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Parabolic Harnack Inequality and Local Limit Theorem for Percolation Clusters

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Abstract

We consider the random walk on supercritical percolation clusters in $\mathbb{Z}^d$. Previous papers have obtained Gaussian heat kernel bounds, and a.s. invariance principles for this process. We show how this information leads to a parabolic Harnack inequality, a local limit theorem and estimates on the Green's function.

Key words: Percolation, random walk, Harnack inequality, local limit theorem.

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1 Introduction

We begin by recalling the definition of bond percolation on $\mathbb{Z}^d$: for background on percolation see [18]. We work on the Euclidean lattice $(\mathbb{Z}^d, \mathbb{E}_d)$, where $d \geq 2$ and $\mathbb{E}_d = \{\{x,y\} : |x-y| = 1\}$. Let $\Omega = \{0,1\}^{\mathbb{E}_d}$, $p \in [0,1]$, and $\mathbb{P} = \mathbb{P}_p$ be the probability measure on Ω which makes $\omega(e)$, $e \in \mathbb{E}_d$ i.i.d. Bernoulli r.v., with $\mathbb{P}(\omega(e) = 1) = p$. Edges e with $\omega(e) = 1$ are called *open* and the *open cluster* $\mathscr{C}(x)$ containing x is the set of y such that $x \longleftrightarrow y$, that is x and y are connected by an open path. It is well known that there exists $p_c \in (0,1)$ such that when $p > p_c$ there is a unique infinite open cluster, which we denote $\mathscr{C}_\infty = \mathscr{C}_\infty(\omega)$.

Let $X = (X_n, n \in \mathbb{Z}_+, P^x_\omega, x \in \mathscr{C}_\infty)$ be the simple random walk (SRW) on $\mathscr{C}_\infty$. At each time step, starting from a point x, the process X jumps along one of the open edges e containing x, with each edge chosen with equal probability. If we write $\mu_{xy}(\omega) = 1$ if $\{x,y\}$ is an open edge and 0 otherwise, and set $\mu_x = \sum_y \mu_{xy}$, then X has transition probabilities

$$P_X(x,y) = \frac{\mu_{xy}}{\mu_x}. \tag{1.1}$$

We define the transition density of X by

$$p^\omega_n(x,y) = \frac{P^x_\omega(X_n = y)}{\mu_y}. \tag{1.2}$$

This random walk on the cluster $\mathscr{C}_\infty$ was called by De Gennes in [12] 'the ant in the labyrinth'.

Subsequently slightly different walks have been considered: the walk above is called the 'myopic ant', while there is also a version called the 'blind ant'. See [19], or Section 5 below for a precise definition.

There has recently been significant progress in the study of this process, and the closely related continuous time random walk $Y = (Y_t, t \in [0,\infty), \tilde{P}^x, x \in \mathscr{C}_\infty)$, with generator

$$\mathscr{L} f(x) = \sum_y \frac{\mu_{xy}}{\mu_x}(f(y) - f(x)).$$

We write

$$q^\omega_t(x,y) = \frac{\tilde{P}^x_\omega(Y_t = y)}{\mu_y} \tag{1.3}$$

for the transition densities of Y. Mathieu and Remy in [20] obtained a.s. upper bounds on $\sup_y q^\omega_t(x,y)$, and these were extended in [2] to full Gaussian-type upper and lower bounds – see [2, Theorem 1.1]. A quenched or a.s. invariance principle for X was then obtained in [25; 7; 21]: an averaged, or annealed invariance principle had been proved many years previously in [14].

The main result in this paper is that as well as the invariance principle, one also has a local limit theorem for $p^\omega_n(x,y)$ and $q^\omega_t(x,y)$. (See [17], XV.5 for the classical local limit theorem for lattice r.v.) For $D > 0$ write

$$k^{(D)}_t(x) = (2\pi t D)^{-d/2} e^{-|x|^2/2Dt}$$

for the Gaussian heat kernel with diffusion constant D.

Theorem 1.1. *Let X be either the 'myopic' or the 'blind' ant random walk on $\mathscr{C}_\infty$. Let $T > 0$. Let $g^\omega_n : \mathbb{R}^d \to \mathscr{C}_\infty(\omega)$ be defined so that $g^\omega_n(x)$ is a closest point in $\mathscr{C}_\infty(\omega)$ to $\sqrt{n}x$. Then there exist* *constants a, D (depending only on d and p, and whether X is the blind or myopic ant walk) such that* $\mathbb{P}$*-a.s. on the event* $\{0 \in \mathscr{C}_\infty\}$,

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^d} \sup_{t\geq T} \left| n^{d/2}\big(p^\omega_{\lfloor nt\rfloor}(0, g^\omega_n(x)) + p^\omega_{\lfloor nt\rfloor+1}(0, g^\omega_n(x))\big) - 2a^{-1}k^{(D)}_t(x) \right| = 0. \tag{1.4}$$

For the continuous time random walk Y we have

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^d} \sup_{t\geq T} \left| n^{d/2} q^\omega_{nt}(0, g^\omega_n(x)) - a^{-1}k^{(D)}_t(x) \right| = 0, \tag{1.5}$$

where the constants a, D are the same as for the myopic ant walk.

We prove this theorem by establishing a parabolic Harnack inequality (PHI) for solutions to the heat equation on $\mathscr{C}_\infty$. (See [2] for an elliptic Harnack inequality.) This PHI implies Hölder continuity of $p^\omega_n(x,\cdot)$, and this enables us to replace the weak convergence given by the CLT by pointwise convergence. In this paper we will concentrate on the proof of (1.4) – the same arguments with only minor changes give (1.5).

Some of the results mentioned above, for random walks on percolation clusters, have been extended to the 'random conductance model', where μ_{xy} are taken as i.i.d.r.v. in $[0,\infty)$ – see [9; 22; 25]. In the case where the random conductors are bounded away from zero and infinity, a local limit theorem follows by our methods – see Theorem 5.7. If however the μ_{xy} have fat tails at 0, then while a quenched invariance principle still holds, the transition density does not have enough regularity for a local limit theorem – see Theorem 2.2 in [8].

As an application of Theorem 1.1 we have the following theorem on the Green's function $g_\omega(x,y)$ on $\mathscr{C}_\infty$, defined (when $d \geq 3$) by

$$g_\omega(x,y) = \int_0^\infty q^\omega_t(x,y)dt. \tag{1.6}$$

Theorem 1.2. *Let* $d \geq 3$. *(a) There exist constants* $\delta, c_1, \dots c_4$, *depending only on d and p, and r.v.* R_x, $x \in \mathbb{Z}^d$ *satisfying*

$$\mathbb{P}(R_x \geq n | x \in \mathscr{C}_\infty) \leq c_1 e^{-c_2 n^\delta}, \tag{1.7}$$

such that

$$\frac{c_3}{|x-y|^{d-2}} \leq g_\omega(x,y) \leq \frac{c_4}{|x-y|^{d-2}} \qquad \text{if } |x-y| \geq R_x \wedge R_y. \tag{1.8}$$

(b) There exists a constant $C = \Gamma(\frac{d}{2}-1)/(2\pi^{d/2}aD) > 0$ *such that for any* $\varepsilon > 0$ *there exists* $M = M(\varepsilon,\omega)$ *such that on* $\{0 \in \mathscr{C}_\infty\}$,

$$\frac{(1-\varepsilon)C}{|x|^{d-2}} \leq g_\omega(0,x) \leq \frac{(1+\varepsilon)C}{|x|^{d-2}} \quad \text{for } |x| > M(\omega). \tag{1.9}$$

(c) We have

$$\lim_{|x|\to\infty} |x|^{2-d}\mathbb{E}(g_\omega(0,x)|0 \in \mathscr{C}_\infty) = C. \tag{1.10}$$

Remark. While (1.7) gives good control of the tail of the random variables R_x in (1.8), we do not have any bounds on the tail of the r.v. M in (1.9). This is because the proof of (1.9) relies on the invariance principles in [25; 7; 21], and these do not give a rate of convergence.

In Section 2 we indicate how the heat kernel estimates obtained in [2] can be extended to discrete time, and also to variants of the basic SRW X. In Section 3 we prove the PHI for $\mathscr{C}_\infty$ using the 'balayage' argument introduced in [3]. In the Appendix we give a self-contained proof of the key equation in the simple fully discrete context of this section. In Section 4 we show that if the PHI and CLT hold for a suitably regular subgraph $\mathscr{G}$ of $\mathbb{Z}^d$, then a local limit theorem holds. In Section 5 we verify these conditions for percolation, and prove Theorem 1.1. In Section 6, using the heat kernel bounds for q_t^ω and the local limit theorem, we obtain Theorem 1.2.

We write c, c' for positive constants, which may change on each appearance, and c_i for constants which are fixed within each argument. We occasionally use notation such as $c_{1.2.1}$ to refer to constant c_1 in Theorem 1.2.

2 Discrete and continuous time walks

Let $\Gamma = (G, E)$ be an infinite, connected graph with uniformly bounded vertex degree. We write d for the graph metric, and $B_d(x, r) = \{y : d(x, y) < r\}$ for balls with respect to d. Given $A \subset G$, we write ∂A for the external boundary of A (so $y \in \partial A$ if and only if $y \in G - A$ and there exists $x \in A$ with $x \sim y$.) We set $\overline{A} = A \cup \partial A$.

Let μ_{xy} be 'bond conductivities' on Γ. Thus μ_{xy} is defined for all $(x, y) \in G \times G$. We assume that $\mu_{xy} = \mu_{yx}$ for all $x, y \in G$, and that $\mu_{xy} = 0$ if $\{x, y\} \notin E$ and $x \neq y$. We assume that the conductivities on edges with distinct endpoints are bounded away from 0 and infinity, so that there exists a constant C_M such that

$$0 < C_M^{-1} \leq \mu_{xy} \leq C_M \qquad \text{whenever } x \sim y,\ x \neq y. \tag{2.1}$$

We also assume that

$$0 \leq \mu_{xx} \leq C_M, \quad \text{for } x \in G; \tag{2.2}$$

we allow the possibility that $\mu_{xx} > 0$ so as to be able to handle 'blind ants' as in [19]. We define $\mu_x = \mu(\{x\}) = \sum_{y \in G} \mu_{xy}$, and extend μ to a measure on G. The pair (Γ, μ) is often called a *weighted graph*. We assume that there exist $d \geq 1$ and C_U such that

$$\mu(B_d(x, r)) \leq C_U r^d, \quad r \geq 1,\ x \in G. \tag{2.3}$$

The standard discrete time SRW X on (Γ, μ) is the Markov chain $X = (X_n, n \in \mathbb{Z}_+, P^x, x \in G)$ with transition probabilities $P_X(x, y)$ given by (1.1). Since we allow $\mu_{xx} > 0$, X can jump from a vertex x to itself. We define the discrete time heat kernel on (Γ, μ) by

$$p_n(x, y) = \frac{P^x(X_n = y)}{\mu_x}. \tag{2.4}$$

Let

$$\mathscr{L} f(x) = \mu_x^{-1} \sum_y \mu_{xy}(f(y) - f(x)). \tag{2.5}$$

One may also look at the continuous time SRW on (Γ, μ), which is the Markov process $Y = (Y_t, t \in [0, \infty), \tilde{P}^x, x \in G)$, with generator $\mathscr{L}$. We define the (continuous time) heat kernel on (Γ, μ) by

$$q_t(x, y) = \frac{\tilde{P}^x(Y_t = y)}{\mu_x}. \tag{2.6}$$

The continuous time heat kernel is a smoother object that the discrete time one, and is often slightly simpler to handle. Note that p_n and q_t satisfy

$$p_{n+1}(x,y)-p_n(x,y)=\mathscr{L}p_n(x,y), \qquad \frac{\partial q_t(x,y)}{\partial t}=\mathscr{L}q_t(x,y).$$

We remark that Y can be constructed from X by making Y follow the same trajectory as X, but at times given by independent mean 1 exponential r.v. More precisely, if M_t is a rate 1 Poisson process, we set $Y_t = X_{M_t}$, $t \geq 0$. Define also the quadratic form

$$\mathscr{E}(f,g)=\tfrac{1}{2}\sum_x\sum_y \mu_{xy}(f(y)-f(x))(g(y)-g(x)). \tag{2.7}$$

[2] studied the continuous time random walk Y and the heat kernel $q_t(x,y)$ on percolation clusters, in the case when $\mu_{xy}=1$ whenever $\{x,y\}$ is an open edge, and $\mu_{xy}=0$ otherwise. It was remarked in [2] that the same arguments work for the discrete time heat kernel, but no details were given. Since some of the applications of [2] do use the discrete time estimates, and as we shall also make use of these in this paper, we give details of the changes needed to obtain these bounds.

In general terms, [2] uses two kinds of arguments to obtain the bounds on $q_t(x,y)$. One kind (see for example Lemma 3.5 or Proposition 3.7) is probabilistic, and to adapt it to the discrete time process X requires very little work. The second kind uses differential inequalities, and here one does have to be more careful, since these usually have a more complicated form in discrete time.

We now recall some further definitions from [2].

Definition Let C_V, C_P, and $C_W \geq 1$ be fixed constants. We say $B_d(x,r)$ is (C_V,C_P,C_W)–*good* if:

$$C_V r^d \leq \mu(B_d(x,r)), \tag{2.8}$$

and the weak Poincaré inequality

$$\sum_{y\in B_d(x,r)}(f(y)-\overline{f}_{B_d(x,r)})^2\mu_y \leq C_P r^2 \sum_{y,z\in B_d(x,C_W r),z\sim y}|f(y)-f(z)|^2\mu_{yz} \tag{2.9}$$

holds for every $f : B_d(x,C_W r)\to\mathbb{R}$. (Here $\overline{f}_{B_d(x,r)}$ is the value which minimises the left hand side of (2.9)).

We say $B_d(x,R)$ is (C_V,C_P,C_W)–*very good* if there exists $N_B = N_{B_d(x,R)} \leq R^{1/(d+2)}$ such that $B_d(y,r)$ is good whenever $B_d(y,r)\subseteq B_d(x,R)$, and $N_B \leq r \leq R$. We can always assume that $N_B \geq 1$. Usually the values of C_V,C_P,C_W will be clear from the context and we will just use the terms 'good' and 'very good'. (In fact the condition that $N_B \leq R^{1/(d+2)}$ is not used in this paper, since whenever we use the condition 'very good' we will impose a stronger condition on N_B).

From now on in the section we fix $d\geq 2$, C_M, C_V, C_P, and C_W, and take $(\Gamma,\mu)=(G,E,\mu)$ to satisfy (2.3). If $f(n,x)$ is a function on $\mathbb{Z}_+\times G$, we write

$$\hat{f}(n,x)=f(n+1,x)+f(n,x), \tag{2.10}$$

and in particular, to deal with the problem of bipartite graphs, we consider

$$\hat{p}_n(x,y)=p_{n+1}(x,y)+p_n(x,y). \tag{2.11}$$

The following Theorem summarizes the bounds on q and p that will be used in the proof of the PHI and local limit theorem.

Theorem 2.1. *Assume that* (2.1)*,* (2.2) *and* (2.3) *hold. Let $x_0 \in G$. Suppose that $R_1 \geq 16$ and $B_d(x_0,R_1)$ is very good with $N^{2d+4}_{B_d(x_0,R_1)} \leq R_1/(2\log R_1)$. Let $x_1 \in B_d(x_0,R_1/3)$. Let $R\log R = R_1$, $T = R^2$, $B = B_d(x_1,R)$, and $q^B_t(x,y)$, $p^B_n(x,y)$ be the heat kernels for the processes Y and X killed on exiting from B. Then*

$$q^B_t(x,y) \geq c_1 T^{-d/2}, \quad \text{if } x,y \in B_d(x_1,3R/4), \quad \tfrac{1}{4}T \leq t \leq T, \tag{2.12}$$

$$q_t(x,y) \leq c_2 T^{-d/2}, \quad \text{if } x,y \in B_d(x_1,R), \quad \tfrac{1}{4}T \leq t \leq T, \tag{2.13}$$

$$q_t(x,y) \leq c_2 T^{-d/2}, \quad \text{if } x \in B_d(x_1,R/2), \quad d(x,y) \geq R/8, \quad 0 \leq t \leq T, \tag{2.14}$$

and

$$p^B_{n+1}(x,y)+p^B_n(x,y) \geq c_1 T^{-d/2}, \quad \text{if } x,y \in B_d(x_1,3R/4), \quad \tfrac{1}{4}T \leq n \leq T, \tag{2.15}$$

$$p_n(x,y) \leq c_2 T^{-d/2}, \quad \text{if } x,y \in B_d(x_1,R), \quad \tfrac{1}{4}T \leq n \leq T, \tag{2.16}$$

$$p_n(x,y) \leq c_2 T^{-d/2}, \quad \text{if } x \in B_d(x_1,R/2), \quad d(x,y) \geq R/8, \quad 0 \leq n \leq T. \tag{2.17}$$

To prove this theorem we extend the bounds proved in [2] for the continuous time simple random walk on (Γ,μ) to the slightly more general random walks X and Y defined above.

Theorem 2.2. *(a) Assume that* (2.1)*,* (2.2) *and* (2.3) *hold. Then the bounds in Proposition 3.1, Proposition 3.7, Theorem 3.8, and Proposition 5.1– Lemma 5.8 of [2] all hold for $\hat{p}_n(x,y)$ as well as $q_t(x,y)$.*
(b) In particular (see Theorem 5.7) let $x \in G$ and suppose that there exists $R_0 = R_0(x)$ such that $B(x,R)$ is very good with $N^{3(d+2)}_{B(x,R)} \leq R$ for each $R \geq R_0$. There exist constants c_i such that if n satisfies $n \geq R_0^{2/3}$ then

$$p_n(x,y) \leq c_1 n^{-d/2} e^{-c_2 d(x,y)^2/n}, \qquad d(x,y) \leq n, \tag{2.18}$$

and

$$p_n(x,y) + p_{n+1}(x,y) \geq c_3 n^{-d/2} e^{-c_4 d(x,y)^2/n}, \qquad d(x,y)^{3/2} \leq n. \tag{2.19}$$

(c) Similar bounds to those in (2.18)*,* (2.19) *hold for $q_t(x,y)$.*

Remark. Note that we do not give in (b) Gaussian lower bounds in the range $d(x,y) \leq n < d(x,y)^{3/2}$. However, as in [2, Theorem 5.7], Gaussian lower bounds on p_n and q_t will hold in this range of values if a further condition 'exceedingly good' is imposed on $B(x,R)$ for all $R \geq R_0$. We do not give further details here for two reasons; first the 'exceedingly good' condition is rather complicated (see [2, Definition 5.4]), and second the lower bounds in this range have few applications.

Proof. We only indicate the places where changes in the arguments of [2] are needed.

First, let $\mu^0_{xy} = 1$ if $\{x,y\} \in E$, and 0 otherwise. Then (2.1) implies that if $\mathcal{E}^0$ is the quadratic form associated with (μ^0_{xy}), then

$$c_1 \mathcal{E}^0(f,f) \leq \mathcal{E}(f,f) \leq c_2 \mathcal{E}^0(f,f) \tag{2.20}$$

for all f for which either expression is finite. This means that the weak Poincaré inequality for $\mathcal{E}^0$ implies one (with a different constant C_P) for $\mathcal{E}$. Using this, the arguments in Section 3–5 of [2] go through essentially unchanged to give the bounds for the continuous time heat kernel on (Γ,μ).

More has to be said about the discrete time case. The argument in [2, Proposition 3.1] uses the equality

$$\frac{\partial}{\partial t} q_{2t}(x_1, x_1) = -2\mathscr{E}(q_t, q_t).$$

Instead, in discrete time, we set $f_n(x) = \hat{p}_n(x_1, x)$ and use the easily verified relation

$$\hat{p}_{2n+2}(x_1, x_1) - \hat{p}_{2n}(x_1, x_1) = -\mathscr{E}(f_n, f_n). \tag{2.21}$$

Given this, the argument of [2, Proposition 3.1] now goes through to give an upper bound on $\hat{p}_n(x, x)$, and hence on $p_n(x, x)$. A global upper bound, as in [2, Corollary 3.2], follows since, taking k to be an integer close to $n/2$,

$$\begin{aligned} p_n(x, y) = \sum_z p_k(x, z) p_{n-k}(y, z) &\le (\sum_z p_k(x, z)^2)^{1/2} (\sum_z p_{n-k}(y, z)^2)^{1/2} \\ &= p_{2k}(x, x)^{1/2} p_{2n-2k}(y, y)^{1/2}. \end{aligned}$$

To obtain better bounds for x, y far apart, [2] used a method of Bass and Nash – see [5; 23]. This does not seem to transfer easily to discrete time. For a process Z, write $\tau_Z(x, r) = \inf\{t : d(Z_t, x) \ge r\}$. The key bound in continuous time is given in [2, Lemma 3.5], where it is proved that if $B = B(x_0, R)$ is very good, then

$$P^x(\tau_Y(x, r) \le t) \le \frac{1}{2} + \frac{ct}{r^2}, \text{ if } x \in B(x_0, 2R/3), \quad 0 \le t \le cR^2/\log R, \tag{2.22}$$

provided $cN_B^d(\log N_B)^{1/2} \le r \le R$. (Here N_B is the number given in the definition of 'very good'.) Recall that we can write $Y_t = X_{M_t}$, where M is a rate 1 Poisson process independent of X. So,

$$P^x(\tau_X(x, r) < t) P^x(M_{2t} > t) = P^x(\tau_X(x, r) < t, M_{2t} > t) \le P(\tau_Y(x, r) < 2t).$$

Since $P(M_{2t} > t) \ge 3/4$ for $t \ge c$, we obtain

$$P^x(\tau_X(x, r) < t) \le \frac{2}{3} + \frac{c' t}{r^2}. \tag{2.23}$$

Using (2.23) the remainder of the arguments of Section 3 of [2] now follow through to give the large deviation estimate Proposition 3.7 and the Gaussian upper bound Theorem 3.8.

The next use of differential inequalities in [2] is in Proposition 5.1, where a technique of Fabes and Stroock [16] is used. Let $B = B_d(x_1, R)$ be a ball in G, and $\varphi : G \to \mathbb{R}$, with $\varphi(x) > 0$ for $x \in B$ and $\varphi = 0$ on $G - B$. Set

$$V_0 = \sum_{x \in B} \varphi(x) \mu_x.$$

Let $g_n(x) = \hat{p}_n(x_1, x)$, and

$$H_n = V_0^{-1} \sum_{x \in B} \log(g_n(x)) \varphi(x) \mu_x. \tag{2.24}$$

We need to take $n \geq R$ here, so that $g_n(x) > 0$ for all $x \in B$. Using Jensen's inequality, and recalling that $P_X(x,y) = \mu_{xy}/\mu_x$,

$$
\begin{aligned}
H_{n+1} - H_n &= \sum_{x \in B} \log(g_{n+1}(x)/g_n(x))\varphi(x)\mu_x \\
&= \sum_{x \in G} \varphi(x)\mu_x \log\Big(\sum_{y \in G} P_X(x,y) g_n(y)/g_n(x)\Big) \\
&\geq \sum_{x \in G} \varphi(x)\mu_x \sum_{y \in G} P_X(x,y) \log(g_n(y)/g_n(x)) \\
&= \sum_{x \in G} \sum_{y \in G} \varphi(x)\mu_{xy}(\log g_n(y) - \log g_n(x)) \\
&= -\tfrac{1}{2} \sum_{x \in G} \sum_{y \in G} (\varphi(y) - \varphi(x))(\log g_n(y) - \log g_n(x))\mu_{xy}. \qquad (2.25)
\end{aligned}
$$

Given (2.25), the arguments on p. 3071-3073 of [2] give the basic 'near diagonal' lower bound in [2, Proposition 5.1], for $\hat{p}_n(x,y)$. The remainder of the arguments in Section 5 of [2] can now be carried through. □

Proof of Theorem 2.1. This follows from Theorem 2.2, using the fact that Theorem 3.8 and Lemma 5.8 of [2] hold. □

3 Parabolic Harnack Inequality

In this section we continue with the notation and hypotheses of Section 2. Our first main result, Theorem 3.1, is a parabolic Harnack inequality. Then, in Proposition 3.2 we show that solutions to the heat equation are Hölder continuous; this result then provides the key to the local limit theorem proved in the next section.

Let

$$Q(x,R,T) = (0,T] \times B_d(x,R),$$

and

$$Q_-(x,R,T) = [\tfrac{1}{4}T, \tfrac{1}{2}T] \times B_d(x, \tfrac{1}{2}R), \quad Q_+(x,R,T) = [\tfrac{3}{4}T, T] \times B_d(x, \tfrac{1}{2}R).$$

We use the notation $t + Q(x,R,T) = (t, t+T) \times B_d(x,R)$. We say that a function $u(n,x)$ is *caloric* on Q if u is defined on $\overline{Q} = ([0,T] \cap \mathbb{Z}) \times \overline{B}_d(x,R)$, and

$$u(n+1,x) - u(n,x) = \mathscr{L}u(n,x) \quad \text{for } 0 \leq n \leq T-1,\ x \in B_d(x,R). \qquad (3.1)$$

It follows that if $n \geq 1$ then $M_k = u(n-k, X_k)$ is a P^x-martingale for $0 \leq k \leq n \wedge \min\{j : X_j \notin B_d(x,R)\}$. We say the parabolic Harnack inequality (PHI) holds with constant C_H for $Q = Q(x,R,T)$ if whenever $u = u(n,x)$ is non-negative and caloric on Q, then

$$\sup_{(n,x) \in Q_-} \hat{u}(n,x) \leq C_H \inf_{(n,x) \in Q_+} \hat{u}(n,x). \qquad (3.2)$$

The PHI in continuous time takes a similar form, except that caloric functions satisfy

$$\frac{\partial u}{\partial t} = \mathscr{L}u,$$

and (3.2) is replaced by $\sup_{Q_-} u \le C_H \inf_{Q_+} u$.

We now show that the heat kernel bounds in Theorem 2.1 lead to a PHI.

Theorem 3.1. *Let $x_0 \in G$. Suppose that $R_1 \ge 16$ and $B_d(x_0,R_1)$ is (C_V, C_P, C_W)–very good with $N^{2d+4}_{B_d(x_0,R)} \le R_1/(2\log R_1)$. Let $x_1 \in B_d(x_0,R_1/3)$, and $R\log R = R_1$. Then there exists a constant C_H such that the PHI (in both discrete and continuous time settings) holds with constant C_H for $Q(x_1,R,R^2)$.*

Remark. The condition $R_1 = R\log R$ here is not necessarily best possible.

Proof. We use the balayage argument introduced in [3] – see also [4] for the argument in a graph setting. Let $T = R^2$, and write:

$$B_0 = B_d(x_1,R/2), \quad B_1 = B_d(x_1,2R/3), \quad B = B_d(x_1,R),$$

and

$$Q = Q(x_1,R,T) = [0,T] \times B, \quad E = (0,T] \times B_1.$$

We begin with the discrete time case. Let $u(n,x)$ be non-negative and caloric on Q. We consider the space-time process Z on $\mathbb{Z} \times G$ given by $Z_n = (I_n, X_n)$, where X is the SRW on Γ, $I_n = I_0 - n$, and $Z_0 = (I_0, X_0)$ is the starting point of the space time process. Define the réduite u_E by

$$u_E(n,x) = E^x\left(u(n-T_E, X_{T_E}); T_E < \tau_Q\right), \tag{3.3}$$

where T_E is the hitting time of E by Z, and τ_Q the exit time by Z from Q. So $u_E = u$ on E, $u_E = 0$ on Q^c, and since $u(n-k, X_k)$ is a martingale on $0 \le k \le T_E$ we have $u_E \le u$ on $Q - E$.

As the process Z has a dual, the balayage formula of Chapter VI of [10] holds and we can write

$$u_E(n,x) = \int_E p^B_{n-r}(x,y)\nu_E(dr,dy), \quad (n,x) \in Q, \tag{3.4}$$

for a suitable measure ν_E. Here $p^B_n(x,y)$ is the transition density of the process X killed on exiting from B.

In this simple discrete setup we can write things more explicitly. Set

$$Jf(x) = \begin{cases} \sum_{y\in B} \frac{\mu_{xy}}{\mu_y} f(y), & \text{if } x \in B_1, \\ 0, & \text{if } x \in B - B_1. \end{cases} \tag{3.5}$$

Then we have for $x \in B$,

$$u_E(n,x) = \sum_{y\in B} p^B_n(x,y)u(0,y)\mu_y + \sum_{y\in B}\sum_{r=2}^{n} p^B_{n-r}(x,y)k(r,y)\mu_y, \tag{3.6}$$

where for $r \ge 2$

$$k(r,y) = J(u(r-1,\cdot) - u_E(r-1,\cdot))(y). \tag{3.7}$$

See the appendix for a self-contained proof of (3.6) and (3.7).

Since $u = u_E$ on E, if $r \geq 2$ then (3.7) implies that $k(r,y) = 0$ unless $y \in \partial(B - B_1)$. Adding (3.6) for $u(n,x)$ and $u(n+1,x)$, and using the fact that $k(n+1,x) = 0$ for $x \in B_0$, we obtain, for $x \in B_0$,

$$\hat{u}_E(n,x) = \sum_{y \in B_1} \sum_{r=1}^{n} \hat{p}^B_{n-r}(x,y) k(r,y) \mu_y. \tag{3.8}$$

Now let $(n_1, y_1) \in Q_-$ and $(n_2, y_2) \in Q_+$. Since $(n_i, y_i) \in E$ for $i = 1,2$, we have $u_E(n_i, y_i) = u(n_i, y_i)$, and so (3.8) holds. By Theorem 2.1 we have, writing $A = \partial(B - B_1)$,

$$\begin{aligned} \hat{p}^B_{n_2-r}(x,y) &\geq c_1 T^{-d/2} \quad \text{for } x,y \in B_1,\ 0 \leq r \leq T/2, \\ \hat{p}_r(x,y) &\leq c_2 T^{-d/2} \quad \text{for } x,y \in B_1,\ T/4 \leq r \leq T/2, \\ \hat{p}_{n_1-r}(x,y) &\leq c_2 T^{-d/2} \quad \text{for } x \in B_0,\ y \in A,\ 0 < r \leq n_1. \end{aligned}$$

Substituting these bounds in (3.8),

$$\begin{aligned} \hat{u}(n_2, y_2) &= \sum_{y \in B_1} \hat{p}^B_{n_2}(y_2, y) u(0,y) \mu_y + \sum_{y \in A} \sum_{r=2}^{n_2} \hat{p}^B_{n_2-s}(y_2,y) k(r,y) \mu_y \\ &\geq \sum_{y \in B_1} \hat{p}^B_{n_2}(y_2, y) u(0,y) \mu_y + \sum_{y \in A} \sum_{r=2}^{n_1} \hat{p}^B_{n_2-s}(y_2,y) k(r,y) \mu_y \\ &\geq \sum_{y \in B_1} c_1 T^{-d/2} u(0,y) \mu_y + \sum_{y \in A} \sum_{r=2}^{n_1} c_1 T^{-d/2} k(r,y) \mu_y \\ &\geq \sum_{y \in B_1} c_1 c_2^{-1} \hat{p}^B_{n_1}(y_1, y) u(0,y) \mu_y + \sum_{y \in A} \sum_{r=2}^{n_1} c_1 c_2^{-1} \hat{p}^B_{n_1-s}(y_1,y) k(r,y) \mu_y \\ &= c_1 c_2^{-1} \hat{u}(n_1, y_1), \end{aligned}$$

which proves the PHI.

The proof is similar in the continuous time case. The balayage formula takes the form

$$u_E(t,x) = \sum_{y \in B} q^B_t(x,y) u(0,y) \mu_y + \sum_{y \in B_1} \int_0^t q^B_{t-s}(x,y) k(s,y) \mu_y ds, \tag{3.9}$$

where $k(s,y)$ is zero if $y \in B - B_1$ and

$$k(s,y) = J(u(s,\cdot) - u_E(s,\cdot))(y), \quad y \in B_1. \tag{3.10}$$

(See [4, Proposition 3.3]). Using the bounds on q^B_t in Theorem 2.1 then gives the PHI. □

Remark. In [2] an elliptic Harnack inequality (EHI) was proved for random walks on percolation clusters – see Theorem 5.11. Since the PHI immediately implies the EHI, the argument above gives an alternative, and simpler, proof of this result.

It is well known that the PHI implies Hölder continuity of caloric functions – see for example Theorem 5.4.7 of [24]. But since in our context the PHI does not hold for all balls, we give the details of the proof. In the next section we will just use this result when the caloric function u is either $q_t(x,y)$ or $\hat{p}_n(x,y)$.

Proposition 3.2. *Let $x_0 \in G$. Suppose that there exists $s(x_0) \geq 0$ such that the PHI (with constant C_H) holds for $Q(x_0,R,R^2)$ for $R \geq s(x_0)$. Let $\theta = \log(2C_H/(2C_H-1))/\log 2$, and*

$$\rho(x_0,x,y) = s(x_0) \vee d(x_0,x) \vee d(x_0,y). \tag{3.11}$$

Let $r_0 \geq s(x_0)$, $t_0 = r_0^2$, and suppose that $u = u(n,x)$ is caloric in $Q = Q(x_0,r_0,r_0^2)$. Let $x_1,x_2 \in B_d(x_0,\frac{1}{2}r_0)$, and $t_0 - \rho(x_0,x_1,x_2)^2 \leq n_1,n_2 \leq t_0 - 1$. Then

$$|\hat{u}(n_1,x_1) - \hat{u}(n_2,x_2)| \leq c\Big(\frac{\rho(x_0,x_1,x_2)}{t_0^{1/2}}\Big)^{\theta} \sup_{Q_+}|\hat{u}|. \tag{3.12}$$

Proof. We just give the discrete time argument – the continuous time one is almost identical. Set $r_k = 2^{-k}r_0$, and let

$$Q(k) = (t_0 - r_k^2) + Q(x_0,r_k,r_k^2).$$

Thus $Q_+(k) = Q(k+1)$. Let k be such that $r_k \geq s(x_0)$. Let $\hat{v}$ be $\hat{u}$ normalised in $Q(k)$ so that $0 \leq \hat{v} \leq 1$, and $\operatorname{Osc}(\hat{v},Q(k)) = 1$. (Here $\operatorname{Osc}(u,A) = \sup_Q u - \inf_A u$ is the oscillation of u on A). Replacing $\hat{v}$ by $1-\hat{v}$ if necessary we can assume $\sup_{Q_-(k)}\hat{v} \geq \frac{1}{2}$. By the PHI,

$$\tfrac{1}{2} \leq \sup_{Q_-(k)} \hat{v} \leq C_H \inf_{Q_+(k)} \hat{v},$$

and it follows that, if $\delta = (2C_H)^{-1}$, then

$$\operatorname{Osc}(\hat{u},Q_+(k)) \leq (1-\delta)\operatorname{Osc}(\hat{u},Q(k)). \tag{3.13}$$

Now choose m as large as possible so that $r_m \geq \rho(x_0,x,y)$. Then applying (3.13) in the chain of boxes $Q(1) \supset Q(2) \supset \ldots Q(m)$, we deduce that, since $(x_i,n_i) \in Q(m)$,

$$|\hat{u}(n_1,x_1) - \hat{u}(n_2,x_2)| \leq \operatorname{Osc}(\hat{u},Q_m) \leq (1-\delta)^{m-1}\operatorname{Osc}(\hat{u},Q(1)). \tag{3.14}$$

Since $(1-\delta)^m \leq c(r_0/t_0^{1/2})^{\theta}$, (3.12) follows from (3.14) . $\square$

4 Local limit theorem

Now let $\mathscr{G} \subset \mathbb{Z}^d$, and let d denote graph distance in $\mathscr{G}$, regarded as a subgraph of $\mathbb{Z}^d$. We assume $\mathscr{G}$ is infinite and connected, and $0 \in \mathscr{G}$. We define μ_{xy} as in Section 2 so that (2.1), (2.2) and (2.3) hold, and write $X = (X_n, n \in \mathbb{Z}_+, P^x, x \in \mathscr{G})$ for the associated simple random walk on $(\mathscr{G},\mu)$. We write $|\cdot|_p$ for the L^p norm in $\mathbb{R}^d$; $|\cdot|$ is the usual ($p=2$) Euclidean distance.

Recall that $k_t^{(D)}(x)$ is the Gaussian heat kernel in $\mathbb{R}^d$ with diffusion constant $D > 0$ and let $X_t^{(n)} = n^{-1/2}X_{\lfloor nt \rfloor}$. For $x \in \mathbb{R}^d$, set

$$H(x,r) = x + [-r,r]^d, \qquad \Lambda(x,r) = H(x,r) \cap \mathscr{G}. \tag{4.1}$$

In general $\Lambda(x,r)$ will not be connected. Let

$$\Lambda_n(x,r) = \Lambda(xn^{1/2}, rn^{1/2}).$$

Choose a function $g_n : \mathbb{R}^d \to \mathscr{G}$ so that $g_n(x)$ is a closest point in $\mathscr{G}$ to $n^{1/2}x$, in the $|\cdot|_\infty$ norm. (We can define g_n by using some fixed ordering of $\mathbb{Z}^d$ to break ties.)

We now make the following assumption on the graph $\mathscr{G}$ and the SRW X on $\mathscr{G}$. Let $x \in \mathbb{R}^d$.

Assumption 4.1. *There exists a constant $\delta > 0$, and positive constants $D, C_H, C_i, a_{\mathcal{G}}$ such that the following hold.*
(a) (CLT for X). For any $y \in \mathbb{R}^d$, $r > 0$,

$$P^0(X_t^{(n)} \in H(y,r)) \to \int_{H(y,r)} k_t^{(D)}(y')dy'. \tag{4.2}$$

(b) There is a global upper heat kernel bound of the form

$$p_k(0,y) \leq C_2 k^{-d/2}, \quad \textit{for all } y \in \mathcal{G}, k \geq C_3.$$

(c) For each $y \in \mathcal{G}$ there exists $s(y) < \infty$ such that the PHI (3.2) *holds with constant C_H for $Q(y,R,R^2)$ for $R \geq s(y)$.*
(d) For any $r > 0$

$$\frac{\mu(\Lambda_n(x,r))}{(2n^{1/2}r)^d} \to a_{\mathcal{G}} \qquad \textit{as } n \to \infty. \tag{4.3}$$

(e) For each $r > 0$ there exists n_0 such that, for $n \geq n_0$,

$$|x' - y'|_\infty \leq d(x',y') \leq (C_1|x'-y'|_\infty) \vee n^{1/2-\delta}, \quad \textit{for all } x', y' \in \Lambda_n(x,r).$$

(f) $n^{-1/2}s(g_n(x)) \to 0$ as $n \to \infty$.

We remark that for any x all these hold for $\mathbb{Z}^d$: for the PHI see [13]. We also remark that these assumptions are not independent; for example the PHI in (c) implies an upper bound as in (b). For the region $Q(y,R,R^2)$ in (c) the space ball is in the graph metric on $\mathcal{G}$.

We write, for $t \in [0,\infty)$,

$$\hat{p}_t(x,y) = \hat{p}_{\lfloor t \rfloor}(x,y) = p_{\lfloor t \rfloor}(x,y) + p_{\lfloor t \rfloor + 1}(x,y).$$

Theorem 4.2. *Let $x \in \mathbb{R}^d$ and $t > 0$. Suppose Assumption 4.1 holds. Then*

$$\lim_{n\to\infty} n^{d/2}\hat{p}_{nt}(0,g_n(x)) = 2a_{\mathcal{G}}^{-1}k_t^{(D)}(x). \tag{4.4}$$

Proof. Write k_t for $k_t^{(D)}$. Let θ be chosen as in Proposition 3.2. Let $\varepsilon \in (0, \frac{1}{2})$. Choose $\kappa > 0$ such that $(\kappa^\theta + \kappa) < \varepsilon$. Write $\Lambda_n = \Lambda_n(x,\kappa) = \Lambda(n^{1/2}x, n^{1/2}\kappa)$. Set

$$J(n) = P^0\Big(n^{-1/2}X_{\lfloor nt \rfloor} \in \Lambda(x,\kappa)\Big) + P^0\Big(n^{-1/2}X_{\lfloor nt \rfloor + 1} \in \Lambda(x,\kappa)\Big) - 2\int_{\Lambda(x,\kappa)} k_t(y)dy. \tag{4.5}$$

Then

$$\begin{aligned} J(n) = \sum_{z \in \Lambda_n} &\big(\hat{p}_{nt}(0,z) - \hat{p}_{nt}(0,g_n(x))\big)\mu_z \\ &+ \mu(\Lambda_n)\hat{p}_{nt}(0,g_n(x)) - \mu(\Lambda_n)n^{-d/2}a_{\mathcal{G}}^{-1}2k_t(x) &(4.6)\\ &\quad + 2k_t(x)\big(\mu(\Lambda_n)n^{-d/2}a_{\mathcal{G}}^{-1} - 2^d\kappa^d\big) &(4.7)\\ &\qquad + 2\int_{H(x,\kappa)} (k_t(x) - k_t(y))dy &(4.8)\\ = J_1(n) + J_2(n) + J_3(n) + J_4(n). \end{aligned}$$

We now control the terms $J(n)$, $J_1(n)$, $J_3(n)$ and $J_4(n)$. By Assumption 4.1 we can choose n_1 with $n_1^{-\delta} < 2C_1\kappa$ such that, for $n \ge n_1$,

$$|J(n)| \le \kappa^d \varepsilon, \tag{4.9}$$

$$\left| \frac{\mu(\Lambda_n)}{a_{\mathscr{G}}(2n^{1/2}\kappa)^d} - 1 \right| \le \varepsilon < \tfrac{1}{2}, \tag{4.10}$$

$$\sup_{k \ge \frac{1}{2}nt, z \in \mathscr{G}} \hat{p}_k(0,z) \le c_1(nt)^{-d/2}, \tag{4.11}$$

$$s(g_n(x))n^{-1/2} \le 2C_1\kappa. \tag{4.12}$$

We bound $J_1(n)$ by using the Hölder continuity of $\hat{p}$, which comes from the PHI and Proposition 3.2. We begin by comparing Λ_n with balls in the d-metric. Let $n \ge n_1$. By (4.10) $\mu(\Lambda_n) > 0$, so $g_n(x) \in \Lambda_n$. By Assumption 4.1(e) there exists $n_2 \ge n_1$ such that, if $n \ge n_2$ and $y \in \Lambda_n$ then

$$d(y, g_n(x)) \le (C_1|y - g_n(x)|_\infty) \vee n^{1/2-\delta} \le n^{1/2}\big((2C_1\kappa) \vee n^{-\delta}\big) \le 2C_1\kappa n^{1/2}.$$

So, writing $B = B_d(g_n(x), 2C_1\kappa n^{1/2})$, $\Lambda_n \subset B$ when $n \ge n_2$. Thus we have, using (4.10),

$$\begin{aligned} |J_1(n)| &\le \mu(\Lambda_n) \max_{z \in \Lambda_n} |\hat{p}_{nt}(0,z) - \hat{p}_{nt}(0, g_n(x))| \\ &\le 2a_{\mathscr{G}}(2n^{1/2}\kappa)^d \max_{z \in B} |\hat{p}_{nt}(0,z) - \hat{p}_{nt}(0, g_n(x))|. \end{aligned} \tag{4.13}$$

Using Assumption 4.1(c), Proposition 3.2 and then (4.11) and (4.12),

$$\begin{aligned} \max_{z \in B} |\hat{p}_{nt}(0,z) - \hat{p}_{nt}(0, g_n(x))| &\le c\Big(\frac{s(g_n(x)) \vee 2C_1\kappa n^{1/2}}{(nt)^{1/2}}\Big)^{\theta} \sup_{k \ge \frac{1}{2}nt, z \in \mathscr{G}} \hat{p}_k(0,z) \\ &\le c(nt)^{-d/2}\Big(\frac{s(g_n(x))n^{-1/2} \vee 2C_1\kappa}{t^{1/2}}\Big)^{\theta} \\ &\le c_2 t^{-(d+\theta)/2} n^{-d/2} \kappa^{\theta}. \end{aligned} \tag{4.14}$$

Hence combining (4.13) and (4.14)

$$|J_1(n)| \le c_3 t^{-(d+\theta)/2} \kappa^{d+\theta}. \tag{4.15}$$

We now control the other terms. Since $|\nabla k_t(x)| \le c_4 t^{-(d+1)/2}$,

$$|J_4(n)| \le 2|\Lambda(x,\kappa)| c_4(t)(2\kappa) = \kappa^{d+1} c_5(t). \tag{4.16}$$

For $J_3(n)$, using (4.10) and (4.11), if $n \ge n_2$ then

$$\begin{aligned} J_3(n) &= 2k_t(x)\Big|\mu(\Lambda_n)n^{-d/2}a_{\mathscr{G}}^{-1} - 2^d\kappa^d\Big| \\ &= 2k_t(x)2^d\kappa^d \left| \frac{\mu(\Lambda_n)}{a_{\mathscr{G}}(2n^{1/2}\kappa)^d} - 1 \right| \le c_6(t)\kappa^d \varepsilon. \end{aligned}$$

Now write $\widetilde{p}_n = n^{d/2}\hat{p}_{nt}(0, g_n(x))$. Then for $n \ge n_2$

$$\begin{aligned} |J_2(n)| &= \mu(\Lambda_n)|\hat{p}_{nt}(0, g_n(x)) - n^{-d/2}a_{\mathscr{G}}^{-1}2k_t(x)| \\ &= \frac{\mu(\Lambda_n)}{(2n^{1/2}\kappa)^d}(2\kappa)^d|\widetilde{p}_n - 2a_{\mathscr{G}}^{-1}k_t(x)| \ge \tfrac{1}{2}a_{\mathscr{G}}(2\kappa)^d|\widetilde{p}_n - 2a_{\mathscr{G}}^{-1}k_t(x)|. \end{aligned}$$

So,

$$\begin{aligned}\tfrac{1}{2}a_{\mathcal{G}}(2\kappa)^d|\widetilde{p}_n - 2a_{\mathcal{G}}^{-1}k_t(x)| &\le |J(n)| + |J_1(n)| + |J_3(n)| + |J_4(n)| \\ &\le \kappa^d\varepsilon + c_3 t^{-(d+\theta)/2}\kappa^{d+\theta} + c_6(t)\kappa^d\varepsilon + c_5(t)\kappa^{d+1} \\ &\le c_7(t)\kappa^d(\varepsilon + \kappa^\theta + \kappa) \le 2c_7(t)\kappa^d\varepsilon.\end{aligned}$$

Thus for $n \ge n_2$,

$$|\widetilde{p}_n - 2a_{\mathcal{G}}^{-1}k_t(x)| \le c_8(t)\varepsilon, \tag{4.17}$$

which completes the proof. □

Corollary 4.3. *Let $0 < T_1 < T_2 < \infty$. Suppose Assumption 4.1 holds, and in addition that for each $H(y,r)$ the CLT in Assumption 4.1(a) holds uniformly for $t \in [T_1, T_2]$. Then*

$$\lim_{n\to\infty} \sup_{T_1 \le t \le T_2} |n^{d/2}\hat{p}_{nt}(0, g_n(x)) - 2a_{\mathcal{G}}^{-1}k_t^{(D)}(x)| = 0. \tag{4.18}$$

Proof. The argument is the same as for the Theorem; all we need do is to note that the constant $c_8(t)$ in (4.17) can be chosen to be bounded on $[T_1, T_2]$. □

If we slightly strengthen our assumptions, then we can obtain a uniform result in x.

Assumption 4.4. *(a) For any compact $I \subset (0,\infty)$, the CLT in Assumption 4.1(a) holds uniformly for $t \in I$.*
(b) There exist C_i such that

$$\hat{p}_k(0,x) \le C_2 k^{-d/2}\exp(-C_4 d(0,x)^2/k), \qquad \text{for } k \ge C_3 \text{ and } x \in \mathcal{G}. \tag{4.19}$$

(c) Assumption 4.1(c) holds.
(d) Let $h(r)$ be the size of the biggest 'hole' in $\Lambda(0,r)$. More precisely, $h(r)$ is the suprema of the r' such that $\Lambda(y,r') = \emptyset$ for some $y \in H(0,r)$. Then $\lim_{r\to\infty} h(r)/r = 0$.
(e) There exist constants δ, C_1, C_H such that for each $x \in \mathbb{Q}^d$ Assumption 4.1(d), (e) and (f) hold.

Note that in discrete time we have $p_k(0,x) = 0$ if $d(0,x) > k$, so it is not necessary in (4.19) to consider separately the case when $d(0,x) \gg k$.

Theorem 4.5. *Let $T_1 > 0$. Suppose Assumption 4.4 holds. Then*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^d} \sup_{t \ge T_1} |n^{d/2}\hat{p}_{nt}(0, g_n(x)) - 2a_{\mathcal{G}}^{-1}k_t^{(D)}(x)| = 0. \tag{4.20}$$

Proof. As before we write $k_t = k_t^{(D)}$. Set

$$w(n,t,x) = |n^{d/2}\hat{p}_{nt}(0, g_n(x)) - 2a_{\mathcal{G}}^{-1}k_t(x)|.$$

Let $\varepsilon \in (0, \frac{1}{2})$. We begin by restricting to a compact set of x and t. Choose n_1 so that $n_1 T_1 \ge C_3$, and $T_2 > 1 + T_1$ such that

$$2a_{\mathcal{G}}^{-1}k_{T_2}(0) + C_2 T_2^{-d/2} \le \varepsilon.$$

If $t \geq T_2$ then using Assumption 4.1(b), for $n \geq n_1$,

$$w(n,t,x) \leq n^{d/2}\hat{p}_{nt}(0,g_n(x)) + 2a_{\mathscr{G}}^{-1}k_t(x) \leq n^{d/2}C_2(nt)^{-d/2} + 2a_{\mathscr{G}}^{-1}k_t(0) \leq \varepsilon.$$

So we can restrict to $t \in [T_1, T_2]$.

Now choose $R > 0$ so that $h(r) \leq \frac{1}{2}r$ for $r \geq R$. Let $|x| \geq R$ and $t \in [T_1, T_2]$. Then

$$2a_{\mathscr{G}}^{-1}k_t(x) \leq cT_1^{-d/2}\exp(-R^2/2T_2). \tag{4.21}$$

We have $|n^{1/2}x - g_n(x)|_\infty \leq h(|x|n^{1/2}) \leq \frac{1}{2}|x|n^{1/2}$, as $|x|n^{1/2} > R$ for all $n \geq 1$, and hence

$$d(0, g_n(x)) \geq |g_n(x)|_\infty \geq \tfrac{1}{2}|x|n^{1/2}.$$

The Gaussian upper bound (4.19) yields

$$n^{d/2}\hat{p}_{nt}(0,g_n(x)) \leq ct^{-d/2}\exp(-c'|x|^2/t) \leq cT_1^{-d/2}\exp(-c'R^2/T_2). \tag{4.22}$$

We can choose R large enough so the terms in (4.21) and (4.22) are smaller than ε. Thus $w(n,t,x) < \varepsilon$ whenever $t > T_2$ or $|x| > R$, and $n \geq n_1$. Thus it remains to show that there exists n_2 such that for $n \geq n_2$,

$$\sup_{|x| \leq R, T_1 \leq t \leq T_2} w(n,t,x) < \varepsilon.$$

Now let κ be chosen as in the proof of Theorem 4.2, and also such that

$$c_1 T_1^{-(d+\theta)/2}\kappa^\theta < \varepsilon, \tag{4.23}$$

where c_1 is the constant c_3 in (4.15). Let $\eta \in (0,\kappa) \cap \mathbb{Q}$. Set $\mathscr{Y} = \{y \in \eta\mathbb{Z}^d \cap B_R(0)\}$, where $B_R(0)$ is the Euclidean ball centre 0 and radius R. By Theorem 4.2 and Corollary 4.3 for each $y \in \mathscr{Y}$ there exists $n_3'(y)$ such that

$$\sup_{T_1 \leq t \leq T_2} w(n,t,y) \leq \varepsilon \quad \text{ for } n \geq n_3'(y). \tag{4.24}$$

We can assume in addition that $n_3'(y)$ is greater than the $n_2 = n_2(y)$ given by the proof of Theorem 4.2. Let $n_4 = \max_{y \in \mathscr{Y}} n_3'(y)$. Now let $x \in B_R(0)$, and write $y(x)$ for a closest point (in the $|\cdot|_\infty$ norm) in $\mathscr{Y}$ to x: thus $|x - y(x)|_\infty \leq \eta$. Let $n \geq n_4$. We have

$$|n^{d/2}\hat{p}_{nt}(0,g_n(x)) - 2a_{\mathscr{G}}^{-1}k_t(x)| \leq |n^{d/2}\hat{p}_{nt}(0,g_n(x)) - n^{d/2}\hat{p}_{nt}(0,g_n(y(x)))| \tag{4.25}$$

$$+ |n^{d/2}\hat{p}_{nt}(0,g_n(y(x))) - 2a_{\mathscr{G}}^{-1}k_t(y(x))| \tag{4.26}$$

$$+ |2a_{\mathscr{G}}^{-1}k_t(y(x)) - 2a_{\mathscr{G}}^{-1}k_t(x)|, \tag{4.27}$$

and it remains to bound the three terms (4.25), (4.26), (4.27), which we denote L_1, L_2, L_3 respectively. Since $\eta < \kappa$ and $n \geq n_4 \geq n_3(y(x))$, we have the same bound for L_1 as in (4.14), and obtain

$$L_1 = |n^{d/2}\hat{p}_{nt}(0,g_n(x)) - n^{d/2}\hat{p}_{nt}(0,g_n(y(x)))| \leq c_1 t^{-(d+\theta)/2}\eta^\theta \tag{4.28}$$

$$\leq c_1 T_1^{-(d+\theta)/2}\eta^\theta < \varepsilon, \tag{4.29}$$

by (4.23). As $n \geq n_4$ and $y(x) \in \mathscr{Y}$, by (4.24) $L_2 < \varepsilon$. Finally,

$$L_3 = |k_t(x) - k_t(y(x))| \leq \eta d^{1/2}||\nabla k_t||_\infty \leq c\eta T_1^{-(d+1)/2},$$

and choosing η small enough this is less than ε. Thus we have $w(n,t,x) < 3\varepsilon$ for any $x \in B_R(0)$, $t \in [T_1, T_2]$ and $n \geq n_4$, completing the proof of the theorem. $\square$

In continuous time we replace X by Y, $p_k(0,y)$ by $q_t(0,y)$, and modify Assumptions 4.1 and 4.4 accordingly. That is, in both Assumptions we replace the CLT for X in (a) by a CLT for Y, replace p_n in (b) by q_t, and require the continuous time version of the PHI in (c). The same arguments then give a local limit theorem as follows.

Theorem 4.6. *Let $T_1 > 0$. Suppose Assumption 4.4 (modified as above for the continuous time case) holds. Then*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^d} \sup_{t\geq T_1} |n^{d/2} q_{nt}(0, g_n(x)) - a_{\mathcal{G}}^{-1} k_t^{(D)}(x)| = 0. \tag{4.30}$$

5 Application to percolation clusters

We now let $(\Omega, \mathbb{P})$ be a probability space carrying a supercritical bond percolation process on $\mathbb{Z}^d$. As in the Introduction we write $\mathscr{C}_\infty = \mathscr{C}_\infty(\omega)$ for the infinite cluster. Let $\mathbb{P}_0(\cdot) = \mathbb{P}(\cdot | 0 \in \mathscr{C}_\infty)$. Let $x \sim y$. We set $\mu_{xy}(\omega) = 1$ if the edge $\{x,y\}$ is open and $\mu_{xy}(\omega) = 0$ otherwise. In the physics literature one finds two common choices of random walks on $\mathscr{C}_\infty$, called the 'myopic ant' and 'blind ant' walks, which we denote X^M and X^B respectively. For the myopic walk we set

$$\mu^M_{xy} = \mu_{xy}, \quad y \neq x,$$
$$\mu^M_{xx} = 0,$$

and for each $\omega \in \Omega$ we then take $X^M = (X^M_n, n \in \mathbb{Z}_+, P^x_\omega, x \in \mathscr{C}_\infty(\omega))$ to be the random walk on the graph $(\mathscr{C}_\infty(\omega), \mu^M(\omega))$. Thus X^M jumps with equal probability from x along any of the open bonds adjacent to x. The second choice ('the blind ant') is to take

$$\mu^B_{xy} = \mu_{xy}, \quad y \neq x,$$
$$\mu^B_{xx} = 2d - \mu_x,$$

and take X^B to be the random walk on the graph $(\mathscr{C}_\infty(\omega), \mu^B(\omega))$. This walk attempts to jump with probability $1/2d$ in each direction, but the jump is suppressed if the bond is not open. By Theorem 2.2 the same transition density bounds hold for these two processes. Since these two processes are time changes of each other, an invariance principle for one quickly leads to one for the other – see for example [7, Lemma 6.4].

In what follows we take X to be either of the two walks given above. We write $p^\omega_n(x,y)$ for its transition density, and as before we set $\hat{p}^\omega_n(x,y) = p^\omega_n(x,y) + p^\omega_{n+1}(x,y)$. We begin by summarizing the heat kernel bounds on $p^\omega_n(x,y)$.

Theorem 5.1. *There exists $\eta = \eta(d) > 0$ and constants $c_i = c_i(d,p)$ and r.v. $V_x, x \in \mathbb{Z}^d$, such that*

$$\mathbb{P}(V_x(\omega) \geq n) \leq c \exp(-cn^\eta), \tag{5.1}$$

and if $n \geq c|x-y| \vee V_x$ then

$$c_1 n^{-d/2} e^{-c_2|x-y|^2/n} \leq \hat{p}^\omega_n(x,y) \leq c_3 n^{-d/2} e^{-c_4|x-y|^2/n}. \tag{5.2}$$

Further if $n \geq c|x-y|$ then

$$c_1 n^{-d/2} e^{-c_2|x-y|^2/n} \leq \mathbb{E}(\hat{p}^\omega_n(x,y) | x, y \in \mathscr{C}_\infty) \leq c_3 n^{-d/2} e^{-c_4|x-y|^2/n}. \tag{5.3}$$

Proof. This follows from Theorem 2.2(a), and the arguments in [2], Section 6. □

We now give the local limit theorem. As in Section 4 we write $g_n^\omega(x)$ for a closest point in $\mathscr{C}_\infty$ to $n^{1/2}x$, set $\Lambda(x,r) = \Lambda(x,r)(\omega) = \mathscr{C}_\infty(\omega) \cap H(x,r)$, and write $h_\omega(r)$ for the largest hole in $\Lambda(0,r)$.

Theorem 5.2. *Let $T_1 > 0$. Then there exist constants a, D such that $\mathbb{P}_0$-a.s.,*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^d} \sup_{t\geq T_1} |n^{d/2}\hat{p}^\omega_{nt}(0, g_n^\omega(x)) - 2a^{-1}k_t^{(D)}(x)| = 0. \tag{5.4}$$

In view of Theorem 4.5 it is enough to prove that, $\mathbb{P}_0$-a.s., the cluster $\mathscr{C}_\infty(\omega)$ and process X satisfy Assumption 4.4. Note that since we apply Theorem 4.5 separately to each graph $\mathscr{C}_\infty(\omega)$, it is not necessary that the constants C_i in Assumption 4.4 should be uniform in ω – in fact, it is clear that the constant C_3 in (4.19) cannot be taken independent of ω.

Lemma 5.3. *(a) There exist constants δ, $C_.$ such that Assumption 4.4 (a), (b), (c) all hold $\mathbb{P}_0$-a.s.*
(b) Let $x \in \mathbb{R}^d$. Then Assumption 4.1(e) holds $\mathbb{P}_0$-a.s.

Proof. (a) The CLT holds (uniformly) by the invariance principles proved in [25; 7; 21]. Assumption 4.4(b) holds by Theorem 1.1 of [2].

For $x \in \mathbb{Z}^d$, let S_x be the smallest integer n such that $B_d(x,R)$ is very good with $N^{2d+4}_{B_d(x,R)} < R$ for all $R \geq n$. (If $x \notin \mathscr{C}_\infty$ we take $S_x = 0$.) Then by Theorem 2.18 and Lemma 2.19 of [2] there exists $\gamma = \gamma_d > 0$ such that

$$\mathbb{P}(S_x \geq n) \leq c\exp(-cn^\gamma). \tag{5.5}$$

In particular, we have that $S_x < \infty$ for all $x \in \mathscr{C}_\infty$, $\mathbb{P}$-a.s. By Theorem 3.1, the PHI holds for $Q(x,R,R^2)$ for all $R \geq S_x$, and Assumption 4.4(c) holds.
(b) Assumption 4.1(e) holds by results in [2] – see Proposition 2.17(d), Lemma 2.19 and Remark 2 following Lemma 2.19. □

In the results which follow, we have not made any effort to obtain the best constant γ in the various bounds of the form $\exp(-n^\gamma)$.

Lemma 5.4. *With $\mathbb{P}$-probability 1, $\lim_{r\to\infty} h_\omega(r)r^{-1/2} = 0$, and so Assumption 4.4(d) holds.*

Proof. Let M_0 be the random variable given in Lemma 2.19 of [2]. Let $\alpha = 1/4$, and note that $\beta = 1 - 2(1+d)^{-1} > 1/3$. Therefore

$$\mathbb{P}_0(M_0 \geq n) \leq c\exp(-cn^{\alpha/3}),$$

and if $M_0 \leq n$ then the event $D(Q,\alpha)$ defined in (2.21) of [2] holds for every cube of side n containing 0. It follows from this (see (2.20) and the definition of $R(Q)$ on p. 3040 in [2]) that every cube of side greater than n^α in $[-n/2, n/2]^d$ intersects $\mathscr{C}_\infty$. Thus

$$\mathbb{P}_0(h_\omega(n) \geq n^\alpha) \leq c\exp(-cn^{\alpha/3}), \tag{5.6}$$

and using Borel-Cantelli we deduce that $\lim_{r\to\infty} h_\omega(r)r^{-1/2} = 0$ $\mathbb{P}_0$-a.s. □

Lemma 5.5. *Let $x \in \mathbb{R}^d$. With $\mathbb{P}$-probability 1, Assumption 4.1(f) holds.*

Proof. Let $F_n = \{g_n^\omega(x) \in \Lambda_n(x,1)\}$, and $B_n = \{S_{g_n^\omega(x)} > n^{1/3}\}$. If F_n^c occurs, then a cube side n containing $\Lambda_n(x,1)$ has a hole greater than $n^{1/2}$. So, by (5.6)

$$\mathbb{P}(F_m^c) \le ce^{-cn^{1/3}}.$$

Let $Z_n = \max_{z \in \Lambda_n(x,1)} S_x$. Then

$$B_n \subset F_n^c \cup \{Z_n > n^{1/3}\},$$

so using (5.5)

$$\mathbb{P}(B_n) \le ce^{-c'n^{1/3}} + cn^{d/2}e^{-c'n^{\gamma/3}},$$

and by Borel-Cantelli Assumption 4.1(f) follows. □

It remains to prove Assumption 4.1(d). If instead we wanted to control $|\Lambda_n|/(n^{1/2}\kappa)^d$ then we could use results in [11; 15]. Since the arguments for $\mu(\Lambda_n)$ are quite similar, we only give a sketch of the proof.

Lemma 5.6. *Let $x \in \mathbb{R}^d$. There exists $a > 0$ such that with $\mathbb{P}$-probability 1,*

$$\frac{\mu(\Lambda_n(x,r))}{(2n^{1/2}r)^d} \to a \qquad \text{as } n \to \infty, \tag{5.7}$$

and so Assumption 4.1(d) holds.

Proof. For a cube $Q \subset \mathbb{Z}^d$ write $s(Q)$ for the length of the side of Q. Let $\partial_i Q = \partial(\mathbb{Z}^d - Q)$ be the 'internal boundary' of Q, and $Q^0 = Q - \partial_i Q$. Recall that μ_x is the number of open bonds adjacent to x, and set

$$M(Q) = \{x \in Q^0 : x \longleftrightarrow \partial_i Q\}, \qquad V(Q) = \mu(M(Q)).$$

Note that if $x \in Q$ and x is connected by an open path to $\partial_i Q$ then x is connected to $\partial_i Q$ by an open path inside Q. Thus the event $x \in M(Q)$ depends only on the percolation process inside Q. So if Q_i are disjoint cubes, then the $V(Q_i)$ are independent random variables. Let $\mathscr{C}_k$ be a cube of side length k and set

$$a_k = \mathbb{E}k^{-d}V(C_k).$$

By the ergodic theorem there exists a such that, $\mathbb{P}$-a.s.,

$$\lim_{R\to\infty} \frac{V(H(0,R/2))}{R^d} \to a, \quad \mathbb{P}\text{-a.s. and in } L^1. \tag{5.8}$$

In particular, $a = \lim a_k$. Since $\mathscr{C}_\infty$ has positive density, it is clear that $a > 0$.

We have

$$\mu(Q \cap \mathscr{C}_\infty) \le V(Q) + c_1 s(Q)^{d-1}.$$

Let $\varepsilon > 0$. Choose k large enough so that $c_1/k \le \varepsilon$, and $a_k \le a + \varepsilon$.

Now let Q be a cube of side nk, and let Q_i, $i = 1, \dots n^d$ be a decomposition of Q into disjoint sub-cubes each of side k. Then

$$\begin{aligned}(nk)^{-d}\mu(Q \cap \mathscr{C}_\infty) - a_k &\le (nk)^{-d}\sum_i \mu(Q_i \cap \mathscr{C}_\infty) - a_k \\ &\le c_1 k^{-1} + n^{-d}\sum_i (k^{-d}V(Q_i) - a_k).\end{aligned}$$

As this is a sum of i.i.d. mean 0 random variables, it follows that there exists $c_2(k,\varepsilon) > 0$ such that

$$\mathbb{P}((nk)^{-d}\mu(Q \cap \mathscr{C}_\infty) > a + 3\varepsilon) \le \exp(-c_2(k,\varepsilon)n^d). \tag{5.9}$$

The lower bound on $\mu(Q \cap \mathscr{C}_\infty)$ requires a bit more work. We call a cube Q 'm-good' if the event $R(Q)$ given in [1] or p. 3040 of [2] holds, and

$$\mu(\mathscr{C}_\infty \cap Q) \ge (a - \varepsilon)s(Q)^d.$$

Let p_k be the probability a cube of side k is m-good. Then by (2.24) in [1], and (5.8), $\lim p_k = 1$. As in [1] we can now divide $\mathbb{Z}^d$ into disjoint macroscopic cubes T_x of side k, and consider an associated site percolation process where a cube is occupied if it is m-good. We write $\mathscr{C}^*$ for the infinite cluster for this process. Let Q be a cube of side nk, and T_x be the n^d disjoint sub-cubes of side k in Q. Then

$$\mu(\mathscr{C}_\infty \cap Q) \ge \sum_x \mu(\mathscr{C}_\infty \cap T_x) \ge (a-\varepsilon)k^d \#\{x : T_x \in \mathscr{C}^*, T_x \subset Q\}. \tag{5.10}$$

By Theorem 1.1 of [15] we can choose k large enough so there exists a constant $c_3(k,\varepsilon)$ such that

$$\mathbb{P}(n^{-d}\#\{x : T_x \in \mathscr{C}^*, T_x \subset Q\} < 1 - \varepsilon) \le \exp(-c_3(k,\varepsilon)n^{d-1}). \tag{5.11}$$

It follows that

$$\mathbb{P}\big((nk)^{-d}\mu(\mathscr{C}_\infty \cap Q) < a - (1+a)\varepsilon\big) \le \exp(-c_3(k,\varepsilon)n^{d-1}). \tag{5.12}$$

Combining (5.9) and (5.12), and using Borel-Cantelli gives (5.7). □

Proof of Theorem 5.2. By Lemmas 5.3, 5.5 and 5.6 Assumption 4.1 holds for all $x \in \mathbb{Q}^d$, $\mathbb{P}$-a.s., and so also $\mathbb{P}_0$-a.s. Therefore using Lemma 5.3 we have that Assumption 4.4 holds $\mathbb{P}_0$-a.s., so (5.4) follows from Theorem 4.5. □

Proof of Theorem 1.1. The discrete time case is given by Theorem 5.2. For continuous time, since Assumption 4.4 holds $\mathbb{P}_0$-a.s., (1.5) follows from Theorem 4.6. Since a is given by (4.3), and μ is the same for Y and the myopic walk, the constant a in (1.5) is the same as for the myopic walk in (1.4). If Z_t is a rate 1 Poisson process then we can write $Y_t = X_{Z_t}$, and it is easy to check that the CLT for X implies one for Y with the same diffusion constant D. □

As a second application we consider the random conductance model in the case when the conductances are bounded away from 0 and infinity.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. Let $K \ge 1$ and μ_e, $e \in \mathbb{E}_d$ be i.i.d.r.v. supported on $[K^{-1}, K]$. Let also η_x, $x \in \mathbb{Z}^d$ be i.i.d. random variables on $[0,1]$, $F : R^{d+1} \to [K^{-1}, K]$, and $\mu_{xx} = F(\eta_x, (\mu_{x\cdot}))$. For each $\omega \in \Omega$ let $X = (X_n, n \in \mathbb{Z}_+, P^x_\omega, x \in \mathbb{Z}^d)$ be the SRW on $(\mathbb{Z}^d, \mu)$ defined in Section 2, and $p^\omega_n(x,y)$ be its transition density.

Theorem 5.7. *Let $T_1 > 0$. Then there exist constants a, D such that $\mathbb{P}_0$-a.s.,*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^d} \sup_{t\ge T_1} |n^{d/2}\tilde{p}^\omega_{nt}(0, g^\omega_n(x)) - 2a^{-1}k^{(D)}_t(x)| = 0. \tag{5.13}$$

Proof. As above, we just need to verify Assumption 4.4. The invariance principle in [25] implies the uniform CLT, giving (a). Since μ_e are bounded away from 0 and infinity, the results of [13] immediately give the PHI (with $S(x)=1$ for all x) and heat kernel upper bound (4.19), so giving Assumption 4.4(b) and (c), as well as Assumption 4.1(f). As $\mathscr{G}=\mathbb{Z}^d$, Assumption 4.4(d) and Assumption 4.1(e) hold.

It remains to verify Assumption 4.1(d), but this holds by an argument similar to that in Lemma 5.6. □

6 Green's functions for percolation clusters

We continue with the notation and hypotheses of Section 5, but we take $d \geq 3$ throughout this section. The Green's function can be defined by

$$g_\omega(x,y)=\int_0^\infty q_t^\omega(x,y)dt. \tag{6.1}$$

By Theorem 2.2(c) $g_\omega(x,y)$ is $\mathbb{P}$-a.s. finite for all $x,y\in\mathscr{C}_\infty$. We have that $g_\omega(x,\cdot)$ satisfies

$$\mathscr{L}g_\omega(x,y)=\begin{cases}0 & \text{if } y\neq x,\\ -1/\mu_x & \text{if } y=x.\end{cases} \tag{6.2}$$

Since any bounded harmonic function is constant (see [6] or [2, Theorem 4]), these equations have, $\mathbb{P}$-a.s., a unique solution such that $g_\omega(x,y)\to 0$ as $|y|\to\infty$. It is easy to check that the Green's function for the myopic and blind ants satisfy the same equations, so the Green's function for the continuous time walk Y, and the myopic and blind ant discrete time walks are the same.

We write $d_\omega(x,y)$ for the graph distance on $\mathscr{C}_\infty$. By Lemma 1.1 and Theorem 1 of [2] there exist $\eta>0$, constants c_i and r.v. T_x such that

$$\mathbb{P}(T_x\geq n)\leq ce^{-c_1n^\eta}, \tag{6.3}$$

so that the following bounds on $q_t^\omega(x,y)$ hold:

$$q_t(x,y)\leq c_2\exp(-c_3d_\omega(x,y)(1+\log\tfrac{d_\omega(x,y)}{t})),\quad 1\leq t\leq d_\omega(x,y), \tag{6.4}$$

$$q_t(x,y)\leq c_4e^{-c_5d_\omega(x,y)^2/t},\qquad d_\omega(x,y)\leq t, \tag{6.5}$$

$$c_6t^{-d/2}e^{-c_7|x-y|^2/t}\leq q_t^\omega(x,y)\leq c_8t^{-d/2}e^{-c_9|x-y|^2/t},\quad t\geq T_x\vee|y-x|. \tag{6.6}$$

We can and will assume that $T_x\geq 1$ for all x.

Lemma 6.1. *Let $x,y\in\mathscr{C}_\infty$, and $\delta\in(0,1)$. Then*

$$\int_0^{d_\omega(x,y)} q_t^\omega(x,y)dt\leq c_1e^{-c_2|x-y|}, \tag{6.7}$$

$$\int_{d_\omega(x,y)}^{T_x} q_t^\omega(x,y)dt\leq c_3T_xe^{-c_4|x-y|^2/T_x}. \tag{6.8}$$

Proof. Using (6.4) and (6.5) we have

$$\int_0^{d_\omega(x,y)} q_t^\omega(x,y)dt \le \int_0^{d_\omega(x,y)} c\exp(-cd_\omega(x,y))dt \le ce^{-cd_\omega(x,y)},$$
$$\int_{d_\omega(x,y)}^{T_x} q_t^\omega(x,y)dt \le \int_{d_\omega(x,y)}^{T_x} ce^{-cd_\omega(x,y)^2/t}dt \le cT_x e^{-cd_\omega(x,y)^2/T_x},$$

and since $d_\omega(x,y) \ge c|x-y|$ this gives (6.7) and (6.8).

Proposition 6.2. *Let $x, y \in \mathscr{C}_\infty$, with $x \neq y$. Then there exist constants c_i such that*

$$\frac{c_1}{|x-y|^{d-2}} \le g_\omega(x,y) \le \frac{c_2}{|x-y|^{d-2}} \qquad \text{if } |x-y|^2 \ge T_x(1+c_3\log|x-y|). \tag{6.9}$$

Further, for $x, y \in \mathbb{Z}^d$,

$$\frac{c_4}{1\vee|x-y|^{d-2}} \le \mathbb{E}\big(g_\omega(x,y)|x,y\in\mathscr{C}_\infty\big) \le \frac{c_5}{1\vee|x-y|^{d-2}}, \tag{6.10}$$
$$\mathbb{E}\big(g_\omega(x,x)^k|x\in\mathscr{C}_\infty\big) \le c_6(k). \tag{6.11}$$

Proof. Note first that, by (6.6)

$$\int_{T_x}^\infty q_t^\omega(x,y)dt \le \int_0^\infty ct^{-d/2}e^{-c|x-y|^2/t}dt \le c'|x-y|^{2-d}. \tag{6.12}$$

Combining (6.7), (6.8) and (6.12) we obtain

$$g_\omega(x,y) \le c'e^{-c|x-y|} + cT_x e^{-c_6|x-y|^2/T_x} + c|x-y|^{2-d}. \tag{6.13}$$

Taking $c_3 = d/c_6$ gives

$$T_x e^{-c_6|x-y|^2/T_x} \le c|x-y|^2 e^{-d\log|x-y|} \le c|x-y|^{2-d},$$

and this gives the upper bound in (6.9). For the lower bound in (6.9) we note that since $T_x \le |x-y|^2$

$$g_\omega(x,y) \ge \int_{|x-y|^2}^\infty q_t^\omega(x,y)dt \ge \int_{|x-y|^2}^\infty ct^{-d/2}e^{-c|x-y|^2/t}dt = c'|x-y|^{2-d}. \tag{6.14}$$

We now turn to (6.10). Choose k_0 such that $\mathbb{P}(T_x \le k_0) \ge \frac{1}{2}$. Then

$$\mathbb{E}^x g_\omega(x,y) \ge \mathbb{E}^x\Big(\int_{T_x}^\infty q_t^\omega(x,y)dt; T_x \le k_0\Big) \ge \tfrac{1}{2}\int_{k_0}^\infty ct^{-d/2}e^{-c|x-y|^2/t}dt. \tag{6.15}$$

If $|x-y|^2 \ge k_0$, then the final term in (6.15) is bounded below by $c|x-y|^{2-d}$ in the same way as in (6.15), while when $|x-y|^2 \le k_0$ we have

$$\mathbb{E}^x g_\omega(x,y) \ge c\int_{k_0}^\infty ct^{-d/2}e^{-c|x-y|^2/t}dt \ge ce^{-c|x-y|^2/k_0}k_0^{1-d/2} \ge c', \tag{6.16}$$

which gives the lower bound in (6.10). For the averaged upper bound, note first that

$$g_\omega(x,x) = \int_0^\infty q_t(x,x)dt \le cT_x + \int_{T_x}^\infty ct^{-d/2}dt \le c'T_x. \tag{6.17}$$

So for any $k \ge 1$, by (6.3)

$$\mathbb{E}(g_\omega(x,x)^k | x \in \mathscr{C}_\infty) \le c(k)\mathbb{E}(T_x^k | x \in \mathscr{C}_\infty) \le c'(k),$$

proving (6.11), and (taking $k = 1$) the upper bound in (6.10) when $y = x$.

Now let $y \neq x$ and $F = \{|x-y|^2 \le T_x(1 + c_{6.2.3}|x-y|)\}$. Then writing $\mathbb{E}_{xy}(\cdot) = \mathbb{E}(\cdot | x, y \in \mathscr{C}_\infty)$, and using (6.9), (6.17), the fact that $g_\omega(x,y) \le g_\omega(x,x)$ and (6.3),

$$\begin{aligned}
\mathbb{E}_{xy} g_\omega(x,y) &= \mathbb{E}_{xy}(g_\omega(x,y); F) + \mathbb{E}_{xy}(g_\omega(x,y); F^c) \\
&\le c|x-y|^{2-d} + (\mathbb{E}_{xy}(g_\omega(x,y)^2))^{1/2}\mathbb{P}_{xy}(F^c)^{1/2} \\
&\le c|x-y|^{2-d} + (\mathbb{E}_{xy}(g_\omega(x,x)^2))^{1/2} ce^{-c|x-y|^{\eta/3}} \le c'|x-y|^{2-d},
\end{aligned}$$

proving (6.10). □

To prove that $|y|^{d-2}g_\omega(0,y)$ has a limit as $|y| \to \infty$ we use Theorem 1.1. Write $k_t(x) = k_t^{(D)}(x)$, where D is the constant in (1.5).

Lemma 6.3. *Let $\varepsilon > 0$. Then for $\mathbb{P}$-a.a. $\omega \in \Omega_0$ there exists $a > 0$ and $N = N(\varepsilon, \omega)$ such that*

$$|q_t^\omega(0,y) - a^{-1}k_t(y)| \le \varepsilon t^{-d/2} \quad \textit{for all } t \ge N,\ y \in \mathscr{C}_\infty(\omega). \tag{6.18}$$

Proof. By Theorem 1.1. there exists N such that

$$\sup_{x\in\mathbb{R}^d} \sup_{s\ge 1} \left| n^{d/2} q_{ns}^\omega(0, g_n^\omega(x)) - a^{-1}k_s(x) \right| \le \varepsilon \text{ for } n \ge N. \tag{6.19}$$

Let $n = N$, $s = t/n$ and $x = n^{-1/2}y$, so that $g_n(x) = y$. Then noting that $k_s(x) = n^{d/2}k_t(y)$ (6.18) follows. □

Let $|z| = 1$ and

$$C = a^{-1}\int_0^\infty k_t(z)dt = (Da)^{-1}\int_0^\infty (2\pi s)^{-d/2}e^{-1/2s}ds = \frac{\Gamma(\frac{d}{2}-1)}{2\pi^{d/2}aD}. \tag{6.20}$$

Proof of Theorem 1.2. (a) This was proved as Proposition 6.2.

(b) Let $\delta \in (0,1)$, to be chosen later. For $y \in \mathscr{C}_\infty$ we set $t_1 = t_1(y) = \delta|y|^2$, and $t_2 = t_2(y) = |y|^2/\delta$. Then

$$g_\omega(0,y) = \int_0^{t_1} q_t^\omega(0,y)dt + \int_{t_1}^{t_2} q_t^\omega(0,y)dt + \int_{t_2}^\infty q_t^\omega(0,y)dt = I_1 + I_2 + I_3. \tag{6.21}$$

As in Proposition 6.2 we have, using (6.7) and (6.8), that provided $|y| \ge T_0$,

$$I_1 \le ce^{-c|y|} + cT_0 e^{-c|y|^2/T_0} + \int_0^{\delta|y|^2} ct^{-d/2} e^{-c|y|^2/t} dt \tag{6.22}$$

$$\le ce^{-c|y|} + c|y|e^{-c|y|} + c|y|^{2-d} \int_0^{\delta} s^{-d/2} e^{-c_1/s} ds \tag{6.23}$$

$$\le ce^{-c|y|} + c|y|^{2-d} e^{-c_1/2\delta}. \tag{6.24}$$

Also

$$I_3 \le \int_{|y|^2/\delta}^{\infty} ct^{-d/2} e^{-c|y|^2/t} dt = c\delta^{d/2-1}|y|^{2-d}. \tag{6.25}$$

So there exist $M_1 < \infty$ and $\delta > 0$ so that

$$I_1 + I_3 \le \tfrac{1}{2}\varepsilon C|y|^{2-d} \qquad \text{when } |y| \ge M_1. \tag{6.26}$$

Now let $\varepsilon' > 0$, and let $N = N(\varepsilon')$ be given by Lemma 6.3. For I_2 we have, provided $t_1 \ge N$

$$I_2 \le \int_{t_1}^{t_2} (\varepsilon' t^{-d/2} + a^{-1}k_t(y))dt \le c\varepsilon' t_1^{1-d/2} + \int_{t_1}^{t_2} a^{-1}k_t(y)dt$$
$$\le c\varepsilon' \delta^{1-d/2}|y|^{2-d} + C|y|^{2-d}. \tag{6.27}$$

Taking $\varepsilon' = \frac{1}{2}(C/c)\varepsilon\delta^{d/2-1}$ gives the upper bound in (1.9). This bound holds provided $|y| \ge M_1 \vee T_0$ and $\delta|y|^2 \ge N(\varepsilon')$, Thus the upper bound in (1.9) holds provided

$$|y| \ge T_0 \vee M_1 \vee (\delta^{-1}N(\varepsilon'))^{1/2}. \tag{6.28}$$

For the lower bound, note that

$$C|y|^{2-d} - \int_{t_1}^{t_2} a^{-1}k_t(y)dt \le c|y|^{2-d}(e^{-c/\delta} + \delta^{d/2-1}). \tag{6.29}$$

So if (6.28) holds then

$$g_\omega(0,y) \ge I_2 \ge \int_{t_1}^{t_2} (-\varepsilon' t^{-d/2} + k_t(y))dt$$
$$\ge |y|^{2-d}\Big(C - c\varepsilon'\delta^{1-d/2} - e^{-c/\delta} - \delta^{d/2-1}\Big),$$

proving the lower bound in (1.9).

(c) Let $\varepsilon > 0$, and M be as in (a), and $U_0 = T_0(1 + c_{6.2.3}\log|y|)$. Then by Proposition 6.2

$$\mathbb{E}_0 g_\omega(0,y) \le \mathbb{E}_0(g_\omega(0,y); M \le |y|) + \mathbb{E}_0(g_\omega(0,y); U_0 \le |y| < M)$$
$$+ \mathbb{E}_0(g_\omega(0,y); |y| < U_0)$$
$$\le \frac{(1+\varepsilon)C}{|y|^{d-2}} + \frac{c_{6.2.2}}{|y|^{d-2}}\mathbb{P}_0(M > |y|) + (\mathbb{E}_0 g_\omega(0,y)^2)^{1/2}\mathbb{P}_0(U_0 > |y|)^{1/2}$$
$$\le \frac{(1+\varepsilon)C + c_{6.2.2}\mathbb{P}_0(M > |y|)}{|y|^{d-2}} + ce^{-c|y|^{\eta/3}} \tag{6.30}$$

Also

$$\mathbb{E}_0 g_\omega(0,y) \geq \mathbb{E}_0(g_\omega(0,y); M \leq |y|) \geq \frac{(1-\varepsilon)C}{|y|^{d-2}} \mathbb{P}(M \leq |y|). \tag{6.31}$$

Combining (6.30) and (6.31) completes the proof of Theorem 1.2. □

A Appendix

In this appendix, we give a proof of the 'balayage' formula (3.6)-(3.7) used in the proof of the PHI in Section 3.

Let $\Gamma = (G,E)$ and μ be as in Section 2. Let B be a finite subset of G, and $B_1 \subset B$. Write $\overline{B} = B \cup \partial B$. Let $T \geq 1$, and

$$Q = (0,T] \times B, \quad \overline{Q} = [0,T] \times \overline{B}, \quad E = (0,T] \times B_1.$$

Recall that $p_n^B(x,y)$ is the heat kernel for the process X killed on exiting B. Set

$$P^B f(x) = \sum_{y \in B} p_1^B(x,y) f(y) \mu_y, \quad Pf(x) = \sum_{y \in G} p_1(x,y) f(y) \mu_y, \tag{A.1}$$

for any function f on G

For a space-time function $w(r,y)$ we will sometimes write $w_r(y) = w(r,y)$. Let

$$Hw(n,x) = w(n,x) - Pw_{n-1}(x). \tag{A.2}$$

Then w is caloric in a space-time region $F \subset \mathbb{Z} \times G$ if and only if $Hw(n,x) = 0$ for $(n,x) \in F$. Let $\mathscr{D}$ be the set of non-negative functions $v(n,x)$ on $\overline{Q}$ such that $v = 0$ on $\overline{Q} - Q$ and v is caloric on $Q - E$. In particular we have $v(0,x) = 0$ for $v \in \mathscr{D}$.

Lemma A.1. *Let $w(r,y) \geq 0$ on $\overline{Q}$, with $w = 0$ on $\overline{Q} - E$, and let $v = v(n,x)$ be given by*

$$v(n,x) = \begin{cases} \sum_{r=1}^n P_{n-r}^B w_r(x), & \text{if } (n,x) \in Q \\ 0 & \text{if } (n,x) \notin Q. \end{cases} \tag{A.3}$$

Then $v \in \mathscr{D}$, and

$$Hv(n,x) = w(n,x), \quad (n,x) \in Q. \tag{A.4}$$

Proof. It is clear that $v \geq 0$, and that $v = 0$ on $\overline{Q} - Q$. If $x \in B$ then it easy to check that $PP_m^B f(x) = P_{m+1}^B f(x)$. Let $(n,x) \in Q$, so $1 \leq n \leq T$ and $x \in B$. Then

$$\begin{aligned} Hv(n,x) &= \sum_{r=1}^n P_{n-r}^B w_r(x) - P\Big(\sum_{r=1}^{n-1} P_{n-1-r}^B w_r\Big)(x) \\ &= \sum_{r=1}^n P_{n-r}^B w_r(x) - \sum_{r=1}^{n-1} P_{n-r}^B w_r(x) = w_n(x). \end{aligned} \tag{A.5}$$

This proves (A.4), and as $w(n,x) = 0$ when $x \in B - B_1$ we also deduce that v is caloric in $Q - E$, proving that $v \in \mathscr{D}$. □

Lemma A.2. *Let $u, v \in \mathscr{D}$ satisfy $Hu(n,x) = Hv(n,x)$ for $(n,x) \in Q$. Then $u = v$ on $\overline{Q}$.*

Proof. We have $u = v = 0$ on $\overline{Q} - Q$. We write $u_k = u(k,\cdot)$. First note that $u_0 = v_0$. If $u_k = v_k$ and $x \in B$ then

$$u(k+1,x) = Hu(k+1,x) + Pu_k(x) = Hv(k+1,x) + Pv_k(x),$$

so that $u_{k+1} = v_{k+1}$. □

Let Z be the space-time process on $\mathbb{Z} \times G$ given by $Z_n = (I_n, X_n)$, where X is the SRW on Γ, $I_n = I_0 - n$, and $Z_0 = (X_0, I_0)$ is the starting point of Z. We write $\hat{E}^{(n,x)}$ for the law of Z started at (n,x). Let $u(n,x)$ be non-negative and caloric on Q. Then the réduite u_E is defined by

$$u_E(n,x) = \hat{E}^{(n,x)}\big(u(I_{T_E}, X_{T_E}); T_E < \tau_Q\big), \tag{A.6}$$

where

$$T_E = \min\{k \geq 0 : Z_k \in E\}, \quad \tau_Q = \min\{k \geq 0 : Z_k \notin Q\}. \tag{A.7}$$

Lemma A.3. *$u_E \in \mathscr{D}$.*

Proof. If $(n,x) \in \overline{Q} - Q$ then $\hat{P}^{(n,x)}(\tau_Q = 0) = 1$, so $u_E(n,x) = 0$. It is clear from the definition (A.6) that u_E is caloric on $Q - E$, and that $u_E \geq 0$. □

Proposition A.4. *Let $1 \leq n \leq T$. Then*

$$u_E(n,x) = \sum_{y \in B} \sum_{r=1}^{n} p^B_{n-r}(x,y) k(r,y) \mu_y, \tag{A.8}$$

where

$$k(r,y) = \begin{cases} \sum_{z \in B} p^B_1(y,z)(u(r-1,z) - u_E(r-1,z))\mu_z, & \text{if } y \in B_1, \\ 0, & \text{if } y \in B - B_1. \end{cases} \tag{A.9}$$

Proof. Let $k_r(y) = k(r,y)$ be defined by (A.9) for $r \geq 1$. Set

$$v(n,x) = \sum_{r=1}^{n} P^B_{n-r} k_r(x). \tag{A.10}$$

By Lemma A.1 we have $v \in \mathscr{D}$. To prove that $v = u_E$ it is sufficient, by Lemma A.2 to prove that $Hv(n,x) = Hu_E(nx,)$ for $(n,x) \in Q$.

We have $Hv(n,x) = k(n,x)$ on Q by (A.4). If $x \in B - B_1$ then $k(n,x) = 0$, while since u_E is caloric in $Q - E$ we have $Hu_E(n,x) = 0$. If $x \in B_1$ then as $u = u_E$ on E, and u is caloric on Q,

$$\begin{aligned} Hu_E(n,x) &= u_E(n,x) - Pu_E(n-1,x) \\ &= u(n,x) - Pu_E(n-1,x) = Pu(n-1,x) - Pu_E(n-1,x) \\ &= P^B_1(u - u_E)(n-1,x). \end{aligned}$$

So we deduce that $v = u_E$. □

If $y \in B_1$ then the $r = 1$ term of (A.8) can be written

$$\sum_{y\in B} p^B_{n-1}(x,y)\mu_y(\sum_{z\in B} p^B_1(y,z)\mu_x u(0,z)) = \sum_{z\in B} \mu_x u(0,z) p^B_n(x,z), \tag{A.11}$$

so that (A.8) can be rewritten as

$$u_E(n,x) = \sum_{y\in B} p^B_n(x,y)u(0,y)\mu_y + \sum_{y\in B}\sum_{r=2}^{n} p^B_{n-r}(x,y)k(r,y)\mu_y, \tag{A.12}$$

which is the form given in (3.6).

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