic behavior of the quadrangulations

First, we state a Lemma analogous to Lemma 14:

Lemma 26. *Let $(X_i)_{i\geq 1}$ be a sequence of i.i.d. random variables taking their values in $\Sigma_2 = \{1, 2\}$ and let W_n be the word $X_1 \dots X_n$.*
(i) $n^{-1}\Lambda'(W_n) \xrightarrow[n]{(a.s.)} \Lambda'_{\square}$ *where*

$$\Lambda'_{\square} := 1/5 \tag{42}$$

(ii) $\mathbb{P}(|\Lambda'(W_n) - n\Lambda'_{\square}| \geq n^{1/2+u}) \to 0$ *for any $u > 0$.*

Proof. It is proved similarly to Lemma 14 except that here if $(X_i)_{i\geq 1}$ is a sequence of i.i.d r.v uniformly distributed in $\{1, 2\}$ and if $W = X_1X_2 \dots$ then for $N \geq 3$ and $k \geq 2$,

$$\mathbb{P}(\tau_2(W) = 2k + 1) = \mathbb{P}(X_1 \neq X_2, \dots, X_{2k-3} \neq X_{2k-2}, X_{2k-1} = X_{2k}) = \frac{1}{2^k},$$

which means hat $\tau_2(W)$ (and $\tau_i(W) - \tau_{i-1}(W)$ as well) has the same law as $1 + 2\mathsf{Geometric}(1/2)$ whose mean is 5. $\square$

We are now in position to state the main theorem of this part. We need to examine first the weak limit of binary trees. Denote by P^{bin}_{2n+1} the uniform distribution on the set of binary trees with $2n + 1$ nodes. This time $P^{\mathsf{bin}}_{\infty}$ is the distribution of a random infinite tree, build around an infinite line of descent $L^{\mathsf{bin}}_{\infty} = (X(j), j \geq 0)$, where $X(j) = X_1 \dots X_j$ [and (X_i) is a sequence of i.i.d. r.v. uniformly distributed on $\Sigma_2 = \{1, 2\}$] on the neighbors of which are grafted critical GW trees with offspring distribution $\nu_{\mathsf{bin}} = \frac{1}{2}(\delta_0 + \delta_2)$. We sum up in the following Proposition the results concerning the convergence of trees under P^{bin}_{2n+1}.

Proposition 27. *(i) When $n \to +\infty$, P^{bin}_{2n+1} converges weakly to $P^{\mathsf{bin}}_{\infty}$ for the topology of local convergence.*
(ii) The following convergence holds for the GH topology. Under P^{bin}_{2n+1},

$$\left(T, \frac{d_T}{\sqrt{2n}}\right) \xrightarrow[n]{(d)} (\mathcal{T}_{2\mathsf{e}}, d_{2\mathsf{e}}).$$

The first point very similar to Proposition 6 is due to Gillet [23], the second point to Aldous [1].

The results concerning triangulations can be extended to the present model of quadrangulations. In particular, a construction of an infinite map $m^{\square}_{\infty}$ similar to m_{∞} can be done. Following the lines of the triangulation case, one may prove the following result:

Theorem 28. *(i) Under $\mathbb{U}_n^\square$, (m_n) converges in distribution to $m_\infty^\square$ for the topology of local convergence.*
(ii) Under $\mathbb{U}_n^\square$,

$$\left(m_n, \frac{D_{m_n}}{\Lambda'_\square\sqrt{2n}}\right) \xrightarrow[n]{(d)} (\mathcal{T}_{2\mathsf{e}}, d_{2\mathsf{e}}),$$

for the Gromov-Hausdorff topology on compact metric spaces.

Now, the asymptotic behavior of maps under $\mathbb{Q}_k^\square$ is studied again thanks to trees under $\mathbb{Q}_{2K-1}^{\mathsf{bin}} := \mathbb{Q}_K^\square \circ (\Psi_K^\square)^{-1}$ the corresponding distribution on trees. This distribution on $\mathcal{T}_{2K-1}^{\mathsf{bin}}$ is famous in the literature since it corresponds to the distribution of binary search trees. Indeed the insertion in the map m corresponds to an uniform choice of a leaf in the tree $\Psi^\square(m)$ and its transformation into an internal node having two children. Again, using the same tools as those used to treat the asymptotic behavior of trees under $\mathbb{Q}^{\mathsf{ter}}$ (in particular, here the fragmentation is binary, and $Y^u \stackrel{d}{=} (U, 1-U)$ where U is uniform in $[0,1]$), we get the following proposition.

We keep in the following Proposition the notation of Proposition 20 when possible.

Proposition 29. *Let $\mathbf{t}$ be a random tree under the distribution $\mathbb{Q}_{2K+1}^{\mathsf{bin}}$.*
1) We have

$$(4\log n)^{-1/2}\left(|\mathbf{u}| - 4\log n, |\mathbf{v}| - 4\log n\right) \xrightarrow[n]{(d)} (N_1, N_2).$$

2) Conditionally to $(|\mathbf{u}^\star|, |\mathbf{v}^\star|)$ (their lengths) $\mathbf{u}^\star$ and $\mathbf{v}^\star$ are independent random words composed with $|\mathbf{u}^\star|$ and $|\mathbf{v}^\star|$ independent letters uniformly distributed in $\Sigma_2 = \{1,2\}$.
3) For any $a_n \to +\infty$, we have $|\mathbf{w}|/a_n \xrightarrow[n]{proba.} 0$.

The interested reader may find in Mahmoud & Neininger [35, Theorem 2] a different proof of the first assertion, the second one, once again being a consequence of the symmetries of this class of random trees.

Similarly to Theorem 21, we obtain the following theorem.

Theorem 30. *Let M_n be a stack-quadrangulation under $\mathbb{Q}_{2n}^\square$. Let $k \in \mathbb{N}$ and $\mathbf{v}_1, \dots, \mathbf{v}_k$ be k nodes of M_n chosen independently and uniformly among the internal nodes of M_n. We have*

$$\left(\frac{D_{M_n}(\mathbf{v}_i, \mathbf{v}_j)}{4\Lambda'_\square \log n}\right)_{(i,j)\in\{1,\dots,k\}^2} \xrightarrow[n]{proba.} (1_{i\neq j})_{(i,j)\in\{1,\dots,k\}^2}.$$

9 Appendix

9.1 Proof of the Theorems of Section 5

The aim of this section is to prove Theorem 16. Our study of the distance in a stack-triangulation m_n passes via the study of the function Λ on the tree $T = \Psi_n^\triangle(m_n)$. Let $w(r)$ be the rth internal node of T according to the LO ($w(0)$ is the root), and $u(r)$ be the rth internal node of m (the image of $w(r)$ as explained in Proposition 1). For any r and s,

$$|d_m(u(r), u(s)) - \Lambda(w(r), w(s))| \leq 4. \tag{43}$$

Lemma 31. *Under $\mathbb{U}^{\triangle}_{2n}$, the family $\left(\left(\frac{d_{m_n}(ns,nt)}{\Lambda_\triangle\sqrt{3n/2}}\right)_{(s,t)\in[0,1]^2}\right)_n$ is tight on $C[0,1]^2$.*

Proof. We claim first that under $\mathbb{U}^{\mathsf{ter}}_{3n+1}$, the family $\left(n^{-1/2}d_{T^\circ}(ns,nt)\right)_n$ is tight in $C[0,1]^2$, where $d_{T^\circ}(k,j)=d_{T^\circ}(w(k),w(j))$ is the (re-parametrization of the) restriction of the distance in T on its set of internal nodes, and where d_{T° is smoothly interpolated as explained below Theorem 15. Indeed, let $(H^\circ(k))_{k=0,\cdots,n-1}$ where $H^\circ(k)=|w(k)|$ be the height process of the internal nodes of T (interpolated between integer points). In Marckert & Mokkadem [38, Corollary 5], the process given the successive height of the nodes of a fixed degree d (according to the LO) in a Galton-Watson tree conditioned by the size is studied, and is shown to converge to the Brownian excursion, under a suitable rescaling. In a ternary tree, the process giving the successive height of the nodes of degree 3 coincides with H°. Using [38], one check easily that

$$\left(\frac{H^\circ(nt)}{\sqrt{3n/2}}\right)_{t\in[0,1]}\xrightarrow[n]{(d)}(2\mathsf{e}_t)_{t\in[0,1]}\,. \tag{44}$$

An alternative proof pointed out by a referee raises on the following claim: if T is a GW tree under P^{ter} (resp. P^{ter}_{3n+1}), then T° is also a Galton-Watson tree with offspring distribution $\mu(0)=8/27,\mu(1)=12/27,\mu(2)=6/27,\mu(3)=1/27$ (resp. conditioned to have n nodes); the mean of the offspring distribution is 1, and its variance $2/3$ leading to (44) readily. A formal proof of this claim is a bit long, but the idea is simple: a node in T° has degree a if the corresponding node in T has a children having some children, and $3-a$ children who have no child.

Using that for $i\le j$,

$$\left|d_{T^\circ}(w(i),w(j))-(H^\circ(i)+H^\circ(j)-2\min_{k\in\{i,i+1,\dots,j\}}H^\circ(k))\right|\le 2$$

we get that

$$\left(\frac{d_{T^\circ}(ns,nt)}{\sqrt{3n/2}}\right)_{s,t\in[0,1]}\xrightarrow[n]{(d)}(d_{2\mathsf{e}}(s,t))_{s,t\in[0,1]}$$

where the convergence holds in $C[0,1]^2$. This is just a consequence of the continuity of the application $f\mapsto\left[(s,t)\to f(s)+f(t)-2\min_{u\in[s,t]}f(u)\right]$ from $C[0,1]$ onto $C[0,1]^2$. We deduce from this that the sequence $\left(\left(\frac{d_{T^\circ}(ns,nt)}{\sqrt{3n/2}}\right)_{s,t\in[0,1]}\right)_n$ is tight and by (43) and the trivial bound $\Lambda(u,v)\le d_{T^\circ}(u,v)$ for any u and $v\in T^\circ$,

$$\frac{d_{m_n}(ns,nt)}{\sqrt{3n/2}}\le\frac{d_{T^\circ}(ns,nt)}{\sqrt{3n/2}}+4n^{-1/2}$$

and thus the Lemma holds true. $\square$.

The convergence of the finite dimensional distributions in Theorem 16 is a consequence of the following stronger result.

Proposition 32. *Let $0\le s<t\le 1$. When n goes to $+\infty$, under $\mathbb{U}^{\triangle}_{3n+1}$*

$$\left|\frac{d_{m_n}(\lfloor ns\rfloor,\lfloor nt\rfloor)}{\Lambda_\triangle\sqrt{3n/2}}-\frac{d_{T^\circ}(\lfloor ns\rfloor,\lfloor nt\rfloor)}{\sqrt{3n/2}}\right|\xrightarrow[n]{proba.}0.$$

To prove this Proposition we need to control precisely $\Lambda(w(\lfloor ns\rfloor), w(\lfloor nt\rfloor))$; we will show that this quantity is at the first order, and with a probability close to 1, equal to $\Lambda_\triangle d_{T^\circ}(ns,nt)$. This part is largely inspired by the methods developed in a work of the second author [36].

We focus only on the case s,t fixed in $(0,1)$ and $s<t$ (which is the most difficult case). In the following we write ns and nt instead of $\lfloor ns\rfloor$ and $\lfloor nt\rfloor$. Consider $\check{w}_{ns,nt} = w(ns)\wedge w(nt)$, and write

$$w(ns) = \check{w}_{ns,nt} l_0 l_{ns,nt} \quad \text{and} \quad w(nt) = \check{w}_{ns,nt} r_0 r_{ns,nt}, \tag{45}$$

where $l_0 \neq r_0$ (the letters l and r refer to “left” and “right”).

For compactness of notation, set

$$\begin{aligned} \mathrm{Dec}(n) &:= (W_1,W_2,W_3,H_1,H_2,H_3,L,R)\\ &:= (\check{w}_{ns,nt}, l_{ns,nt}, r_{ns,nt}, |\check{w}_{ns,nt}|, |l_{ns,nt}|, |r_{ns,nt}|, l_0, r_0), \end{aligned}$$

Dec standing for ”decomposition”. Even if not recalled in the statements, these variables are considered as random variables under P^{ter}_{3n+1}. Let now $\widetilde{\mathrm{Dec}}$ be the random variable defined by

$$\widetilde{\mathrm{Dec}}(n) := (\tilde{W}_1,\tilde{W}_2,\tilde{W}_3,H_1,H_2,H_3,\tilde{L},\tilde{R})$$

such that, conditionally on $(H_1,H_2,H_3)=(h_1,h_2,h_3)$, the random variables $\tilde{W}_1,\tilde{W}_2,\tilde{W}_3,\tilde{L},\tilde{R}$ are independent and defined by:
– for each $i\in\{1,2,3\}$, $\tilde{W}_i$ is a word with h_i i.i.d. letters, uniformly chosen in $\{1,2,3\}$,
– the variable $(\tilde{L},\tilde{R})$ is a random variable uniform in $I_3=\{(1,2),(1,3),(2,3)\}$.

Definition 8. *Let $(Y_1,Y_2,\dots)$ and $(X_1,X_2,\dots)$ be two sequences of r.v. taking their values in a Polish space S. We say that $\mathbb{P}_{X_n}/\mathbb{P}_{Y_n} \xrightarrow{\star} 1$ or $X_n /\!\!/_\star\, Y_n \to 1$ if for any $\varepsilon>0$ there exists a measurable set A_n^ε and a measurable function $f_n^\varepsilon : A_n^\varepsilon \mapsto \mathbb{R}$ satisfying $\mathbb{P}_{X_n} = f_n^\varepsilon \mathbb{P}_{Y_n}$ on A_n^ε, such that $\sup_{x\in A_n^\varepsilon} |f_n^\varepsilon(x)-1| \underset{n}{\to} 0$ and such that $\mathbb{P}_{Y_n}(A_n^\varepsilon)\geq 1-\varepsilon$ for n large enough.*

The following lemma is proved in [36, Lemma 16][1].

Lemma 33. *Assume that $X_n /\!\!/_\star Y_n \to 1$ then:*
• *If $Y_n \xrightarrow[n]{(d)} Y$ then $X_n \xrightarrow[n]{(d)} Y$.*
• *Let (g_n) be a sequence of measurable functions from S into a Polish space S'. If $X_n /\!\!/_\star\, Y_n \to 1$ then $g_n(X_n) /\!\!/_\star\, g_n(Y_n) \to 1$*

The main step in the proof of Proposition 32 is the following Proposition.

Proposition 34. *When $n\to+\infty$, $\mathrm{Dec}(n) /\!\!/_\star \widetilde{Dec}(n) \to 1$.*

Assume that this proposition holds true and let us end the proof of Proposition 32.

Proof of Proposition 32. From Proposition 34 and Lemma 33, we deduce

$$(H_2,H_3,W_2,W_3) /\!\!/_\star (H_2,H_3,\tilde{W}_2,\tilde{W}_3) \to 1.$$

[1]In[36, Lemma 16] the function g_n is assumed to be continuous, but only the measurability is needed

Since $(3n/2)^{-1/2}\left(H_2, H_3, \Lambda(\tilde{W}_2), \Lambda(\tilde{W}_3)\right)$ converges in distribution to

$$(2\mathsf{e}_s - m_{2\mathsf{e}}(s,t), 2\mathsf{e}_t - m_{2\mathsf{e}}(s,t), \Lambda_\triangle(2\mathsf{e}_s - m_{2\mathsf{e}}(s,t)), \Lambda_\triangle(2\mathsf{e}_t - m_{2\mathsf{e}}(s,t))) \tag{46}$$

thanks to (44) and Lemma 14 (and also Lemma 35 below which ensures that $H_i \in [M^{-1}, M]\sqrt{n}$ with probability arbitrary close to 1, if M is chosen large enough, leading to a legal using of Lemma 14). We then deduce by the first assertion of Lemma 33 that

$$(3n/2)^{-1/2}\left(H_2, H_3, \Lambda(W_2), \Lambda(W_3)\right)$$

converges also in distribution to the random variable described in (46).
Since $|d_T((w(ns), w(nt)) - (H_2 + H_3)| \leq 2$ and $|\Lambda(w(ns), w(ns)) - (\Lambda(\tilde{W}_2) + \Lambda(\tilde{W}_3))| \leq 1$, we have

$$|\Lambda_\triangle d_T((w(ns), w(nt)) - \Lambda(w(ns), w(ns))| \leq |\Lambda_\triangle(H_2 + H_3) - (\Lambda(\tilde{W}_2) + \Lambda(\tilde{W}_3))| + cte$$

which implies together with what precedes

$$n^{-1/2}\left|\Lambda_\triangle d_T(w(ns), w(nt)) - \Lambda(w(ns), w(ns))\right| \xrightarrow[n]{proba.} 0. \quad \square$$

It only remains to show Proposition 34. The absolute continuity $\mathbb{P}_{\mathrm{Dec}(n)} \prec \mathbb{P}_{\widetilde{\mathrm{Dec}}(n)}$ comes from the inclusion of the (discrete) support of $\mathrm{Dec}(n)$ in that of $\widetilde{\mathrm{Dec}}(n)$.

For any word $w = w_1 \dots w_k$ with letters in $\{1, 2, 3\}$ define

$$N_1(w) = \sum_{j=1}^{k}(w_i - 1) \text{ and } N_2(w) = \sum_{j=1}^{k}(3 - w_i).$$

Seeing w as a node in a tree, $N_1(w)$ and $N_2(w)$ give the number of nodes at distance 1 on the left (resp. on the right) of the branch $[\![\varnothing, w]\!]$. Set

$$\begin{aligned} A_{n,M} &= \{(w_1, w_2, w_3, h_1, h_2, h_3, l, r) \mid h_1, h_2, h_3 \in \sqrt{n}[M^{-1}, M], \\ &\quad (w_1, w_2, w_3) \in J_{h_1} \times J_{h_2} \times J_{h_3}, (l, r) \in I_3\}, \end{aligned}$$

where for any $h > 0$, $J_h = \left\{\mathsf{a} \in \Sigma_3^h \mid (N_1(\mathsf{a}), N_2(\mathsf{a})) \in \left[h - h^{2/3}, h + h^{2/3}\right]^2\right\}$.

Lemma 35. *For any $\varepsilon > 0$, there exists $M > 0$ such that for n large enough*

$$\mathbb{P}_n(\widetilde{Dec(n)} \in A_{n,M}) \geq 1 - \varepsilon.$$

Proof. The convergence of the rescaled height process to $2\mathsf{e}$ (as stated in (44)) implies that the vector $(3n/2)^{-1/2}(H_1, H_2, H_3)$ converges in distribution to $(m_{2\mathsf{e}}(s,t), 2\mathsf{e}_s - m_{2\mathsf{e}}(s,t), 2\mathsf{e}_t - m_{2\mathsf{e}}(s,t))$. Since a.s. $m_{2\mathsf{e}}(s,t) < 2\min(\mathsf{e}_s, \mathsf{e}_t)$, and a.s $m_{2\mathsf{e}}(s,t) > 0$ (if $s, t \notin \{0, 1\}$), for any $\varepsilon > 0$ there exists $M > 0$ such that

$$\mathbb{P}\big(\{m_{2\mathsf{e}}(s,t), 2\mathsf{e}_s - m_{2\mathsf{e}}(s,t), 2\mathsf{e}_t - m_{2\mathsf{e}}(s,t)\} \subset (M^{-1}, M)\big) \geq 1 - \varepsilon.$$

By the Portmanteau theorem, and taking into account the normalisation, for any $\varepsilon > 0$,

$$\liminf P^{\mathsf{ter}}_{3n+1}(H_i \in [M^{-1}, M]\sqrt{n}, i \in \{1,2,3\}) \geq 1 - \varepsilon \quad \text{for } M \text{ large enough.} \tag{47}$$

Let $W[h]$ be a random word with h i.i.d. letters uniform in Σ_3. For $h \in \mathbb{N}$, by symmetry $N_1(W[h])$ and $N_2(W[h])$ have the same law, and there exists $c_1 > 0, c_2 > 0$, s.t

$$\mathbb{P}\left(W[h] \notin J_h\right) \leq c_1 \exp(-c_2\, h^{1/3}).$$

Indeed the number x_i of letters i in $W[h]$ is binomial $B(h, 1/3)$ distributed, and the Hoeffding inequality leads easily to this result ($N_1(h) = x_2 + 2x_3$ which is in mean $h/3 + 2h/3 = h$). $\square$

To prove Proposition 34, we now evaluate $\mathbb{P}(\mathrm{Dec}(n) = x)/\mathbb{P}(\widetilde{\mathrm{Dec}}(n) = x)$ for any $x \in A_{n,M}$. The number of ternary trees from $\mathcal{T}^{\mathsf{ter}}_{3n+1}$ satisfying

$$\mathrm{Dec}(n) = (w_1, w_2, w_3, |w_1|, |w_2|, |w_3|, l, r)$$

for some prescribed words w_1, w_2, w_3 and $(l, r) \in I_3$ is equal to the number of 3-tuples of forests as drawn on Figure 14. The first forest F_1 has $S_1(w_1, w_2, w_3, l, r) = N_1(w_1) + N_1(w_2) + l - 1$ roots and since $w(ns)$ is the $ns + 1$th internal nodes (not counted in F_1) and since the branch $[\![\varnothing, w(ns)]\!]$ contains $|w_1| + |w_2| + 2$ internal nodes, F_1 has $n_1(w_1, w_2, w_3, l, r) = ns - |w_1| - |w_2| - 1$ internal nodes (and then $3n_1 + S_1$ nodes). The second forest F_2 has $S_2(w_1, w_2, w_3, l, r) = 3 + N_2(w_2) + N_1(w_3) + (r - l - 1)$ roots (the 3 comes from the fact that $w(ns)$ is an internal node), and $n_2(w_1, w_2, w_3, l, r) = nt - ns - |w_3| - 1$ internal nodes. Finally the third forest F_3 has $S_3(w_1, w_2, w_3, l, r) = 3 + N_2(w_3) + N_2(w_1) + 3 - r$ roots and $n_3(w_1, w_2, w_3, l, r) = n - nt - 1$ internal nodes.

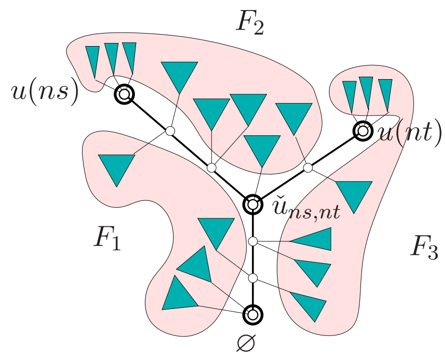

Figure 14: On this example $w_1 = 321$, $w_2 = 12$,$w_3 = 2$, $l = 1, r = 3$, $S_1 = 4$, $S_2 = 8$, $S_3 = 7$.

Before going further, we recall that under P^{ter} all trees in $\mathcal{T}^{\mathsf{ter}}_{3n+1}$ have the same weight $3^{-n}(2/3)^{2n+1}$ since they have n internals nodes and $2n+1$ leaves. Let $F_k = (T(1), \ldots, T(k))$ be a forest composed with k independent GW trees with distribution P^{ter}, and let $|F_k| = \sum_{i=1}^{k} |T(i)|$ be the total number of nodes in F_k. By the rotation/conjugation principle,

$$P^{\mathsf{ter}}(|F_k| = m) = \frac{k}{m} q(m, k)$$

where $q(m, k) = \mathbb{P}(Z_m = -k)$ where $Z := (Z_i)_{i \geq 0}$ is a random walk starting from 0, whose increment value are -1 or 2 with respective probability $2/3$ and $1/3$.

Lemma 36. *For any words w_1, w_2, w_3 on the alphabet Σ_3 and $(l,r) \in I_3$, we have*

$$\begin{aligned} P^{\mathsf{ter}}_{3n+1}\left((W_1, W_2, W_3, L, R) = (w_1, w_2, w_3, l, r)\right) &= \frac{P^{\mathsf{ter}}\left(|F^i_{S_i}| = 3n_i + S_i, i \in \{1,2,3\}\right)}{3^{|w_1|+|w_2|+|w_3|+3} P^{\mathsf{ter}}(\mathcal{T}^{\mathsf{ter}}_{3n+1})} \\ &= \frac{\prod_{i=1}^{3} \frac{S_i}{3n_i+1} q_{3n_i+S_i, S_i}}{3^{|w_1|+|w_2|+|w_3|+3} P^{\mathsf{ter}}(\mathcal{T}^{\mathsf{ter}}_{3n+1})} \end{aligned}$$

where the F^i are independent GW forests with respective number of roots the $S_i := S_i(w_1, w_2, w_3, l, r)$'s, and $n_i = n_i(w_1, w_2, w_3, l, r)$ for any $i \in \{1,2,3\}$.

Note 6. Notice that if $|w_i| = h_i$ for every i, for any $l, r \in \{1,2,3\}$, then

$$P^{\mathsf{ter}}_{3n+1}\left(\mathrm{Dec}(n) = (w_1, w_2, w_3, h_1, h_2, h_3, l, r)\right) = P^{\mathsf{ter}}_{3n+1}\left((W_1, W_2, W_3, L, R) = (w_1, w_2, w_3, l, r)\right).$$

Proof. Notice that there is a hidden condition here since (L,R) are well defined only when $u(ns)$ is not an ancestor of $u(nt)$ (which happens with probability going to 0).
The proof follows a counting argument, together with the remark that all the trees in $\mathcal{T}^{\mathsf{ter}}_{3n+1}$ have the same weight. The term $(1/3)^{|w_1|+|w_2|+|w_3|+3}$ comes from the $|w_1| + |w_2| + |w_3| + 3$ internal nodes on the branches $[\![\varnothing, w(ns)]\!] \cup [\![\varnothing, w(nt)]\!]$. □

We now evaluate $P^{\mathsf{ter}}_{3n+1}\left(\widetilde{\mathrm{Dec}}(n) = (w_1, w_2, w_3, h_1, h_2, h_3, l, r)\right)$ for $(w_1, w_2, w_3) \in \Sigma_3^{h_1} \times \Sigma_3^{h_2} \times \Sigma_3^{h_3}$ and $(l,r) \in I_3$. The variable $\widetilde{\mathrm{Dec}}(n)$ is defined conditionally on (H_1, H_2, H_3). We have

$$\begin{aligned} P^{\mathsf{ter}}_{3n+1}\left((H_1, H_2, H_3) = (h_1, h_2, h_3)\right) &= \sum \frac{P^{\mathsf{ter}}\left(|F^i_{S'_i}| = 3n'_i + S'_i, i \in \{1,2,3\}\right)}{3^{|w'_1|+|w'_2|+|w'_3|+3} P^{\mathsf{ter}}(\mathcal{T}^{\mathsf{ter}}_{3n+1})} \\ &= \sum \frac{\prod_{i=1}^{3} \frac{S'_i}{3n'_i+S'_i} q(3n'_i + S'_i, S'_i)}{3^{|w'_1|+|w'_2|+|w'_3|+3} P^{\mathsf{ter}}(\mathcal{T}^{\mathsf{ter}}_{3n+1})} \end{aligned}$$

where $S'_i := S_i(w'_1, w'_2, w'_3, l', r')$'s, $n'_i = n_i(w'_1, w'_2, w'_3, l', r')$ and where the sum is taken on $(w'_1, w'_2, w'_3) \in \Sigma_3^{h_1} \times \Sigma_3^{h_2} \times \Sigma_3^{h_3}$ and $(l', r') \in I_3$. The term $3^{-|w'_1|-|w'_2|-|w'_3|-3}$ comes from the internal nodes of the branch $[\![\varnothing, w(ns)]\!] \cup [\![\varnothing, w(nt)]\!]$. In other words

$$P^{\mathsf{ter}}_{3n+1}\left((H_1, H_2, H_3) = (h_1, h_2, h_3)\right) = \frac{\mathbb{E}\left(\prod_{i=1}^{3} \frac{\mathbf{S}_i}{3\mathbf{n}_i + \mathbf{S_i}} q(3\mathbf{n}_i + \mathbf{S}_i, \mathbf{S}_i)\right)}{3^2 P^{\mathsf{ter}}(\mathcal{T}^{\mathsf{ter}}_{3n+1})} \tag{48}$$

where $\mathbf{S}_i$ and $\mathbf{n}_i$ are the r.v. S_i and n_i when the w_i are words with h_i i.i.d. letters, uniform in Σ_3 and $(\mathbf{l}, \mathbf{r})$ is uniform in I_3. Finally, by conditioning on the H_i's, we get

$$P^{\mathsf{ter}}_{3n+1}\left(\widetilde{\mathrm{Dec}}(n) = (w_1, w_2, w_3, h_1, h_2, h_3, l, r)\right) = \frac{P^{\mathsf{ter}}_{3n+1}\left((H_1, H_2, H_3) = (h_1, h_2, h_3)\right)}{3^{|w_1|+|w_2|+|w_3|+1}}$$

and

$$\frac{P^{\mathsf{ter}}_{3n+1}\left(\mathrm{Dec}(n) = (w_1, w_2, w_3, h_1, h_2, h_3, l, r)\right)}{P^{\mathsf{ter}}_{3n+1}\left(\widetilde{\mathrm{Dec}}(n) = (w_1, w_2, w_3, h_1, h_2, h_3, l, r)\right)} = \frac{\prod_{i=1}^{3} \frac{S_i}{3n_i+S_i} q(3n_i + S_i, S_i)}{\mathbb{E}\left(\prod_{i=1}^{3} \frac{\mathbf{S}_i}{3\mathbf{n}_i + \mathbf{S}_i} q(3\mathbf{n}_i + \mathbf{S}_i, \mathbf{S}_i)\right)}. \tag{49}$$

This quotient may be uniformly approached for $(w_1, w_2, w_3, h_1, h_2, h_3, l, r) \in A_{n,M}$ thanks to a central local limit theorem applied to the random walk Z:

$$\sup_{l \in -n+3\mathbb{N}} \left| \frac{\sqrt{n}}{3} \mathbb{P}(Z_n = l) - \frac{1}{\sqrt{4\pi}} \exp\left(-\frac{l^2}{4n} \right) \right| \xrightarrow[n]{} 0,$$

since the increment of Z are centered and have variance 2. This gives easily an equivalent for the numerator of (49) (since $q(m,k) = \mathbb{P}(Z_m = -k)$). For the denominator, split the expectation with respect to (w'_1, w'_2, w'_3) belonging to $J_{h_1} \times J_{h_2} \times J_{h_3}$ or not. The first case occurs with probability close to 1, and the local central limit theorem provides the same asymptotic that the numerator. The second case provides an asymptotic with a smaller order (notice that the fact that $0 \le N_1(w) \le 2|w|$ simplifies the use of the central local limit theorem) and we get for any $\varepsilon > 0$,

$$\left| \frac{P^{\mathsf{ter}}_{3n+1}\left(\mathrm{Dec}(n) = (w_1, w_2, w_3, h_1, h_2, h_3, l, r)\right)}{P^{\mathsf{ter}}_{3n+1}\left(\widetilde{\mathrm{Dec}}(n) = (w_1, w_2, w_3, h_1, h_2, h_3, l, r)\right)} - 1 \right| \le \varepsilon$$

on $A_{n,M}$ for n large enough. □

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