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VARIABLY SKEWED BROWNIAN MOTION

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Abstract
Given a standard Brownian motion B, we show that the equation

$$X_t = x_0 + B_t + \beta(L_t^X)\,, \quad t \geq 0\ ,$$

has a unique strong solution X. Here L^X is the symmetric local time of X at 0, and β is a given differentiable function with $\beta(0) = 0$, $-1 < \beta'(\cdot) < 1$. (For linear $\beta(\cdot)$, the solution is the familiar skew Brownian motion).

1 Introduction

In this paper we consider the following stochastic differential equation:

$$X_t = x_0 + B_t + \beta(L_t^X), \qquad t \geq 0. \tag{1.1}$$

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Here $B = \{B_t, t \geq 0\}$ is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, and β is a (fixed) differentiable function, which satisfies $\beta(0) = 0$, $-1 < \beta'(x) < 1$. We seek a solution pair (X, L^X) to (1.1) where $X = \{X_t, t \geq 0\}$ is a continuous semimartingale, adapted to $(\mathcal{F}_t)$ and $L^X = \{L_t^X, t \geq 0\}$ is the symmetric local time of X at 0.
The special case $\beta(x) = \beta_0 x$ was introduced by Harrison and Shepp [HS], who proved that the unique strong solution to (1.1) is skew Brownian motion (see Itô and McKean [IM], Walsh [W1]). Note that the extreme cases $\beta' \equiv +1$ and $\beta' \equiv -1$ give rise to reflected Brownian motion.
A related stochastic differential equation, introduced by Weinryb [We], is

$$X_t = x_0 + B_t + \int_0^t \alpha(s) d\widehat{L}_s^0, \qquad t \geq 0,$$

where α is a given deterministic function and $\widehat{L}^0$ is here the nonsymmetric local time of X at 0. In [We] it was shown that a unique strong solution exists if $|\alpha| \leq \frac{1}{2}$.
Our motivation for studying equation (1.1) arose from continuous-time multi-armed bandits – see [KM1, KM2, M]. Let φ be a monotone strictly increasing function with $\varphi(0) = 0$. Given two independent one-dimensional Brownian motions W_1 and W_2, consider the multi-parameter time change $Z_i(t) = W_i(T_i(t))$, where $T_1(t) + T_2(t) = t$, $t \geq 0$, and the processes $T_i(t)$ are chosen so that $T_1(\cdot)$ increases only at times t when $\varphi(W_1(T_1(t))) > W_2(T_2(t))$, while $T_2(\cdot)$ increases if the reversed inequality applies.
Multi-parameter time changes of this kind are called *strategies* in the context of multi-armed bandits, and *optional increasing paths* [W2] in the theory of multi-parameter processes. (The pair $(T_1(t), T_2(t))$ allocates play between the two 'bandits' W_1 and W_2.)
To see the relation between this and (1.1), assume for simplicity that $x_0 = 0$ and consider the time-changed process $Z = (Z_1, Z_2)$ as a stochastic process in the plane (see Figure 1). When the process Z is above the curve $C = \{(x_1, x_2) \mid x_2 = \varphi(x_1)\}$, then only T_1 is increasing, and so Z moves horizontally. Similarly Z moves vertically below C. The motion on C is not quite so obvious, but it turns out that Z crawls upwards along C at the local time rate, while performing excursions away from it.
Let X be the distance from Z to the curve C, in the following sense. When Z is above the curve the distance is measured horizontally, and given a negative sign; otherwise, the distance is measured vertically and given a positive sign. Then, working on the filtration generated by Z, we prove in Section 2 that X is a solution to (1.1). This proves weak existence for the equation (1.1): weak because both the 'solution' X and the driving process B are constructed from the pair (W_1, W_2).
For those readers that are not familiar with the theory of multi armed bandits and multi parameter time changes, we provide another weak solution, which follows along the lines of the construction in [W1] of linearly skewed Brownian motion. We would like to thank the referee for pointing out that this construction can be carried easily to our situation.
In Section 3, we prove pathwise uniqueness of the solution to (1.1). We will show there that any two solutions yield a third solution whose local time at zero dominates the local times of both original solutions. This leads to pathwise uniqueness, using the fact that L^X corresponding to any solution to (1.1) must be the local time of a reflected Brownian motion. Weak existence, combined with pathwise uniqueness, establishes existence and uniqueness of a strong solution to (1.1), via the classical result of Yamada and Watanabe.
Recall from [RY], Chapter VI, the definition of the non-symmetric local times $\widehat{L}_t^a$ of a semimartingale X, and of the left limits (in a) $\widehat{L}_t^{a-}$. The symmetric local time of X at 0 is defined

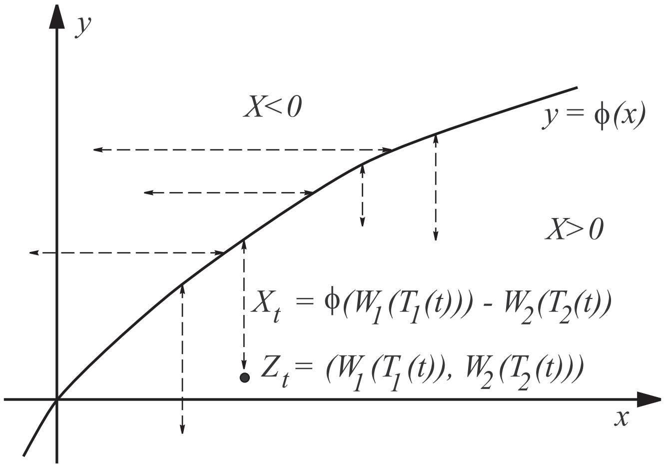

Figure 1: The process Z moves horizontally(vertically) above(below) the curve $y = \varphi(x)$.

by

$$L_t^X = \frac{1}{2}(\widehat{L}_t^0 + \widehat{L}_t^{0-}),$$

and satisfies the following version of Tanaka's formula:

$$|X_t| = |X_0| + \int_0^t \widetilde{\operatorname{sgn}}\,(X_s)dX_s + L_t^X,$$

where

$$\widetilde{\operatorname{sgn}}\,x = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0\ . \end{cases}$$

2 A Weak Solution

We begin with a lemma concerning the solutions to (1.1).

Lemma 2.1 *(a) Let (X, L) satisfy*

$$X_t = x_0 + B_t + \beta(L_t), \quad t \geq 0,$$

and suppose that L is a process of locally finite variation with $1_{(X_s \neq 0)} dL_s = 0$. Then $|X_t|$ is a reflecting Brownian motion.
(b) The processes $\widehat{L}^0$, $\widehat{L}^{0-}$ and L^X are related by:

$$\widehat{L}_t^0 = L_t^X + \beta(L_t), \quad \widehat{L}_t^{0-} = L_t^X - \beta(L_t).$$

Proof. (a) Apply the symmetric version of Tanaka's formula to X:

$$|X_t| = |x_0| + \int_0^t \widetilde{\operatorname{sgn}}\,(X_s)dX_s + L_t^X.$$

As $\widetilde{\operatorname{sgn}}(0) = 0$ and $1_{(X_s \neq 0)} dL_s = 0$,

$$\int_0^t \widetilde{\operatorname{sgn}}(X_s) dX_s = \int_0^t \widetilde{\operatorname{sgn}}(X_s) dB_s + \int_0^t \widetilde{\operatorname{sgn}}(X_s) \beta'(L_s) dL_s = \int_0^t \widetilde{\operatorname{sgn}}(X_s) dB_s.$$

Thus

$$|X_t| = |x_0| + \int_0^t \widetilde{\operatorname{sgn}}(X_s) dB_s + L_t^X. \tag{2.1}$$

Since X and B differ by a process of finite variation

$$\int_0^t 1_{(X_s=0)} d\langle B\rangle_s = \int_0^t 1_{(X_s=0)} d\langle X\rangle_s = 0,$$

so that the stochastic integral in (2.1) is a Brownian motion, V say. Hence (see [RY], Exercise VI.1.16), $L_t^{|X|} = -\inf_{s\leq t} V_s$, so we can write

$$|X_t| = |x_0| + V_t - \inf_{s\leq t} V_s.$$

This implies that $|X|$ is a reflected Brownian motion.
(b) By [RY], Theorem VI.1.7,

$$\widehat{L}_t^0 - \widehat{L}_t^{0-} = 2\int_0^t 1_{(X_s=0)} dX_s = 2\beta(L_t).$$

Since $2L_t^X = \widehat{L}_t^{0-} + \widehat{L}_t^0$, the second pair of equalities is clear. □

We now construct two weak solutions to (1.1). The first one is straight forward and is close in nature to Walsh's construction of (linearly) skewed Brownian motion in [W1], the second, as described in the introduction, arises from multi armed bandits. Since it is trivial to construct a solution to (1.1) up to the time of the first hit by X of 0, in what follows we take $x_0 = 0$.
Given β we wish to construct a function φ such that

$$\text{if } y = \varphi(u) + u \quad \text{ then } \beta(y) = \varphi(u) - u. \tag{2.2}$$

This is easy to do: let $y(u)$ be the unique real y such that $u = \frac{1}{2}(y - \beta(y)$ – unique since the function $y - \beta(y)$ is strictly increasing. Now define

$$\varphi(u) = \frac{1}{2}\big(y(u) + \beta(y(u))\big). \tag{2.3}$$

It is easy to verify that φ is increasing and is in C^1 if $\beta \in C^1$. With φ defined above, let

$$r_\beta(x,l) = \begin{cases} 2\varphi'(l)x & \text{if } x \geq 0 \\ 2x & \text{if } x < 0 \end{cases} \tag{2.4}$$

Let $(B_t, \mathcal{F}_t, P_x : x \in \Re)$ be a Brownian motion, and (L_t^0) its local time at 0. Let T_t be the time change associated with the increasing process

$$A_t = 4\int_0^t ((\varphi'(L_s^0))^2 1_{\{B_s\geq 0\}} + 1_{\{B_s<0\}}) ds.$$

That is

$$T_t = inf\{s : A_s > t\}$$

Note that by the continuity of $\varphi^{'}$, $A_t < \infty$ for $t < \infty$, and $A_\infty = \infty$, so that $0 < T_t < \infty$ for $t < \infty$.

Let

$$X_t^\beta = r_\beta(B_{T_t}, L_{T_t}^0). \tag{2.5}$$

Proposition 2.2 *X_t^β satisfies (1.1)*

Proof

By [RY] Proposition VI (4.3)

$$\begin{aligned} r_\beta(B_t, L_t^0) &= 2\int_0^t (\varphi^{'}(L_s^0)1_{\{B_s\geq 0\}} + 1_{\{B_s<0\}})dB_s + \int_0^t(\varphi^{'}(L_s^0) - 1)dL_s^0 \\ &= 2\int_0^t (\varphi^{'}(L_s^0)1_{\{B_s\geq 0\}} + 1_{\{B_s<0\}})dB_s + \varphi(L_t^0) - L_t^0. \end{aligned}$$

$$\begin{aligned} \mid r_\beta(B_t, L_t^0) \mid &= 2\int_0^t (\varphi^{'}(L_s^0)1_{\{B_s\geq 0\}} - 1_{\{B_s<0\}})dB_s + \int_0^t(\varphi^{'}(L_s^0) + 1)dL_s^0 \\ &= 2\int_0^t (\varphi^{'}(L_s^0)1_{\{B_s\geq 0\}} - 1_{\{B_s<0\}})dB_s + \varphi(L_t^0) + L_t^0. \end{aligned}$$

Note that the square variation of the semimartingale $r_\beta(B_t, L_t^0)$ is equal to A_t. Thus the square variation of $r_\beta(B_{T_t}, L_{T_t}^0)$ is $A_{T_t} = t$. It follows from from the above equations that

$$W_t = 2\int_0^{T_t} (\varphi^{'}(L_s^0)1_{\{B_s\geq 0\}} + 1_{\{B_s<0\}})dB_s$$

is a Brownian motion, that $L_t = \varphi(L_{T_t}^0) + L_{T_t}^0$ is carried by $\{t : r_\beta(B_{T_t}, L_{T_t}^0) = 0\}$, and is the symmetric local time of $r_\beta(B_{T_t}, L_{T_t}^0)$ at 0, and that by (2.2)

$$r_\beta(B_{T_t}, L_{T_t}^0) = W_t + \beta(L_t)$$

as required. □

Our second weak solution originates from the theory of multi armed bandits, and since this was our motivation to study the problem we shall describe it briefly. The framework is that of general multi-armed bandits, but here we introduce only concepts that are directly relevant to our problem. The complete set-up can be found in [KM1],[KM2], or [M].

Let $(W_1(t))$ and $(W_2(t))$ be two independent Brownian motions started at 0. Let $(\mathcal{F}_t^1)$ and $(\mathcal{F}_t^2)$ be their respective filtrations, completed and right-continuous as usual. Set $S = \mathbb{R}_+^2$, and introduce the multi-parameter filtration $(\mathcal{F}_s)$, given by

$$\mathcal{F}_s = \mathcal{F}_{s_1}^1 \vee \mathcal{F}_{s_2}^2, \quad s = (s_1, s_2) \in S. \tag{2.6}$$

An S-valued stochastic process $T(t) = (T_1(t), T_2(t))$, $t \geq 0$, is called a *strategy* if it has the following properties: $T(0) = 0$, $T_1(t)$ and $T_2(t)$ are nondecreasing in $t \geq 0$,

$$T_1(t) + T_2(t) = t, \quad t \geq 0, \tag{2.7}$$

and

$$\{T_1(t) \leq s_1,\ T_2(t) \leq s_2\} \in \mathcal{F}_s\ , \forall s = (s_1, s_2) \in S. \tag{2.8}$$

We look for a strategy $T(t) = (T_1(t), T_2(t))$ that does the following:
$T_1(t)$ increases at rate 1 if $\varphi(W_1(T_1(t)) < W_2(T_2(t))$, and
$T_2(t)$ increases at rate 1 if $\varphi(W_1(T_1(t))) > W_2(T_2(t))$.
The existence of such a strategy follows for example from [M]. We also have from there that this strategy is unique – but we will not need this. Here is an outline of the construction. Define

$$D = \{(s_1, s_2) \in S : \varphi(\sup_{u_1 \le s_1} W_1(s_1)) > \sup_{u_2 \le s_2} W_2(s_2)\}.$$

It is clear that the closure $\overline{D}$ of D has the following three properties:

(i) $\{(s_1, 0) : s_1 \ge 0\} \in \overline{D}$,

(ii) $(s_1, s_2) \in \overline{D} \Rightarrow \{(u_1, u_2) : u_1 \ge s_1,\ 0 \le u_2 \le s_2\} \in \overline{D}$,

(iii) $\{s \in \overline{D}\} \in \mathcal{F}_s$.

By Theorem 2.7 of [W2], the northwest boundary of $\overline{D}$ can be parametrized as a strategy $T = (T_1, T_2)$, with respect to the filtration $(\mathcal{F}_s)$, which is the one we are seeking: T_1 increases at rate 1 when

$$W_2(T_2(t)) > \varphi(W_1(T_1(t)) \ ,$$

and T_2 increases at rate 1 when $W_2(T_2(t)) < \varphi(W_1(T_1(t))$. (In the language of [M], such a strategy *follows the leader between* $\varphi(W_1)$ *and* W_2).
With the strategy T as above, define $\mathcal{G}_t = \mathcal{F}_{T(t)}$, and let

$$Z_1(t) = W_1(T_1(t)) \ , \quad Z_2(t) = W_2(T_2(t)), \quad t \ge 0, \tag{2.9}$$

$$B_t = Z_1(t) - Z_2(t), \quad t \ge 0. \tag{2.10}$$

It is clear that Z_i and $(Z_i)^2 - T_i$ are continous $(\mathcal{G}_t)$ martingales, so that $\langle Z_i \rangle_t = T_i(t)$. Therefore (B_t) is a $(\mathcal{G}_t)$ Brownian motion, since it is a continuous martingale with quadratic variation $\langle B \rangle_t = T_1(t) + T_2(t) = t$.
Write $Z_i^+(t) = \sup_{s \le t} Z_i(s)$. As $T(t)$ is on the boundary of D we must have

$$\varphi(Z_1^+(t)) = Z_2^+(t).$$

Write $U_t = Z_1^+(t)$; if Z_t is not on the curve $C = \{(x, y) : y = \varphi(x)\}$, then it will return to C at the point $(U_t, \varphi(U_t))$. So

$$U_t = Z_1(t) \vee \varphi^{-1}(Z_2(t)). \tag{2.11}$$

Note that U is constant on each excursion of Z away from the curve C, and that U is increasing. The signed horizontal/vertical distance of (Z_1, Z_2) from the curve C is given by

$$X_t = (\varphi(U_t) - Z_2(t)) - (U_t - Z_1(t)). \tag{2.12}$$

and $X_t|$ the horizontal/vertical distance of (Z_1, Z_2) from C is given by

$$|X_t| = \varphi(U_t) - Z_2(t) + U(t) - Z_1(t). \tag{2.13}$$

Define

$$L_t = \varphi(U_t) + U_t. \tag{2.14}$$

Note that all these processes are semimartingales (with respect to $(\mathcal{G}_t)$), and that L is increasing (since U is).

Theorem 2.3 *The pair (X, L), given by (2.12)–(2.14) solves equation (1.1):*

$$X_t = B_t + \beta(L_t), \quad t \geq 0, \tag{2.15}$$

and (L_t) is the symmetric local time of (X_t) at 0.

Proof. Using (2.2) we have from (2.14) that

$$\beta(L_t) = \varphi(U_t) - U_t.$$

So by (2.10)

$$B_t + \beta(L_t) = Z_1(t) - Z_2(t) + \varphi(U_t) - U_t = X_t,$$

proving that the pair (X, L) satisfies the equation (2.15).
Further, arguing as for B_t above, $\tilde{B}_t = -W_1(T_1(t)) - W_2(T_2(t))$ is a Brownian motion with respect to $(\mathcal{G}_t)$ and

$$|X_t| = \tilde{B}_t + L_t. \tag{2.16}$$

Our result will follow from the Tanaka formula once we note that

$$\tilde{B}_t = -(Z_1(t) + Z_2(t)) = -\int_0^t \widetilde{\operatorname{sgn}}\,(X_s) dB_s = -\int_0^t \widetilde{\operatorname{sgn}}\,(X_s) dX_s. \tag{2.17}$$

□

Remark 2.4 Note that for the second weak solution it is enough to require that β is differentiable, rather than C^1 as is required for the first weak solution.

3 Strong Uniqueness

Theorem 3.1 *There exists a unique strong solution to (1.1).*

The proof of Theorem 3.1 uses the following lemma.

Lemma 3.2 *Let X be a continuous semimartingale, let A be of integrable variation and let $Y = X + A$. Then*

$$1_{(A_s=0)} dL_s^X = 1_{(A_s=0)} dL_s^Y.$$

Remark 3.3 *The statement above with random measures is equivalent to saying that for any bounded previsible H*

$$\int_0^t H_s 1_{(A_s=0)} dL_s^X = \int_0^t H_s 1_{(A_s=0)} dL_s^Y.$$

Proof of Lemma 3.2. Suppose first that $A_t \geq 0$ for all t. Then $Y \vee X = Y$, and so, since $Y - X = A$ has zero local time, by Exercise (1.21)(c) of [RY]

$$d\widehat{L}_s^0(Y) = 1_{(X_s<0)} d\widehat{L}_s^0(Y) + 1_{(Y_s \leq 0)} d\widehat{L}_s^0(X).$$

Hence since $A_s = 0$ and $Y_s = 0$ implies $X_s = 0$, we have

$$1_{(A_s=0)} d\widehat{L}_s^0(Y) = 1_{(A_s=0)} d\widehat{L}_s^0(X).$$

If A is not non-negative, let $A = A^+ - A^-$, where $A^+ = A \vee 0$, $A^- = (-A) \vee 0$. Then set $Z = Y + A^- = X + A^+$. By the calculation above, and as $A_s = 0$ implies $A_s^{\pm} = 0$,

$$1_{(A_s=0)} d\widehat{L}_s^0(Z) = 1_{(A_s=0)} d\widehat{L}_s^0(X) = 1_{(A_s=0)} dL_s^0(Y).$$

Thus the lemma holds for the non-symmetric local times $\widehat{L}^0$. Using the identity $\widehat{L}^{-0}(X) = \widehat{L}^0(-X)$ it also holds for the left local times $\widehat{L}^{0-}$, and so, by addition, for the symmetric local times L^X and L^Y. □

Proof of Theorem 3.1. To prove pathwise uniqueness, assume that there exist two strong solutions (X_t^1) and (X_t^2) to (1.1), and let (L_t^1), (L_t^2) be their respective symmetric continuous local times. Define a new process (Y_t) by

$$Y_t = x_0 + B_t + \beta(L_t^1 \vee L_t^2) = \begin{cases} X_t^1 & \text{if} \quad L_t^1 > L_t^2, \\ X_t^2 & \text{if} \quad L_t^1 \leq L_t^2. \end{cases}$$

We shall prove that $(Y, L^1 \vee L^2)$ solves (1.1). To do this, we need to show that $L^1 \vee L^2$ is the symmetric local time at 0 of Y. First note that since $t \to L_t^1 \vee L_t^2$ is continuous, so is $t \to Y_t$. Thus if Y switches from X^1 to X^2, this can happen only at a point of increase of L_t^2, that is, when $X_t^2 = 0$. Since at the time t of such a switch $L_t^1 = L_t^2$, X_t^1 must be equal to 0 as well, and thus $Y_t = 0$. We can write $Y_t = X_t^1 + A_t$, where

$$A_t = (\beta(L_t^2) - \beta(L_t^1)) 1_{(L_t^2 > L_t^1)}.$$

Note that A is of integrable variation. Therefore by Lemma 3.2

$$1_{(A_s=0)} dL_s^Y = 1_{(A_s=0)} dL_s^1.$$

Since $L_s^2 \leq L_s^1$ implies $A_s = 0$, it follows that

$$1_{(L_s^2 \leq L_s^1)} dL_s^Y = 1_{(L_s^2 \leq L_s^1)} dL_s^1$$

Interchanging X^1 and X^2, and multiplying by the previsible process $1_{(L_s^2 > L_s^1)}$ we deduce

$$1_{(L_s^2 > L_s^1)} dL_s^Y = 1_{(L_s^2 > L_s^1)} dL_s^2.$$

Hence,

$$\begin{aligned} L_t^Y &= \int_0^t 1_{(L_s^2 \leq L_s^1)} dL_s^Y + \int_0^t 1_{(L_s^2 > L_s^1)} dL_s^Y \\ &= \int_0^t 1_{(L_s^2 \leq L_s^1)} dL_s^1 + \int_0^t 1_{(L_s^2 > L_s^1)} dL_s^2 = L_t^1 \vee L_t^2. \end{aligned}$$

Since X^1, X^2 and Y all satisfy (1.1), using Lemma 2.1 we deduce that $|X^1|$, $|X^2|$ and $|Y|$ are all reflected Brownian motions. Thus $L_t^Y = L_t^1 \vee L_t^2$, L_t^1 and L_t^2 are all local times of reflected Brownian motions, and so all have the same distribution. So $EL_t^1 = EL_t^2 = EL_t^1 \vee L_t^2$, which implies that $L_t^1 = L_t^2$ for all t, a.s. This in turn implies that $X_t^1 = X_t^2$ for all t, so that pathwise uniqueness holds for (1.1).

Pathwise uniqueness, together with the existence of a weak solution now implies, using the Yamada-Watanabe Theorem, the existence of a unique strong solution, that is a solution adapted to the filtration of the driving Brownian motion B. (Although stated for an ordinary SDE, their argument, as it appears in [RY] IX 1.7, for example, carries over to our situation with almost no changes). □

Remark 3.4 Uniqueness in law of the solution to (1.1) is also part of the Yamada- Watanabe result. It may also be derived independently, as in [We], by noting that if (X_t) solves (1.1) and $g_t(\lambda) = E(e^{i\lambda X_t})$, then $g_t(\lambda)$ satisfies

$$g_t(\lambda) = e^{i\lambda x_0} - \frac{\lambda^2}{2}\int g_s(\lambda)ds + i\lambda h(t),$$

where $h(t) = E(\beta(L_t))$ and (L_t) is the symmetric local time of reflected Brownian motion started at x_0.

Remark 3.5 While Theorem 3.1 proves that the solution X of (1.1) is $\mathcal{F}^B$ adapted, and so a functional of the driving Brownian motion B, it does not give any procedure for the construction of X from B. We may compare this with the case of reflecting Brownian motion (i.e. $\beta(x) = x$), where (if $x_0 = 0$) then $X_t = B_t - \inf_{s\le t} B_s$.
However, it is a general principle in the theory of stochastic differential equations that if pathwise uniqueness holds, then any 'reasonable' approximation scheme will converge in probability to the solution X. See Jacod and Memin [JM], Kurtz and Protter [KP]. The argument outlined below comes from Lemma 5.5 of [KP].
Suppose $X_t^n = F_n(B,t)$ are (adapted) functionals of B such that $\{(X^n, B), n \ge 1\}$ is tight in the Skorohod topology, and any weak limit point (X', B) gives a solution X' to (1.1). Then this also applies to $\{(X^n, B, X^m, B), n \ge 1, m \ge 1\}$. If (X', B, X'', B) is a weak limit point, then as X' and X'' are both solutions of (1.1), we have $X' = X''$. Hence $X^n - X^m$ converges in law to 0, and so (X^n) is a Cauchy sequence in probability. Thus (X^n) converges in probability to a solution of (1.1).

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